Master's thesis

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Introduction to gravitational waves

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Abstract

In this thesis we derive the Einstein field equations from the Einstein-Hilbert action using a variational principle. We then look at the weak-field limit of these equations through a linear approximation. Through a gauge fixing procedure, we find two different polarizations of gravitational waves. Next, we derive how gravitational waves affect stationary objects in different gauges, and see how the choice of gauge is connected to our choice of coordinate system. We then derive the stress-energy tensor of gravitational waves by extending our weak-field approximation to second order in the metric. By separating the background metric from the gravitational waves, we see that the stress-energy tensor of gravitational waves arises naturally on an averaged form. Briefly, we discuss the possibility of gravity being mediated by spin-2 particles in flat spacetime, and we show how simple assumptions lead to the full non-linear form of gravity and the principle of general covariance. Lastly, we derive the formulas for emission of gravitational waves to lowest order in the velocity of the source, and apply this to a binary system following Newtonian orbits.

Preface

This thesis is the concluding work on the master program in Applied physics and mathematics from NTNU (Norwegian University of Science and Technology). The work lasted one semester, and was the continuation of a smaller project previous fall. To keep this thesis self-contained and to avoid extensive cross-referencing, I have reused parts of the previous assignment, namely chapters 1 through 5 in this thesis, as well as Appendix A.

The main purpose of this thesis is to develop an understanding of gravitational waves. A weakness I found in the literature that exists on gravitational waves is that often the derivations are brief and mathematical, thereby not addressing the physical understanding. Therefore, as a student of the subject, most of my time on this thesis has been spent wrestling with concepts and trying to connect the mathematics of gravitational waves to physical understanding. I have therefore tried to include as much physical intuition in my derivations as possible, and answer questions that the reader might have, especially when these answers are not easily found in the literature.

I am very grateful for the help of Prof. Jens Oluf Andersen, who was my supervisor. He gave me structure and a direction, as well as freedom for me to pursue the ideas that I found most interesting. He also read my work along the way and challenged my arguments, which proved to be very valuable. Thank you for taking the time.

Lastly, I want to include a quote sometimes attributed to Enrico Fermi, which sums up my process of studying gravitational waves

> *“Before I came here, I was confused about this subject. Having listened to your lecture, I am still confused – but on a higher level.“*

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Chapter 1

Introduction

The special theory of relativity was published by Albert Einstein in 1905. It was a landmark development in our understanding of the fabric of our universe. Here Einstein introduced the equivalence principle stating that the laws of physics should be equal in any inertial frame, and that the speed of light in vacuum is constant. This gave an interpretation of time- and length-dilation that radically changed our perception of the fabric of reality [1]. Einstein then spent the next ten years developing the general theory of relativity, which was published in 1915. Here Einstein extended the equivalence principle to the principle of general covariance, stating that the laws of physics should have the same form in any coordinate system. Einstein managed to incorporate acceleration into the theory, and gravity was reinterpreted not as a force, but rather as a result of the geometry of spacetime. Gravitational waves (GW) are one of the important predictions of general relativity. These waves are ripples in the fabric of spacetime, stretching and contracting lengths in space as they pass. The effects of these waves are minuscule, and detecting them require incredible sensitivity. In 2015, one hundred years after the formulation of the general theory of relativity, the first direct observation of gravitational waves were made by the Laser Interferometer Gravitational-wave Observatory (LIGO) [2].

In 1905 Henri Poincar was the first physicist to suggest the existence of GWs, and argued that as gravity propagates at the speed of light, there should exist waves in the gravitational field in analogy to electromagnetic waves. Ten years later Einstein published his theory of general relativity, where these waves were one of the predictions. For a long time after this publication and the initial predictions by Einstein, there were confusion surrounding the interpretation of gravitational waves. This included questions regarding whether the waves were physical or just a remnant of coordinate choice. The confusion was cleared up in the 1950s, and physicists also concluded that gravitational waves carry energy.

The efforts to measure gravitational waves began in the 1960s, with resonant mass antennas, pioneered by Joseph Weber [3]. He constructed large aluminum cylinders, about 153 cm in length, 66 cm in diameter and weighing 3 tons, later dubbed Weber bars. The idea was that as gravitational waves pass the cylinder, the cylinder is stretched and contracted. If the wave has a frequency close to the resonant frequency of the cylinder, the vibration will resonate and hopefully grow to a stable amplitude that is large enough to be measured and distinguished from noise. Weber claimed in 1969 to have measured gravitational waves using Weber bars. A year later he claimed to have measured several more GWs emitted from the center of our galaxy. The detection frequency and magnitude of these results implied that around 1000 solar masses of energy were converted into gravitational waves every year, much higher than the estimated upper limits for our galaxy. It thus became increasingly clear that Weber's results were not credible. Other groups of physicists have developed the concept of resonant mass antennas in later years, but have not been able to obtain measurements of gravitational waves. Parallel to this development, there has been another approach to measuring GWs using laser interferometry [4].

The first evidence of gravitational waves did not come in the form of a direct observation, but rather as an indirect measurement through cosmological observations. Hulse and Taylor made an important discovery in 1974 [5]. They discovered the first binary pulsar, a neutron star that radiates electromagnetic radiation (a pulsar) and a neutron star rotating orbiting a common mass center. They were able to deduce this by measuring the arrival of the pulses from the pulsar. The pulses had a period of 59 milliseconds, but there seemed to be small deviations from the expected arrival time with a fixed period of 7.75 hours. They realized that this was the predicted result if the pulsar was orbiting another stellar object. Using this information, they could derive the existence of another neutron star, and were able to estimate their masses, the radii of their orbits, and they could monitor the orbital periods. Over time the orbits have been shown to gradually contract and the stars slow down. This is the result of the energy loss due to emission of gravitational waves, which is predicted by the general theory of relativity. Measurements of this binary system have been shown to be in good agreement with calculations, and this observation was the first empirical test of the effects of gravitational waves. Hulse and Taylor received the 1993 Nobel Prize in Physics for this work, their discovery and analysis of this system, "for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation."

On September 14, 2015 the first direct observation of gravitational waves was made. The discovery was made by LIGO, using laser interferometry. The experimental method builds on the concept of the Michelson interferometer, famously used in the Michelson-Morley experiment in 1887 to show that the speed of light does not depend on the velocity of Earth. This result discredited a popular theory that light propagated in a medium called the aether, which was believed to move independently of Earth [6]. Because the speed of light in vacuum is constant, the path-time of light will be changed when space is stretched or contracted. This can be measured using an advanced Michelson interferometer. A laser beam is split into two different paths by a beam splitter, reflected by mirrors, and recombined onto a detector. The detector then measures the interference pattern of the two beams. Waves with opposite phase vanish, and the interference is effectively a measure of the difference in the two path lengths of the laser beams. In the upper part of Figure 1.1 an interferometer is shown, where the light interferes constructively. When a gravitational wave passes, the interferometer is stretched and contracted by the GW and the path lengths are changed. This change in path length can be measured. In the lower part of Figure 1.1 a gravitational wave, shown in yellow, stretches one of the arms of the interferometer such that the interference changes. The LIGO experiment uses a setup based on this interferometer, where a laser beam is split into two 4 km long tunnels that are placed perpendicular to each other, with mirrors reflecting the light. The mirrors are adjusted such that the light interferes destructively, and any deviation from this can be measured. A gravitational wave that passes in the direction of one tunnel will stretch and contract the other tunnel, the path length of the light will change, which will change the phase difference of the light. This will show up on the detector as small beats of light, which is what they try to detect.

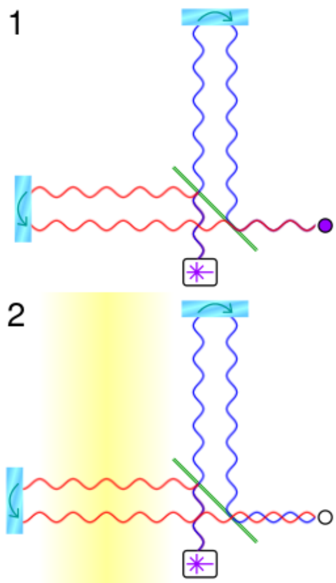

Figure 1.1: This figure shows the concept of interferometric gravitational sensing, with an interferometer with and without a gravitational wave. See full text for explanation. The figure is taken from https://en.wikipedia.org/wiki/LIGO

The successful detection of a gravitational wave is a tremendous achievement. There are two main challenges to overcome to be able to measure gravitational waves; one, to achieve high enough sensitivity, and two, to be able to distinguish signal from noise. Gravitational waves have a very weak effect. The first gravitational wave detected had a maximal strain of $\sim 10^{-21}$, where strain is the relative change in lengths caused by the wave. This means that the lengths of the 4 km long tunnels were changed by no more than a thousandth of the width of a proton, approximately two attometres. The interferometers therefore need to be incredibly sensitive. The LIGO experiment had to try to maximize their sensitivity in every way possible. In the two tunnels of 4 km, they have mirrors sending the laser light back and forth about 280 times, effectively increasing the path length of each tunnel to 1120 km. Another important factor of the sensitivity of the interferometer, was the intensity of the laser beam. The strength of the laser determines the photon count, and thus the resolution of the interferometer. The smallest quanta of light is a photon, and the resolution of the detection is therefore limited to only integer numbers of photons. By increasing the total photon count by having a stronger laser, one photon corresponds to a smaller fraction of the total photon count, and thus smaller deviations can be detected. LIGO uses a sophisticated set of mirrors to recycle power, and achieves a laser beam that shines at 750 kilowatts, enough power to supply 1000 households [2].

The second challenge is to distinguish signal from measuring noise. Because the interferometers have to be extremely sensitive to be able to detect a gravitational wave, it also means they will pick up any vibration in the ground. This can be caused by events such as nearby traffic or earthquakes far away. It is therefore important that the interferometer is isolated from the environment, and a combination of active and passive damping. For the active dampening, LIGO uses sensors to detect ground movements and then uses active counter motion to keep their mirrors motion free. This counteracts vibrations in the ground, but does not affect the detection of gravitational waves, as these waves do not move the mirrors, but rather changes the length of the whole tunnel. For the passive dampening, LIGO's reflective mirrors (often called test masses) together with stabilizing weights are suspended from four 0.4 mm thick fused-silica (glass) fibres. This helps prevent motion not cancelled out by the noise-cancelling system from affecting the mirrors. Another source of noise is air or dust in the laser beam. Air refracts light, and can therefore spread the laser beam as well as influence the optical path length of the light, and dust can scatter the light if it drifts into the beam or onto the mirrors. It is therefore important to have as little air as possible where the laser beams are. LIGO has created one of the largest and purest sustained vacuums in the world, with a pressure of one-trillionth that of atmospheric pressure. As a final effort to reduce the effects of noise, LIGO made two separate interferometers. One is located in Livingston, Louisiana and the other in Hanford, Washington. The distance between the two facilities is around 1900 km, or 0.01 light seconds. As sources of gravitational waves are of cosmological distances from Earth, both facilities should detect the wave. The GW signal from both detectors should be of the same strength and shape, and with a maximal separation in time of 10 milliseconds. By comparing the measurements from both facilities, LIGO can make sure that any GW they measure is not just caused by local vibrations or instrument noise. Having two spatially separated detectors can also be used to give some information about the direction of the wave, as difference in detection time depends on the direction of the wave. When the two facilities show a match between their signal, the proposed wave is then compared to a library of wave signals generated by simulations of cosmological events believed to produce gravitational waves. This is both a test to check if the signal is actually a gravitational wave, as well as it gives information about the source of the wave. Figure 1.2 shows a simplified version of the interferometers, their positions on the map of the USA, as well as a plot of the measuring noise. To date there are 11 recorded measurements of gravitational waves, all made by LIGO [2].

Any mass that accelerates creates gravitational waves, which includes any accelerating mass here Earth. Only very large masses can create waves large enough to detect, and we therefore look to space. There are several known sources of gravitational waves. One important source is what is called a binary system. A binary system consists of two massive objects, like black holes, neutron stars or white dwarfs, orbiting each other. The objects will lose energy due to the emission of GWs, caused by the circular motion, and they will spiral inward and eventually merge. This inspiral will release a GW lasting a short time, from fractions of seconds for black holes to minutes for neutron stars, of increasing frequency. The first ever detected gravitational wave came from such a system, where two black holes spiralled and merged. The signal lasted 0.2 seconds with a frequency from 35 to 250 Hz. [7]. So far all directly detected GWs comes from binary systems of either two black holes or two neutron stars. Another source of gravitational waves are spinning masses, for example neutron stars. Bumps or imperfections on the surface will create GWs as the star rotates. These waves will be of continuous amplitude and frequency, and are expected to be found with frequencies in the milliseconds. Supernovas are also expected to produce GWs. When the core collapses large amounts of mass (1 - 100 solar masses) are moved at relativistic speeds. The asymmetry of this collapse will create gravitational waves. This process is however not well understood, and the magnitude and frequency is hard to predict [8]. These waves are also expected to have frequencies in the millisec-

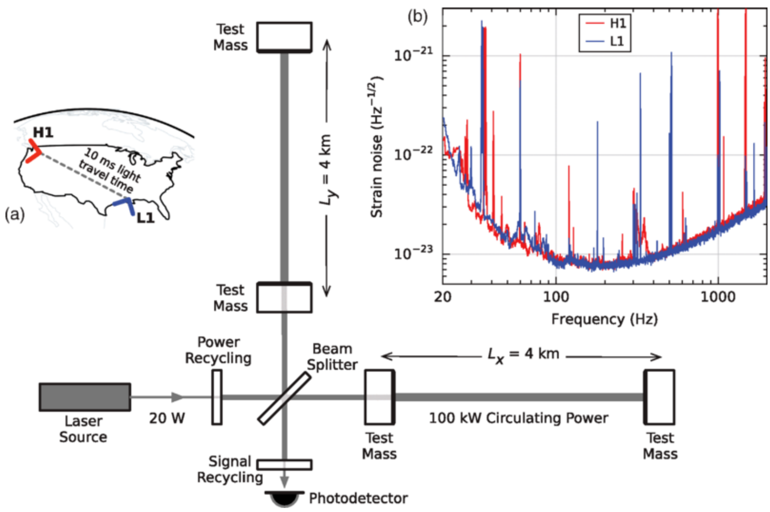

Figure 1.2: a) The locations of the gravitational observatories where the interferometers are placed. b) A plot showing the instrument noise of the interferometers depending on the frequency of the gravitational wave. The spikes are the result of resonant frequencies in the components of the interferometers. c) A figure showing a simplified version of the interferometers. Figure is taken from https://en.wikipedia.org/wiki/LIGO

onds range. The first time after the Big Bang is also expected to have created gravitational waves. These are called primordial GWs, and come from the rapid expansion in the cosmic inflation in the earliest universe. Figure 1.3 shows the expected frequencies of the above mentioned sources, as well as detection methods that can be used. More about these detection methods can be found in reference [9].

Astronomy has traditionally relied on electromagnetic radiation. The advances of the study of gravitational waves can provide complementary information about the events that we can already study, as well as allow us to observe and study systems that are impossible to detect with electromagnetic radiation. There is a possibility that we will also find gravitational waves from phenomena or systems that we did not even know about. The detection of the first gravitational waves is an exciting step in the direction of new knowledge. There are currently plans for several new gravitational observatories of equal or higher sensitivity than LIGO, which will allow more frequent observations and the ability to determine direction and distance to the source, as well as observations at different frequencies.

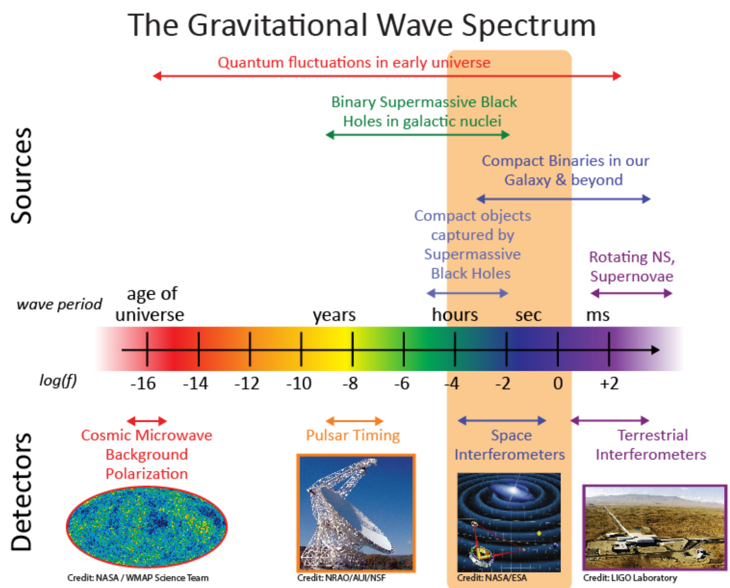

Figure 1.3: This figure shows some sources of gravitational waves and their expected frequency, and some detection methods and what in what range of frequencies they operate. The LISA detection frequency is highlighted. Figure is taken from [9].

Chapter 2

Deriving the Einstein field equations from variational principles

In this chapter we want to derive the Einstein field equations using variational principles. In the first section we will use the Einstein-Hilbert action, and vary the action with respect to the metric. In the second section we will follow the Pallatini approach, and let the action depend on the metric and the connection coefficients independently.

2.1 Einstein field equations from the Hilbert-Action

We begin by assuming that we can describe spacetime as a pseudo-Riemannian manifold, with signature $(1, 3)$. See A.4 for a short introduction. In addition we also assume that the manifold is torsionless, with torsion defined by Eq. (A.9).

We want to investigate the geometry of spacetime, described by the metric $g_{\mu\nu}$. One important concept from Newtonian physics is the principle of stationary action. For a mechanical system, the action S is defined as a time integral of the Lagrangian L

$$S = \int_{t_1}^{t_2} L dt. \tag{2.1}$$

The principle of stationary action then states that the system follows a trajectory such that the variation of the action satisfies

$$\delta S = 0, \tag{2.2}$$

meaning a small perturbation of the system does not change the action integral. We assume that this principle holds for the geometry of spacetime, and using the Lagrange density $\mathcal{L}$, we can reformulate the action as an integral over a region of spacetime Ω, as

$$S = \int_{\Omega} \mathcal{L}\, d^4x. \tag{2.3}$$

We then can develop the field equations for the geometry of spacetime, by looking at small variations in the metric $g_{\mu\nu}$, and requiring that the variation of the action vanishes.

To begin to tackle this problem, we first need to find a suitable Lagrangian. Our Lagrangian must depend on the metric, and because the action is a physical quantity, it must also as give us an action integral that is coordinate invariant. In A.5 we discuss the transformational properties of integrals in spacetime, and show that we can use $\sqrt{|g|}d^4x$ as the coordinate invariant measure, where $g = \det(g_{\mu\nu})$. This means we can construct invariant integrals by letting our integrand consist of only coordinate invariant terms. The simplest such term we can construct, that also depends on the metric tensor, is the Ricci scalar R, defined by Eq. (A.6). We also choose to add a constant term, Λ, and scale our Lagrangian by a constant factor κ. We then arrive at the Einstein-Hilbert action, where we have used the conventional factors in front of the constants, given by

$$S_{\text{EH}} = \int \frac{1}{2\kappa} (R - 2\Lambda) \sqrt{-g} d^4x, \tag{2.4}$$

where we used that $|g| = -g$, which holds on a manifold with an odd number of timelike dimensions.

The Einstein-Hilbert action has no dependence on matter fields, so we add a term $\mathcal{L}_M$ containing this dependence. The action then becomes

$$S = \int \left[\frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_M \right] \sqrt{-g} d^4x. \tag{2.5}$$

We next consider a small variation of the inverse metric $\delta g^{\mu\nu}$, and require that the action is stationary,

$$\begin{aligned} 0 &= \delta S \\ &= \int \left[\frac{1}{2\kappa} \left(\frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} - \frac{2\Lambda \delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x \\ &= \int \left[\frac{1}{2\kappa} \left(\frac{R - 2\Lambda}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta R}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \sqrt{-g} d^4x. \end{aligned} \tag{2.6}$$

As the variation $g^{\mu\nu}$ is arbitrary, the integrand must vanish everywhere, giving us the following field equation

$$\frac{R - 2\Lambda}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta R}{\delta g^{\mu\nu}} = \kappa \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}. \tag{2.7}$$

The right-hand side define the Hilbert stress-energy tensor $T_{\mu\nu}$ as in [10, p. 75], in the following way

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2 \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M, \tag{2.8}$$

where we have used the relation

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{2.9}$$

which is given in Ref. [11, p. 117]. Now we need to rewrite the term containing the variation of the Ricci scalar. The variation of the Ricci scalar is given by

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \tag{2.10}$$

Now we need to relate $\delta R_{\mu\nu}$ to the variation $\delta g^{\mu\nu}$. Using the definition of the Riemann tensor $R^{\rho}{}_{\sigma\mu\nu}$, given in Eq. (A.4), the variation becomes

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \delta\Gamma^{\rho}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\rho}_{\mu\sigma} + \delta\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \delta\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma}. \tag{2.11}$$

The Christoffel symbols $\Gamma^{\rho}{}_{\mu\nu}$ are defined in Eq. (A.7). The variation $\delta\Gamma^{\rho}{}_{\mu\nu}$ is the difference between two connections. If we look at the transformation law given in Eq. (A.8), we see that the second term containing only the coordinate basis cancels, and the variation of the Christoffel symbols therefore transforms as a tensor. We can then construct the covariant derivative using Eq. (A.16),

$$\nabla_\mu(\delta\Gamma^{\rho}_{\nu\sigma}) = \partial_\mu(\delta\Gamma^{\rho}_{\nu\sigma}) + \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\delta\Gamma^{\rho}_{\nu\lambda}. \tag{2.12}$$

Using that the Christoffel symbols are symmetric in the lower indices when we have assumed no torsion, we see that we can express the variation of the Riemann tensor as the difference of two such terms,

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \nabla_\mu(\delta\Gamma^{\rho}_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^{\rho}_{\mu\sigma}). \tag{2.13}$$

We may now obtain the variation of the Ricci tensor by contracting two of the indices, and we get

$$\delta R_{\mu\nu} = \delta R^{\gamma}{}_{\mu\gamma\nu} = \nabla_\gamma\left(\delta\Gamma^{\gamma}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^{\gamma}_{\mu\gamma}\right). \tag{2.14}$$

Using the metric compatibility explained in A.3, given by

$$\nabla_\gamma g^{\mu\nu} = 0, \tag{2.15}$$

we get

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\gamma(g^{\mu\nu}\delta\Gamma^{\gamma}_{\mu\nu} - g^{\mu\gamma}\delta\Gamma^{\lambda}_{\mu\lambda}). \tag{2.16}$$

This is just a divergence on the form $\nabla_\mu A^\mu$. Considering again Eq. (2.6), we see that this term is exactly on the form of the divergence theorem for a Riemannian manifold given in Eq. (A.15), and will therefore only contribute a boundary term when integrated. Requiring that the variation of the metric, and its derivatives, vanishes on the boundary, the contribution vanishes, and we can use

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu}. \tag{2.17}$$

Inserting Eq. (2.17) into Eq. (2.7), and again using Eq. (2.9), we obtain

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.18}$$

which is Einstein's field equations. The LHS of Eq. (2.18) is often simplified using the Einstein tensor, $G_{\mu\nu}$, defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{2.19}$$

giving us

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.20}$$

The value of κ can be determined by considering the Newtonian limit, and is found to be $\kappa = 8\pi G$, where G is the Newtonian constant of gravitation. Λ is called the cosmological constant, and describes the vacuum energy density. $|\Lambda|^{-1/2}$ is of dimension length, and on lengths or times small compared to this quantity, the effects of Λ are negligible. Experiments suggest a small positive value: $\Lambda^{-1/2} \sim 10^9$ light years. This is of the same order of magnitude as the observable universe. We can therefore set $\Lambda = 0$ when we are not discussing cosmology, which we will assume unless otherwise stated for the remainder of this thesis.

2.2 The Palatini Approach

In the derivation of the Einstein field equations in the last section, we put two important restrictions on the structure of spacetime. Firstly, we assumed that the manifold is torsionless, and secondly, that the connection coefficients satisfy the metric compatibility given by Eq. (2.15) in A.3. We can relax these restrictions slightly, and assume that the metric compatibility no longer holds. This means that the only constraints on the connection coefficients now are that the torsion vanishes. Now the connection coefficients and the metric are independent fields, which lead to new field equations. We will see that this leads to the same field equations as in the previous section, and that the resulting connection coefficients satisfy the metric compatibility.

In the previous section we considered the variation of the action from a variation in the inverse metric $g^{\mu\nu}$, were the connection coefficients depended on the metric by Eq. (A.7). Now the connection coefficients are given by Eq. (A.4) and are independent of $g^{\mu\nu}$. The Lagrangian we start with is still the same as in the last section, and we assume that the matter part of the Lagrangian only depends on the metric. The important distinction in the variation of the action, is that the variation of the Ricci tensor $\delta R_{\mu\nu}$ is now a function only of the connection coefficients, as seen from Eq. (A.5). The variation of the action can then be separated into terms depending on $g^{\mu\nu}$ and $R_{\mu\nu}$ in the following way

$$\begin{aligned}\delta S &= \int \left[\frac{1}{2\kappa}\left(R - 2\Lambda\right)\delta\sqrt{-g} + \frac{1}{2\kappa}(\sqrt{-g}\delta R) + \mathcal{L}_M \delta\sqrt{-g}\right] d^4x \\ &= \int \left[\frac{1}{2\kappa}\left(-\frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + R_{\mu\nu}\right) + \frac{1}{2}T_{\mu\nu}\right]\delta g^{\mu\nu}\sqrt{-g}d^4x \\ &\quad + \frac{1}{2\kappa}\int \left[g^{\mu\nu}\delta R_{\mu\nu}\right]\sqrt{-g}d^4x. \end{aligned} \tag{2.21}$$

As the two variations are independent, both integrals must vanish. The first integral leads exactly to Einstein's field equations, as in the previous section. The second integral depends on the variation of the Ricci tensor, and we can use Eq. (2.14), which has only assumed no torsion, to get

$$\delta S = \frac{1}{2\kappa}\int \left[\nabla_\gamma(\delta\Gamma^\gamma_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\gamma_{\mu\gamma})\right]\sqrt{-g}g^{\mu\nu}d^4x. \tag{2.22}$$

Through an integration by parts, we have

$$\begin{aligned}\delta S =& \frac{1}{2\kappa}\int \nabla_\nu\left[g^{\mu\gamma}\delta\Gamma^\nu_{\mu\gamma} - g^{\mu\nu}\delta\Gamma^\gamma_{\mu\gamma}\right]\sqrt{-g}d^4x \\ &- \frac{1}{2\kappa}\int\sqrt{-g}\left[\nabla_\nu(g^{\mu\gamma})\delta\Gamma^\nu_{\mu\gamma} - \nabla_\nu(g^{\mu\nu})\delta\Gamma^\gamma_{\mu\gamma}\right]d^4x\end{aligned} \tag{2.23}$$

As in the section above, the first integral is by the divergence theorem only a boundary term, which vanishes when we require the variation to vanish on the boundary. The second integral can be rearranged as follows,

$$\begin{aligned}\delta S &= \frac{1}{2\kappa}\int\sqrt{-g}\left[\nabla_\nu(g^{\mu\nu})\delta\Gamma^\gamma_{\mu\gamma} - \nabla_\nu(g^{\mu\gamma})\delta\Gamma^\nu_{\mu\gamma}\right]d^4x \\ &= \frac{1}{2\kappa}\int\sqrt{-g}\left[\nabla_\lambda(g^{\mu\lambda})\delta^\gamma_\nu\delta\Gamma^\nu_{\mu\gamma} - \nabla_\nu(g^{\mu\gamma})\delta\Gamma^\nu_{\mu\gamma}\right]d^4x \\ &= \frac{1}{2\kappa}\int\sqrt{-g}\left[\nabla_\lambda(g^{\mu\lambda})\delta^\gamma_\nu - \nabla_\nu(g^{\mu\gamma})\right]\delta\Gamma^\nu_{\mu\gamma}d^4x.\end{aligned} \tag{2.24}$$

$\delta\Gamma^\nu_{\mu\gamma}$ is symmetric in the lower indices, as there is no torsion. $\nabla_\nu(g^{\mu\gamma})$ is also symmetric in μ and γ. The term $\nabla_\lambda(g^{\mu\lambda})\delta^\gamma_\nu$ can be made symmetric in μ and γ by addition of $\nabla_\lambda(g^{\gamma\lambda})\delta^\mu_\nu$, and antisymmetric by subtracting the same term. This means we can split this term into a symmetric tensor $S^{\mu\gamma}_\nu$, and an antisymmetric tensor $A^{\mu\gamma}_\nu$. Eq. (2.24) then becomes

$$\delta S = \frac{1}{2\kappa}\int\sqrt{-g}\left[S^{\mu\gamma}_\nu + A^{\mu\gamma}_\nu\right]\delta\Gamma^\nu_{\mu\gamma}d^4x, \tag{2.25}$$

where

$$S^{\mu\gamma}_\nu = \frac{1}{2}\left[\nabla_\lambda(g^{\mu\lambda})\delta^\gamma_\nu + \nabla_\lambda(g^{\gamma\lambda})\delta^\mu_\nu\right] - \nabla_\nu(g^{\mu\gamma}), \tag{2.26}$$

$$A^{\mu\gamma}_\nu = \frac{1}{2}\left[\nabla_\lambda(g^{\mu\lambda})\delta^\gamma_\nu - \nabla_\lambda(g^{\gamma\lambda})\delta^\mu_\nu\right]. \tag{2.27}$$

A symmetric tensor times an antisymmetric tensor is antisymmetric, and $A^{\mu\gamma}_\nu$ therefore does not contribute, as we sum over μ and γ. This gives us

$$\delta S = \frac{1}{2\kappa}\int\sqrt{-g}\left(S^{\mu\gamma}_\nu\right)\delta\Gamma^\nu_{\mu\gamma}d^4x. \tag{2.28}$$

Since the variation of the connection coefficients are arbitrary, $S^{\mu\gamma}_\nu$ must vanish, or

$$\frac{1}{2}\left[\nabla_\lambda(g^{\mu\lambda})\delta^\gamma_\nu + \nabla_\lambda(g^{\gamma\lambda})\delta^\mu_\nu\right] - \nabla_\nu(g^{\mu\gamma}) = 0. \tag{2.29}$$

Setting $\nu \neq \gamma$ and $\nu \neq \mu$ gives $\nabla_\nu(g^{\mu\gamma}) = 0$, and all terms where the derivative does not match an index in the metric vanishes. Setting $\nu = \gamma \neq \mu$, gives $\frac{1}{2}\nabla_\lambda g^{\mu\lambda} - \nabla_\nu g^{\mu\gamma}$.

By varying $\nu = \gamma$, we see that all terms where the derivative share an index with them metric vanishes. Therefore, the covariant derivative of $g^{\mu\nu}$ is equal to zero, which also gives $\nabla_\gamma g_{\mu\nu} = 0$. By Eq. (A.16), we then have

$$0 = \nabla_\gamma g_{\mu\nu} = \partial_\gamma g_{\mu\nu} - \Gamma^\lambda_{\mu\gamma} g_{\lambda\nu} - \Gamma^\lambda_{\gamma\nu} g_{\mu\lambda}, \tag{2.30}$$

and

$$\partial_\gamma g_{\mu\nu} = \Gamma^\lambda_{\mu\gamma} g_{\lambda\nu} + \Gamma^\lambda_{\gamma\nu} g_{\mu\lambda}. \tag{2.31}$$

This is the exact equation that leads to the equation for the Christoffel symbols given in Eq. (A.7), which can be shown by calculating $\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ki}$ and using the symmetry of the lower indices of the connection coefficient. Thus, we have shown that the principle of least action for the connection exactly implies the metric compatibility of the connection. It is worth noting that this is only valid when we assume a torsionless manifold and Einstein-Hilbert action. For a general action, the Palatini approach does not necessarily lead to a metric compatible connection.

Chapter 3

The weak-field limit of the Einstein equations

The nonlinearity of the Einstein field equations make them generally hard to solve. However, in some cases we are dealing with weak gravitational fields, and we can then consider spacetime as almost flat. We let the metric be given by the Minkowski metric of flat spacetime, plus a small perturbation metric $h_{\mu\nu}$, in the following way

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.1}$$

with $|h_{\mu\nu}| \ll 1$. When the perturbation is small, we can as a good approximation only consider terms up to first order of the perturbation. We call this the weak-field limit. We note here that the derivatives of $h_{\mu\nu}$ in a source free space is equal to $h_{\mu\nu}$ times a frequency, which we will see later, and are therefore also first order in $h_{\mu\nu}$. Our metric needs to satisfy $g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}_{\nu}$, which to first order in $h_{\mu\nu}$ gives us

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{3.2}$$

The inverse perturbation can be calculated,

$$\begin{aligned} h^{\mu\nu} &= g^{\mu\rho}g^{\nu\sigma}h_{\rho\sigma} \\ &= \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}, \end{aligned} \tag{3.3}$$

where the second line is achieved by only considering first order terms. This demonstrates that we can raise and lower indices in the weak-field approximation using the unperturbed Minkowski metric, $\eta_{\mu\nu}$. We can now calculate the Christoffel symbols from Eq. (A.7). The derivatives of the Minkowski metric are always zero, and only terms depending on $h_{\mu\nu}$ remain. We obtain

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}) = \frac{1}{2}(h^{\rho}{}_{\mu,\nu} + h^{\rho}{}_{\nu,\mu} - h_{\mu\nu}{}^{,\rho}), \tag{3.4}$$

to first order in $h_{\mu\nu}$. The Riemann tensor is defined by Eq. (A.4). In the weak field limit the last two terms are of second order in $h_{\mu\nu}$, and therefore disregarded. This gives

$$\begin{aligned} R^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} \\ &= \frac{1}{2}(h^{\rho}{}_{\nu,\mu\sigma} + h^{\rho}{}_{\sigma,\mu\nu} - h_{\nu\sigma,\mu}{}^{\rho}) - \frac{1}{2}(h^{\rho}{}_{\mu,\nu\sigma} + h^{\rho}{}_{\sigma,\nu\mu} - h_{\mu\sigma,\nu}{}^{\rho}) \\ &= \frac{1}{2}(h^{\rho}{}_{\nu,\mu\sigma} + h_{\mu\sigma,\nu}{}^{\rho} - h_{\nu\sigma,\mu}{}^{\rho} - h^{\rho}{}_{\mu,\nu\sigma}). \end{aligned} \tag{3.5}$$

The Ricci tensor is defined by Eq. (A.5). Using Eq. (3.5), we get

$$\begin{aligned} R_{\mu\nu} &= R^{\rho}{}_{\mu\rho\nu} \\ &= \frac{1}{2}(h^{\rho}{}_{\nu,\rho\mu} + h_{\rho\mu,\nu}{}^{\rho} - h_{\nu\mu,\rho}{}^{\rho} - h^{\rho}{}_{\rho,\nu\mu}) \\ &= \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h^{\rho}{}_{\mu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right), \end{aligned} \tag{3.6}$$

where $h = h^{\rho}{}_{\rho}$ and $\Box = \partial^\rho\partial_\rho$. The Ricci scalar, defined by Eq. (A.6), is then

$$R = g^{\mu\nu}R_{\mu\nu} = \eta^{\mu\nu}R_{\mu\nu} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h. \tag{3.7}$$

The Einstein tensor, given in Eq. (2.19), for the weak-field limit then becomes

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h^{\rho}{}_{\mu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\left[\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h\right]\right). \tag{3.8}$$

Inserting this into Eq. (2.20), gives us

$$\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h^{\rho}{}_{\mu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\left(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h\right) = 2\kappa T_{\mu\nu}. \tag{3.9}$$

We can multiply Eq. (3.9) with $\eta^{\mu\nu}$, obtaining

$$2\kappa T^{\mu}{}_{\mu} = 2\partial_\mu\partial_\rho h^{\rho\mu} - 2\Box h - \delta^{\mu}_{\mu}\left(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h\right), \tag{3.10}$$

from which we can calculate the trace of $T_{\mu\nu}$,

$$T = T^{\mu}{}_{\mu} = -\frac{1}{\kappa}(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h). \tag{3.11}$$

Thus we can rewrite Eq. (3.9) as

$$\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h^{\rho}{}_{\mu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h = 2\kappa\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) = 2\kappa\bar{T}_{\mu\nu} \tag{3.12}$$

where we have defined the bar notation, which we will call "trace-reversed" as $\bar{T}^{\rho}{}_{\rho} = -T^{\rho}{}_{\rho}$. We can compare this equation to Eq. (3.6), which immediately gives us the following relation,

$$R^{(1)}_{\mu\nu} = \kappa\bar{T}_{\mu\nu}, \tag{3.13}$$

where the superscript signifies the term of linear order in h. We can also rewrite Eq. (3.9) in terms of the trace-reversed $\bar{h}_{\mu\nu}$,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{3.14}$$

giving us

$$\partial_\mu\partial_\rho\bar{h}^{\rho}{}_{\nu} + \partial_\nu\partial_\rho\bar{h}^{\rho}{}_{\mu} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} = 2\kappa T_{\mu\nu}. \tag{3.15}$$

Chapter 4

Gauge freedom in linearized gravity

In this chapter we want to investigate the gauge freedom in Eq. (3.15), and fix a gauge that will simplify our calculations. We begin with a general discussion of Green's functions and how they relates to gauge symmetry. We then use the formulation developed to fix the gauge for linearized gravity.

4.1 Green's functions and gauge symmetry

A Green's function $G(x, s)$ is defined as a function that satisfies

$$LG(\mathbf{x}, \mathbf{s}) = \delta(\mathbf{s} - \mathbf{x}), \tag{4.1}$$

for a linear differential operator $L = L(\mathbf{x})$ in n-dimensions, where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{s} = (s_1, \dots, s_n)$. We suppress the vector notation in the following for simplicity. This property can be used to solve differential equations on the form

$$Lu(x) = f(x) \tag{4.2}$$

by setting

$$u(x) = \int G(x, s) f(s) ds. \tag{4.3}$$

Eq. (4.3) a solution to Eq. (4.2), which is verified by direct insertion, and using that L commutes with integration,

$$\begin{aligned} Lu(x) &= \int LG(x, s) f(s) ds \\ &= \int \delta(s - x) f(s) ds \\ &= f(x). \end{aligned} \tag{4.4}$$

The Green's function therefore defines an integral transform T_G that maps a function f to another function $T_G f$, in the following way:

$$(T_G f)(x) = \int G(x,s) f(s) ds. \tag{4.5}$$

This integral transform T_G is the right inverse of the differential operator L, as

$$L(T_G f)(x) = f(x), \tag{4.6}$$

which is already demonstrated in Eq. (4.4). We can define a kernel K of the differential operator L as the space of all functions $k(x)$ that satisfies

$$Lk(x) = 0. \tag{4.7}$$

If the kernel K is nontrivial, meaning $K \neq \{k(x) = 0\}$, then the Green's functions of L are not unique. This is because when there exists some function $k(x)$ for which $Lk(x) = 0$, we can construct two different Green's functions that both satisfy Eq. (4.2),

$$\begin{aligned} u(x) &= \int G(x,s) f(s) ds, \\ u'(x) &= \int G'(x,s) f(s) ds, \end{aligned} \tag{4.8}$$

where

$$u'(x) = u(x) + k(x), \text{ for } k(x) \in K. \tag{4.9}$$

This is equivalent to saying that L is not invertible. To make L invertible we need to establish a one-to-one relation, making sure there exists a unique Green's function for our system. We therefore have to deal with the kernel K. The kernel lives in the space of all differentiable functions. Due to the linearity of the differential operator, we know that K itself is linear. This means that K is spanned by a (possibly infinite) set of linearly independent basis functions. We can separate these into two different types of basis functions; functions that affect the boundary conditions, which will be called global, and functions that does not affect boundary conditions, which will be called local. As the basis functions are independent, specifying the boundary conditions of the system will completely determine the value of all global functions, whereas all local functions are still undefined. Now we let Eq. (4.2) be the equation of motion for some physical system, where $f(x)$ is the source of a field $u(x)$. The kernel K is the space of all solutions to $Lu(x) = 0$. K may contain both functions that correspond to physical observables and functions corresponding to gauge symmetry. Here we use the word gauge symmetry as any non-physical symmetry of the system[1], e.g. choice of coordinate system, unobservable variables, etc. From classical physics we know we can uniquely determine the physical observables of $u(x)$ if we know the source field $f(x)$ and all boundary conditions (both in space and time). This is just another way to say that classical equations of motion are deterministic. A classical wave has to have a source, and cannot spontaneously come into existence by

[1] Some authors only use the word gauge symmetry when there is local gauge symmetry.

itself. We can therefore conclude that all functions in K corresponding to physically observable phenomena have to affect the boundary conditions, and therefore be described by purely global functions. Let us consider a system with only global functions in K. Before we have chosen the boundary conditions, there exists many Green's functions, as shown in Eq. (4.8). By letting the boundary conditions be a constraint on the system, only a single function from the kernel survives, we will call it $k_f(x)$, and the solution to the system is $u_f(x) = u(x) + k_f(x)$. Now the kernel of L is trivial, because there exists no nontrivial function $k(x)$ we can add that satisfies $Lk(x) = 0$ and also does not change the boundary conditions. These boundary conditions have a corresponding Green's function $G_f(x, s)$ that satisfies Eq. (4.1). This Green's function will also satisfy the additional constraint that any function transformed by the corresponding integral transformation, Eq. (4.5), satisfies the boundary conditions.

Now we want to include local functions in K in our discussion. Apart from functions corresponding to physical observables, there can exist gauge symmetry in our theory. The functions corresponding to gauge freedom can be both local and global. A global gauge freedom is dealt with in the same way as global physical freedom; by enforcing boundary conditions. Local gauge freedom, on the other hand, has the unique property that it makes the propagation of the system non-deterministic. From our discussion above we saw that, without local gauge symmetry, we could find a unique Green's function for a physical system when we defined the boundary condition. This Green's function works as a propagator, taking in initial conditions and sources, and creates the field. The local gauge freedom makes it impossible to define a general propagator, as there are multiple possible solutions to the propagated system which only differ by local gauge transformations, and does not affect the boundary conditions. To continue we therefore have to fix the local gauge freedom, i.e. the value of the local functions in K. To do this we impose some constraint that selects a function from the set of local functions in K. This will become clear when we consider examples.

Before we move on, a couple things should be noted. For a physical system, functions corresponding to physical observables and gauge freedom are always linearly independent. This follows from the fact that we require that the gauge freedom cannot change any physical observable. When we start from Eq. (4.2), with no boundary conditions and no gauge choices, the kernel K of L is spanned by possible global physical functions, global gauge freedom, and local gauge freedom. Due to linearity of K, the fixing of either one does not affect the freedom of the others, and we can chose to fix them in any order we want.

4.2 Fixing the gauge in linearized gravity

We now return to linearized gravity, and the wave equation for $\bar{h}_{\mu\nu}$ given in Eq. (3.15). We will use language developed in the previous section to investigate and fix the gauge freedom in this theory. The gauge freedom arises from the fact that we can make coordinate transformations, $x^\mu \to x^\mu + \xi^\mu(x)$, where $\xi^\mu(x)$ satisfies $|\xi^\mu{}_{,\nu}| \ll 1$, that leads to changes in the metric $h_{\mu\nu}$. This freedom has both a local and a global part, which we will see explicitly. We want to create a general equation for $\bar{h}_{\mu\nu}$, and not impose global boundary conditions, and thus we want to deal with the local gauge freedom first. After that we will use the global gauge freedom to impose constraints on our boundary conditions, which will put $h_{\mu\nu}$ in a particularly nice form.

The gauge freedom is given by the coordinate transformation

$$x^\mu \to x^\mu + \xi^\mu(x), \qquad |\xi^\mu{}_{,\nu}| \ll 1. \tag{4.10}$$

This transformation has a local freedom, because the derivatives $\xi^\mu(x)$, can be chosen independently at every point in spacetime. This coordinate change is essentially a perturbation of the coordinate system, where every point in our coordinate system is perturbed by a small vector, given by the function ξ^μ. To make sure that all lengths and angles in our system are preserved, the metric has to change according to the new coordinate system. Thus, the new metric is a perturbation of the old metric. Under this transformation, a general metric $g_{\mu\nu}$ transforms as

$$\begin{aligned} g'^{\mu\nu}(x') &= \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} g^{\rho\sigma}(x) \\ &= (\delta^\mu_\rho + \xi^\mu{}_{,\rho})(\delta^\nu_\sigma + \xi^\nu{}_{,\sigma}) g^{\rho\sigma}(x) \\ &= g^{\mu\nu} + \xi^{\mu,\nu} + \xi^{\nu,\mu}. \end{aligned} \tag{4.11}$$

In the wield approximation we have $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, which inserted into the above transformation gives

$$\begin{aligned} \eta^{\mu\nu} - h'^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + \xi^{\mu,\nu} + \xi^{\nu,\mu} \\ h'^{\mu\nu} &= h^{\mu\nu} - \xi^{\mu,\nu} - \xi^{\nu,\mu}. \end{aligned} \tag{4.12}$$

We can lower the indices, obtaining

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}. \tag{4.13}$$

We know that the entries of $h'_{\mu\nu}$ are small, as we have required that all the derivative of ξ^μ are small. We see that both our initial and transformed coordinate systems have metrics that can be described by Eq. (3.1), satisfying the weak field approximation. These coordinate systems are therefore equally valid, and we have no physical grounds to chose one over the other. This freedom is purely a gauge freedom. The two different coordinate systems describe the same manifold, and we can show explicitly that this is true by calculating the change in the Riemann tensor. Using Eqs. (3.4), (4.13) and the first order terms of (A.4), we get that the infinitesimal change in the Riemann tensor is

$$\begin{aligned} R'_{\mu\nu\rho\sigma} - R_{\mu\nu\rho\sigma} &= \frac{1}{2}(\partial_\rho\partial_\nu\partial_\mu\xi_\sigma + \partial_\rho\partial_\nu\partial_\sigma\xi_\mu + \partial_\sigma\partial_\mu\partial_\nu\xi_\rho + \partial_\sigma\partial_\nu\partial_\mu\xi_\rho + \partial_\sigma\partial_\mu\partial_\rho\xi_\nu \\ &\quad - \partial_\rho\partial_\nu\partial_\mu\xi_\sigma - \partial_\rho\partial_\nu\partial_\sigma\xi_\mu - \partial_\sigma\partial_\mu\partial_\nu\xi_\rho - \partial_\sigma\partial_\nu\partial_\mu\xi_\rho - \partial_\sigma\partial_\mu\partial_\rho\xi_\nu) \\ &= 0, \end{aligned} \tag{4.14}$$

as expected. The RHS of Eq. (3.15) can be expressed by Eq. (2.18), where we already have set $\Lambda = 0$, which to first order in $h_{\mu\nu}$ gives us

$$2\kappa T_{\mu\nu} = 2R_{\mu\nu} - R\eta_{\mu\nu}. \tag{4.15}$$

The expressions of $R_{\mu\nu}$ and R are fully determined by the Riemann tensor, and as the Riemann tensor is invariant under the above mentioned coordinate transformation, $\kappa T_{\mu\nu}$ is also invariant. Thus, we have shown that the transformation given in Eq. (4.13) is a gauge symmetry. We also note here that this infinitesimal coordinate transformation has no Newtonian counterpart. Under this transformation the local properties of the coordinate system are changed, and the only reason this can be done is because this information can be put into the metric. In Newtonian physics we do not use a metric function, and we therefore do not have this option.

We can rewrite Eq. (3.15) as

$$L_{\mu\nu}^{\alpha\beta}\left(\bar{h}_{\alpha\beta}\right) = 2\kappa T_{\mu\nu}, \tag{4.16}$$

where

$$L_{\mu\nu}^{\alpha\beta} = \left(\delta_\nu^\beta \partial_\mu \partial^\alpha + \delta_\mu^\beta \partial_\nu \partial^\alpha - \delta_\mu^\alpha \delta_\nu^\beta \Box - \eta_{\mu\nu} \partial^\alpha \partial^\beta\right) \tag{4.17}$$

is our differential operator. We now have our wave equation on the form of Eq. (4.2). The differential operator in Eq. (4.17) has a nontrivial kernel K, which is spanned by basis functions corresponding to possible wave solutions and gauge symmetry. For any function $k_{\alpha\beta}$ from the kernel K, we have

$$L_{\mu\nu}^{\alpha\beta}\left(k_{\alpha\beta}\right) = 0. \tag{4.18}$$

We want to deal with the local gauge freedom, which means fixing the part of K spanned by local functions. We can use the transformational property of $\bar{h}_{\mu\nu}$, which from Eqs. (3.14) and (4.11) are

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu} \xi_\rho{}^{,\rho}. \tag{4.19}$$

This allows us to choose what is called the harmonic gauge, defined by

$$\partial^\nu \bar{h}_{\mu\nu} = 0, \tag{4.20}$$

or, expressed for the normal perturbative metric,

$$\partial^\lambda h_{\alpha\lambda} = \frac{1}{2} \partial_\alpha h. \tag{4.21}$$

We can demonstrate that we can always use the harmonic gauge. Suppose our metric is not in the harmonic gauge. Can we find a $\xi^\mu(x)$ that transforms our system into the harmonic gauge? For a transformation of coordinate systems, by taking the derivative ∂^ν of Eq. (4.19) we have

$$\begin{aligned} \partial^\nu \bar{h}'_{\mu\nu} &= \partial^\nu \bar{h}_{\mu\nu} - \partial^\nu \partial_\nu \xi_\mu - \partial^\nu \partial_\mu \xi_\nu + \partial_\mu \partial^\rho \xi_\rho \\ &= \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu. \end{aligned} \tag{4.22}$$

We can choose a transformation that satisfies

$$\Box \xi_\mu = \partial^\nu \bar{h}_{\mu\nu}. \tag{4.23}$$

This is a wave equation that always has solutions, and we can therefore always choose the harmonic gauge. These solutions are not unique, which we will see later. What we have done here is that we have chosen the derivatives of ξ^μ to satisfy a local constraint, given by the harmonic gauge. With this gauge choice only one term on the LHS of Eq. (3.15) remains, and our wave equation simplifies to

$$\Box\bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}. \tag{4.24}$$

We know that the harmonic gauge is a strong enough constraint to remove all local gauge freedom, as there are no local functions in the kernel of our new differential operator $L' = \Box$. Therefore, we have now made a gauge choice that deals with all the local gauge freedom in our system. This means that Eq. (4.24) can be solved uniquely once the boundary conditions are specified. We will see that there is still a global gauge freedom, but this freedom affects the boundary conditions. We will return to the global gauge freedom when we consider gravitational wave solutions in the next chapter.

4.3 Comparison to gauge fixing in electrodynamics

To better understand the gauge choices we have just made, we will briefly compare it to the gauge fixing scheme in electrodynamics, which is easier to understand. For a gauge transformation all physical observables are invariant. In linearized gravity the invariant observable is the Riemann curvature tensor. In electrodynamics the invariant observable is the stress-energy tensor $F^{\mu\nu}$, given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{4.25}$$

The wave equation for A^μ, in a general gauge, is given by

$$\left(\Box\delta^\nu_\mu - \partial^\nu\partial_\mu\right) A^\mu = J^\mu. \tag{4.26}$$

Eq. (4.26) is on the form of (4.2), where our linear operator is given by $L^\nu_\mu = \left(\Box\delta^\nu_\mu - \partial^\nu\partial_\mu\right)$. This linear operator has a nontrivial kernel, which consists in part of gauge symmetry. In electrodynamics the gauge symmetry is easily seen by observing that that $F^{\mu\nu}$ is invariant under the transformation $A^\mu \to A^\mu + \partial^\mu\varphi$. We can verify that $\partial^\mu\varphi$ is in the kernel of L^ν_μ by insertion,

$$L^\nu_\mu(\partial^\mu\varphi) = \left(\Box\delta^\nu_\mu - \partial^\nu\partial_\mu\right)\partial^\mu\varphi = 0. \tag{4.27}$$

We observe that this is a local gauge symmetry, which by our discussion above means that the propagation of A^μ is not deterministic. We therefore need to fix the gauge. We observe that we can choose the Lorentz gauge, defined by

$$\partial_\alpha A^\alpha = 0, \tag{4.28}$$

by choosing a φ that satisfies

$$\Box\varphi = -\partial_\alpha A^\alpha. \tag{4.29}$$

This is a wave equation that always have multiple solutions. In this gauge the second term in the differential operator L^ν_μ vanishes. We can thus remove this term, and our differential operator becomes invertible, which means we can solve for A^μ. Still there is a gauge freedom left in our system. We see that a transformation $A^\mu \to A^\mu + \partial^\mu\varphi'$, where

φ' satisfies $\Box\varphi' = 0$, is still permitted, and does not affect the Lorentz gauge condition. The possible φ' are on the form

$$\varphi' = \text{Re}\left(Ae^{ik_\sigma x^\sigma}\right), \tag{4.30}$$

where A is a constant and k_σ a null vector. We see that φ' clearly affects the boundary conditions, and thus this is a global gauge symmetry.

From this short discussion of gauge freedom in electrodynamics, we can draw very clear comparisons to the gauge fixing procedure in linearized gravity. Both theories include a non-invertible linear operator in the equation of motion, due to local gauge symmetry. The process of fixing the local gauge is by choosing a transformation that ensures that all but one of the terms in the linear operator are zero. This gives us a new linear operator that is invertible. Also in both theories there is a global gauge symmetry still left, which can be used to put constraints on the boundary conditions (the global gauge symmetry in linearized gravity is discussed in the next chapter). The reason why the gauge fixing procedure in linearized gravity seems more complicated, is because the physical freedom and the gauge symmetry appear mixed. All the entries of $h_{\mu\nu}$ have gauge freedom and physical freedom. However, they are linearly independent, as already discussed. In electrodynamics the separation between gauge freedom and physical freedom is much more evident, as A^μ has physically observable entries only on the off-diagonal, while the diagonal entries only correspond to gauge freedom. We can thus conclude that the gauge freedom in linearized gravity and electrodynamics, and the gauge fixing procedures, are very similar in nature.

Chapter 5

Gravitational wave solutions in vacuum

Now we return to linearized gravity. We still need to fix the global gauge, and to do this we will consider gravitational wave solutions in vacuum. The gauge we will choose can only be chosen in vacuum. In vacuum Eq. (4.24) reduces to

$$\Box \bar{h}_{\mu\nu} = 0. \tag{5.1}$$

We look for solutions on the form

$$\bar{h}_{\mu\nu} = \text{Re}\left(H_{\mu\nu} e^{ik_\rho x^\rho}\right), \tag{5.2}$$

where $H_{\mu\nu}$ is a constant symmetric complex matrix, and k^μ is a wave vector. We suppress Re in our notation in the following for simplicity. Inserting Eq. (5.2) into our wave equation, we see this is a solution if

$$k_\mu k^\mu = 0. \tag{5.3}$$

This implies that k^μ is a null vector, which means that the wave propagates at the speed of light relative to the background Minkowski space. Imposing the harmonic gauge condition gives

$$k^\nu H_{\mu\nu} = 0, \tag{5.4}$$

and we see that the wave is transverse. We now consider the gauge freedom that is left in our formulation. We want to find a coordinate transformation that preserves the harmonic gauge. If we choose a four-vector ξ^μ that satisfies

$$\Box \xi_\mu = 0, \tag{5.5}$$

the harmonic gauge condition is still satisfied, as seen by Eq. (4.23). We see that we have a freedom to choose a coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$, with ξ given by

$$\xi_\mu(x) = \text{Re}\left(iX_\mu e^{ik_\rho x^\rho}\right). \tag{5.6}$$

Here k^μ is a null vector, which ensures that Eq. (4.23) is satisfied, and X_μ is a real-valued constant four-vector. We see that this is a global gauge freedom, as X_μ is constant. Using Eq. (4.19), we see that our residual gauge freedom is then

$$H'_{\mu\nu} = H_{\mu\nu} + k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k^\rho X_\rho. \tag{5.7}$$

Eq. (5.7) is a transformation with four degrees of freedom, given by the four components of X_μ. This gauge choice affects our boundary conditions, as it is global. We can choose 4 constraints on our boundary conditions, which will in turn fix the global gauge. We select four functions of $H_{\mu\nu}$ that we will use to fix the gauge: $\frac{1}{2}H, H_{01}, H_{02}$, and H_{03}. By using Eq. (5.7) we find that these components transform in following way:

$$\begin{pmatrix} \frac{1}{2}H' \\ H'_{01} \\ H'_{02} \\ H'_{03} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}H \\ H_{01} \\ H_{02} \\ H_{03} \end{pmatrix} + \begin{pmatrix} -\omega & -k_1 & -k_2 & -k_3 \\ k_1 & -\omega & 0 & 0 \\ k_2 & 0 & -\omega & 0 \\ k_3 & 0 & 0 & -\omega \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}, \tag{5.8}$$

where $\omega = -k_0 = k^0$. The transformation matrix is invertible, so we can choose

$$\begin{pmatrix} \frac{1}{2}H \\ H_{01} \\ H_{02} \\ H_{03} \end{pmatrix}' = 0 \tag{5.9}$$

as a gauge condition, by setting

$$\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = - \begin{pmatrix} -\omega & -k_1 & -k_2 & -k_3 \\ k_1 & -\omega & 0 & 0 \\ k_2 & 0 & -\omega & 0 \\ k_3 & 0 & 0 & -\omega \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2}H \\ H_{01} \\ H_{02} \\ H_{03} \end{pmatrix}. \tag{5.10}$$

Thus, we have fully fixed X^μ, which means that there is no more gauge freedom in our system. By Eq. (4.20) and (5.4), this gauge choice satisfies

$$H_{0\mu} = 0, \tag{5.11}$$

and

$$H = H^\rho{}_\rho = 0. \tag{5.12}$$

The last conditions shows that the waves are traceless. We already mentioned the waves are transverse, and we therefore call this the transverse-traceless (TT) gauge. Now our gauge is completely fixed. The gauge we have chosen has only traceless solutions. $\bar{h}_{\mu\nu} = h_{\mu\nu}$ follows, and we can drop the bar notation. We also see that by choosing this gauge, we have imposed constraints on our possible boundary conditions, given in Eq. (5.9). These constraints are however only on the global gauge, and not the physical situations it allows. All physical boundary conditions are still available and can be expressed in the harmonic gauge [1]. Our symmetric constant matrix $H_{\mu\nu}$ originally had 10 independent entries. The

[1] As long as we are in vacuum. The gauge conditions have to be modified to allow for matter.

choice of harmonic gauge, Eq. (5.4), gave four constraints, bringing us down to six independent entries. The gauge condition of the TT gauge, Eq. (5.9), remove four more, bringing us down to two independent degrees of freedom. We have now exhausted the gauge freedom in the system, which means that the kernel of L that also satisfies the boundary constraints, now only consists of functions describing physical waves. This means that we have established a one-to-one relation between $h_{\mu\nu}$ and the boundary conditions, and we can therefore solve for $h_{\mu\nu}$ when these are specified. This concludes our gauge fixing process.

5.1 Polarizations of Gravitational waves

After fully fixing the gauge, we are left with two degrees of freedom in $H_{\mu\nu}$. This freedom corresponds to two different polarizations. We can demonstrate this by considering a wave on the form given by Eq. (5.2), with a wave vector $k^\mu = \omega(1, 0, 0, 1)$. For this wave vector, the harmonic gauge, Eq. (4.20), together with Eq. (5.11), gives the condition

$$H_{3\nu} = H_{\nu 3} = 0, \tag{5.13}$$

and we see that only H_{11}, H_{22}, H_{12} and H_{21} are nonzero. Imposing now that $H_{\mu\nu}$ is traceless and symmetric, gives

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.14}$$

Here we clearly see two distinct polarizations, H_+ and $H_\times$, which, together with the wave frequency ω, completely characterizes a plane wave traveling in the x^3-direction. For a wave traveling in a general direction $\hat{\mathbf{n}}$, the spatial part of $H_{\mu\nu}$ can be expressed as [12, Ch. 1.2]

$$H_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij}, \tag{5.15}$$

where the polarization tensors are defined as

$$e^+_{ij}(\hat{\mathbf{n}}) = \hat{\mathbf{u}}_i \hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i \hat{\mathbf{v}}_j, \qquad e^\times_{ij}(\hat{\mathbf{n}}) = \hat{\mathbf{u}}_i \hat{\mathbf{v}}_j + \hat{\mathbf{u}}_j \hat{\mathbf{v}}_i, \tag{5.16}$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are unit vectors orthogonal to the propagation vector $\hat{\mathbf{n}}$ and each other.

Chapter 6

Effects of gravitational waves on test masses

In this chapter we want to discuss how gravitational waves interact with masses. We will use the term test mass to describe a freely falling mass which is small compared to the metric, so that the metric can be considered uniform over the whole mass. In the previous chapters we have discussed the gauge dependence of GWs. In this chapter we will see that our choice of gauge corresponds to a choice of coordinate system, which is important in deriving the equations of motions for a test mass. This chapter is largely based on [12, Ch.1].

6.1 Equation of geodesic deviation

In Newtonian physics, an object has zero acceleration in the absence of external forces. In general relativity, the equivalent relation is called the *geodesic equation* (a derivation can be found in [13, Ch. 8]), and is given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{6.1}$$

This equation describes the motion of a test mass in a curved space, in the absence of non-gravitational forces. The path such a particle follows is called a geodesic.

We now want to look at two nearby test masses, and how their separation vector changes as we increase the proper time, as a function of the metric. Let the initial separation vector be given by ξ^μ, and using Eq. (6.1), we get

$$\frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x+\xi)\frac{d(x^\nu + \xi^\nu)}{d\tau}\frac{d(x^\rho + \xi^\rho)}{d\tau} = 0. \tag{6.2}$$

We let $|\xi^\mu|$ be small compared to the scale of variation of the metric. We now take the difference between Eqs. (6.1) and (6.2), and expand to the first order in ξ, giving us

$$\begin{aligned} 0 &= \frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x)\left(\frac{d\xi^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \frac{dx^\rho}{d\tau}\frac{d\xi^\nu}{d\tau}\right) + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} \\ &= \frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} \end{aligned} \tag{6.3}$$

To write it in a more elegant way, we introduce the covariant derivative along a curve $x^\mu(\sigma)$, in the following way

$$\frac{D}{d\sigma} \equiv \frac{dx^\mu}{d\sigma}\nabla_\mu. \tag{6.4}$$

Choosing $\sigma = \tau$ and using Eq. (A.16), the covariant derivative of a vector field $V^\mu(x)$ along $x(\tau)$ is

$$\frac{DV^\mu}{d\tau} = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}V^\nu\frac{dx^\rho}{d\tau}. \tag{6.5}$$

We can use this to simplify Eq. (6.3) which, after some algebra and by applying Eq. (A.4), is equal to

$$\frac{D^2\xi^\mu}{d\tau^2} = -R^\mu{}_{\nu\rho\sigma}\xi^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}, \tag{6.6}$$

which is the *equation of geodesic deviation*. This equation describes how the separation vector between two nearby geodesics change as a function of curvature, expressed by the Riemann tensor. It is important to note that ξ^μ is a coordinate length, not a proper length, and therefore depends on the choice of coordinates. The gauge freedom in the theory is therefore included in ξ^μ and $R^\mu{}_{\nu\rho\sigma}$, as both the coordinate length as well as the Riemann tensor depend on the choice of coordinate system.

6.2 Reference frame and the TT-gauge

We have already derived the TT-gauge, and the simple form the gravitational wave solutions have in this gauge. Now we want to connect the TT-gauge to a reference frame, which we will call the TT-frame. To do this we want to know what it means physically to be in the TT-gauge. We answer this question by considering the geodesic of a test mass initially at rest. We use the spatial part of Eq. (6.1), evaluated $\tau = 0$, giving us

$$\begin{aligned} \left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} &= -\left[\Gamma^i_{\nu\sigma}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right]_{\tau=0} \\ &= -\left[\Gamma^i_{00}(x)\left(\frac{dx^0}{d\tau}\right)^2\right]_{\tau=0}. \end{aligned} \tag{6.7}$$

Here $i = \{1, 2, 3\}$, and in the second line we have used that the initial velocity, $dx^i/d\tau$, is zero by assumption. The TT-gauge is valid in the linearized theory, so the Christoffel symbol is given by Eq. (3.4), giving us

$$\Gamma^i_{00}(x) = \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00}). \tag{6.8}$$

However, in the TT-gauge we have $h_{0i} = h_{00} = 0$, and therefore Eq. (6.8) vanishes. Thus the RHS of Eq. (6.7) is zero, which means that $dx^i/d\tau$ remains at zero, from which it follows that

$$\frac{d^2x^i}{d\tau^2} = 0 \tag{6.9}$$

for any τ. We see that in the TT-frame, any test mass initially at rest stays at rest even when a gravitational wave passes, at least to linear order in $h_{\mu\nu}$. This means that the coordinate system can be defined by a letting a grid of free falling test masses define the coordinate points. From this it follows that the coordinate length between two free falling masses, initially at rest, never changes. We can show this by explicit calculation using Eq. (6.3), evaluated at $\tau = 0$. As our test masses are initially at rest, we have $dx^i/d\tau = 0$, and we get

$$\left.\frac{d^2\xi^\mu}{d\tau^2}\right|_{\tau=0} = -\left[2\Gamma^\mu_{0\rho}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{00}\right]_{\tau=0}. \tag{6.10}$$

$\Gamma^0_{0\rho} = \Gamma^0_{00} = 0$ in the TT-gauge. We already showed that $\Gamma^\mu_{00} = 0$ in the TT-gauge, and we are left with the term depending on $\Gamma^\mu_{0\rho}$. In the TT-gauge, we also have $\Gamma^0_{0\rho} = 0$. Together with $\Gamma^i_{00} = 0$, this means we can exchange both μ and ρ with spatial indices, $j, i \in \{1, 2, 3\}$. Now reviewing Eq. (3.4) for the remaining term in Eq. (6.10), we have

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} = -\left[\partial_0 h_{ij}\frac{d\xi^i}{d\tau}\right]_{\tau=0}. \tag{6.11}$$

At $\tau = 0$ we have $d\xi^i/d\tau = 0$. This means there is no acceleration, and $d\xi^i/d\tau$ remains at zero. Therefore the separation vector ξ^μ is a constant, as we have previously argued for.

6.3 Proper length in the TT-frame

Until now we have dealt with the coordinate lengths in the TT-frame, and we have seen that the coordinate length is constant for stationary, free falling objects. Next we want to calculate the proper length between our free falling test masses. We consider two masses separated by an infinitesimal separation vector. We place the first test mass at the origin, and one placed at $(0, dx, dx, 0)$, giving us the separation vector $\xi^\mu = (0, dx, dy, 0)$. We let a gravitational wave propagating in the x^3-direction pass. The metric in the TT-frame can be expressed in the following form (see Eqs. (5.2) and (5.14)),

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos(k^3x^3 - k^0t). \tag{6.12}$$

The differential proper length ds is given by

$$ds = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}. \tag{6.13}$$

At $x^\mu = 0$ we have $\cos(k^3x^3 - k^0t) = 1$, and the infinitesimal proper length is then

$$ds = \sqrt{(1 + H_+)dx^2 + (1 - H_+)dy^2 + (2H_\times)dxdy}. \tag{6.14}$$

Inserting for the cases for $dy = 0$ and $dx = 0$, gives us

$$ds_x = \sqrt{1 + H_+}dx \tag{6.15}$$

$$ds_y = \sqrt{1 - H_+}dy, \tag{6.16}$$

and we see that the H_+-polarization stretches the proper length in the x-direction, and contracts the proper lengths in the y-direction. Similarly, the $H_\times$-polarization stretches the proper length in the xy-direction, and contracts the proper length in the negative xy-direction. This is shown in Figures 6.1 and 6.2.

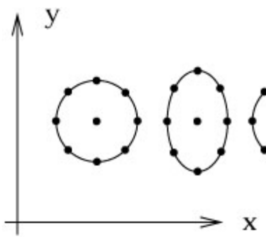

Figure 6.1: Diagram showing the effects of an H_+-polarized wave on a ring of particles. Time evolution is to the right, and the wave propagates in the z-direction. Figure courtesy of Sean Carroll, [14].

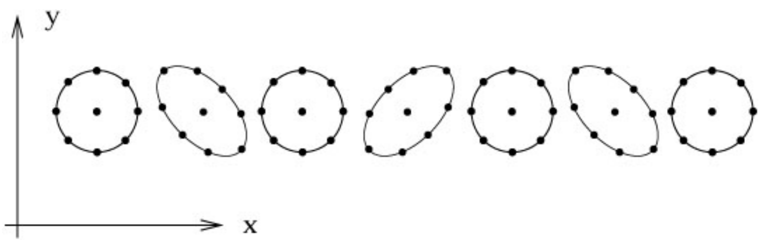

Figure 6.2: Diagram showing the effects of an $H_\times$-polarized wave on a ring of particles. Time evolution is to the right, and the wave propagates in the z-direction. Figure courtesy of Sean Carroll, [14].

Before we proceed on, it is appropriate that we discuss the results we have obtained so far. We have seen that the TT-gauge corresponds to a coordinate system defined by free falling masses. A freely falling object will therefore not change its coordinate position due to a passing gravitational wave. This is counter-intuitive from a Newtonian perspective. From Newtonian physics we are used to gravity as an accelerating force, and we would therefore expect the test masses to be accelerated. We will see that this does indeed happen if we use a different frame of reference, although the nature of the acceleration will be different from that of a particle in a stationary gravitation fields. In the discussion above we let our reference frame be decided by the TT-gauge choices, which means the reference frame is perturbed to account for the change in proper length between the free falling masses. Another reference point we can chose is a rigid ruler. Do we have something that approximates a rigid ruler? Yes. The physical laws that determine atomic spacing in normal materials, the Coulomb force and the nuclear forces, depend on the proper length, and are orders of magnitude stronger than the gravitational force. Thus the length of a physical object will be dominated by non-gravitational forces, and the proper length will stay fixed. Therefore the next frame we will investigate is the detector frame, which is spanned by idealized rigid rulers. This is essentially the coordinate system spanned by markers in a lab, and is therefore more intuitive to work in. In this frame we expect particles that are free to move to be accelerated by gravitational waves.

6.4 Defining the detector frame

To construct the detector frame we will evoke some results from differential geometry. It is always possible to perform a change of coordinates so that all the Christoffel symbols vanish at the origin. We then let the spatial origin of our coordinate system follow the geodesic of a fictitious test mass placed in the origin; this is called a free falling coordinate system. We then use three spatial orthogonal basis vectors, with a coordinate length defined as the proper length along the vector, and let the coordinate time be defined by the proper time of the geodesic, i.e. $t = \tau$. Lastly, we attach gyroscopes to the basis vectors, which ensures the coordinate system rotates with a free falling system. This guarantees that our coordinate system is a Local Lorentz frame at any time, where the Christoffel symbols vanish at the spatial origin at any time [12, Ch.1].

We now restrict our coordinate system to a sufficiently small region of space, and we can calculate the metric in orders of the spatial component x^i. To first order the metric is just the Minkowski metric of flat spacetime, which follows from the fact that we have chosen a coordinate system where all the Christoffel symbols vanish at the spatial the origin. The line element $ds^2(x^i)$ in this coordinate system can be found, and is given in [12, Eq. 1.87] to second order in x^i, as

$$\begin{aligned} ds^2 = -dt^2\left[1 + R_{0i0j}x^i x^j\right] - 2dtdx^i\left(\frac{2}{3}R_{0jik}x^j x^k\right) \\ +dx^i dx^j\left[\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l\right] + \mathcal{O}\left[(x^i)^3\right], \end{aligned} \tag{6.17}$$

with the Riemann tensor evaluated at $x^\mu = (\tau, 0)$. In this frame we can investigate what happens with a free falling test mass, initially at rest at $(0, x^i)$, where x^i small, under the influence of a gravitational wave. We again use Eq. (6.3). The Christoffel symbols are zero in the vicinity of $(t, 0)$, so we immediately get

$$\frac{d^2x^\mu}{d\tau^2} = -x^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho}(0)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}. \tag{6.18}$$

We know that the time derivatives of the Christoffel symbols in the vicinity of the spatial origin are zero, as we have $\Gamma^\rho_{\mu\nu}(t, 0) = 0$ for any t. We also have $R^i{}_{0j0} = \partial_j\Gamma^i_{00} - \partial_0\Gamma^i_{j0} = \partial_j\Gamma^i_{00}$. Inserting this into (6.18) gives us

$$\frac{d^2x^i}{d\tau^2} = -R^i{}_{0j0}x^j\left(\frac{dx^0}{d\tau}\right)^2. \tag{6.19}$$

The proper time τ relates to the coordinate time as $d\tau^2 = dt^2 + \mathcal{O}(h^2)$. This follows from a linear dependence on h of the velocity introduced by a gravitational wave. If we restrict ourselves to linear order in h, Eq. (6.19) becomes

$$\ddot{x}^i = -R^i{}_{0j0}x^j. \tag{6.20}$$

We now want to evaluate the Riemann tensor. In the linearized theory of gravity the Riemann tensor is *invariant*, as seen by Eq. (4.14). This means that no components of the Riemann tensor changes under a gauge transformation, and we are thus permitted to calculate the Riemann tensor in any gauge we like. For simplicity, we chose the TT-gauge. From Eq. (A.4), and using that only the purely spatial part of $h_{\mu\nu}$ is non-zero in the TT-gauge, Eq. (6.20) simplifies to

$$\ddot{x}^i = \frac{1}{2}\ddot{h}^{TT}_{ij}x^j. \tag{6.21}$$

From this equation we see that in the detector frame, the gravitational wave can be observed as a Newtonian acceleration, or equivalently as a force through $\vec{F} = m\vec{a}$, where the acceleration depends on the spatial separation vector x^j. The form of the equation mirrors Newtonian gravity, but at the same time shows how different the effects of gravitational waves are from a static gravitational field. The acceleration from a GW has no intrinsic direction, and is uniform in space, on lengths small compared to the wavelength. The direction and magnitude depends on the separation from where you are measuring, i.e. the spatial origin of the coordinate system. The gravitational force from the GW is *intrinsic* in space, whereas .

To fully understand what this means, and also shed some light on the detector frame, we want to do a thought experiment. We consider two free falling objects, A and B, separated by a rigid object of length L. We assume that object L is stationary in the coordinate system. In panel 1 of Figure 6.3 we have shown the initial situation. We now let a gravitational wave pass our system, with a propagation vector normal to the paper plane, and consider a time where the axis along our system is stretched. We place the origin of our detector frame in point A. Here the Christoffel symbols vanish, and A therefore stays in the center during free fall. We want to consider what happens to the coordinates of the free falling mass B. From Eq. (6.21), we see that mass B has accelerated out compared to the rigid ruler of length L. This is shown in panel 2. The distance d is proportional to the length L and the strain (defined as $\Delta L/L$) of the gravitational wave. In panel 3 we have set the same requirements, just assuming we place mass B in the center of our detector frame instead. We see that panel 2 and panel 3 show physically different scenarios, and can therefore not describe the same physical situation. Therefore it seems our argumentation is flawed. The reason for this error is that the rigid ruler is not in free fall in point A and B, which we used as an assumption. When the GW passes, the Coulomb forces in the ruler act against the forces from the GW, preventing a change in the length of the ruler. The center of the ruler is in free fall, as the Coulomb forces from the two arms cancel, but as you go out to either side, this is no longer the case. At the ends of the ruler, i.e. point A and B, the Coulomb forces only act to one side, pulling the ends of the ruler towards to center. Therefore, point A and point B are not in free fall. We have to place the origin of our free falling coordinate system in the center of mass of L, and we get panel 4, which shows the correct state of the system when a GW passes. Thus we see that a rigid objects break the translational symmetry of Eq. (6.21), which follows from the requirement of having the origin of the coordinate system follow a geodesic.

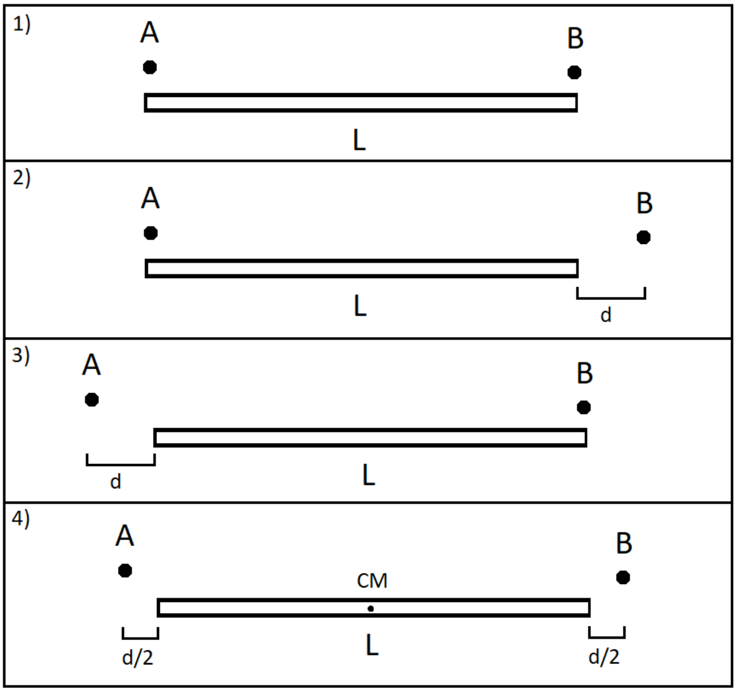

Figure 6.3: This figure highlights the restriction of placing the origin, in constructing the detector frame, see main text for details.

The above discussion of the detector frame is relevant for detection of gravitational waves on earth. However there are some challenges that we need to discuss. We derived Eq. (6.21) for a freely falling, non-rotating coordinate system. A Lab on the Earth, however, is not in free fall, and also rotates relative to a local gyroscope, as demonstrated by Foucault pendulum. Furthermore, there are time varying changes in the static gravitational field due to distance to other stellar objects, such as the Sun and the Moon. Also, a Lab on the Earth is connected to the Earth, which is a large object, which to some degree is rigid. We will deal with these challenges in two steps. First, we will consider how Eq. (6.21) changes when the center of the coordinate system to a rigid object, i.e. having a lab placed in point A in the situation shown in Figure 6.3. Second, we will briefly describe how we can select the contribution from GWs from a myriad of other sources.

We want to consider what happens when we shift the spatial origin of the lab frame. In the lab frame, the spatial center is placed in the center of mass, which means calculations will follow Eq. (6.21). We now make a coordinate transformation by shifting the spatial center, transforming our coordinates by $x'^{\mu} = x^{\mu} + \xi^{\mu}$, where $\xi = (0, \xi^i)$ is a constant spatial vector. The constancy of ξ^i shows that the center of the transformed coordinate system is connected to the center of the old coordinate system by a rigid ruler. The shift of the spatial origin does not change under the effect of GWs. We see that we can modify Eq. (6.21) to be used in the transformed coordinate system, and by simple insertion we have

$$\ddot{x}'^i = \frac{1}{2}\ddot{h}_{ij}^{TT}(x'^j + \xi^j). \tag{6.22}$$

We see that there is a contribution from the deviation from the center of mass. However, if we consider a relative acceleration of two free falling masses in the shifted coordinate system, the contribution from $\frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j$ cancels, and we can use Eq. (6.21) in this case. Thus for an Earth based lab, this issue is dealt with by having two test masses, and measuring the relative distance between them.

The second challenge we have for an Earth based detector are all the (gravitational and inertial acceleration) contributions from other sources than GWs. We already know that gravitational waves are very weak, and we expect the effect of the other sources to be much stronger than that of gravitational waves. The solution is quite simple. We start by suspending the free test masses in pendulums. This restricts the radial motion of the test masses, thus cancelling the effects of the stationary gravitational field. It also allows the test masses to move freely, for low deviations, in the plane parallel to the Earth. Now, all the different contributions, i.e. Coriolis force, centrifugal acceleration, positions of the Moon, etc., as well as the GWs, will affect the position of the test masses. Luckily, other than GWs, all these contributions vary slowly in time, with a frequency lower than a few Hz, while the typical frequency of detectable GWs are in the order of 100 Hz. We can therefore consider the position of the test masses due to all other sources as a the base position, and look for high-frequency variation from this position. This is achieved by looking only at the high-frequency components of the Fourier transformed signal.

From this discussion we see how the ground-based interferometer from the introduction, shown in Fig. 1.2, can successfully detect gravitational waves, by hanging its mirrors in pendulums and Fourier transforming the signal. We have also successfully connected the gauge choice of the gravitational wave, with a coordinate choice used for detection, where we can calculate their expected effects.

Chapter 7

The stress-energy tensor of gravitational waves

An aspect of gravitational wave theory that has been the subject of much debate, is the energy carried by gravitational waves. This is due to the difficulty of separating the effects of the choice of coordinate system from physical effects, as well as the problem of localizing the energy. In the following discussion we will see how we can derive the energy of GWs, and overcome the challenges mentioned here.

7.1 Second-order expansion of the metric around flat spacetime

As previously noted, the Einstein field equations are non-linear. Thus far we have used a linear approximation in h, and we want to see what happens if we include second order terms. This means that we will allow our metric to be expanded as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu}, \tag{7.1}$$

where the components of $h^{(2)}_{\mu\nu}$ are of order h^2. We want to find the Einstein tensor of this metric, up to second order in h. We can write the Einstein tensor into terms of different powers of h. The second order term will consist of a term quadratic in h and a term linear in $h^{(2)}$. Thus we have

$$G_{\mu\nu} = G^{(1)}_{\mu\nu}[h] + G^{(2)}_{\mu\nu}[h] + G^{(1)}_{\mu\nu}[h^{(2)}] + \mathcal{O}(h^3), \tag{7.2}$$

where the superscript denotes linear or quadratic dependence on the argument. The first order term of the Einstein tensor is just what we calculated in the weak-field approximation, given in Eq. (3.8). The calculation of $G^{(1)}_{\mu\nu}[h^{(2)}]$ is exactly the same as that of $G^{(1)}_{\mu\nu}[h]$ with a different argument, and we can simply make the substitution $h \to h^{(2)}$ in Eq. (3.8).

To calculate $G^{(2)}_{\mu\nu}[h]$, we start by expanding R and $R_{\mu\nu}$ in the same way as we did for $G_{\mu\nu}$, obtaining

$$R_{\mu\nu} = R^{(1)}_{\mu\nu}[h] + R^{(2)}_{\mu\nu}[h] + R^{(1)}_{\mu\nu}[h^{(2)}] + \mathcal{O}(h^3), \tag{7.3}$$

and

$$R = R^{(1)}[h] + R^{(2)}[h] + R^{(1)}[h^{(2)}] + \mathcal{O}(h^3). \tag{7.4}$$

We can now calculate $G^{(2)}_{\mu\nu}[h]$ by inserting Eqs. (7.3) and (7.4) into Eq. (2.19), and collecting the terms quadratic in h, giving us

$$G^{(2)}_{\mu\nu}[h] = R^{(2)}_{\mu\nu}[h] - \frac{1}{2}R^{(1)}[h]h_{\mu\nu} - \frac{1}{2}R^{(2)}[h]\eta_{\mu\nu}. \tag{7.5}$$

From the definition of the Ricci scalar, Eq. (A.6), we get

$$R^{(2)}[h] = \eta^{\mu\nu}R^{(2)}_{\mu\nu}[h] - h^{\mu\nu}R^{(1)}_{\mu\nu}[h], \tag{7.6}$$

where the minus sign comes from the inverse form of the metric.

We now consider the vacuum. $h_{\mu\nu}$ satisfies the linear Einstein field equations in the vacuum, which means $G^{(1)}[h] = 0$. Requiring that the Einstein tensor vanish therefore means that the terms quadratic in h have to cancel against the terms linear in $h^{(2)}$, in the following way,

$$G^{(2)}_{\mu\nu}[h] + G^{(1)}_{\mu\nu}[h^{(2)}] = 0. \tag{7.7}$$

In the vacuum we have $R^{(1)}_{\mu\nu} = 0$, which can be seen by Eq. (3.6). Thus, by using this and Eqs. (7.5) and (7.6), we obtain

$$G^{(2)}_{\mu\nu}[h] = R^{(2)}_{\mu\nu}[h] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h]\eta_{\mu\nu}. \tag{7.8}$$

Now, Eq. (7.7) can be rewritten as

$$G^{(1)}_{\mu\nu}[h^{(2)}] = -\left(R^{(2)}_{\mu\nu}[h] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h]\right). \tag{7.9}$$

This effectively is an equation of motion for $h^{(2)}_{\mu\nu}$ that depends on the linear term $h_{\mu\nu}$. Thus, this equation shows that $h^{(2)}_{\mu\nu}$ is a non-linear correction term to the linear Einstein field equations. Furthermore, by comparison to Eq. (2.18), we see that the RHS of Eq. (7.9) acts as a matter field for $h^{(2)}_{\mu\nu}$. It is therefore natural to define $t_{\mu\nu}$ as

$$\begin{aligned} t_{\mu\nu} &= -\frac{1}{\kappa}G^{(2)}_{\mu\nu}[h] \\ &= \frac{1}{\kappa}\left(R^{(2)}_{\mu\nu}[h] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h]\right). \end{aligned} \tag{7.10}$$

With $h'_{\mu\nu} = h_{\mu\nu} + h^{(2)}_{\mu\nu}$, we have

$$G^{(1)}_{\mu\nu}[h'] = \kappa\left(T_{\mu\nu} + t_{\mu\nu}\right) \tag{7.11}$$

to first order in $h'_{\mu\nu}$, which means $t_{\mu\nu}$ affects the metric like an stress-energy tensor.

We next consider the contracted Bianchi identity on the form given in Eq. (B.2),

$$g^{\mu\rho}\nabla_\rho G_{\mu\nu} = 0. \tag{7.12}$$

We can expand this in orders of h, using

$$g^{\mu\rho} = \eta^{\mu\rho} - h^{\mu\rho} + \mathcal{O}(h^2), \tag{7.13}$$

$$\nabla_\rho = \partial_\rho + \Gamma[h] + \mathcal{O}(h^2), \tag{7.14}$$

where $\Gamma[h]$ denotes some dependence on the Christoffel symbols of order h, as well as our expansion of $G_{\mu\nu}$ given in Eq. (7.2). To first order in h, Eq. (7.12) is

$$\partial^\mu G^{(1)}_{\mu\nu}[h] = 0. \tag{7.15}$$

The Bianchi identity holds for an arbitrary metric that is metric compatible, and we have not yet imposed any restrictions on h. Therefore, Eq. (7.15) holds to first order for an arbitrary metric perturbation, i.e. we can choose $h = h^{(2)}$. The second order-term of Eq. (7.12) is

$$\partial^\mu \left(G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h] \right) + \Gamma[h] G^{(1)}[h] = 0. \tag{7.16}$$

Here the last term has dependence on the Christoffel symbols and the first-order Einstein tensor. Now we impose the requirement that $h_{\mu\nu}$ satisfy the linear Einstein equations, which means that the last term vanishes. Furthermore, the term $G^{(1)}_{\mu\nu}[h^{(2)}]$ is also zero, which follows from Eq. (7.15). Thus, (7.16) reduces to

$$\partial^\mu G^{(2)}_{\mu\nu}[h] = \partial^\mu t_{\mu\nu} = 0. \tag{7.17}$$

We now know that $t_{\mu\nu}$ is symmetric, quadratic in the metric perturbation $h_{\mu\nu}$, and conserved when $h_{\mu\nu}$ satisfies the linear Einstein equations in vacuum. In addition, it serves as a source for the second order corrections to the metric, and is therefore a natural candidate for the stress-energy tensor of the linearized gravitational field. However, we will see that this statement requires more consideration.

7.2 Gauge dependence of $t_{\mu\nu}$

To understand whether $t_{\mu\nu}$ can be a suitable candidate for the stress-energy tensor for GWs, we need to consider its gauge dependence. We start by discussing what we mean by gauge dependence. The principle of general covariance states that the form of the laws of physics should be invariant under general diffeomorphisms, i.e. coordinate transformations. Einstein's field equations equates the stress-energy tensor $T_{\mu\nu}$ with the Einstein tensor $G_{\mu\nu}$, constructed entirely from the metric $g_{\mu\nu}$ and its derivatives. Under a coordinate change $x^\mu \to x'^\mu$ the tensors $G_{\mu\nu}$ and $T_{\mu\nu}$ transform covariantly, i.e. as

$$T'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} T_{\rho\sigma}. \tag{7.18}$$

Because $T_{\mu\nu}$ and $G_{\mu\nu}$ transform in the same way, Einstein's field equations hold in all coordinate systems. We have already seen how, in linearized gravity, $h_{\mu\nu}$ follows a different transformation law (see Eq. (4.13)). We call this a gauge dependence because we can change the form of $h_{\mu\nu}$ through a coordinate transformation, i.e. to satisfy $\partial^\mu h_{\mu\nu} = 0$. However, the quantities corresponding to physical observables, such as the stress-energy tensor $T_{\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$, always transform as in Eq. (7.18). This is a requirement that follows from the principle of general covariance, to keep the form of the physical laws the same in all coordinate systems. Thus, for $t_{\mu\nu}$ to be a valid stress-energy tensor, we need it to transform according to Eq. (7.18). From Eq. (7.10), we see that $t_{\mu\nu}$ depends on the quadratic part of the Ricci tensor. Eq. (A.5) gives the expression for the Ricci tensor, and by only keeping terms that are quadratic in h, we get the expression for $R^{(2)}_{\mu\nu}[h]$. The terms with derivatives of the Christoffel symbols are quadratic in h, so they do not contribute. We obtain

$$R^{(2)}_{\mu\nu}[h] = \Gamma^{\gamma}_{\gamma\lambda}[h]\Gamma^{\lambda}_{\mu\nu}[h] - \Gamma^{\gamma}_{\nu\lambda}[h]\Gamma^{\lambda}_{\gamma\mu}[h]. \tag{7.19}$$

Now we use the equation for the Christoffel symbols in the linear theory, given in Eq. (3.4), giving us

$$\begin{aligned} R^{(2)}_{\mu\nu}[h] = \frac{1}{2}h^{\rho\sigma}h_{\rho\sigma,\mu\nu} + \frac{1}{4}h_{\rho\sigma,\mu}h^{\rho\sigma}{}_{,\nu} + h^{\rho}{}_{\nu}{}^{,\sigma}h_{\mu[\rho,\sigma]} - h^{\rho\sigma}h_{\sigma(\nu,\mu)\rho} \\ + \frac{1}{2}\partial_\sigma(h^{\rho\sigma}h_{\mu\nu,\rho}) - \frac{1}{4}h_{\mu\nu,\rho}h^{,\rho} - (h^{\rho\sigma}{}_{,\sigma} - \frac{1}{2}h^{,\rho})h_{\rho(\nu,\mu)}, \end{aligned} \tag{7.20}$$

where we have introduced the following notation,

$$A_{(\nu,\mu)} = \frac{1}{2}(A_{\nu,\mu} + A_{\mu,\nu}), \tag{7.21}$$

$$A_{[\nu,\mu]} = \frac{1}{2}(A_{\nu,\mu} - A_{\mu,\nu}). \tag{7.22}$$

We know that we can always make a coordinate transformation that sets the Christoffel symbols to zero at a point, i.e. using Riemann normal coordinates. By looking at Eqs. (7.10) and (7.19), this means that for any point in spacetime, we can make a gauge transformation that sets $t_{\mu\nu} = 0$ at that point. The same thing can not be said of a tensor that transforms as (7.18). Therefore we know that $t_{\mu\nu}$ does not transform as a covariant tensor, and we say that $t_{\mu\nu}$ is gauge dependent. This disqualifies $t_{\mu\nu}$ as a stress-energy tensor. We can make this point clearer by considering Eq. (7.11) again. Here we see that $h_{\mu\nu} + h^{(2)}_{\mu\nu}$ satisfies the linearized Einstein field equations for a stress-energy tensor given by $T_{\mu\nu} + t_{\mu\nu}$. However, the LHS of Eq. (7.11) is gauge invariant, while the RHS is not gauge invariant due to $t_{\mu\nu}$. Thus, promoting $t_{\mu\nu}$ to the stress-energy tensor of GWs in linearized theory leads to breaking the principle of general covariance. However, we have already shown that $t_{\mu\nu}$ fulfills many of the requirements of an stress-energy tensor, and we will therefore pursue the idea further.

7.3 Gravitational waves in curved spacetime

In a pursuit to understand the full meaning of Eq. (7.10) it is necessary to take a more sophisticated approach. In the above section we looked at small perturbations of the metric about flat spacetime. As a first approximation, this is a very useful approach. However, by allowing gravitational waves to carry energy, we are allowing the waves to curve the background on which they are propagating. Above, this extra curvature was expressed by $h^{(2)}_{\mu\nu}$. By allowing the background metric to be perturbed by the GWs we will gain new insight into the problem.

In this section I will discuss some of the important ramifications of allowing a general background metric, without going into detail of the mathematical derivations. A full discussion can be found in [12, Ch. 1].

We define our metric as

$$g_{\mu\nu} = g^B_{\mu\nu} + h_{\mu\nu}, \tag{7.23}$$

where $g^B_{\mu\nu}$ is a slowly changing background metric, and $h_{\mu\nu}$ is our familiar perturbation. Here we meet one of the important distinctions we have to make. There is in general no unambiguous way to make the separation of Eq. (7.23). When we looked at perturbations about flat spacetime, the separation was automatic, whereas here we need to define which part of the metric is *background* and which part is the GW. To be able to distinguish the perturbations from the background, we need that $g^B_{\mu\nu}$ varies slowly compared to $h_{\mu\nu}$, either spatially or temporally. To express this condition, we can use L_B as the scale on which $g^B_{\mu\nu}$ changes and f_B as the maximal frequency of its Fourier decomposition, and λ as the typical scale where $h_{\mu\nu}$ changes and f as its typical frequency. The separation can be made if either one of the following two conditions are met,

$$\lambda \ll L_B, \tag{7.24}$$

$$f \gg f_B. \tag{7.25}$$

It is most common to discuss the case where Eq. (7.24) is met, and we will assume that this condition is met here, although both cases are equivalent for the purpose of this discussion. We choose a coordinate system where the the components of $g^B_{\mu\nu}$ are of order 1, which can always be done for a region of space, and require that the components of $h_{\mu\nu}$ are small compared to those of $g^B_{\mu\nu}$. We will see that this latter condition follows from Eq. (7.24).

To find the equations of motion for $h_{\mu\nu}$ we follow a similar strategy to the discussion above, and expand the Ricci tensor in orders of h, as

$$R_{\mu\nu} = R^B_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathcal{O}(h^3). \tag{7.26}$$

Here $R^B_{\mu\nu}$ depends only on $g^B_{\mu\nu}$, $R^{(1)}_{\mu\nu}$ is linear in h, and $R^{(2)}_{\mu\nu}$ is quadratic in h. We note that we have no term corresponding to $R^{(1)}_{\mu\nu}[h^{(2)}]$ in Eq. (7.3), as $h^{(2)}_{\mu\nu}$ is in this formulation incorporated into $g^B_{\mu\nu}$. Furthermore, we recast the Einstein's field equations in the following form,

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{7.27}$$

which is obtained by calculating the trace of $T_{\mu\nu}$, and eliminating the term depending on the Ricci scalar from the LHS of Eq. (2.18). Now we look at the terms in Eq. (7.26), from the perspective of our separation condition Eq. (7.24). We want to separate the terms in accordance with the scale at which they vary. $R^B_{\mu\nu}$ varies slowly as it depends only on a slowly varying background metric. $R^{(1)}_{\mu\nu}$ depends linearly on $h_{\mu\nu}$, and therefore varies quickly. $R^{(2)}_{\mu\nu}$ depends quadratically on $h_{\mu\nu}$, and therefore has both high- and low-frequency modes, where the low-frequency modes can be formed as $\sim h_{\mu\nu}h_{\sigma\rho}$, with $\mathbf{k}_1 \simeq -\mathbf{k}_2$. Thus, we combine Eqs. (7.26) and (7.27) into the following two equations,

$$R^B_{\mu\nu} = -\Big[R^{(2)}_{\mu\nu}\Big]^{\text{Low}} + \kappa\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)^{\text{Low}}, \tag{7.28}$$

and

$$R^{(1)}_{\mu\nu} = -\Big[R^{(2)}_{\mu\nu}\Big]^{\text{High}} + \kappa\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)^{\text{High}}. \tag{7.29}$$

For the case where $T_{\mu\nu} = 0$, we see that in Eqs. (7.28) and (7.29) we have equated terms of different order in h. This is fine as we have an auxiliary small expansion parameter, namely the ratio λ/L_B (or f_B/f in the case of Eq. (7.25)), which can compensate for the smallness of h. Furthermore, this means that the relative strength of these parameters are decided by the Einstein field equations. We can find this relation, by comparing orders of magnitude.

The expressions for $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are given in [12, Sect. 1.4], as

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\big(\nabla^{B\alpha}\nabla^B_\mu h_{\nu\alpha} + \nabla^{B\alpha}\nabla^B_\nu h_{\mu\alpha} - \nabla^{B\alpha}\nabla^B_\alpha h_{\mu\nu} - \nabla^B_\mu\nabla^B_\nu h\big), \tag{7.30}$$

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}g^{B\rho\sigma}g^{B\alpha\beta}\Big[&\frac{1}{2}\nabla^B_\mu h_{\rho\alpha}\nabla^B_\nu h_{\sigma\beta} + (\nabla^B_\rho h_\nu)(\nabla^B_\sigma h_{\mu\beta} - \nabla^B_\beta h_{\mu\sigma}) \\ &+ h_{\rho\alpha}(\nabla^B_\nu\nabla^B_\mu h_{\sigma\beta} + \nabla^B_\beta\nabla^B_\sigma h_{\mu\nu} - \nabla^B_\beta\nabla^B_\nu h_{\mu\sigma} - \nabla^B_\beta\nabla^B_\mu h_{\nu\sigma}) \\ &+ (\frac{1}{2}\nabla^B_\alpha h_{\rho\sigma} - \nabla^B_\rho h_{\alpha\sigma})(\nabla^B_\nu h_{\mu\beta} + \nabla^B_\mu h_{\nu\beta} - \nabla^B_\beta h_{\mu\nu})\Big], \end{aligned} \tag{7.31}$$

where ∇^B denotes the covariant derivative with respect to the background metric $g^B_{\mu\nu}$. The exact form of these equations is not relevant for this discussion, but we are interested in the form of the terms. We see that $R^{(2)}_{\mu\nu}$ contains both terms of order $(\partial h)^2$ and $h\partial^2 h$, and we assume that the projection onto low-frequency modes will be of order $(\partial h)^2$. In orders of magnitude Eq. (7.30) then reads

$$R^B_{\mu\nu} \sim (\partial h)^2. \tag{7.32}$$

This expressed that the background curvature $R^B_{\mu\nu}$ depends quadratically on h. As a quick note, we can compare this to our expansion in flat spacetime, where we found Eq. (7.9) as an equation of motion for $h^{(2)}_{\mu\nu}$, which also depends quadratically on h. This implies that $h^{(2)}_{\mu\nu}$ does indeed contain information on how the background metric is curved by the metric perturbation $h_{\mu\nu}$, which we already expected. What has happened to $h^{(2)}_{\mu\nu}$ in this section is that its low-frequency modes have been included in $g^B_{\mu\nu}$, and its high-frequency modes are now a included in $h_{\mu\nu}$.

Continuing our analysis of the orders of magnitude, we know that $g^B_{\mu\nu}$ changes on the length scale L_B, and thus we have

$$\partial g^B_{\mu\nu} \sim \frac{1}{L_B}, \tag{7.33}$$

while $h_{\mu\nu}$ changes over the length λ which gives

$$\partial h_{\mu\nu} \sim \frac{h}{\lambda}. \tag{7.34}$$

$R^B_{\mu\nu}$ is contracted with $g^B_{\mu\nu}$, and from Eq. (A.4) we therefore have

$$R^B_{\mu\nu} \sim \partial^2 g^B_{\mu\nu} \sim \frac{1}{L_B^2}. \tag{7.35}$$

Combining now Eqs. (7.32), (7.34) and (7.35), we obtain

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda}\right)^2, \tag{7.36}$$

or,

$$h \sim \frac{\lambda}{L_B}. \qquad \text{(curvature determined by GWs).} \tag{7.37}$$

This holds for $T_{\mu\nu} = 0$. If we allow matter sources, these contributions will be much larger than those of the GWs, and we will have $1/L_B^2 \sim$ (matter contrib.) $\gg h^2/\lambda^2$, which gives

$$h \ll \frac{\lambda}{L_B}. \qquad \text{(curvature determined by matter).} \tag{7.38}$$

From this we see that the condition that the order of h is much smaller then that of g^B follows naturally from the requirement that $\lambda/L_B \ll 1$, which is needed to separate out the GW from the background. We also see that in the linearized theory, where we set $1/L_B$ to zero, we can not satisfy $h \lesssim \lambda/L_B$. Strictly speaking, this expansion is therefore not valid, as the background can not be flat if there is a gravitational wave present there. It is however, still a useful calculation that can help us gain some understanding of the system.

7.4 Obtaining the correct stress-energy tensor of gravitational waves

We now return to the expansion about flat spacetime from section 7.1, with the new-found knowledge from our discussion above. Most importantly we have seen that only the low-frequency modes of $R^{(2)}_{\mu\nu}$ contribute to the curvature of the background metric in the full theory. As the stress-energy tensor is the source of curvature in general relativity, we can assign the low-frequency modes of $R^{(2)}_{\mu\nu}$ as the stress-energy tensor of GWs. Considering again the flat spacetime approach, only the low-frequency modes of Eq. (7.9) should contribute to background curvature, and thus be a part of the stress-energy tensor. We therefore expect to obtain a correct stress-energy tensor of the GWs if we project Eq. (7.10) onto the low-frequency modes. This can be done by averaging over several wavelengths, as the low-frequency modes will be practically constant, and the high-frequency modes will average out to zero. We therefore state the correct form of $t_{\mu\nu}$ as,

$$t_{\mu\nu} = \frac{1}{\kappa}\left\langle R^{(2)}_{\mu\nu}[h] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h]\right\rangle, \tag{7.39}$$

where $\langle \ldots \rangle$ signify an average over several wavelengths. We will not show it here, but this can indeed be shown to be gauge invariant.

Above we saw that if we do not average $t_{\mu\nu}$ it is not gauge invariant, which theoretically brings up several questions. The gauge dependence arises locally from the requirement of a length scale. To make an unambiguous separation of a wave from a background metric, we need to define a length scale on which the wave varies, and we need to look at a section of spacetime where this length scale is defined. Locally, this can not be done, and the GW and the background metric is mixed under a gauge transformation. However, when we move to a macroscopic description, where the separation between the GWs and background metric is clear, this gauge dependence disappears. It is important to note that the averaged form of $t_{\mu\nu}$ was not something we imposed to deal with the gauge dependence, but rather it came out naturally in section 7.3, as a result of the short wave condition given in Eq. (7.24).

Another important interpretation of the averaged form of Eq. (7.39), is the inability to localize the energy of a GW. It is not possible to state exactly where in the wave the energy is carried, but rather we can state the total energy in a region of some wavelengths. This also follows because locally we cannot separate a gravitational wave from the background metric.

7.5 Energy of a GW in the TT-gauge.

Now we want to calculate $t_{\mu\nu}$, as defined in Eq. (7.39), in the TT-gauge. In Eq. (7.20) we gave the expression for $R^{(2)}_{\mu\nu}[h]$ in a general gauge. From [15, Ch. 7.6] we see that, in the TT-gauge, this reduces to

$$\begin{aligned} R^{TT(2)}_{\mu\nu}[h] = \frac{1}{2} h^{\rho\sigma}_{TT} \partial_\mu \partial_\nu h^{TT}_{\rho\sigma} + \frac{1}{4} (\partial_\mu h^{TT}_{\rho\sigma}) \partial_\nu h^{\rho\sigma}_{TT} + \frac{1}{2} \eta^{\rho\lambda} (\partial^\sigma h^{TT}_{\rho\nu}) \partial_\sigma h^{TT}_{\lambda\mu} \\ - \frac{1}{2} (\partial^\sigma h^{TT}_{\rho\nu}) \partial^\rho h^{TT}_{\sigma\mu} - h^{\rho\sigma}_{TT} \partial_\rho \partial_{(\mu} h^{TT}_{\nu)\sigma} + \frac{1}{2} h^{\rho\sigma}_{TT} \partial_\sigma \partial_\rho h^{TT}_{\mu\nu} . \end{aligned} \tag{7.40}$$

Under the averaging operator $\langle \ldots \rangle$, a total derivative gives zero, as a boundary term. This follows because the metric varies . This gives the following relation,

$$\begin{aligned} \langle A(\partial_\mu B) \rangle + \langle (\partial_\mu A) B \rangle = \langle \partial_\mu (AB) \rangle = 0, \\ \langle A(\partial_\mu B) \rangle = - \langle (\partial_\mu A) B \rangle . \end{aligned} \tag{7.41}$$

The by applying Eq. (7.41) on the last three terms of Eq. (7.40), we get terms on the form $\partial^\sigma h_{\sigma\alpha} \partial h$. In the TT-gauge we have that $\partial^\mu h^{TT}_{\mu\nu} = 0$, and therefore the last three terms of Eq. (7.40) vanish. This leaves us with

$$\left\langle R^{TT(2)}_{\mu\nu} \right\rangle = -\frac{1}{4} \left\langle (\partial_\mu h^{TT}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{TT}) + 2\eta^{\rho\lambda} (\Box h^{TT}_{\rho\nu}) h^{TT}_{\lambda\mu} \right\rangle . \tag{7.42}$$

We also know that $h^{TT}_{\mu\nu}$ satisfies the wave equation in the TT-gauge, given in Eq. (4.24). Setting $T_{\mu\nu} = 0$ in vacuum leads to $\Box h^{TT}_{\mu\nu} = 0$, and we obtain

$$\left\langle R^{TT(2)}_{\mu\nu} \right\rangle = -\frac{1}{4} \left\langle (\partial_\mu h^{TT}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{TT}) \right\rangle . \tag{7.43}$$

Now we can take the trace of this expression to find the Ricci scalar,

$$\begin{aligned}\left\langle \eta^{\mu\nu} R_{\mu\nu}^{TT(2)} \right\rangle &= -\frac{1}{4}\left\langle \eta^{\mu\nu}(\partial_\mu h_{\rho\sigma}^{TT})(\partial_\nu h_{TT}^{\rho\sigma})\right\rangle \\ &= \frac{1}{4}\left\langle (\Box h_{\rho\sigma}^{TT}) h_{TT}^{\rho\sigma}\right\rangle \\ &= 0,\end{aligned} \tag{7.44}$$

where we have used the relation in Eq. (7.41) in the second line. Returning now to our expression for $t_{\mu\nu}$, Eq. (7.39) can be expressed in the TT-gauge as,

$$t_{\mu\nu} = \frac{1}{4\kappa}\left\langle (\partial_\mu h_{\rho\sigma}^{TT})(\partial_\nu h_{TT}^{\rho\sigma})\right\rangle. \tag{7.45}$$

For a single wave, with a perturbation metric given by $H_{\mu\nu}\cos(k_\rho x^\rho)$ (see Eq. (5.2)), the stress-energy tensor is given as

$$t_{\mu\nu} = \frac{1}{4\kappa} k_\mu k_\nu H_{\rho\sigma} H^{\rho\sigma}\left\langle \sin^2(k_\rho x^\rho)\right\rangle. \tag{7.46}$$

The sine term averages to $1/2$. With the same wave vector as we have previously used, $k_\mu = \omega(-1, 0, 0, 1)$, and using Eq. (5.14) as the polarization matrix $H_{\mu\nu}$, we get

$$H_{\rho\sigma} H^{\rho\sigma} = 2(H_+^2 + H_\times^2). \tag{7.47}$$

The final expression for the $t_{\mu\nu}$ becomes

$$t_{\mu\nu} = \frac{2\omega^2}{4\kappa}(H_+^2 + H_\times^2)\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{7.48}$$

Chapter 8

Gravity as a field theory

Up until now we have treated gravity as a geometrical theory, starting from Einstein's postulate that space and time exist as a Riemannian manifold curved by mass and energy. This theory has enjoyed enormous success and its predictions, such as the existence of gravitational waves, have been confirmed by experiments. The important fact that motivated Einstein to recast gravity as a geometric theory, was that the inertial mass involved with acceleration was exactly equal to the gravitational mass involved with the gravitational force. Einstein took this equality seriously, and postulated that gravitational force and acceleration were two sides of the same coin, where gravitational force is the result of acceleration associated with curvature. In his theory gravitational mass is simply inertial mass, and the equality is explained.

In the century since Einstein, other branches of physics have emerged, and notable among them is quantum field theory (QFT). This framework have provided striking accuracy in experiments regarding particles, and is our best attempt at describing the world on the smallest scales. It is therefore natural to try to unify gravity with QFT. As it stands, general relativity is fundamentally different from a quantum field theory, as it treats spacetime itself as a dynamical field, whereas QFTs treat spacetime strictly as a background over which the field propagates. A natural question is therefore if it is possible to formulate gravity using the same framework - as a field theory living in flat spacetime. We then require that the geometric interpretation should fall out as a result of the theory. Thus, we forget that gravity has a geometric interpretation, which means that the curved Riemann-manifold of general relativity no longer exist, and instead we formulate gravity as the result of a quantum field living in Minkowski space. We will see that we are able to retrieve the full non-linear structure of gravity from only simple assumptions, and also that the geometry of gravity emerges naturally. One of the earliest physicists to follow this approach was Richard Feynman. He held a lecture series on this subject at Caltech in the academic year of 1962/63, which later turned into a book, [16]. In this chapter I will partly follow Feynman's derivation, but also loan arguments from chapter 2 of [12].

8.1 Gravity mediated by a massless spin-2 boson

We want to postulate a new fundamental quantum field which can give rise to the gravitational force. We know from Newtonian physics, that in the non-relativistic limit gravity is an attractive force coupling to the mass. We want our field to couple to a local quantity, and we can easily make a guess as to what this is - the stress-energy tensor $T_{\mu\nu}$. Its 00-component is the energy density, whose spatial integral in the non-relativistic limit gives the mass. It also satisfies the Lorentz transformation properties of special relativity. Thus, $T_{\mu\nu}$ seems like a good choice.

In quantum field theories interactions are mediated by bosons of integer spin. We know that odd spin leads to a theory where likes repel, and even spin gives an attractive theory. We must therefore consider spins $0, 2, 4$, etc [16, Ch. 3.1]. A spin-0 field can only consistently couple to $T_{\mu\nu}$ by the trace $T^{\mu}{}_{\mu}$, because we require rotational invariance. However, this would mean that photons, which have an anti-symmetric stress-energy tensor $F^{\mu\nu}$, does not couple to gravity. We know that this is not true from experiments of light rays being deflected by gravity. We can therefore exclude spin-0, and instead focus our attention on spin-2, and hope that we do not have to consider more complicated theories.

To formulate a theory with a spin-2 field, we need a spin-2 representation of the Lorentz group. The smallest tensor that contains a spin-2 representation is a traceless symmetric tensor $S_{\mu\nu}$, which under rotations decompose as,

$$S_{\mu\nu} \in \mathbf{0} \otimes \mathbf{1} \otimes \mathbf{2}, \tag{8.1}$$

where $\otimes$ denotes the direct sum, and $\mathbf{0}, \mathbf{1}$ and $\mathbf{2}$ are representations of a given spin, i.e. $\mathbf{1}$ is a vector representation. The trace of a tensor transforms as a scalar, so a general symmetric tensor $h_{\mu\nu}$ decomposes as $h_{\mu\nu} \in \mathbf{0} \otimes (\mathbf{0} \otimes \mathbf{1} \otimes \mathbf{2})$. We will use $h_{\mu\nu}$ as a starting point. We know that a massive spin-s particle has $2s + 1$ degrees of freedom, meaning a symmetric tensor $h_{\mu\nu}$ has five degrees of freedom corresponding to the spin-2 representation, three for the spin-1 and the remaining two degrees of freedom for the two scalars. This means a gravitational field should have at most five degrees of freedom. However, this number can be reduced. We know that gravity is a long range force, and we still have not found a bound on the range of the interaction. Therefore it is natural to assume that gravity is mediated by a massless particle, as the mass of the boson gives a factor $e^{-\alpha m r}$ in the interaction, where α is a constant. A massless particle always has two helicity states, $\pm s$. This means that only two of the degrees of freedom in $h_{\mu\nu}$ are physical.

8.2 Constructing a Lagrangian

Now we want to construct a Lagrangian using the above field $h_{\mu\nu}$. We immediately note that we need to have a gauge freedom in the Lagrangian which eliminates the eight extra degrees of freedom in $h_{\mu\nu}$. We could have guessed at the gauge symmetry, and used that symmetry as a constraint in determining the Lagrangian, but instead we will impose the following constraint. We will assume that energy conservation holds, which is expressed as

$$\partial_\mu T^{\mu\nu} = 0, \tag{8.2}$$

where $T^{\mu\nu}$ is the stress-energy tensor. We have not yet assigned an energy to the field $h_{\mu\nu}$. This means that $T^{\mu\nu}$ does not include the energy carried by the $h_{\mu\nu}$-field itself, but rather is the stress-energy tensor from all other fields. Therefore, as long as $T_{\mu\nu}$ is independent of $h_{\mu\nu}$, requiring that the total energy is conserved, we need to also include a term depending on $h_{\mu\nu}$. If we assume that the energy in the $h_{\mu\nu}$-field is at least of second order in the field, we have

$$\partial_\mu T^{\mu\nu} = \mathcal{O}(h^2), \tag{8.3}$$

where the form of the correction is not yet found. This correction will become important later, when we derive the full non-linear form of gravity. However, our first goal is only to find the terms in the Lagrangian of lowest order in $h_{\mu\nu}$. We will therefore disregard the correction terms for now, and reintroduce them later.

To construct a Lagrangian for the free field we write down all the terms quadratic in $h_{\mu\nu}$ with two derivatives, which are

$$\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}\ , \quad \partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho}\ , \quad \partial_\nu h^{\mu\nu}\partial^\rho h_{\mu\rho}\ , \quad \partial_\nu h^{\mu\nu}\partial_\nu h\ , \quad \partial^\mu h\partial_\mu h. \tag{8.4}$$

The terms above are related to those on the form $h\partial\partial h$ by a single integration by parts, so we can disregard these as surface terms. We also see that the second and third terms are related by partial integration, by swapping two derivatives. We choose to only keep the second term. The general free field action can therefore be written on the form

$$S_{\text{free}} = \int d^4x \left[a_1\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + a_2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} + a_3\partial_\nu h^{\mu\nu}\partial_\nu h + a_4\partial^\mu h\partial_\mu h\right]. \tag{8.5}$$

Our job is to specify the constants a_1, a_2, a_3, a_4, and we want to make use of energy conservation from Eq. (8.2). The next step is therefore to add an interaction term to our action. We already decided we wanted to couple our field to $T^{\mu\nu}$, so the most natural guess for an interaction term is $-\kappa h_{\mu\nu}T^{\mu\nu}$, where κ is a coupling constant. This looks very familiar to the electromagnetic analogue $-A_\mu j^\mu$ where j^μ also satisfies $\partial_\mu j^\mu = 0$. Note that we have disregarded a self-coupling term here. $T^{\mu\nu}$ does not contain the energy of the $h_{\mu\nu}$-field, and we should also add an interaction term of some higher order of $h_{\mu\nu}$, but again we choose to disregard higher order corrections for now. The action is given by the following integral,

$$S = \int d^4x \left[a_1\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + a_2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} + a_3\partial_\nu h^{\mu\nu}\partial_\nu h + a_4\partial^\mu h\partial_\mu h - \kappa h_{\mu\nu}T^{\mu\nu}\right]. \tag{8.6}$$

Now we vary the action as defined in Eq. (8.6) with respect to $h^{\alpha\beta}$, and require that the variation vanishes. Because $h_{\alpha\beta}$ is symmetric, we can only require that the variation of Eq. (8.6) vanish with respect to the symmetric part of $\delta h_{\alpha\beta}$. The symmetric part is given by $\delta h'_{\alpha\beta} = 1/2(\delta h_{\alpha\beta} + \delta h_{\beta\alpha})$, which we will use in calculations. This gives

$$\begin{aligned} a_1\left(\partial^\rho h^{\mu\nu}\frac{\delta\partial_\rho h_{\mu\nu}}{\delta h'^{\alpha\beta}} + \partial_\rho h_{\mu\nu}\frac{\delta\partial^\rho h^{\mu\nu}}{\delta h'^{\alpha\beta}}\right) + a_2\left(\partial_\rho h_{\mu\nu}\frac{\delta\partial^\nu h^{\mu\rho}}{\delta h'^{\alpha\beta}} + \partial^\nu h^{\mu\rho}\frac{\delta\partial_\rho h_{\mu\nu}}{\delta h'^{\alpha\beta}}\right) + \\ a_3\left(\partial_\nu h^{\mu\nu}\frac{\delta\partial_\nu h}{\delta h'^{\alpha\beta}} + \partial_\nu h\frac{\delta\partial_\nu h^{\mu\nu}}{\delta h'^{\alpha\beta}}\right) + a_4\left(\partial^\mu h\frac{\delta\partial_\mu h}{\delta h'^{\alpha\beta}} + \partial_\mu h\frac{\delta\partial^\mu h}{\delta h'^{\alpha\beta}}\right) - \\ \kappa T^{\mu\nu}\frac{\delta h_{\mu\nu}}{\delta h'^{\alpha\beta}} = 0. \end{aligned} \tag{8.7}$$

The operators δ and ∂ commute, which means we can flip the order and then use partial integration to move the partial derivatives, which introduces a minus sign. We obtain

$$2a_1\Box h_{\alpha\beta}+a_2(h_{\alpha\rho,\beta}{}^{\rho}+h_{\beta\rho,\alpha}{}^{\rho})+a_3(h^{\rho}{}_{\rho,\alpha\beta}+\eta_{\alpha\beta}h^{\mu\nu}{}_{,\mu\nu})+2a_4\eta_{\alpha\beta}\Box h=-\kappa T_{\alpha\beta}. \quad (8.8)$$

Now we can differentiate Eq. (8.8) by ∂_β, and use the law of energy conservation as expressed in Eq. (8.2), giving us

$$2a_1\Box h^{\alpha\beta}{}_{,\beta}+a_2\Box h^{\alpha\rho}{}_{,\rho}+a_2h^{\beta\rho,\alpha}{}_{\rho\beta}+a_3\Box h^{,\alpha}+a_3h^{\mu\nu,\alpha}{}_{\mu\nu}+2a_4\Box h^{,\alpha}=0. \quad (8.9)$$

This equation should hold for a general $h_{\mu\nu}$. And as the terms are linearly independent in general, we can collect terms of equal $h_{\mu\nu}$-dependence, and set the coefficients to zero. This gives us the following relations

$$\begin{aligned} 2a_1+a_2&=0\\ a_2+a_3&=0\\ a_3+2a_4&=0. \end{aligned} \quad (8.10)$$

We can choose normalization and set $a_1=\frac{1}{2}$, which fixes the rest of the coefficients as

$$a_1=\frac{1}{2} \qquad a_2=-1 \qquad a_3=1 \qquad a_4=-\frac{1}{2}. \quad (8.11)$$

Thus, requiring our field to satisfy energy conservation to the first order in $h_{\mu\nu}$ uniquely fixes the Lagrangian, apart from a normalizing factor. The Lagrangian for the free field is given by

$$S_{\text{free}}=\frac{1}{2}\int d^4x\left[\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}-2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho}+2\partial_\nu h^{\mu\nu}\partial_\nu h-\partial^\mu h\partial_\mu h\right]. \quad (8.12)$$

The derivation above was done by disregarding correction terms corresponding to the energy in the $h_{\mu\nu}$-field and therefore also self-interactions. It is possible to follow the same derivation while including higher-order terms. However, due to linear independence, you end up still having to cancel out the same linear terms, and therefore end up with the same coefficients.

We noted above that our Lagrangian must have a gauge symmetry to remove spurious degrees of freedom. Now that we have constructed a Lagrangian, this is a good check to see if our Lagrangian can be correct. We find the following transformation,

$$h'_{\mu\nu}=h_{\mu\nu}-\xi_{\mu,\nu}-\xi_{\nu,\mu}, \quad (8.13)$$

which leaves the Lagrangian unchanged. From chapter 5 we already know that this gauge symmetry removes eight degrees of freedom (together with boundary conditions). This means that we have two physical degrees of freedom in $h_{\mu\nu}$, which is what we want for a massless particle with helicities ± 2.

To give a unique solution to the variation of the action integral, we add a gauge fixing term to break the gauge symmetry. We choose $-(\partial^\nu h_{\mu\nu}-\frac{1}{2}\partial_\mu h)^2$, giving

$$\begin{aligned} S_{\text{gf}}&=\int d^4x-\left(\partial^\nu h_{\mu\nu}\partial_\rho h^{\mu\rho}-\frac{1}{2}\partial_\mu h\partial_\rho h^{\mu\rho}-\frac{1}{2}\partial^\nu h_{\mu\nu}\partial^\mu h+\frac{1}{4}\partial_\mu h\partial^\mu h\right)\\ &=\int d^4x\left(\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho}-\partial_\nu h^{\mu\nu}\partial_\mu h+\frac{1}{4}\partial^\mu h\partial_\mu h\right), \end{aligned} \quad (8.14)$$

where we have swapped the derivatives using partial integration in the second line. The action becomes

$$\begin{aligned} S =& S_{\text{free}} + S_{\text{gf}} + S_{\text{int}} \\ =& \int d^4x \left[\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial^\mu h \partial_\mu h - \kappa h_{\mu\nu}T^{\mu\nu} \right]. \end{aligned} \tag{8.15}$$

This action gives, from the variational principles, the equations of motion [12, Ch. 2]

$$\Box\bar{h}_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{8.16}$$

which we recognize as the wave equation of the weak-field approximation in the harmonic gauge, as seen in Eq. (4.24).

8.3 Deriving full gravity

We now want to review the simplifications we made in the section above, namely disregarding self-interactions. We will see that the inclusion of self-interaction leads to a series of extra terms in the Lagrangian, that if summed properly gives us the correct non-linear form of gravity.

We start by showing the need for including self-interactions in our theory. Eq. (8.2) can be written on the form,

$$\frac{d}{dt}\int_V d^3x\, T^{00} = -\int_V d^3x\, \partial_i T^{0i}, \tag{8.17}$$

expressing that the change of energy in a volume is due only to the flow of the matter field in or out of the volume. This can not hold for a theory where the $h_{\mu\nu}$-field interacts dynamically with matter. If the $h_{\mu\nu}$-field carry energy then that would violate Eq. (8.17). Therefore we suspect that Eq. (8.2) is not exact, which we already stated above. To remedy the situation, we assign a second order energy contribution from the $h_{\mu\nu}$-field, and call it $t^{(2)}_{\mu\nu}$. We expect the energy carried by the $h_{\mu\nu}$-field to act as a source for $h_{\mu\nu}$ in the same way as $T_{\mu\nu}$, and thus we make the replacement

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + t^{(2)}_{\mu\nu}, \tag{8.18}$$

in the wave Eq. (8.16). This gives us the new equation of motion,

$$\Box\bar{h}_{\mu\nu} = -\kappa(T_{\mu\nu} + t^{(2)}_{\mu\nu}). \tag{8.19}$$

Here $t^{(2)}_{\mu\nu}$ is a tensor constructed by terms quadratic in $h_{\mu\nu}$. Choosing the harmonic gauge, $\partial^\mu h_{\mu\nu} = 0$, together with Eq. (8.19) implies the relation

$$\partial^\mu(T_{\mu\nu} + t^{(2)}_{\mu\nu}) = 0. \tag{8.20}$$

This is now the new energy conservation, and we see that $t^{(2)}_{\mu\nu}$ is the leading order correction term in Eq. (8.3). We now want to alter our action integral so that Eq. (8.19) is the solution achieved by variation. We have an extra term on the symbolic form $\sim \kappa\partial h\partial h$, and thus we need to add a term $\sim \kappa h\partial h\partial h$ to the action to obtain it by variation. Explicitly, we can write the addition to the action as

$$S_3 = \kappa \int d^4x \; h_{\mu\nu} A^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} \partial_\rho h_{\alpha\beta} \partial_\sigma h_{\gamma\delta}, \tag{8.21}$$

where $A^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}$ is constructed from flat metric factors.

If we now check our Lagrangian for gauge symmetry, we will see that the term

$$L_3 = -\kappa h_{\mu\nu} t^{(2)\mu\nu} \tag{8.22}$$

from S_3 is not gauge invariant. Under a gauge transformation as given in Eq. (8.13), L_3 gives an extra term, as

$$L_3' = L_3 + \kappa \partial_{(\mu}\xi_{\nu)} t^{(2)\mu\nu}. \tag{8.23}$$

We know that the gauge symmetry is essential to remove non-physical degrees of freedom from $h_{\mu\nu}$. Luckily we can change the gauge transformation to

$$h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \kappa\mathcal{O}(h\partial\xi). \tag{8.24}$$

Here the symbolic term $\kappa\mathcal{O}(h\partial\xi)$ is defined to cancel the extra term in Eq. (8.23) to order $\mathcal{O}(k)$, thus reintroducing the gauge symmetry.

However, the we are not done yet. Once we add the term S_3 to the action, Noether's theorem produces another contribution to the energy of the $h_{\mu\nu}$-field, now cubic in $h_{\mu\nu}$ and linear in κ, as $\sim \kappa t^{(3)}_{\mu\nu}$. We make the exchange

$$T_{\mu\nu} + t^{(2)}_{\mu\nu} \to T_{\mu\nu} + t^{(2)}_{\mu\nu} + \kappa t^{(3)}_{\mu\nu}, \tag{8.25}$$

and get a new equation of motion,

$$\Box \bar{h}_{\mu\nu} = -\kappa(T_{\mu\nu} + t^{(2)}_{\mu\nu} + \kappa t^{(3)}_{\mu\nu}). \tag{8.26}$$

Thus, we have to alter the action again, adding a term on the form $S_4 \sim \kappa^2 h^2 \partial h \partial h$, and the gauge transformation changes to cancel the κ^2 term. The S_4 term in turn creates another contribution the stress-energy tensor through Noether's current, and the iteration procedure continues indefinitely. We therefore obtain

$$\Box \bar{h}_{\mu\nu} = -\kappa(T_{\mu\nu} + t^{(2)}_{\mu\nu} + \kappa t^{(3)}_{\mu\nu} + \kappa^2 t^{(4)}_{\mu\nu} + \ldots), \tag{8.27}$$

and the gauge symmetry

$$h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \kappa\mathcal{O}(h\partial\xi) + \kappa^2\mathcal{O}(h^2\partial\xi) + \ldots. \tag{8.28}$$

It is possible to calculate the exact result of this iterative procedure, and this is done i.e. by Deser [17]. The result is that we obtain the full non-linear Einstein-Hilbert action, with the gauge symmetry as the class of general coordinate transformations.

8.4 Geometry from a field theory

In this chapter we have derived gravity as a field theory in flat spacetime, mediated by a massless spin-2 particle. With only assuming energy conservation, this has led us to a full non-linear theory of gravity. As we know, gravity can also be approached from a geometric viewpoint, and it is a natural next step to try to unify the two approaches. Our next goal is therefore to find how the geometric interpretation emerges from a field theory in flat spacetime.

The path of a particle is described by minimizing the proper length of the particle in flat spacetime,

$$-\int \sqrt{(ds)^2} = -\int \sqrt{dx_\mu dx^\mu} = -\int d\alpha \sqrt{\frac{dx_\mu}{d\alpha}\frac{dx^\mu}{d\alpha}}. \tag{8.29}$$

In geometric gravity, the effect of gravity is included through the metric $g_{\mu\nu}$, but we are working in a strictly flat spacetime, where the metric is the Minkowski metric. We therefore need to include the effects of gravity in a different way. In electrodynamics we can obtain the equations of motion for a charged particle by varying the following integral,

$$-\frac{m}{2}\int d\alpha \left(\frac{dx_\mu}{d\alpha}\right)\left(\frac{dx^\mu}{d\alpha}\right) - e\int d\alpha\, A_\mu \left(\frac{dx^\mu}{d\alpha}\right). \tag{8.30}$$

After some calculation, this gives the equation of motion,

$$m\frac{d^2 x_\mu}{d\alpha^2} = eF_{\mu\nu}\left(\frac{dx^\nu}{d\alpha}\right), \tag{8.31}$$

where $F_{\mu\nu}$ is the curl of A_μ [16, Ch. 4.6]. As in electrodynamics the field is included by coupling $A_{\mu\nu}$ to a four-velocity $dx^\mu/d\alpha$. We guess that the tensor $T^{\mu\nu}$ is simply two such vectors, and set

$$T^{\mu\nu} = m_0 \left(\frac{dx^\mu}{d\alpha}\right)\left(\frac{dx^\nu}{d\alpha}\right). \tag{8.32}$$

Here we also included the normalization constant m_0, so that the 00-component correctly gives the energy density. The Lagrangian action integral now becomes,

$$m_0\left[-\frac{1}{2}\int d\alpha \left(\frac{dx_\mu}{d\alpha}\right)\left(\frac{dx^\mu}{d\alpha}\right) - \lambda\int d\alpha\, h_{\mu\nu}\left(\frac{dx^\mu}{d\alpha}\right)\left(\frac{dx^\nu}{d\alpha}\right)\right]. \tag{8.33}$$

We can introduce a new tensor to simplify the expression,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\lambda h_{\mu\nu}, \tag{8.34}$$

and our Lagrangian integral becomes

$$-\frac{m_0}{2}\int d\alpha\, g_{\mu\nu}\left(\frac{dx^\mu}{d\alpha}\right)\left(\frac{dx^\nu}{d\alpha}\right). \tag{8.35}$$

We recognize that this is the same Lagrangian that we know from geometric gravity, where $g_{\mu\nu}$ is the metric, and the solution is

$$\frac{d}{d\alpha}\left(g_{\sigma\nu}\frac{dx^\nu}{d\alpha}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{dx^\sigma}\frac{dx^\mu}{d\alpha}\frac{dx^\nu}{d\alpha} = 0, \tag{8.36}$$

or equivalently,

$$g_{\sigma\nu}\frac{d^2x^\nu}{d\alpha^2} = -\Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\alpha}\frac{dx^\nu}{d\alpha}. \tag{8.37}$$

If we now interpret $g_{\mu\nu}$ as a metric, and not simply a tensor created from $\eta_{\mu\nu}$ and $h_{\mu\nu}$, we obtain the geometric interpretation of gravity.

Thus, we see that the geometry of gravity emerges naturally from a theory of a massless spin-2 field that couples uniformly to the stress-energy tensor. The flat spacetime described by $\eta_{\mu\nu}$ is no longer observable, because *all matter* now moves in an effective Riemann manifold described by $g_{\mu\nu}$. This in turn gives us the principle of general covariance. We have therefore showed that a spin-2 theory seems like a promising candidate for explaining gravity.

Chapter 9

Gravitational wave generation in linearized theory

In this section we will derive the equations describing the generation of gravitational waves in the linearized theory. For the sources we consider, we assume that the relative speeds of the internal masses are small compared to the speed of light. Furthermore, we let the distance to the source be much larger than the radius of the source. We begin with the linearized Einstein equation in the harmonic gauge (see Eq. (4.24)),

$$\Box \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}. \tag{9.1}$$

This equation can be solved using Green's functions, as described in section 4.1. We let $G(t, \mathbf{x}; t', \mathbf{x}')$ be the Green's function of $\Box$, satisfying

$$\int dt' d\mathbf{x}' \Box G(t, \mathbf{x}; t', \mathbf{x}') = \delta(t - t')\delta(|\mathbf{x} - \mathbf{x}'|). \tag{9.2}$$

This means we can solve Eq. (9.1) by setting

$$\bar{h}_{\mu\nu} = -2\kappa \int dt' d\mathbf{x}' G(t, \mathbf{x}; t', \mathbf{x}') T_{\mu\nu}(t', \mathbf{x}'). \tag{9.3}$$

This integral is a summation of contributions from the stress-energy tensor $T_{\mu\nu}$ from different points in spacetime, with a correlation function given by $G(t, \mathbf{x}; t', \mathbf{x}')$. In principle the integral is over all of spacetime, however we want to calculate the contribution from the source, so the spatial integral only need to cover the extent of the source (we assume $T_{\mu\nu} = 0$ outside of the source). Furthermore, causality require that $t' \leq t$ for the Green's function to be non-zero. The Green's function of $\Box$ is well known, and can be expressed as

$$G(t, \mathbf{x}; t', \mathbf{x}') = -\frac{\delta(t' - [t - |\mathbf{x} - \mathbf{x}'|])}{4\pi |\mathbf{x} - \mathbf{x}'|}. \tag{9.4}$$

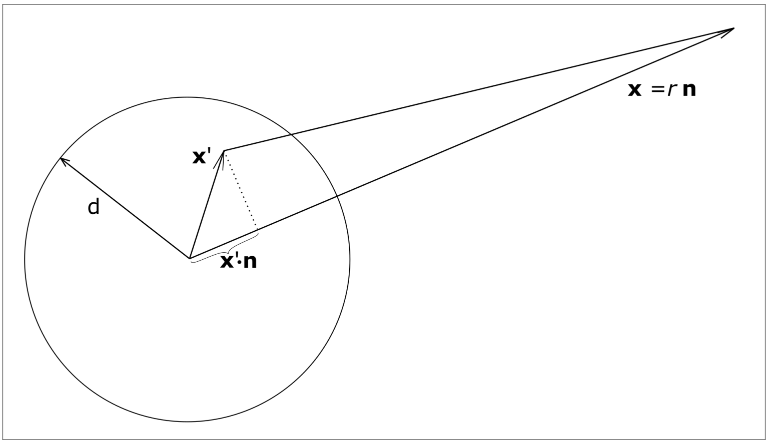

Figure 9.1: A graphical illustration of the integral in Eq. (9.6). See text for details.

We see that the delta function restricts contributions to coordinate points satisfying $t - t' = |\mathbf{x} - \mathbf{x}'|$, showing that the gravitational interaction propagate at the speed of light. Performing the integral over t' in Eq. (9.3), we obtain

$$\bar{h}_{\mu\nu} = \frac{\kappa}{2\pi} \int d\mathbf{x}' \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{9.5}$$

The radiative degrees of freedom are contained in the spatial part of the metric. This follows from the fact that we can always choose a gauge where all the nonzero components of $h_{\mu\nu}$ are contained in h_{ij}. Thus we only need to find the spatial part of the metric. We can write

$$\bar{h}^{ij} = \frac{\kappa}{2\pi} \int d\mathbf{x}' \frac{T^{ij}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \tag{9.6}$$

where we also have raised the spatial indices. We define r as the average distance to source from **x**. In Fig 9.1 we have illustrated the situation. The source is contained in the circular region of radius d, and we are calculating the field far away at a point **x**. The integral region in Eq. (9.6) is over the source of radius d. Because we have that the source is small compared to the distance r, we have $|\mathbf{x} - \mathbf{x}'| \simeq r$ for the whole source, which we can use in the denominator of Eq. (9.6). This allows us to move the factor $1/r$ out side the integral. For the time argument of T^{ij} we can also make the same approximation, giving us

$$T^{ij}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}') \simeq T^{ij}(t - r, \mathbf{x}'). \tag{9.7}$$

The error we introduce in the time argument by this approximation is maximally $|\mathbf{x}'|/c$. We have required that the speeds in the source are small compared to the speed of light, and thus T^{ij} does not change significantly on this time scale. Eq. (9.6) can then be written as

$$\bar{h}^{ij} = \frac{\kappa}{2\pi r} \int d\mathbf{x}' T^{ij}(t - r, \mathbf{x}'). \tag{9.8}$$

Next we need to derive some relations for the stress-energy tensor. We start from energy conservation, which is expressed as $\partial_\mu T^{\nu\mu} = 0$. This can be split into

$$\partial_0 T^{00} = -\partial_i T^{0i}, \tag{9.9}$$
$$\partial_0 T^{i0} = -\partial_j T^{ij}. \tag{9.10}$$

Using these relations, we have

$$\begin{aligned}\partial_0^2 T^{00} &= \partial_0(-\partial_i T^{0i}) \\ &= \partial_i(-\partial_0 T^{0i}) \\ &= \partial_i \partial_j T^{ij}.\end{aligned} \tag{9.11}$$

Now we multiply both sides of Eq. (9.11) by $x^i x^j$. The LHS becomes

$$\partial_0^2 T^{00} x^i x^j = \partial_0^2 (T^{00} x^i x^j). \tag{9.12}$$

To rewrite the RHS, we observe that

$$\begin{aligned}\partial_k \partial_l (T^{kl} x^i x^j) &= \partial_k \left(\partial_l T^{kl} x^i x^j + T^{ki} x^j + T^{kj} x^i\right) \\ &= \partial_k \partial_l T^{kl} x^i x^j + \partial_l T^{li} x^j + \partial_l T^{lj} x^i + \partial_k T^{ki} x^j + \partial_k T^{kj} x^i + 2T^{ij} \\ &= \partial_k \partial_l T^{kl} x^i x^j + 2\partial_k \left(T^{ki} x^j + T^{kj} x^i\right) - 2T^{ij},\end{aligned} \tag{9.13}$$

giving us

$$\partial_k \partial_l T^{kl} x^i x^j = \partial_k \partial_l (T^{kl} x^i x^j) - 2\partial_k \left(T^{ki} x^j + T^{kj} x^i\right) + 2T^{ij}. \tag{9.14}$$

Combining the LHS and RHS, we have

$$\partial_0^2 (T^{00} x^i x^j) = \partial_k \partial_l (T^{kl} x^i x^j) - 2\partial_k \left(T^{ki} x^j + T^{kj} x^i\right) + 2T^{ij}. \tag{9.15}$$

We can now isolate the last term T^{ij} and perform a spatial integral over some source in the following way,

$$\begin{aligned}\int d\mathbf{x}' T^{ij} &= \int d\mathbf{x}' \left[\frac{1}{2}\partial_0^2 (T^{00} x'^i x'^j) - \frac{1}{2}\partial_k \partial_l (T^{kl} x'^i x'^j) + 2\partial_k \left(T^{ki} x'^j + T^{kj} x'^i\right)\right] \\ &= \int d\mathbf{x}' \frac{1}{2}\partial_0^2 (T^{00} x'^i x'^j) \\ &= \frac{1}{2}\partial_0^2 \int d\mathbf{x}' T^{00} x'^i x'^j \\ &= \frac{1}{2}\partial_0^2 \int d\mathbf{x}' \rho x'^i x'^j,\end{aligned} \tag{9.16}$$

where, in going from the first to the second line, we have dropped the last two terms as divergence terms. We also define the second moment I_{ij} for the mass distribution, as

$$I^{ij}(t) = \int d\mathbf{x}' \rho(t, \mathbf{x}) x'^i x'^j. \tag{9.17}$$

Inserting Eqs. (9.16) and (9.17) into Eq. (9.8), we obtain

$$\bar{h}_{ij} = \frac{\kappa}{4\pi r}\partial_0^2 I_{ij}(t-r). \tag{9.18}$$

We now want to obtain h_{ij} in the transverse-traceless gauge, and one way to achieve this is by a projection operator. Following the derivation in [12, Ch. 1.2], we start by defining the tensor

$$P_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - n_i n_j. \tag{9.19}$$

This tensor is symmetric, it satisfies $n^i P_{ij} = 0$, so it is transverse, and it is a projector, satisfying $P_{ik}P^k{}_j = P_{ij}$. Thus, we can create a transverse projector as $P_{ik}P_{jl}$. However, to obtain a transverse-traceless projector, we must also subtract the trace. The trace of $P_{ik}P_{jl}I_{jk}$ is $P_{kl}I_{kl}$, and thus we can subtract $\frac{1}{2}P_{ij}P_{kl}$ from our projector, making sure the projection is traceless. The factor $\frac{1}{2}$ comes from the trace of P_{ij} which is 2. Thus, we construct the tensor

$$\Lambda_{ijkl}(\hat{\mathbf{n}}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}, \tag{9.20}$$

which is traceless in the (i, j) and the (k, l) indices,

$$\Lambda_i{}^i{}_{kl} = \Lambda_{ijk}{}^k = 0. \tag{9.21}$$

This tensor also satisfies $\Lambda_{ijkl}\Lambda^{kl}{}_{mn} = \Lambda_{ijmn}$ and is transverse in all indices, and is therefore our traceless-transverse projector. With this tensor we can project out the TT-part of h_{ij} along a direction $\hat{\mathbf{n}}$ in the following way

$$h_{ij}^{TT} = \Lambda_{ij}{}^{kl}(\hat{\mathbf{n}})h_{kl}. \tag{9.22}$$

We now define the quadrupole moment tensor $\mathcal{I}_{ij}$ as the traceless part of I_{ij},

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I. \tag{9.23}$$

Returning to Eq. (9.18), and inserting our TT-projector and the quadrupole moment, we arrive at

$$h_{ij}^{TT} = \frac{\kappa\Lambda_{ij}{}^{kl}}{4\pi r}\ddot{\mathcal{I}}_{kl}(t-r), \tag{9.24}$$

where we have used that we can interchange I_{ij} with $\mathcal{I}_{ij}$, as they only differ by a trace which is zero under the Λ_{ijkl} projector. Thus we have derived, to leading order in $1/r$, how the generation of gravitational waves depend on the mass quadrupole momentum. We also see that the generation of GWs in a given direction depends solely on the mass quadrupole moment projected into the transverse plane. Longitudinal motion does not contribute at first order in $1/r$. In addition, the generation depends on the second time derivative of $\mathcal{I}_{ij}$, which means that non-accelerating masses do not contribute to the generation of GWs to first order in $1/r$.

Chapter 10

Binary system as a source for gravitational waves

Next we want to apply our formalism from last chapter on a physical system. We will calculate the gravitational waves generated by a simple binary system. The binary we consider will consist of two equal masses in circular orbits around a shared center of mass, and assume weak gravitation so that Newtonian orbits are a good approximation. We let the radii of the masses be small compared to the orbit, and we can consider them as point masses in this calculation. The position of the masses in the xy-plane is given by $(x, y) = \pm(R\cos\omega t, R\sin\omega t)$. A graphic illustration is given in Fig 10.1. To find the emission of GWs, we know from Eq. (9.24) that we need to find the quadrupole moment I_{ij} of the binary at the retarded time $(t-r)$. For simplicity we let t denote the retarded time $t_{\text{ret}} = (t-r)$. Eq. (9.17) gives us,

$$I_{xx} = 2mR^2\cos^2(\omega t), \tag{10.1}$$
$$I_{yy} = 2mR^2\sin^2(\omega t), \tag{10.2}$$
$$I_{xy} = 2mR^2\cos(\omega t)\sin(\omega t), \tag{10.3}$$
$$I_{zi} = 0, \tag{10.4}$$

where ω is the orbital frequency. We can rewrite Eqs. (10.1-10.3) to depend on $2\omega t$ by using the following trigonometric relations:

$$\cos(2\theta) = 2\cos(\theta) - 1, \tag{10.5}$$
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \tag{10.6}$$
$$\sin^2(\theta) = 1 - \cos^2(\theta). \tag{10.7}$$

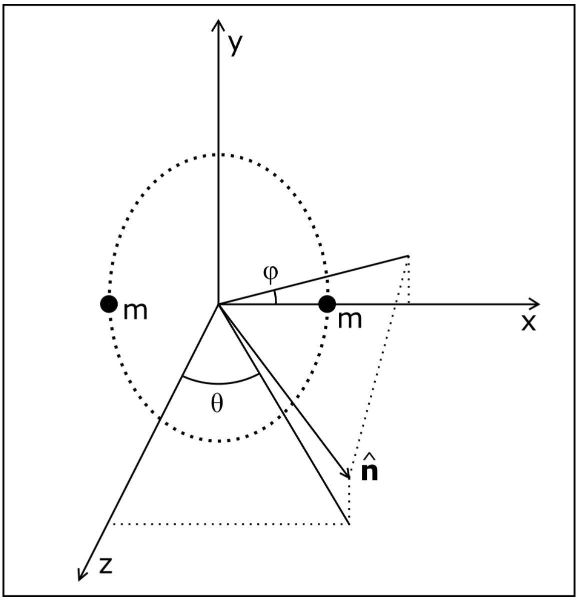

Figure 10.1: A graphic illustration of the binary. The masses orbit in the xy-plane. The angles θ and ϕ are the angles of $\hat{\mathbf{n}}$ projected into the zx- and xy-planes

We obtain

$$I_{xx} = mR^2 \cos(2\omega t) + mR^2 \tag{10.8}$$

$$I_{yy} = -mR^2 \cos(2\omega t) + mR^2 \tag{10.9}$$

$$I_{xy} = mR^2 \sin(2\omega t). \tag{10.10}$$

Only the time-dependent terms contribute. From last chapter we have Eq. (9.24), which states

$$h_{ij}^{TT} = \frac{\kappa \Lambda_{ijkl}}{4\pi r} \ddot{\mathcal{I}}_{kl}(t). \tag{10.11}$$

We can use Let use the traceless $\mathcal{I}_{ij}$ and I_{ij} interchangeably, as the time-independent part of I_{ij} is traceless. Let us first calculate the gravitational waves generated in the direction perpendicular to the orbital plane. In Fig. 10.1, this corresponds to $\theta = \phi = 0$. Here $\ddot{I}_{ij}$ is already both traceless and transverse, and thus $\Lambda_{ijkl}\ddot{I}_{kl} = \ddot{I}_{ij}$. Inserting this into Eq. (9.24), we get

$$h_{ij}^{TT} = \frac{\kappa}{4\pi r} \ddot{I}_{ij}(t). \tag{10.12}$$

By performing the time derivatives, we easily obtain the polarizations (see Eq. (5.14)),

$$h_+ = -\frac{\kappa m R^2 \omega^2}{\pi r} \cos(2\omega t), \qquad h_\times = \frac{\kappa m R^2 \omega^2}{\pi r} \sin(2\omega t). \tag{10.13}$$

We see that the two polarizations have the same amplitude, with a phase difference of $\pi/2$. This corresponds to a circular polarization.

We can find the GW along a direction $\hat{\mathbf{n}} = (\sin\theta, 0, \cos\theta)$, with $\phi = 0$. The calculation is done in Appendix B.2. The result is

$$h_+ = \frac{\kappa m R^2\omega^2}{\pi r}(\frac{1}{2}\sin^2\theta - 1)\cos(2\omega t), \qquad h_\times = \frac{\kappa m R^2\omega^2}{\pi r}\cos\theta\sin(2\omega t). \quad (10.14)$$

Thus we see that along the x-axis (for $\theta = \pi/2$ and $\phi = 0$) we have only $+$-polarization, and the amplitude is half of each of the polarizations for the wave perpendicular to the plane. As energy flux scales as amplitude squared, the energy flux in the orbital plane is $1/8$ of that in the perpendicular direction, and the GWs are linearly polarized. To find the GWs in a general direction, we can use Eq. (10.14) and then rotate by an angle ϕ in the xy-plane, using the rotational properties of GWs given in [12, p.12],

$$h_+ \to h_+\cos(2\phi) - h_\times\sin(2\phi), \quad (10.15)$$

$$h_\times \to h_+\sin(2\phi) + h_\times\cos(2\phi). \quad (10.16)$$

We want to understand the logic behind Eq. (10.14). The first thing we note is that the generation of GWs depends on twice the orbital frequency. This is an expected result, as a gravitational wave is invariant under a rotation of π around the axis of propagation (as seen in Eqs. (10.15-10.16)). Thus the phase of the GW has to complete two periods while the masses orbit once.

To understand the polarizations, we must consider the equation of the quadrupole moment Eq. (9.17). We already saw in the previous chapter that the wave generation depends on the mass projected into the plane transverse to the propagation. We can investigate this further. I_{xx} can be found by projecting the mass onto the x-axis, and then calculating $\int m|x|^2dx$. Similarly, I_{yy} is found by a projection of the mass to the y-axis, and I_{xy} and I_{yx} are found by projections onto the $(y = x)-$ and $(y = -x)$-axes. Time derivatives of I_{xx} and I_{yy} make up the $+$-polarization, while time derivatives of I_{xy} and I_{yx} make up the $\times$-polarization. We see that the axes of the $+$- and $\times$-polarizations are rotated $\pi/4$ compared to each other, and we thus expect a phase difference of $\Delta\omega = \pi/4$ between them. Combining this with the dependence of the GW on twice the orbital frequency, we see that the polarizations have a phase difference of $\pi/2$, which we see in Eq. (10.14). Let us now consider the factors of $(\frac{1}{2}\sin^2\theta - 1)$ for the $+$-polarization and $\cos\theta$ for the $\times$-polarization. The $+$-polarization is made up by two contributions. The component projected onto the y-axis, the I_{yy}, is unchanged when θ change, while the component along the x-axis changes as $I_{xx}(\theta) \propto \cos^2\theta$. Thus for the $+$-polarization we have $h_+ \propto (\frac{1}{2} + \frac{1}{2}\cos^2(\theta)) = -(\frac{1}{2}\sin^2\theta - 1)$, which we wanted to explain. Next up is the factor $\cos(\theta)$ for the $\times$-polarization. Here both components $I_{xy} = I_{yx}$ scale as $\propto \cos\theta$. Here the cosine is not squared, as the length l projected onto the $(x = y)$-axis scales as $\propto \sqrt{\cos\theta}$, and squaring this as $\propto m|l|^2$ gives us $h_\times \propto \cos\theta$.

10.1 Energy loss due to GW emission

We now want to calculate the total energy of the emitted gravitational waves. The energy-stress tensor of a GW is given by Eq. (7.45), as

$$t_{\mu\nu} = \frac{1}{4\kappa}\left\langle(\partial_\mu h_{ij}^{TT})(\partial_\nu h^{ij}_{TT})\right\rangle. \quad (10.17)$$

To find the total energy flux, we first need the energy flux in a given direction, which is contained in the $0i$-components of $t_{\mu\nu}$. If we let $n^\mu = (-1, \mathbf{n})$ be a four-vector parallel to the wave vector k^μ. We have the following relation,

$$n^\mu t_{\mu\nu} = 0, \tag{10.18}$$

which follows from the factor $n^\mu k_\mu = 0$, where the k_μ comes out from the derivative applied in $\partial_\mu h_{ij}^{TT}$ (see $t_{\mu\nu}$ on the form given in (7.46)). Thus, the energy of GWs in a direction **n** is given by,

$$n^i t_{i0} = -t_{00}, \tag{10.19}$$

which is simply a splitting of the temporal and spatial components of Eq. (10.18). We can calculate t_{00} explicitly by using Eq. (9.24),

$$\begin{aligned} t_{00} &= \frac{1}{4\kappa}\left\langle \dot{h}_{ij}^{TT}\dot{h}_{TT}^{ij}\right\rangle \\ &= \frac{\kappa}{64\pi^2 r^2}\left\langle \Lambda_{ijlk}\Lambda^{ij}{}_{uv}\dddot{\mathcal{I}}_{lk}\dddot{\mathcal{I}}_{uv}\right\rangle . \end{aligned} \tag{10.20}$$

Writing out the projection, and integrating over the full unit sphere, gives the total energy emitted per unit time, which is the GW-luminosity, L_{GW}, of the binary system. The projector is given by Eq. (9.19). Using that $P_{ij}\mathcal{I}^{ij} = -n_i n_j \mathcal{I}^{ij}$, from it being traceless, and $P_i{}^j P_{jk} = P_{ik}$, we obtain

$$L_{\mathrm{GW}} = \frac{\kappa}{32\pi^2}\int_{S^2}\left\langle \frac{1}{2}\dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij} - n_i n_k \dddot{\mathcal{I}}^{ij}\dddot{\mathcal{I}}^{k}{}_{j} + \frac{1}{2}n_i n_j n_k n_l \dddot{\mathcal{I}}^{ij}\dddot{\mathcal{I}}^{kl}\right\rangle d\Omega. \tag{10.21}$$

We now use the following relations, given in [12, p.105],

$$\frac{1}{4\pi}\int_{S^2} d\Omega = 1, \tag{10.22}$$

$$\frac{1}{4\pi}\int_{S^2} n_i n_j \, d\Omega = \frac{1}{3}\delta_{ij} \tag{10.23}$$

$$\frac{1}{4\pi}\int_{S^2} n_i n_j n_k n_l \, d\Omega = \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} + \delta_{ik}\delta_{lj}). \tag{10.24}$$

The full luminosity follows,

$$L_{\mathrm{GW}} = \frac{\kappa}{40\pi}\left\langle \dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij}\right\rangle . \tag{10.25}$$

Calculating $\left\langle \dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij}\right\rangle$ explicitly from Eqs. (10.8-10.10) gives

$$\begin{aligned} L_{\mathrm{GW}} &= \frac{\kappa}{40\pi}\left((\dddot{\mathcal{I}}_{xx})^2 + (\dddot{\mathcal{I}}_{yy})^2 + (\dddot{\mathcal{I}}_{xy})^2 + (\dddot{\mathcal{I}}_{yx})^2\right) \\ &= \frac{16\kappa}{5\pi} m^2 R^4 \omega^6. \end{aligned} \tag{10.26}$$

Wee see that the luminosity depends strongly on the angular frequency, as $\propto \omega^6$. Because $\omega \propto v/R$, with v being the speed of the masses, we also have $L_{GW} \propto v^6/R^2$. If we now were to consider elliptical orbits, we would expect an eccentric orbit be more luminous, as the maximum velocity is higher, and the luminosity scales so depends so strongly on the speed.

10.2 The change in the orbit due to GWs

We will now show how the orbit of a binary system changes over time, and we will follow the derivation given in [18]. From the luminosity relation given in Eq. (10.26) we can derive the change in the orbit using relations from Newtonian mechanics, as we are working in the low-gravity limit. We the following relation for Newtonian orbits,

$$R^3 = \frac{m}{4\omega^2}, \tag{10.27}$$

which allows us to eliminate R. If we also rewriting in terms of the GW frequency, $\omega_{gw} = 2\omega$, Eq (10.26) becomes

$$L_{\text{GW}} = \frac{\kappa}{20\pi}(m\omega_{gw})^{10/3}. \tag{10.28}$$

The energy radiated from the binary system has to come from the orbital energy, given for a binary as

$$E_{\text{orb}} = -m\omega^2 R^2. \tag{10.29}$$

Energy conservation imposes $\dot{E} = -L_{\text{GW}}$, which gives us, after again using Eq. (10.27),

$$\dot{\omega}_{gw} = \frac{\kappa}{10\pi} m^{5/3} \omega_{gw}^{11/3}. \tag{10.30}$$

We see that the change in the frequency of the gravitational wave only depends on the frequency of the signal and the masses in the binary. Thus, using only the signal measured of a gravitational wave, we can infer the distance to the binary. From the signal we measure the chirp rate $\dot{\omega}_{gw}$ and the frequency ω_{gw} directly. By Eq. (10.30) we can then infer the mass m, and from this the intrinsic luminosity from Eq. (10.28). Comparing then the intrinsic luminosity of the binary with the amplitude of the signal, we can find the distance r the signal has travelled. Even if the masses are not equal, this still holds, but we have to exchange the mass m with the chirp mass $\mathcal{M} = \mu^{5/3} M^{2/5}$, where μ is the reduced mass, $\mu = \frac{m_1 m_2}{m_1 + m_2}$, and M is the total mass, $M = m_1 + m_2$ [18].

Chapter 11

Conclusion and outlook

In this thesis we have developed the basic machinery for treating gravitational waves of low magnitude. We started by deriving the Einstein field equations using variational principles, assuming the Einstein-Hilbert action. By allowing the metric connection to be a field independent of the metric, we showed that the Levi-Civita connection follows from the Einstein-Hilbert action. The Einstein field equations were then linearized by expanding the metric to first order around flat spacetime. In the linear theory, the perturbation metric initially has ten independent degrees of freedom. To remove nonphysical degrees of freedom, a gauge fixing scheme was introduced where we separated local and global freedoms. We found that only two of the degrees of freedom in the perturbation metric corresponds to physical freedoms. Of the remaining eight degrees of freedom, four correspond to global coordinate choices and affect the boundary conditions, and the remaining four depend on a local gauge choice. We compared this gauge fixing scheme with the gauge fixing scheme of electrodynamics, and concluded that they are similar in structure. The gravitational wave solutions in the linear theory were then found for the vacuum, and its two polarizations.

Next, we investigated how GWs interact with test particles in different frames of reference, connecting the gauge choices with coordinate systems. We considered the free falling frame, where the coordinate value of free falling masses are unchanged, and the lab frame, where the coordinates are spanned by rigid rulers. The forces of GWs to first order in these frames were found, and the relevant forces of a GW on an interferometric detector on the Earth were discussed.

To find the energy carried by gravitational waves, the metric was expanded to second order around a slowly varying background curvature. We found that the first order metric perturbation acts as a source for the second order perturbation. Splitting the second order perturbation into a high- and a low-frequency part, we found that the source of the high-frequency part is gauge dependant, while the source of the low-frequency part corresponds to the lowest order stress-energy tensor of GWs. The stress-energy tensor of gravitational waves thus emerged naturally as an average over several wavelengths.

We briefly investigated the possibility that gravity is the result of a field theory in flat spacetime. We started with a massless spin-2 particle, which satisfies the basic requirements of gravity, being a long-range and attractive force. By only assuming energy conservation we found its free Lagrangian. This field was then coupled to the full stress-energy tensor, which lead to an infinite series of higher order self-interaction terms. This series is possible to sum analytically. The result gave us the full non-linear structure of gravity and the principle of general covariance. This essentially showed that a massless spin-2 theory in flat spacetime is a viable candidate for describing relativistic gravity.

Lastly, the equation for the emission of gravitational waves for slowly moving sources was derived. We saw that only the motion transverse to the wave vector contributes, and that to first order in the velocity the emission depends on the change in the quadrupole moment of the mass. The magnitude and polarization of the GWs emitted by a binary system following Newtonian orbits were found in all directions. We also found the rate of energy loss of the binary, and the evolution of its orbital frequency.

Throughout this thesis we have considered situations where various approximations have been valid, namely we have used linearized gravity in the weak-field limit and considered only slowly moving sources. However for many real-life physical systems we would need more advanced methods of calculation, especially for sources of strong GWs. A natural extension of this thesis is therefore to develop machinery for more general GW sources, considering terms of higher order in the velocity, relativistic corrections and effects from strong gravitational fields. Another topic that would be interesting to pursue is the propagation of GWs. Due to the self-interaction of GWs, the wave-form changes as they propagate, depending on the magnitude of the wave and the background curvature. This also means that for strong gravitational waves we can not use the wave solutions from the linearized theory. This is interesting to study, as it would give a better understanding of the non-linearity of GWs, and give insight into how GWs propagate on larger scales.

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Appendices

Appendix A

A.1 Notation and Conventions

The following notation conventions are used in this work.
Units This thesis uses natural units, with $c = 1$; $\hbar = 1$; $\mu_0 = 1$.
Differentiation Partial derivatives are written in the following ways

$$\frac{\partial}{\partial x^a} = \partial_a = \ _{,a} \tag{A.1}$$

Indices Greek letters $(\mu, \nu, \rho, \ldots)$ are used for indices that can take values from $\{0, 1, 2, 3\}$, while Latin letters $(i, j, k, \ldots)$ are used for indices that take values from $\{1, 2, 3\}$.
Einstein summation Einstein summation convention is assumed unless otherwise stated, which means repeated lower and upper indices are summed over, i.e.:

$$x^i x_i = \sum_i x^i x_i \tag{A.2}$$

Metric signature We use the metric signature where the Minkowski metric of flat space-time takes the following form:

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \tag{A.3}$$

Abbreviations LHS and RHS are used as abbreviations for left-hand side and right-hand side.

A.2 Tensor Definitions

The Riemann curvature tensor is defined as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}. \tag{A.4}$$

The Ricci tensor is a contraction of the Riemann tensor, given by

$$R_{\mu\nu} = R^{\gamma}{}_{\mu\gamma\nu} = \partial_\gamma \Gamma^{\gamma}{}_{\nu\mu} - \partial_\nu \Gamma^{\gamma}{}_{\gamma\mu} + \Gamma^{\gamma}{}_{\gamma\lambda}\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\gamma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\gamma\mu}, \tag{A.5}$$

where the terms $\Gamma^{\gamma}{}_{\mu\nu}$ are defined below. The Ricci scalar, in turn, is a contraction of the Ricci tensor with the metric, in the following way

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{A.6}$$

The Christoffel symbols $\Gamma^{\rho}{}_{\mu\nu}$ describe the unique connection that is both metric compatible, as described in A.3, and torsionless, defined by Eq. (A.9), and is given by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}). \tag{A.7}$$

This connection is called the Levi-Civita connection. It is important to note that the Christoffel symbols are not tensors, as their transformation law include an extra term. Under a coordinate change, the Christoffel symbols transform as

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \frac{\partial \tilde{x}^{\rho}}{\partial x^{\lambda}} \frac{\partial x^{\gamma}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\nu}} \Gamma^{\lambda}_{\gamma\sigma} + \frac{\partial^2 x^{\lambda}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}} \frac{\partial \tilde{x}^{\rho}}{\partial x^{\lambda}}. \tag{A.8}$$

The torsion tensor $T^{\rho}{}_{\mu\nu}$ describes the twist along a curve in a manifold, and is given by the following equation, where $\Gamma^{\rho}_{\mu\nu}$ are the connection coefficients,

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}. \tag{A.9}$$

A.3 Metric Compatibility

A requirement on the metric connection is that it is metric compatible. This requirement says the metric has to be covariantly constant, given by the equation

$$\nabla_c g_{ab} = \nabla_c g^{ab} = 0. \tag{A.10}$$

A connection satisfying Eq. (A.10) ensures lengths and angles are invariant under parallel transport. It also ensures we can introduce in all of spacetime local coordinate systems where the laws of special relativity are valid, and consistently connect these by parallel transports [10, p. 42].

A.4 Characteristics of a pseudo-Riemannian manifold

In general relativity spacetime is represented by a pseudo-Riemannian manifold, which is a generalization of a Riemannian manifold where the metric tensor $g_{\mu\nu}$ does not need to be positive definite. Here we will only briefly discuss a few of its characteristics. The signature of a pseudo-Riemannian manifold is given as (n, m), where n and m denotes the number of negative and positive eigenvalues in the metric tensor. Alternatively we say that we have n timelike and m spacelike dimensions. The signature of a manifold is invariant of coordinate system. A Riemannian manifold is also equipped with a connection, given by the connection coefficients $\Gamma^{\rho}{}_{\mu\nu}$. The torsion on a Riemannian manifold is defined by Eq. (A.9), which means that the connection coefficients of a torsionless manifold are symmetric in the lower indices. Riemannian manifolds also have the important property that we can define a covariant derivative, as given by Eq. (A.16), that satisfies the Leibniz rule and transforms as a tensor. A complete definition of a pseudo-Riemannian manifold can be found in any book on differential or Riemannian geometry.

A.5 Integrals in spacetime

To be able to use integration in spacetime for physically meaningful quantities, we need to look at the transformational properties of integrals in an n-dimensional pseudo-Riemannian manifold. We first consider $d^n x$, which transforms as

$$d^n \tilde{x} = \left| \det\left(\frac{\partial \tilde{x}}{\partial x} \right) \right| d^n x. \tag{A.11}$$

We want to find a measure on our manifold that is coordinate invariant. To accomplish this, we consider the transformation law of $\sqrt{|g|}$, where g is the determinant of the metric tensor. We have the following transformation law

$$\sqrt{|\tilde{g}|} = \sqrt{|\det(\tilde{g}_{\mu\nu})|} = \sqrt{\left| \det\left(\frac{\partial x^\rho}{\partial \tilde{x}^\mu} g_{\rho\sigma} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} \right) \right|} = \left| \det\left(\frac{\partial x}{\partial \tilde{x}} \right) \right| \sqrt{|g|}. \tag{A.12}$$

If we combine this with $d^n x$, the determinants cancel, and we get a coordinate invariant measure $\sqrt{|g|} d^n x$, that satisfies

$$\sqrt{|\tilde{g}|} d^n \tilde{x} = \sqrt{|g|} d^n x. \tag{A.13}$$

This implies that the integral of a tensor with respect to this measure, has the same transformation law as the original tensor, and the integral itself is therefore a tensor. For a tensor $T^{i \ldots n}_{i \ldots m}$, the following holds

$$\begin{aligned} \tilde{A}^{i \ldots n}_{i \ldots m} &= \int_{\tilde{V}} \tilde{T}^{i \ldots n}_{i \ldots m} \sqrt{|\tilde{g}|} d^n \tilde{x} \\ &= \left(\frac{\partial \tilde{x}^i}{\partial x^{i'}} \cdots \frac{\partial \tilde{x}^n}{\partial x^{n'}} \right) \left(\frac{\partial x^{j'}}{\partial \tilde{x}^j} \cdots \frac{\partial x^{m'}}{\partial \tilde{x}^m} \right) \int_V T^{i' \ldots n'}_{i' \ldots m'} \sqrt{|g|} d^n x. \end{aligned} \tag{A.14}$$

This measure can also be used to generalize the divergence theorem to pseudo-Riemannian manifolds. The derivation is involved and beyond the scope of this text, but can be found in [11], which on page 113 gives the following equation, assuming the Levi-Civita connection,

$$\int_M \nabla_a X^a \sqrt{|g|} d^n x = \int_{\partial M} n_a X^a \sqrt{|h|} d^{n-1} x, \tag{A.15}$$

where X^a is a vector field on M, ∇ is the covariant derivative, h denotes the metric on the boundary ∂M, and n is the outward pointing normal on ∂M. The important thing to note from this equation is that $\nabla_a X^a$ is the curved manifold equivalent of a divergence, and that the RHS is just a boundary term. We also readily see that these integrals are invariant, as the LHS transforms as $\nabla_a X^a$, which is a scalar.

A.6 Generalized Derivatives

We define the covariant derivative ∇_c of a tensor $T^{a \ldots}_{b \ldots}$ as

$$\nabla_c T^{a \ldots}_{b \ldots} = \partial_c T^{a \ldots}_{b \ldots} + \Gamma^a_{dc} T^{d \ldots}_{b \ldots} + \ldots - \Gamma^d_{bc} T^{a \ldots}_{d \ldots} - \ldots \tag{A.16}$$

The covariant derivative of a tensor transforms as a regular tensor.

Appendix B

B.1 Bianchi Identities

The Bianchi identity is given by

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\sigma R_{\mu\nu\lambda\rho} + \nabla_\rho R_{\mu\nu\sigma\lambda} = 0. \tag{B.1}$$

The contracted Bianchi identity is given as

$$\nabla_\mu R^\mu{}_\nu - \nabla_\nu R = 0. \tag{B.2}$$

From Eq. (B.2), the following relation for the Einstein tensor can be derived,

$$g^{\mu\rho}\nabla_\rho G_{\mu\nu} = 0. \tag{B.3}$$

B.2 Generation of GWs in a general direction

Lets calculate the GWs in the direction along

$$\hat{\mathbf{n}} = (\sin\theta, 0, \cos\theta). \tag{B.4}$$

Here θ is the angle between $\hat{\mathbf{n}}$ and the normal vector of the orbital plane, and for $\theta = 0$, $\hat{\mathbf{n}}$ points along the x-axis. We want to calculate $\mathcal{I}_{ij}(\hat{\mathbf{n}})$, which is the traceless-transverse projection of I_{kl} along $\hat{\mathbf{n}}$. We have

$$\begin{aligned}\mathcal{I}_{ij}(\hat{\mathbf{n}}) &= \Lambda_{ijkl} I_{kl} \\ &= P_{ik}P_{jl}I_{kl} - \frac{1}{2}P_{ij}P_{kl}I_{kl} \\ &= (PIP)_{ij} - \frac{1}{2}P_{ij}\,\mathrm{Tr}\,(PI)\,,\end{aligned} \tag{B.5}$$

where we in the last line have used that P_{ij} is symmetric. $P_{ij}(\hat{\mathbf{n}})$ is given in Eq. (9.19), giving us

$$P_{ij}(\hat{\mathbf{n}}) = \begin{pmatrix} \cos^2\theta & 0 & -\cos\theta\sin\theta \\ 0 & 1 & 0 \\ -\cos\theta\sin\theta & 0 & \sin^2\theta \end{pmatrix}_{ij}. \tag{B.6}$$

By using that $I_{yy} = -I_{xx}$ and $I_{xy} = I_{yx}$, we find by careful calculation

$$(PIP)_{ij} = \begin{pmatrix} I_{xx}\cos^4\theta & I_{xy}\cos^2\theta & -I_{xx}\cos^3\theta\sin\theta \\ I_{xy}\cos^2\theta & -I_{xx} & -I_{xy}\cos\theta\sin\theta \\ -I_{xx}\cos^3\theta\sin\theta & -I_{xy}\cos\theta\sin\theta & I_{xx}\cos^2\theta\sin^2\theta \end{pmatrix}_{ij}, \tag{B.7}$$

$$\frac{1}{2}P_{ij}\,\mathrm{Tr}\,(PI) = \frac{1}{2}\left(-\sin^2\theta\right)\begin{pmatrix} 1-\sin^2\theta & 0 & -\cos\theta\sin\theta \\ 0 & 1 & 0 \\ -\cos\theta\sin\theta & 0 & 1-\cos^2\theta \end{pmatrix}_{ij}. \tag{B.8}$$

Calcucenterg Eq. (B.5) explicitly gives

$$\mathcal{I}_{ij}(\hat{\mathbf{n}}) = \begin{pmatrix} -I_{xx}\cos^2\theta(-\cos^2\theta + \frac{1}{2}\sin^2\theta) & I_{xy}\cos^2\theta & I_{xx}\cos\theta\sin\theta(-\cos^2\theta+\frac{1}{2}\sin^2\theta) \\ I_{xy}\cos^2\theta & I_{xx}(-1+\frac{1}{2}\sin\theta) & -I_{xy}\cos\theta\sin\theta \\ I_{xx}\cos\theta\sin\theta(-\cos^2\theta+\frac{1}{2}\sin^2\theta) & -I_{xy}\cos\theta\sin\theta & -I_{xx}\sin^2\theta(-\cos^2\theta-\frac{1}{2}\sin^2\theta) \end{pmatrix}_{ij} \tag{B.9}$$

We can now split this matrix by Splitting it by the two polarization matrices for the wave vector $\hat{\mathbf{n}}$. We chose the two orthonormal vectors in the transverse plane as

$$\hat{\mathbf{u}} = (0,1,0) \quad \text{and} \quad \hat{\mathbf{v}} = (\cos\theta, 0, \sin\theta). \tag{B.10}$$

Using Eq. (5.16), we find

$$e^{+}_{ij}(\hat{\mathbf{n}}) = \begin{pmatrix} -\cos^2\theta & 0 & \cos\theta\sin\theta \\ 0 & I_{xx} & 0 \\ \cos\theta\sin\theta & 0 & -\sin^2\theta \end{pmatrix}_{ij}, \tag{B.11}$$

$$e^{\times}_{ij}(\hat{\mathbf{n}}) = \begin{pmatrix} 0 & \cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix}_{ij}, \tag{B.12}$$

and we readily see that we have

$$\mathcal{I}_{ij}(\hat{\mathbf{n}}) = I_{xx}\left(-1+\frac{1}{2}\sin^2\theta\right)e^{+}_{ij}(\hat{\mathbf{n}}) + I_{xy}\cos\ e^{\times}_{ij}(\hat{\mathbf{n}}). \tag{B.13}$$

Here we clearly see the prefactors depending on θ, which is $(-1+\frac{1}{2}\sin^2\theta)$ for the $+$-polarization, and $\cos\theta$ for the $\times$-polarization.

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