Master's thesis

2019

Magnus Liland

Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

Magnus Liland

Counterparty Credit Exposure in Interest Rate Derivatives

Investigating the Effect of Collateralization

June 2019

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of
Science and Technology

Counterparty Credit Exposure in Interest Rate Derivatives

Investigating the Effect of Collateralization

Magnus Liland

Master of Science in Physics and Mathematics
Submission date: June 2019
Supervisor: Jacob Laading

Norwegian University of Science and Technology
Department of Mathematical Sciences

Preface

This thesis was carried out at the Department of Mathematical Sciences at the Norwegian University of Science and Technology (NTNU) during the Spring of 2019. The thesis concludes a 5-year engineering degree in physics and mathematics with a specialization in industrial mathematics and statistics, leading to the degree Master of Science (M.Sc.).

I want to thank my supervisor Jacob Laading, for the constructive feedback and valuable information he provided throughout the process.

Magnus Liland
Trondheim, Norway
June 2019

Abstract

This paper concerns the application of a stochastic interest model based on the LIBOR market models (LMM) to evaluate the fair price and risk associated with interest rate derivatives. After establishing fundamental concepts related to finance and risk management, the functionality of the interest rate model is investigated by inspection of the generated distributions and pricing of zero-coupon bonds. Furthermore, the model is employed to evaluate two common interest rate derivatives, namely interest rate floor and swap contracts. The contracts are priced for a number of different fixed rates, and the associated market risk is quantified through the risk measures *value at risk* (VAR) and *expected shortfall* (ES).

Interest rate derivative transactions are commonly bilateral and subject to counterparty credit risk. This form of risk and its significance in the financial markets are discussed, in addition to how to construct a framework to evaluate the counterparty credit exposure. Increased regulation requires flexibility to account for collateralization, as the majority of bilateral trades entered into by financial intuitions utilize this form of risk mitigation practice. The implemented framework is employed to assess the counterparty credit exposure associated with an interest rate swap contract. Moreover, the effect of collateralization is investigated by evaluating the impact of each of the standardized collateral agreement parameters.

The role of stochastic modeling in the financial industry is discussed, particularly in reference to risk assessment and accounting for the impact of collateralization. Furthermore, the performance of the implemented stochastic interest rate model is reviewed, with emphasis on the consequences of the underlying assumptions. In particular, the use of a log-normal simulation scheme causes a non-negative and asymmetric interest rate distribution and will be debated in detail.

Sammendrag

Denne rapporten omhandler bruken av en stokastisk rentemodell, basert på LIBOR market models (LMM), til å bestemme prisen til rentederivater, samt risikoen assosiert ved derivatene. Etter å ha etablert grunnleggende konsept relatert til finans og risikostyring, blir funksjonaliteten til rentemodellen undersøkt ved inspeksjon av de genererte rente-fordelingene, samt prising av nullkupongobligasjoner. Videre blir modellen brukt til å evaluere to vanlige rentederivater; rentegulv-kontrakter og fastrente-kontrakter. Kontraktene er priset for en rekke forskjellige faste renter, og den tilhørende risikoen er kvantifisert ved bruk av risikomålene *value at risk* og *expected shortfall*.

En stor andel av rentederivater er bilaterale, og dermed gjenstand for motpartsrisiko. Denne formen for risiko, samt dens betydning i dagens finansmarkeder, blir diskutert, i tillegg til hvordan en kan konstruere et rammeverk for å evaluere motpartseksponering. Økende grad av regulering gjør at brorparten av bilaterale handler som inngås av finansielle institusjoner bruker sikkerhetsstillelse for å redusere risikoen. Dermed er det nødvendig at et rammeverk som skal evaluere motpartseksponering kan ta høyde for denne formen for risikostyring. Det implementerte rammeverket blir benyttet til å estimere motpartseksponeringen i en fastrentekontrakt. Videre blir effekten av sikkerhetsstillelse undersøkt ved å evaluere effekten til hver av de standardiserte parameterne som definerer sikkerhetsavtalen.

Rollen til stokastisk modellering i finansindustrien blir diskutert, særlig i forbindelse med risikoevaluering og evnen til å modellere sikkerhetsstillelse. Videre blir ytelsen til rentemodellen vurdert med vekt på konsekvensene av de underliggende antagelsene. Bruken av en log-normal simuleringsalgoritme gir ikke-negativ og asymmetrisk rente-fordeling, og dette valget vil bli nøye diskutert.

Contents

Preface i

Abstract iii

Sammendrag v

Contents vii

1 Introduction 1

2 Financial concepts 5
2.1 Financial markets . . . 5
2.1.1 Efficient market hypothesis . . . 5
2.1.2 Arbitrage . . . 5
2.1.3 Financial assets . . . 6
2.1.4 Financial derivatives . . . 7
2.2 Interest rates . . . 8
2.2.1 Bonds . . . 9
2.2.2 Interest rate derivatives . . . 10
2.3 Risk . . . 11
2.3.1 Volatility . . . 11
2.3.2 Hedging . . . 12

3 Stochastic interest rate models 13
3.1 Stochastic interest rates . . . 13
3.1.1 One-factor models . . . 13
3.1.2 Bond pricing with stochastic interest rate . . . 14
3.1.3 The market price of risk . . . 15
3.2 The LIBOR market models (LMM) . . . 16
3.2.1 Mathematical formulation . . . 16
3.2.2 Simulation . . . 18
3.2.3 Calibration and volatility structure . . . 18
3.2.4 Pricing of interest rate derivatives . . . 22

4 Risk management 23
4.1 Risk measures . . . 23
4.1.1 Value at risk . . . 23
4.1.2 Expected shortfall . . . 24
4.2 Counterparty credit risk . . . 24
4.2.1 Counterparty credit exposure . . . 25
4.2.2 Counterparty risk measures . . . 29
4.2.3 Counterparty risk management . . . 29
4.2.4 Collateralization . . . 31

5 Data presentation 37
5.1 Historical data . . . 37
5.2 Initial rates and yield curve . . . 40

6 Results and discussion **43**
6.1 Volatility structure . . . 43
6.2 Rate distribution . . . 44
6.3 Pricing of interest rate derivatives . . . 48
6.3.1 Pricing of zero-coupon bonds . . . 48
6.3.2 Pricing of interest rate floor contracts . . . 49
6.4 Evaluation of interest rate swap contract market risk . . . 51
6.5 Evaluation of counterparty credit exposure . . . 53
6.6 Quantification of collateralization impact . . . 55
6.6.1 Independent amount . . . 56
6.6.2 Threshold . . . 57
6.6.3 Minimum transfer amount . . . 58
6.6.4 Margin frequency . . . 59
6.6.5 Margin period of risk . . . 60

7 Conclusion **63**

References **65**

Appendices **69**
A Wiener process . . . 69
B Itô's lemma . . . 69
C Monte Carlo . . . 69

1 Introduction

A widely supported hypothesis in financial economics says that asset prices fully represent all available information and are traded at their fair value. Further, it says that the market reacts instantaneously to new information [33]. As the future is largely unknown, so are future price developments. This motivates the application of stochastic modeling to account for the uncertainty in the future.

The stochastic nature of financial instruments causes difficulty in predicting future returns, but perhaps of greater importance, it causes the probability of the actual return deviating from the expected return to be significant. This is essentially what in finance is referred to as *risk*. Hence, the ability to quantify the risk associated with a financial instrument is an important attribute inherent in a stochastic model.

In this paper, the focus is the application of statistics and stochastic models in the field of quantitative finance, in particular, modeling of interest rates. It is worth noting that all models inevitably are imperfect representations of reality, as accurately accounting for the complex nature of human behavior by a mathematical model is simply not possible [14]. Regardless, stochastic models have proven valuable and may provide useful information despite its shortcomings. In this respect, it is of great importance to be fully aware of a model's assumptions and limitations, in order to apply it where it is appropriate and not overestimate its capabilities.

The interest rate is the cost of borrowing money, and naturally, it has a profound effect on the entire economy. Low interest rates cause borrowing to be cheap and stimulates growth, while high interest rates cause borrowing to be expensive and generally constraints growth [19]. The market nominal of financial instruments dependent on the interest rate is enormous, just OTC (over-the-counter) interest rate derivatives accounted for roughly $450 *trillion* in 2017 [25]. Naturally, the importance of market consistent interest rate models able to accurately predict and measure risk associated with future developments is not to be understated.

The modeling of interest rates is more complex than the modeling of asset prices for numerous reasons; the most prominent reason is the vast majority of interconnected and dependent interest rates. There exist interest rates for a number of maturities, and they are heavily correlated. If one would simply overlook this dynamic and model one interest rate by one source of randomness, the model would greatly simplify the interaction between the rates, and the model would be rendered useless in most practical cases.

A substantial part of this paper is concerned with the formulation, implementation, and application of the LIBOR market models (LMM), a class of multi-factor interest rate models capable of capturing the complex correlation structure evident in the interest rate market. The implemented model is calibrated to historical correlations in the NIBOR rates and employed to assess the fair price of interest rate derivatives, as well as the associated risk. The derivatives considered are zero-coupon bonds, interest rate floor contracts, and interest rate swap contracts.

Further, the model is employed to assess the counterparty credit exposure associated with interest rate derivatives. This is a fundamentally different type of risk and requires a more sophisticated modeling framework than what is needed to describe market risk (changes in market variables). This type of risk played a central role in the economic crisis of 2008 [42], which is commonly referred to as the *credit crisis*. This destructive display of mismanaging of counterparty credit risk demonstrates the importance of such models, in addition to effective risk mitigation practices.

An immensely common risk mitigation practice is *collateralization*, which entails making other financial assets (for instance cash) available in the event of a default. This practice is widespread; hence, in order to construct a realistic framework for credit exposure, one typically needs to account for collateralization. Collateral usually reduces the exposure significantly; however, there are multiple, sometimes subtle, aspects that must be considered in order to properly assess the true extent of risk reduction. In this paper, the counterparty credit exposure associated with an interest rate swap contract is evaluated with and without the presence of collateral, by that quantifying the impact of collateralization.

The report commences by introducing basic financial concepts, which are fundamental when establishing reasonable constraints on the interest rate models. It includes the concept of arbitrage, the log-normal random walk for asset prices, interest rates, bonds, financial derivatives, and concepts related to risk.

The next section introduces stochastic interest rate models, starting with the simple one-factor models, discussing their applications and shortcomings. Subsequently, pricing of bonds and other interest rate derivatives is discussed, before moving on to the multi-factor interest rate models, introducing the LIBOR market models (LMM). The section concludes with a description of the procedure for volatility calibration and how to price interest rate derivatives by a Monte Carlo approach.

Section 4 concerns the last piece of theory needed before embarking on implementing and employing the interest rate model, namely risk management. As previously stated, quantifying the risk associated with financial instruments is one of the main applications of a stochastic model, and in this section two heavily used risk measures are introduced, namely value at risk (VAR) and expected shortfall (ES). Further, the concept of counterparty credit risk is formally introduced, including how to measure the risk and common practices employed to manage it. Particular emphasis is put on the practice of collateralization and how to account for this effect in a modeling framework.

The next section presents the historical data employed to calibrate the model. In addition to a plot displaying the historical NIBOR rates, certain descriptive statistics are presented to describe common tendencies in the rates. Further, the initial rates used for the simulation are presented, accompanied by the initial yield curve, in order to convey how one would expect the simulated rates to behave.

Section 6 presents the results and associated discussion. The distribution of rates returned by the model is evaluated with the true realizations of rates as a reference, before moving on to pricing interest rate derivatives and quantifying the risk. Thereafter, the counterparty credit exposure associated with an interest rate swap contract is evaluated. The effect of collateralization is investigated by an individual study of each of the standardized parameters in a collateral agreement. Lastly follows a conclusion in Section 7, in which the main results are summarized and discussed.

2 Financial concepts

2.1 Financial markets

A **financial market** is a broad term describing any marketplace where trading of securities, including assets, bonds, currencies, and derivatives, occurs [47]. There are many kinds of financial markets, to mention a few important ones:

(i) **Stock markets**, where buying, selling and issuance of shares of publicly held companies take place;

(ii) **Bond markets**, which deals in government and other bonds;

(iii) **Currency markets** or **foreign exchange markets**, where currencies are bought and sold;

(iv) **Commodity markets**, where physical assets such as gold, oil and electricity are traded;

(v) **Futures** and **options markets**, on which the derivative products are traded.

2.1.1 Efficient market hypothesis

An important theory in modern financial economics is the **efficient market hypothesis (EHM)**. Although there are several different versions of the hypothesis, the main consequence coincides: *At any time an asset will reflect all available information and trade at its fair value, implying it is impossible to outperform the market by expert stock selection or market timing.* Hence, current prices reflect all available information, and they will only shift as a response to new information. As future information is unknown, this theory implies that asset prices are random. Further, according to this theory, the only way to earn returns greater than the market, is by taking on greater risk [33].

2.1.2 Arbitrage

Related to the efficient market hypothesis is the concept of **arbitrage**, which is fundamental in the theory of financial derivative pricing and hedging. Arbitrage is exploiting market inefficiencies to make an *instantaneous, risk-free* profit. An example of arbitrage would be if the same asset was traded at two different prices, as one in such a case could make an instantaneous risk-free profit by buying at the lowest price and selling at the highest price. In efficient markets, such opportunities cannot exist for a significant length of time before prices move to eliminate them [45].

The concept of arbitrage can be formulated mathematically and serves as a tool when constructing statistical models to describe financial instruments. Let V_t denote the value of a portfolio at time t. If the value satisfies

$$P(V_t \geq 0) = 1 \quad \text{and} \quad P(V_t \neq 0) > 0, \tag{2.1}$$

where $V_0 = 0$, then there exists an arbitrage opportunity. Equation 2.1 essentially says that there exists an opportunity to start without any money, and accumulate a non-negative amount with a non-zero probability, with no chance of occurring a loss. When constructing a model for a financial instrument one impose a no-arbitrage condition, meaning (2.1) cannot hold.

2.1.3 Financial assets

A **financial asset** is a tangible asset that gets its value from a contractual claim, for instance, cash, stocks, or a bank deposit. An important subset of financial assets is **financial securities**. A financial security describes a negotiable financial instrument that holds some type of monetary value [40]. It may represent an ownership position in a publicly traded company (via stocks), a creditor relationship with a government institution or a corporation (via bonds), or rights to ownership such as an option.

The main focus in this report is the modeling of interest rates; however, it may be helpful to cover a common model for asset prices, as the construction of a stochastic interest rate model follows the same rationale. The EHM essentially says the asset price is random and cannot be predicted, and as a result, the price of an asset is often modeled as a random walk. Suppose at time t the price is S. Consider a small subsequent time interval $\mathrm{d}t$, during which S changes to $S + \mathrm{d}S$. The most common model decomposes the change into two parts - a predictable and deterministic change and a random change,

$$\mathrm{d}S = u(t, S)\mathrm{d}t + w(t, S)\mathrm{d}X, \tag{2.2}$$

where $\mathrm{d}X$ is a sample from a normal distribution with zero mean and variance $\mathrm{d}t$, i.e. $\mathrm{d}X = \phi\sqrt{\mathrm{d}t}$, where ϕ is standard normal. This is known as a **Wiener process** (Appendix A). Note that (2.2) does not satisfy (2.1), implying that the model is arbitrage free by construction.

A widely used model on the form (2.2) is the log-normal random walk,

$$\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}X, \tag{2.3}$$

where μ is called the drift and σ is called the volatility. This model is used when deriving the Black-Scholes formula [6], and is hence fundamental for the theory concerning derivative pricing.

Firstly, note that equation (2.3) does not refer to the past history of the asset price; the next asset price depends solely on today's price (*Markov property*). Secondly, consider the expectation of $\mathrm{d}S$

$$\mathrm{E}(\mathrm{d}S) = \mathrm{E}(\mu S\mathrm{d}t + \sigma S\mathrm{d}X) = \mu S\mathrm{d}t, \tag{2.4}$$

meaning on average the next value for S is higher than the old by an amount $\mu S\mathrm{d}t$. Thirdly, the variance of $\mathrm{d}S$ is

$$\mathrm{Var}(\mathrm{d}S) = \mathrm{E}(\mathrm{d}S^2) - \mathrm{E}(\mathrm{d}S)^2 = \mathrm{E}(\sigma^2 S^2\mathrm{d}X^2) = \sigma^2 S^2\mathrm{d}t, \tag{2.5}$$

meaning the standard deviation is proportional to the volatility σ.

By employing an important result, **Itô's lemma** [26] (Appendix B), to a function $f(S)$ one gets the expression

$$\mathrm{d}f = \sigma S\frac{\mathrm{d}f}{\mathrm{d}S}\mathrm{d}X + \left(\mu S\frac{\mathrm{d}f}{\mathrm{d}S} + \frac{1}{2}\sigma^2 S^2\frac{\mathrm{d}^2 f}{\mathrm{d}S^2}\right)\mathrm{d}t, \tag{2.6}$$

for a small change in f, $\mathrm{d}f$. For $\sigma = 0$, the change in S is deterministic, and equation (2.3) models an exponential price development, motivating the choice $f(S) = \log S$. Inserting into (2.6), and integrating yields

$$\begin{aligned} \mathrm{d}f &= \sigma \mathrm{d}X + \left(\mu - \frac{1}{2}\sigma^2\right) \mathrm{d}t \\ \implies \quad S_t &= S_0 \exp\left\{\sigma X_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right\}. \end{aligned} \tag{2.7}$$

That is, S_t/S_0 is distributed according to a log-normal distribution with parameters μ and σ^2 on the form,

$$\log\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right\}. \tag{2.8}$$

Empirically this model has proven quite successfully, although not without rightful criticism. Daily log-returns tend to have heavier tails than what is expected from (2.8), and in addition, realizations close to the mean are observed more frequently than one would expect. Formally, log-returns are known to be *leptokurtic*, meaning the occurrence of outliers (extreme outcomes) is more frequent than in a normal distribution [12] [41]. In spite of these problems, it is widely accepted to be the best *simple* model and is heavily used in practice.

2.1.4 Financial derivatives

A **financial derivative** is a financial security with a value that is reliant upon or derived from an underlying asset or group of assets, usually referred to as simply the **underlying** [18].

An important subgroup of derivatives is **options**. An option is a financial contract which gives the holder of the contract the right to buy or sell an underlying asset, often at a predetermined price (**exercise price** or **strike price**) and a predetermined time (**expiry**). The simplest form of options is the European call and put options. A call option gives the holder of the contract the right to buy the underlying asset, for instance, a stock, for the exercise price E, at the expiry date T. Similarly, a put option gives the holder the right to sell for E at T. Let S be the price of the underlying asset at expiry T, then, with a simple arbitrage argument, the payoff from a call option $C(S,T)$ and put option $P(S,T)$ are

$$C(S,T) = \max(S - E, 0); \tag{2.9}$$

$$P(S,T) = \max(E - S, 0). \tag{2.10}$$

As one would expect, the payoff from a call option is increasing with the price of the underlying asset, while the payoff from a put option is decreasing.

Options have two primary uses: hedging, which will be discussed shortly, and speculation. The speculation aspect is simply that an investor buys options essentially to make bets on the movement of the underlying asset, making money, if correct. The holder of an option has the possibility for an arbitrarily large payoff, with the loss limited to the initial premium. In addition, buying an option on an asset rather than the asset itself will increase the payout in percentage of the original investment, as the return of an option responds in an exaggerated way to changes in the underlying asset. This is known as **gearing**.

2.2 Interest rates

The interest rate is an important quantity in finance, although it is not itself considered a financial asset as it cannot be bought or sold directly. However, there exists a wide variety of interest rate derivative products, and the price of other financial assets is highly dependent on the interest rate.

The interest rate is the cost of borrowing capital, or equivalently, the compensation for setting money to the disposal of others. That is, the interest rate is the amount charged by the lender to the borrower for the use of assets, and it is naturally dependent on certain characteristics of the loan, mainly the time horizon and riskiness [24].

When pricing short term assets, for instance, a stock option, the interest rate is mainly assumed to be deterministic, often even constant. In reality, interest rates are not deterministic, but when pricing short term assets, the error associated with this simplification is usually less than 2% [47].

Almost all financial theory assumes the existence of **risk-free** investments that give a guaranteed return with no chance of default. A good approximation to such an investment is a government bond or a deposit in a sound bank. Following the concept of arbitrage, the greatest risk-free return than one can make on a portfolio of assets is the same as the return if the equivalent amount of cash was placed in a bank.

The intuitive rationale for this concept is that if there existed an opportunity to make a greater risk-free return, sensible investors would exploit this opportunity. By a supply-and-demand argument that would increase the price to the point that the return would be equal to that from the risk-free interest.

For valuing derivatives, the most important concept concerning interest rates is that of **present value** or **discounting**. How much is a guaranteed amount E received at the future time T worth? The answer is found by *discounting* the future value, E, using continuously compounded interest. With a constant, risk-free interest rate r, money in the bank $M(t)$ grows according to

$$\begin{aligned} \frac{\mathrm{d}M}{M} &= r\mathrm{d}t \\ \implies M &= c\mathrm{e}^{rt}. \end{aligned} \tag{2.11}$$

As $M = E$ at $t = T$,

$$M = E\mathrm{e}^{-r(T-t)}, \tag{2.12}$$

or if the interest rate is a known function of time $r(t)$, then

$$M = E\exp\left(-\int_t^T r(s)\mathrm{d}s\right). \tag{2.13}$$

Interest rates are commonly *simple*, rather than *continuously compounded*. A simple interest rate means that if L denotes the rate for an accrual period of length δ (1/4 for three months, 1/2 for six months, etc.), then the interest earned over the period is δL. A *forward* simple rate is similar. Fix δ and consider a maturity T. The forward rate $L(0,T)$ is the rate set at time 0 for the future interval $[T, T+\delta]$. That is, entering into a contract at time 0 to borrow 1 at time T, the interest made at time $T+\delta$ is $\delta L(0,T)$. These differences may seem unimportant, but their significance will become apparent later on.

2.2.1 Bonds

A **bond** is a debt instrument used to raise capital, mainly issued by governments and larger companies. The issuer borrows a fixed amount, called the **principal**, from the buyer of the bond (the lender). At a predetermined time in the future, the **maturity date**, the lender is repaid the principal with interest. It is common that a bond also has annual or semiannual interest payments, called **coupons**. If the premium, or any of the coupons are not repaid, the issuer of the bond is said to **default** on its debt.

In general, it can be stated that the price of a bond with a given maturity depends on two things, the interest rate and the riskiness of the loan. For an essentially riskless bond, like most government bonds, the price is only dependent on the interest rate. For simplicity, consider a bond with no interest payments, called a **zero-coupon bond**, with principle payment Z at maturity T, and let $r(t)$ be the expected short rate for $0 < t < T$. For this simple case, the price of the bond, $B(t,T)$, is simply the discounted principle payment,

$$B(t,T) = Z \exp\left(-\int_t^T r(s)\mathrm{d}s\right). \tag{2.14}$$

In the market, the short interest rate $r(t)$ is not directly observable but can be derived from market prices of zero-coupon bonds with different maturities. For simplicity, let the principle be one, $Z = 1$, then by the fundamental theorem of calculus equation (2.14) gives

$$\begin{aligned} r(T) &= -\frac{\partial}{\partial T}\log(B(t,T)) \\ &= -\frac{1}{B(t,T)}\frac{\partial B}{\partial T}. \end{aligned} \tag{2.15}$$

By collecting the prices for a number of bonds with different maturities, equation (2.15) can be used to deduce the expected short interest rate going forward. Note that if the interest rate is positive (which it usually is), (2.15) also suggest that

$$\frac{\partial B}{\partial T} < 0, \tag{2.16}$$

meaning that the longer the lifetime of a bond, the less it is worth.

Closely related to both bonds and interest rates is the **yield curve**, which is another measure of the future values of the interest rates. Consider zero-coupon bonds with premium payment 1, $B(T,T) = 1$, and define

$$Y(t,T) = -\frac{\log\left(B(t,T)\right)}{T-t}. \tag{2.17}$$

Hence, $Y(t,T)$ is the average yield on the bond from time t to maturity T. An advantage with this measure is that it does not require the value to be differentiable with respect to T.

The yield curve is commonly shown as a function of $T - t$ (time to maturity), and the most common form is increasing - implying it is more rewarding to tie up money for a long time than for a short time. The development of the yield curve is reported frequently in financial news as it is known to reflect the market's expectation concerning several economic aspects, including expected GDP growth [1], the possibility for a regression [15], as well as the state of the overall economy going forward [8].

2.2.2 Interest rate derivatives

There exists a wide variety of derivatives relying on the interest rate, including bond options, floors, caps, and swaps, to mention a few. A **bond option** is in structure, essentially the same as a stock option, with the modification that the underlying asset is a bond rather than a stock. The pricing of a bond option is more complex than the pricing of a stock option, as the payoff at expiry is dependent on the bond price at that time. As a consequence, the final condition for the pricing equation is a random variable, adding another layer of uncertainty. Market practice is therefore often to model the bond price as a log-normal random walk instead, which is not unreasonable if it is not too close to maturity [46].

Interest rate cap and **interest rate floor** contracts are commonly used to hedge against rising and falling interest rates, respectively. A cap is a contract that guarantees to its holder that an otherwise floating interest rate will not exceed a specified amount, while a floor in an analogous way guarantees the holder that the interest rate will not go below a specified amount. A typical cap contract involves multiple possible payments at times t_i (for instance each quarter), called **caplets**, on the form

$$Z\alpha_i \max(r_{l_i} - r_c, 0), \tag{2.18}$$

where Z is the principal, r_{l_i} is the floating rate at time t_i, r_c is the caped rate and α_i a time count fraction corresponding to the form of r_L (on a yearly contract, if r_l is annual rate and the caplets are paid every quarter, then α_1 would be 1/4 for instance). A cap is thus a sum of many caplets on the form (2.18). Each caplet can be thought of as a European call option on the floating rate r_L with strike price r_c, and is *increasing* in value with increasing interest rate.

In a similar fashion, a floor is made up of a sum of **floorlets** on the form,

$$Z\alpha_i \max(r_f - r_{l_i}, 0), \tag{2.19}$$

and can be thought of as a put option on the interest rate. An interest rate floor contract is *decreasing* in value as the interest rate increases and thereby *increasing* in value with declining interest rate.

An **interest swap contract** is where two parties agree to exchange payments based on two different interest rates, often a fixed and a floating interest rate. A swap is thus essentially long a cap and short a floor contract, with cash flow at each payment time proportional to

$$\max(r_{l_i} - r_c, 0) - \max(r_c - r_{l_i}, 0) = r_{l_i} - r_c, \tag{2.20}$$

implying that the buyer of the floating rate payments profits when the floating rate exceeds the fixed rate and takes on losses if it declines below.

2.3 Risk

Risk is a broad term essentially describing the chance that an investment's actual return will differ from the expected return. A fundamental idea in finance is the relationship between risk and return. The greater amount of risk an investor is willing to take, the greater the potential return [17].

Risk is commonly described as being of two types: specific and non-specific. The latter is also called market risk or systematic risk. Specific risk is the component of risk associated with a single asset (or a sector of the market), whereas non-specific risk is associated with factors affecting the whole market. It is possible to diversify away specific risk by having a portfolio with a large number of assets from different sectors; however, it is not possible to diversify away the non-specific risk. It is commonly said that specific risk is not rewarded and that only taking on greater non-specific risk should be rewarded by a greater return [47].

Risk can further be divided into categories depending on the source of uncertainty, for instance, market risk (changes in market variables), operational risk (faults in human or operational practices), credit risk (default on debt), etc. One may consult, for instance, Wilmott for an exhaustive categorization of risk types [46] [45]. This will be discussed further in the context of risk management and counterparty credit risk in Section 4.

2.3.1 Volatility

Volatility is a statistical measure of the dispersion of returns from a given security. Commonly, the higher the volatility, the riskier the asset. In section 2.1.3, the standard deviation of the return was proportional to the volatility of the asset, σ. In this model, the volatility was assumed to be deterministic and constant; however, in reality, the volatility is stochastic [46].

The volatility is not directly observable in the market and is therefore estimated. There are two main ways of estimating the volatility of an asset. The **implied volatility** is the volatility expected by the market based on the current pricing of options on the asset. This is done by employing the solution of the Black-Scholes equation (or another relevant closed-form equation) to obtain the expected volatility based on market prices. This is the volatility quoted in the VIX-index, which measures the markets overall expected future volatility, commonly referred to as the *fear index* [16].

The second way of estimating the volatility is using the **historical volatility**. This is done by calculating the standard deviation of the past returns of an asset and using this to estimate the volatility going forward. The implied volatility is typically larger than the historical, as the implied volatility is based on the inversion of an equation building on (2.8), which, as previously discussed, underestimates the probability of extreme events. Market prices will naturally reflect the true behavior of the assets, and consequently, the volatility obtained through inversion of the Black-Scholes equation will typically be higher than the historical.

Generally, when pricing derivatives, it is preferred to use the implied volatility [30]. The reason is that it, to a greater extent, represents the *current* state. The volatility is affected by recent events, and if using the implied volatility, these are appropriately accounted for as they are reflected in current market prices. If, however, one would use the historical volatility, these events would be marginalized.

2.3.2 Hedging

Hedging is making investments, to reduce the risk of adverse price movements in an asset. This is usually done by taking an offsetting position, for instance in an option, thereby taking advantage of the correlation between the asset price and option price. Hedging is in many ways analogous to taking out an insurance policy. Inherent in hedging is a risk-reward trade-off - it reduces potential risk, but it also chips away from potential gains [4].

A common example of a hedge that employs the correlation between the asset and its options, is the **delta-neutral** approach. Let V be the value of an option, and S the value of the underlying asset and introduce Δ as the change in V caused by a change in S, i.e.

$$\Delta = \frac{\partial V}{\partial S}. \tag{2.21}$$

Now consider a portfolio of n European call options, each with value $C(S,t)$. This portfolio is sensitive to a change in the underlying asset by $n\Delta$. Introduce a short position in $n\Delta$ of the underlying asset to the portfolio, and let Π be the value of the portfolio,

$$\Pi = nC(S,t) - \Delta nS. \tag{2.22}$$

The portfolio is now hedged against movement in the underlying asset, as

$$\begin{aligned} \frac{\partial \Pi}{\partial S} &= n\frac{\partial C(S,t)}{\partial S} - n\Delta\frac{\partial S}{\partial S} \\ &= n\Delta - n\Delta = 0. \end{aligned} \tag{2.23}$$

Hence, a change in the underlying asset causes no change in the value of the portfolio. However, the portfolio is still sensitive to changes in other relevant parameters. Note that Δ is *not* constant over S, so one needs to continuously sell and buy assets to maintain the hedge.

3 Stochastic interest rate models

3.1 Stochastic interest rates

As briefly discussed in section 2.2.1, the future interest rates are unknown, but the expected future rates can be derived from bond prices. However, the *realized* interest rates can vary greatly from the expected, stemming from the stochastic nature of the interest rates. The stochastic modeling of interest rates is therefore of great importance in order to better understand and predict future developments, as well as to measure risk associated with the stochastic behavior.

There exists a wide variety of such interest models, with different complexities. The simplest, the one-factor models, account for one source of uncertainty and are usually used to model one particular interest rate. The more complex models, the multi-factor models, account for multiple sources of uncertainty and can be used to model several different interest rates with different maturities. Firstly, the one-factor models will be discussed before moving on to a more complex model, introducing the **LIBOR market models (LMM)**.

3.1.1 One-factor models

As the future interest rate is not known and considered uncertain, it is natural to model the interest rate as a random variable. The **one-factor models** usually study the behavior of the interest rate for the shortest possible deposit, commonly referred to as the **spot rate**. The stochastic models discussed by Wilmott in [46] are on the form

$$\mathrm{d}r = w(r,t)\mathrm{d}X + u(r,t)\mathrm{d}t, \tag{3.1}$$

where $\mathrm{d}r$ is the change in the spot rate over a time increment $\mathrm{d}t$, and $\mathrm{d}X$ is a Wiener process.

The functional form of $w(r,t)$ and $u(r,t)$ determines the characteristics of the spot rate, and experience shows that they must have a more complex form than constant coefficients (like for the asset price model (2.3)). For practical purposes, consider specific forms for u and w, which have the most general form compatible with a particular tractable class of solutions [47], namely

$$w(r,t) = \sqrt{\alpha(t)r - \beta(t)}, \tag{3.2}$$

$$u(r,t) = -\gamma(t)r + \eta(t) + \lambda(r,t)\sqrt{\alpha(t)r - \beta(t)}, \tag{3.3}$$

for functions α, β, γ, η and λ (which will be discussed further in section 3.1.3). By imposing suitable restrictions on these functions, one can ensure that the interest rate has specific economic properties, for instance, non-negative rates and mean-reverting behavior. The most popular choices have specific properties and are often named after their inventors. These models will however, not be discussed in further detail.

As the price of zero-coupon bonds is dependent on the spot rate, a one-factor model builds up an entire yield curve, essentially from the choice of parameters. If using historical data to estimate the parameters, the resulting yield curve will not necessarily (extremely seldom) fit with the market yield curve. As it would be naive to trust the model yield curve over the one observed in the market, **yield curve fitting** is often applied to one-factor models. This is done by allowing one or more of the parameters in the model to be time-dependent and choosing the form such that the yield curve from the model coincides with the market curve.

A clear advantage of yield curve fitting is that the bond prices, returned by the model, match the market prices, which makes it market consistent with respect to bonds. However, this approach has some drawbacks as the model parameters have to be calibrated frequently in order to capture the high curvature of the short end of the yield curve. One-factor models are known to be quite inconsistent, largely from the problems concerning the yield curve fitting.

Another problem with the one-factor models is that all bond prices are related to the same random factor, and consequently, bond prices are far more correlated in this simplified framework than they are in reality. The importance of the maturity of the bond is underestimated, as the ability to capture the dynamics concerning multiple interest rates are severely limited in the one-factor model. This is a consequence of the fact that the development of the yield curve is very constrained, as one factor drives the development of the entire curve.

3.1.2 Bond pricing with stochastic interest rate

In section 2.2.1, the procedure for pricing risk-free bonds with deterministic future interest rate, $r(t)$, was covered. This procedure is naturally considerably more complex when one relaxes the assumption about a deterministic future interest rate and instead models the interest rate as a stochastic variable.

A derivative is in general, dependent on a variable with uncertain future outcomes, an option is, for instance, dependent on the price and characteristics of the underlying asset. In a similar fashion, a bond is dependent on a stochastic variable, the spot rate, and its characteristics. A complication is that in this case there is no underlying asset to hedge with, as there is when pricing an option. Consider a portfolio with value Π, consisting of long a bond with value V_1 and maturity T_1 and short an amount Δ of a bond with value V_2 and maturity T_2,

$$\Pi = V_1 - \Delta V_2. \tag{3.4}$$

Similar to when deriving the Black-Scholes formula [6], consider the change in Π over a time increment $\mathrm{d}t$, employing Itô's lemma,

$$\mathrm{d}\Pi = \frac{\partial V_1}{\partial t}\mathrm{d}t + \frac{\partial V_1}{\partial r}\mathrm{d}r + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}\mathrm{d}t - \Delta\left(\frac{\partial V_2}{\partial t}\mathrm{d}t + \frac{\partial V_2}{\partial r}\mathrm{d}r + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\mathrm{d}t\right). \tag{3.5}$$

The only stochastic variable in the expression above is $\mathrm{d}r$, so by choosing $\Delta = \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}$ the stochastic term disappears, and the change in Π becomes purely deterministic. By appealing to the concept of arbitrage, this risk-free return should be the same as the return from the risk-free interest rate,

$$\begin{aligned}\mathrm{d}\Pi &= \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \frac{\partial V_1/\partial r}{\partial V_2/\partial r}\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right)\right)\mathrm{d}t \\ &= r\Pi\mathrm{d}t \\ &= r\left(V_1 - \frac{\partial V_1/\partial r}{\partial V_2/\partial r}V_2\right)\mathrm{d}t\end{aligned} \tag{3.6}$$

By gathering all terms with V_1 on one side, and all terms with V_2 on the other side, one obtains

$$\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1\right)\Big/\frac{\partial V_1}{\partial r} = \left(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2\right)\Big/\frac{\partial V_2}{\partial r}\ . \tag{3.7}$$

The right-hand side of the equation above is a function of T_1, while the left-hand side is a function of T_2. This implies that both sides are independent of the maturity date, and therefore it can be written,

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - rV\right)\Big/\frac{\partial V}{\partial r} = a(r,t), \tag{3.8}$$

dropping the subscript on V, for some function $a(r,t)$. The function $a(r,t)$ is often written on the form

$$a(r,t) = w(r,t)\lambda(r,t) - u(r,t), \tag{3.9}$$

for some function $\lambda(r,t)$ and the functions $u(r,t)$ and $w(r,t)$ as in (3.1). This will prove convenient later. Inserting for $a(r,t)$, the pricing equation for the zero-coupon bond is

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0. \tag{3.10}$$

In order to solve this differential equation, the final condition

$$V(r,T) = Z, \tag{3.11}$$

is imposed, simply stating that as the bond is risk-free, the principle is repaid in full at maturity. Further, in order to solve the equation, the functions u, w and λ need to be specified, in addition to the boundary conditions in r. This is dependent on the interest rate model employed and will not be discussed in further detail.

3.1.3 The market price of risk

To give an interpretation of the function $\lambda(t,r)$ consider an unhedged position in one bond with maturity T. The change in the value of the bond over a time increment $\mathrm{d}t$ is

$$\mathrm{d}V = w\frac{\partial V}{\partial r}\mathrm{d}X + \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r}\right)\mathrm{d}t. \tag{3.12}$$

From equation (3.10), the factor multiplied by $\mathrm{d}t$ can be substituted, and the change in the bond value can be written

$$\begin{aligned} &\mathrm{d}V = w\frac{\partial V}{\partial r}\mathrm{d}X + \left(w\lambda\frac{\partial V}{\partial r} + rV\right)\mathrm{d}t \\ \implies\quad &\mathrm{d}V - rV\mathrm{d}t = w\frac{\partial V}{\partial r}(\mathrm{d}X + \lambda\mathrm{d}t). \end{aligned} \tag{3.13}$$

Note that the left-hand side is the difference between the change in the bond value $\mathrm{d}V$ and the return associated with risk-free interest r, indicating that this difference *not* zero. In fact, this difference is not zero as there is some risk associated with the bond. The right-hand side contains two terms - a deterministic term in $\mathrm{d}t$ and a stochastic term in $\mathrm{d}X$. The expected value of the excess return above the risk-free rate is the deterministic term, hence it may be interpreted as the excess return above the risk-free rate for accepting a certain level of risk. As it is proportional to λ, this function is called the **market price of risk**.

3.2 The LIBOR market models (LMM)

The **LIBOR market model (LMM)** is stochastic framework constructed to describe the *forward* LIBOR rates (LIBOR is an abbreviation for the London Inter-Bank Offered Rate). It is also known as the BGM model (Brace Gatarek Musiela model), in reference to the names of some of the inventors [35].

LIBOR rates are *simple*, meaning that if L denotes the rate for an accrual period of length δ (1/4 for three months, 1/2 for six months, etc.), then the interest earned over the period is δL. The LMM framework aims to model the simple *forward* LIBOR rates. Fix δ and consider a maturity T. The forward rate $L(0,T)$ is the rate set at time 0 for the future interval $[T, T+\delta]$. That is, entering into a contract at time 0 to borrow 1 at time T, the interest made at time $T+\delta$ is $\delta L(0,T)$.

3.2.1 Mathematical formulation

The relation between the rates and bond prices are fundamental in the construction of an interest rate model, as the quantities are so closely linked. Hence, the relation between the variables should be used to ensure that the model is reasonable with respect to observed bond prices. One can obtain the following identity between the forward rates and bond prices,

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)} \tag{3.14}$$

Consider a finite set of maturities, referred to as **tenor dates**, $0 = T_0 < T_1 < \cdots < T_M < T_{M+1}$, and let $\delta_i = T_{i+1} - T_i$, $i = 0, \ldots, M$. Let $B_n(t)$ denote the price of a bond maturing at T_n at time t, $0 \le t \le T_n$, and similarly, let $L_n(t)$ be the forward rate of time t for $[T_n, T_{n+1}]$. Rewriting (3.14) yields

$$L_n(t) = \frac{B_n(t) - B_{n+1}(t)}{\delta_n B_{n+1}(t)}, \quad 0 \le t \le T_n, \quad n = 0, 1, \ldots, M. \tag{3.15}$$

The relationship between the bond prices and the forward rates are highlighted in Figure 1.

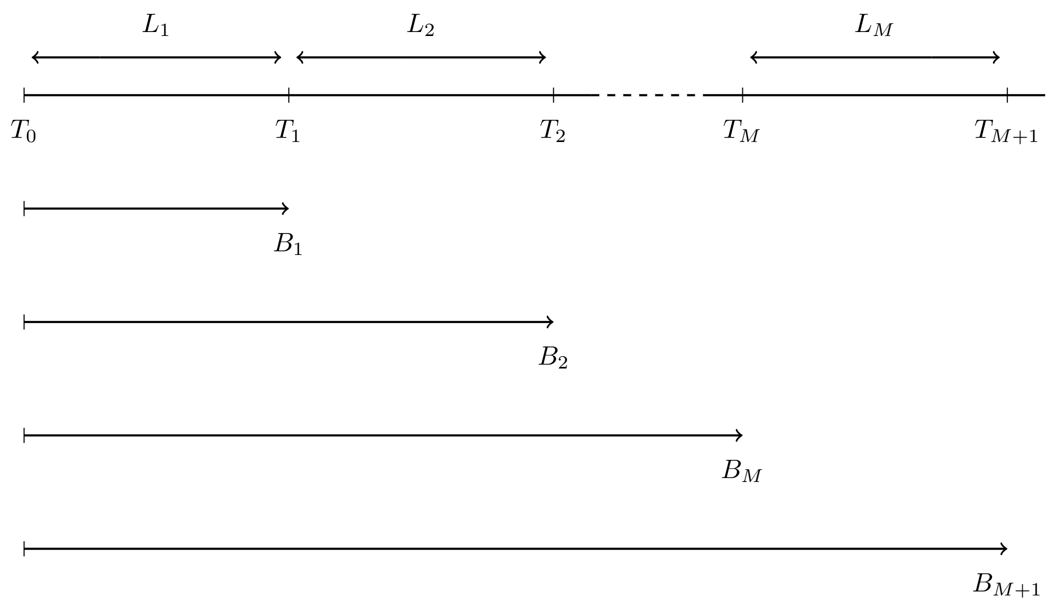


Figure 1: Illustration of the forward rate term structure

Equation 3.15 implies that bond prices determine the forward rates, however the converse is not entirely the case. For t equal a tenor date T_i equation (3.15) implies

$$\implies \quad B_n(T_i) = \prod_{j=i}^{n-1} \frac{1}{1+\delta_j L_j(T_i)}, \quad n = i+1, \dots, M+1. \tag{3.16}$$

However, at an arbitrary time t, the rates do *not* directly determine the bond prices as the discount factor is not entirely specified by the LIBOR rates. For instance, let $T_i < t < T_{i+1}$ and consider $B_n(t)$ for some $n < i+1$. Consider pricing with (3.16), and note that LIBOR rates does not specify the discount factor from T_{i+1} to t. Let $\eta(t)$ be the unique integer such that

$$T_{\eta(t)-1} \leq t \leq T_{\eta(t)}, \tag{3.17}$$

meaning it gives the index of the next tenor date. With this notation

$$B_n(t) = B_{\eta(t)}(t) \prod_{j=\eta(t)}^{n-1} \frac{1}{1+\delta_j L_j(T_i)}, \quad 0 \leq t \leq T_n, \tag{3.18}$$

thus $B_{\eta(t)}(t)$ is the missing factor needed to fully determine the bond price by the LIBOR rates.

Consider a model on the form

$$\frac{\mathrm{d}L_n(t)}{L_n(t)} = \mu_n(t)\mathrm{d}t + \sigma_n(t)^\top \mathrm{d}W(t), \quad 0 \leq t \leq T_n, \quad n = 1, \dots, M. \tag{3.19}$$

The coefficients μ_n and σ_n may depend on the current rates $(L_1(t), \ldots, L_M(t))$ and the time t. The procedure for determining the coefficients is covered by Jamshidian in [27], and by non-arbitrage considerations with respect to the bond prices he obtains,

$$\mu_n(t) = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t) \sigma_n(t)^\top \sigma_j(t)}{1 + \delta_j L_j(t)}. \tag{3.20}$$

The procedure for determining the volatility structure is somewhat more complicated and will be covered in detail in Section 3.2.3.

3.2.2 Simulation

In order to simulate from this model, a discretization is required, as simulation in continuous time is, in general, not feasible. It is only necessary to discretize time, as the maturity argument is already in a discrete set. Let the time grid be $0 = t_0 < \cdots < t_m < t_{m+1}$, in which the tenor dates are usually included. Simulation of the model can be done in numerous ways, for instance by an Euler scheme,

$$\hat{L}_n(t_{t+1}) = \hat{L}_n(t_i) + \mu_n(\hat{L}(t_i), t_i)\hat{L}_n(t_i)[t_{i+1} - t_i] + \hat{L}_n(t_i)\sqrt{t_{i+1} - t_i}\sigma_n(t_i)^\top Z_{i+1}, \tag{3.21}$$

with

$$\mu_n(\hat{L}_n(t_i), t_i) = \sum_{j=\eta(t_i)}^{n} \frac{\delta_j \hat{L}_j(t_j) \sigma_n(t_i)^\top \sigma_j(t_j)}{1 + \sigma_j \hat{L}_j(t_i)}, \tag{3.22}$$

and where $Z_1, Z_2, \ldots,$ are independent d-dimensional $\mathcal{N}(0, I)$ vectors. The hats are used to identify discretized variables. Given an initial set of bond prices $B_1(0), \ldots, B_{M+1}(0)$ the initial values are

$$\hat{L}_n(0) = \frac{B_n(0) - B_{n+1}(0)}{\delta_n B_{n+1}(0)}, \quad n = 1, \ldots, M. \tag{3.23}$$

An alternative is applying the Euler scheme to $\log L_n$, which is particularly attractive in the case of deterministic σ_n, as L_n would be close to log-normal in this case. In addition, it does not allow for negative values, which generally is a desired property. The iteration procedure is then

$$\begin{aligned} \hat{L}_n(t_{i+1}) = \hat{L}_n(t_i) \times \\ \exp\left(\left[\mu_n(\hat{L}(t_i), t_i) - \frac{1}{2} \|\sigma_n(t_i)\|^2 \right] (t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i}\ \sigma_n(t_i)^\top Z_{i+1} \right). \end{aligned} \tag{3.24}$$

Note that one would only generate $\hat{L}_n(t_i)$ for $n \geq \eta(t_i)$, as $\hat{L}_n$ cease to be meaningful for $t_i > T_n$, since the rate is set at T_n and relevant for the interval $[T_n, T_{n+1}]$.

3.2.3 Calibration and volatility structure

In the previous sections, there has been no discussion about how to determine the volatility term. In the following section, the procedure for estimating the volatility term in the LMM setting will be covered, based on the discussions by Glasserman in [20] and by Jamshidian in [27].

The variables $\sigma_n(t)$ describe two aspects - the level of volatility in the forward rates and the correlations between the different rates. Firstly, consider the overall level of volatility. Given the market price of a caplet for the interval $[T_n, T_{n+1}]$, one can by inverting the Black formula [5] calculate the implied volatility v_n. Hence, for the model to return a caplet price consistent with the market price, the volatility σ_n can be any deterministic $\mathbb{R}^d$ function satisfying,

$$\frac{1}{T_n}\int_0^{T_n} \|\sigma_n(t)\|^2 \mathrm{d}t = v_n^2. \tag{3.25}$$

By imposing this constraint on all σ_j, one ensures that the model is calibrated to market caplet prices.

As the interest rates over accrual periods shorter than $[T_i, T_{i+1}]$ are not specified, it is customary to restrict attention to functions constant between tenor dates. Let each σ_n be right-continuous and denote the value over the interval $[T_i, T_{i+1})$ by $\sigma_n(T_i)$. Suppose for the time being that $d = 1$, implying that each σ_n is scalar valued. In this particular case, it may be convenient to think of the volatility structure as specified through a lower-triangular matrix on the form

$$\begin{bmatrix} \sigma_1(T_0) & & & \\ \sigma_2(T_0) & \sigma_2(T_1) & & \\ \vdots & \vdots & \ddots & \\ \sigma_M(T_0) & \sigma_M(T_1) & ... & \sigma_M(T_{M-1}) \end{bmatrix} \tag{3.26}$$

As each $L_n(t)$ cease to be meaningful for $t > T_n$, the upper half of the matrix is irrelevant. Note that

$$\int_0^{T_n} \sigma_n^2(t)\mathrm{d}t = \sigma_n^2(T_0)\delta_0 + \sigma_n^2(T_1)\delta_1 + \dots + \sigma_n^2(T_{n-1})\delta_{n-1}, \tag{3.27}$$

hence (3.25) further constraints the sum of squares along each row in the matrix.

If $\sigma_n(t)$ is only dependent on n and t through the difference $T_n - t$, the volatility structure is said to be *stationary*. In this case, the matrix given in (3.26) takes the form

$$\begin{bmatrix} \sigma(1) & & & \\ \sigma(2) & \sigma(1) & & \\ \vdots & \vdots & \ddots & \\ \sigma(M) & \sigma(M-1) & ... & \sigma(1) \end{bmatrix} \tag{3.28}$$

where $\sigma(i)$ can be thought of as the volatility of a forward rate i periods away from maturity. Note that in this case, the number of variables equals the number of caplets maturities, and calibration to additional instruments requires introduction of additional factors or non-stationarity.

In a multi-factor model ($d \geq 2$), the entries $\sigma_n(T_i)$, $i = 1, \dots, n$, in (3.26) are replaced by the norms $\|\sigma_n(T_i)\|$, as each $\sigma_n(T_i)$ is now a vector. With $\sigma_n(t)$ piece-wise constant,

$$\int_0^{T_n} \|\sigma_n(t)\|^2 \mathrm{d}t = \|\sigma_n(T_0)\|^2\delta_0 + \|\sigma_n(T_1)\|^2\delta_1 + \dots + \|\sigma_n(T_{n-1})\|^2\delta_{n-1}, \tag{3.29}$$

implying that the extended model does *not* provide increased flexibility in matching implied volatilities from caplet prices. The potential value of a multi-factor model is inherent in its ability to capture correlations between forward rates of different maturities. From (3.24) it follows that

$$\rho(\log L_j(t), \log L_k(t)) \approx \frac{\sigma_k(t)^\top \sigma_j(t)}{\|\sigma_k(t)\| \; \|\sigma_j(t)\|}, \tag{3.30}$$

where $\rho(\cdot,\cdot)$ is used to denote the correlation. A common choice is choosing the volatility structure such that it matches historical correlations.

In markets such as the Norwegian, where caps are rather illiquid assets, calibration of the volatility terms based on the method proposed above proves difficult. As a consequence, models are commonly calibrated to match historical correlations, in particular, if the purpose of the model is not pricing of interest rate derivatives and market consistency is not of the utmost importance. A way to conduct such a calibration is to choose a stationary structure, such that the volatility for rate L_n is only dependent on time to maturity,

$$\sigma_n(t) = \sigma(T_n - t). \tag{3.31}$$

Further, for an iteration scheme on the form (3.24), calculate the covariance matrix Σ for the log-differences in the forward rates, and compute the spectral decomposition

$$\Sigma = V \Lambda V^{-1}, \tag{3.32}$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_M)$ is a diagonal matrix with increasing eigenvalues along the diagonal. The number of factors in the model can then be chosen to be such that it explains a satisfactory amount of the variability in the data. This is done by employing the result that the first d factors explain a fraction

$$\frac{\sum_{i=1}^{d} \lambda_i}{\sum_{i=1}^{M} \lambda_i} \tag{3.33}$$

of the variability in the original data. Typically, one will choose d such that it explains a certain percentage of the variability, e.g. 90 %. If the number of factors is d, the volatility factors are

$$\sigma_n^i = \sqrt{\lambda_i} V_{ni}, \quad \text{for } i = 1, \ldots, d, \quad n = 1, \ldots, M, \tag{3.34}$$

where σ_n^i denotes the ith volatility factor a forward rate T_n from maturity. Note that this implies approximate compliance with the relation (3.30) (exact compliance if the number of factors equal the number of rates, $d = M$).

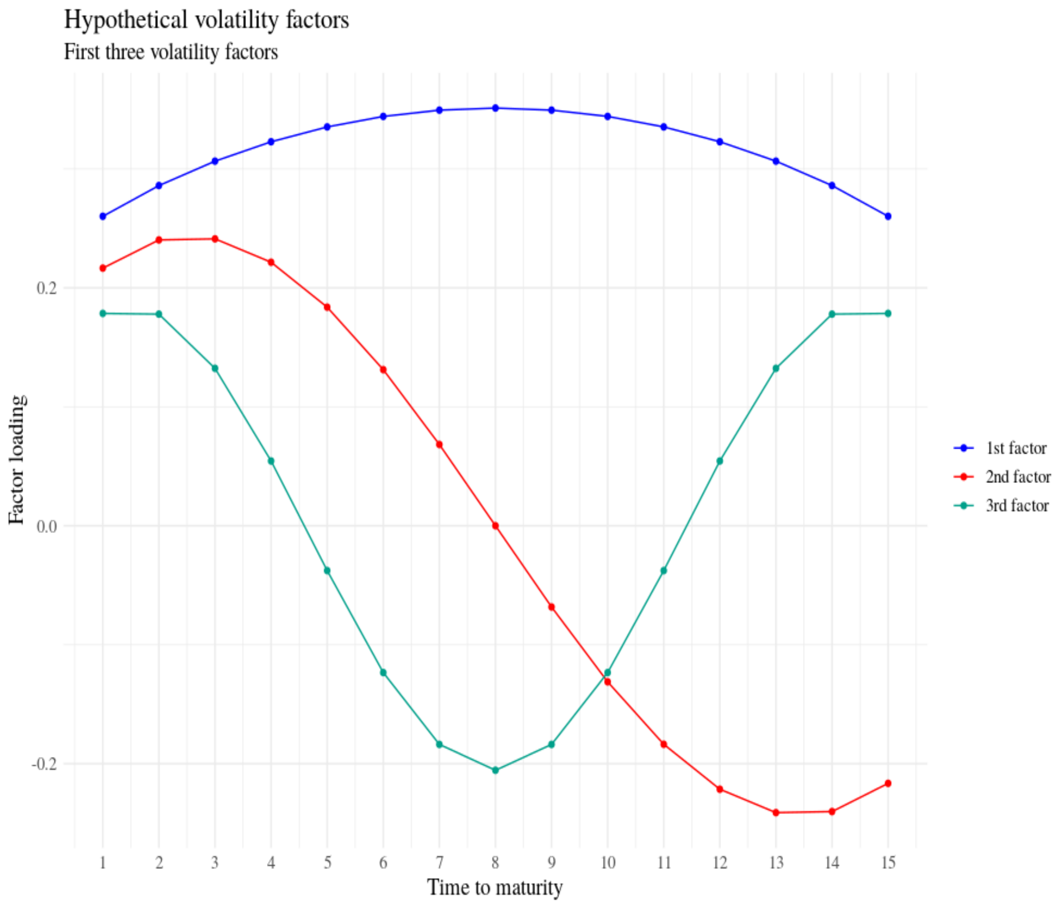


Figure 2: Hypothetical volatility factors

This type of factor analysis typically produces the qualitative features of the volatility factors displayed in Figure 2. The first volatility factor is commonly relatively constant across maturities, and as it causes all rates to shift in the same direction, it represents a parallel shift in the yield curve. The second factor typically shifts from negative values to positive values (or vice versa) and thereby causes the short and long end of the yield curve to move in opposite directions. The third factor moves the short and long end in the same direction while moving the middle in the opposite direction, thereby causing the yield curve to steepen or flatten.

These qualitative features should be present in the volatility factors of a multi-factor interest model and serve as a primary sanity check [38]. However, they will typically not be so distinct, and deviations may occur. The volatility factors displayed in Figure 2 are calculated from the hypothetical covariance matrix expressed by

$$\Sigma_{ij} = 0.5^2 \exp\left(-\frac{1}{2}\sqrt{|i-j|}\right), \quad i,j = 1,\ldots,15. \tag{3.35}$$

3.2.4 Pricing of interest rate derivatives

The main application of a multi-factor interest rate model is to evaluate the fair price, and corresponding uncertainty, of interest rate derivatives. In particular, the uncertainty is of interest, as it may be applied to measure and manage the risk associated with the derivatives. This will be discussed further in Section 4.

Depending on the procedure for determining the volatility structure, the model will be market consistent with respect to different financial instruments. For some derivatives, there exist close-form solutions [34], however, in the following, emphasis will be on obtaining the estimates through Monte Carlo approach (Appendix C).

The forward rates are initialized according to (3.15), and as the model is derived with non-arbitrage considerations, the model should return bond prices consistent with the initial values. Hence, the pricing of zero-coupon bonds by simulation serves as a useful verification tool. The procedure to determine $B_n(0)$ by simulation is to generate $\hat{L}_1(T_1), \hat{L}_2(T_2), \ldots, \hat{L}_{n-1}(T_{n-1})$ from the model, and calculate the present value of the bond by

$$\hat{B}_n = \prod_{j=0}^{n-1} \frac{1}{1+\delta_j \hat{L}_j(T_j)}. \tag{3.36}$$

Averaging over many realizations will give the expected value of the bond, and hence an estimate of the fair price.

If the volatility structure is calibrated as suggested for the Norwegian market, the model will *not* necessarily be market consistent for other interest rate derivatives. However, it may be employed to give a reasonable estimate of the fair price and to determine the uncertainty in the expected payoff.

Consider a derivative dependent on $L_1, L_2, \ldots, L_n$ with payoff $g(L(T_n))$ at time T_n. In order to price this derivative, simulate to time T_n and calculate the present value of the payoff

$$g(\hat{L}(T_n)) \prod_{j=0}^{n-1} \frac{1}{1+\delta_j \hat{L}_j(T_j)}. \tag{3.37}$$

Again, averaging over many independent replications, results in an estimate of the price of the derivative at time 0.

4 Risk management

The process of identification, analysis and, finally, acceptance (or mitigation) of uncertainty related to investments is, in the financial industry, referred to as **risk management**. As mentioned in section 2.3, risk is a broad term essentially describing how likely the return is to deviate from the expectation. Although risk generally provokes predominantly negative associations, in the financial industry, it is necessary and inseparable from performance according to the efficient market hypothesis [47].

The amount of risk one is willing to accept, depends on the risk tolerance, which is an expression of the capacity to tolerate risk based on specific financial circumstances and the propensity to do so. In this section, the main focus is describing the different types of risk and how to quantify the risk through risk measures. The practical application of the measures includes managing the risk through risk mitigation practices such as netting and collateralization. How such practices may be implemented, in addition to how their effect may be quantified, is also discussed.

4.1 Risk measures

Central in risk management is the use of **risk measures**, which are statistical measures quantifying the risk associated with an investment, for instance, determining the amount of capital necessary to satisfy regulative requirements.

If X is a stochastic variable (e.g. the daily log-returns on a portfolio), then $\rho(X)$ is said to be a **coherent risk measure** if it satisfies:

(i) **Sub-additivity:** $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
Meaning that the risk associated with two portfolios together cannot be larger than the sum of the individual portfolios (diversification principle).

(ii) **Monotonicity:** If $X \leq Y$ in every scenario, then $\rho(X) \leq \rho(Y)$.
That is, if the return of Y is larger than X in every scenario, then Y is the riskier portfolio.

(iii) **Positive homogeneity:** If $\alpha > 0$, then $\rho(\alpha X) = \alpha\rho(X)$.
Implying that if the value of the portfolio doubles, so does the associated risk.

(iv) **Translation invariant:** If $\gamma > 0$, then $\rho(X+\gamma) = \rho(X) - \gamma$.
Meaning that adding cash (a risk-free return) to a portfolio, lowers the associated risk.

These assumptions are reasonable to impose on a risk measure, as inconsistencies may cause non-intuitive results. Hence, coherent risk measures are generally preferred over alternative risk measures [2].

4.1.1 Value at risk

Value at risk (VAR) is a risk measure heavily used in practice and is a statistical technique to quantify the risk associated with an asset over a specific time frame. More precisely, if X

is the return on a portfolio over a given time period, then $\mathrm{VAR}_\alpha(X)$ is defined as the number corresponding to the lower (1-α)-quantile of the distribution of X,

$$\mathrm{VAR}_\alpha(X) = \inf\{x \in \mathbb{R} : F_X(x) > 1 - \alpha\}. \tag{4.1}$$

The intuitive interpretation is the loss occurring with probability 1-α. Although this measure is widely used and easy to grasp, it has certain downsides, in particular, the violation of the sub-additivity property. As the value at risk measure does not consider the tail of the distribution, merely the quantile, it is rather easy to construct portfolios to which the value at risk returns misguiding results, which could discourage diversification [31]. Despite not being a coherent risk measure, it is simple and easily interpretable, and therefore still an industry favorite.

4.1.2 Expected shortfall

Expected shortfall (ES) (also referred to as conditional value at risk) is a slightly more complex risk measure, derived from the value at risk. Expected shortfall quantifies the *tail risk* of an investment, by a weighted average of outcomes in the tail beyond the value at risk cutoff. Mathematically it is defined as

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VAR}_\gamma(X)\mathrm{d}\gamma. \tag{4.2}$$

Meaning, it is the expected loss, given that the loss exceeds the value at risk. Contrary to the value at risk, the expected shortfall is a coherent risk measure, as it considers the entire lower tail rather than simply the lower quantile. Hence, it provides more information about the potential loss than the value at risk. However, it is more challenging to get a reasonable estimate for the expected shortfall as it demands excessive knowledge of the tail, which generally is the least explored part of the distribution. In order to convey a comprehensive measure of the risk, the ES is often used in combination with the VAR.

4.2 Counterparty credit risk

Counterparty credit risk describes the risk arising from the possibility that the counterparty does not honor its contractual obligations, i.e., defaults on amounts owned on a derivative transaction. Only contracts that are privately negotiated between counterparties, for instance, swap contracts, are subject to counterparty risk. Securities traded on an exchange are not affected by this type of risk, as the exchange guarantees the cash flows promised by the derivative.

An important feature inherent in the concept of counterparty credit risk is that the risk is bilateral; each party has credit risk concerns with respect to the other party. This is a subtle, but important distinction from what is usually referred to as credit risk, where the party lending money takes on credit risk while the other party (the lender) takes no such risk.

Although counterparty risk is in one sense a specific form of credit risk, the significance is greater than what such a description might suggest. Counterparty risk requires knowledge of a wide specter of different forms of risk, as it is manifested as a combination of credit risk and other types of risk [21];

(i) **Market risk**. The interaction between market and credit risk has long been associated with counterparty credit risk. The counterparty credit exposure is essentially the market value of the contract, and as market risk describes the risk arising from changes in market variables, the exposure to a particular counterparty is naturally subject to said changes.

(ii) **Market liquidity risk**. In the case of a defaulting counterparty, the collateral (if applicable, covered in Section 4.2.4) and the position need to be liquidated and closed out. This process is subject to market liquidity risk, which refers to the risk that the asset cannot be traded quickly enough in the market without impacting the price.

(iii) **Funding liquidity risk**. If the parties of a contract are required to post collateral (discussed in Section 4.2.4), this may cause liquidity issues if the changes in the contract valuation are significant enough. This may further question the creditworthiness of the counterparty.

(iv) **Operational risk**. The process of managing counterparty risk relies on practices such as netting and collateralization (Section 4.2.3), which introduce operational risk from the probability that these practices are poorly or insufficiently handled.

(v) **Systemic risk**. Market participants are closely linked, and the default of one may provoke a chain reaction causing other participants to default. Moreover, the use of central counterparties (Section 4.2.3) introduces greater systemic risk to the market, due to the possibility that they themselves might fail.

Furthermore, the interaction of these risks plays a defining role in the nature of counterparty risk. The ability to quantify and manage counterparty risk is crucial for the growth and stability of the financial markets - the effect of the opposite; poor understanding and management of counterparty risk is demonstrated by the financial crisis of 2008 [28].

Commonly, counterparty credit risk is divided into three parts; **probability of default**, **loss given default** and **counterparty credit exposure**. The two former describe the probability that the counterparty defaults and the loss given that it defaults, respectively. An exhaustive framework for counterparty credit risk would incorporate all these parts, as outlined by Chatterjee in [10]. However, this thesis exclusively focuses on the last component of counterparty credit risk, namely counterparty credit exposure.

4.2.1 Counterparty credit exposure

The counterparty credit exposure (often referred to as simply exposure) is the amount one could potentially lose in the event that the counterparty defaults. If at any point the counterparty defaults, the exposure is the current value of the remaining future payments. More precisely, it is the replacement cost of the contract. The value of derivative contracts can fluctuate significantly over time; hence, it may be useful to consider the exposure at future times, referred to as potential credit exposure. The current exposure is known, the market value of the contract, while the future exposure is uncertain.

A defining characteristic of exposure is the fact that a positive valuation of the derivative corresponds to a claim on a defaulting counterparty, while in the event of a negative valuation, one cannot walk away from the contractual obligations. Meaning, if a party is owed money and their counterparty defaults they will incur a loss, however, in the reverse situation they will not gain from the default as they will not be released from their liability [21].

Determining the counterparty credit exposure entails calculating the potential loss if the counterparty defaults at some time. By convention, any recovery value in the event of a default is ignored, and consequently, the exposure is the loss as defined by the replacement cost of the derivative. Note that exposure is conditional on counterparty default [9].

Consider an interest rate derivative subject to counterparty credit risk, for instance, an interest rate swap contract. Denote the value of the derivative V_t at time t, for $0 \leq t \leq T$. The value of the derivative may take both positive or negative values, depending on the particular derivative - a swap may for instance yield cash flows in both directions depending on the movements in the underlying interest rate.

As previously mentioned, the counterparty credit exposure is the replacement cost of the contract, i.e., the market value of the contract. The procedure for obtaining the potential exposure involves generating realizations of the underlying risk factors at future points in time. For interest rate derivatives, this entails generating the relevant interest rates. Further, in order to determine the potential exposure, one needs to calculate the price of the derivative for each realization of the underlying risk factors. In the absence of a closed form solution, pricing the derivative requires a new simulation according to (3.37). This procedure is highlighted in Figure 3.

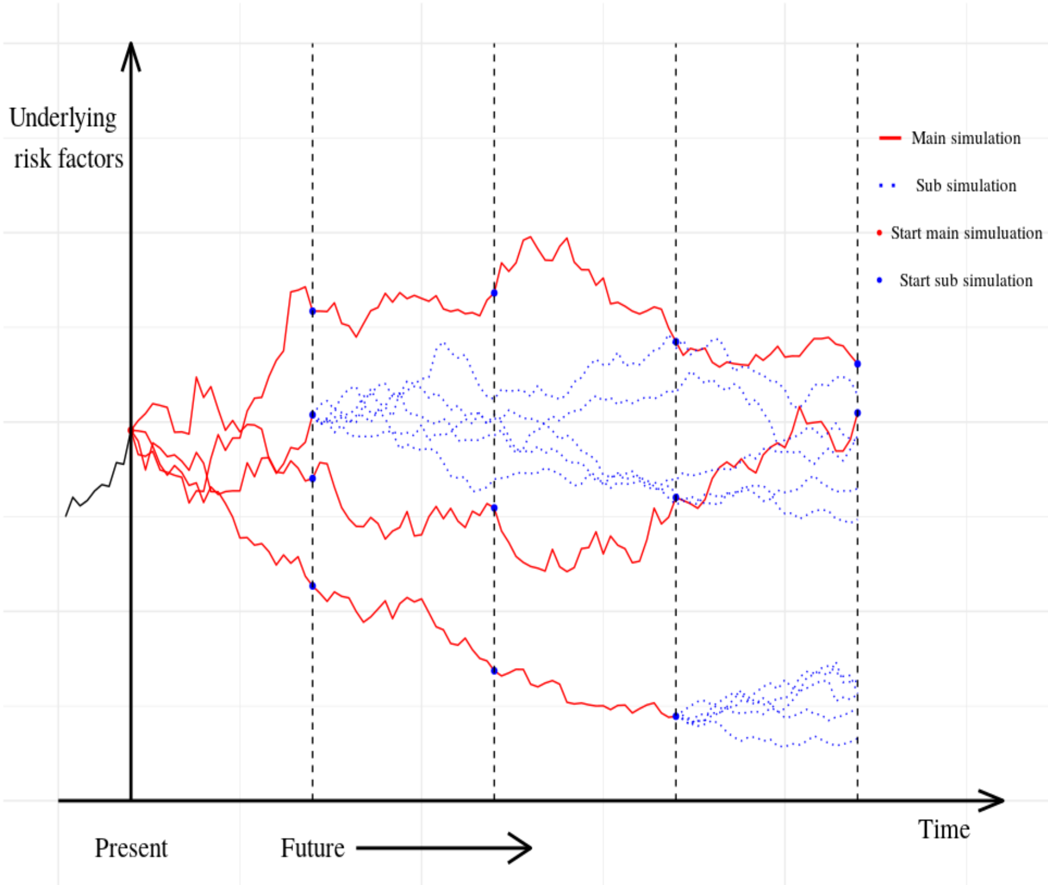


Figure 3: Illustration of the simulation procedure to obtain the counterparty credit exposure in the absence of a closed form solution

The red lines in Figure 3 indicate the generated scenarios of the underlying risk factors, while the blue lines indicate the simulations required to obtain the price of the derivative. Note that sub simulations start from where the relevant main simulation ends, at each relevant time indicated by the vertical black lines. There is no need to calculate the value of the derivative for other times, and consequently, there is no need for sub simulations. Finally, note that the main simulations continue until the maturity T, and the red lines are removed prior to the sub simulations purely to avoid the graph from becoming overpopulated.

From the derivative values generated by the procedure above, one can construct the empirical price distribution at each of the predetermined time points. From the distribution one extracts the desired statistical quantities (discussed in Section 4.2.2). A schematic illustration of the procedure is presented in Figure 4, also highlighting the end product when aggregating the results for the different time points.

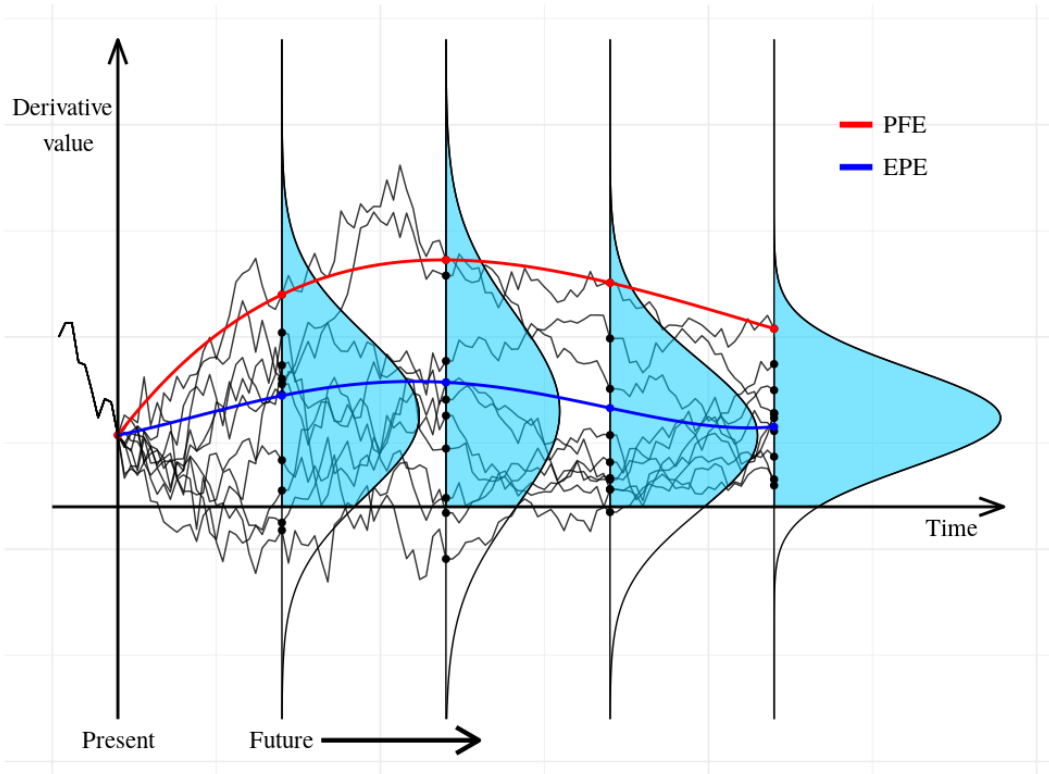


Figure 4: Illustration of the construction of the potential counterparty credit exposure from the simulated derivative values

The mathematical definitions of the red and blue line, and their practical implications are discussed in Section 4.2.2. For the time being, it is sufficient to think of them as describing some aspect of the counterparty credit exposure. The lines indicating the derivative value are purely for illustration purposes, as one would not calculate the derivative value at every time point, merely at the predetermined time points of interest, as previously discussed.

The general form of the exposure profile is descriptive of the typical counterparty credit exposure inherent in an interest rate swap contract. The current exposure is known, and naturally, the initial value of all the simulated derivative values coincide at this point. The potential exposure increases as the future value of the derivative is unknown. The uncertainty in the underlying risk factors is typically largest at maturity, as the uncertainty increases the further from the initial condition one considers. However, the exposure is commonly not largest at maturity, despite the uncertainty in the underlying being the largest. Simultaneously with the increase in uncertainty, payments are exchanged, and consequently, the magnitude of the outstanding payments decrease. As a result, although the uncertainty increases, the exposure typically decreases slightly close to maturity.

4.2.2 Counterparty risk measures

As outlined in Section 4.2.1, the simulated values of the derivative, V_t, are used to construct the empirical distribution. Further, one may extract the statistical quantities of interest, which describe certain aspects of the counterparty credit exposure. The metrics used to describe the counterparty credit exposure are fundamentally different from the risk measures discussed in Section 4.1. While the risk measures discussed previously describe some aspect of *loss*, the measures used for exposure are describing *gain*, as this is the amount at risk if the counterparty defaults and renders the contract worthless [48].

A frequently employed metric is the **expected positive exposure (EPE)**, defined as

$$\text{EPE}_t = \text{E}(\max(V_t, 0)) = \text{E}(V_t^+). \tag{4.3}$$

The expectation is taken over the positive valuations, as the case of a default in the event of a negative valuation will not result in a loss.

Potential future exposure (PFE) aims to account for the worst potential exposure one could have a certain time in the future, defined as

$$\text{PFE}_{\alpha,t} = \inf\{x \in \mathbb{R} : P(V_t \leq x) \geq \alpha\}. \tag{4.4}$$

Note that the definition is analogous to the definition of VAR (4.1), with the modification that PFE is associated with a gain rather than a loss. For instance, the PFE at confidence level $\alpha = 99\%$ is the exposure that would be exceeded with a probability no more than $1-\alpha = 1\%$.

As PFE is defined as the upper quantile of the distribution, the measure is subject to the same criticism as the value at risk, namely that it violates the sub-additivity property. Consequently, one can construct portfolios to which the PFE fails to accurately capture the associated risk, by that yielding misleading results.

A possible approach to sidestep this problem is to use the expected shortfall instead; however, this comes at the added expense of generating a large sample to obtain an accurate estimate. Expected shortfall is generally employed when it is convenient to take into account events of significant magnitude that can occur with only a very small probability. A typical example is credit derivatives (discussed in Section 4.2.3), where the default of the party protected by the derivative is a low probability event, albeit with significant impact. In this report, the exposure is described through EPE and PFE.

4.2.3 Counterparty risk management

After having computed the exposure, the next step is to employ it to effectively manage the counterparty risk. A straightforward manner to mitigate the risk with regards to a particular counterparty is to restrict the exposure, for instance, based on the value of the PFE, by not doing any business which would force the PFE above a certain threshold. These limits will reflect the quality of the counterparty, in addition to the credit risk appetite towards the particular counterparty. This is also a natural approach to control the exposure to, for instance, a given geographical region or business type [21].

Restricting the magnitude of the exposure to a particular counterparty does not reduce the exposure with regards to contracts already agreed upon. One way of actively reducing this exposure is **netting**. Netting is legally offsetting negative and positive positions with the same counterparty, thereby, in the event of default, giving additional benefits at the expense of other creditors. More precisely, if a company has n contracts with a counterparty, each with value v_t^i at time t, $i = 1, \ldots, n$. Then, rather than considering the exposure of each individual contract and then adding the individual exposures, the contracts are aggregated prior to calculating the exposure, so one would rather consider

$$\sum_{i=1}^{n} v_t^i \leq \sum_{i=1}^{n} v_t^{i+}. \tag{4.5}$$

Clearly, the exposure of the netted position is less or equal to the exposure of the un-netted position. Ignoring the operational costs, netting is essentially free. However, the impact of netting is finite and heavily dependent on the type of transactions [9]. For instance, netting has a more significant effect if the contracts are negatively correlated than if they are positively correlated.

Collateralization is a way to reduce counterparty credit risk more significantly, and in theory, even eliminate it entirely. However, it is associated with considerable operational costs, as well as giving rise to other risks, including liquidity risk and operational risk. Collateral may be posted as cash or other liquid assets and is used as protection against the default of the counterparty. In the event of default, the loss is partially or completely covered, and, consequently, the potential exposure is reduced by means of collateralization. The reason to include a collateral agreement in a transaction is naturally to reduce the exposure, for instance, with the objective to do more business with the particular counterparty without exceeding the exposure limit. The impact of collateralization is further discussed in Section 4.2.4, in addition to how to account for this effect in a modeling framework.

The use of **central counterparties (CCP)** is another way to mitigate counterparty credit risk, or rather, move the exposure to a central counterparty with high credit rating. In effect, this offers advantages such as risk reduction and operational efficiencies, as all the exposure is gathered towards one counterparty. The CCP agrees to meet the contractual obligations if one of the counterparties in the trade defaults, hence the counterparty credit risk is severely limited by the introduction of a CCP in most cases. However, CCPs potentially allow asymmetric information flow, in addition to creating greater systematic risk due to the possibility that they themselves may default.

Counterparty credit risk may also be mitigated in another manner, by hedging through the purchase of **credit derivatives**. There exists a wide variety of credit derivatives, many of them infamous from their central role in the economic meltdown of 2008 [42], including credit default swaps (CDS) and collateralized debt obligations (CDO). In this case, the derivatives were employed as speculative vehicles and magnified the overall risk in the market. Despite the destructive display in the economic crisis, credit derivatives may be useful in reducing (or even removing) the counterparty credit risk associated with a portfolio.

A **credit default swap (CDS)** is designed to transfer the credit exposure of a contract from the buyer of the CDS to the seller. The seller agrees that in the event the debt issuer (original counterparty) defaults, the buyer will be reimbursed the outstanding payments in the security. Thereby, the counterparty credit concerns are in a way transferred to the seller of the CDS, and typically, these concerns are small as the seller tends to have high credit-worthiness. In a sense, CDS is constructed such that credit risk can be borne by those in the best position to bear it. A CDS is usually constructed such that the buyer makes payments to the seller over the lifetime of the security, analogous to an insurance policy. The pricing of a CDS, and credit derivatives in general, relies on a framework similar to the one explained to evaluate the counterparty credit exposure, with emphasis on the extreme outcomes [22] [23]. In addition, the pricing procedure will have to incorporate the probability of default and loss given default, which will not be discussed in this paper.

4.2.4 Collateralization

In this section, the quantification of credit exposure is discussed under the presence of collateral. The use of collateral is widespread, as increasing regulation requires all bilateral trades between financial counterparties to be collateralized [11]. In addition, an increasing number of bilateral trades with other parties are collateralized as this reduces the associated risk and generally induces a better deal for both parties [36]. Hence, in order to construct a realistic framework for credit exposure, one typically needs to account for collateralization. Collateral usually reduces the exposure significantly; however, there are multiple, sometimes subtle, aspects that must be considered in order to properly assess the true extent of risk reduction. Note that collateral often is referred to as **margin**.

While the cash-flows in a contract may be yearly, the contract is typically valued far more frequently, often on a daily basis. Collateral may be posted such that the net exposure is zero, according to the current value of the contract. However, setting up a collateral agreement is more complex than what such a simple explanation might suggest, as there are multiple considerations that need addressing to ensure that the agreement mitigates the risk to a satisfactory extent, while not requiring excessive operational mandating. These considerations translate into collateral agreement parameters that define the impact of the collateralization. A brief introduction to the parameters, which correspond to the ones described in the standardized legal protocol CSA (Credit Support Annex) provided by ISDA (International Swaps and Derivatives Association) [43], is presented in Table 1.

Parameter	**Description**
Margin frequency	The frequency of margin exchanges, i.e. how often one is required to post collateral.
Independent margin	Initial collateral posted independent of the market value, and independent of any subsequent collateralization.
Threshold	Value of the contract for which collateral shall start to be exchanged. Below this value, no collateral is required.
Minimum transfer amount	Limits the need to honor margin calls below a certain threshold, aiming to reduce operational costs.
Eligible collateral	Type of collateral required, cash being the most common.
Bilateral or unilateral	Determining if both parties or just one party is required to post collateral. Note that the above parameters may be different for the two parties involved.

Table 1: Collateral agreement parameters.

Despite the contract being valued rather frequently, it is typically impractical to post collateral equally often, as that would induce higher operational costs, in addition to potential liquidity issues. Rather, one decides on a **margin frequency**, with the time between margin calls referred to as the **re-margin period**. In an exhaustive framework, one would need to make further considerations in order to capture the operational dynamics in a margin call [21]. This includes:

(i) **Valuation.** This represents the time to compute the exposure and current market value of the collateral.

(ii) **Receiving collateral.** This represents the delay due to the counterparty processing the collateral request.

(iii) **Settlement period.** Depending on the type of collateral, there will be a settlement period, implying the collateral will not be received immediately. For cash, the settlement period is typically very short, while other types of assets may be subject to longer periods.

(iv) **Grace period.** In the event the relevant collateral is not received following a valid collateral call, there may be a grace period before the counterparty is deemed to be in default.

(v) **Liquidation.** Finally, it will be necessary to liquidate the collateral (if it is not cash), in order to close out the position.

If these periods can be assumed negligible in comparison to the re-margin period, the defining quantity is the margin frequency, as illustrated in Figure 5.

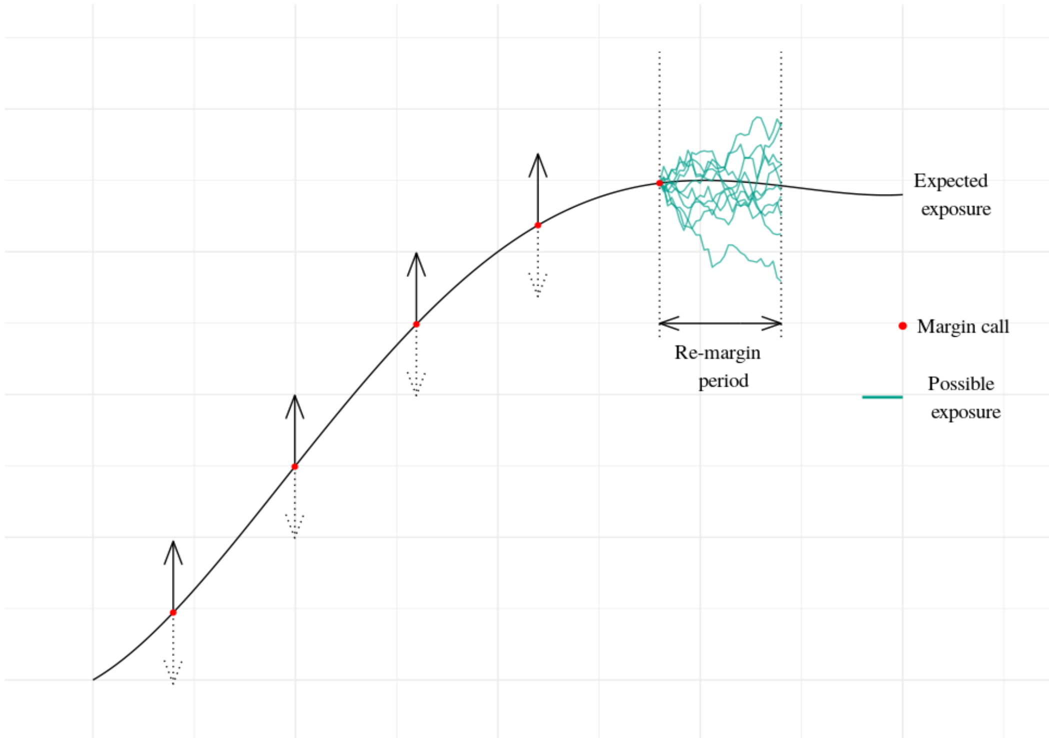


Figure 5: Illustration of the impact of the margin frequency on a collateralized exposure.

As Figure 5 suggests, the exposure of a collateralized contract is determined by the volatility in the re-margin period, as the position is assumed to well collateralized at each margin call. In a case where the threshold and MTA are 0, the exposure is entirely a reflection of the changes between margin calls.

However, simply considering the re-margin period is typically a somewhat too crude assumption, as the margin frequency tends to be rather high. Consequently, in order to realistically capture the effect of collateralization, the periods mentioned previously need to be considered. The period from the margin call to the collateral may be accounted for in the calculation of the exposure is referred to as the **close-out period**. In this framework it is split into **settlement period** and **liquidation period**, as illustrated in Figure 6.

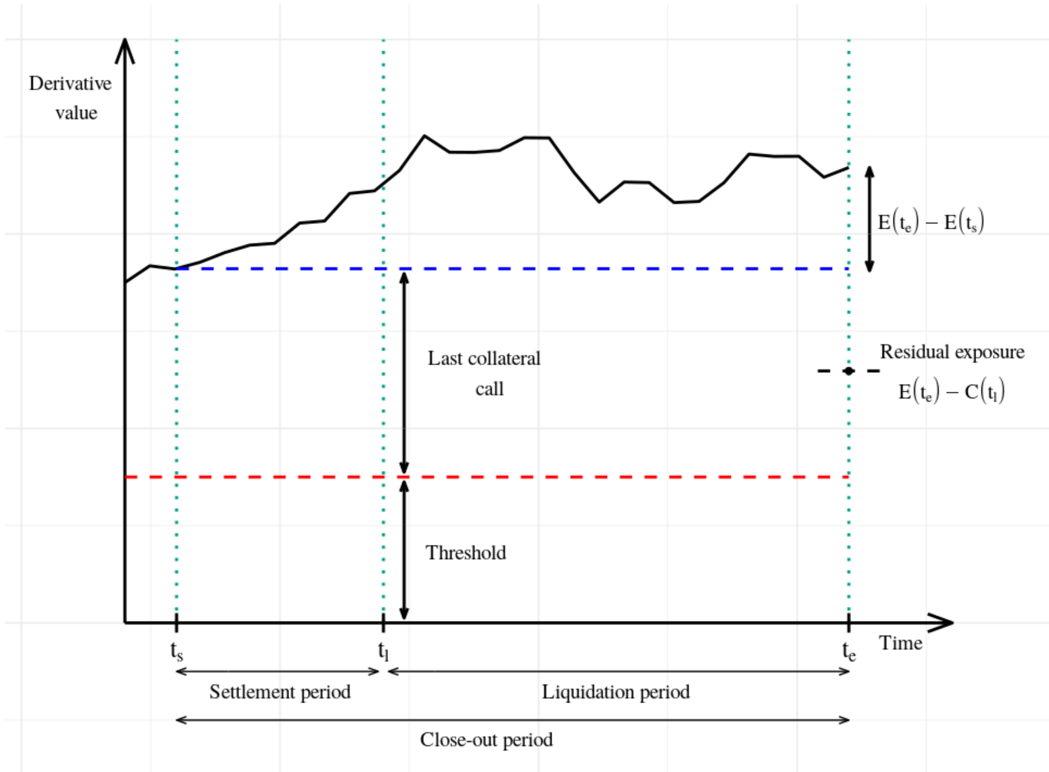


Figure 6: Illustration of the close-out period and its impact on the collateralized exposure.

The period starts at t_s at which point the counterparty may have become insolvent and ends at t_e at which point the collateral has been liquidated, and the position has been closed out. The intermediate time point t_l indicates the end of the settlement period, which is from the margin call until the collateral is received. The liquidation period represents the time it takes to liquidate the collateral and close out the position, in which period it is assumed that no payments are exchanged. However, as the position is not closed, the exposure is subject to risk accumulation over the entire close-out period, as illustrated in Figure 6.

Commonly, the settlement period is significantly shorter than the liquidation period. In fact, if the collateral is cash, the settlement period may even be negligible. The length of the settlement period is of particular significance in relation to derivative transaction cash-flows, giving rise to what is referred to as **collateral settlement risk** [29]. In this paper, however, the collateral is assumed to be cash, and so the close-out period is essentially the liquidation period. The entire close-out period is typically assumed to be around ten business days and is referred to as the **margin period of risk (MPOR)**.

Mechanisms aiming to reduce the operational burden give rise to the parameters **threshold** and **minimum transfer amount**. The former describes the exposure for which collateral shall start to be exchanged, while exposures below the threshold are deemed too low to require risk mitigation through collateralization. In the case the exposure exceeds the threshold, only the incremental exposure is posted as collateral. The latter describes a threshold for each individual posting of collateral, thereby limiting the need to honor margin calls below this threshold, hence avoiding frequent transfers of insignificant amounts. Both of these represent the balance between risk mitigation and operational burden.

As long as the counterparty posting the collateral is not in default, it remains the owner of the collateral from an economic point of view. Hence, coupon payments, dividends, and other cash flows must be passed on by the receiver of the collateral. However, in the case of an immediate margin call, the collateral receiver will typically keep the minimum component of the cash flow in order to remain appropriately collateralized.

From a modeling perspective, the impact of collateral may be accounted for after the simulation of the exposure, assuming the collateral agreement depends only on the exposure and not on other market variables. This assumption is typically valid as long as the collateral does not depend on currency translations, which will not be considered in this paper. This assumption will greatly simplify the analysis of the collateralized exposure.

Consider an uncollateralized exposure at time t of E_t, with the amount of collateral being held denoted C_{t_m}, where t_m represents the time of the last margin call ($t_m \leq t$). Generally, there are two sources of risk for a collateralized exposure:

(i) The risk stemming from an increase in exposure in-between margin calls, corresponding to $E_t > E_{t_m}$ (as illustrated in Figure 5).

(ii) The risk stemming from imperfect collateralization due to terms in the collateral agreement (threshold, minimum transfer amount, etc.), corresponding to the event $E_{t_m} > C_{t_m}$.

If the close-out period is non-negligible, the considerations are analogous but somewhat more tedious. This emphasizes the fact that the treatment of collateral is path-dependent, as the exposure is dependent on the amount of collateral posted in the past, as is the incremental amount of collateral required.

5 Data presentation

5.1 Historical data

The interest model implemented in this paper aims to model the NIBOR (Norwegian Inter-Bank Offered Rate) of different maturities, employing an LMM framework. In total, 1806 historical NIBOR realizations from 02.01.2012 to 11.03.2019 for a range of maturities are used for model calibration. An overview of the historical data is displayed in Figure 7, accompanied by descriptive statistics in Table 2.

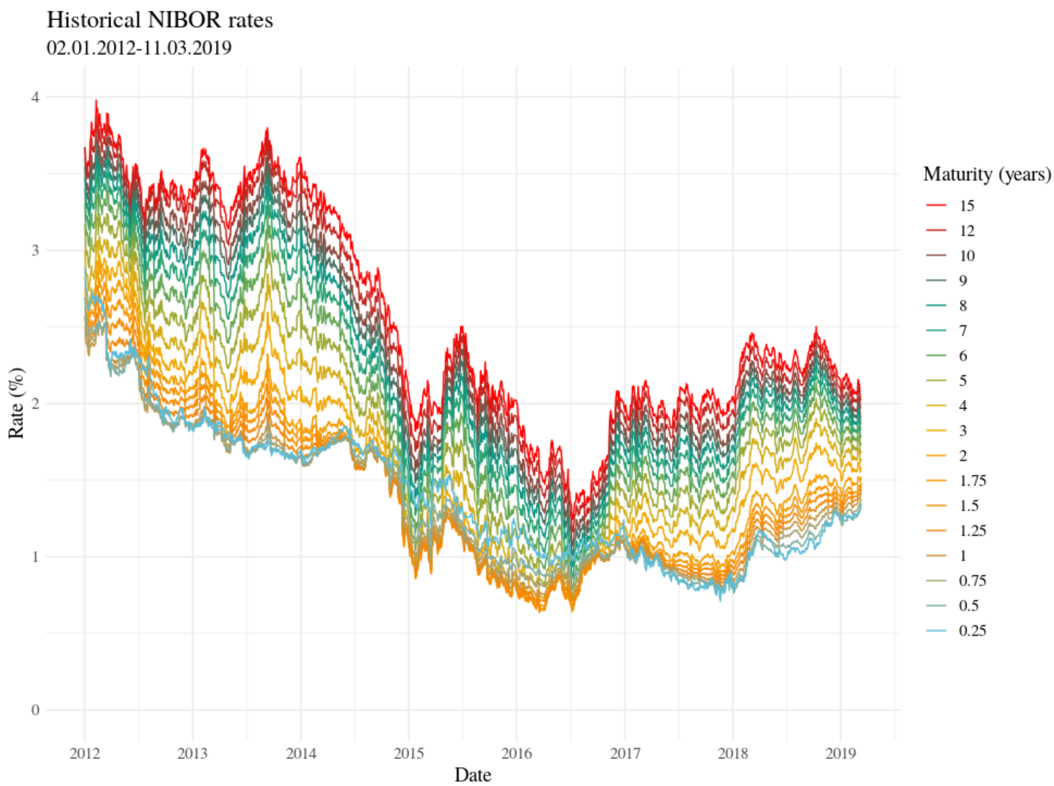


Figure 7: Historical NIBOR 02.01.2012 to 11.03.2019

Maturity	**Average rate** (%)	**Volatility** (%)
3 months	1.42	0.46
6 months	1.37	0.45
9 months	1.36	0.45
1 year	1.37	0.46
1.25 years	1.39	0.49
1.5 years	1.42	0.52
1.75 years	1.45	0.55
2 years	1.49	0.57
3 years	1.60	0.59
4 years	1.73	0.62
5 years	1.86	0.64
6 years	1.99	0.66
7 years	2.11	0.68
8 years	2.21	0.69
9 years	2.29	0.70
10 years	2.37	0.71
12 years	2.47	0.72
15 years	2.55	0.72

Table 2: Descriptive statistics for the historical NIBOR 02.01.2012 to 11.03.2019

Note that the interest rate generally is increasing in maturity, which is what one would expect - higher yields are typically demanded for tying up money for a longer time. Further, the spread in interest rates is quite large, indicating that the market expects the interest rates to increase. In fact, since the financial crisis in mid-2008, the market has expected an increase in the interest rates, with a continually increasing yield curve. However, the much-anticipated rise in the interest rates took longer than expected, and the interest rate offered by the Norwegian Central Bank reached an all-time low 2016 through 2018 [3]. The interest rate was raised from the record-low values in September 2018, and raised again in March 2019. It is projected to increase further in the coming years, however, compared to historical rates, still at what one would consider moderate levels.

Perhaps somewhat counter-intuitively, the volatility seems to be increasing with maturity. Historically, the shorter rates typically vary more than the longer rates, which tend to be rather stable. A plausible reason for this abnormal feature is that as the market has been expecting an increase in the interest rates that is yet to happen, the short rates have been rather stable at low values, while the long rates have been more varying, reacting to macroeconomic trends and indicators from the central bank.

The rates quoted in Table 2 are simple, meaning that if the maturity is δ, and the rate is r, the interest earned on one unit of currency is $r\delta$. They are however not *forward*, hence a transformation is required before applying the LMM framework discussed in 3.2.

These rates are assumed *risk-free*, meaning there is no chance of default. In reality, this is not the case, and presumably, the presence of default risk is reflected in the rates. However, this risk is considered negligible, as is customary in these sorts of frameworks. As a consequence, the rates determine the price of a riskless zero-coupon bond for each of the maturities. To calculate the price, a slight modification to (2.14) is required, to account for the fact that the rates are simple and not continuously compounded. Denote by r_T the simple NIBOR rate for a loan of length T, then the price of a zero-coupon bond with maturity T, B_T, needs to satisfy

$$\begin{aligned} B_T(1 + r_T T) &= 1 \\ \implies B_T &= \frac{1}{1 + r_T T} \end{aligned} \tag{5.1}$$

Further, the forward rates follow readily from (3.15).

The derivative contracts considered in this paper do not have maturities exceeding 10 years. Consequently, rates with maturities exceeding 10 years will be discarded as they do not affect the value of the derivatives. The relevant forward rates, with descriptive statistics, are presented in Table 3.

Rate	**Start year** (T_i)	**Stop year** (T_{i+1})	**Average rate** (%)	**Volatility** (%)
L_0	0	0.25	1.42	0.46
L_1	0.25	0.50	1.31	0.45
L_2	0.50	0.75	1.34	0.46
L_3	0.75	1.00	1.35	0.48
L_4	1.00	1.25	1.46	0.61
L_5	1.25	1.50	1.52	0.65
L_6	1.50	1.75	1.59	0.69
L_7	1.75	2.00	1.79	0.68
L_8	2.00	3.00	1.75	0.61
L_9	3.00	4.00	2.01	0.64
L_{10}	4.00	5.00	2.23	0.66
L_{11}	5.00	6.00	2.38	0.65
L_{12}	6.00	7.00	2.47	0.62
L_{13}	7.00	8.00	2.51	0.58
L_{14}	8.00	9.00	2.51	0.55
L_{15}	9.00	10.00	2.47	0.52

Table 3: Descriptive statistics for the historical forward rates 02.01.2012 to 11.03.2019

As evident in Table 3, the forward rate L_i is relevant for the interval $[T_i, T_{i+1}]$. Hence, a reasonable interpretation is that L_i represents the expected rate at time T_i. Meaning, one can draw a similar conclusion from this table, namely that it is expected that the interest rates will increase, as the further from today one considers, the higher the expected rate tends to be. Note that L_0 is not simulated, and its primary purpose is to serve as a discount factor to obtain the present value when pricing interest rate derivatives by (3.37).

Depending on the utilization of the forward rates, certain rates may be added by means of interpolation, while others may be removed. Typically, one would only include forward rates that are relevant for the particular derivative contract in question. For instance, if the forward rates are used to evaluate the counterparty credit exposure of a 3-year swap contract with interest payments every three months, one would add rates at 2.25, 2.50 and 2.75 years, while removing all rates with maturity larger than three years.

5.2 Initial rates and yield curve

For the purpose of evaluating the performance of the model, the interest rate data from January 2016 to March 2019 is removed from the data set. In this way, the output from the model may be compared with the true realizations of the interest rates. The initial interest rates will be as of 04.01.2016, quoted in Table 4.

Maturity	**Rate** (%)
3 months	1.12
6 months	0.95
9 months	0.87
1 year	0.82
1.25 years	0.79
1.5 years	0.77
1.75 years	0.76
2 years	0.79
3 years	0.86
4 years	0.98
5 years	1.13
6 years	1.29
7 years	1.43
8 years	1.55
9 years	1.66
10 years	1.76

Table 4: Initial rates (04.01.2016)

Commonly the rates for a specific date are displayed in the form of the yield curve, as in Figure 8, as the raw data typically is somewhat harder to process. However, the significance of certain initial rates will become apparent when evaluating the value of swap contracts.

A natural way to inquire on whether or not the valuation of a swap contract is correct is to compare the swap rate that results in a fair value of 0 to the initial rate with the same maturity as the swap contract. Any discrepancy would result in an arbitrage opportunity, as both rates essentially represent the same derivative transaction. This follows as the rates are simple, and therefore one may consider it equal to a swap contract with the same rate and maturity, and thus simple rates are often referred to as swap rates, as they are fixed for the duration of the loan.

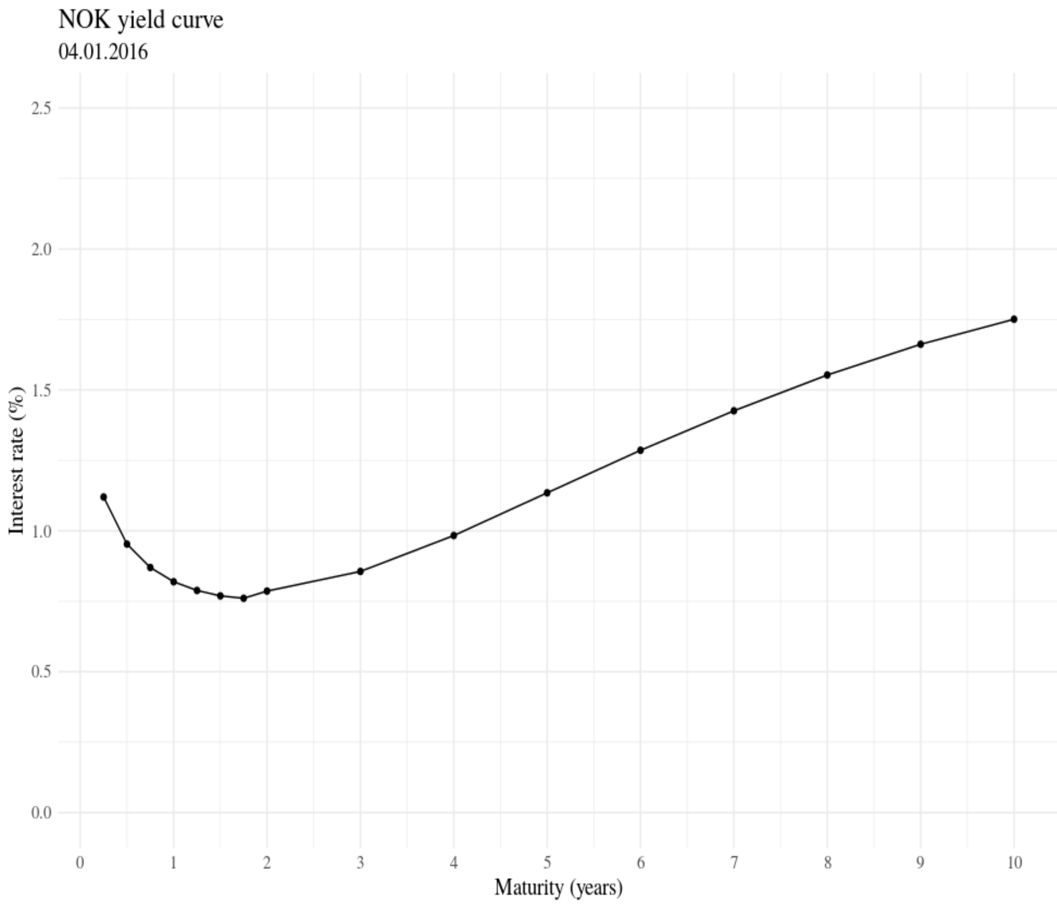


Figure 8: Initial yield curve (04.01.2016)

In Figure 8 it is apparent that the short end of the yield curve is *inverted* [13], meaning that tying up money for a longer time is *not* necessarily rewarded by a higher return. A common way to interpret the yield curve is to consider a specific maturity, for instance, the 1-year rate, and follow the yield curve to the right to deduce what the expectation is going forward. For instance, in this case, the yield curve implies that the 1-year rate was expected to decrease slightly in the short term, before increasing in the longer term. The rates with a longer maturity, however, were expected to increase. For instance, the 5-year rate is slightly lower than the rates with higher maturity, and one would therefore expect it to increase slightly going forward. These features of the yield curve may be useful to keep in mind when evaluating the performance of the model.

6 Results and discussion

6.1 Volatility structure

In order to simulate an algorithm on the form (3.24), the volatility structure needs to be determined. The factors are determined as described in Section 3.2.3. The volatility factors for the forward rate are typically harder to interpret, and although they do exhibit some of the qualitative features from Figure 2, it is less apparent than for the original NIBOR rates. The first three volatility factors derived from the covariance matrix for the original NIBOR rates are displayed in Figure 9.

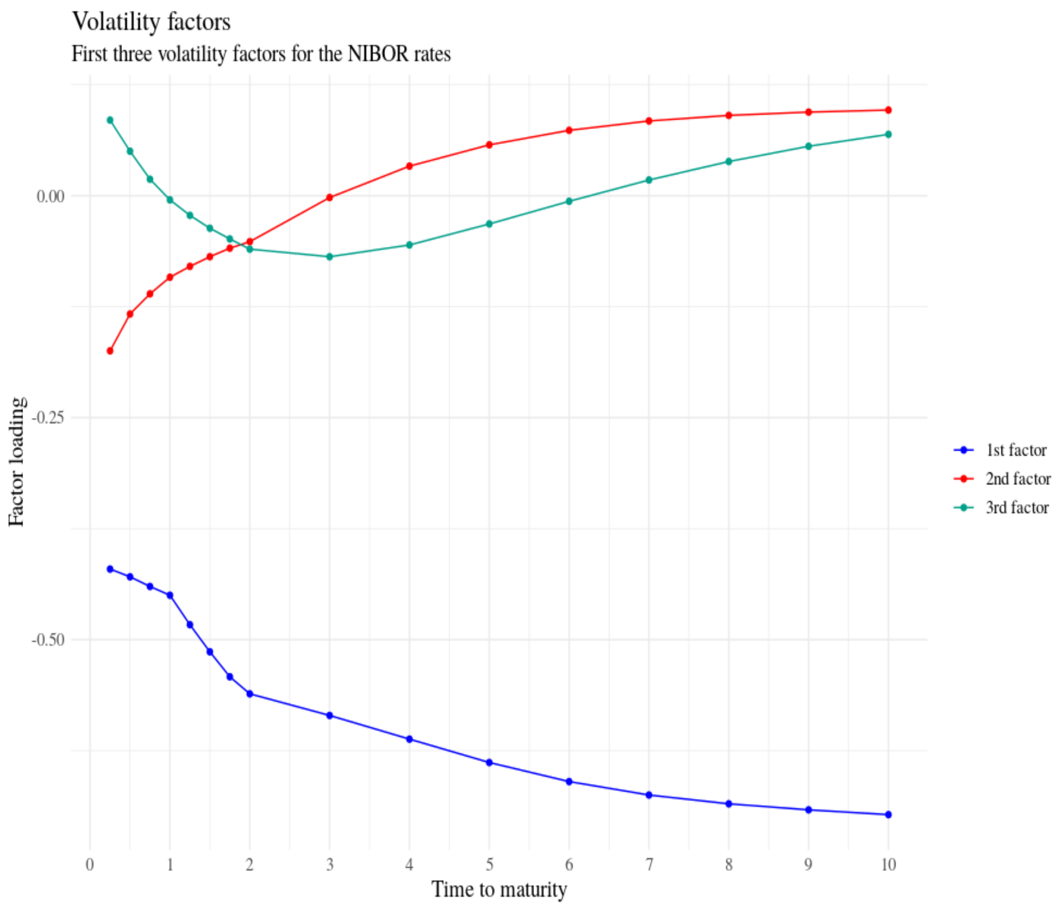


Figure 9: Volatility factors for the NIBOR rates

Observe that these factors, to a large extent, exhibit the same qualitative features as the hypothetical case discussed in section 3.2.3. The first factor represents a shift, with equal signs for all maturities. The second factor represents steeping or flatting of the curve, shifting the long and short end in opposite directions. Finally, the third factor changes the sign twice and represents a twisting of the curve. Note that the features are not quite as distinct as in the hypothetical case; however, the qualitative interpretations coincide, which is the most important aspect.

In order to determine the number of factors necessary for the model to account for a satisfactory amount of the variability, consider the cumulative sum of the eigenvalues, as suggested in Section 3.2.3, displayed in Table 5. Note that this is the eigenvalues of the covariance matrix for the log-differences of the forward rates, which is the relevant volatility structure for a simulation on the form (3.24), which will be implemented in this paper.

Number of factors d	**Eigenvalue**	**Value** (10^{-4})	$\sum_{i=1}^{d} \lambda_i / \sum_{i=1}^{M} \lambda_i$
1	λ_1	35.32	59 %
2	λ_2	10.01	76 %
3	λ_3	6.17	87 %
4	λ_4	1.95	90 %
5	λ_5	1.78	93 %
6	λ_6	0.95	95 %
7	λ_7	0.72	97 %
8	λ_8	0.65	98 %
9	λ_9	0.52	98 %
10	λ_{10}	0.38	99 %
11	λ_{11}	0.31	99 %
12	λ_{12}	0.24	99 %
13	λ_{13}	0.20	100 %
14	λ_{14}	0.13	100 %
15	λ_{15}	0.11	100 %
16	λ_{16}	0.06	100 %

Table 5: Eigenvalues and cumulatively explained variability

A suitable number of factors seems to be around 4, which will explain 90% of the covariance structure observed between the forward rates. In practice, this is a trade-off between goodness of fit and simplicity. As the model will never be *perfect*, one could argue that a lower number of factors would also be acceptable in many situations, as it would improve the running time and still account for a decent amount of the variability. However, as the primary purpose, in this case, is to build a solid model and the running time is not a great concern, the number of factors is chosen to be 4.

6.2 Rate distribution

A natural approach for evaluating the performance of the implemented interest model is to generate the distribution of different rates at different times and see where the true rate falls in this distribution. Ideally, the realization of the rate should be in the distribution generated, as this is the range deemed most probable by the model.

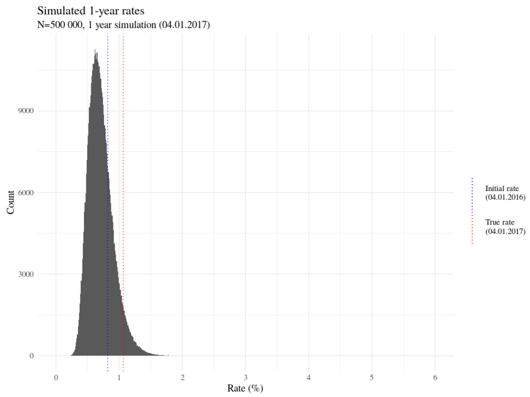
For the purpose of this exercise, assume that the date is January 4th 2016 and consider the 1-year rate and 5-year rate at 04.01.2017, 03.01.2018 and 03.01.2019 (approximately one year, two years and three years from 04.01.16). The true rates are quoted in Table 6.

Date / Rate	04.01.2016	04.01.2017	03.01.2018	03.01.2019
1-year rate (%)	0.82	1.07	0.84	1.26
5-year rate (%)	1.13	1.38	1.42	1.66

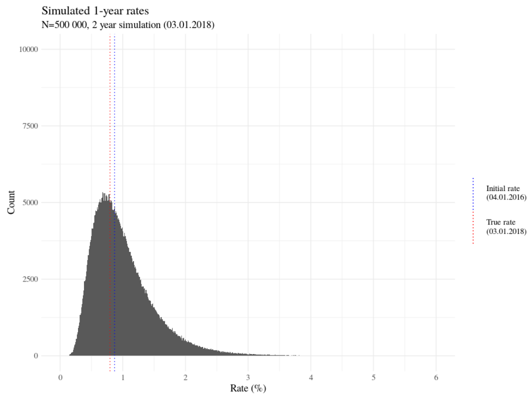
Table 6: True NIBOR rates

Observe that the 5-year rate increases steadily, although just slightly, from 2016 to 2019. The 1-year rate also increases, however, not steadily, as it first increases, thereafter declines in 2018, before again increasing by 2019. Another important aspect to consider when evaluating the simulated rates is the degree to which it has incorporated the market expectation, as discussed with the aid of the yield curve in Section 5.2.

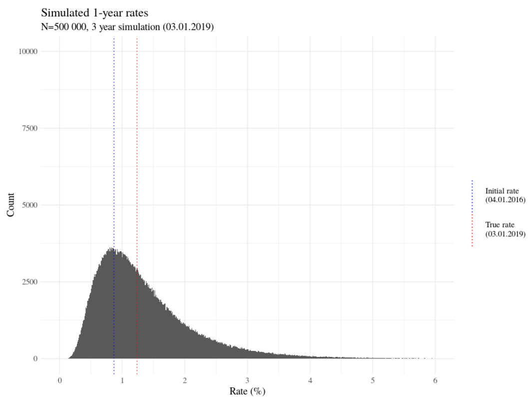
The distributions generated for the 1-year rates are displayed in Figure 10, for the three different cases introduced above. By a simulation scheme on the form (3.24), the rates are distributed according to a log-normal distribution, which is apparent from Figure 10d. Further, this scheme ensures non-zero interest rates, and this feature is also clearly visible in the histograms.



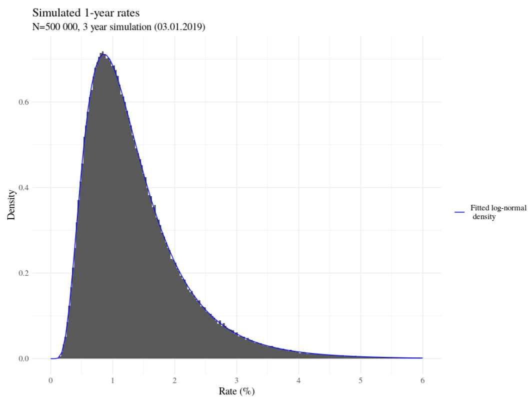
(a) 1 year simulation



(b) 2 year simulation



(c) 3 year simulation

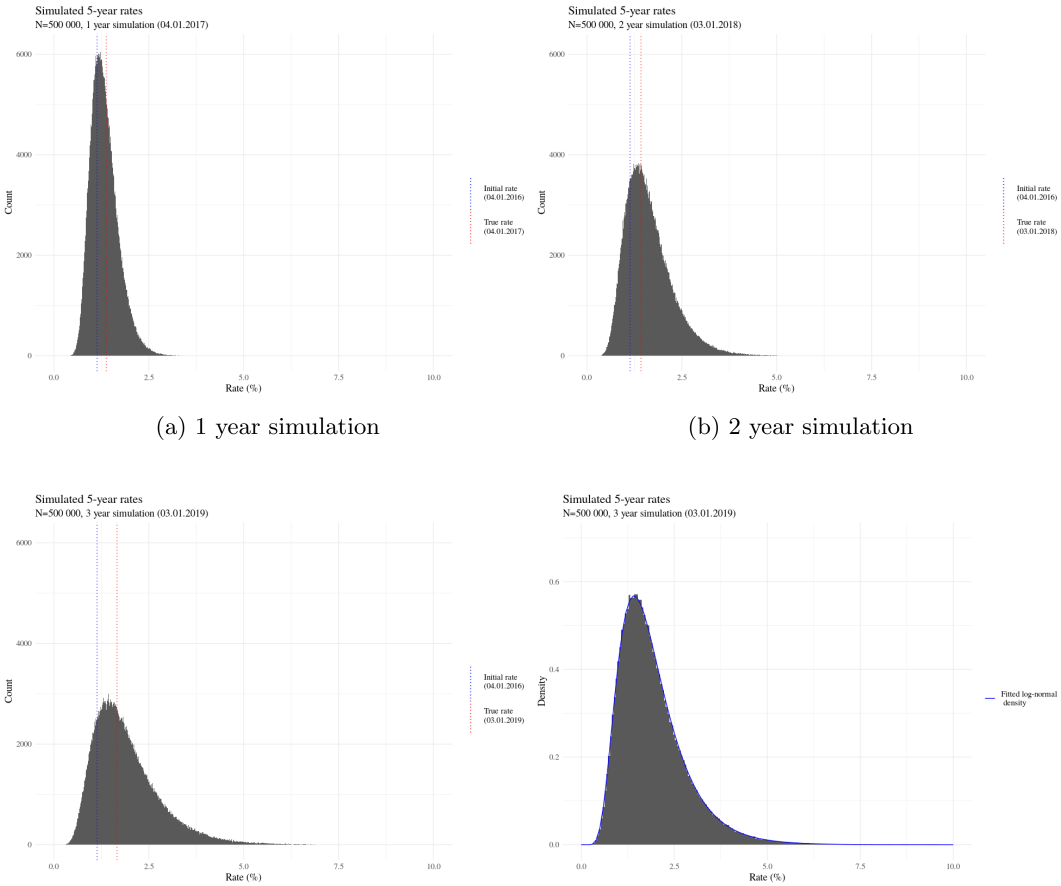


(d) Fitted log-normal density

Figure 10: 1-year rate for $N = 500\ 000$ simulations for different times in the future

In terms of performance, it seems that the realizations of the 1-year rate fall well inside what the model considers the most probable futures rates, in particular for the two latter cases. In the first case, the realization of the 1-year rate, somewhat surprisingly perhaps, increases rather than decreases, as one would expect from the yield curve. For this reason, the realization, in this case, is in the upper tail of the distribution.

It can also be noted that the realizations from the model become increasingly spread as the length of the simulation increases. For the one year simulation, the realizations are concentrated close to the initial rate, while for the longer simulations, the histograms are shifted longer from the initial rate, in addition to having considerably thicker tails. The interpretation is simply that the uncertainty in the future interest rate is significant, in particular for the more extended simulations.



(a) 1 year simulation

(b) 2 year simulation

(c) 3 year simulation

(d) Fitted log-normal density

Figure 11: 5-year rate for $N = 500\ 000$ simulations for different times in the future

Displayed in Figure 11 are similar visualizations for the 5-year rate, namely the 5-year rates generated by the model one year, two years, and three years in the future. The histograms exhibit similar qualitative features as the 1-year rate - a log-normal distribution with increasing standard deviation. Contrary to the 1-year rate, the 5-year rate was expected to increase following January 2016, and hence the model predicts an increased 5-year in all three cases.

Also, for the 5-year rate, the true rate falls within what the model considers the most probable outcomes. The 5-year rate did, in fact, increase slightly in the years following January 2016, as expected, and so the predictions provided by the model coincide to a great extent with the true realizations.

6.3 Pricing of interest rate derivatives

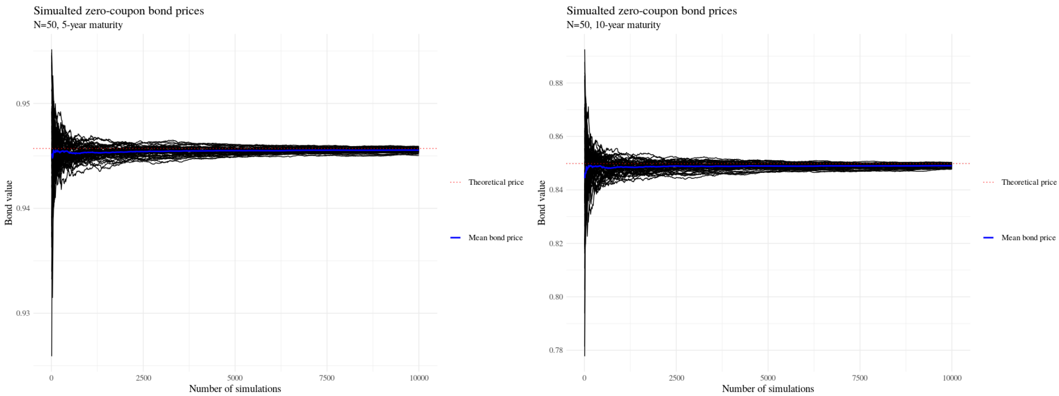
6.3.1 Pricing of zero-coupon bonds

As previously stated, if the algorithm is implemented correctly, then pricing zero-coupon bonds by simulation should yield prices consistent with the initial bond prices. For the purpose of pricing, the algorithm is simulated 500 000 times, and displayed in Table 7 are the resulting prices for the maturities considered in the model.

Maturity (years)	**Theoretical price**	**Simulated price**
0.25	0.9972	0.9972
0.50	0.9953	0.9953
0.75	0.9935	0.9935
1.00	0.9919	0.9919
1.25	0.9902	0.9903
1.50	0.9886	0.9886
1.75	0.9869	0.9869
2.00	0.9845	0.9849
3.00	0.9750	0.9753
4.00	0.9622	0.9621
5.00	0.9463	0.9455
6.00	0.9284	0.9267
7.00	0.9092	0.9046
8.00	0.8895	0.8837
9.00	0.8698	0.8632
10.00	0.8510	0.8479

Table 7: Theoretical and simulated zero-coupon bond prices

As expected, the prices obtained through simulation are very similar to the theoretical prices, with small deviations. The deviations are larger for the longer bonds, which is reasonable as they are dependent on a larger number of stochastic variables, inducing larger variance. The source of the error is both the discretization of the model and Monte Carlo simulation error. By increasing the number of simulations, the error is reduced. This is illustrated in Figure 12.



(a) Bond with 5-year maturity

(b) Bond with 10-year maturity

Figure 12: Price of zero-coupon bonds as a function of the number of simulations

It is clear from the figures above that the bond prices converge with an increasing number of simulations, as one would expect. For a low number of simulations, the deviation is rather large, particularly for the 10-year bond. As this estimation is based on the Monte Carlo principle, the convergence is of order $\mathcal{O}(n^{-\frac{1}{2}})$ (Appendix C), and consequently, the convergence stagnates, as evident in the plots. Regardless, the simulated bond prices do converge to the theoretical price with an increasing number of simulations. However, in order to obtain estimates fully consistent with the theoretical prices, a large number of simulations are required, in combination with a small time step, causing the running time to be quite long. Nevertheless, it seems reasonable to conclude that the algorithm is functioning as it should.

6.3.2 Pricing of interest rate floor contracts

The last few years have been characterized by low interest rates, and despite an increasing yield curve, the rates have been stable, or even falling, at abnormally low levels. Consequently, interest rate floors have become ever more popular derivatives, insuring against interest rates falling below a certain threshold.

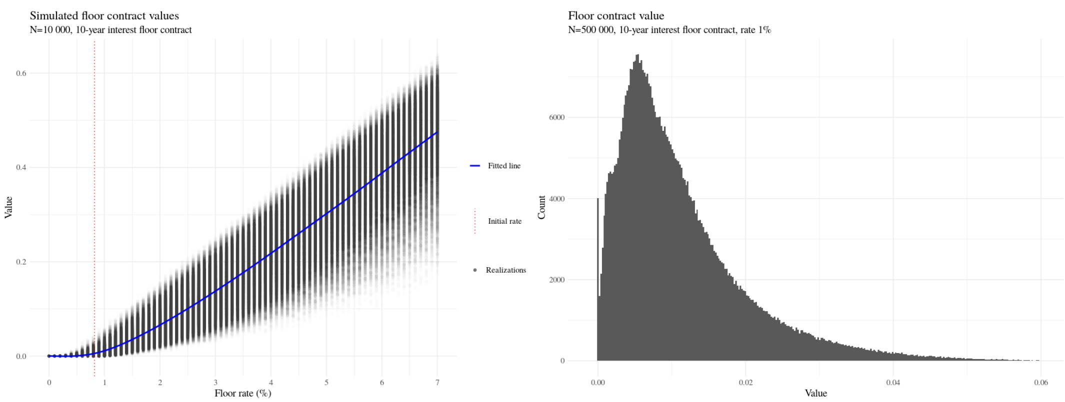
Consider an interest rate floor derivative with strike rate r_f for each of the accrual periods $[T_i, T_{i+1}]$, $i = 0, \dots, 9$. That is, the payoff at time T_i is

$$\max(r_f - L_i(T_i), 0). \tag{6.1}$$

The value of such an interest floor contract is the sum of the discounted payoffs at each T_i, according to (3.37). To give an interpretation of the contract described, consider a bank issuing a loan of size one at time T_0, being repaid the loan at time T_{10}. The bank receives yearly interest on the loan at the floating rate, but no down-payments, hence the size of the loan is 1 for the duration of the contract. Buying a floor contract on the form described insures the issuer of the loan that the interest payments will not fall below r_f, being reimbursed the difference if the floating rate goes below. Calculating the value as specified, yields the expected present value of these payoffs, as a fraction of the initial loan.

As previously stated, floor contracts can be thought of as a collection of European put options on the interest rate with strike r_f, with the modification that the contract is dependent on multiple maturities T_i, $i = 0, \dots, 9$, rather than just one. In an analogous way as with the put option, the buyer of a floor contract has the possibility for an arbitrarily large payoff, with the loss limited to the initial premium. Hence, the risk associated with the derivative is largely on the seller's side.

Displayed in Figure 13 are simulated values for a 10-year floor contract with different floor rates, as well as a histogram of the simulated floor values with a strike rate of 1 %, in Figure 13a and 13b respectively.



(a) Value for different floor rates

(b) Value floor contract with $r_f = 1\%$

Figure 13: Simulations of 10-year interest rate floor contract

Clearly visible from Figure 13a is the fact that the value of the floor contract is increasing with the strike rate, which is reasonable as it receives a payoff when the observed rate is below the strike rate. Essentially, the higher this strike rate is, the more likely it is that the observed rate is below this threshold. Further, the uncertainty in the value is increasing with the strike price, which is consistent with the observations in Section 6.2. As more realizations will be below the strike for higher rates, the uncertainty in the floor contract value essentially reflects the uncertainty in the future rates. In fact, as the floor rate increases, the value of the floor contract approaches the value of a swap contract with the same rate. This follows as the rate is almost certain to be below the floor rate, and consequently, the contract reflects the exchange of a floating rate $L_i(T_i)$, $i = 0, \dots, 9$, with a fixed rate r_f. The same conclusion may be reached from (2.20), as the value of the corresponding cap contract is 0. That this is the case is confirmed in the next section, in Figure 14a.

Figure 13b exhibits similar features as the rate distributions discussed in Section 6.2, namely a log-normal like form. Note that (6.1) implies the value of the contract is 0 if $L_i(T_i) < r_f$ for all i, resulting in a significant chance that it will expire worthless. The positive evaluations essentially represent the distribution of the rates above the strike rate, and is unsurprisingly similar to the rate distribution to the right of the strike rate, in this particular case a log-normal like distribution.

The outcome that the contract expires worthless is naturally more likely for smaller strikes, which is apparent from Figure 13a. Further, one can observe that a floor contract with a strike rate 0% is deemed worthless by the model, as the probability of negative interest rates through (3.24) is 0. This is perhaps a drawback of such a model, as negative interest rates have occurred, and a floor contract with near 0 % strike rate should not be offered free of charge [37].

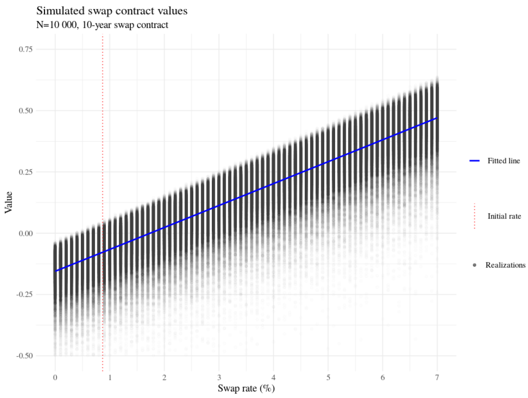
6.4 Evaluation of interest rate swap contract market risk

To demonstrate how the algorithm may be employed to evaluate the risk associated with an interest rate derivative, consider an interest rate swap contract, swapping a floating interest rate with a fixed rate. From the standpoint of the party receiving the fixed payments, the contract has a positive payoff if the floating rate is below the fixed rate, and analogously, a negative payoff if the floating rate is above the fixed rate. If the contract is valid for $[T_i, T_{i+1}]$, $i = 0, \ldots, 9$, the payoff at the end of the contract is

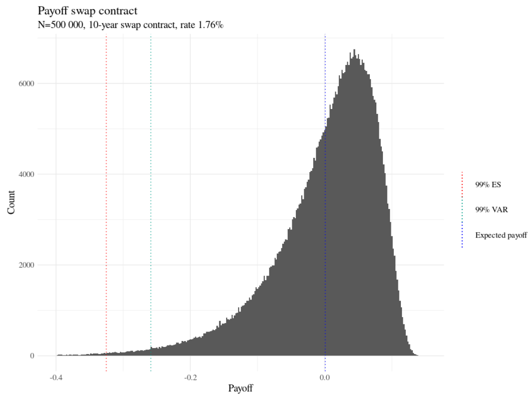
$$\sum_{n=0}^{9} \left[(r_s - L_n(T_n)) \prod_{j=0}^{n-1} \frac{1}{1 + \delta_j \hat{L}_j(T_j)} \right], \tag{6.2}$$

where r_s is the fixed swap rate. The interpretation of this contract is similar to the floor contract, namely that interest payments are exchanged at each time T_i, but no down-payments, implying that the size of the loan remains unchanged for the duration of the contract. Hence, the interpretation of the value is the same; the expected present value of all the payoffs as a fraction of the initial loan. Unlike the floor contract, there is significant risk involved for both sides of the deal and is thus a more symmetric contract with regards to the risk associated.

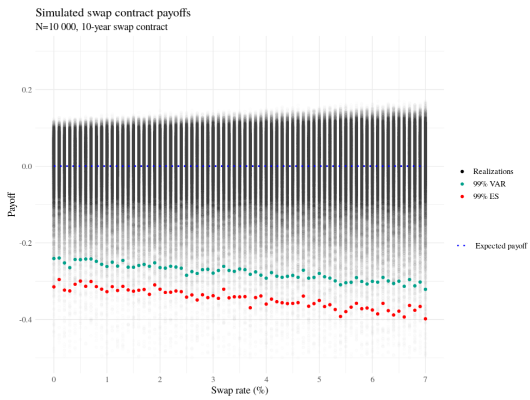
The payoff of the contract is simulated for a number of different swap rates, and the result is presented in Figure 14.



(a) Value for different swap rates



(b) Swap contract payoff with $r_s = 1.76\%$



(c) Payoff with risk measures for different swap rates

Figure 14: Simulations of 10-year interest rate swap contract

For low swap rates the party *receiving* the fixed rate would demand a premium, while for high swap rates the party *paying* the fixed rate would demand a premium. This feature is clear from Figure 14a, which also reveals that a swap rate around 1.75 % would be such that neither party would require a premium, as the fair value is 0 in this case. This rate is significantly higher than the initial rate of 0.84 %, again reflecting that the interest rates are expected to increase in the long term. In fact, the swap rate that induces a fair price of 0 should coincide with the initial 10-year rate, as the contrary would account for an arbitrage opportunity. The initial 10-year rate is 1.76%, so this provides further confidence that the algorithm is functioning as it should.

In figure 14c are the corresponding *payoffs*, that is, what remains after the fair premium is paid (or received). Note that when the contract is paid for according to its fair value, the expected payoff is 0. There is, however, some risk involved, and clearly visible from the plot is the fact that the payoff is quite likely to deviate from the expectation. The green and red points indicate the estimated value at risk and expected shortfall, respectively. Note the discrepancies in the risk measures across swap rates, in particular for the expected shortfall. As the value essentially reflects the shifted distribution $r_s - L_i(T_i)$, there is no reason these should not coincide. This illustrates an important issue concerning the estimation of risk measures, namely that it requires large sample sizes to obtain consistent estimates.

Displayed in Figure 14b is a histogram of the simulated payoffs for a swap rate at 1.76%, the rate such that neither party demands a premium, which is commonly how swap contracts are constructed. This highlights the feature present in Figure 14c, namely that the distribution of the payoffs is non-symmetric. This causes the potential loss for the party receiving the fixed payments to be significantly larger than for the party paying them. This is a feature inherent in a simulation scheme on the form (3.24), as this causes the distribution to be log-normal. Such a distribution does not allow negative values, yet high positive values occur with a relatively high frequency. This attribute was evident when pricing floor contracts, and also, in this case, it causes somewhat counter-intuitive results.

An important takeaway from these observations is that such a simulation scheme may cause one to believe that it is more favorable to be on one side of the deal than the other. However, if the rates are in fact more symmetrically distributed, this may be misleading. Consequently, this could quite easily lead to a mismanaging of risk in such a situation, as the possible downside is significantly larger than one has accounted for. Nevertheless, empirically, high interest rates tend to occur with a higher frequency than close-to-zero and negative interest rates, supporting the use of a log-normal distribution.

6.5 Evaluation of counterparty credit exposure

In this section, the algorithm is employed to construct the potential counterparty credit exposure over the lifetime over an interest rate derivative subject to counterparty credit risk. Consider an interest rate swap contract similar to the one described in the previous section, with the modification that it is a 3-year swap contract with quarterly payments. The swap rate is set equal to the initial 3-year rate, such that the contract is *at-the-money* (meaning the market value is 0). The potential exposure is calculated every day over the lifetime of the contract with the approach outlined in Section 4.2.1. The exposure is from the point of view of the party receiving the floating rate, which from the previous section is known to have the largest upside, and therefore, the largest potential exposure. The resulting exposure profile is displayed in Figure 15.

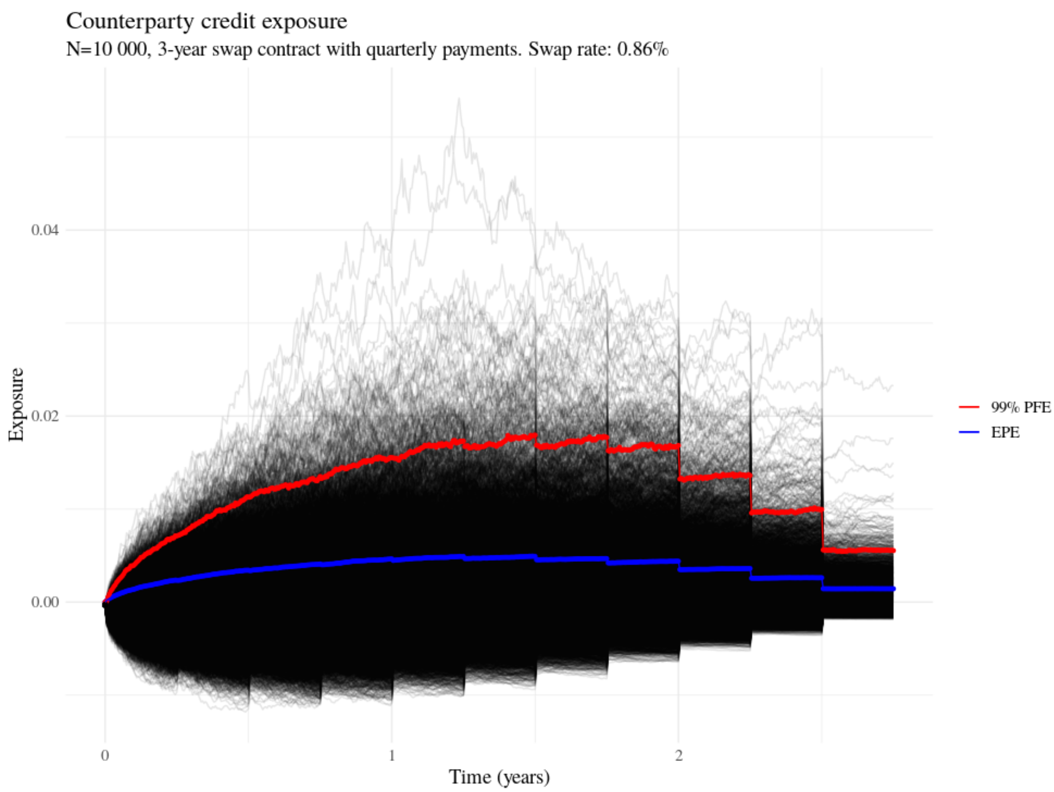


Figure 15: Simulated counterparty credit exposure for 3-year swap contract with quarterly interest payments.

The form of the exposure profile coincides to a great extent to the illustrative case presented in Section 4.2.1. Further, the form suggests that the distribution of the interest rate is rather heavy-tailed, evident from the number of extreme outcomes. This is consistent with the observation made in the previous section, a consequence of the log-normal simulation scheme (3.24).

Further, the asymmetry in the distribution of the rates is clearly visible also in Figure 15. As the interest rates are bounded below by 0, the negative exposure (that is, the exposure for the counterparty), is bounded below by some threshold, which is apparent from the figure. However, the potential exposure on the other side of the deal has no such bound.

The times of the interest payments are quite evident in the exposure profile, and, in general, cause the exposure to decrease in absolute value. This follows as the magnitude of outstanding payments decreases, and so does the (absolute) value of the contract. For the first year or so, the payment dates are not visible in the positive exposures, as the reduction in outstanding payments is countered by an increasing interest rate. However, towards the end of the contract, the distribution of the rates is stabilizing, in addition to less stochastic variables affecting the values, and consequently, the effect of the interest payments is clear also for the positive exposures.

6.6 Quantification of collateralization impact

In this section, the exposure of the interest rate swap contract considered in the previous section will be evaluated under the presence of collateral. The most common type of collateral is cash, and as it has the added benefit of not requiring additional modeling, eligible collateral is assumed to be cash. Further, the collateral agreement is assumed to be bilateral, meaning both parties are required to post collateral. Additionally, all CSA parameters are the same for both parties (threshold, minimum transfer amount, etc.)

The effect of each CSA parameter is investigated in the subsequent sections. In order to best highlight the effect of each parameter, the other parameters are chosen such that the impact of the parameter in question is most apparent.

From the previous section, it was clear that the magnitude of the exposure is rather low; a few percent of the notional value. The collateralized exposure will naturally be even smaller, and to avoid working with such small numbers, the magnitude of the loan is assumed to be 100 million NOK. Consequently, the exposure will be in units of million NOK.

For each parameter, the EPE and 99% PFE will be presented for different choices of the parameter value, and therefore it may be useful to have the uncollateralized exposure as a reference, displayed in Figure 16.

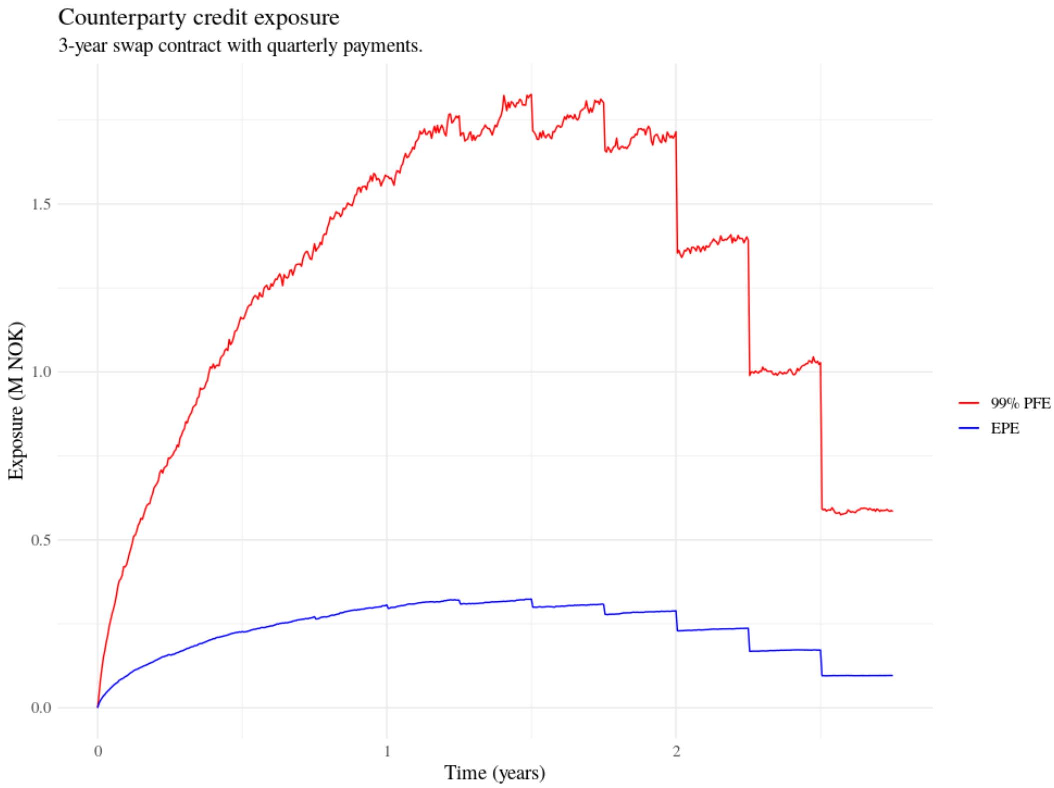
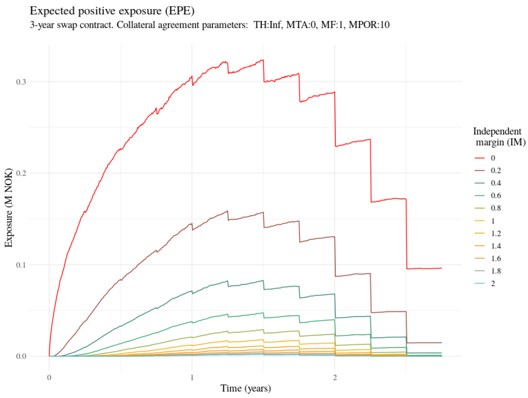


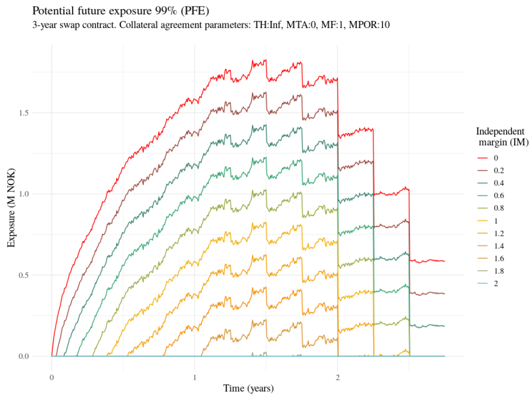
Figure 16: Simulated EPE and 99% PFE for uncollateralized 3-year swap contract.

6.6.1 Independent amount

The independent margin represents the amount of collateral posted by both parties at the initiation of the deal. It remains unchanged for the duration of the transaction. The EPE and 99% PFE for different values of the independent margin is presented in Figure 17a and 17b, respectively.



(a) Expected positive exposure (EPE)



(b) 99% potential future exposure (PFE)

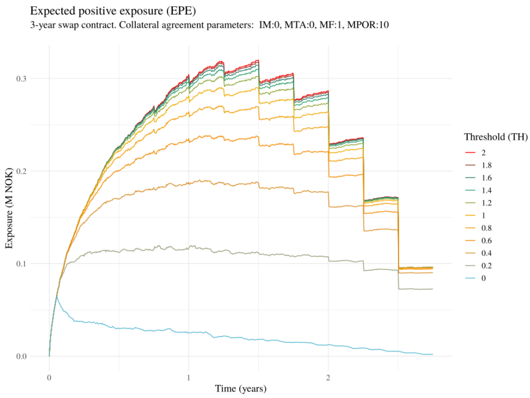
Figure 17: EPE and 99% PFE for different values of the independent margin.

From Figure 17 it is clear that an independent margin of 0 is the equivalent to an uncollateralized exposure (as the threshold is set to infinity, causing no additional collateral). The PFE represents the upper quantile of the exposure, and consequently, the reduction in PFE compared to the uncollateralized exposure is equal to the magnitude of the independent amount. This follows as every realization of the exposure is reduced by the size of the independent margin, including the upper quantile. A common practice is therefore to set the independent margin equal to the maximum PFE (often referred to as **effective potential future exposure**), thereby essentially eliminating the counterparty credit exposure [36]. In such a case, the deal will be subject to a non-zero exposure with a probability of less than 1% (if matched to the 99% PFE).

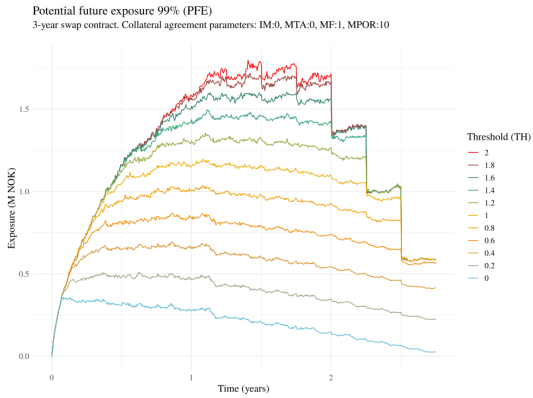
The majority of the realizations of the exposure, displayed in Figure 15, are close to zero. Hence, introducing an independent margin of only 0.2 million will reduce a significant amount of the realizations to zero, causing a substantial decrease in the EPE. By increasing the independent margin further, the EPE is decreased, however, by less than the increase in the independent margin. As only a very few realizations are affected by increasing the independent margin further, any subsequent increase results in only a very slight reduction in EPE, apparent from Figure 17a.

6.6.2 Threshold

The threshold parameter describes the value of the exposure for which collateral shall start to be exchanged. Only the incremental exposure above the threshold is required as collateral. The EPE and 99% PFE for different values for the threshold are presented in Figure 18a and 18b, respectively.



(a) Expected positive exposure (EPE)



(b) 99% potential future exposure (PFE)

Figure 18: EPE and 99% PFE for different values of the threshold

From Figure 18 it is clear that the overall effect from the threshold parameter is as one would expect - a lower threshold reduces the exposure more than a higher threshold. The effect of increasing the threshold on the EPE, displayed in Figure 18a, is similar to what was observed with regards to increasing the independent amount. That is, increasing the threshold from 0 to 0.2 has a more significant impact than increasing the threshold from 1.8 to 2, as a larger portion of the realizations is affected by the former. Moreover, there is practically no difference in a threshold of, for instance, 1.8 and 2 as so few realizations are above either threshold.

The effect of the threshold is perhaps more visual on the PFE, as the measure is not concerned with the number of realizations affected, merely by the upper quantile of the realizations. Consequently, as the threshold is increased, the PFE is shifted by roughly the magnitude of the increase. In the cases where the uncollateralized exposure is below the threshold, further increasing it causes no change in the PFE. This is apparent from the last couple of months in Figure 18b, where the PFE for all thresholds above 0.6 coincides.

OTC interest rate transactions between financial counterparties are required to have a threshold of 0 by the European Market Infrastructure Regulation (EMIR) [11], which also applies to Norwegian financial institutions. Hence, although the counterparties may consider an exposure below a certain threshold to be non-significant, they are enforced by regulative entities to collateralize all exposures. Consequently, relieving the operational burdens inherent in a collateral agreement has to be done by other means, for instance, by agreeing on a high independent amount.

6.6.3 Minimum transfer amount

The minimum transfer amount represents the balance between risk mitigation and operational workload, as it limits the need to honor margin calls below a certain threshold. As collateral cannot be transferred in amounts smaller than this threshold, it will typically cause an increasing exposure to be slightly under-collateralized while a decreasing exposure will be slightly over-collateralized, as the minimum transfer amount also applies to the return of collateral. This effect is visualized in Figure 19 for the EPE and 99 % PFE with different values for the MTA parameter.

(a) Expected positive exposure (EPE)

(b) 99% potential future exposure (PFE)

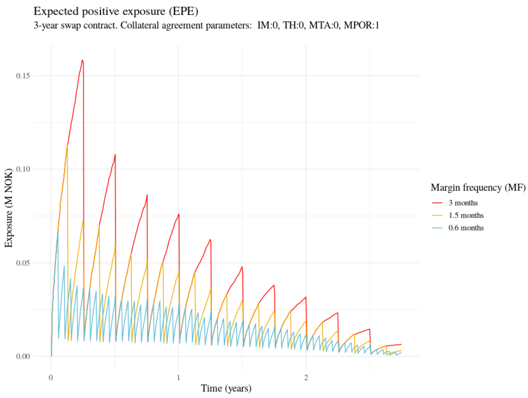
Figure 19: EPE and 99% PFE for different values of the minimum transfer amount.

The minimum transfer amount and the threshold are additive in the sense that the exposure must exceed the sum of the two before any collateral is exchanged, and this feature is evident in both the EPE and the PFE. In particular, for the PFE it is evident that the exposures spread after reaching the threshold 0.5, as the actual threshold before collateral is required is 0.5 (the threshold parameter) plus the minimum transfer amount. Towards the end of the contract, the exposures for the different MTA values are not necessarily in order of lowest to highest, as the exposure may be increasingly over-collateralized as a result of a high MTA.

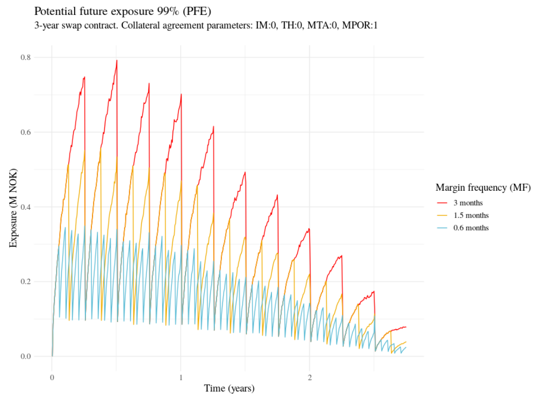
The overall effect of the minimum transfer parameter is like one might expect, a low MTA results in a lower exposure, while inducing higher operational costs as essentially all margin calls must be honored. On the other hand, a high MTA causes a higher exposure but relieves some operational burden as there will be fewer collateral exchanges. Whether lower exposure or lower operational burden is most desirable will be dependent on the particular transaction in question. By the same regulations mentioned in the previous section (EMIR [11]), transactions between financial counterparties are required to have an MTA under 500 000 EUR. Keep in mind that derivative contracts between financial counterparties may quite easily exceed 100M NOK, so although it is not really applicable to the swap contract considered in this paper, this constraint limits the flexibility to choose the MTA freely.

6.6.4 Margin frequency

The margin frequency determines how often collateral calls are made. As discussed in Section 4.2.4, the collateralized exposure is largely a reflection of the volatility in the re-margin period. In order to make this effect most visual, the MPOR is set equal to 1 day. The exposure is calculated for three choices of the margin frequency, once every 3, 1.5, and 0.6 months. As the interest payments are made quarterly, these choices will be such that collateral is posted once, twice and five times per interest payment, respectively. The resulting EPE and 99 % PFE are displayed in Figure 20.



(a) Expected positive exposure (EPE)



(b) 99% potential future exposure (PFE)

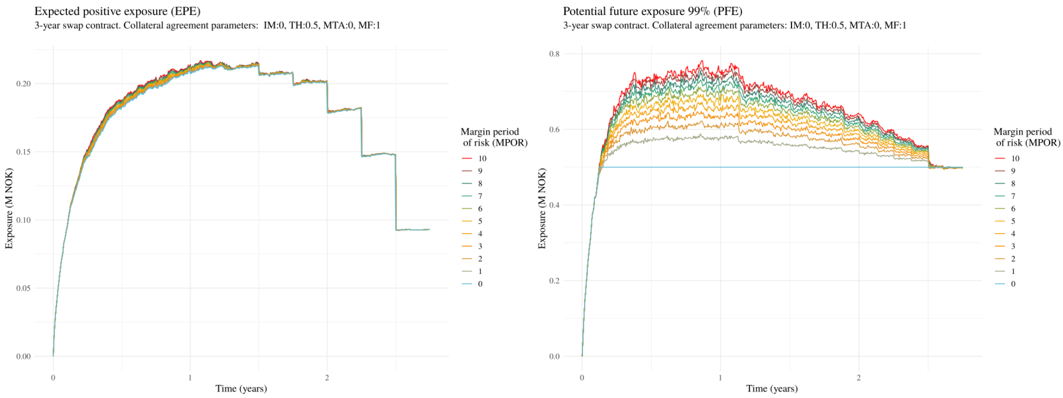
Figure 20: EPE and 99% PFE for different values of the margin frequency.

Note that the exposures for each of the margin frequencies share considerable similarities, despite appearing rather different. For all three cases, the exposure increase following a margin call, at which the exposure is virtually zero (not exactly zero as there is an MPOR of 1 day). The increase following a margin call is more substantial at the beginning of the contract than towards the end. This is a consequence of three features that have been discussed previously. Firstly, the number of outstanding payments is naturally largest at the initiation of the contract, and thereafter decreasing every three months. Secondly, in the beginning, the value is dependent on a larger number of stochastic variables which induces a more substantial variance. Lastly, towards the end of the contract, the distribution of the underlying interest rates is more stable than early on, causing the changes in between margin calls to be less significant.

As one would expect, the exposure is largest for the lowest margin frequency, as this implies few margin calls, while the exposure is lowest for the highest margin frequency, as this causes the position to be properly collateralized more often. Hence, if the objective is to construct a collateral agreement such that the exposure is the lowest, a natural approach would be to agree on a high margin frequency. Typically, derivative contracts are valued on a daily basis for accounting purposes, and commonly one agrees on a margin frequency equally often, and instead limits the operational costs by other means, for instance by a high independent amount or a high threshold. In fact, trades between financial counterparties are obliged to have a daily margin frequency by EMIR [11].

6.6.5 Margin period of risk

The margin period of risk (MPOR) describes the period from a margin call until, in the cause of a defaulting counterparty, the position is closed out. MPOR is not a parameter in the sense that it is agreed upon in the collateral agreement, rather is decided for modeling purposes in order to realistically model the collateralized exposure. Nevertheless, it may be useful to investigate how the exposure is affected by this choice. The EPE and 99% PFE for different values of the MPOR is presented in Figure 21a and 21b, respectively.



(a) Expected positive exposure (EPE)

(b) 99% potential future exposure (PFE)

Figure 21: EPE and 99% PFE for different values of the margin period of risk.

The effect of MPOR on the exposure is in some ways similar to the margin frequency, in the sense that the longer it is, the larger the resulting exposure is. This follows as a consequence of the fact that risk is accumulating over the length of the MPOR, as discussed in Section 4.2.4. This effect is apparent in Figure 21b, on the 99 % PFE, where the larger MPOR consistently results in a larger exposure. This effect is not as clear in EPE, which follows from the fact that the majority of the realizations are below the threshold, and therefore, the effect of MPOR is naturally not as prominent.

Commonly, the margin frequency is rather high, implying that it is unrealistic to consider the MPOR negligible. The typical choice is to set the MPOR equal to 10 business days. This is a rather conservative choice, as the actual time to settle the collateral and liquidate the position in most cases may be significantly shorter. However, the general approach is to err on the side of caution in these frameworks, so one rather overestimates the risk than underestimates it, as the latter has far more dramatic consequences [36]. When determining the MPOR when modeling the exposure for a specific derivative with a specific counterparty, the MPOR may be adjusted to account for the liquidity of the derivative and the quality of the counterparty.

7 Conclusion

The application of statistical methods in finance is nearly inevitable, as it is hard to consider models describing future developments without including stochastic terms unless the future is absolutely certain, which it seldom is. All models simplify reality, and thus, the accuracy is not necessarily ideal. This highlights an important point regarding financial models, namely that the assumptions to a great extent determine the scope of the model. Hence, a substantial task included in the construction of a stochastic model is communicating the consequences inherent in the underlying assumptions.

A great part of this paper concerned the implementation and demonstration of an interest rate model based on the LMM. Based on the distribution of the rates and the pricing of zero-coupon bonds, it is reasonable to conclude that the algorithm is functioning as it should. The realized rates were within the range deemed the most probable by the model, and the simulated rates reflected the market expectation present in the yield curve. Moreover, the simulated bond prices were consistent with the theoretical values.

Further, when employed to price interest rate derivatives, namely interest rate floors and swaps, it returned qualitatively reasonable results. The model was calibrated to match historical correlation patterns in the Norwegian interest rate market. It was not calibrated to be market consistent to other interest rate derivatives than zero-coupon bonds, and hence no comparison to relevant market prices was conducted. Nevertheless, the model may be employed to assess qualitative features and the associated risk, as its ability to reflect possible future developments in the interest rate seemed to be satisfactory. Furthermore, when evaluating the 10-year swap contract, the swap rate that results in a fair price of zero was consistent with the initial 10-year rate. Although this does not guarantee that the pricing of swap contracts with other swap rates is market consistent, this is less important, as swap contracts commonly are constructed such that neither party pays a premium at the initiation.

Some skepticism regarding its functionality is however justified, particularly with regards to its limited flexibility to incorporate negative interest rates. In the current low interest rate environment, this is a considerable drawback, as interest rates have been known to go negative [37]. A possible extension of the model is, therefore, to incorporate this possibility, which can be achieved by employing the iteration scheme (3.21) rather than (3.24). This comes at the expense of some of the desired properties inherent in the latter; hence it is not immediately clear whether this is favorable. Another possibility is to perform a transformation on the rates, and by that move the support below zero. This would keep the asymmetry in the rates, which based on empirical data is desirable, while having some non-zero probability of negative rates. However, it is not clear how large this probability should be, neither what the lower bound on the rates should be. Regardless, in the current low interest rate environment, the model should allow for negative interest rates, and consequently, the model would need some adjusting before evaluating derivatives that are sensitive to the probability of negative interest rates, such as floor contracts.

The financial markets are, to some extent, synonymous with "ups and downs", as they, after all, concern themselves with the trading of stochastic variables. However, after certain displays of catastrophic "downs", such as the economic crisis of 2008, that either partly or entirely is to blame on the financial industry, there has been increasing regulation and supervision by the authorities. Consequently, the demand for sophisticated stochastic models able to describe the risk associated with portfolios and derivatives has seen a similar upsurge.

Collateralization as a risk mitigation practice in the financial industry is widespread, and presently the majority of bilateral trades employ a collateral agreement - and the use is increasing [36]. Consequently, the ability to realistically capture the effect of collateral when assessing the risk is crucial. Although the collateralization practice reduces the counterparty credit risk, the demand for sophisticated models is *not* reduced. On the contrary, in order to be able to accurately capture the effect of collateralization, the models need to be more complex than if the objective is to describe the uncollateralized exposure. For the collateralization practice to actually reduce the risk in the market, rather than magnify it by misrepresenting the true impact, the effect has to be modeled accurately. Consequently, the role of stochastic modeling in finance has become increasingly important.

In this paper, a framework accounting for the impact of collateralization was implemented. The main purpose was to investigate the effect of each of the CSA parameters, both qualitatively and quantitatively. The results seemed intuitively reasonable, and the results were as one might expect. Certain simplifying assumptions were made, including assuming the collateral was posted as cash and that the settlement period was zero. Both assumptions hold in most cases; however, in order to construct an exhaustive framework, one would have to introduce some flexibility to account for cases where these assumptions do not hold. This does require some additional modeling, particularly to allow for alternative eligible collateral. The fundamental modeling framework would remain the same, and based on the results presented, it seems reasonable to conclude that it captures the effect of collateralization in an accurate manner.

References

[1] Andrew Ang, Monika Piazzesi, and Min Wei. *What does the yield curve tell us about GDP growth?* Journal of Econometrics, Volume 131, Issue 2: 359-403, 2006.

[2] Philippe Artzner et al. *Coherent Measures of Risk*. Journal of Mathematical Finance, Volume 9, Issue 3: 203-228, 1999.

[3] Norges Bank. *Pengepolitikk: Styringsrenten*. `https://www.norges-bank.no/pengepolitikk/Styringsrenten/`. Accessed: 24.11.2018.

[4] Andrew Beattie. *Hedging basics: What is a hedge?* `https://www.investopedia.com/articles/optioninvestor/07/hedging-intro.asp`. Accessed: 14.09.2018.

[5] Fischer Black. *The pricing of commodity contracts*. Journal of Financial Economics, Volume 3, Issue 2: 167-179, 1972.

[6] Fisher Black and Myron Scholes. *The Pricing of Options and Corporate Liabilities*. The Journal of Political Economy, Volume 81, no. 3: 637-654, 1973.

[7] A. Brace, D. Gatarek, and M. Musiela. *The market model of interest rate dynamics*. Mathematical Finance, Volume 7, Issue 2: 127-155, 1997.

[8] John Y. Campbell. *Some Lessons from the Yield Curve*. Journal of Economic Perspectives, Volume 9, No. 3: 129-152, 1995.

[9] Giovanni Cesari et al. *Modelling, Pricing, and Hedging Counterparty Credit Exposure*. Springer Finance, 2009.

[10] Somnath Chatterjee. *Modelling Credit Risk*. Centre for Central Banking Studies, Bank of England, 2015.

[11] European Commission. *European Market Infrastructure Regulation (EMIR)*. `https://ec.europa.eu/info/business-economy-euro/banking-and-finance/financial-markets/post-trade-services/derivatives-emir_en`. Accessed: 14.05.2019.

[12] Charles J. Corrado and Tie Su. *Skewness and Kurtosis in SP Index Returns Implied by Option Prices*. Journal of Finacial Research, Volume 10, Issue 2: 175-192, 1996.

[13] Paul F. Cwik. *The Inverted Yield Curve and the Economic Downturn*. New Perspectives on Political Economy, Volume 1, No. 1: 1–37, 2005.

[14] Emanuel Derman and Paul Wilmott. *The Financial Modelers' Manifesto*. `https://wilmott.com/financial-modelers-manifesto/`. Accessed: 02.12.2018.

[15] Arturo Estrella and Frederic S. Mishkin. *The Yield Curve as a Predictor of U.S. Recessions*. Current Issues in Economics and Finance, Volume 2, No. 7: 1-6, 1996.

[16] Chicago Board Option Exchange. *CBOE VIX-Index*. `http://www.cboe.com/vix`. Accessed: 02.11.2018.

[17] Eugene F. Fama and James D. MacBeth. *Risk, Return, and Equilibrium: Empirical Tests*. Journal of Political Economy, Volume 81, No. 3: 607-636, 1973.

[18] Jean Fogler. *What is a derivative?* `https://www.investopedia.com/ask/answers/12/derivative.asp`. Accessed: 14.09.2018.

[19] Francesco Giavazzi and Alessia Blanchard. *Macroeconomics: A European Perspective*. Pearson Education Limited, 2010.

[20] Paul Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer New York, 2003.

[21] John Gregory. *Counterparty Credit Risk*. John Wiley and Sons Ltd, 2010.

[22] John C. Hull and Alan D. White. *Valuing Credit Default Swaps I: No Counterparty Default Risk*. NYU Working Paper No. FIN-00-021, 2000.

[23] John C. Hull and Alan D. White. *Valuing Credit Default Swaps II: Modeling Default Correlations*. The Journal of Derivatives, Volume 8, Issue 3: 12-21, 2001.

[24] *Interest rates*. `https://www.investopedia.com/terms/i/interestrate.asp`. Accessed: 12.09.2018.

[25] BIS (Bank of International Settlements). *OTC derivatives statistics at end-December 2017*. `https://www.bis.org/publ/otc_hy1805.htm`. Accessed: 29.11.2018.

[26] Kiyosi Itô. *Stochastic Integral*. Proceedings of the Imperial Academy Tokyo, Volume 20: 519-524, 1944.

[27] Farshid Jamshidian. *LIBOR and swap market models and measures,* Finance and Stochastics, Volume 1, Issue 3: 293-330, 1997.

[28] Philippe Jorion. *Risk Management Lessons from the Credit Crisis*. Blackwell Publishing Ltd, 2009.

[29] Charles M. Kahn, James McAndrews, and William Roberds. *Settlement Risk under Gross and Net Settlement*. Journal of Money, Credit and Banking, Volume 35, No. 4: 591-608, 2003.

[30] Siem Jan Koopman, Borus Jungbacker, and Eugenie Hol. *Forecasting daily variability of the SP 100 stock index using historical, realised and implied volatility measurements*. Journal of Empirical Finance, Volume 12, Issue 3: 445-475, 2005.

[31] Andreas Krause. *Exploring the Limitations of Value at Risk*. The Journal of Risk Finance, Volume 4, Issue 2: 19-28, 2003.

[32] Nick Lioudis. *What is arbitrage?* `https://www.investopedia.com/ask/answers/what-is-arbitrage/`. Accessed: 10.09.2018.

[33] Burton G. Malkiel. *The Efficient Market Hypothesis and Its Critics*. Journal of Economic Perspectives, Volume 17, No. 1: 59-82, 2003.

[34] K.R. Miltersen, K. Sandmann, and D. Sondermann. *Closed-form solutions for term structure derivatives with lognormal interest rates*. Journal of Finance, Volume 52, Issue 1: 409-430, 1997.

[35] Marek Musiela and Marek Rutkowski. *Martingale Methods in Financial Modelling*. Springer-Verlag Berlin Heidelberg, 2005.

[36] DNB Markets - Risk Quantification. *Private communication*. 2019.

[37] Jana Randow and Simon Kennedy. *Negative Interest Rates*. `https://www.bloomberg.com/quicktake/negative-interest-rates`. Accessed: 29.11.2018.

[38] R. Rebonato. *Modern Pricing of Interest-Rate Derivatives: The LIBOR Market Model and Beyond*. Princeton University Press, New Jersey., 2002.

[39] *Risk*. `https://www.investopedia.com/terms/r/risk.asp`. Accessed: 11.09.2018.

[40] *Security*. https://www.investopedia.com/terms/s/security.asp. Accessed: 11.09.2018.

[41] Abdullah Sheikh and Hongtao Qiao. *Non-normality of Market Returns*. J.P. Morgan Asset Management, 2009.

[42] Rene M Stulz. *Credit Default Swaps and the Credit Crisis*. Journal of Economic Perspectives, Volume 24, Issue 1: 73-92, 2010.

[43] International Swaps and Derivatives Association (ISDA). *Credit support annex (CSA)*. https://www.isda.org/tag/credit-support-annex-csa/. Accessed: 26.02.2019.

[44] Paul Wilmott. *Frequently Asked Questions in Quantitative Finance*. John Wiley and Sons Ltd, 2007.

[45] Paul Wilmott. *Paul Wilmott Introduces Quantitative Finance*. John Wiley and Sons Ltd, 2001.

[46] Paul Wilmott. *Paul Wilmott on Quantitative Finance*. John Wiley and Sons Ltd, 2006.

[47] Paul Wilmott, Sam Howison, and Jeff Dewynne. *The Mathematics of Financial Derivatives*. Cambridge University press, 1995.

[48] Steven H. Zhu and Micheal Pykhtin. *A Guide to Modeling Counterparty Credit Risk*. GARP Risk Review, July/August, 2007.

Appendices

A Wiener process

A **Wiener process** is a continuous-time stochastic process named in honor of American mathematician Norbert Wiener. It is often referred to as a standard Brownian motion due to the historical connection to the physical process known as Brownian movement. Wiener processes frequently occur in a number of different fields, including economics, quantitative finance, biology, and physics. A Wiener process, $\{W_t\}_{t=0}^{\infty}$, is characterized by the following four properties;

(i) $W_0 = 0$ almost surely, i.e. $P(W_0 = 0) = 1$;

(ii) W has *independent increments*: for every $t > 0$, future increments $W_{t+u} - W_t$, $u \geq 0$, are independent of the past increments W_s, $s < t$;

(iii) W has *Gaussian increments*: $W_{t+u} - W_t \sim \mathcal{N}(0, u)$

(iv) W is continuous in t with probability one.

B Itô's lemma

Itô's lemma relates small change in a function of a random variable, to a change in the random variable itself. It can be heuristically derived by employing the Taylor series expansion of the function and serves in a sense as the stochastic counterpart to Taylor's theorem. The lemma is particularly useful in mathematical finance, as the development of financial assets commonly is described through stochastic variables.

Let X be a random variable described through the stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}B_t = \mu_t\mathrm{d}t + \sigma_t\mathrm{d}B_t, \tag{B.1}$$

where B_t is a Wiener process. Further, let f be a smooth function of X, then, by Itô's lemma, a small change in f, $\mathrm{d}f$, is expressed by

$$\mathrm{d}f = \sigma_t\frac{\partial f}{\partial x}\mathrm{d}B_t + \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t. \tag{B.2}$$

C Monte Carlo

The **Monte Carlo method** is a broad class of algorithms relying on the repeated random sampling of some underlying stochastic variable in order to obtain a numerical result. Essentially, rather than solving some integral analytically (which may not even be possible in most cases), one approximate this integral by the empirical equivalent. Consider

$$\alpha = E(g(X)) = \int_{\mathbb{R}} g(x)f_X(x)\mathrm{d}x, \tag{C.1}$$

where $g(X)$ is the desired function of the underlying stochastic variable (for instance the mean $g(X) = X$ or the variance $g(X) = (X - \mathrm{E}(X))^2$), and $f_X(x)$ is the density function. By sampling $x_1, \ldots, x_n$ from f_X one approximate (C.1) by

$$\hat{\alpha}_n = \frac{1}{n} \sum_{i=1}^{n} g(x_i). \tag{C.2}$$

By the strong law of large numbers [20],

$$P\left(\lim_{n \to \infty} \alpha_n = \alpha\right) = 1. \tag{C.3}$$

Moreover, the estimation error $\hat{\alpha}_n - \alpha$ is normally distributed with mean 0 and variance σ_ϵ, which may be estimated by

$$\hat{\sigma}_\epsilon = \frac{1}{n(n-1)} \sum_{i=1}^{n} (g(x_i) - \hat{\alpha}_n)^2. \tag{C.4}$$

The order of convergence for an algorithm relying on the Monte Carlo principle is $\mathcal{O}(n^{-\frac{1}{2}})$. Hence, the convergence rate is rather slow. Nevertheless, the Monte Carlo approach has significant advantages, most importantly convergence independent of problem dimension. Furthermore, the procedure of generating scenarios of the underlying stochastic variable induce both flexibility and insight to the dynamics of the modelling [20].