Journal of Algebra 546 (2020) 119–134

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Maximal τ_d-rigid pairs

Karin M. Jacobsen [a,*,1], Peter Jørgensen [b]

[a] *Norwegian University of Science and Technology, Department of Mathematical Sciences, Sentralbygg 2, Gløshaugen, 7491 Trondheim, Norway*
[b] *School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne NE1 7RU, United Kingdom*

A R T I C L E   I N F O

Article history:
Received 30 April 2019
Available online 14 November 2019
Communicated by David Hernandez

MSC:
16G10
18E10
18E30

Keywords:
d-Abelian category
$(d+2)$-Angulated category
Higher homological algebra
Maximal d-rigid object
Maximal τ_d-rigid pair

A B S T R A C T

Let $\mathscr{T}$ be a 2-Calabi–Yau triangulated category, T a cluster tilting object with endomorphism algebra Γ. Consider the functor $\mathscr{T}(T,-) : \mathscr{T} \to \operatorname{mod}\Gamma$. It induces a bijection from the isomorphism classes of cluster tilting objects to the isomorphism classes of support τ-tilting pairs. This is due to Adachi, Iyama, and Reiten.

The notion of $(d+2)$-angulated categories is a higher analogue of triangulated categories. We show a higher analogue of the above result, based on the notion of maximal τ_d-rigid pairs.

0. Introduction

In triangulated categories, the notions of *cluster tilting objects* (introduced in [4, p. 583]) and *maximal rigid objects* have recently been extensively investigated. They

* Corresponding author.
E-mail addresses: kjacobsen@math.uni-bielefeld.de (K.M. Jacobsen), peter.jorgensen@ncl.ac.uk (P. Jørgensen).
URL: http://www.staff.ncl.ac.uk/peter.jorgensen (P. Jørgensen).
[1] Current address: Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany.

https://doi.org/10.1016/j.jalgebra.2019.10.046
0021-8693/© 2019 Published by Elsevier Inc.

frequently coincide, by [22, thm. 2.6], and they are closely linked to the notion of *support τ-tilting pairs* in abelian categories (introduced in [1, def. 0.3]). Indeed, there is often a bijection between the cluster tilting objects in a triangulated category and the support τ-tilting pairs in a suitable (abelian) module category, see [1, thm. 4.1].

This paper investigates the analogous theory in $(d+2)$-angulated and d-abelian categories, which are the main objects of higher homological algebra, see [8, def. 2.1] and [15, def. 3.1]. Several key properties from the classic case do not carry over. For example, cluster tilting objects are maximal d-rigid, but the converse is rarely true. Moreover, the higher analogue of support τ-rigid pairs permit a bijection to the maximal d-rigid objects, but not to the cluster tilting objects.

For further reading in higher homological algebra a number of references have been included in the bibliography, see [3], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].

Let k be an algebraically closed field, $d \geqslant 1$ an integer, $\mathscr{T}$ a k-linear Hom-finite $(d+2)$-angulated category with split idempotents, see [8, def. 2.1]. Assume that $\mathscr{T}$ is $2d$-Calabi–Yau, see [21, def. 5.2], and let Σ^d denote the d-suspension functor of $\mathscr{T}$.

Cluster tilting and maximal d-rigid objects. An object $X \in \mathscr{T}$ is *d-rigid* if $\operatorname{Ext}^d_{\mathscr{T}}(X, X) = 0$. We recall three important definitions.

Definition 0.1 *([21, def. 5.3]).* An object $X \in \mathscr{T}$ is *Oppermann–Thomas cluster tilting in $\mathscr{T}$* if:

(i) X is d-rigid.
(ii) For any $Y \in \mathscr{T}$ there exists a $(d+2)$-angle

$$X_d \to \cdots \to X_0 \to Y \to \Sigma^d X_d$$

with $X_i \in \operatorname{add} X$ for all $0 \le i \le d$.

Definition 0.2. An object $X \in \mathscr{T}$ is *d-self-perpendicular in $\mathscr{T}$* if

$$\operatorname{add} X = \{\, Y \in \mathscr{T} \mid \operatorname{Ext}^d_{\mathscr{T}}(X, Y) = 0 \,\}.$$

Definition 0.3. An object $X \in \mathscr{T}$ is *maximal d-rigid in $\mathscr{T}$* if

$$\operatorname{add} X = \{\, Y \in \mathscr{T} \mid \operatorname{Ext}^d_{\mathscr{T}}(X \oplus Y, X \oplus Y) = 0 \,\}.$$

Our first main result is:

Theorem A. *X is Oppermann–Thomas cluster tilting $\Rightarrow$ X is d-self-perpendicular $\Rightarrow$ X is maximal d-rigid.*

We prove this in Theorem 1.1. Of equal importance is that the implications cannot be reversed in general, see Remark 1.2. In particular, when $d \geqslant 2$, the class of maximal d-rigid objects is typically strictly larger than the class of Oppermann–Thomas cluster tilting objects, in contrast to the classic case $d = 1$ where the two classes usually coincide, see [22, thm. 2.6].

Maximal τ_d-rigid pairs. Let $T \in \mathscr{T}$ be an Oppermann–Thomas cluster tilting object and let $\Gamma = \operatorname{End}_{\mathscr{T}}(T)$. Recall the following result.

Theorem 0.4 *([14, thm. 0.6]). Consider the essential image $\mathscr{D}$ of the functor $\mathscr{T}(T,-) : \mathscr{T} \to \operatorname{mod}\Gamma$. Then $\mathscr{D}$ is a d-cluster tilting subcategory of $\operatorname{mod}\Gamma$. There is a commutative diagram, as shown below, where the vertical arrow is the quotient functor and the diagonal arrow is an equivalence of categories:*

$$\begin{array}{ccc} \mathscr{T} & \xrightarrow{\mathscr{T}(T,-)} & \mathscr{D} \\ {\scriptstyle \overline{(-)}}\downarrow & \nearrow{\scriptstyle \sim} & \\ \mathscr{T}/\operatorname{add}\Sigma^d T. & & \end{array}$$

The category $\mathscr{D}$ is a d-abelian category by [15, thm. 3.16]. It has a d-Auslander–Reiten translation τ_d, which is a higher analogue of the classic Auslander–Reiten translation τ, see [12, sec. 1.4.1]. A module $M \in \mathscr{D}$ is called *τ_d-rigid* if $\operatorname{Hom}_\Gamma(M, \tau_d M) = 0$.

Remark 0.5. The classic add-proj-correspondence holds, as $\mathscr{T}(T,-)$ restricts to an equivalence $\operatorname{add} T \to \operatorname{proj}\Gamma$. The functor also restricts to an equivalence $\operatorname{add} ST \to \operatorname{inj}\Gamma$. [14, lem. 2.1]

It is natural to ask if $\mathscr{D}$ permits a higher analogue of the τ-tilting theory of [1]. We will not answer this question, but will instead introduce the following definitions inspired by it.

Definition 0.6. A pair (M, P) with $M \in \mathscr{D}$ and $P \in \operatorname{proj}\Gamma$ is called a *τ_d-rigid pair in $\mathscr{D}$* if M is τ_d-rigid and $\operatorname{Hom}_\Gamma(P, M) = 0$.

Definition 0.7. A pair (M, P) with $M \in \mathscr{D}$ and $P \in \operatorname{proj}\Gamma$ is called a *maximal τ_d-rigid pair in $\mathscr{D}$* if it satisfies:

(i) If $N \in \mathscr{D}$ then

$$N \in \operatorname{add} M \Leftrightarrow \begin{cases} \operatorname{Hom}_\Gamma(M, \tau_d N) = 0, \\ \operatorname{Hom}_\Gamma(N, \tau_d M) = 0, \\ \operatorname{Hom}_\Gamma(P, N) = 0. \end{cases}$$

(ii) If $Q \in \operatorname{proj} \Gamma$, then

$$Q \in \operatorname{add} P \Leftrightarrow \operatorname{Hom}_\Gamma(Q, M) = 0.$$

A maximal τ_d-rigid pair is a τ_d-rigid pair.

Our second main result is:

Theorem B. *If each indecomposable object of $\mathscr{T}$ is d-rigid, then there is a bijection*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{maximal d-rigid objects in } \mathscr{T} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{maximal } \tau_d\textit{-rigid pairs in } \mathscr{D} \end{array} \right\}.$$

We prove this in Section 3. If $d = 1$, then (M, P) is a maximal τ_1-rigid pair if and only if it is a support τ-tilting pair in the sense of [1, def. 0.3(b)], see [1, def. 0.3, prop. 2.3, and cor. 2.13]. Hence Theorem B is a higher analogue of the bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{cluster tilting object in } \mathscr{T} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{support } \tau\text{-tilting pairs in } \operatorname{mod} \Gamma \end{array} \right\}$$

which exists by [1, thm. 4.1] when $\mathscr{T}$ is triangulated, i.e. in the case $d = 1$. However, when $d \geqslant 2$, we do not think of maximal τ_d-rigid pairs as support τ_d-tilting pairs. The reason is that by Theorem B, maximal τ_d-rigid pairs are linked to maximal d-rigid objects in higher angulated categories. As remarked above, this class is typically strictly larger than the class of Oppermann–Thomas cluster tilting objects when $d \geqslant 2$.

Note that [19] makes an approach to higher support tilting theory.

This paper is organised as follows: Section 1 proves Theorem A, Section 2 investigates the precise relation between Hom spaces in $\mathscr{T}$ and $\mathscr{D}$, Section 3 proves Theorem B, and Section 4 gives an example.

Setup 0.8. Throughout the paper we use the following notation:

- k: An algebraically closed field.
- D: The duality functor $\operatorname{Hom}_k(-, k)$.
- $\mathscr{T}$: A k-linear, Hom-finite, $(d+2)$-angulated category with split idempotents. We assume that $\mathscr{T}$ is $2d$-Calabi–Yau, that is $\mathscr{T}(X, Y) \cong \mathrm{D}\mathscr{T}(Y, \Sigma^{2d} X)$ naturally in $X, Y \in \mathscr{T}$.

Σ^d: The d-suspension functor on $\mathscr{T}$.
T: An Oppermann–Thomas cluster tilting object in $\mathscr{T}$.
$\overline{(-)}$: The canonical functor $\mathscr{T} \to \mathscr{T} / \operatorname{add} \Sigma^d T$, whose target is the naive quotient category of $\mathscr{T}$ modulo the morphisms which factor through an object in $\operatorname{add} \Sigma^d T$.
Γ: The endomorphism ring $\operatorname{End}_{\mathscr{T}}(T)$.
ν_Γ: The Nakayama functor on $\operatorname{mod} \Gamma$.
τ_d: The d-Auslander–Reiten translation on $\operatorname{mod} \Gamma$.
$\mathscr{D}$: The essential image of the functor $\mathscr{T}(T, -) : \mathscr{T} \to \operatorname{mod} \Gamma$.

1. Proof of Theorem A

Theorem 1.1. *Let $X \in \mathscr{T}$ be given.*

(i) There are implications

$$\begin{gathered} X \text{ is Oppermann–Thomas cluster tilting} \\ \Downarrow \\ X \text{ is } d\text{-self-perpendicular} \\ \Downarrow \\ X \text{ is maximal } d\text{-rigid} \\ \Downarrow \\ X \text{ is } d\text{-rigid.} \end{gathered}$$

(ii) If each indecomposable object in $\mathscr{T}$ is d-rigid, then

$$X \text{ is } d\text{-self-perpendicular} \Leftrightarrow X \text{ is maximal } d\text{-rigid.}$$

Proof. (i), the first implication: Suppose X is Oppermann–Thomas cluster tilting. We must prove the equality in Definition 0.2, and the inclusion $\subseteq$ is clear. For the inclusion $\supseteq$, suppose $\operatorname{Ext}^d_{\mathscr{T}}(X, Y) = 0$. Then each morphism $X_0 \to \Sigma^d Y$ with $X_0 \in \operatorname{add} X$ is zero. This applies in particular to the $(d+2)$-angle $X_d \to \cdots \to X_0 \to \Sigma^d Y \to \Sigma^d X_d$ with $X_i \in \operatorname{add} X$, which exists since X is Oppermann–Thomas cluster tilting. But then the morphism $\Sigma^d Y \to \Sigma^d X_d$ is a split monomorphism, and applying Σ^{-d} gives a split monomorphism $Y \to X_d$ proving $Y \in \operatorname{add} X$.

(i), the second implication: Suppose that X is d-self-perpendicular. We must prove the equality in Definition 0.3, and the inclusion $\subseteq$ is clear. For the inclusion $\supseteq$, suppose $\operatorname{Ext}^d_{\mathscr{T}}(X \oplus Y, X \oplus Y) = 0$. Then in particular, $\operatorname{Ext}^d_{\mathscr{T}}(X, Y) = 0$, whence $Y \in \operatorname{add} X$.

(i), the third implication: This is clear.

(ii): Suppose that each indecomposable object in $\mathscr{T}$ is d-rigid. Because of part (i), it is enough to prove the implication $\Leftarrow$ in (ii), so suppose that X is maximal d-rigid. We must prove the equality in Definition 0.2, and $\subseteq$ is clear.

For the inclusion $\supseteq$, observe that $\{ Y \in \mathscr{T} \mid \operatorname{Ext}^d_{\mathscr{T}}(X, Y) = 0 \}$ is closed under direct sums and summands by additivity of Ext. Hence it is enough to suppose that Y is an indecomposable object in this set and prove $Y \in \operatorname{add} X$. However, $\operatorname{Ext}^d_{\mathscr{T}}(X,Y) = 0$ implies $\operatorname{Ext}^d_{\mathscr{T}}(Y,X) = 0$ because $\mathscr{T}$ is $2d$-Calabi–Yau, and $\operatorname{Ext}^d_{\mathscr{T}}(Y,Y) = 0$ by assumption. Finally, X is d-rigid by part (i), so $\operatorname{Ext}^d_{\mathscr{T}}(X,X) = 0$. Combining these equalities shows $\operatorname{Ext}^d_{\mathscr{T}}(X \oplus Y, X \oplus Y) = 0$, and $Y \in \operatorname{add} X$ follows. □

Remark 1.2. The implications in Theorem 1.1(i) cannot be reversed in general:

- An example of a d-self-perpendicular object X which is not Oppermann–Thomas cluster tilting is given in Section 4. In fact, the objects in the last three rows of Fig. 4 are such examples. The example was originally given in [21, p. 1735].
- An example of a maximal d-rigid object which is not d-self-perpendicular can be obtained by combining proposition 2.6 and corollary 2.7 in [5]. These results give a maximal 1-rigid object which is not cluster tilting, but in the triangulated setting of [5], cluster tilting is equivalent to 1-self-perpendicular, see [5, bottom of p. 963].
- Finally, an example of a d-rigid object which is not maximal d-rigid is the zero object, as soon as $\mathscr{T}$ has a non-zero d-rigid object.

We end the section by observing that Theorem 1.1(ii) can be applied to an important class of categories.

Proposition 1.3. *Let Λ be a d-representation finite algebra, $\mathscr{O}_\Lambda$ the $(d+2)$-angulated cluster category associated to Λ in [21, thm. 5.2]. Then each $X \in \mathscr{O}_\Lambda$ satisfies*

$$X \text{ is } d\text{-self-perpendicular} \Leftrightarrow X \text{ is maximal } d\text{-rigid.}$$

Proof. Each indecomposable in $\mathscr{O}_\Lambda$ is d-rigid by [21, Lemma 5.41], so the equivalence follows from Theorem 1.1(ii). □

2. A dimension formula for $\operatorname{Ext}^d_{\mathscr{T}}$

Recall from Setup 0.8 that T is a fixed Oppermann–Thomas cluster tilting object in $\mathscr{T}$, and that $\mathscr{T}$ is $2d$-Calabi–Yau, that is, $\mathscr{T}(X,Y) \cong \mathrm{D}\mathscr{T}(Y, \Sigma^{2d}X)$ naturally in $X, Y \in \mathscr{T}$.

Lemma 2.1. *There is a natural isomorphism*

$$\nu_\Gamma \mathscr{T}(T,T') \cong \mathscr{T}\big(T, \Sigma^{2d}(T')\big)$$

for $T' \in \operatorname{add} T$.

Proof. By the $2d$-Calabi-Yau property we have

$$\mathscr{T}\big(T, \Sigma^{2d}(T')\big) \cong \mathrm{D}\mathscr{T}(T',T).$$

By [14, Lemma 2.2(i)],

$$\mathrm{D}\mathscr{T}(T',T) \cong \mathrm{D}\mathrm{Hom}_\Gamma\left(\mathscr{T}(T,T'),\mathscr{T}(T,T)\right) = \mathrm{D}\mathrm{Hom}_\Gamma\left(\mathscr{T}(T,T'),\Gamma\right).$$

Finally, by definition we have

$$\mathrm{D}\mathrm{Hom}_\Gamma\left(\mathscr{T}(T,T'),\Gamma\right) = \nu_\Gamma\mathscr{T}(T,T'),$$

see [2, def. III.2.8]. □

Lemma 2.2. *If $X \in \mathscr{T}$ has no non-zero direct summands in* $\operatorname{add}\Sigma^d T$*, then there exists a $(d+2)$-angle*

$$T_d \to \cdots \to T_0 \to X \to \Sigma^d T_d$$

in $\mathscr{T}$ with the following properties: Each T_i is in $\operatorname{add} T$*, and applying the functor $\mathscr{T}(T,-)$ gives a complex*

$$\mathscr{T}(T,T_d) \to \cdots \to \mathscr{T}(T,T_0) \to \mathscr{T}(T,X) \to 0$$

which is the start of the augmented minimal projective resolution of $\mathscr{T}(T,X)$.

Proof. Given X, there exists a $(d+2)$-angle

$$\Sigma^{-d}X \to T_d \to \cdots \to T_0 \to X$$

with each T_i in $\operatorname{add} T$ by Definition 0.1. Since X has no non-zero direct summands in $\operatorname{add}\Sigma^d T$, the first morphism in the $(d+2)$-angle is in the radical of $\mathscr{T}$. By dropping trivial summands of the form $T' \xrightarrow{\cong} T'$, we can assume that so are the other morphisms except the last morphism.

By [8, prop. 2.5(a)], applying the functor $\mathscr{T}(T,-)$ gives an exact sequence

$$\mathscr{T}(T,\Sigma^{-d}X) \to \mathscr{T}(T,T_d) \to \cdots \to \mathscr{T}(T,T_0) \to \mathscr{T}(T,X) \to \mathscr{T}(T,\Sigma^d T_d) = 0.$$

By Theorem 0.4, applying the functor $\mathscr{T}(T,-)$ is, up to isomorphism, just to apply a quotient functor, and this preserves radical morphisms. So in the exact sequence each morphism, except possibly $\mathscr{T}(T,T_0) \to \mathscr{T}(T,X)$, is in the radical of $\operatorname{mod}\Gamma$. This proves the claim of the lemma. □

Lemma 2.3. *If $X \in \mathscr{T}$ has no non-zero direct summands in* $\operatorname{add}\Sigma^d T$*, then there is a natural isomorphism*

$$\tau_d\mathscr{T}(T,X) \cong \mathscr{T}(T,\Sigma^d X).$$

Proof. As X has no non-zero direct summands in $\operatorname{add}\Sigma^d T$, we can consider the $(d+2)$-angle from Lemma 2.2. Apply $\mathscr{T}(T,-)$ to get the following part of an augmented minimal projective resolution in $\operatorname{mod}\Gamma$:

$$\mathscr{T}(T,T_d) \to \cdots \to \mathscr{T}(T,T_0) \to \mathscr{T}(T,X) \to 0.$$

Using the Nakayama functor and Lemma 2.1 we get the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_d\mathscr{T}(T,X) & \longrightarrow & \nu_\Gamma\mathscr{T}(T,T_d) & \longrightarrow & \cdots & \longrightarrow & \nu_\Gamma\mathscr{T}(T,T_0) \\
 & & & & \downarrow \wr & & & & \downarrow \wr \\
0 & \longrightarrow & \mathscr{T}(T,\Sigma^d X) & \longrightarrow & \mathscr{T}(T,\Sigma^{2d}T_d) & \longrightarrow & \cdots & \longrightarrow & \mathscr{T}(T,\Sigma^{2d}T_0)
\end{array}$$

The top sequence is exact by the definition of τ_d, see [12, sec. 1.4.1]. The bottom sequence is exact because it is obtained by applying $\operatorname{Hom}_{\mathscr{T}}(T,-)$ to a $(d+2)$-angle in $\mathscr{T}$, see [8, prop. 2.5(a)]. The first term of the bottom sequence is actually $\mathscr{T}(T,\Sigma^d T_0)$, but this is zero. Since we have $d \geq 1$, the diagram implies

$$\tau_d\mathscr{T}(T,X) \cong \mathscr{T}(T,\Sigma^d X). \quad \square$$

We write $[\operatorname{add}T](X,Y) = \{\, f \in \mathscr{T}(X,Y) \mid f \text{ factors through an object of } \operatorname{add}T \,\}$.

Lemma 2.4. *There is a natural isomorphism*

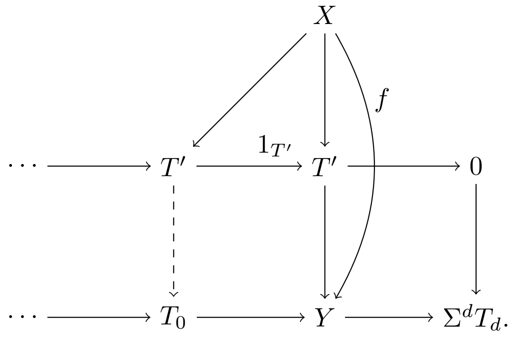
$$\mathrm{D}[\operatorname{add}T](X,Y) \cong \operatorname{Hom}_{\mathscr{T}/\operatorname{add}\Sigma^d T}(\overline{Y}, \overline{\Sigma^{2d}X})$$

for $X, Y \in \mathscr{T}$.

Proof. Pick a $(d+2)$-angle in $\mathscr{T}$:

$$T_d \to \ldots \to T_0 \to Y \to \Sigma^d T_d,$$

with $T_i \in \operatorname{add}T$. Use $\mathscr{T}(X,-)$ to obtain the morphism $\Psi : \mathscr{T}(X,T_0) \to \mathscr{T}(X,Y)$. This is a homomorphism of k-vector spaces, hence we can talk about the image of Ψ. We first note that any morphism f in the image of Ψ must factor through $\operatorname{add}T$. Now suppose $f \in \mathscr{T}(X,Y)$ factors through $T' \in \operatorname{add}T$. We have the following commutative diagram, where the lower row is a part of the $(d+2)$-angle above:



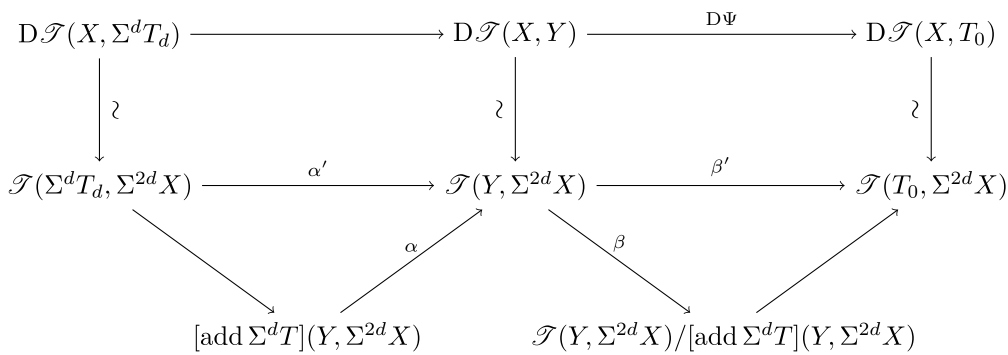
The dashed arrow exists by completing the commutative square to a morphism of $(d+2)$-angles. We conclude that $f \in \operatorname{Im}\Psi$. Hence

$$\operatorname{Im}\Psi = [\operatorname{add} T](X, Y).$$

We now return to the long exact sequence

$$\cdots \to \mathscr{T}(X, T_0) \xrightarrow{\Psi} \mathscr{T}(X, Y) \to \mathscr{T}(X, \Sigma^d T_d) \to \cdots .$$

Using the duality functor D and Serre duality we get the following diagram with exact rows:



Analogous to the above discussion, the space $[\operatorname{add} \Sigma^d T](Y, \Sigma^{2d} X)$ is the image of the map α'. Hence α is the kernel of β' and $\mathrm{D}\Psi$ (by isomorphism). The morphism β is by definition the cokernel of α, and $\mathscr{T}(Y, \Sigma^{2d} X)/[\operatorname{add} \Sigma^d T](Y, \Sigma^{2d} X)$ is thus the image of $\mathrm{D}\Psi$. Thus we have

$$\begin{aligned} \mathrm{D}[\operatorname{add} T](X, Y) &\cong \mathrm{D} \operatorname{Im}\Psi \cong \operatorname{Im} \mathrm{D}\Psi \cong \mathscr{T}(Y, \Sigma^{2d} X)/[\operatorname{add} \Sigma^d T](Y, \Sigma^{2d} X) \\ &\cong \operatorname{Hom}_{\mathscr{T}/\operatorname{add} \Sigma^d T}(\overline{Y}, \overline{\Sigma^{2d} X}). \quad \square \end{aligned}$$

Lemma 2.5. *Suppose $X, Y \in \mathscr{T}$. Then we have a short exact sequence*

$$0 \to \mathrm{D}\operatorname{Hom}_{\mathscr{T}/\operatorname{add} \Sigma^d T}(\overline{Y}, \overline{\Sigma^d X}) \to \operatorname{Ext}^d_{\mathscr{T}}(X, Y) \to \operatorname{Hom}_{\mathscr{T}/\operatorname{add} \Sigma^d T}(\overline{X}, \overline{\Sigma^d Y}) \to 0.$$

Proof. By the definition of the quotient functor we have a short exact sequence

$$0 \to [\operatorname{add} \Sigma^d T](X, \Sigma^d Y) \to \mathscr{T}(X, \Sigma^d Y) \to \operatorname{Hom}_{\mathscr{T}/\operatorname{add}\Sigma^d T}(\overline{X}, \overline{\Sigma^d Y}) \to 0.$$

We have $[\operatorname{add} \Sigma^d T](X, \Sigma^d Y) \cong [\operatorname{add} T](\Sigma^{-d} X, Y)$. By Lemma 2.4 we have

$$[\operatorname{add} T](\Sigma^{-d} X, Y) \cong \mathrm{D}\operatorname{Hom}_{\mathscr{T}/\operatorname{add}\Sigma^d T}(\overline{Y}, \overline{\Sigma^{2d}\Sigma^{-d} X}) \cong \mathrm{D}\operatorname{Hom}_{\mathscr{T}/\operatorname{add}\Sigma^d T}(\overline{Y}, \overline{\Sigma^d X}).$$

We also know that $\mathscr{T}(X, \Sigma^d Y) \cong \operatorname{Ext}^d_{\mathscr{T}}(X, Y)$, so the conclusion follows. □

Lemma 2.6. *Suppose $X, Y \in \mathscr{T}$ have no non-zero direct summands in* $\operatorname{add}\Sigma^d T$*. Then we have a short exact sequence*

$$\begin{aligned} 0 &\to \mathrm{D}\operatorname{Hom}_\Gamma\big(\mathscr{T}(T, Y), \tau_d \mathscr{T}(T, X)\big) \to \operatorname{Ext}^d_{\mathscr{T}}(X, Y) \\ &\to \operatorname{Hom}_\Gamma\big(\mathscr{T}(T, X), \tau_d \mathscr{T}(T, Y)\big) \to 0. \end{aligned}$$

Proof. Consider the short exact sequence from Lemma 2.5. By Theorem 0.4 we know that

$$\mathrm{D}\operatorname{Hom}_{\mathscr{T}/\operatorname{add}\Sigma^d T}(\overline{Y}, \overline{\Sigma^d X}) \cong \mathrm{D}\operatorname{Hom}_\Gamma\big(\mathscr{T}(T, Y), \mathscr{T}(T, \Sigma^d X)\big).$$

Applying Lemma 2.3 we have

$$\mathrm{D}\operatorname{Hom}_\Gamma\big(\mathscr{T}(T, Y), \mathscr{T}(T, \Sigma^d X)\big) \cong \mathrm{D}\operatorname{Hom}_\Gamma\big(\mathscr{T}(T, Y), \tau_d \mathscr{T}(T, X)\big).$$

Similarly we can show $\operatorname{Hom}_{\mathscr{T}/\operatorname{add}\Sigma^d T}(\overline{X}, \overline{\Sigma^d Y}) \cong \operatorname{Hom}_\Gamma\big(\mathscr{T}(T, X), \tau_d \mathscr{T}(T, Y)\big)$. □

The map defined next will eventually induce the equivalence of Theorem B.

Definition 2.7. For each $X \in \mathscr{T}$, pick an isomorphism $X \cong X' \oplus X''$ such that X' has no non-zero direct summands in $\operatorname{add}\Sigma^d T$ and $X'' \in \operatorname{add}\Sigma^d T$. Let

$$\Delta(X) = \big(\mathscr{T}(T, X'), \mathscr{T}(T, \Sigma^{-d} X'')\big).$$

This is a pair of Γ-modules where $\mathscr{T}(T, X')$ is in $\mathscr{D}$ and $\mathscr{T}(T, \Sigma^{-d} X'')$ is in $\operatorname{proj}\Gamma$.

Proposition 2.8. *Given $X, Y \in \mathscr{T}$, set $(M, P) = \Delta(X)$ and $(N, Q) = \Delta(Y)$, where Δ is the map in Definition 2.7. Then*

$$\begin{aligned} \dim_k \operatorname{Ext}^d_{\mathscr{T}}(X, Y) = &\dim_k \operatorname{Hom}_\Gamma(M, \tau_d N) + \dim_k \operatorname{Hom}_\Gamma(N, \tau_d M) \\ &+ \dim_k \operatorname{Hom}_\Gamma(P, N) + \dim_k \operatorname{Hom}_\Gamma(Q, M). \end{aligned}$$

Proof. By additivity of Ext we have

$$\begin{aligned}\operatorname{Ext}^d_{\mathscr{T}}(X,Y) &\cong \operatorname{Ext}^d_{\mathscr{T}}(X' \oplus X'', Y' \oplus Y'') \\ &\cong \operatorname{Ext}^d_{\mathscr{T}}(X',Y') \oplus \operatorname{Ext}^d_{\mathscr{T}}(X',Y'') \oplus \operatorname{Ext}^d_{\mathscr{T}}(X'',Y') \oplus \operatorname{Ext}^d_{\mathscr{T}}(X'',Y'').\end{aligned}$$

As T is d-rigid, we see that $\operatorname{Ext}^d_{\mathscr{T}}(X'',Y'') = 0$, and hence we have

$$\dim \operatorname{Ext}^d_{\mathscr{T}}(X,Y) = \dim \operatorname{Ext}^d_{\mathscr{T}}(X',Y') + \dim \operatorname{Ext}^d_{\mathscr{T}}(X',Y'') + \dim \operatorname{Ext}^d_{\mathscr{T}}(X'',Y'). \quad (2.1)$$

From Lemma 2.6 we have the short exact sequence:

$$\begin{aligned}0 &\to \operatorname{DHom}_\Gamma\big(\mathscr{T}(T,Y'), \tau_d \mathscr{T}(T,X')\big) \to \operatorname{Ext}^d_{\mathscr{T}}(X',Y') \\ &\to \operatorname{Hom}_\Gamma\big(\mathscr{T}(T,X'), \tau_d \mathscr{T}(T,Y')\big) \to 0,\end{aligned}$$

which means that

$$\begin{aligned}\dim \operatorname{Ext}^d_{\mathscr{T}}(X',Y') &= \dim_k \operatorname{Hom}_\Gamma\big(\mathscr{T}(T,X'), \tau_d \mathscr{T}(T,Y')\big) \\ &\quad + \dim_k \operatorname{Hom}_\Gamma\big(\mathscr{T}(T,Y'), \tau_d \mathscr{T}(T,X')\big) \\ &= \dim_k \operatorname{Hom}_\Gamma(M, \tau_d N) + \dim_k \operatorname{Hom}_\Gamma(N, \tau_d M). \quad (2.2)\end{aligned}$$

We see that

$$\begin{aligned}\operatorname{Ext}^d_{\mathscr{T}}(X'',Y') &\cong \mathscr{T}(X'', \Sigma^d Y') \cong \mathscr{T}(\Sigma^{-d} X'', Y') \cong \operatorname{Hom}_\Gamma\big(\mathscr{T}(T, \Sigma^{-d} X''), \mathscr{T}(T,Y')\big) \\ &\cong \operatorname{Hom}_\Gamma(P,N).\end{aligned}$$

The third isomorphism follows from [14, Lemma 2.2(i)] and the fact that $\Sigma^{-d} X'' \in \operatorname{add} T$. Similarly,

$$\operatorname{Ext}^d_{\mathscr{T}}(X',Y'') \cong \operatorname{DExt}^d_{\mathscr{T}}(Y'',X') \cong \operatorname{DHom}_\Gamma(Q,M).$$

Thus we have

$$\dim \operatorname{Ext}^d_{\mathscr{T}}(X'',Y') = \dim_k \operatorname{Hom}_\Gamma(P,N) \quad (2.3)$$

$$\dim \operatorname{Ext}^d_{\mathscr{T}}(X',Y'') = \dim_k \operatorname{Hom}_\Gamma(Q,M). \quad (2.4)$$

Substituting (2.2), (2.3), and (2.4) into (2.1) gives the result. □

As a consequence we have:

Corollary 2.9. *Given $X, Y \in \mathscr{T}$, set $(M,P) = \Delta(X)$ and $(N,Q) = \Delta(Y)$. Then*

$$\begin{aligned}&\operatorname{Ext}^d_{\mathscr{T}}(X,Y) = 0 \Leftrightarrow \\ &\operatorname{Hom}_\Gamma(M, \tau_d N) = \operatorname{Hom}_\Gamma(N, \tau_d M) = \operatorname{Hom}_\Gamma(P,N) = \operatorname{Hom}_\Gamma(Q,M) = 0.\end{aligned}$$

3. Proof of Theorem B

The following results use the map Δ from Definition 2.7.

Lemma 3.1. *Given $X, Y \in \mathscr{T}$, set $(M, P) = \Delta(X)$ and $(N, Q) = \Delta(Y)$. Then $Y \in \operatorname{add} X$ if and only if $N \in \operatorname{add} M$ and $Q \in \operatorname{add} P$.*

Proof. Let $X \cong X' \oplus X''$ be the decomposition from Definition 2.7, where X' has no non-zero direct summands from $\operatorname{add} \Sigma^d T$ while X'' is in $\operatorname{add} \Sigma^d T$. We have $(M, P) = \big(\mathscr{T}(T, X'), \mathscr{T}(T, \Sigma^{-d} X'')\big)$. Similarly, $(N, Q) = \big(\mathscr{T}(T, Y'), \mathscr{T}(T, \Sigma^{-d} Y'')\big)$.

The condition $Q \in \operatorname{add} P$ is equivalent to $Y'' \in \operatorname{add} X''$ by the add-proj-correspondence, (see Remark 0.5). The condition $N \in \operatorname{add} M$ is equivalent to $Y' \in \operatorname{add} X'$ by Theorem 0.4 because X', Y' have no non-zero direct summands in $\operatorname{add} \Sigma^d T$. The result follows. $\square$

Lemma 3.2. *The category $\mathscr{T}$ is skeletally small. The map Δ induces a bijection*

$$\delta : \operatorname{iso} \mathscr{T} \to \operatorname{iso} \mathscr{D} \times \operatorname{iso} \operatorname{proj} \Gamma, \tag{3.1}$$

where iso *denotes the set of isomorphism classes of a skeletally small category.*

Proof. Let Iso denote the class of isomorphisms of a category. For a skeletally small category $\mathscr{C}$ we have that $\operatorname{Iso} \mathscr{C} = \operatorname{iso} \mathscr{C}$. Note that since a module category over a ring is skeletally small, we have that $\mathscr{D}, \operatorname{proj} \Gamma \subseteq \operatorname{mod} \Gamma$ are skeletally small.

It is clear that Δ induces a well-defined map of the form

$$\delta' : \operatorname{Iso} \mathscr{T} \to \operatorname{iso} \mathscr{D} \times \operatorname{iso} \operatorname{proj} \Gamma.$$

To see that δ' is injective, argue like the proof of Lemma 3.1, replacing membership of add with isomorphism.

It follows that $\mathscr{T}$ is skeletally small. We can thus replace δ' with the map δ from (3.1).

To see that δ is surjective, let (M, P) be a pair with $M \in \mathscr{D}$ and $P \in \operatorname{proj} \Gamma$. By Theorem 0.4 there is an object $X' \in \mathscr{T}$ with no non-zero direct summands in $\operatorname{add} \Sigma^d T$ such that $M \cong \mathscr{T}(T, X')$. By the add-proj correspondence, see Remark 0.5, there is an object $X'' \in \operatorname{add} \Sigma^d T$ such that $P \cong \mathscr{T}(T, \Sigma^{-d} X'')$. Setting $X = X' \oplus X''$ gives $(M, P) \cong \Delta(X)$. $\square$

Lemma 3.3. *If $X \in \mathscr{T}$ is d-self-perpendicular, then $(M, P) = \Delta(X)$ is a maximal τ_d-rigid pair.*

Proof. Let $N \in \mathscr{D}$ and $Q \in \operatorname{proj} \Gamma$ be given. By Lemma 3.2, there is an object $Y \in \mathscr{T}$ such that $(N, Q) \cong \Delta(Y)$. Then

$$\begin{aligned}
&N \in \operatorname{add} M \text{ and } Q \in \operatorname{add} P \\
&\Leftrightarrow Y \in \operatorname{add} X \\
&\Leftrightarrow \operatorname{Ext}^d_{\mathscr{T}}(X, Y) = 0 \\
&\Leftrightarrow \operatorname{Hom}_\Gamma(M, \tau_d N) = \operatorname{Hom}_\Gamma(N, \tau_d M) = \operatorname{Hom}_\Gamma(P, N) = \operatorname{Hom}_\Gamma(Q, M) = 0,
\end{aligned}$$

where the equivalences, respectively, are by Lemma 3.1, Definition 0.2, and Corollary 2.9.

The conditions of Definition 0.7 are recovered by setting $Q = 0$ respectively $N = 0$. □

Lemma 3.4. *Let $X \in \mathscr{T}$ be given. If $(M, P) = \Delta(X)$ is a maximal τ_d-rigid pair, then X is d-self-perpendicular.*

Proof. Let $Y \in \mathscr{T}$ be given and set $(N, Q) \cong \Delta(Y)$. Then

$$\begin{aligned}
&\operatorname{Ext}^d_{\mathscr{T}}(X, Y) = 0 \\
&\Leftrightarrow \operatorname{Hom}_\Gamma(M, \tau_d N) = \operatorname{Hom}_\Gamma(N, \tau_d M) = \operatorname{Hom}_\Gamma(P, N) = \operatorname{Hom}_\Gamma(Q, M) = 0 \\
&\Leftrightarrow N \in \operatorname{add} M \text{ and } Q \in \operatorname{add} P \\
&\Leftrightarrow Y \in \operatorname{add} X,
\end{aligned}$$

where the equivalences, respectively, are by Corollary 2.9, Definition 0.7, and Lemma 3.1. □

Theorem 3.5. *Recall that the map Δ from Definition 2.7 induces the bijection $\delta : \operatorname{iso} \mathscr{T} \to \operatorname{iso} \mathscr{D} \times \operatorname{iso} \operatorname{proj} \Gamma$ from Lemma 3.2.*

(i) δ restricts to a bijection

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{d-rigid objects in } \mathscr{T} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \tau_d\textit{-rigid pairs in } \mathscr{D} \end{array} \right\}.$$

(ii) δ restricts further to a bijection

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{d-self-perpendicular objects in } \mathscr{T} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{maximal } \tau_d\textit{-rigid pairs in } \mathscr{D} \end{array} \right\}.$$

Proof. (i): Consider $X \in \mathscr{T}$ and set $(M, P) = \Delta(X)$. Then

$$\operatorname{Ext}^d_{\mathscr{T}}(X, X) = 0 \Leftrightarrow \operatorname{Hom}_\Gamma(M, \tau_d M) = 0 \text{ and } \operatorname{Hom}_\Gamma(P, M) = 0$$

by Corollary 2.9, so the result follows.

(ii): See Lemmas 3.3 and 3.4. □

Proof of Theorem B (from the introduction). Combine Theorems 3.5(ii) and 1.1(ii). □

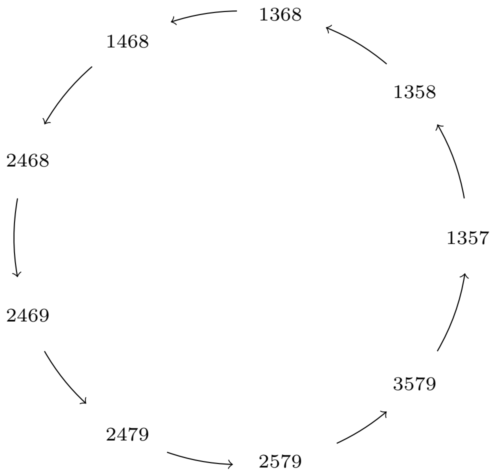


Fig. 1. The AR quiver of the 5-angulated category $\mathscr{T}$.

4. An example

In this section we let $d = 3$ and $\mathscr{T} = \mathscr{O}_{A_2^3}$. This is the 5-angulated (higher) cluster category of type A_2, see [21, def. 5.2, sec. 6, and sec. 8]. The indecomposable objects can be identified with the elements of the set

$$^{\circlearrowleft}\mathbf{I}_9^3 = \{ 1357, 1358, 1368, 1468, 2468, 2469, 2479, 2579, 3579 \},$$

see [21, sec. 8]. The AR quiver of $\mathscr{T}$ is shown in Fig. 1. By [21, thm. 5.5 and sec. 8], the object

$$T = 1357 \oplus 1358 \oplus 1368 \oplus 1468$$

is Oppermann–Thomas cluster tilting.

If $X, Y \in \mathscr{T}$ are indecomposable objects, then

$$\mathscr{T}(X, Y) = \begin{cases} k & \text{if } Y \text{ is } X \text{ or its immediate successor in the AR quiver,} \\ 0 & \text{otherwise,} \end{cases}$$

see [21, prop. 6.1 and def. 6.9]. It follows that $\Gamma = \operatorname{End}_{\mathscr{T}}(T) = kQ/I$, where

$$Q = 1 \to 2 \to 3 \to 4$$

and I is the ideal generated by all compositions of two consecutive arrows. The action of the functor $\mathscr{T}(T, -) : \mathscr{T} \to \operatorname{mod}\Gamma$ on indecomposable objects is shown in Fig. 2, where $P(q)$ and $I(q)$ denote the indecomposable projective and injective modules associated to the vertex $q \in Q$. Note that the essential image of $\mathscr{T}(T, -)$ is

X	1357	1358	1368	1468	2468	2469	2479	2579	3579
$\mathscr{T}(T,X)$	$P(4)$	$P(3)$	$P(2)$	$P(1)$	$I(1)$	0	0	0	0

Fig. 2. The action of the functor $\mathscr{T}(T,-):\mathscr{T}\to \operatorname{mod}\Gamma$.

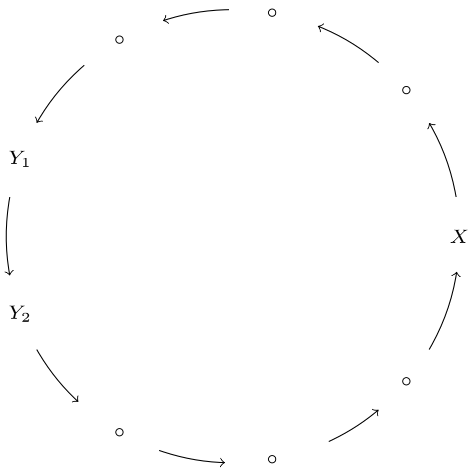


Fig. 3. The functor $\operatorname{Ext}^3_{\mathscr{T}}(X,-)$ is non-zero on Y_1 and Y_2. It is zero on every other indecomposable object.

Maximal 3-rigid object X	Maximal τ_3-rigid pair $\Delta(X)$
$1357\oplus1358\oplus1368\oplus1468$	$(\Gamma,0)$
$1358\oplus1368\oplus1468\oplus2468$	$(\mathrm{D}\Gamma,0)$
$1368\oplus1468\oplus2468\oplus2469$	$\big(P(2)\oplus P(1)\oplus I(1),P(4)\big)$
$1468\oplus2468\oplus2469\oplus2479$	$\big(P(1)\oplus I(1),P(4)\oplus P(3)\big)$
$2468\oplus2469\oplus2479\oplus2579$	$\big(I(1),P(4)\oplus P(3)\oplus P(2)\big)$
$2469\oplus2479\oplus2579\oplus3579$	$(0,\Gamma)$
$2479\oplus2579\oplus3579\oplus1357$	$\big(P(4),P(3)\oplus P(2)\oplus P(1)\big)$
$2579\oplus3579\oplus1357\oplus1358$	$\big(P(4)\oplus P(3),P(2)\oplus P(1)\big)$
$3579\oplus1357\oplus1358\oplus1368$	$\big(P(4)\oplus P(3)\oplus P(2),P(1)\big)$
$1357\oplus1468\oplus2479$	$\big(P(4)\oplus P(1),P(3)\big)$
$1358\oplus2468\oplus2579$	$\big(P(3)\oplus I(1),P(2)\big)$
$1368\oplus2469\oplus3579$	$\big(P(2),P(4)\oplus P(1)\big)$

Fig. 4. These are all the basic maximal 3-rigid objects of $\mathscr{T}$ and their corresponding maximal τ_3-rigid pairs in $\mathscr{D}$.

$$\mathscr{D}=\operatorname{add}\{\,P(4),P(3),P(2),P(1),I(1)\,\}.$$

This is a 3-cluster tilting subcategory of $\operatorname{mod}\Gamma$ and hence it is 3-abelian.

The 3-suspension functor Σ^3 acts on the AR quiver by moving four steps clockwise. Combined with our knowledge of Hom, this shows that if X is a fixed indecomposable object in $\mathscr{T}$, then the indecomposable objects Y with $\operatorname{Ext}^3_{\mathscr{T}}(X,Y)\neq 0$ are precisely the two objects furthest from X in the AR quiver, see Fig. 3.

Based on this, we can compute all basic 3-self-perpendicular objects in $\mathscr{T}$, and by Proposition 1.3 they coincide with the basic maximal 3-rigid objects in $\mathscr{T}$. For each such object X, there is a maximal τ_3-rigid pair $\Delta(X)=\big(\mathscr{T}(T,X'),\mathscr{T}(T,\Sigma^{-3}X'')\big)$ by Theorem B. See Fig. 4. Note that the first nine objects in Fig. 4 are Oppermann–Thomas cluster tilting, but the three last objects are not.

Acknowledgment

This work was supported by EPSRC grant EP/P016014/1 "Higher Dimensional Homological Algebra". Karin M. Jacobsen is grateful for the hospitality of Newcastle University during her visit in October 2018.

References

[1] T. Adachi, O. Iyama, I. Reiten, τ-Tilting theory, Compos. Math. 150 (2014) 415–452.
[2] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras, Vol. 1, Techniques of Representation Theory, London Math. Soc. Stud. Texts, vol. 65, Cambridge University Press, Cambridge, 2006.
[3] P.A. Bergh, M. Thaule, The axioms for n-angulated categories, Algebr. Geom. Topol. 13 (2013) 2405–2428.
[4] A.B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2006) 572–618.
[5] A.B. Buan, R.J. Marsh, D.F. Vatne, Cluster structures from 2-Calabi–Yau categories with loops, Math. Z. 265 (2010) 951–970.
[6] F. Fedele, Auslander-Reiten $(d+2)$-angles in subcategories and a $(d+2)$-angulated generalisation of a theorem by Brüning, J. Pure Appl. Algebra 223 (2019) 3554–3580.
[7] F. Fedele, d-Auslander–Reiten sequences in subcategories, preprint, arXiv:1808.02709, 2018.
[8] C. Geiss, B. Keller, S. Oppermann, n-angulated categories, J. Reine Angew. Math. 675 (2013) 101–120.
[9] M. Herschend, O. Iyama, n-representation-finite algebras and twisted fractionally Calabi–Yau algebras, Bull. Lond. Math. Soc. 43 (2011) 449–466.
[10] M. Herschend, O. Iyama, Selfinjective quivers with potential and 2-representation-finite algebras, Compos. Math. 147 (2011) 1885–1920.
[11] O. Iyama, Cluster tilting for higher Auslander algebras, Adv. Math. 226 (2011) 1–61.
[12] O. Iyama, Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories, Adv. Math. 210 (2007) 22–50.
[13] O. Iyama, S. Oppermann, Stable categories of higher preprojective algebras, Adv. Math. 244 (2013) 23–68.
[14] K.M. Jacobsen, P. Jørgensen, d-Abelian quotients of $(d+2)$-angulated categories, J. Algebra 521 (2019) 114–136.
[15] G. Jasso, n-Abelian and n-exact categories, Math. Z. 283 (2016) 203–759.
[16] G. Jasso, J. Külshammer, Higher Nakayama algebras I: construction, preprint, arXiv:1604.03500, 2016.
[17] G. Jasso, J. Külshammer, Nakayama-type phenomena in higher Auslander–Reiten theory, in: Representations of Algebras, in: Contemp. Math., vol. 705, American Mathematical Society, Providence, RI, 2018, pp. 79–98.
[18] G. Jasso, S. Kvamme, An introduction to higher Auslander–Reiten theory, Bull. Lond. Math. Soc. 51 (2019) 1–24.
[19] J. McMahon, Higher support tilting I: higher Auslander algebras of linearly oriented type A, preprint, arXiv:1808.05184, 2018.
[20] Y. Mizuno, A Gabriel-type theorem for cluster tilting, Proc. Lond. Math. Soc. (3) 108 (2014) 836–868.
[21] S. Oppermann, H. Thomas, Higher-dimensional cluster combinatorics and representation theory, J. Eur. Math. Soc. (JEMS) 14 (2012) 1679–1737.
[22] Y. Zhou, B. Zhu, Maximal rigid subcategories in 2-Calabi–Yau triangulated categories, J. Algebra 348 (2011) 49–60.