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Optimal Investment in Research and Development: A Real Option Approach

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Problem description

This paper studies the optimal level of investment in research and development under uncertainty with the use of real option theory.

Preface

This thesis was written during the Spring of 2019 as the final part of a five-year Master of Science degree at the Norwegian University of Science and Technology (NTNU). The degree specializes in Financial Engineering at the Department of Industrial Economics and Technology Management. I would like to thank my supervisors Verena Hagspiel and Tord Olsen for excellent guidance throughout the project.

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Glenn-Endré Østensen

Sammendrag

Denne oppgaven ser på det optimale investeringsnivået i forskning og utvikling for et firma som har muligheten til å gjennomføre et teknologibytte. Oppgaven modellerer investeringen i forskning og utvikling ved å påvirke hoppstørrelsen i en Poisson-hoppeprosess. Vi undersøker både en deterministisk og stokastisk hoppstørrelse. Oppgaven finner at et optimum kun eksisterer hvis det er en avtagende avkastning på investeringen i forskning og utvikling. Vi finner også at det er optimalt for firmaet å øke investeringen i forskning og utvikling når innovasjonsnivået i markedet øker. Oppgaven viser også at det optimale handlingsmønsteret for å investere i forskning og utvikling ikke er monoton når investeringskostnaden, ved å ta i bruk ny teknologi, øker.

Abstract

This paper looks at the optimal investment level in research and development for a firm that holds an option to make a technology switch. The paper models the investment in R&D by affecting the jump size in a Poisson jump process. We investigate both a deterministic and stochastic jump size. The paper conducts a sensitivity analysis that finds that for the optimum to be well defined there has to be a diminishing return on investment in R&D. We find that it is optimal for the firm to invest more in R&D as the level of innovation in the market increases. The paper also finds that the optimal policy for investing in R&D is not monotonic when the investment cost of adopting new technology changes.

1 Introduction

Growth has become the epitome of modern civilization and is present in many forms, e.g. an increase in population, an increase in the amount of available resources, or development of new technology. Growth is one of the key parameters we use to judge if a company is successful or not. How then can a company sustain its growth, or even better, how can they increase the rate of growth itself? It can be argued that technology has held center stage when it comes to the growth and creation of new companies the last century. The increasing amount of money invested in research and development (R&D) indicates that this is no secret to large companies. The question that then arises is how much a company should invest in R&D. This is a difficult question to answer, partially due to the high levels of uncertainty related to conducting R&D.

An industry where this has become especially true is in the industry of micro processors. The industry is particularly known for a consistent and high level of development. This progress is so consistent that it has been given its own law, the well known Moore's law. In resent years the level of innovation in the industry has declined, tending away from Moore's Law (Thornhill, 2018). This is in part due to the physical limitations that has started to affect the development of new micro processors. In short, the transistors have become as small as quantum physics allow them to be, at least with the current level of technology. This has resulted in the pursuit of alternative technologies. One promising lead are quantum computers that utilizes the properties of atoms to conduct calculations. This search for alternative technologies stands as an excellent example of some of the challenges that are involved with R&D. Companies like Google and IBM have invested massive amounts into the development of quantum computers (Mosvitch, 2019), but it is still unclear if the technology is even practical and economically sound compared to traditional transistor processors. This makes optimizing the amount invested in R&D hard to determine, not only is it difficult to guarantee the results from the R&D venture, but the value of the resulting technology can itself be uncertain.

This paper uses an analytical approach to address some of the aspects and challenges a firm might encounter when trying to decide how much they should invest in R&D. We treat the opportunity to adopt new technology as a real option, and let the firm influence the expected value of the option through R&D. The paper expands on the model presented in Huisman (2001) by allowing for an investment in R&D. The model uses a Poisson Jump process to imitate a changing technological progress that the firm can utilize. There seems to be two clear approaches when trying to account for R&D when using a Poisson jump process as a basis. The first is to affect the arrival rate through R&D, this has been explored in the preliminary work, and some of the results are presented in Appendix A. The work concluded in two surprising findings: The firms willingness to invest in R&D increases if the level of free innovation in the market increases. We also found a non-monotonic behavior when the cost of adopting new technology changed. In this paper we look at the alternative method where we let the investment in R&D affect the jump size in the Poisson process. Our first model let the firm invest some amount into R&D that directly changes the jump size in the process. In our second model we let the firm invest some amount into R&D to affect the expected jump size, where the jump size follows an exponential distribution.

Huisman (2001) uses both decision theoretic models and game theoretic models. This paper does not look at the game theoretic aspects of investing in R&D. An alternative way of modeling investment in technology is presented in a paper by Kwon (2010). The paper looks at a firm that faces a declining market. The market is modelled using Brownian motion with a negative drift. The firm holds an option to exit the market at any time, and a one-time option to boost the profit rate by investing in a technology improvement. A paper by Hagspiel et al. (2016) models a declining market similarly to Kwon (2010). The model differ by also including the opportunity to optimize the level of production capacity. This is similar to our paper as it also deals with an optimization problem simultaneously as an optimal stopping problem. Our model includes the one-time option to adopt new technology, but our model does not include an option to exit the market. We assume that the supply is always equal to the demand, and let the profit of the firm only be reliant on the technology level. A paper by Hagspiel et al. (2015) looks at technology adoption when the firm faces uncertainty of timing of new technology improvements. The paper models the technology level using a jump process like in Huisman (2001), but extends the existing model by allowing for the arrival rate to change after a jump in the Poisson process.

The paper finds that an increase in the base level of innovation in the market results in a higher investment level in R&D. This means that the our models predicts that innovation will result in more innovation. Another finding of the paper is a non-monotonic behavior of the firm when the cost of adopting new technology increases. Our model predicts that the firm would increase the investment level in R&D until the investment cost of adoption reaches a certain level. The firm would then reduce the size of the investment in R&D.

The paper is structured as follows. Section 2 derives a model that lets the firm investment in R&D to affect a deterministic jump size in the technology process. Section 3 presents the results from this model and the apparent implication and predicted behaviour of the firm. Section 4 derives the model where the jump sizes follow exponential distribution. Section 5 looks at the results from this model and its implications. Section 6 concludes on the findings of this paper.

2 Model I: Deterministic jump size

Huisman (2001) presents a model where the technology development is modeled using a Poisson jump process. We utilize the same model, but let the firm invest in R&D at the beginning of the project. The investment in R&D affects the Poisson jump process and consequently the value of the project. In the first model we treat the jump size as a constant determined by the investment size in R&D while deriving the option value and optimal investment level. The main focus of this paper is to look at the effects of changing the initial parameters and how this shapes the behavior of the firm.

One could imagine different approaches of modelling the affects of investment in R&D with the use of the model presented in Huisman (2001) as a basis. The choice of letting the investment change the jump size, and not any other parameters, seems reasonable in some industries. Companies that release new products every year often release new products that are in direct competition to their existing products. The model could therefore be seen as the value of single product launch. The arrival rate of the possible advancements are therefore already set by the firm, but how innovative the new product is, can be affected by the firm through R&D. The model therefore let the investment in R&D affect how large the jump in innovation becomes. The choice of not adopting a new technology after a jump can easily be defended if the current product is too profitable, and firm therefore waits. The results of any R&D venture is not lost, but the firm has not taken advantage of its potential.

There seems to be two major reasons for a firm to conduct R&D when we exclude R&D into supportive elements, e.g. reducing production costs, better marketing techniques, more efficient supply chains. The first is the possibility of releasing new products to the customers as a response to a declining market, as modeled in Kwon, 2010 and Hagspiel, Huisman, Kort, et al., 2016. The other is to get a competitive advantage over other firms to capture or maintain market share. The two are not necessarily exclusive, but might result in very different levels on investment in R&D. Both rely on increasing or maintaining the profit flow from customers, but the difference might be seen as a firm that holds a monopoly position and one that co-exists with other firms.

Huisman (2001) also looks at models with a game theoretical approach, accounting for interactions with other firms. The presence of other firms could affect investment level in several ways, one being patent protection. If the firm was the first to create a new technology or advancement, the firm could apply for patent protection, stopping other firms from copying the results of their R&D venture. It should be noted that patents do not always give the desired protection, and many countries have a tendency to ignore patent protection (Love, 2012). The other case would be where another firm made advances in the same technology, and then applying for a patent. The result could be catastrophic, giving the competitor massive market power. Either by demanding high licensing fees to use their technology, or by not allowing competitors to use the technology at all.

The micro processor industry provides a good example. The industry is for all practical purposes a duopoly when it comes to providing commercial processors to the consumer market (Jeyaratnam, 2019). The power balance between Intel and AMD, the two main competitors, changed when Intel launched a new series of processors in 2006. Intel used this position to greatly inflate its prices, as AMD did not have the technology to compete. AMD has since then used massive amounts of money to develop a new line of processors which they launched in 2019 (Lee, 2019). The new processors are predicted to heavily impact the market share of Intel in the coming years. This example shows how important research and development can be in order to maintain market share. The example also shows a case where a competitor is not able to utilize the technology developed by the competitor, or is at least not able to utilize it to a large extent. Intel experienced a long period with nearly the power of a monopoly. We therefore treat the sudden renewed competition by AMD in 2019 as an unexpected market penetration. Since it can be extremely hard to predict new market penetrations, and since the game theoretical aspect is outside the scope of this paper, we assume that the firm is a monopoly. We do look at the affect of a base level of innovation in the market. This could be seen as R&D preformed by other firms and could account for some affects we suspect to see from the presence of other firms. We also look at a case where the investment cost accounts for the current technology level by increasing the cost of R&D. This will influence the amount of R&D conducted, but we will not look at the action as a response to other firms actions, where one could assume that there exists a Nash equilibrium in the level of R&D performed.

2.1 Assumptions

We start by deriving the model presented in Huisman (2001), with the option to invest some amount in R&D to affect a deterministic jump size in the Poisson jump process.

We look at a risk-neutral monopolistic firm that holds an option to invest in new technology. The firms profit function is only dependent on the technology level, and produces products with the current technology level $\zeta(t)$. The firms technology level is unchanged until it decides to pay an investment cost I and adopt a new technology level. We assume that the firm can only adopt new technology once, and consequently have to produce with that technology level forever. The technology development over time is modeled using a Poisson jump process and denoted $\theta(t)$. The firm is currently producing at technology level $\zeta_0 \leq \theta_0$, where $\theta(0) = \theta_0$. The firm has an instantaneous profit function $\pi(\zeta(t))$, where $\pi : R_+ \to R$. We also assume that the firm discounts with the risk-free rate r. Our extension also allows the firm to invest some amount I_{RD} into R&D, the investment changes the jump size in the Poisson jump process. The size of the jump is denoted $u(k)$, where k is a metric that represent the level of R&D performed. The shape of $u(k)$ and I_{RD} will be discussed later in the paper.

The Poisson jump process can mathematically be stated as:

$$d\theta(t) = \begin{cases} u(k) & \text{with probability} \quad \lambda dt \\ 0 & \text{with probability} \quad (1 - \lambda dt) \end{cases} \tag{1}$$

The arrival rate λ determines the likelihood of a jump occurring in any time interval dt. The size of lambda is assumed to be constant and non-zero.

The complete problem can then be stated as:

$$F(\theta(t), k|\zeta_0) = \sup_{\tau,k} \mathbf{E} \left(\int_0^{\tau} \pi(\zeta_0) e^{-rt} dt + \int_{\tau}^{\infty} \pi(\theta(\tau)) e^{-rt} dt - I e^{-r\tau} - I_{RD}(k) \right) \tag{2}$$

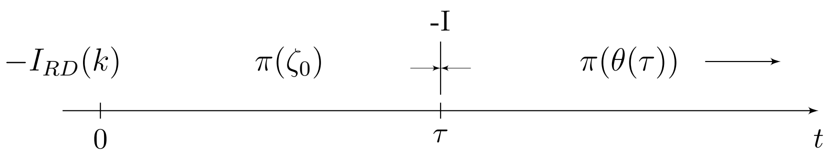

Figure 1: Timeline of project investments and profit flow.

Equation (2) shows how the firm pays an initial amount into R&D and by that affects the jump size. The equation then shows the expected value of the project from producing with the current technology level ζ_0 until the firm invests amount I at time τ. The firm then produces with the new technology level $\theta(\tau)$ forever. Figure (1) shows the timing of the investment in R&D, profit flow, and the adoption of the new technology level.

2.2 Deriving option value

Before we can find the optimal value of the effort parameter k, we need to derive the value function of the project.

Let θ^* be the technology level for which the firm is indifferent between investing in the new technology and keeping the current technology. Meaning that the firm will invest for all technology levels $\theta(t) > \theta^*$, and refrain from investing for any level $\theta(t) < \theta^*$. For this model we assume that it is possible to update technology only once and that the firm produces with the new technology level forever. Thereafter the termination payoff can be stated as:

$$V(\zeta) = \int_{t=0}^{\infty} \pi(\zeta) e^{-rt} dt = \frac{\pi(\zeta)}{r}, \tag{3}$$

which is equal to the discounted profit from producing with technology level ζ forever.

When the current technology level θ becomes higher than the optimal technology level θ^*, the firm pays the investment cost I and starts producing with the new technology. This leads to the following value function in the stopping region:

$$\Omega(\theta) = V(\theta) - I \qquad \theta \geq \theta^* \tag{4}$$

The value of the total project solves the following maximization problem:

$$F(\theta(t), \zeta_0) = \max\left(\Omega(\theta(t)), \pi(\zeta_0)dt + e^{-rdt}\mathbf{E}(F(\theta(t+dt)|\theta(t))\right) \tag{5}$$

The first term is the termination payoff. The second term is the value where the firm receives the current profit $\pi(\zeta_0)$ and not stopping at time t but continuing and acting optimal for time $t+dt$ and onward. This follows from the Bellman principle of optimization. The second term in the maximization problem for the continuation region $\theta < \theta^*$. This leads us the following equation for the continuation region:

$$F(\theta(t), \zeta_0) = \pi(\zeta_0)dt + e^{-rdt}\mathbf{E}(F(\theta(t+dt)|\theta(t))) \tag{6}$$

We let dt go towards zero, rearranging and setting $e^{-rdt} \approx (1 - rdt)$.

$$F(\theta(t), \zeta_0) = \pi(\zeta_0)dt + r\mathbf{E}(F(\theta(t+dt)|\theta(t))dt + \mathbf{E}(F(\theta(t+dt)|\theta(t))) \tag{7}$$

$$r\mathbf{E}(F(\theta(t+dt)|\theta(t))dt = \pi(\zeta_0)dt + \mathbf{E}(F(\theta(t+dt)|\theta(t) - F(\theta(t), \zeta_0) \tag{8}$$

$$rF(\theta(t+dt)|\theta(t))dt = \pi(\zeta_0)dt + \mathbf{E}(dF(\theta(t+dt))|\theta(t)) \tag{9}$$

$$rF(\theta(t), \zeta_0) = \pi(\zeta_0) + \lim_{dt \to 0} \frac{1}{dt}\mathbf{E}(dF(\theta, \zeta_0)) \tag{10}$$

Equation (10) is the resulting Bellman equation that F must satisfy. The size of the jump in the Poisson process is equal to $u(k)$ and the result of an investment in R&D. This results in two scenarios for the continuation region. In the first scenario, the resulting technology level after the jump is not large enough to trigger investment: $\{\theta | \theta < \theta^* - u(k)\}$. In the second scenario, the jump will trigger investment in the new technology level: $\{\theta|\theta^* - u(k) \leq \theta < \theta^*\}$. We have to solve the cases separately. Notice that the threshold of the region will be determined by the size of the investment in R&D due to the change in the jump size.

The equation includes a stochastic component that we solve by applying Itô's Lemma:

$$dF(\theta(t),t) = \frac{\partial F(\theta(t),t)}{\partial t}dt + \frac{\partial F(\theta(t),t)}{\partial \theta(t)}d\theta(t) + \frac{1}{2}\frac{\partial^2 F(\theta(t),t)}{(\partial \theta(t))^2}(d\theta(t))^2 \tag{11}$$

Since $\theta(t)$ follows a Poisson jump process, the third term in the equation can be set to zero as $(d\theta(t))^2 \approx 0$ when dt goes to zero. We start in the continuation region where $\theta(t) < \theta^* - u(k)$:

$$\frac{\partial F(\theta(t),t)}{\partial t}dt = 0 \quad \text{and} \quad \frac{\partial F(\theta(t),t)}{\partial \theta(t)}d\theta(t) = \lambda dt[F(\theta + u, \zeta_o) - F(\theta, \zeta_0)] \tag{12}$$

That gives:

$$dF(\theta(t),t) = \lambda dt[F(\theta + u, \zeta_0) - F(\theta, \zeta_0)] \tag{13}$$

Which gives us the following equation for the total value:

$$rF(\theta(t), \zeta_0) = \pi(\zeta_0) + \lambda(F(\theta + u(k), \zeta_0) - F(\theta, \zeta_0)) \tag{14}$$

The solution of equation (14) is equal to:

$$F(\theta, \zeta_0) = c\left(\frac{\lambda}{\lambda + r}\right)^{-\frac{\theta}{u(k)}} + \frac{\pi(\zeta_0)}{r} \tag{15}$$

where c is a constant that can be found by applying the value matching condition at $\theta = \theta^* - u(k)$. We solve the second part of the continuation region, $\theta^* - u(k) \leq \theta < \theta^*$, by applying Itô's Lemma again:

$$\frac{\partial F(\theta(t),t)}{\partial t}dt = 0 \quad \textit{and} \quad \frac{\partial F(\theta(t),t)}{\partial \theta(t)}d\theta(t) = \lambda dt(V(\theta + u(k)) - I - F(\theta, \zeta_0)) \tag{16}$$

Which gives us the equation:

$$rF(\theta(t), \zeta_0) = \pi(\zeta_0) + \lambda(V(\theta + u(k)) - I - F(\theta, \zeta_0)) \tag{17}$$

Rewriting gives:

$$F(\theta(t), \zeta_0) = \frac{\pi(\zeta_0)}{r + \lambda(k)} + \frac{\lambda}{r + \lambda}(V(\theta + u(k)) - I) \tag{18}$$

We know that the value in the two regions has to be equal at $\theta = \theta^* - u$:

$$\frac{\pi(\zeta_0)}{r + \lambda} + \frac{\lambda}{r + \lambda}(V(\theta^* - u(k) + u(k)) - I) = c\left(\frac{\lambda}{\lambda + r}\right)^{-\frac{\theta^* - u(k)}{u(k)}} + \frac{\pi(\zeta_0)}{r} \tag{19}$$

Solving this equation for c gives:

$$c = \left(\frac{\lambda}{\lambda + r}\right)^{\frac{\theta^*}{u(k)}} (V(\theta^*) - V(\zeta_0) - I) \tag{20}$$

Finally, we can express the total value in the three cases. The cost of investing in R&D occurs at $t = 0$ at a cost of $I_{RD}(k)$. As the investment in R&D occurs at the beginning of the project the cost is simply deducted from the total value of the project in the different cases.

$$F(\theta, k|\zeta_0) = \begin{cases} \left(\frac{\lambda}{\lambda+r}\right)^{\frac{\theta^* - \theta}{u(k)}} (V(\theta^*) - V(\zeta_0) - I) + V(\zeta_0) - I_{RD}(k) & \text{if} \quad \theta < \theta^* - u(k) \\ \frac{\pi(\zeta_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}(V(\theta + u(k)) - I) - I_{RD}(k) & \text{if} \quad \theta^* - u(k) \leq \theta < \theta^* \\ \frac{\pi(\theta)}{r} - I - I_{RD}(k) & \text{if} \quad \theta \geq \theta^* \end{cases} \tag{21}$$

As mentioned earlier, the threshold θ^* determines which value region we are currently in. We can find the threshold by value matching. We therefore equate the second and third region and let $\theta(t) = \theta^*$:

$$\frac{\pi(\zeta_0)}{r + \lambda} + \frac{\lambda}{r + \lambda}\left(\frac{\pi(\theta^* + u(k))}{r} - I\right) - I_{RD}(k) = \frac{\pi(\theta^*)}{r} - I - I_{RD}(k) \tag{22}$$

$$G_I(k) = \frac{\pi(\zeta_0)}{r + \lambda} + \frac{\lambda}{r + \lambda}\left(\frac{\pi(\theta^* + u(k))}{r} - I\right) - \frac{\pi(\theta^*)}{r} + I = 0. \tag{23}$$

$G_I(k)$ is an implicit function that gives us the threshold. We do need to define a profit function π in order to solve the equation. We will define a profit function and functions for $u(k)$ and $I_{RD}(k)$ in the next section. Note that θ^* is dependent on the jump size $u(k)$, implying that the threshold itself is dependent on the k, i.e. the level of effort put into R&D . Also note that the threshold is not dependent on the cost of R&D. This is because $I_{RD}(k)$ is incurred at the beginning and is considered sunk for the problem of optimal adoption of the technology.

In Appendix B we have also derived the expected time until we reach the optimal technology level:

$$\mathbf{E}[\tau] = \frac{n^*(k)}{\lambda} \tag{24}$$

Where $\mathbf{E}(\tau)$ is the expected time to reach the optimal technology level θ^*, and n^* is the number of jumps needed before reaching the optimal technology level. The appendix also show that the variance is given by:

$$Var[\tau] = \frac{n^*(k)}{\lambda^2} \tag{25}$$

2.3 Finding the optimal effort level k

The goal of this section is to find the optimal level of investment in R&D. To be able to find the optimum we need to define three functions: The profit function π, the response of the jump size on the level of effort, and the cost of conducting R&D.

2.3.1 The Profit Function

A good model should be flexible to a wide variety of profit functions. This paper does not take a deeper look at different profit functions, but introduces the same as the one presented in Huisman (2001), as this makes comparisons to the benchmark model easier. It also removes the effects of using a different profit function. The profit function is therefore:

$$\pi(\zeta) = \varphi\zeta^b \tag{26}$$

With:

$$\varphi = (1-a)\left(\frac{a}{w}\right)^{\frac{a}{1-a}} p^{\frac{1}{1-a}} \tag{27}$$

And:

$$b = \frac{1}{1-a} \tag{28}$$

Where $a \in (0,1)$ is the output elasticity, p is the output price and w is the input price. These parameters are seen as exogenously given and constant for our model. ζ can simply be replaced with θ and θ^* for their respective cases, meaning that the technology level is the only variable directly affecting the profit.

The reader should note that the number of solutions to the implicit function of the threshold θ^* is dependent on the output elasticity a.

2.3.2 Return on effort into R&D

One of the most central parts of this model is the functional form of $u(k)$. The function expresses what the firm should expect from conducting R&D. In our first model we assume that the jump size after the firm has conducted R&D is deterministic. This makes it possible to observe how the value of the investment changes with a change in the level of R&D. The downside is the lack of uncertainties around conducting R&D itself, this means that the first model only accounts for the insecurities surrounding the value of the resulting R&D. The second model uses a jump size that follows a stochastic distribution, accounting for the fact that the result from conducting R&D can be more or less successful.

The functional form should reflect that the investment in R&D results in a direct change in the jump size. Let us first look at a possible candidate:

$$u(k) = \frac{k^{\beta}}{10} \tag{29}$$

Equation (29) shows how the effort k put into R&D is affected by two parameters that determines the return on conducting R&D. The β-parameter makes it possible to account for diminishing return ($\beta < 1$), constant return to scale ($\beta = 1$), or return to scale ($\beta > 1$). We will assume that there are a diminishing return to scale ($\beta < 1$) which partially is supported by Shephard and Färe (1974). This seems reasonable as one should expect the complexity of organizing an R&D venture to increase with the effort level k. There are probably a finite amount of researchers, scientists, and engineers within a certain field of technology. For some industries it is therefore reasonable to assume that the competence level of the labor is diminishing with the number of employees. This assumes that the firm employs the best worker first, the second best second, and so on. This is not necessarily the case for industries with a large available work force. But there are also other factors that might result in a diminishing return on investments.

Notice how the jump size is divided by a factor of 10. This is not random, and serves two purposes. The first purpose is to assure a diminished return on R&D in the region of the resulting jump size $0 < u \leq 1$. The scaling also ensures that the effort parameter is in the same range as the one used in the preliminary work presented in Appendix A.

This function for the jump size $u(k)$ does not account for the possibility of the return on R&D being higher in the beginning. The quality of the employees could be increasing with the level of investment in the beginning as the firm could attract better staff with higher salaries. This could be modelled using a type of Sigmoid function, either as the jump size itself, or an inverse Sigmoid function for the β-parameter. This introduces several challenges when it comes to solving for the optimum. The model also becomes very sensitive to the shape of the Sigmoid function, making it harder to interpret the effects of other parameter changes.

The current form of $u(k)$ results in a jump size of zero if the firm restrains from investing in R&D. This is reasonable if the firm is a monopoly and is not dependent on any other technologies but the one developed by the firm. This seems unlikely as most technologies can utilize breakthroughs in other fields. We can include some base level of innovation as a part of the jump size that is independent from the effort of the firm. This base level of innovation could also be a result from R&D conducted by other firms, but that is available to competitors. We account for all available development not conducted by the firm as u_0:

$$u(k) = \frac{k^\beta}{10} + u_0 \tag{30}$$

2.3.3 Cost of conducting R&D

The former section established a functional form of the jump size function $u(k)$. The function does not account for any changes in how the cost develop when we increase the amount of effort k. We saw how increasing the amount of labourers working on R&D could result in a diminishing return on effort. One should also most likely experience an increase in the cost when increasing the effort level. It can be hard to define technological progress in a consistent hierarchy, and it can be even harder to quantify technological progress. This makes it difficult to relate a resulting jump size directly to the cost. The cost function should therefor relate the value added by R&D to the costs. We will later see how a small change in the jump size results in a large increase in the project value.

One cost function would be:

$$I_{RD}(k) = ck \tag{31}$$

The constant c accounts for any linear cost relationship. This makes the cost scalable to the other parameters. This cost structure does not account for any other effects that might influence the cost of conducting R&D. The first being the current technology level $\theta(t)$. The complexity of conducting R&D in any fields seems to be rising. One example is the ever decreasing size of the transistors in a micro processor. This results in the need for advanced electron microscopes in order to develop and validate the new processors. This again results in an increasing cost of equipment as the technology develop. The following cost function accounts for this by making the cost dependent on the current technology level. We also assume that the increase in cost is a result of the gap between the current technology level available in the market θ_0, and the technology level currently in use by the firm ζ_0. Note that we assess the cost with the initial value of the two parameters as the investment is made at the beginning:

$$I_{RD}(k|\theta_0, \zeta_0) = ck + v\left(\frac{\theta_0}{\zeta_0} - 1\right)k \tag{32}$$

Equation (32) relates the cost of the technology gap as a fraction between θ_0 and ζ_0, with the assumption that $\theta_0 \geq \zeta_0$. There is no penalty if the current technology level in use is equal ζ_0 is equal to the available technology level θ_0. We choose a separate parameter for the cost of the technology gap as the actual numerical value of the technology level most likely is different from the expected increase in costs.

2.3.4 Optimal effort level k^*

We have now defined all functions necessary to search for an optimal level of R&D, denoted k^*, given a set of input parameters. The optimal level is given when we maximize the total value of the project. The value is defined by the value function in equation (21), and has three distinct regions as previously mentioned. The optimum is therefore dependent on finding the threshold θ^* in order to determine which value region that is currently valid. This creates several challenges when it comes to solving for an optimum. The first being the thresholds dependency on the order of the profit function, resulting in several possible solutions. The second challenge is that the changing value function is dependent on the threshold function itself. The third is the nonlinear optimization problem that results from deriving the optimum. These factors forces us to use numerical methods to find the optimum as there is no known analytical solution. We introduce numerical values for the parameters in the next section in order to solve for the optimum.

3 Results: Model I

In order to use Huisman (2001) as benchmark model, we introduce the same parameter values to our extended model. The parameters are: $a = \frac{1}{2}$, $p = 200$, $w = 50$, $r = 0.1$, $\lambda = 1$, and $I = 1600$. Resulting in $\varphi = 200$ and $b = 2$ for the profit function. We let $\zeta_0 = \theta_0$ in all cases, except where we specifically want to observe the effect of a gap in technology level. The base level of innovation is set equal to $u_0 = 1$, as we expect some innovation in the market that is not dependent on investments in R&D done by the firm. This also makes it easier to observe the difference from the benchmark model.

We start by deriving the values for the benchmark model with a jump size of $u = 0.1$, and $\zeta_0 = \theta_0 = 1$. We first calculate the optimal technology level θ^* which is equal to 2.703. The total value is then, F= 4172.52. We also calculate that 18 jumps are needed to reach the optimal technology level, this results in $\mathbf{E}(t^*) = 18$ as the arrival rate is equal to one.

Let us look at the existence of an optimal level of investment in R&D before we look at how changing the initial parameters change the behavior of the firm. If we look closely at the value function, we can observe that it is strictly increasing with an increase in the jump size. The exception is when the technology level θ_0 is larger than the threshold θ^*. It should also be mentioned that θ^* is strictly increasing with the jump size. Figure 2 shows how the value of the project and the threshold increases with the jump size, any affects from investing in R&D is not included, i.e. $c = 0$.

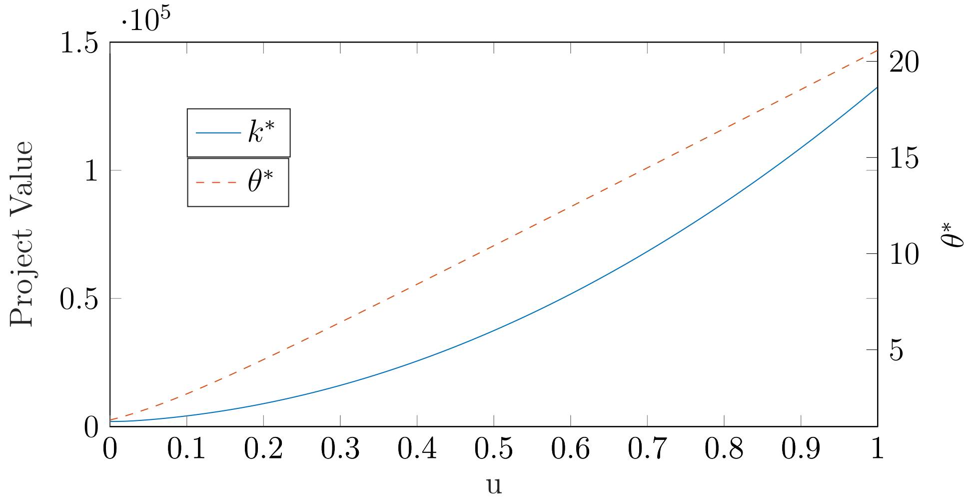

Figure 2: Project value and threshold for different values of u. Other parameters are: r = 0.1, a = 0.5, w = 50, p = 200, $\lambda = 1$, $\zeta_0 = 1$, I=1600, $\theta_0 = 1$, and c=0

A key insight from figure 2 is how the increase in value is nonlinear. An optimum exists where the marginal gain of preforming R&D is equal to the marginal cost. An optimum would therefore only be possible if there are diminishing return on R&D, assuming a linear cost structure. The value of the β-parameter therefore determines if there is an optimum or not. There are little empirical evidence for what the return on R&D should be, only that it is reasonable to assume that there is a diminished return, meaning $\beta < 1$. We choose a marginal cost of R&D of $c = 200$, this results in an investment cost of $I_{RD}(k) = 200$ when $k = 1$. This is around 10% of the project value of the benchmark model given the same parameters. This is similar to the percentages of what larger technology firms uses on R&D in comparison to total revenue (Truong, 2016). β is chosen to be equal to 0.3 in cases where we do not look at the effect of changing the β-parameter.

3.1 Sensitivity analysis

We will now look at how the optimum effort level k^* changes when changing the input parameters. The following parameter changes will be investigated: The cost of adopting new technology I, the base level of innovation in the market u_0, a gap between θ_0 and ζ_0, and the marginal cost of conducting R&D c. The reader should note that some of the graphs use scientific notation, this is simply due to practicalities.

Effect of a changing the investment cost I
The first parameter we observe is the investment cost of adopting new technology I. This cost is incurred when the firm chooses to adopt the new technology level, implying that the technology level has reached the threshold θ^*. In essence the firm tries to time when the future value of producing with a new technology level surpasses the discounted value of the investment cost I. By delaying the adoption, the firm postpones I, reducing its present value due to discounting.

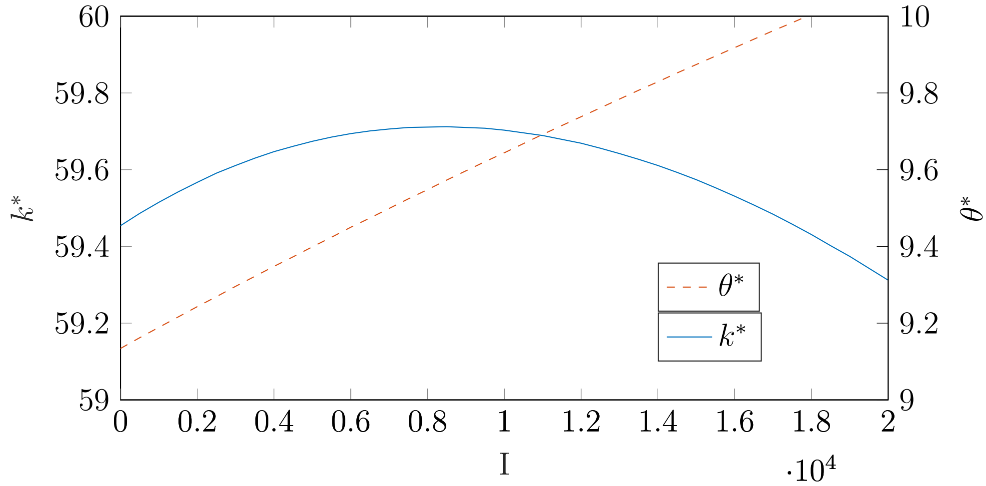

Figure 3: Optimal investment level in R&D for different values of I. Other parameters are: r = 0.1, a = 0.5, w = 50, p = 200, $\lambda = 1$, $\zeta_0 = 1$, $\theta_0 = 1$, $\beta = 0.3$, and $c = 200$

Figure 3 shows a non-monotonic behavior of the optimal investment level in R&D when changing the investment cost I. This strengthens the findings in the preliminary work presented in Appendix A. We can observe that changing the investment level I only triggers minor changes in the optimal investment level, but there is still a clear non-monotonic behavior. This means that in the beginning the firm increases its investment in R&D as the investment cost increases. This is the optimal strategy for the firm until the investment cost reaches around $I = 8500$, for this parameter set. At that point the value gained from discounting the investment cost I is larger than the added value of waiting for a higher technology level. The result is a diminish optimal level of effort k^*. The optimal threshold is still increasing, but the value comes from discounting the investment cost over time, not reaching a higher technology level.

We can also observe that θ^* only changes from about 9.1 to 10.1. The effect on the threshold is monotonic and increasing with the investment cost. This means that the value of waiting increases, this is in part due to the higher marginal value of discounting the investment cost. Said differently, the higher investment cost makes it valuable to discount it over a longer period of time.

Effect of changing the base level of innovation u_0

The base level of innovation in the market u_0 could affect the optimal investment level k^* in three ways. The first being a reduction in the investment in R&D. This could be a result of the firm reaching the optimal jump size with a lower investment in R&D, utilizing the free innovation in the market. The second outcome could be that the base level on innovation would not affect the optimal investment of the firm. The firm would still benefit from the increased innovation in the market, resulting in a higher project value. But the marginal gain of investing in R&D would remain unaffected by a higher base level of innovation, resulting in a constant investment level. The third outcome would be where an increased base level of innovation would result in a higher investment level in R&D. This would mean that the added value would incentives the firm to invest even more.

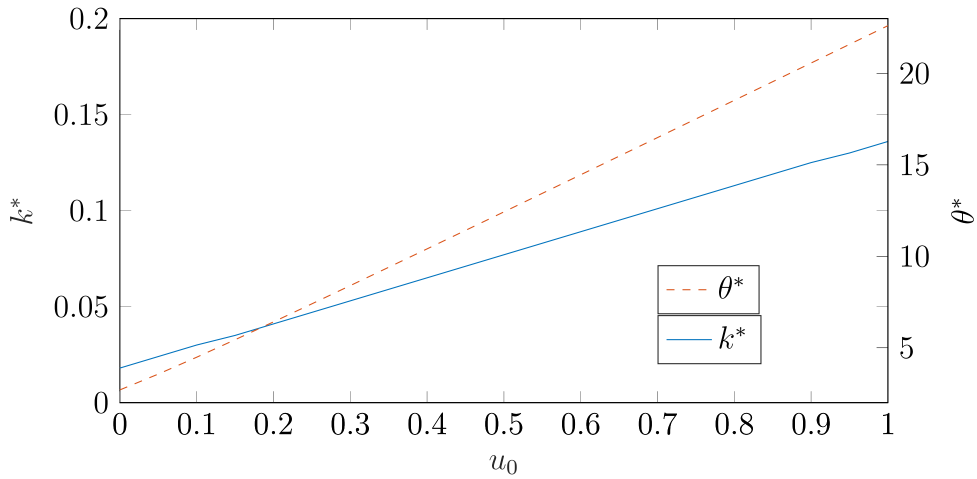

Figure 4: Optimal investment level in R&D for different values of u_0. Other parameters are: r = 0.1, a = 0.5, w = 50, p = 200, $\lambda = 1$, $\zeta_0 = 1$, $\theta_0 = 1$, $\beta = 0.3$, and $c = 200$

Figure 4 shows how the willingness to invest in R&D increases with base level of innovation in the market, accordance with the third prediction. This means that the marginal gain of conducting R&D is increasing with the base level of innovation. We can also observe that the technology threshold θ^* is increasing with an increase in u_0. This again means that the firm is willing to wait for a higher technology level before adopting a new technology. The increased technology threshold increases the value of the project, resulting in a higher marginal gain from investing in R&D. The marginal cost c of conducting R&D is constant, the optimal investment level k^* is therefore increased. This is a result of the nonlinear increase in project value when increasing the jump size as shown in figure 2. The diminishing return on R&D eventually creates an effort level where the marginal cost is equal to the marginal gain. This conclusion is strengthened by the findings in the preliminary work, presented in Appendix A.

Effect of changing the gap between θ_0 and ζ_0

We assume that the current technology level in use is $\zeta_0 = 1$. The increase in the starting technology level in the market θ_0 results in higher R&D costs as modelled with the cost function presented in 2.3.3. The marginal cost v of the technology gap is unknown. We therefore look at three different values.

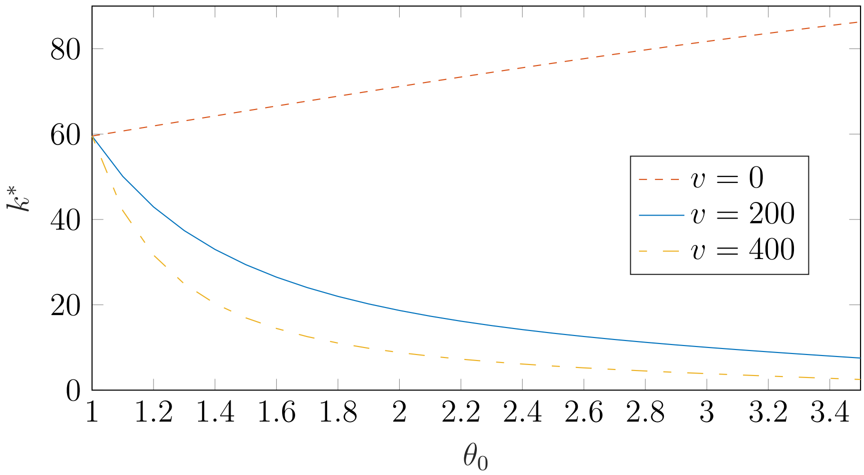

Figure 5: Optimal investment level in R&D with changing technology level θ_0, for $v = 0, 200$ and 400. Other parameters are: r = 0.1, u_0 = 0.1, a = 0.5, w = 50, p = 200, λ = 1, ζ_0 = 1, I=1600, and c = 200.

Figure 5 shows that the marginal cost of the technology gap v dictates the behavior of the firm. The firm wishes to invest increasingly more in R&D if there are no penalties for a gap in technology. This is in line with the findings of increasing the base level of innovation in the market. The increase in the starting technology level θ_0 has a positive effect on project value when $v = 0$, resulting in a higher marginal gain from investing in R&D.

We can observe that the optimal strategy changes when the firm incurs a penalty due to the technology gap. An increase in the marginal cost of the technology gap results in a decrease in the willingness to conduct R&D, as seen by the change between $v = 0$, $v = 200$, and $v = 400$. We can observe that an increase in the technology gap results in a larger increase in the total marginal cost of R&D when $v = 200$, making the firm wanting to reduce the effort level in R&D. An increase in the technology gap now reduces the project value.

Effect of changing the marginal cost of investing in R&D

Finally we will observe how a change in the marginal cost of conducting R&D affects the willingness to invest in R&D.

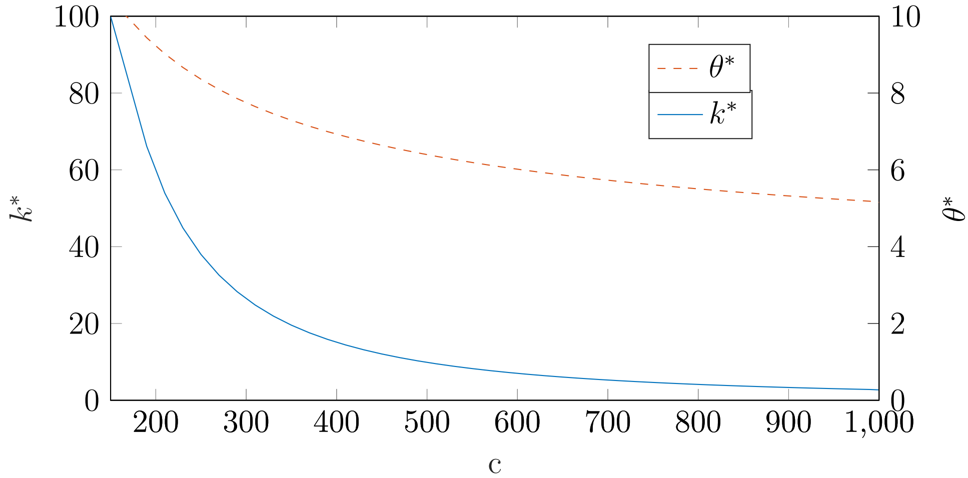

Figure 6: Optimal investment level in R&D for different values of c. Other parameters are: r = 0.1, a = 0.5, w = 50, p = 200, $\lambda = 1$, $\zeta_0 = 1$, $\theta_0 = 1$, $\beta = 0.3$

Figure 6 shows how an increasing marginal cost results in lower willingness to invest in R&D. The marginal cost c lowers the total gain from conducting R&D, this is a direct result of the diminishing return on R&D. The diminishing return makes the cost of conducting R&D quickly surpass the gained value. One can also observe that the optimal technology threshold flattens and seems to approach a constant value. This value is simply the value as if the firm did not conduct any R&D, as that is precisely the optimal strategy if the marginal costs is to high.

4 Model II: Stochastic Jump Size

Model I has the disadvantage that the firm can exactly determine how the investment in R&D affects the jump size. There are still uncertainties present in model I since the arrival rate of the jumps are stochastic. But this resulted in a model that only accounted for uncertainties outside the influence of the firm. We therefore introduce an alternative model that accounts for the uncertainties surrounding the results from an investment in R&D. We now let the jump size follow some stochastic distribution and let the firm invest in the expected jump size instead of the jump size itself. The first question that arise is what type of distribution the jump size should follow.

4.1 Assumptions

4.1.1 Distribution of the Jump Size u

Huisman (2001) presents two models that uses a stochastic distribution for the jump size. The first follows a uniform distribution, and the second follows an exponential distribution. The goal of model II is to include the uncertainties related to the success of the R&D venture. Let us start by evaluating the uniform distribution.

With the uniform distribution there are an equal probability for all values of u within a domain. Let us say that the domain is $u = [a, b]$. This would mean that there is an equal probability for the jump size being any value from a to b. The mean of the distribution is stated as:

$$E(u) = \frac{a+b}{2} \tag{33}$$

One could imagine that the firm could invest some amount into R&D in order to influence the mean by changing a and b. A natural start for a could be zero, meaning that there is a chance that the R&D venture failed completely. The challenges is to set the upper limit of the jump size. This could be done by relating it to the Poisson process, limiting the jumps to be smaller than some fraction of the current technology level. Using a uniform distribution seems to have several flaws. The most apparent being the assumption that there is an equal probability for any jump size to occur within the domain. It seems unreasonable to assume that it is an equal probability of the R&D venture to fail completely, or for getting the biggest breakthrough possible. A real life firm would most likely learn something from any venture, and the probability of making a revolutionary step should be highly unlikely.

We are therefore looking for a distribution that makes large jumps less probable. One could expect that the jump size followed a normal distribution. The firm could then affect the distribution by changing the mean and variance. A challenge with the normal distribution is that it is symmetric around the mean, resulting in a positive probability for negative jump sizes. One could limit the negative values by simply equation them to zero, but this would create a skewness that changed depending on the mean and variance.

Another candidate would be an exponential distribution. The distribution has only positive values and a diminishing probability for larger values. Another positive feature is the number of parameters that has to be determined. We only need to know the rate parameter μ in order to define the distribution. We could then imagine that the firms investment in R&D affected this one parameter, influencing the distribution of the expected jump size. The exponential distribution as a density function:

$$f(u|\mu) = \begin{cases} \mu e^{-\mu k} & \text{if} \quad u \geq 0 \\ \\ 0 & \text{if} \quad u < 0 \end{cases} \tag{34}$$

And a expected mean of:

$$E(u) = \frac{1}{\mu} \tag{35}$$

And variance of:

$$var(u) = \frac{1}{\mu^2} \tag{36}$$

4.1.2 Return on Effort Into R&D

We let the investment in R&D affect the expected jump size from the exponential distribution. This means that the firm can not guarantee the resulting jump size, only the expected size. Equation (35) shows that the expected mean is the inverse of the rate parameter. This means that an increase in the rate parameter would result in a lower expected jump size. We want to observe a larger expected jump size when the firm invest in R&D, we therefore let investments affect the inverse of the rate parameter. Resulting in a rise of the expected jump size if effort into R&D is increased. This also makes it easier to compare model I and II. The result from an effort level k is equal too:

$$\mu(k) = \frac{1}{k^\beta + \mu_0} \tag{37}$$

We also include the possibility of a base level of innovation μ_0. We could include a base level for the jump size, but this would complicate the derivation of the value function. The downside is that we most likely can not directly observe the difference between changing the base level of innovation in the two models. This is because we add a fixed amount to the jump size in model I u_0, and a higher expected jump size in model II μ_0. Notice how we account for a base level in innovation by adding the value under the fraction, this to insure that an increased base level of innovation results in a higher expected jump size. We include the β-parameter for the same reason as in model I. The parameter let us account for a diminishing return on R&D, which we will later show to be a requirement in order for an optimum k^* to exist.

4.2 Deriving Option Value

The value function derived for model I is no longer valid. This is because the jump size is no longer deterministic. We perform the same derivation as in Huisman (2001), but include the option of investing in R&D. The value function changes in several ways from the derivation of model I, the first being only two value regions. The reduction from three to two value regions is because there is always a positive probability that the next jump will be large enough to exceed the threshold value. The Bellman equation for model II is:

$$
\begin{aligned}
F(\theta|\zeta_0) =& \frac{\pi(\zeta)}{r+\lambda} + \frac{\lambda}{r+\lambda}\int_{u=0}^{\theta^*-\theta} F(\theta+u,\zeta_0)\mu(k)e^{-\mu(k)u}du \\
&+ \frac{\lambda}{r+\lambda}\int_{\theta^*-\theta}^{\infty}(V(\theta+u)-I)\mu(k)e^{-\mu(k)u}du
\end{aligned} \tag{38}
$$

The first part is the discounted value of producing with current technology level ζ forever. The second expression is the discounted expected value of the project before the threshold θ^* is reached, and the value of continuing to act optimally. The third expression is the discounted expected value of adopting the new technology, paying the investment cost I, and producing with the adopted technology level forever. Note that this is done by adding all the expected jumps and their probability, and multiplying by the value of the project in each case.

The Bellman equation results in a differential equation that is solved in the appendix. By solving the differential equation we get:

$$
\begin{aligned}
F(\theta|\zeta_0) =& \frac{\lambda}{r+\lambda}e^{\frac{\mu(k)(r\theta+\lambda\theta^*)}{r+\lambda}}\int_{u=\theta^*}^{\infty}(V(u)-I)\mu(k)e^{-\mu(k)u}du \\
&+ (1-\frac{\lambda}{r+\lambda}e^{\frac{\mu(k)r(\theta-\theta^*)}{r+\lambda}})V(\zeta_0)
\end{aligned} \tag{39}
$$

The second region is equal to the one in model I. It is simply the expected pay-off value of the project. We also assume that the investment into R&D is made at the beginning of the project, meaning we can subtract the cost from the value function:

$$
F(\theta,k|\zeta_0) = \begin{cases}
\frac{\lambda}{r+\lambda}e^{\frac{\mu(k)(r\theta+\lambda\theta^*)}{r+\lambda}}\int_{u=\theta^*}^{\infty}(\frac{\pi(u)}{r}-I)\mu(k)e^{-\mu(k)u}du+ & \\
(1-\frac{\lambda}{r+\lambda}e^{\frac{\mu(k)r(\theta-\theta^*)}{r+\lambda}})\frac{\pi(\zeta_0)}{r} - I_{RD}(k) & \text{if} \quad \theta < \theta^* \\
\frac{\pi(\theta)}{r} - I - I_{RD}(k) & \text{if} \quad \theta \geq \theta^*
\end{cases} \tag{40}
$$

The threshold θ^* can be found in the same ways as in model I by value matching at $\theta = \theta^*$. This results in the implicit function:

$$\frac{\pi(\zeta_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}e^{\mu(k)\theta^*} + \int_{u=\theta^*}^{\infty} (\frac{\pi(u)}{r} - I)\mu(k)e^{-\mu(k)u}du = \frac{\pi(\theta^*)}{r} - I \tag{41}$$

$$G_{II}(k) = \frac{\pi(\zeta_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}e^{\mu(k)\theta^*} + \int_{u=\theta^*}^{\infty} (\frac{\pi(u)}{r} - I)\mu(k)e^{-\mu(k)u}du - \frac{\pi(\theta^*)}{r} + I = 0 \tag{42}$$

4.3 Finding the Optimal Effort level k

We need a functional form of the profit function and cost function in order to solve for the optimum effort level. We therefore introduce the same profit and cost functions as in model I, making it easier to compare the two models. Finding the optimal investment level in R&D is even more complicated in model II. The threshold function $\theta*$ includes an integral that only has a numerical solution. Finding an optimal investment level is therefore only possible with the use of numerical methods. The next section will introduce numerical values to the parameters and search for the optimal investment level in R&D through the use of numerical methods.

5 Results: Model II

We start by reintroducing the parameters used in model I for reference. The parameters are: $a = \frac{1}{2}$, $p = 200$, $w = 50$, $r = 0.1$, $\lambda = 1$, and $I = 1600$. Resulting in $\varphi = 200$ and $b = 2$ for the profit function. We let $\zeta_0 = \theta_0$ in all cases, again except where we specifically want to observe the effect of a gap in technology level. We set the marginal cost of conducting R&D higher than in model I, $c = 2000$. The parameter is much higher in model II as the marginal increase in project value is higher when increasing the expected value of the jump size in model II, rather than changing the actual jump size in model I.

The main goal is to observe the effects of introducing uncertainty into the investment in R&D. The firm has to not only account for a changing technology level, but also that the results from investment in R&D are stochastic.

5.1 Sensitivity analysis

Figure 7 is created using a Monte Carlo simulation. The simulation utilizes the optimal technology threshold derived in equation (42) to time the technology adoption. The simulation is performed 10 000 times and the 90%-confidence interval is derived from the resulting data.

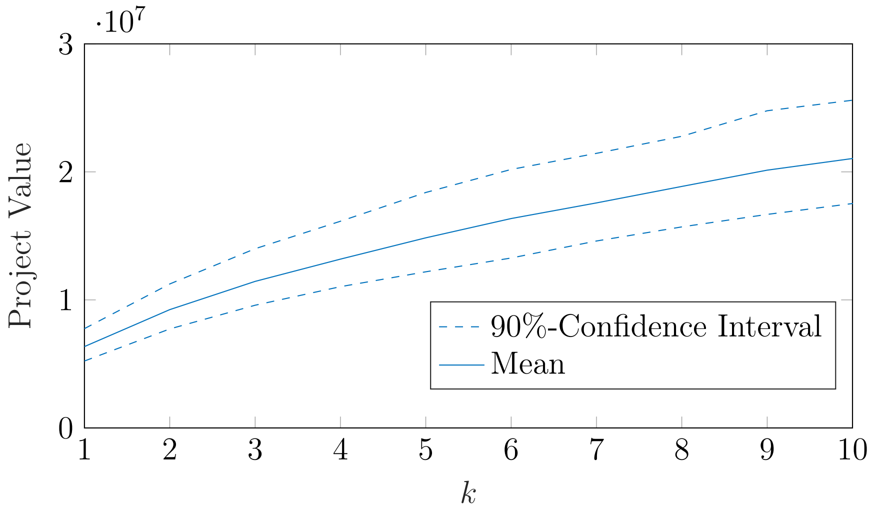

Figure 7: Monte Carlo simulation of the confidence interval of the project value. Other parameters are: r = 0.1, μ_0 = 0, a = 0.5, w = 50, p = 200, λ = 1, ζ_0 = 1, I=1600, θ_0 = 1, c = 2000, and μ_0=0.1.

The simulation shows that the size of the investment in R&D affects the size of the confidence interval. This can also be observed by the variance, equation (36), of the expected jump size in the exponential distribution. The variance is dependent on the inverse of the squared rate parameter. The investment in R&D affects the inverse of the rate parameter, resulting in an increasing variance as the effort level in R&D increases. This seems reasonable as an increase in the R&D effort results in a higher expected jump size as the likelihood of larger advancements from the R&D venture has increased. The R&D venture could still be unsuccessful resulting in larger range of probable outcomes. The firm should therefore expect higher uncertainties as the investment level in R&D increases.

Effect of a changing the investment cost I

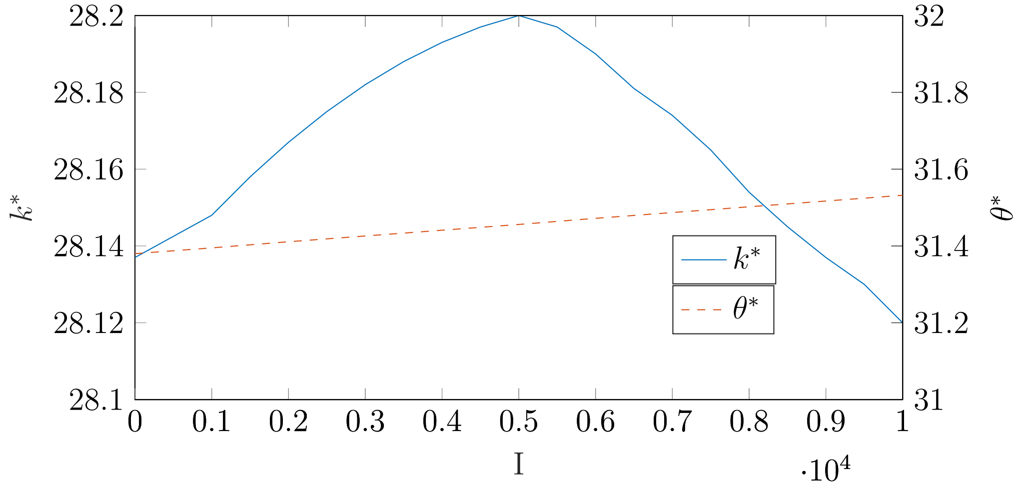

Figure 8: Optimal investment level in R&D when changing the adoption cost I. Other parameters are: r = 0.1, a = 0.5, w = 50, p = 200, λ = 1,β = 0.3, ζ_0 = 1, θ_0 = 1, c = 2000.

Figure 8 displays the same non-monotonic behavior has observed in model I. One should notice that the change in the optimal effort level K^* is very small. The small change in optimal effort level is because the project value is very sensitive to changes in the rate parameter. The change in project value is relatively much larger than the change in the adoption cost I. Resulting in only minor changes in the optimum effort level.

Effect of changing the base level of innovation μ_0
We can not effect the jump size directly in model II, only the expected jump size. This creates a different affect on the value of the project compared to model I. The reader should note that the expected jump size decreases with an increase in the rate parameter. This is accounted for by adding the base level of innovation in the denominator in equation (37), resulting in a higher expected jump size when μ_0 increases.

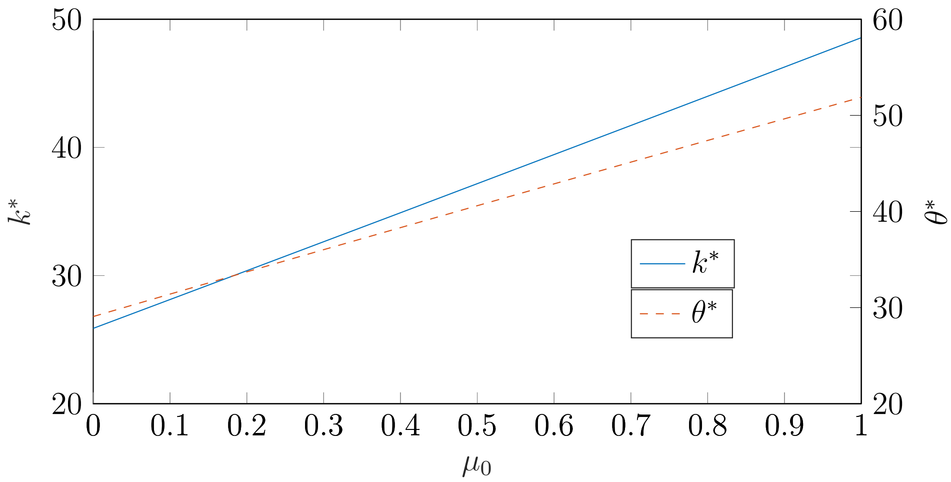

Figure 9: Optimal investment level in R&D for different values of μ_0. Other parameters are: r = 0.1, a = 0.5, w = 50, p = 200, $\lambda = 1$,$\beta = 0.3$, $\zeta_0 = 1$, I=1600, $\theta_0 = 1$, c = 2000.

Figure 9 shows the same trend as we observed in model I. The added value to the project results in the firm wanting to invest more into R&D. This seems to be a result of the same affects observed in model I, where the marginal gain from investing in R&D is increasing with the project value. Again, the diminishing return to R&D results in the existence of an optimum as the marginal cost is of conducting R&D constant.

Effect of changing the gap between θ_0 and ζ_0
We assume that the current technology level in use is $\zeta_0 = 1$. The increase in the starting technology level θ_0 results in higher R&D costs as modelled with the cost function presented in 2.3.3. The cost function is equal for both models.

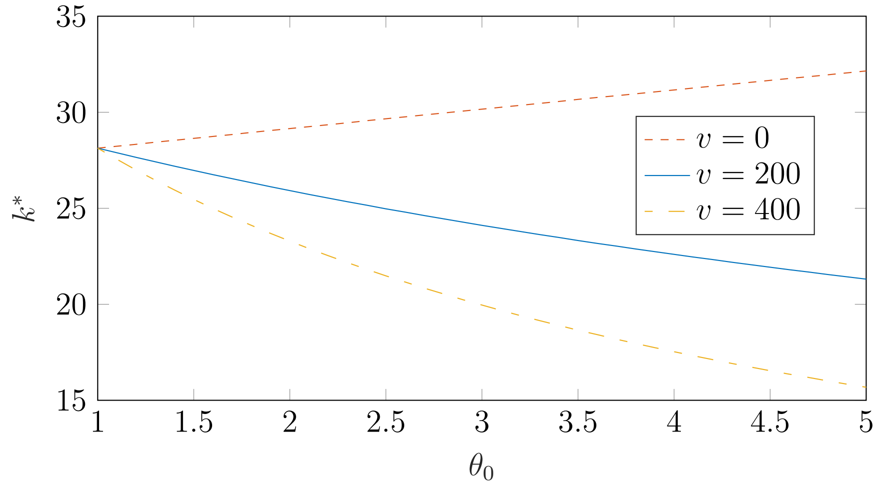

Figure 10: Optimal investment level in R&D when changing the gap between θ_0 and ζ_0. Other parameters are: r = 0.1, a = 0.5, w = 50, p = 200, $\lambda = 1$,$\beta = 0.3$, $\zeta_0 = 1$, I=1600, c = 2000, μ_0=0.1, and v =200.

Figure 10 shows how the firm responds in the same way in both model I and II. A higher starting technology level in the market θ_0 increases the value of the project if there are no penalties for the gap in technology between θ_0 and ζ_0. An increase in the marginal cost of the technology gap results in a reduction in the optimal effort level k^*.

Effect of changing the marginal cost of conducting R&D

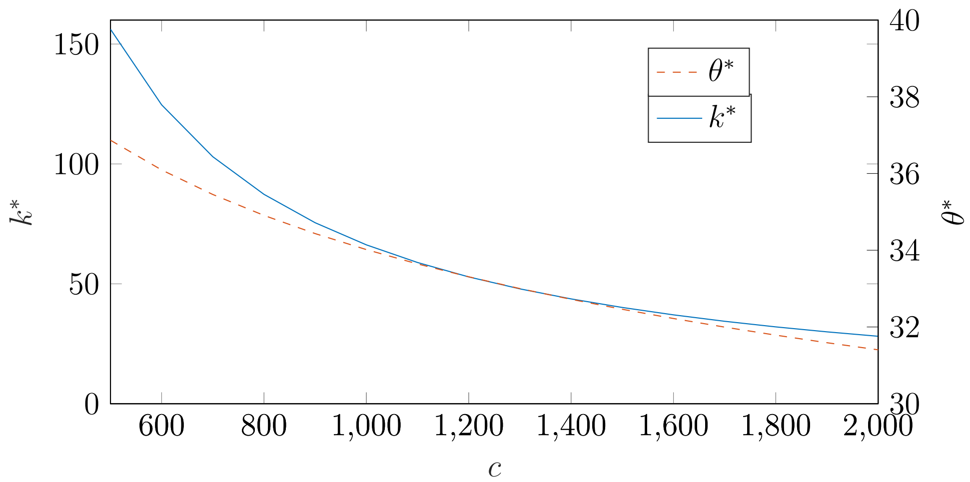

Figure 11: Optimal investment level in R&D for different values the marginal cost c. Other parameters are: r = 0.1, a = 0.5, w = 50, p = 200, $\lambda = 1$,$\beta = 0.3$, $\zeta_0 = 1$, I=1600, $\theta_0 = 1$, and μ_0=0.1.

Figure 11 shows how an increasing marginal cost results in lower optimal effort parameter k^*. This again supports earlier findings , and is a result of the same affects observed in model I.

6 Conclusion

This paper looks at a value maximizing firm considering to invest in R&D. The firm's technology affects its profits, and an investment in R&D can affect the size of the technological innovations available for the firm. The firm can invest in new technology that results from the R&D venture or that is available in the market. The technology level is modeled using a Poisson jump process where the investment in R&D are modeled in two separate ways. The first model lets the firm directly change the jump size in the Poisson jump process by investing in R&D. The second model assumes that an investment in R&D affects the expected jump size in the Poisson jump process, with an exponentially distributed jump size. The paper looks at candidates for both the return on performing R&D, and the expected costs. The paper also looks at the case where the cost of performing R&D is dependent on the gap between the technology in use by the firm, and the available technology in the market.

The paper finds that an optimum only exists when there are diminishing returns on R&D. This is a result of the ever increasing value of the option to wait for new technology with an increasing jump size. This is the case for both models. We also find that an increase in the base level of innovation in the market results in a higher investment level in R&D. This means that our models predicts that innovation will result in more innovation. The models also predict that if the technology gap between the firms current technology level and the one available in the market becomes too high, the firm would reduce its investments in R&D. This assuming an increased cost of performing R&D if the firm has gotten too far behind the current technology level.

Another important finding is the non-monotonic behavior of the firm when the cost of adopting new technology increases. Our model predicts that the firm would increase the investment level in R&D until the investment cost of adoption reaches a certain level. The firm would then reduce the size of the investment in R&D. The paper also find instances where it is not optimal for the firm to invest in R&D. This is the case where the starting technology is high enough to trigger instantaneous investment in the new technology and any level of effort k would reduce the total value.

The paper showed that an optimal effort level k^* exists for a wide range of input parameters, even though we did not find an analytical solution. There are several modifications to the current models that should be investigated. A real life firm would most likely be subject to running costs while performing R&D. This would mean that the cost of conducting R&D would be dependent on the expected time before the R&D venture was successful or abandoned. One could also investigate different distributions for the expected jump size. The exponential distribution will most likely penalize the firm too hard for low efforts put into R&D.

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Appendix A: Modelling R&D by Affecting the Arrival Rate

In the preliminary work we let the investment in R&D affect the arrival rate of the Poisson Jump process. We assumed a similar relationship too that presented in model I:

$$\lambda(k) = k^{\beta} + \lambda_0 \tag{43}$$

And:

$$I_{RD}(k) = ck \tag{44}$$

Resulting in a value function of:

$$F(\theta, k|\zeta_0) = \begin{cases} \left(\frac{\lambda(k)}{\lambda(k)+r}\right)^{\frac{\theta^*-\theta}{u}} (V(\theta^*) - V(\zeta_0) - I) + V(\zeta_0) - I_{RD}(k) & \text{if} \quad \theta < \theta^*(k) - u \\ \frac{\pi(\zeta_0)}{r+\lambda(k)} + \frac{\lambda(k)}{r+\lambda(k)}(V(\theta+u) - I) - I_{RD}(k) & \text{if} \quad \theta^*(k) - u \leq \theta < \theta^*(k) \\ V(\theta) - I - I_{RD}(k) & \text{if} \quad \theta \geq \theta^*(k) \end{cases} \tag{45}$$

The two most interesting results from the preliminary work were:

- The model predicts that firm increases its investment in R&D with an increase in the base level of innovation. Depicted in figure 12.
- The optimal policy for investing in R&D is not monotonic when the investment cost of adopting new technology changes. Depicted in figure 13.

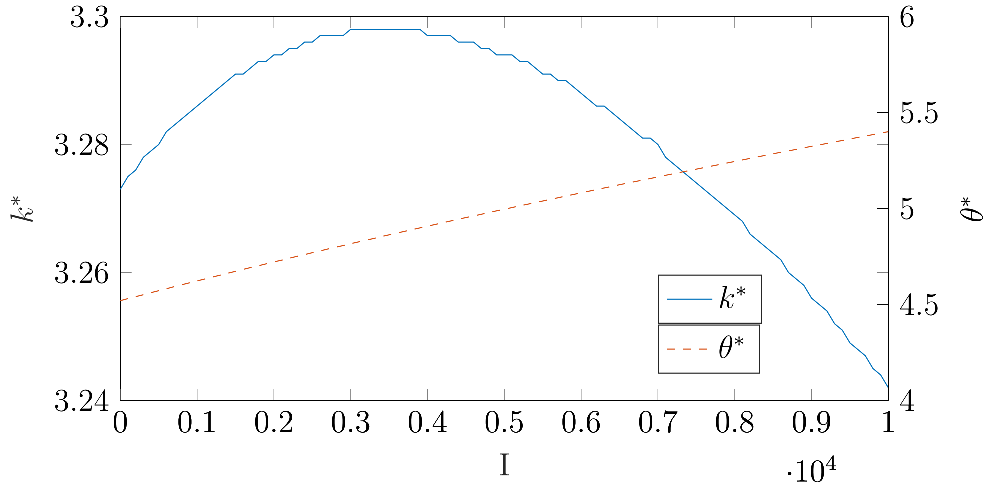

Figure 12: Optimal k and technology threshold for different investment cost I in adoption of new technology. With r = 0.1, u = 0.1, a = 0.5, w = 50, p = 200, λ_0 = 1, ζ_0 = 1, β = 0.1, θ_0 = 1, c = 200.

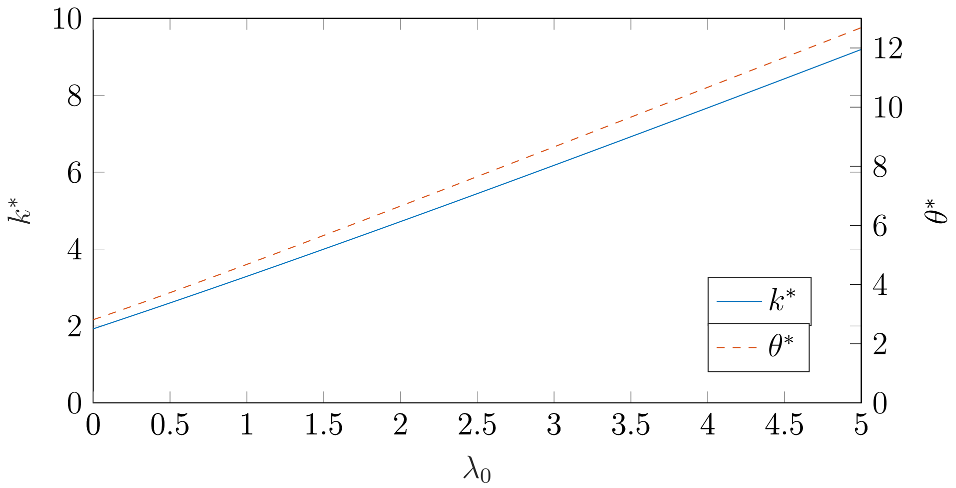

Figure 13: Optimal k and technology threshold for different values of base level of innovation, λ_0. With u = 0.1, r = 0.1, I = 1600, a = 0.5, w = 50, p = 200, θ_0= 1, ζ_0 = 1, β = 0.1, c = 200.

Appendix B: Proof of Expected Adoption Time τ

We know that the number of jumps before we invest in the new technology has to be the difference between the optimal technology level and the starting technology level, divided by the jump size $u(k)$, plus one:

$$n^* = \left\lfloor \frac{\theta^* - \theta}{u(k)} \right\rfloor + 1 \tag{46}$$

The fraction is always rounded down as the number of jumps needed has to be an integer. We know that a jump occurs with a probability of λdt and has a Poisson distribution, we can use this to calculate the expected adoption time τ. The following derivation is equal to the one presented in Huisman (2001), but also include the extensions in our model.

We are deriving the expression:

$$\mathbf{E}[\tau] = \int_{t=0}^{\infty} (1 - P(\tau \leq t))dt. \tag{47}$$

This is the same as the sum of the probability of the n jumps past n^*:

$$\mathbf{E}[\tau] = \int_{t=0}^{\infty} (1 - \sum_{n=n^*}^{\infty} P(N(t) = n))dt. \tag{48}$$

This is equal to:

$$= \int_{t=0}^{\infty} \sum_{n=0}^{n^*-1} P(N(t) = n)dt = \int_{t=0}^{\infty} \sum_{n=0}^{n^*-1} e^{-\lambda t} \frac{(\lambda t)^n}{n!} dt. \tag{49}$$

The Second term is replaced with probability function for the Poisson jump process. The integral is solved by observing that the expression is equal to the density function of the gamma distribution:

$$= \sum_{n=0}^{n^*-1} \frac{1}{\lambda} \int_{t=0}^{\infty} \frac{\lambda^{n+1} t^n}{n!} e^{-\lambda t} dt = \frac{n^*}{\lambda}. \tag{50}$$

Equation (37) show the final expression for the expected time to reach the optimal technology level θ^*. The variance of the expected time can be stated as:

$$Var(\tau) = \mathbf{E}[(\tau)^2] - (\mathbf{E}[\tau])^2. \tag{51}$$

The first term can be found in the same way as the derivation of the expected time to optimal technology level. And is equal to:

$$\mathbf{E}[(\tau)^2] = \frac{n^*(n^*+1)}{\lambda^2} \tag{52}$$

Giving:

$$Var(\tau) = \frac{n^*(n^*+1)}{\lambda^2} - \left(\frac{n^*}{\lambda}\right)^2 = \frac{n^*}{\lambda^2}. \tag{53}$$

Appendix C: Solution to the Differential Equation in Model II

The solution to the differential equation is presented in Huisman (2001), p.38:

$$\begin{aligned} F(\theta|\zeta_0) =& \frac{\pi(\zeta)}{r+\lambda} + \frac{\lambda}{r+\lambda}\int_{u=0}^{\theta^*-\theta} F(\theta+u,\zeta_0)\mu(k)e^{-\mu(k)u}du \\ &+ \frac{\lambda}{r+\lambda}\int_{\theta^*-\theta}^{\infty}(V(\theta+u)-I)\mu(k)e^{-\mu(k)u}du \end{aligned} \tag{54}$$

We then define the derivatives:

$$\frac{\delta F(x)}{\delta x} = f(x) \quad and \quad \frac{\delta G(x)}{\delta x} = g(x) \tag{55}$$

Let us also define:

$$\frac{\delta h(x)}{\delta x} = g(x-\infty)e^{\frac{\lambda}{r+\lambda}x} \tag{56}$$

We can rewrite the differential equation as:

$$\frac{\delta F(x)}{\delta x} = \frac{\pi(\zeta)}{r+\lambda} + \frac{\lambda}{r+\lambda}(F(\theta^*) - F(x) + G(x+\infty) - G(\theta^*)) \tag{57}$$

Differentiation gives:

$$\frac{\delta^2 F(x)}{\delta x^2} = -\frac{\lambda}{r+\lambda}\left(\frac{\delta F(x)}{\delta x} - \frac{\delta G(x+\infty)}{\delta x}\right) \tag{58}$$

With:

$$\frac{\delta F(x)}{\delta x} = \left(\int \frac{\lambda}{r+\lambda}\frac{\delta G(x+\infty)}{\delta x} e^{\frac{\lambda}{r+\lambda}x} dx + c\right) e^{-\frac{\lambda}{r+\lambda}x} \tag{59}$$

c is an unknown constant. Combining the previous equations gives:

$$F(x) = \int \left(\int \frac{\lambda}{r+\lambda}\frac{\delta G(x+\infty)}{\delta x} e^{\frac{\lambda}{r+\lambda}x} dx + c\right) e^{-\frac{\lambda}{r+\lambda}x} dx \tag{60}$$

Equation (56) gives:

$$F(x) = -\left(h(x) + \frac{c}{\frac{\lambda}{r+\lambda}}\right) e^{-\frac{\lambda}{r+\lambda}} + G(x+\infty) \tag{61}$$

Finally we get an expression for the c:

$$c = \left(\frac{\pi(\zeta)}{r+\lambda} + \frac{\lambda}{r+\lambda}\int_{y=0}^{\infty} g(\theta^* + y)dy\right) e^{\frac{\lambda}{r+\lambda}\theta^*} - \frac{\lambda}{r+\lambda}h(\theta^*) \tag{62}$$

The final solution is found when substituting equation (62) into equation (59).