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Rickard's Morita theorem for derived categories

Master's thesis in Mathematical Sciences

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Abstract

We give, in full detail, two different proofs of Rickard's Morita theorem for derived categories. In the first proof we use a modified double chain complex to construct an equivalence between the derived categories directly. After that, we develop the theory of derived categories of differential graded algebras, which we then use to give an alternate proof of the theorem.

Sammendrag

Vi gir, i full detalj, to ulike bevis for Rickards moritateorem for deriverte kategorier. I det første beviset bruker vi et modifisert dobbeltkjedekompleks for å konstruere en ekvivalens mellom de deriverte kategoriene direkte. Deretter utleder vi teorien rundt deriverte kategorier for differensielt graderte algebraer, som vi så bruker for å gi et alternativt bevis for teoremet.

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Contents

Introduction

After its introduction in 1958, the concept of Morita equivalence proved to be a powerful addition to the ring theory toolbox. Following huge developments in the theory of derived categories during the seventies and eighties, mathematicians wondered if there existed a similar result for derived categories. That is, one that related the derived categories of rings in the same way Morita equivalence related their module categories. This question was settled by Jeremy Rickard in [Ric89], who used the theory of tilting complexes to give a condition for when the derived categories of two rings are equivalent. Bernhard Keller later gave an alternate proof of the theorem, using the theory of differential graded algebras.

Motivation

In the study of rings, many of the properties we are interested in for a given ring are determined by its module category. Thus, if two rings have equivalent module categories, we can learn things about one by studying the other, which might be easier to work with. This is part of the reason why Morita equivalence is so important, because it gives a necessary and sufficient condition for when two rings have equivalent module categories. It even gives an explicit description of the functors which produce the equivalence.

In a similar way, rings whose derived categories are equivalent share some properties, so being able to determine when we have such equivalence could be helpful. Rickard's theorem states that the following is a necessary and sufficient condition for two rings to have equivalent derived categories: That there exists a tilting module over one of the rings, such that the other ring is isomorphic to the endomorphism ring of that tilting module.

It is easy to show that equivalent derived categories implies the existence of a tilting module, so the hard part is showing the other direction. In his original proof, Rickard did it by constructing the equivalence directly. For a given complex in the derived category of one ring, he used the tilting complex to make a corresponding double complex of sorts, and by forming the total complex he got a complex in the derived category of the other ring. Here are two reasons why this is a good approach:

- It doesn't require any additional theory. If you know enoguh homological algebra to understand the statement of the theorem, you don't need any more theory to be able to understand the proof.

- Every step in the proof is clearly motivated. The idea of the proof is basically to try a natural way of constructing an equivalence between the two derived categories, seeing where it breaks down, and then modifying the construction until it does work.

One problem with this approach is that the construction of the functor is very specialized for this particular situation. Thus, the construction isn't really that useful in a wider mathematical context. This is the reason Keller's alternate proof is interesting, because it employs a much more general technique to prove the theorem.

In his proof, Keller uses the theory of unbounded derived categories to show that a slightly altered version of Rickard's theorem holds if we have a complex of bimodules over both rings. He then introduces the concept of differential graded algebras, shows how to obtain derived categories in the differential graded case, and defines total derived functors between those categories. Finally, he uses these tools to construct a bimodule complex, allowing him to apply the previous result to prove Rickard's theorem.

Overview

The goal of this thesis is to present two different proofs of Rickard's theorem for derived Morita equivalence. In **chapter 1** we define tilting complexes and state Rickard's theorem. The first proof is given in **chapter 2**, and it is based on [Kö98]. **Chapter 3** is used to define differental graded algebras, as well as other related structures (like differential graded modules, and the their homotopy category). We then use these constructions in **chapter 4** to give the other proof of Rickard's theorem, which follows [Kel98]. **Chapter 5** contains an example meant to illustrate how Rickard's theorem can be used in practice. Finally, the appendix contains some results we use, but whose proofs we didn't want to include in the main text.

Notation and conventions

Throughout this thesis, Λ and Γ are rings (associative, with 1). The category of (left) Λ-modules is denoted $Mod - \Lambda$, with $Proj - \Lambda$ denoting the subcategory of projective Λ-modules. Lowercase first letter, i.e. $mod - \Lambda$ and $proj - \Lambda$, indicates their respective subcategories of finitely generated objects. The category of free Γ-modules is denoted $free - \Gamma$. If T is a complex, then $Sum - T$ denotes the category of direct sums of copies of T, and $Add - T$ is the category of arbitrary direct sums of direct summands of T (finite direct sums give $add - T$). If we let $\mathcal{C}$ be an abelian category, then $\boldsymbol{C}(\mathcal{C})$ is the category of unbounded chain complexes of objects in $\mathcal{C}$, and $\boldsymbol{K}(\mathcal{C})$ is the homotopy category of $\mathcal{C}$. In particular, we are interested in the following subcategories of the homotopy category of Λ-modules (denoted $\mathbf{K}(\Lambda)$ for simplicity):

- $\boldsymbol{K}(Proj - \Lambda)$ - unbounded complexes of projective Λ-modules.
- $\mathbf{D}(\Lambda)$ - the unbounded derived category of $Mod - \Lambda$.

- $\mathbf{K}^{-}(Proj - \Lambda)$ - right bounded complexes of projective Λ-modules, up to homotopy.
- $\mathbf{D}^{b}(\Lambda)$ - bounded complexes in the derived category, that is, complexes with bounded homology.
- $\mathbf{K}^{b}(Proj - \Lambda)$ - bounded complexes of projective Λ-modules, up to homotopy.
- $\operatorname{per} \Lambda$ - the category of perfect complexes, meaning bounded complexes of finitely generated Λ-modules, up to homotopy. Could also be written as $\boldsymbol{K}^{b}(proj - \Lambda)$.

Each of these categories is a full subcategory of the category above it, and showing this is straight forward. The proof in chapter 2 relies heavily on double complexes and similar constructions, and throughout we will use the convention that our double complexes are defined with anticommutative squares, instead of commutative squares. The reason for this is that it removes the alternating sign from the differential we get when forming the total complex, which will greatly simplify our constructions.

The intended reader

This thesis is written to be understandable to someone who has completed an introductory course in homological algebra, to the point where they are comfortable with triangulated and derived categories, left and right derived functors, double complexes and total complexes, and the homotopy category of a module category. They should also be familiar with some basic notions of category theory, such as (co)limits, (co)products and adjoint functors. Any necessary theory beyond this will be properly introduced. No prior knowledge of differential graded algebras is required, nor any familiarity with tilting complexes.

Chapter 1

Rickard's theorem for derived Morita equivalence

Before we state Rickard's celebrated theorem, we will present a prior theorem due to Happel. This serves as motivation, because Happel's theorem relies on tilting modules, and Rickard's theorem (which is a generalization of Happel's theorem) relies on tilting complexes (which is a generalization of tilting modules).

1.1 Tilting modules and a prior theorem

A few years before Rickard published his proof, Happel [Hap87] presented a theorem which was essentially a special case of Rickard's derived Morita theorem. He showed that derived equivalence of finite dimensional algebras could be determined by looking at the endomorphism ring of a so-called tilting module over one of the algebras.

Definition 1.1.1. Let Λ be a finite dimensional algebra. A *tilting module* T over Λ is a finitely generated left Λ-module which satisfies the following conditions:

1. The projective dimension of T is zero or one.
2. T has no self-extensions, meaning $\mathrm{Ext}^i_\Lambda(T,T) = 0$ for $i \neq 0$.
3. There is a natural number m such that there exists an exact sequence $0 \to \Lambda \to T_1 \to T_2 \to \cdots \to T_m \to 0$, where each T_i is a direct summand of a finite direct sum of copies of T.

This definition allows us to state the previously mentioned theorem properly. We will omit the proof here.

Theorem 1.1.2. *Let Λ and Γ be two finitely generated algebras, and let T be a tilting module over Λ. If $\Gamma \simeq End_\Lambda(T)$, then $\boldsymbol{D}^b(\Lambda)$ and $\boldsymbol{D}^b(\Gamma)$ are equivalent as triangulated categories.*

Proof. See [Hap87]. □

The main result in Rickard's theorem is very similar to theorem 1.1.2, with one important difference. Instead of relying on the existence of a tilting module, it requires the existence of a tilting complex, which is a more general kind of object. As a result of this, Rickard's theorem gives a necessary *and* sufficient condition for when two rings are derived equivalent (also notice that we don't require them to be finite dimensional).

1.2 Tilting complexes and Rickard's theorem

We will now give the definition of a tilting complex, and then we will state Rickard's theorem. As mentioned above, tilting complexes are a generalization of tilting modules.

Definition 1.2.1. For a given ring Λ, a *tilting complex* T over Λ is an object in $\operatorname{per}\Lambda$ (bounded complexes of finitely generated modules up to homotopy) which satisfies the following conditions:

1. for all $i \neq 0$, the set $\operatorname{Hom}_{\mathbf{D}^b(\Lambda)}(T, T[i])$ of homomorphisms in $\mathbf{D}^b(\Lambda)$ vanishes,

2. the category $add(T)$ generates $\operatorname{per}\Lambda$ as a triangulated category. That is, the smallest full triangulated subcategory of $\operatorname{per}\Lambda$ which contains $add(T)$ and is closed under extensions, is the whole of $\operatorname{per}\Lambda$.

A complex $T \in \operatorname{per}\Lambda$ satisfying condition (1), but not necessarily (2), is sometimes called a partial tilting complex.

Rickard's theorem for derived Morita equivalence

Theorem 1.2.2 (Rickard)**.** *Let Λ and Γ be two rings. Then the following conditions are pairwise equivalent:*

1. *the triangulated categories $\boldsymbol{K}^-(Proj-\Lambda)$ and $\boldsymbol{K}^-(Proj-\Gamma)$ are equivalent*

2. *the triangulated categories $\boldsymbol{D}^b(\Lambda)$ and $\boldsymbol{D}^b(\Gamma)$ are equivalent*

3. *the triangulated categories $\boldsymbol{K}^b(Proj-\Lambda)$ and $\boldsymbol{K}^b(Proj-\Gamma)$ are equivalent*

4. *the triangulated categories $per\,\Lambda$ and $per\,\Gamma$ are equivalent*

5. *there exists a tilting complex* T *over* Λ *such that* $\Gamma \simeq End_{\boldsymbol{D}^b(\Lambda)}(T)^{op}$, *the endomorphism ring of* T.

Throughout this thesis, whenever we refer to *Rickard's theorem*, or somehting to that effect, it is this theorem we are refering to. The proof of this theorem is given in the next chapter, with an alternate proof being presented in chapter 4.

Chapter 2

Proof using a modified double complex

This chapter follows the proof given in Steffen König's paper *Rickard's Fundamental Theorem* [Kö98], which is based on [Ric89]. Compared to the other proof we will present, this is more conceptually straight forward, but the method is not as useful as a general tool.

2.1 Strategy of the proof

Most of the work in this proof goes into showing that the existence of a tilting complex over Λ with endomorphism ring isomorphic to Γ implies that we have an equivalence $\mathbf{K}^{-}(Proj-\Lambda) \simeq \mathbf{K}^{-}(Proj-\Gamma)$. To construct this equivalence, we will use a sort of modified double complex. We then show that such an equivalence restricts nicely down to each of the subcategories, and finally, that an equivalence between per Λ and per Γ implies the existence of the desired tilting complex.

Lemma 2.1.1. *1. The functor* $Hom(T,-): \boldsymbol{K}^{-}(Proj-\Lambda) \to Mod-\Gamma$ *restricts to an equivalence* $Sum-T \to Free-\Gamma$.

2. The inclusion $\boldsymbol{K}^{-}(Free-\Gamma) \to \boldsymbol{K}^{-}(Proj-\Gamma)$ *is an equivalence of triangulated categories.*

Proof. 1) By assumption, we have that $\Gamma \simeq \mathrm{Hom}(T,T)$. From this we get that $\bigoplus_{i\in I} \mathrm{Hom}(T,T) \simeq \bigoplus_{i\in I} \Gamma \in Free-\Gamma$, which means that we can view elements of $Free-\Gamma$ as direct sums of copies of $\mathrm{Hom}(T,T)$. There is a natural homomorphsim $\bigoplus_{i\in I} \mathrm{Hom}(T,T) \to \mathrm{Hom}(T, \bigoplus_{i\in I} T)$, given by sending a tuple of endomorphisms $(\varphi_i)_{i\in I}$ to the map $\Big(t \mapsto \big(\varphi_i(t)\big)_{i\in I}\Big)$. Observe that if any φ_i is nonzero, then $(\varphi_i)_{i\in I}$ will not be sent to zero in $\mathrm{Hom}(T, \bigoplus_{i\in I} T)$, so the homomorphism is injective. To see that it is in fact an isomorphism, notice that since T is a bounded complex of finitely generated Λ-modules, all elements of $\mathrm{Hom}(T, \bigoplus_{i\in I} T)$ are fixed by the values they take on a finite number of elements of the terms of T. This means that each element of $\mathrm{Hom}(T, \bigoplus_{i\in I} T)$ is the image of some element of $\bigoplus_{i\in I} \mathrm{Hom}(T, T)$, which shows surjectivity. Thus, we have an isomorphism $\mathrm{Hom}(T, \bigoplus_{i\in I} T) \simeq \bigoplus_{i\in I} \mathrm{Hom}(T, T) \simeq \bigoplus_{i\in I} \Gamma$, which shows that $\mathrm{Hom}(T, -)$ gives an equivalence $Sum - T \to Free - \Gamma$.

2) For any complex in $\mathbf{K}^-(Proj - \Gamma)$ we want to find an isomorphic complex in $\boldsymbol{K}^-(Free - \Gamma)$. Let $X \in \mathbf{K}^-(Proj - \Gamma)$ be the following complex, where i is the highest nonzero degree

$$\cdots \xrightarrow{d} P^{i-2} \xrightarrow{d} P^{i-1} \xrightarrow{d} P^i \xrightarrow{0} 0$$

Since P^i is projective, we know that there exists a $Q^i \in Proj - \Gamma$ such that $P^i \bigoplus Q^i \simeq \Gamma^{n_i}$ for some $n_i \in \mathbb{N}$. By theorem A.1.1, we can now add the trivial direct summand $Q^i \xrightarrow{1} Q^i$ to the complex X, and get the isomorphic complex

$$\cdots \xrightarrow{d} P^{i-2} \xrightarrow{\binom{d}{0}} P^{i-1} \bigoplus Q^i \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)} \underset{\Gamma^{n_i}}{\overset{P^i \bigoplus Q^i}{\|}} \xrightarrow{0} 0$$

Now, since $P^{i-1} \bigoplus Q^i$ is projective, we can again find a $Q^{i-1} \in Proj - \Gamma$ such that $(P^{i-1} \bigoplus Q^i) \bigoplus Q^{i-1} \simeq \Gamma^{n_{i-1}}$ for some $n_{i-1} \in \mathbb{N}$. Then we add the trivial direct summand $Q^{i-1} \xrightarrow{1} Q^{i-1}$ to get the isomorphic complex

$$\cdots \xrightarrow{d} P^{i-2} \bigoplus Q^{i-1} \xrightarrow{\binom{d}{0}} \underset{\Gamma^{n_{i-1}}}{\overset{P^{i-1} \bigoplus Q^i \bigoplus Q^{i-1}}{\|}} \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)} \Gamma^{n_i} \xrightarrow{0} 0$$

Because all terms in the complex X are projective, we can continue this process indefinitely to the left, and by induction we get a complex that is isomorphic to X, where all the terms are free Γ-modules. This shows that $\boldsymbol{K}^-(Free - \Gamma)$ and $\mathbf{K}^-(Proj - \Gamma)$ are equivalent as triangulated categories. □

By combining the two statements of lemma 2.1.1, we get an equivalence between $\mathbf{K}^-(Proj - \Gamma)$ and $\mathbf{K}^-(Sum - T)$. Our current goal is to construct a functor from $\mathbf{K}^-(Proj - \Gamma)$ to $\mathbf{K}^-(Proj - \Lambda)$, so the next step will be to find a functor $\mathbf{K}^-(Sum-T) \to \mathbf{K}^-(Proj-\Lambda)$. Notice that, since $T \in \operatorname{per}\Lambda$, the objects of $Sum-T$ are built up of objects from $\operatorname{per}\Lambda \subset \mathbf{K}^-(Proj-\Lambda)$. This means that the objects of $\mathbf{K}^-(Sum-T)$ can be viewed as double complexes over Λ, that is, complexes of complexes Λ-modules. This gives us an idea for how to create functors $\mathbf{K}^-(Sum-T) \leftrightarrow \mathbf{K}^-(Proj-\Lambda)$.

($\to$): Let $X \in \mathbf{K}^-(Sum-T)$, so X is a complex with a direct sum of copies of T in each degree. From this X we must construct a complex over Λ. A natural candidate would be to view X as a double complex over Λ, and form the total complex. But there is a problem with this solution. The differential in the total complex is defined using both the differential d in T and the differential δ in X (viewed as a complex over $Sum-T$), and relies on both of them squaring to zero. But since X is in the homotopy category, we generally only have that δ^2 is homotopic to zero, not equal. This means that also the differential in the double complex will square to something homotopic to zero, rather than to zero. So we need to modify our double complex in such a way that we are able to form the total complex. More precisely, we will define a construction which moves the error of our differential to maps of higher and higher degrees. Because T is bounded, this error term will eventually be zero. In order to do this, we will need the assumption that $\operatorname{Hom}_{\mathbf{D}^b(\Lambda)}(T, T[i]) = 0$ for $i \neq 0$, meaning that T has no self-extensions. This will be used in the construction of the functor we call F.

($\leftarrow$): Given a complex L over Λ, we must find a corresponding complex in $\mathbf{K}^-(Sum-T)$. As we have seen, complexes over $Sum-T$ are double complexes over Λ (up to homotopy). So our goal is to create a '$Sum-T$-resolution' of the complex L in $\mathbf{K}^-(Proj-\Lambda)$, a complex in $\mathbf{K}^-(Sum-T)$ which is homotopy equivalent to L. From this, we can define a functor G, which is right adjoint to F. In order to prove that F and G are mutually inverse equivalences, we will need the assumption that $add(T)$ generates $\operatorname{per}\Lambda$.

2.2 Construction of F

Our aim now is to construct a functor $F\colon \mathbf{K}^-(Proj-\Gamma) \to \mathbf{K}^-(Proj-\Lambda)$. Since we have already shown that $\mathbf{K}^-(Proj-\Gamma) \simeq \mathbf{K}^-(Sum-T)$, we will focus on finding a functor $\mathbf{K}^-(Sum-T) \to \mathbf{K}^-(Proj-\Lambda)$. Like we said above, it doesn't work to simply form the total complex of $X \in \mathbf{K}^-(Sum-T)$ as a double complex over Λ. Since the square of the differential in $\mathbf{K}^-(Sum-T)$ is not equal (only homotopic) to zero, it's not even a double complex. So what we will do is to modify the construction to get something similar to a double complex, for which we can actually form the total complex. The following example shows the problem we run into when trying to form the double complex with no modifications, and it illustrates how we will try to solve it.

Example 2.2.1. Let Q be the quiver

$$\begin{array}{ccc} a & \xrightarrow{\ \alpha\ } & b \\ \big\downarrow{\scriptstyle\gamma} & & \big\downarrow{\scriptstyle\beta} \\ c & \xrightarrow{\ \delta\ } & d, \end{array}$$

and let $C = kQ/(\beta\alpha - \delta\gamma)$, the algebra corresponding to a commutative square. We want to find a tilting complex T over C, and then look at a complex made of shifted copies of T. We start by finding the indecomposable projective left modules of C, or more precisely, their composition series

$$P_a = \begin{smallmatrix} & a & \\ b & & c \\ & d & \end{smallmatrix},\ P_b = \begin{smallmatrix} b \\ d \end{smallmatrix},\ P_c = \begin{smallmatrix} c \\ d \end{smallmatrix},\ P_d = d.$$

We notice that the simple module S_a has no self-extensions, which means it is a partial tilting complex. Since this is preserved by quasi-isomorphisms, we can instead look at a projective resolution of S_a, for example:

$$0 \to P_d \xrightarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} P_b \oplus P_c \xrightarrow{(1\ 1)} P_a \to 0.$$

Lets call this complex T. If we now look at an example of a complex in $\mathbf{K}^-(Sum - T)$, and try to form the total complex of it, we will see where the problem arises. Let $T_1 = T$, $T_2 = T[1]$ and $T_3 = T[2]$, and take the complex $0 \to T_1 \to T_2 \to T_3 \to 0$, given by:

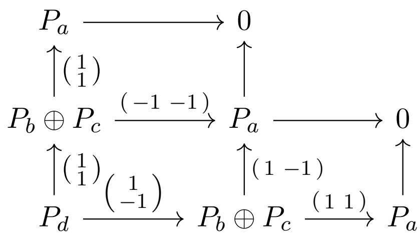
$$\begin{array}{lccccccccc}
T_3: & P_d & \xrightarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} & P_b \oplus P_c & \xrightarrow{(1\ 1)} & P_a & \longrightarrow & 0 & & \\
\big\uparrow & \big\uparrow & & \big\uparrow{\scriptstyle\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} & & \big\uparrow{\scriptstyle(1\ 1)} & & \big\uparrow & & \\
T_2: & 0 & \longrightarrow & P_d & \xrightarrow{\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)} & P_b \oplus P_c & \xrightarrow{(-1\ -1)} & P_a & \longrightarrow & 0 \\
\big\uparrow & & & \big\uparrow & & \big\uparrow{\scriptstyle\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} & & \big\uparrow{\scriptstyle(1\ -1)} & & \big\uparrow \\
T_1: & & & 0 & \longrightarrow & P_d & \xrightarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} & P_b \oplus P_c & \xrightarrow{(1\ 1)} & P_a
\end{array}$$

Notice that all of the squares in this diagram anticommute (which is how we define double complexes), and that the two vertical maps $T_1 \to T_2$ and $T_2 \to T_3$ only differ in signs. So in order to see that this actually is a complex in $\mathbf{K}^-(Sum - T)$, we must show that the differential squared is null-homotopic. The composition of two vertical maps is obviously zero everywhere, except for the composition

$$P_d \xrightarrow{(1,1)} P_b \oplus P_c \xrightarrow{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} P_a,$$

which goes from T_1 to T_3. To see that this composition is null-homotopic, we simply observe that it factors through $P_b \oplus P_c \to P_a$ in T_3 by the map $(1, 1)$. So $T_1 \to T_2 \to T_3$ is in fact in $\mathbf{K}^-(Sum - T)$.

Now let's try to form the total complex of this system, and see how its differential d behaves. Recall that the components of the total complex are given by taking the direct sum of diagonals in the double complex. If we call the horizontal and vertical differentials in the double complex d_0 and d_1 respectively, then $d_0 + d_1$ is the differential in the total complex.[1] We now look at the bottom copy of P_d, which is a direct summand in degree 0 of the total complex. Starting in this copy of P_d, the differential squared gives the following diagram



We see that there are 4 compositions $P_d \to P_a$, so we can write d^2 as the sum $(d_0 + d_1)^2 = d_0^2 + d_1 d_0 + d_0 d_1 + d_1^2$. The first term is 0, because d_0 is actually a differential. The second and third term cancel each other by anticommutativity. Thus, we are left with composition d_1^2, which we clearly see is not zero. What we have shown is that d_1^2 is homotopic to zero, in other words that there exists a map $h\colon P_d \oplus (P_b \oplus P_c) \to (P_b \oplus P_c) \oplus P_a$ of degree $(-1, 2)$ such that $-d^2 = hd_0 + d_0 h$. We rename h to d_2, and try to see if redefining the differential to $d = d_0 + d_1 + d_2$ will fix our problem. In this case, we get that $d^2 = (d_0 + d_1 + d_2)^2 = d_0^2 + d_0 d_1 + d_1 d_0 + d_1^2 + d_0 d_2 + d_2 d_0$. As before, the first term is zero, and the second and third cancel each other. By construction, d_1^2 is canceled by $d_0 d_2 + d_2 d_0 = -d_1^2$. Note that we in general could have nonzero terms $d_1 d_2$, $d_2 d_1$ and d_2^2, but they are maps of vertical degree 3 and 4, which in this example means that any nonzero component in the diagram will be sent to zero by them. Thus, they are all zero in d^2.

Adding this up, we get that $d^2 = 0$ and hence d can work as a differential. So we can form a sort of 'modified total complex' of $T_1 \to T_2 \to T_3$, where the terms are the diagonals as usual, but with $d = d_0 + d_1 + d_2$ as the differential, instead of $d_0 + d_1$. The reason this works in our case is that the complex in $\mathbf{K}^-(Sum - T)$ we are looking at only has three nonzero terms, so the terms $d_1 d_2$, $d_2 d_1$ and d_2^2 are actually zero. In general, however, that may not be the case.

We see in this example what the problem with the differential is. When we define the differential as a sum of maps, and square it, we end up with some nonzero terms. We try to fix this by adding 'correction maps' (given by

[1] This is why we use the anticommutative definition of double complexes. When double complexes are defined with commutative squares, the differential in degree l of the total complex is $d_0 + (-1)^l d_1$, which would be harder to work with in our case.

homotopy) to the differential, to ensure that the nonzero maps are cancelled out in d^2. This pushes the problem into another degree (higher vertical degree and lower horizontal degree), where we generally can't say which maps become zero. However, we will now see that because T is assumed to be a tilting complex, this method actually ends up working.

First of all, since $\mathrm{Hom}_{\mathbf{D}^b(\Lambda)}(T, T[i]) = 0$ for all $i \neq 0$, we have that all maps $T \to T[i]$ are null-homotopic. This means that no matter what the degree of the maps are, we can always find homotopy maps to cancel them out with. Secondly, since T is a bounded complex, shifting it far enough to the left or right will ensure that no nonzero degrees of the shifted complex will overlap with the nonzero degrees of T. This means that for n large enough, any map $T \to T[i]$ will be zero for all $i > n$. Which in turn ensures that the method outlined above will terminate at some point. Think of it like this: we iteratively find correction maps to deal with the terms that don't go to zero, which pushes the problem 'up and to the left' in the diagram. This corresponds to finding maps $T \to T[i]$ for increasing values of i. Eventually, it will be pushed so far to the left that the map must be zero. Then all terms in the differential squared are either zero or they will cancel each other out by construction, and we are done.

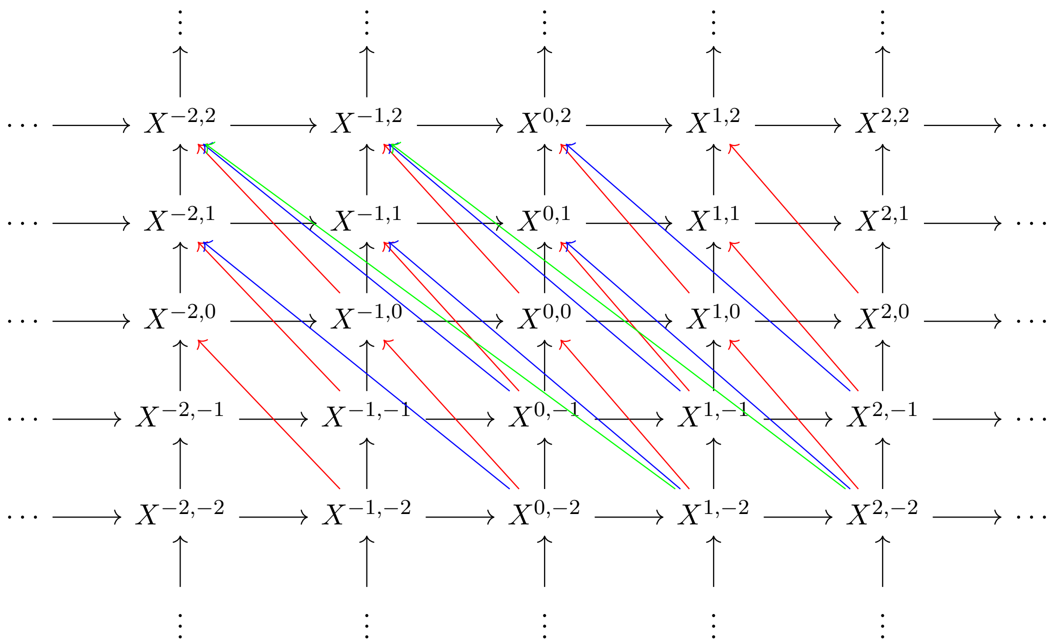
We will now formalize the idea. To do this, we create a new category, which will be a modification of the category of double complexes over Λ. This is essentially done by adding all the correction maps to each complex. After defining the category, we will show how we can embed $\boldsymbol{C}^{-}Sum - T$ into it, and how it works with taking total complexes. Combining the embedding with taking the total complex gives a functor $\boldsymbol{C}^{-}Sum - T \to \mathbf{K}^{-}(Proj{-}\Lambda)$, which we will show factors through $\mathbf{K}^{-}(Sum - T)$. This induced functor will be our F.

Before we define the intermideate category, let's look closer at the structure of objects in $\boldsymbol{C}^{-}(Sum - T)$. A complex $X \in \boldsymbol{C}^{-}(Sum - T)$ is a double complex, that is, a complex of complexes over $\operatorname{per}\Lambda$. This means that X is graded in two directions. It is graded as a complex over $Sum - T$, and each degree of that has the grading given by T as a complex. So we have two sets of differentials, one with degree $(1, 0)$ and one with degree $(0, 1)$. Let's call the differential in T for d, and say that d has degree $(1, 0)$. Then the differential in X as a complex in $\boldsymbol{C}^{-}(Sum - T)$, which we'll call δ, has degree $(0, 1)$. Notice that since T really is a complex, we have that $d^2 = 0$. On the other hand, δ is given by maps between shifted copies of T, which are only defined up to homotopy (since T is a tilting complex). This means that δ^2 is not equal to zero (although it is null-homotopic). Finally, morphisms between complexes in $\boldsymbol{C}^{-}(Sum - T)$ are maps of degree zero with respect to the grading of $\boldsymbol{C}^{-}(Sum - T)$, which corresponds to maps of degree $(0, 0)$ in the associated bigraded objects. We are now ready to modify this construction and create the category we need, which we will call $G(\Lambda)$.

Definition 2.2.2. The category $G(\Lambda)$ is defined as follows

- The objects of $G(\Lambda)$ are bigraded projective Λ-modules $X^{*,*}$, together with a family $(d_i)_{i\in\mathbb{N}}$ of graded endomorphisms of degree $(1-i,i)$ such that $\sum_{0\leq i\leq n} d_i d_{n-1} = 0$ for each $n \in \mathbb{N}$.
- A morphism from $X^{*,*}$ to $Y^{*,*}$ in $G(\Lambda)$ is a family $(\alpha_i)_{i\in\mathbb{N}}$ of maps of degree $(-i,i)$ such that for each $n \in \mathbb{N}$, we have the equality $\sum_{0\leq i\leq n} \alpha_i d_{n-1} = \sum_{0\leq i\leq n} d_i \alpha_{n-1}$.

The following is meant to help with visualizing what the definition is saying. An object $X \in G(\Lambda)$ is on this form



Here, the horizontal maps belong to d_0, the vertical maps belong to d_1, the red maps to d_2, the blue maps to d_3, the green maps to d_4, and this pattern continues. Be aware that maps of all degrees $(1-i,i)$ exist for all terms, even though they are not shown above. The diagram is just meant to help with visualization. Now let's look at the condition $\sum_{0\leq i\leq n} d_i d_{n-1} = 0$ for all $n \in \mathbb{N}$. Written out for each n, this becomes

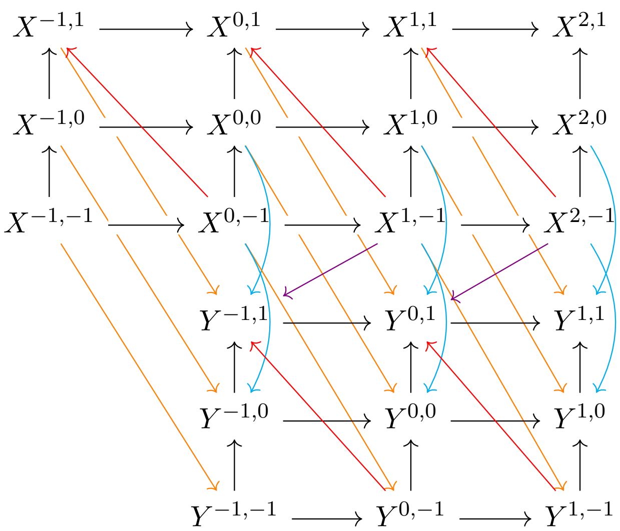
$$\begin{aligned}
d_0 d_0 &= 0 && \text{degree: } (2,0)\\
d_0 d_1 + d_1 d_0 &= 0 && \text{degree: } (1,1)\\
d_0 d_2 + d_1 d_1 + d_2 d_0 &= 0 && \text{degree: } (0,2)\\
d_0 d_3 + d_1 d_2 + d_2 d_1 + d_3 d_0 &= 0 && \text{degree: } (-1,3)\\
&\vdots
\end{aligned}$$

All possible compositions of two arrows appear in this sum. Notice that if we start in the component $X^{i,j}$ and apply any composition of exactly two maps, we end up somewhere on the diagonal with total degree $i + j + 2$. In other words, we end in a component on the form $X^{i+2-k,j+k}$, for some $k \in \mathbb{N}$. Each of the sums above contain all compositions of two maps of the given combined degree. So the statement is essentially that if we take the sum of all possible paths from one component to a given component on the diagonal which is two steps above it, we get zero.

Now let's look at morphisms in $G(\Lambda)$, and try to wrap our minds around how they work. Again, a morphism from $X^{*,*}$ to $Y^{*,*}$ in $G(\Lambda)$ is given by a family of maps $(\alpha_i)_{i\in\mathbb{N}}$ of degree $(-i,i)$, such that $\sum_{0\le i\le n} \alpha_i d_{n-1} = \sum_{0\le i\le n} d_i \alpha_{n-1}$ for all $n \in \mathbb{N}$. Writing this out for each n, we get

$$
\begin{aligned}
\alpha_0 d_0 &= d_0 \alpha_0 && \text{degree: } (1,0)\\
\alpha_0 d_1 + \alpha_1 d_0 &= d_0 \alpha_1 + d_1 \alpha_0 && \text{degree: } (0,1)\\
\alpha_0 d_2 + \alpha_1 d_1 + \alpha_2 d_0 &= d_0 \alpha_2 + d_1 \alpha_1 + d_2 \alpha_0 && \text{degree: } (-1,2)\\
&\vdots
\end{aligned}
$$

To aid in visualizing what this means, here is a (sligthly horrifying) partial diagram. It shows the maps from a (3×4)-section of $X^{*,*}$ to a (3×3)-section of $Y^{*,*}$. In reality, there would be way more arrows both into and out of each component, but we tried to keep it simple. The black and red arrows are as above, and the orange, cyan and violet arrows are α_0, α_1 and α_2, respectively.



In the context of this diagram, the above equalities basically say this: When you go from a component of $X^{*,*}$ to one of $Y^{*,*}$, following arrows in $X^{*,*}$ and then passing to $Y^{*,*}$ is the same as first passing to $Y*,*$ and follow the arrows there. More precisely, this is true if you take the sum of all possible such paths between the given components. For example, the first equality says that following a horizontal arrow and then an orange arrow in the diagram, is the same as first following an orange arrow and then a horizontal one. The second equality says that (horizontal then blue)+(vertical then orange) is the same as (orange then vertical)+(blue then horizontal). Likewise, the third equality gives that (red then orange)+(vertical then blue)+(horizontal then violet) is equal to (orange then red)+(blue then vertical)+(purple then horizontal), and this pattern continues.

Observation 2.2.3. We can form the total complex of an object in $G(\Lambda)$, and in this case, $\sum_i d_i$ really is a differential. To see that $(\sum_i d_i)^2 = 0$, we observe that $(\sum_i d_i)^2$ is the sum of all possible combinations $d_i d_j$ (where $i = j$ is allowed). If we sort these into groups where $i + j$ is constant, the construction of $G(\Lambda)$ ensures that each of the groups will be zero, so the entire sum is zero.

$$(\sum_{i\in\mathbb{N}} d_i)^2 = \overbrace{d_0 d_0}^{=0} + \overbrace{d_0 d_1 + d_1 d_0}^{=0} + \overbrace{d_0 d_2 + d_1 d_1 + d_2 d_0}^{=0} + \cdots = 0$$

Now that we (hopefully) have some idea of how the category $G(\Lambda)$ works, our next goal is to find an embedding $\boldsymbol{C}^-(Sum - T) \hookrightarrow G(\Lambda)$. Remember that we are trying to construct a functor $\mathbf{K}^-(Sum - T) \to \mathbf{K}^-(Proj - \Gamma)$ by passing through $G(\Lambda)$. Embedding $\boldsymbol{C}^-(Sum - T)$ into $G(\Lambda)$ is one step in this construction, since we already have a functor from $G(\Lambda)$ to $\mathbf{K}^-(Proj - \Lambda)$ (which is to form the total complex). Given an object X in $\boldsymbol{C}^-(\Lambda)$, we need to find a corresponding object in $G(\Lambda)$, and the same is true for morphisms. Notice that X is already bigraded as a Λ-module, it just doesn't have all the maps d_i. This means that we can send X to the object $X^{*,*}$ in $G(\Lambda)$, given that we find maps that function as the d_i's. What we do have, is the horizontal map $d\colon X^{i,j} \to X^{i+1,j}$ and the vertical map $\delta\colon X^{i,j} \to X^{i,j+1}$. We set $d_0 = d$ and $d_1 = (-1)^{i+j}\delta$. The rest of the d_i's correspond to the correction maps from the example, and we define them inductively. To do so we'll need the following lemma:

Lemma 2.2.4. *Let X and Y be objects in $\boldsymbol{C}^-(Sum - T)$, regarded as bigraded objects as above. For any graded map $\alpha\colon X \to Y$ of degree (p, q), where $p \neq 0$ and α commutes with d, there exists a graded map h of degree $(p - 1, q)$, such that $\alpha = d_0 h + h d_0$.*

Proof. If we fix the second degree, say we set it equal to i, then α gives a graded map of degree p from one sum of copies of T to another. In each summand, this is the same as a map of chain complexes from T to $T[p]$. Since T is assumed to be a tilting complex, we know that $\mathrm{Hom}_{\mathbf{D}^b(\Lambda)}(T, T[p]) = 0$

for all $p \neq 0$, which is equivalent to all chain maps $T \to T[p]$ being null-homotopic. This means we can find a homotopy map $T \to T[p]$ of degree -1, which gives the map h that we are looking for. $\square$

Now we are ready to construct the d_i's. The base case for the induction is covered by observing that $d_1^2 = d\delta - \delta d = 0$ by commutativity of the squares. Let's assume that for all $i \in \{0, \ldots, k\}$, we have d_i such that $\sum_{0 \leq i \leq k} d_i d_{k-i} = 0$. What we want is to define a map d_{k+1} such that $\sum_{0 \leq i \leq k+1} d_i d_{k+1-i} = 0$. Notice that this sum can be written as

$$\sum_{i=0}^{k+1} d_i d_{k+1-i} = d_0 d_{k+1} + d_{k+1} d_0 + \sum_{i=1}^{k} d_i d_{k+1-i}.$$

We now define a map α to be equal to the negative of the last term above, that is

$$\alpha = -\sum_{i=1}^{k} d_i d_{k+1-i}.$$

Our goal now is to apply lemma 2.2.4 to this map α. All of the terms in the sum are maps of degree $(1-k, k+1)$, so for $k \geq 2$ the degree condition is satisfied. Thus, for α to satisfy the lemma we must show that it commutes with d_0. We calculate $d_0\alpha$ and αd_0, in order to show that they are equal:

$$\begin{aligned} d_0\alpha = d_0\Big(-\sum_{i=1}^{k} d_i d_{k+1-i}\Big) &= -\sum_{i=1}^{k} d_0 d_i d_{k+1-i} \\ &= -\sum_{i=1}^{k} \left(-\sum_{j=1}^{i} d_j d_{i-j}\right) d_{k+1-i} \\ &= \sum_{i=1}^{k} \sum_{j=1}^{i} d_j d_{i-j} d_{k+1-i} \end{aligned}$$

$$\begin{aligned} \alpha d_0 = \Big(-\sum_{i=1}^{k} d_i d_{k+1-i}\Big) d_0 &= -\sum_{i=1}^{k} d_i d_{k+1-i} d_0 \\ &= -\sum_{i=1}^{k} d_i \left(-\sum_{j=1}^{k+1-i} d_{k+1-i-j} d_j\right) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k+1-i} d_i d_{k+1-i-j} d_j \end{aligned}$$

The middle step in the first calculation comes from applying the equality $d_0 d_i = -\sum_{1\leq j\leq i} d_j d_{i-j}$, which is true by the induction hypothesis. Likewise, $d_{k+1-i}d_0 = -\sum_{1\leq j\leq k+1-i} d_{k+1-i-j}d_j$ gives the middle step in the second calculation. The question now boils down to whether

$$\sum_{i=1}^{k}\sum_{j=1}^{i} d_j d_{i-j} d_{k+1-i} \stackrel{?}{=} \sum_{i=1}^{k}\sum_{j=i}^{k+1-i} d_i d_{k+1-i-j} d_j.$$

To see that this equality holds, observe that we can arrange the first double sum as a triangle-shaped list in the following way. The i-th row in the triangle is equal to the sum $\sum_{1\leq j\leq i} d_j d_{i-j} d_{k+1-i}$, which is the i-th term of the double sum. The trick now is to notice that the columns in the triangle correspond exactly to the sums in each term of the other double sum, which shows that the equality holds. This means that α commutes with d_0.

$$\begin{array}{llll}
d_1 d_0 d_k & & & \\
+\, d_1 d_1 d_{k-1} & +d_2 d_0 d_{k-1} & & \\
+\, d_1 d_2 d_{k-2} & +d_2 d_1 d_{k-2} + d_3 d_0 d_{k-2} & & \\
 & \vdots & & \\
+\, d_1 d_i d_{k-i} & +d_2 d_{i-1} d_{k-i} + d_3 d_{i-2} d_{k-i} & +\cdots + d_{i-1} d_0 d_{k-i} & \\
 & \vdots & & \\
+\, d_1 d_{k-1} d_1 & +d_2 d_{k-2} d_1 + d_3 d_{k-3} d_1 & +\cdots + d_{i-1} d_{k+1-i} d_1 & +\cdots + d_k d_0 d_1
\end{array}$$

By the above discussion, we see that α satisfies the conditions in lemma 2.2.4. Hence there is a map h such that $\alpha = d_0 h + h d_0$. We now simply define $d_{k+1} \coloneqq h$, so that

$$d_0 d_{k+1} + d_{k+1} d_0 = d_0 h + h d_0 = -\sum_{i=1}^{k} d_i d_{k+1-i}.$$

If we plug this back into the sum we started with, we get

$$\begin{aligned}
\sum_{i=0}^{k+1} d_i d_{k+1-i} &= d_0 d_{k+1} + d_{k+1} d_0 + \sum_{i=1}^{k} d_i d_{k+1-i} \\
&= -\sum_{i=1}^{k} d_i d_{k+1-i} + \sum_{i=1}^{k} d_i d_{k+1-i} = 0.
\end{aligned}$$

This concludes the induction, which shows that sending $X \in \boldsymbol{C}^-(Sum-T)$ to $X^{*,*}$ in $G(\Lambda)$ works.

Now let's look at how to send morphisms in $\boldsymbol{C}^-(Sum-T)$ to morphisms in $G(\Lambda)$. The argument is fairly similar to that for objects. We start with a map $\alpha\colon X \to Y$ which is graded of degree $(0,0)$, and we want a family of maps $(\alpha_i)_{i\in\mathbb{N}}$ of degree $(-i,i)$ which satisfy the necessary commutativity property. We start by setting $\alpha_0 = \alpha$, and then we proceed by induction. Assume that for $i \in \{0,\dots,k-1\}$ we have (α_i) such that

$$\sum_{i=0}^{k-1} \alpha_i d_{k-1-i} = \sum_{i=0}^{k-1} d_i \alpha_{k-1-i}.$$

We want to show that we then can define a map α_k such that

$$\sum_{i=0}^{k} \alpha_i d_{k-i} = \sum_{i=0}^{k} d_i \alpha_{k-i}.$$

By rearranging the terms, we get the equivalent equation

$$\alpha_k d_0 - d_0 \alpha_k = \sum_{i=1}^{k} d_i \alpha_{k-i} - \sum_{i=1}^{k} \alpha_{k-i} d_i.$$

We will use lemma 2.2.4 to perform the induction, and in order to do so, we define

$$\gamma = \sum_{i=1}^{k} d_i \alpha_{k-i} - \sum_{i=1}^{k} \alpha_{k-i} d_i = \sum_{i=1}^{k} (d_i \alpha_{k-i} - \alpha_{k-i} d_i).$$

We will not apply the lemma to γ directly, but to a slight modification. In fact, we can't apply the lemma to γ, since it doesn't commute with d_0. We actually have that $d_0\gamma = -\gamma d_0$, as we will see now by explicit computation:

$$\begin{aligned}
\gamma d_0 &= \sum_{i=1}^{k}(d_i\alpha_{k-i} - \alpha_{k-i}d_i)d_0 = \sum_{i=1}^{k} d_i\alpha_{k-i}d_0 - \alpha_{k-i}d_id_0 \\
&\overset{*}{=} \sum_{i=1}^{k} d_i\alpha_{k-i}d_0 - \alpha_{k-i}(-d_0d_i - \sum_{j=1}^{i-1} d_jd_{i-j}) \\
&= \left(\sum_{i=1}^{k} d_i\alpha_{k-i}d_0 + \alpha_{k-i}d_0d_i\right) + \sum_{i=1}^{k}\sum_{j=1}^{i-1}\alpha_{k-i}d_jd_{i-j} \\
&\overset{**}{=} \left(\sum_{i=1}^{k}(d_i\alpha_{k-i}d_0 + \left(d_0\alpha_{k-i} + \sum_{j=1}^{k-i} d_j\alpha_{k-i-j} - \alpha_{k-i-j}d_j\right)d_i\right) + \sum_{i=1}^{k}\sum_{j=1}^{i-1}\alpha_{k-i}d_jd_{i-j} \\
&= \left(\sum_{i=1}^{k}(d_i\alpha_{k-i}d_0 + d_0\alpha_{k-i}d_i\right) + \left(\sum_{i=1}^{k}\sum_{j=1}^{k-i} d_j\alpha_{k-i-j}d_i - \alpha_{k-i-j}d_jd_i\right) \\
&\qquad + \sum_{i=1}^{k}\sum_{j=1}^{i-1}\alpha_{k-i}d_jd_{i-j} \\
&= \left(\sum_{i=1}^{k}(d_i\alpha_{k-i}d_0 + d_0\alpha_{k-i}d_i\right) + \sum_{i=1}^{k}\left(\sum_{j=1}^{k-i} d_j\alpha_{k-i-j}d_i - \sum_{j=1}^{k-i}\alpha_{k-i-j}d_jd_i\right. \\
&\qquad \left. + \sum_{j=1}^{i-1}\alpha_{k-i}d_jd_{i-j}\right) \\
&\overset{***}{=} \left(\sum_{i=1}^{k}(d_i\alpha_{k-i}d_0 + d_0\alpha_{k-i}d_i\right) + \sum_{i=1}^{k}\sum_{j=1}^{k-i} d_j\alpha_{k-i-j}d_i
\end{aligned}$$

$$
\begin{aligned}
d_0\gamma &= d_0 \sum_{i=1}^{k}(d_i\alpha_{k-i} - \alpha_{k-i}d_i) = \sum_{i=1}^{k} d_0 d_i \alpha_{k-i} - d_0\alpha_{k-i}d_i \\
&\overset{*}{=} \sum_{i=1}^{k}(-d_i d_0 - \sum_{j=1}^{i-1} d_j d_{i-j})\alpha_{k-i} - d_0\alpha_{k-i}d_i \\
&= -\left(\sum_{i=1}^{k} d_0\alpha_{k-i}d_i + d_i d_0 \alpha_{k-i}\right) - \sum_{i=1}^{k}\sum_{j=1}^{i-1} d_j d_{i-j}\alpha_{k-i} \\
&\overset{**}{=} -\left(\sum_{i=1}^{k} d_0\alpha_{k-i}d_i + d_i\left(\alpha_{k-i}d_0 + \sum_{j=1}^{k-i}\alpha_{k-i-j}d_j - d_j\alpha_{k-i-j}\right)\right) - \sum_{i=1}^{k}\sum_{j=0}^{i} d_j d_{i-j}\alpha_{k-i} \\
&= -\left(\sum_{i=1}^{k} d_0\alpha_{k-i}d_i + d_i\alpha_{k-i}d_0\right) - \left(\sum_{i=1}^{k}\sum_{j=1}^{k-i} d_i\alpha_{k-i-j}d_j - d_i d_j \alpha_{k-i-j}\right) \\
&\qquad - \sum_{i=1}^{k}\sum_{j=0}^{i} d_j d_{i-j}\alpha_{k-i} \\
&= -\left(\sum_{i=1}^{k} d_0\alpha_{k-i}d_i + d_i\alpha_{k-i}d_0\right) - \sum_{i=1}^{k}\left(\sum_{j=1}^{k-i} d_i\alpha_{k-i-j}d_j - \sum_{j=1}^{k-i} d_i d_j\alpha_{k-i-j}\right. \\
&\qquad \left. + \sum_{j=0}^{i} d_j d_{i-j}\alpha_{k-i}\right) \\
&\overset{***}{=} -\left(\sum_{i=1}^{k} d_0\alpha_{k-i}d_i + d_i\alpha_{k-i}d_0\right) - \sum_{i=1}^{k}\left(\sum_{j=1}^{k-i} d_i\alpha_{k-i-j}d_j\right)
\end{aligned}
$$

In both of the derivations above, the unmarked equals signs are simply rearranging the expressions, whil the marked equals signs mean the following:

* means we rearrange and apply the equality $\sum_{j=0}^{i} d_j d_{i-j} = 0$.

** means we rearrange and apply the equality $\sum_{j=0}^{k-i} \alpha_{k-i-j}d_j = \sum_{j=0}^{k-i} d_j\alpha_{k-i-j}$, which holds by the induction hypothesis, since $i \geq 1$.

*** is given by the fact that when i and j run through all possible values, the last two sums cancel each other out perfectly.

As we can see, the result is that $d_0\gamma = -\gamma d_0$. Notice that γ is a graded map of degree $(1-k,k)$, so we can define $\tilde{\gamma} = (-1)^{p+q}\gamma$ for $\gamma\colon X^{p,q} \to Y^{p+1-k,q+k}$. Then $\tilde{\gamma}$ is a graded map of degree $(1-k,k)$ which commutes with d_0, so we can apply lemma 2.2.4. Thus we can find a map h such that $\tilde{\gamma} = hd_0 + d_0h$. Finally, we set $\alpha_k := (-1)^{p+q-1}h$ when h is a map from

$X^{p,q}$. This ensures that $(-1)^{p+q}\gamma = \tilde{\gamma} = d_0 h + h d_0 = (-1)^{p+q-1} d_0 \alpha_k + (-1)^{p+q} \alpha_k d_0$, which in turn means that $\gamma = \alpha_k d_0 - d_0 \alpha_k$. Thus we have that

$$\sum_{i=0}^{k} d_i \alpha_{k-i} = d_0 \alpha_k + \sum_{i=1}^{k} d_i \alpha_{k-i} = (\alpha_k d_0 - \gamma) + (\gamma + \sum_{i=1}^{k} \alpha_{k-i} d_i) = \sum_{i=0}^{k} \alpha_{k-i} d_i,$$

which is what we needed to finish the induction. Hence we conclude that a map α in $\boldsymbol{C}^{-}(Sum - T)$ gives rise to a map $(\alpha_i)_{i \in \mathbb{N}}$ in $G(\Lambda)$.

The last thing we need to determine, is what happens when we have a homotopy in $\boldsymbol{C}^{-}(Sum - T)$, and then go to $G(\Lambda)$. Assume that α is a null-homotopic map in $\boldsymbol{C}^{-}(Sum - T)$, so we know that we can find a map h_0 such that $\alpha = d_0 h_0 + h_0 d_0$. We also know that α is sent to a family (α_i) of maps in $G(\Lambda)$, with $\alpha_0 = \alpha$. We can use α_0 and h_0 as the start of an induction argument similar to the one above, in order to construct a homotopy in $G(\Lambda)$. Assume that we already have h_i for $i \in \{0, \ldots, k-1\}$ such that we can write α_n as $\sum_{0 \leq i \leq n} (d_i h_{n-i} + h_i d_{n-i})$ for each $n \in \{0, \ldots, k-1\}$. We then need to find h_k such that $\alpha_k = \sum_{0 \leq i \leq k} (d_i h_{k-i} + h_i d_{k-i})$. This is done by checking that $\left(\alpha_k - \sum_{0 \leq i \leq k} (d_i h_{k-i} + h_i d_{k-i})\right)$ commutes with d_0, and then applying lemma 2.2.4 to it. We skip the details here.

In total, we conclude that we can define a functor

$$\Phi\colon \boldsymbol{C}^{-}(Sum - T) \to \mathbf{K}^{-}(Proj - \Lambda),$$

given by first going to $G(\Lambda)$ and then taking total complexes. Again, what we really want is a functor $\mathbf{K}^{-}(Sum - T) \to \mathbf{K}^{-}(Proj - \Lambda)$, and to get that we'll factor Φ.

Proposition 2.2.5. *The functor Φ factors through the natural functor*

$$\boldsymbol{C}^{-}(Sum - T) \to \boldsymbol{K}^{-}(Sum - T),$$

and the resulting functor

$$\boldsymbol{K}^{-}(Sum - T) \to \boldsymbol{K}^{-}(Proj - \Lambda)$$

is a triangle functor.

Proof. By construction, we know that every map $\alpha\colon X \to Y$ in $\boldsymbol{C}^{-}(Sum - T)$ gives rise to a distingusihed triangle

$$X \xrightarrow{\alpha} Y \to Z \to X[1]$$

in $\mathbf{K}^{-}(Sum - T)$, where Z is the mapping cone of α. To prove the proposition, we must show that for any map α, this triangle is sent to a distinguished triangle in $G(\Lambda)$ (which is the definition of functor between triangulated categories being a triangle functor). Since Z is the cone of α, we explicitly know what it looks like, and we can use that to compute its bigraded structure. Let X be a complex given as $\cdots \to X^n \to X^{n+1} \to \cdots$, and Y a complex given as $\cdots \to Y^n \to Y^{n+1} \to \cdots$. To keep things simple, we call both differentials d. We then have that Z is

$$\cdots \to Y^n \oplus X^{n+1} \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ \alpha & d \end{smallmatrix}\right)} Y^{n+1} \oplus X^n \to \cdots$$

Now we must find a corresponding object in $G(\Lambda)$, and since we already have a bigrading on Z, we just need to find the family of morphisms which form the differential. It turns out that we can do this so that for $Z^{i,j} = Y^{i,j} \oplus X^{i,j+1}$, the differerntial in $Z^{*,*}$ is on the form

$$\begin{pmatrix} d_k & 0 \\ (-1)^{i+j}\alpha_{k-1} & d_k \end{pmatrix} : Z^{i,j} \to Z^{i+1-k,j+k} \quad \text{(where } \alpha_{-1} = 0 \text{ by definition).}$$

What we need to check now, is that applying Φ to Z gives the same complex as applying Φ to $\alpha\colon X \to Y$ and then taking the mapping cone (up to isomorphism). Notice that the complexes $\Phi(Z)$ and the cone of $\Phi(\alpha)$ have the same terms, and that the difference only is in the differentials. If we say that $a = \sum_i \alpha_i$, then the differential in the cone of $\Phi(\alpha)$ is $\left(\begin{smallmatrix} d & 0 \\ a & d \end{smallmatrix}\right)$. From the construction above, we see that $\Phi(Z)$ has $\left(\begin{smallmatrix} d & 0 \\ (-1)^n a & d \end{smallmatrix}\right)$ as differential in degree n. This means that we can define an isomorphism between the two complexes quite easily. We simply fix the terms coming from Y, and multiply the terms coming from X by powers of -1. □

Combining this result with the equivalence $\mathbf{K}^-(Proj-\Gamma) \to \mathbf{K}^-(Sum-T)$, we immediately get the following theorem:

Theorem 2.2.6. *Let Λ be a ring, and let T be a bounded complex of finitely generated projective Λ-modules. Suppose that $Hom_{\boldsymbol{K}^b(Proj-\Lambda)}(T, T[n]) = 0$ holds for all $n < 0$. Let Γ be the endomorphism ring of T in $\boldsymbol{K}^b(Proj - \Lambda)$. Then there is a triangle functor $F\colon \boldsymbol{K}^-(Proj - \Gamma) \to \boldsymbol{K}^-(Proj - \Lambda)$ which sends Γ to T and bounded complexes to bounded complexes.*

2.3 F is a full embedding

Now that we have defined the functor F, our next step will be to prove that F is a full embedding. In other words, we will prove the theorem:

Theorem 2.3.1. *The functor $F\colon \boldsymbol{K}^-(Proj - \Gamma) \to \boldsymbol{K}^-(Proj - \Lambda)$ is fully faithful.*

Proof. First of all, F sends a free Γ-module first to a direct sum of copies of T and then to the associated total complex. In other words, if $X \simeq \bigoplus_{i\in I} \Gamma$ is some free Γ-module, then $FX \simeq \bigoplus_{i\in I} T$. This means, in particular, that if X and Y are free Γ-modules of rank one (so $X \simeq \Gamma \simeq Y$), then F induces an isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_\Gamma(X,Y) & \overset{F}{\longmapsto} & \mathrm{Hom}_\Lambda(T,T) \\ \wr\| & & \wr\| \\ \Gamma & & \Gamma \end{array}$$

If we now keep $X \simeq \Gamma$, and let $Y \simeq \bigoplus_{i\in I}\Gamma$ be any free Γ-module, the fact that the covariant hom-functor commutes with direct sums gives that

$$\mathrm{Hom}_\Gamma(\Gamma, \bigoplus_{i\in I}\Gamma) \simeq \bigoplus_{i\in I}\mathrm{Hom}_\Gamma(\Gamma,\Gamma) \simeq \bigoplus_{i\in I}\mathrm{Hom}_\Lambda(T,T) \simeq \mathrm{Hom}(T, \bigoplus_{i\in I} T).$$

The contravariant hom-functor turns coproducts into products, so if we now let both $X = \bigoplus_{j\in J}\Gamma$ and Y be arbitrary free Γ-modules, we get

$$\mathrm{Hom}_\Gamma(\bigoplus_{j\in J}\Gamma, Y) \simeq \prod_{j\in J}\mathrm{Hom}_\Gamma(\Gamma,Y) \simeq \prod_{j\in J}\mathrm{Hom}_\Lambda(T,FY) \simeq \mathrm{Hom}(\bigoplus_{j\in J} T, FY).$$

This shows that $\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}(FX,FY)$ holds for all free Γ-modules X,Y. In addition, all complexes of projective modules that have only one nonzero degree can be written as a direct summand of some shift of a free Γ-module. And since F is a triangle functor, it preserves direct summands and shifts. The only possible problem is that even if X is a complex that is concentrated in one degree, FX will generally not be concentrated in one degree. However, since T is a tilting complex, we know that shifted endomorphisms of T are zero. Thus, we get that $\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}(FX,FY)$ for all X,Y in $\mathbf{K}^-(Proj-\Gamma)$ that are concentrated in one degree.

We will now extend this result, and show that it also holds for all bounded complexes of projectives. In other words, we show that F is fully faithful when it is restricted to $\mathbf{K}^b(Proj-\Gamma)$. To do so, we will use induction on the number of nonzero terms in X and Y. Finally, we will use a colimit argument to extend the statement to all of $\mathbf{K}^-(Proj-\Gamma)$. We start by showing that any complex X with n nonzero terms can be placed in a distinguished triangle $V \to X \to W \rightsquigarrow$ where V and W both have fewer than n nonzero terms. If we have that X is the complex

$$X = (0 \to X_1 \to X_2 \to \cdots \to X_n \to 0),$$

and set

$$V = (0 \to \cdots \to 0 \to X_n \to 0),$$

then the mapping cone of the inclusion $V \hookrightarrow X$ is the complex

$$(0 \to X_1 \to \cdots \to X_{n-1} \oplus X_n \to X_n \to 0).$$

By removing a trivial summand (theorem A.1.1), we see that this complex is homotopy equivalent to the following complex, which will be our W:

$$W = (0 \to X_1 \to X_2 \to \cdots \to X_{n-1} \to 0 \to 0).$$

We see that both V and W have fewer than n nonzero terms, and $V \to X \to W \rightsquigarrow$ is a distinguished triangle by construction, since W is the cone of $V \to X$. This, together with the 5-lemma, allows us to show by induction that F is fully faithful on $\mathbf{K}^b(Proj - \Gamma)$. Assume that we have shown it for complexes with $n-1$ or fewer nonzero terms. If we apply $\mathrm{Hom}(-, Y)$ to the triangle $V \to X \to W \rightsquigarrow$ given above, we get a long exact sequence, and the same is true for applying $\mathrm{Hom}(-, FY)$ to the triangle $FV \to FX \to FW \rightsquigarrow$. We then get the following diagram, where the squares commute and the rows are exact:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{Hom}(W,Y) & \longrightarrow & \mathrm{Hom}(X,Y) & \longrightarrow & \mathrm{Hom}(V,Y) & \longrightarrow & \mathrm{Hom}(W[1],Y) & \longrightarrow \\
 & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr & \\
\longrightarrow & \mathrm{Hom}(FW,FY) & \longrightarrow & \mathrm{Hom}(FX,FY) & \longrightarrow & \mathrm{Hom}(FV,FY) & \longrightarrow & \mathrm{Hom}(FW[1],FY) & \longrightarrow
\end{array}$$

The isomorphisms come from the iduction hypothesis, since V and W both have fewer than n nonzero terms. The 5-lemma now implies that the map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$ also is an isomorphism (the same is true for all shifts). This concludes the induction argument.

We will now extend the result to all of $\mathbf{K}^-(Proj - \Gamma)$, and to do so, we will need the following construction. Note that we start by viewing X as a regular right bound complex, not up to homotopy. A complex X in $\boldsymbol{C}^-(Proj - \Gamma)$ is given as

$$X = (\cdots \to X_{-n-1} \to X_{-n} \to X_{-n+1} \to \cdots \to X_N \to 0)$$

for some $N \in \mathbb{Z}$. We define the truncated complex $X(n)$ by removing all values of X below degree $-n$, that is, we set $X_i = 0$ for all $i < -n$:

$$X(n) := (\cdots \to 0 \to X_{-n} \to X_{-n+1} \to \cdots \to X_N \to 0).$$

We can, without loss of generality, assume that $N = 0$. Now consider the directed system

$$X(0) \hookrightarrow X(1) \hookrightarrow \cdots \hookrightarrow X(n) \hookrightarrow X(n+1) \hookrightarrow \cdots,$$

given by the obvious inclusions $\iota_{i,j} \colon X(i) \hookrightarrow X(j)$ for $i \leq j$. Since each $X(n)$ contains the rightmost $n+1$ nonzero terms of X, it is clear that $\varinjlim X(n) = X$. Next, we form the direct sum $\bigoplus_{n\in\mathbb{N}} X(n)$, and look at the endomorphism $\bigoplus_{n\in\mathbb{N}} X(n) \to \bigoplus_{n\in\mathbb{N}} X(n)$ generated by sending (x_i) to $(x_i - \iota_{i,j}(x_i))$. The cokernel of this map is $\varinjlim X(n)$ (proposition 7.94 in [Rot02, p 506]), which means that we can form the short exact sequence

$$0 \to \bigoplus_{n\in\mathbb{N}} X(n) \to \bigoplus_{n\in\mathbb{N}} X(n) \to X \to 0.$$

This induces a distinguished triangle in $\mathbf{K}^-(Proj-\Gamma)$

$$\bigoplus_{n\in\mathbb{N}} X(n) \to \bigoplus_{n\in\mathbb{N}} X(n) \to X \to \bigoplus_{n\in\mathbb{N}} X(n)[1].$$

We now apply the functor $\mathrm{Hom}_{\mathbf{K}^-(Proj-\Gamma)}(-,Y)$ to this triangle, to get a long exact sequence. Recall that coproducts in the first coordinate of a hom-set can be pulled out, but then they become products, so we have that $\mathrm{Hom}_{\mathbf{K}^-(Proj-\Gamma)}(\bigoplus_{n\in\mathbb{N}} X(n), Y) \simeq \prod_{n\in\mathbb{N}} \mathrm{Hom}_{\mathbf{K}^-(Proj-\Gamma)}(X(n),Y)$. This means we can write the long exact sequence as the following, where we write $\mathrm{Hom}_{\mathbf{K}^-(Proj-\Gamma)}(A,B)$ as (A,B) to save space:

$$\cdots \longrightarrow \prod_{n\in\mathbb{N}}(X(n),Y[-1]) \longrightarrow \prod_{n\in\mathbb{N}}(X(n),Y[-1]) \longrightarrow$$

$$\longrightarrow (X,Y) \longrightarrow \prod_{n\in\mathbb{N}}(X(n),Y) \longrightarrow \prod_{n\in\mathbb{N}}(X(n),Y) \longrightarrow \cdots$$

Note that generally $F(X(n)) \neq (FX)(n)$, because FX might have something nonzero in degree i, even though $X_i = 0$. We denote $F(X(n))$ as $FX(n)$. Since T is a bounded complex, the way F is constructed ensures that $FX(n)$ will be bounded. It also means that, for n large enough, the rightmost part of $X(n)$ and the rightmost part of X will be mapped to the same thing by F. Hence we see that $\varinjlim FX(n) = FX$. Thus we can do a similar construction as above for FX, and get the long exact sequence

$$\cdots \longrightarrow \prod_{n\in\mathbb{N}}(FX(n),Y[-1]) \longrightarrow \prod_{n\in\mathbb{N}}(FX(n),Y[-1]) \longrightarrow$$

$$\longrightarrow (FX,FY) \longrightarrow \prod_{n\in\mathbb{N}}(FX(n),Y) \longrightarrow \prod_{n\in\mathbb{N}}(FX(n),Y) \longrightarrow \cdots$$

The functorality of F ensures that we get a morphism of complexes from the first long exact sequence to the second long exact sequence. Since $X(n)$ is bounded for each n, our previous result, together with the fact that products preserve isomorphisms, shows that the maps between the product terms must be isomorphisms. That is, we have an isomorphism $\prod(X(n),F[i]) \xrightarrow{\sim} \prod(FX(n),FY[i])$ for all $i \in \mathbb{Z}$. Now, let's look at the subdiagram given by the five terms of each long exact sequence shown above, together with the morphisms between them. This is a commutative diagram with two exact rows of five terms each, where all the vertical morphisms except the middle one are isomorphisms. Thus, by the five lemma, the middle morphism must also be an isomorphism. But the middle morphism is

$\mathrm{Hom}_{\mathbf{K}^-(Proj-\Gamma)}(X,Y) \to \mathrm{Hom}_{\mathbf{K}^-(Proj-\Lambda)}(FX,FY)$, so this concludes our proof that F is fully faithful.

□

2.4 Construction of G

Now that we have shown that we have a fully faithful functor $F\colon \mathbf{K}^-(Proj-\Gamma) \to \mathbf{K}^-(Proj-\Lambda)$, our next goal is to construct a functor $G\colon \mathbf{K}^-(Proj-\Lambda) \to \mathbf{K}^-(Proj-\Gamma)$ which is right adjoint to F. Recall that we have an equivalence between $\mathbf{K}^-(Proj-\Gamma)$ and $\mathbf{K}^-(Sum-T)$, so what we actually need is to find a way to send complexes in $\mathbf{K}^-(Proj-\Lambda)$ to complexes of sums of copies of T. In other words, we must find a way to construct some kind of 'T-resolutions' of complexes in $\mathbf{K}^-(Proj-\Lambda)$.

We start by noting that, because T is a bounded complex, any complex X in $\mathbf{K}^-(Proj-\Lambda)$ can be shifted so far to the left that there are no nonzero morphisms from T to the shifted X. This is because X is right bounded, so for large enough n there will be no overlap of nonzero degrees in T and $X[n]$. Now, let N be the smallest number such that

$$\mathrm{Hom}(T, X[n]) = 0 \text{ for all } n > N.$$

We define $X^{(0)}$ to be equal to $X[N]$. This means that $\mathrm{Hom}(T, X^{(0)}) \neq 0$, but for any positive shift i we have that $\mathrm{Hom}(T, X^{(0)}[i]) = 0$. We now want an object $S^{(0)}$ in $Sum-T$, together with a morphism $\alpha\colon S^{(0)} \to X^{(0)}$, such that $\mathrm{Hom}(T, S^{(0)})$ is mapped surjectively onto $\mathrm{Hom}(T, X^{(0)})$ via α. For example, we can take the object $\bigoplus_{f\in\mathrm{Hom}(T,X^{(0)})} T$, which has one copy of T for each map from T to $X^{(0)}$. Then α could be the map $S^{(0)} \to X^{(0)}$ where the component map from the f-th summand is just f. This map clearly gives a surjection from $\mathrm{Hom}(T, S^{(0)})$ to $\mathrm{Hom}(T, X^{(0)})$, given by taking the inclusion morphisms into each summand of $S^{(0)}$. We take $S^{(0)}$ and α to be the first term of our 'T-resolution'. To continue the construction, we find a complex $X^{(1)}$ which completes α to a distinguished triangle:

$$X^{(-1)} \longrightarrow S^{(0)} \xrightarrow{\alpha} X^{(0)} \longrightarrow X^{(-1)}[1].$$

We now repeat the process for $X^{(-1)}$, i.e. find an object $S^{(-1)}$ in $Sum-T$ and a morphism $\beta\colon S^{(-1)} \to X^{(-1)}$ which maps $\mathrm{Hom}(T, S^{(-1)})$ surjectively to $\mathrm{Hom}(T, X^{(-1)})$. Then we find a complex $X^{(-2)}$ that completes this morphism to a distinguished triangle. The general process is

- Take the object $X^{(i)}$.
- Find an object $S^{(i)} \in Sum-T$, and a map $S^{(i)} \to X^{(i)}$ that induces a surjection on the hom-sets.

- Complete the map to a distingusihed triangle:
  $X^{(i-1)} \to S^{(i)} \to X^{(i)} \rightsquigarrow$
- Repeat for $X^{(i-1)}$.

By continuing this process, we end up with a collection $(S^{(i)})_{i\leq 0}$ of objects in $Sum - T$. We define $d\colon S^{(i)} \to S^{(i+1)}$ as the composition of the maps $S^{(i)} \to X^{(i)}$ and $X^{(i)} \to S^{i+1}$ coming from the triangles in the construction. Note that $d^2 = 0$, because it contains the composition of two consecutive maps in a distinguished triangle, which in turn means that (S^i, d) is a complex of objects in $Sum - T$. We call $(S^{(i)}, d)$ a T-resolution of X.

It is not obvious that this construction gives a functor, since it's not clear that it has a unique result. We will now prove that it does. Observe that $\mathrm{Hom}(T, X^{(i)}[n]) = 0$ when $i \leq N$ and $n > 0$. We prove this by downward induction on i. For $i = 0$, the assertion is true because $X^{(0)} = X[N]$ and we chose N such that $\mathrm{Hom}(T, X[j]) = 0$ for all $j > N$. Now assuming the assertion holds for $i = k$, we want to show that it holds for $i = k - 1$. From the construction of the T-resolution we get the distingushed triangle $X^{(k-1)} \to S^{(k)} \to X^{(k)} \rightsquigarrow$, and by applying the functor $\mathrm{Hom}(T, -)$ to it we get a long exact sequence. By the induction hypothesis we know that $\mathrm{Hom}(T, X^{(k)}[n]) = 0$ for all $n > 0$. Because $S^{(k)}$ is in $Sum - T$ and shifted endomorphisms of T vanish (since T is a tilting complex), we also have that $\mathrm{Hom}(T, S^{(k)}[n]) = 0$ for all $n \neq 0$. This means that for $n \geq 2$ there will be zeros on both sides of $\mathrm{Hom}(T, X^{(k-1)}[n])$ in the long exact sequence, which means that it must itself be zero. For $n = 1$, we get this part of the long exact sequence

$$\mathrm{Hom}(T, S^{(k)}) \twoheadrightarrow \mathrm{Hom}(T, X^{(k)}) \to \mathrm{Hom}(T, X^{(k-1)}[1]) \longrightarrow \mathrm{Hom}(T, S^{(k)}[1]) = 0,$$

where the first arrow is a surjection because of how we constructed $S^{(k)}$. By exactness, we get that $\mathrm{Hom}(T, X^{(k-1)}[1]) = 0$, and since we have already shown it for all larger shifts, the induction is complete.

We will now state and prove a proposition which shows that taking T-resolution gives a functor. In the following, R is the image of a T-resolution of X under the natural quotient map $\boldsymbol{C}^-(Sum - T) \to \mathbf{K}^-(Sum - T)$.

Proposition 2.4.1. *Let X be an object in $\boldsymbol{K}^-(Proj - \Lambda)$ and R the image in $\boldsymbol{K}^-(Proj - \Gamma)$ of a T-resolution of X. Then there exists a homomorphism $\alpha : FR \to X$ which induces, for any complex Q, an isomorphism $Hom(FQ, FR) \simeq Hom(FQ, X)$.*

Proof. The idea here is to use induction on length to prove the proposition for all bounded complexes, and the pass to limits to prove it for complexes that are unbounded to the left. In order to do so, we will use the octahedral axiom for triangulated categories to create a distinguished triangle where one of the maps satisfies the wanted conditions. To keep things simple, we assume $N = 0$ in the following.

Claim: For each $n > 0$, there exists a distinguished triangle

$$X^{(-n)}[n-1] \xrightarrow{u_{n-1}} F(R(n-1)) \xrightarrow{v_{n-1}} X \longrightarrow X^{(-n)}[n]$$

satisfying the condition that the following diagrams commute:

$$\begin{array}{ccc} FR_{-n}[n-1] & \xrightarrow{f_{n-1}} & X^{(-n)}[n-1] \\ & \searrow^{d_{n-1}} & \downarrow u_{n-1} \\ & & F(R(n-1)) \end{array} \qquad \begin{array}{ccc} F(R(n-2)) & \xrightarrow{g_{n-1}} & F(R(n-1)) \\ & \searrow^{v_{n-2}} & \downarrow v_{n-1} \\ & & X \end{array}$$

Here, d_{n-1} is the differential in FR, given as a map from $FR_{-n}[n-1]$ (the complex with the $-n$-th term of FR in position $(-n+1)$, and 0 everywhere else) to $F(R(n-1))$. The map f_{n-1} is taken from the n-th distinguished triangle used in the construction of the T-resolution of X:

$$X^{(-n-1)}[-n-1] \longrightarrow FR_{-n}[n-1] \xrightarrow{f_{n-1}} X^{(-n)}[n-1] \longrightarrow X^{(-n-1)}[-n].$$

The map g_{n-1} is the obvious map from $F(R(n-2))$ to $F(R(n-1))$, given by the inclusion $R(n-2) \hookrightarrow R(n-1)$.

We prove the claim by induction on n. For $n = 1$, we first notice that $R(0)$ is isomorphic to $S^{(0)}$ as a complex in $\mathbf{K}^-(Sum - T)$, which from the way F is constructed means that $F(R(0)) = S^{(0)}$. The second diagram is trivially satisfied, since $R(-1) = 0$, so we just need to check the first one. To see that $FR_{-1}[0] \to F(R(0))$ factors through $X^{(-1)}[0] = X^{(-1)}$, we use the fact that $FR_{-1}[0] = FR_{-1} = S^{(-1)}$. Actually, the factoring comes from the construction of the T-resolution, and is given as $FR_{-1} = S^{(-1)} \to X^{(-1)} \to S^{(0)} = F(R(0))$. This takes care of the base case of the induction.

Now, assume that we have constructed a triangle satisfying the given conditions for $m < n$. Then the solid part of the following diagram will commute, with the two rows and the first column being distinguished triangles.

$$\begin{array}{ccccccc} FR_{-n}[n-1] & \xrightarrow{f_{n-1}} & X^{(-n)}[n-1] & \longrightarrow & X^{(-n-1)}[n] & \longrightarrow & FR_{-n}[n] \\ \| & & \downarrow u_{n-1} & & \downarrow u_n & & \| \\ FR_{-n}[n-1] & \xrightarrow{d_{n-1}} & F(R(n-1)) & \xrightarrow{g_n} & F(R(n)) & \longrightarrow & FR_{-n}[n] \\ & & \downarrow v_{n-1} & & \downarrow v_n & & \downarrow f_{n-1}[1] \\ & & X & = & X & \longrightarrow & X^{(-n)}[n] \\ & & \downarrow & & \downarrow & & \\ & & X^{(-n)}[n] & \longrightarrow & X^{(-n-1)}[n+1] & & \end{array}$$

The octahedral axiom then implies that the dashed arrows form a distinguished triangle, and that they commute with the rest of the diagram. This is the triangle we want, we just need to show that the maps fit in the above diagrams. The bottom dashed arrow shows that $v_{n-1} = v_n \circ g_n$, so $F(R(n-1)) \to X$ factors through $F(R(n))$ like it should.
To show that the composition

$$FR_{-n-1}[n] \xrightarrow{f_n} X^{(-n-1)}[n] \xrightarrow{u_n} F(R(n))$$

is equal to the differential in FR, we look at the composition

$$FR_{-n-1}[n] \xrightarrow{f_n} X^{(-n-1)}[n] \xrightarrow{u_n} F(R(n)) \to FR_{-n}[n].$$

From the octahedral diagram, this composition is equal to

$$FR_{-n-1}[n] \xrightarrow{f_n} X^{(-n-1)}[n] \to FR_{-n}[n],$$

which by definition is the differential in a T-resolution of X. If we now apply the functor $\mathrm{Hom}(FR_{-n-1}[n], -)$ to the distinguished triangle

$$FR_{-n} \longrightarrow F(R(n-1)) \longrightarrow F(R(n)) \rightsquigarrow$$

we get a long exact sequence. We have that $\mathrm{Hom}(FR_{-n-1}[n], F(R(n-1))) = 0$, there is no overlap between the degrees in which $FR_{-n-1}[n]$ and $F(R(n-1))$ are nonzero, so we know that the sequence

$$\begin{array}{l} \mathrm{Hom}(FR_{-n-1}[n], F(R(n-1))) \\ \quad \| \\ \quad 0 \longrightarrow \mathrm{Hom}(FR_{-n-1}[n], F(R(n))) \longrightarrow \mathrm{Hom}(FR_{-n-1}[n], FR_{-n}[n]) \end{array}$$

is exact. In other words, the map

$$\mathrm{Hom}(FR_{-n-1}[n], F(R(n))) \to \mathrm{Hom}(FR_{-n-1}[n], FR_{-n}[n])$$

is injective. We have already shown that the composition

$$u_n \circ f_n \in \mathrm{Hom}(FR_{-n-1}[n], F(R(n-1)))$$

is sent to the differential in FR in $\mathrm{Hom}(FR_{-n-1}[n], FR_{-n}[n])$. Now, since d_n simply is the differential in FR represented as a map in $\mathrm{Hom}(FR_{-n-1}[n], F(R(n)))$, it is also sent to the differential in FR. And thus, by injectivity, we have that $u_n \circ f_n = d_n$. This concludes the proof of the claim.

Now assume that we have a right bounded complex X. Notice that the claim implies that we get a map $v_n: FR(n) \to X$ for all n. If we now write FR as the colimit of bounded complexes $FR = \varinjlim FR(n)$, the universal property of colimits gives us a map $FR \to X$. Let $\tilde{X}$ be the cone of this map. For any i, the complexes FR and $FR(i)$ are identical in degrees higher than $-i$. This means that for any given p, all maps from $T[p]$ to FR factor uniquely through $FR(i)$ when i is large enough. From the observation immediately preceding the statement of the proposition, we see that $\mathrm{Hom}(T[p], X^{(-i-1)}[i]) = 0$ for i sufficiently large. Then, the long exact sequence given by applying $\mathrm{Hom}(T[p], -)$ to the distinguished triangle from the preceding claim shows that

$$\mathrm{Hom}(T[p], FR(i)) \longrightarrow \mathrm{Hom}(T[p], X)$$

is an isomorphism for sufficiently large i. In fact, since Hom is an additive functor, we can replace $T[p]$ by any element of $add(T)$ and still have isomorphism. By then applying the functor $\mathrm{Hom}(-, X^{(-i-1)}[i])$ to distinguished triangles of the form $A \to B \to C \rightsquigarrow$, where $A, B \in add(T)$ and C is the cone of the map $A \to B$, we know that A and B will be sent to 0. So the 2-out-of-3-property tells us that $\mathrm{Hom}(C, X^{(-i-1)}[i]) = 0$, and consequently that the isomorphism above holds for cones as well as for elements in $add(T)$. This shows that the composition

$$\mathrm{Hom}(*, FR) \longrightarrow \mathrm{Hom}(*, FR(i)) \longrightarrow \mathrm{Hom}(*, X)$$

is an isomorphism for all $*$ in the triangulated category generated by $add(T)$, provided that i is sufficiently large. Now, since $\tilde{X}$ is the cone of the map $FR \to X$, this isomorphism means that $\mathrm{Hom}(*, \tilde{X}) = 0$. If we take any complex $Q \in \mathbf{K}^-(Proj - \Gamma)$, then $FQ(i)$ will be in the triangulated category generated by $add(T)$, so $\mathrm{Hom}(FQ(i), \tilde{X}) = 0$ for all i. By the Mittag-Leffler condition, we have that the map $\mathrm{Hom}(FQ, \tilde{X}) \to \varprojlim \mathrm{Hom}(FQ(i)), \tilde{X})$ is an isomorphism. Thus, we get that $\mathrm{Hom}(FQ, \tilde{X}) = 0$, and consequently that

$$\mathrm{Hom}(FQ, FR) \longrightarrow \mathrm{Hom}(FQ, X)$$

is an isomorphism. This concludes the proof of the proposition.

□

The above proposition implies that the image in $\mathbf{K}^-(Proj - \Gamma)$ of a T-resolution of X is unique (up to isomorphism). To see this, take two images of T-resolutions of X, say R and R'. From the proposition we get maps $\alpha: FR \to X$ and $\beta: FR' \to X$ such that $\mathrm{Hom}(FQ, FR) \simeq \mathrm{Hom}(FQ, X)$ and $\mathrm{Hom}(FQ, FR') \simeq \mathrm{Hom}(FQ, X)$ for any complex Q.

By setting $Q = R$ and $Q = R'$, we can find maps γ and δ such that the following diagrams commute.

$$\begin{array}{ccc} FR & \xrightarrow{\gamma} & FR' \\ & \searrow^{\alpha} & \downarrow^{\beta} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} FR' & \xrightarrow{\delta} & FR \\ & \searrow^{\beta} & \downarrow^{\alpha} \\ & & X \end{array}$$

From this we get that $\alpha = \beta\gamma = \alpha\delta\gamma$, which means that $\alpha(id_{FR} - \delta\gamma) = 0$. By the proposition, we get in particular that α is sent to an isomorphism by $\mathrm{Hom}(FR, -)$. This means that $(id_{FR} - \delta\gamma)$ must be zero, since it is sent to zero by $\mathrm{Hom}(FR, \alpha)$, and thus $\delta\gamma = id_{FR}$. The same argument for β shows that $\gamma\delta = id_{FR'}$, and consequently, $FR \simeq FR'$.

Corollary 2.4.2. *The functor F has a right adjoint G. In particular, F commutes with arbitrary direct sums.*

Proof. The preceding argument shows that it is well-defined to send an object X of $\mathbf{K}^-(Proj - \Lambda)$ to the image in $\mathbf{K}^-(Proj - \Gamma)$ of some T-resolution of X. If we let Q be any object in $\mathbf{K}^-(Proj - \Gamma)$, we get an isomorphism

$$\varphi : \mathrm{Hom}(FQ, X) \xrightarrow{\sim} \mathrm{Hom}(FQ, FGX) \xrightarrow{\sim} \mathrm{Hom}(Q, GX)$$

where the first isomorphism comes from the proposition, and the second comes from F being fully faithful. This means that G is a right adjoint to F, given that it actually is a functor. To check that it is, we must show that it preserves identity and commutes with composition. We set $Q \coloneqq GX$, and thus get a morphism $\eta_X : FGX \to X$, which is the unique morphism that is sent by φ to the identity on GX. Then, for any morphism $\alpha : Y \to Z$ in $\mathbf{K}^-(Proj - \Lambda)$, we can define $G(\alpha)$ to be $\varphi(\alpha\eta_Y)$. By the functorality of F we get that φ is natural in the first variable, which implies that $G(id_Y) = \varphi(id_Y\eta_Y) = \varphi(\eta_Y) = id_{GY}$ and $G(\beta\alpha) = \varphi(\beta\alpha\eta_Y) = \varphi(\beta\eta_Z)\varphi(\alpha\eta_Y) = G(\beta)G(\alpha)$. This proves the functorality of G, so G is indeed a right adjoint functor to F.

□

Remark 2.4.3. Because the construction of G only involves the hom-functor and forming distinguished triangles, we see that G is a triangle functor.

2.5 Proof of theorem 1.2.2

Now we are almost ready to prove Rickard's Morita theorem. We just need to check three things first:

(1) If $add(T)$ generates $\mathrm{per}\,\Lambda$ as a triangulated category, then F is dense.

(2) The categorical structure of the various subcategories can be used to force an equivalence of $\boldsymbol{K}^-$ to restrict to an equivalence of $\boldsymbol{D}^b$, and so on.

(3) If F is an equivalence, then the image of Γ is a tilting complex.

Proof of (1)

If we assume that T is a tilting complex, then we know from theorem 2.3.1 that F is fully faithful. So showing that F is dense is all we need in order to conclude that it is an equivalence. To do this, we must take an arbitrary object $Y \in \mathbf{K}^-(Proj - \Lambda)$ and find an object $X \in \mathbf{K}^-(Proj - \Gamma)$ such that $Y = FX$. Given an object Y we know that there always exists an adjunction map $FGY \to Y$ (given by the counit of adjunction). We complete this map to a distinguished triangle $FGY \to Y \to Z \rightsquigarrow$. By applying G, which is a triangle functor, we get the distingusihed triangle $GFGY \to GY \to GZ \rightsquigarrow$. A property of the counit is that the composition $G \to GFG \to G$ is the identity (see for example theorem A.6.2 in [Wei94]). This means that the first map in the triangle is an isomorphism, which means that GZ is zero. This is obvously true for all shifts as well, and it implies that $FGZ[i] = 0$ for all i. Which in turn means that $\text{Hom}(T, Z[i]) = 0$, by the following isomorphisms

$$\begin{aligned} 0 = \text{Hom}(T, FGZ[i]) &\simeq \text{Hom}(FGT, FGZ[i]) \\ &\simeq \text{Hom}(GT, GZ[i]) \\ &\simeq \text{Hom}(FGT, Z[i]) \\ &\simeq \text{Hom}(T, Z[i]) \end{aligned}$$

where the second isomorphism comes from F being fully faithful, and the third one is given by F and G being an adjoint pair. We get the first and last isomorphisms from the fact that, by the construction of the functors, $FGT \simeq T$ (note that the T-resolution of T is itself). Thus there are no nonzero maps from T to $Z[i]$ for any i. The same is true for sums of summands of shifted copies of T, which means there are no maps from the triangulated category generated by $add - T$ to Z. Now, because Λ and all shifted copies of it lie in that category, this means that Z must be zero. Hence the map $FGY \to Y$ in the triangle we started with is an isomorphism, and we see that Y, which we chose arbitrarily, is isomorphic to $F(GY)$. This concludes the proof that F is dense.

Proof of (2)

We will now show that an equivalence $\mathbf{K}^-(Proj - \Lambda) \to \mathbf{K}^-(Proj - \Gamma)$ restricts to an equivalence $\mathbf{D}^b(\Lambda) \to \mathbf{D}^b(\Gamma)$, which itself restricts to an equivalence $\mathbf{K}^b(Proj - \Lambda) \to \mathbf{K}^b(Proj - \Gamma)$, which finally restricts to an equivalence $\mathrm{per}\,\Lambda \to \mathrm{per}\,\Gamma$.

Proposition 2.5.1. *An object X in $\boldsymbol{K}^-(Proj - \Lambda)$ lies up to isomoprhism in the subcategory $\boldsymbol{D}^b(\Lambda)$ (meaning it has bounded homology) if and only if for all objects Y in $\boldsymbol{K}^-(Proj - \Lambda)$ there exists an integer $N(X,Y)$ such that $Hom(Y, X[n]) = 0$ for all $n < N(X,Y)$.*

Thus any equivalence of triangulated categories between $\boldsymbol{K}^-(Proj - \Lambda)$ and $\boldsymbol{K}^-(Proj - \Gamma)$ induces an equivalence between $\boldsymbol{D}^b(\Lambda)$ and $\boldsymbol{D}^b(\Gamma)$.

Proof. An object X in $\mathbf{D}^b(\Lambda)$ satisfies the condition. To see why, notice that X is isomorphic in $\mathbf{D}^b(\Lambda)$ to a complex X' with zero in all small enough degrees. This means that the nonzero degrees of X' can be shifted away from the nonzero degrees of Y, giving only zero maps.

Conversely, if X is not in $\mathbf{D}^b(\Lambda)$ it has unbounded homology. If we set $Y = \Lambda$, we have that

$$\mathrm{H}^n(X) \simeq \mathrm{Hom}(\Lambda, \mathrm{H}^n(X)) \simeq \mathrm{H}^n\mathrm{Hom}(\Lambda, X) \simeq \mathrm{Hom}(\Lambda, X[n]).$$

So if $\mathrm{H}^n(X)$ is nonzero for some n, then $\mathrm{Hom}(\Lambda, X[n])$ is also nonzero. But since X has unbound homology $\mathrm{H}^n(X)$ will be nonzero for arbitrarily small n, which means there can't exist a number $N(X,\Lambda)$ such that $\mathrm{Hom}(\Lambda, X[n]) = 0$ for all $n < N(X,\Lambda)$.

Now, to see that this implies that the equivalence restricts down to $\mathbf{D}^b(\Lambda) \to \mathbf{D}^b(\Gamma)$, assume that we have an equivalence $G\colon \mathbf{K}^-(Proj - \Lambda) \to \mathbf{K}^-(Proj - \Gamma)$. Then we know that $\mathrm{Hom}(Y, X[n]) \simeq \mathrm{Hom}(GY, GX[n])$ for all $X, Y \in \mathbf{K}^-(Proj - \Lambda)$. So if there exists a number $N(X,Y)$ such that $\mathrm{Hom}(Y, X[n]) = 0$ for all $n < N(X,Y)$, there will exist a number $N(GX, GY)$ such that $\mathrm{Hom}(GY, (GX)[n]) = 0$ for all $n < N(GX, GY)$. Thus, the equivalence sends the subcategory $\mathbf{D}^b(\Lambda)$ to $\mathbf{D}^b(\Gamma)$, so it restricts as wanted. □

Proposition 2.5.2. *An object X in $\boldsymbol{D}^b(\Lambda)$ lies in $\boldsymbol{K}^b(Proj - \Lambda)$ (meaning it is a bounded complex) if and only if for all $Y \in \boldsymbol{D}^b(\Lambda)$ the set $Hom(X, Y[i])$ is zero for large i.*

Thus any equivalence of triangulated categories between $\boldsymbol{D}^b(\Lambda)$ and $\boldsymbol{D}^b(\Gamma)$ induces an equivalence between $\boldsymbol{K}^b(Proj - \Lambda)$ and $\boldsymbol{K}^b(Proj - \Gamma)$.

Proof. Let X be a complex in $\mathbf{K}^b(Proj - \Lambda)$ and Y a complex in $\mathbf{D}^b(\Lambda)$, meaning X is a bounded complex and Y is a (right bounded) complex with bounded homology. Then $\mathrm{Hom}(X, Y[i])$ will be zero for large enough i, since the highest nonzero degree of Y will be shifted past the lowest nonzero degree of X. For the other direction, we will show the contrapositive statement, namely that for an unbounded complex X' in $\mathbf{D}^b(\Lambda)$ there exists some complex Y such that there are nonzero morphisms from X' to all shifts of Y. We note that we can replace X' with an isomorphic complex X which has projective terms. Since X has bounded homology, we know that $\operatorname{Im} d_{i-1} = \operatorname{Ker} d_i$ for sufficiently small i.

As a step in the proof, we will now show that if $\operatorname{Ker} d_i$ is projective for some small i, then X must be bounded. To see this, notice that from degree $i-1$ in X, we get the short exact sequence

$$0 \to \operatorname{Ker} d_{i-1} \to X_{i-1} \to \operatorname{Im} d_{i-1} \to 0.$$

For small i we know that $\operatorname{Im} d_{i-1} = \operatorname{Ker} d_i$, so if $\operatorname{Ker} d_i$ is projective, the short exact sequence ends in a projective module, which means it is split exact. This implies that $X_{i-1} \simeq \operatorname{Ker} d_{i-1} \oplus \operatorname{Ker} d_i$, which means that $\operatorname{Ker} d_{i-1}$ is a direct summand of the projective module X_{i-1}, and hence projective. Repeating the same argument for $\operatorname{Ker} d_{i-1}$ shows that $X_{i-2} \simeq \operatorname{Ker} d_{i-2} \oplus \operatorname{Ker} d_{i-1}$, and if we keep going, we get that X is isomorphic to the complex

$$\cdots \to \operatorname{Ker} d_{i-3} \oplus \operatorname{Ker} d_{i-2} \to \operatorname{Ker} d_{i-2} \oplus \operatorname{Ker} d_{i-1} \to \operatorname{Ker} d_{i-1} \oplus \operatorname{Ker} d_i \to X_i \to \cdots$$

We can now remove trivial summands (theorem A.1.1) from all terms below degree i, and end up with the isomorphic complex

$$\cdots \to 0 \to 0 \to \operatorname{Ker} d_i \to X_i \to X_{i+1} \to \cdots .$$

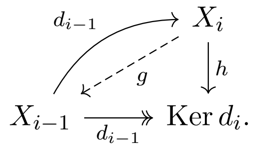
So if $\operatorname{Ker} d_i$ is projective for some small i, then X is a bounded complex (up to isomoprhism). This means that if X is not isomorphic to a bounded complex, then there are infinitely many i for which $\operatorname{Ker} d_i$ is not projective. We will now use this to find a complex Y such that $\operatorname{Hom}(X, Y[i]) \neq 0$ for all i.

First of all, notice that because X is right bounded, $\operatorname{Ker} d_i$ is trivially projective for all i large enough. This means that for there to be infinitely many i with $\operatorname{Ker} d_i$ not projective, such i must appear in arbitrarily small degrees. We choose infinitely many such i from below where the homology stops occuring in X (remember that X has bounded homology), so $\operatorname{Im} d_{i-1} = \operatorname{Ker} d_i$ for all i. Now we let Y be the direct sum of all these non-projective kernels $\operatorname{Ker} d_i$, viewed as a complex concentrated in degree zero. To see that this choice of Y ensures that $\operatorname{Hom}(X, Y[i])$ is nonzero for all i, observe the following: For each i we have a map $f \colon X \to Y[i]$, given by

$$\begin{array}{ccccccccc}
X = \cdots & \longrightarrow & X_{i-2} & \xrightarrow{d_{i-2}} & X_{i-1} & \xrightarrow{d_{i-1}} & X_i & \longrightarrow & \cdots \\
f = & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d_{i-1}} & & \downarrow{\scriptstyle 0} & & \\
Y[i] \supset \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{Ker} d_i & \longrightarrow & 0 & \longrightarrow & \cdots ,
\end{array}$$

since $\operatorname{Im} d_{i-1} = \operatorname{Ker} d_i$. The only possible problem now is that f might be null-homotopic. But if that is the case, then there exists an $h \colon X_i \to \operatorname{Ker} d_i$

such that $d_{i-1} = hd_{i-1}$. Notice that d_{i-1} is surjective on $\operatorname{Im} d_{i-1} = \operatorname{Ker} d_i$, and recall that X_i is projective. By the lifting property we then know there exists a dashed map g such that $h = d_{i-1}g$, making the following diagram commutative



(2.1)

Combining the two expressions, we get that $h = d_{i-1}g = hd_{i-1}g$. Now notice that $d_{i-1} = hd_{i-1}$ being surjective on $\operatorname{Ker} d_i$ implies h is also surjective on $\operatorname{Ker} d_i$, so h restricted to $\operatorname{Im} d_{i-1} = \operatorname{Ker} d_i$ is an isomorphism. Thus, when we restrict to $\operatorname{Ker} d_i$ we can remove h from both sides of $h = hd_{i-1}g$, and get that $id_{\operatorname{Ker} d_i} = d_{i-1}g$. In other words, d_{i-1} is a split epimorphism from X_{i-1} to $\operatorname{Ker} d_i$. This means that the (canonical) short exact sequence

$$0 \to \operatorname{Ker} d_{i-1} \to X_{i-1} \xrightarrow{d_{i-1}} \operatorname{Ker} d_i \to 0$$

is split exact. This implies that $\operatorname{Ker} d_i$ is projective, since it is the last term in a split exact sequence. But $\operatorname{Ker} d_i$ is non-projective by assumption, so this is a contradiction. Thus, f can't be null-homotopic, which means that there is a nonzero map $X \to Y[i]$. This concludes the proof of the "if and only if" part. Now we easily see that this implies an equivalence between $\mathbf{D}^b(\Lambda)$ and $\mathbf{D}^b(\Gamma)$ restricts to an equivalence between $\mathbf{K}^b(Proj - \Lambda)$ and $\mathbf{K}^b(Proj - \Gamma)$. Simply notice that since $\operatorname{Hom}(X, Y[i]) \simeq \operatorname{Hom}(GX, GY[i])$, one of them is zero for large i if and only if the other one is. □

Proposition 2.5.3. *An object X in $\boldsymbol{K}^b(Proj - \Lambda)$ lies in $per\,\Lambda$ (has finitely generated terms) if and only if the functor*

$$Hom(X, -)\colon \boldsymbol{K}^b(Proj - \Lambda) \to Ab$$

commutes with arbitrary direct sums. Thus any equivalence of triangulated categories between $\boldsymbol{K}^b(Proj - \Lambda)$ and $\boldsymbol{K}^b(Proj - \Gamma)$ induces an equivalence between $per\,\Lambda$ and $per\,\Gamma$.

Proof. To see that $\operatorname{Hom}(X, -)$ commutes with arbitrary direct sums when X lies in $\operatorname{per} \Lambda$, we use the same argument as in the proof of lemma 2.1.1. Let $(Y_i)_{i\in I}$ be a collection of complexes in $\mathbf{K}^b(Proj - \Lambda)$ for some index set I. Then there is a natural homomorphsim $\bigoplus_{i\in I} \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}(X, \bigoplus_{i\in I} Y_i)$, given by sending a tuple of endomorphisms $(\varphi_i)_{i\in I}$ to the map $\left(x \mapsto \big(\varphi_i(x)\big)_{i\in I}\right)$. Observe that the only way $\left(x \mapsto \big(\varphi_i(x)\big)_{i\in I}\right)$ can be the zero map is if all φ_i are zero, so the morphism between them is injective. To see that it actually is an isomorphism, we use the fact that X is a bounded complex of finitely generated Λ-modules. This means that a map in $\mathrm{Hom}(X, \bigoplus_{i\in I} Y_i)$ is fixed by where it sends a finite number of elements from each of the finitely many terms of X. Consequently, for each element of $\mathrm{Hom}(X, \bigoplus_{i\in I} Y_i)$ we can find some element of $\bigoplus_{i\in I} \mathrm{Hom}(X, Y_i)$ which is mapped to that element. We conclude that the map is surjective, and hence an isomorphism. This shows that $\mathrm{Hom}(X, -)$ commutes with arbitrary direct sums when $X \in \mathrm{per}\,\Lambda$.

For the other direction, let's assume that $\mathrm{Hom}(X, -)$ commutes with arbitrary direct sums for $X \in \mathbf{K}^b(Proj - \Lambda)$. We will show that all terms of X are finitely generated, by induction on the number n of nonzero terms. If $n = 0$ then X is the zero complex, which does have finitely generated terms, so the assertion is true. Now we assume it is true for all complexes of length less than or equal to n, and consider the complex $X = X_0 \to X_1 \to \cdots \to X_n$. We denote by Y the n-term complex we get by cutting off X_0 from X, that is $Y = X_1 \to \cdots \to X_n$. Now we can form a distinguished triangle $Y \to X \to X_0 \rightsquigarrow$, and for any Z we may apply $\mathrm{Hom}(-, Z)$ to this triangle to get a long exact sequence. Then, the five lemma implies that if Y and X_0 have finitely generated terms, then so does X.

The only problem now is that we haven't checked what happens when X_0 is not finitely generated. First of all, note that since X_0 is projective we can add some trivial summand to X (theorem A.1.1) to get an isomorphic complex where X_0 is replaced by a free object $\Lambda^{(I)}$, for some index set I. Thus we get that

$$\mathrm{Hom}(X, X_0) \simeq \mathrm{Hom}(X, \Lambda^{(I)}) \simeq \mathrm{Hom}(X, \Lambda^{(I)}).$$

By definition, only finitely many of the maps in the last coproduct above are nonzero. Now denote by α the natural map $X \to X_0$ which is the identity on X_0 and 0 everywhere else. The above implies that $\alpha \in \mathrm{Hom}(X, X_0)$ is homotopic to a map, say β, with finitely generated image. If we call the homotopy s, we can write this as a diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & X_0 & \xrightarrow{d_0} & X_1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle 1} & \swarrow{\scriptstyle s} & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X_0 & \longrightarrow & 0 & \longrightarrow & \cdots .
\end{array}$$

This means that we can decompose X_0 as $Y \oplus Y'$ where Y is finitely generated and containing the image of β. Since $1_{X_0} = sd_0$, we can write the projection $X_0 \xrightarrow{p} Y'$ as $X_0 \xrightarrow{d_0} X_1 \xrightarrow{s} X_0 \xrightarrow{p} Y'$. By restricting to Y', we see that this implies that X contains a direct summand $Y' \to Y'$ in degree zero and one. This is a trivial summand, so by theorem A.1.1, removing it from X yields an isomorphic complex. In degree zero of the resulting complex we only have Y, which is finitely generated. Thus, we can apply the induction as we did above, which completes the proof.

Now all we need to show is that the equivalence restricts as it should. To see that it does, we must show that $\mathrm{Hom}(X,-)$ commutes with arbitrary sums if and only if $\mathrm{Hom}(GX,-)$ does so. Note that we have the following

$$\begin{aligned}\mathrm{Hom}(G,\bigoplus_i GY_i) &\simeq \mathrm{Hom}(GX,G(\bigoplus_i Y_i))\\ &\simeq \mathrm{Hom}(X,\bigoplus_i Y_i)\\ &\simeq \bigoplus \mathrm{Hom}(X,Y_i)\\ &\simeq \bigoplus \mathrm{Hom}(GX,GY_i)\end{aligned}$$

The first isomorphism holds because G is right adjoint, so it commutes with colimits. The second and fourth isomorphism come from the fact that G is fully faithful. The third isomorphism is given by the assumption that $\mathrm{Hom}(X,-)$ commutes with arbitrary direct sums. □

Proof of (3)

Proof. Assume that we have an equivalence $F\colon \mathbf{K}^-(Proj{-}\Gamma) \to \mathbf{K}^-(Proj{-}\Lambda)$. By (2), this restricts down to an equivalence $F\colon \operatorname{per}\Gamma \to \operatorname{per}\Lambda$. We will now show that $F(\Gamma)$, the image of F under Γ, is a tilting complex over Λ. First note that Γ is sent by F to an object in $\operatorname{per}\Lambda$, since Γ is contained in $\operatorname{per}\Gamma$. Hence $F(\Gamma)$ is a bounded complex of finitely generated projective Λ-modules. Moreover, Γ generates $\operatorname{per}\Gamma$ (by definition), so F being an equivalence implies that $F(\Gamma)$ generates $\operatorname{per}\Lambda$. Lastly, observe that $\mathrm{Hom}(\Gamma,\Gamma[n]) = 0$ when $n \neq 0$, since Γ is viewed as a complex concentrated in degree 0. When F is an equivalence, this property is carried over to $F(\Gamma)$. Thus, $F(\Gamma)$ satisfies all properties of being a tilting complex over Λ. □

The proof of theorem 1.2.2 now follows from the construction of the functors F and G, together with the results (1), (2) and (3).

Chapter 3

Differential graded algebras

Differential graded algebras, or dg algebras, are graded algebras endowed with a differential morphism. In this chapter we will give some central definitions, and present some of the theory surrounding dg algebras. We will apply this theory in chapter 4, where we give another proof of Rickard's Morita theorem.

3.1 Definitions and examples

Definition 3.1.1. This is how we define a graded structure on rings, algebras and modules.

- A ($\mathbb{Z}$-)*graded ring* is a ring which can be written as a direct sum of abelian groups, $R = \bigoplus_{i\in\mathbb{Z}} R^i$, such that $R^iR^j \subseteq R^{i+j}$ for all $i, j \in \mathbb{Z}$.
- A ($\mathbb{Z}$-)*graded algebra* is an algebra which is graded as a ring.
- A ($\mathbb{Z}$-)*graded module* is a right module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ over a graded ring R, such that $M^iR^j \subseteq M^{i+j}$.
- A morphism $f\colon M \to N$ between graded modules is called a *graded morphism of degree d*, and it is a collection of morphisms between the underlying modules such that $f(M^i) \subseteq N^{i+d}$.

Note: we also have what is called a *bigraded* module, which is a graded module in which each degree is itself a graded module. In a bigraded module, we refer to degrees by pairs of numbers, where (i, j) means the component which is in degree i of the graded module in degree j of the bigraded module. The degrees of maps are also given as pairs of numbers.

Definition 3.1.2 (dg algebra)**.** Let k be a commutative ring. We define a *dg algebra* (differential graded k-algebra) as a $\mathbb{Z}$-graded associative k-algebra

$$\Lambda = \bigoplus_{p\in\mathbb{Z}} \Lambda^p$$

with a differential $d\colon \Lambda \to \Lambda$ which is k-linear and graded of degree 1 (so $d\Lambda^p \subset \Lambda^{p+1}$), and satisfies the graded Leibniz rule

$$d(ab) = (da)b + (-1)^p adb,\ \forall a \in \Lambda^p,\ \forall b \in \Lambda.$$

We do not impose any finiteness conditions on Λ. We define dg modules in a similar way as dg algebras.

Example 3.1.3. Any 'ordinary' k-algebra Γ can be viewed as a dg algebra Λ concentrated in one degree, that is

$$\Lambda^n = \begin{cases} \Gamma, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

The converse also holds, that any dg algebra which is concentrated in one degree corresponds to an 'ordinary' algebra.

Example 3.1.4. Let Γ be a k-algebra, and let M and X be chain complexes of Γ-modules. We can define a new complex $\mathcal{H}om_\Gamma(X, M)$, which is given by the followng

$$\mathcal{H}om_\Gamma(X, M)^n = \prod_{-p+q=n} \mathrm{Hom}_\Gamma(X^p, M^q)$$

$$(df)(x) = d(f(x)) - (-1)^n f(dx), f \in \mathcal{H}om_\Gamma(X, M)^n.$$

Then $\Lambda = \mathcal{H}om_\Gamma(M, M)$ has a natural graded structure and a differential. This means that Λ is a dg algebra. Be aware that there generally will be non-vanishing components in arbitrarily small *and* arbitrarily large degrees of Λ, even if $M^i = 0$ for all $i \gg 0$.

Definition 3.1.5 (dg module)**.** A *dg Λ-module* (differential graded module over Λ) is a $\mathbb{Z}$-graded right Λ-module

$$M = \bigoplus_{p \in \mathbb{Z}} M^p$$

with a differential $d\colon M \to M$ which is k-linear and graded of degree 1, and satisfies the graded Leibniz rule

$$d(ma) = d(m)a + (-1)^p ad(m),\ \forall m \in M^p,\ \forall a \in \Lambda$$

A morphism $f\colon M \to N$ of dg Λ-modules is a graded morphism of degree 0 from M to N as graded Λ-modules which commutes with the differentials.

Example 3.1.6. For any dg algebra Λ which is concentrated in one degree (so it is an 'ordinary' algebra Γ in degree zero, and 0 in all other degrees), the category of dg Λ-modules is equal to the category of chain complexes of right Γ-modules.

Example 3.1.7. If Γ is a k-algebra, M is a complex of right Γ-modules and $\Lambda = \mathcal{H}om_\Gamma(M, M)$, then M becomes a left Λ-module by the action $(f^i)(M^j) = (f^i(m^i))$

3.2 The homotopy category

A morphism $f\colon M \to N$ of dg modules is called *null-homotopic* if there exists a graded morphism $r\colon M \to N$ of degree -1, such that $f = dr + rd$. We define the *homotopy category* $\mathbf{K}(\Lambda)$, whose objects are the dg Λ-modules, and whose morphisms $\bar{f}$ are equivalence classes of dg module morphisms modulo null-homotopic morphisms. We will show that $\mathbf{K}(\Lambda)$ is a triangulated category, and to do so we need two more definitions.

Definition 3.2.1. The *suspension functor* $[1]\colon \mathbf{K}(\Lambda) \to \mathbf{K}(\Lambda)$ is defined by

$$\big(M[1]\bigcap)^p = M^{p+1}$$

$$d_{M[1]} = -d_M$$

$$\mu_{M[1]}(m,a) = \mu_M(m,a)$$

for $m \in M, a \in \Lambda$, where μ_M and $\mu_{M[1]}$ are the multiplication maps in the respective modules.

Definition 3.2.2. A *standard triangle* of $\mathbf{K}(\Lambda)$ is a sequence

$$L \xrightarrow{\bar{f}} M \xrightarrow{\bar{g}} Cf \xrightarrow{\bar{h}} L[1],$$

where $f\colon L \to M$ is a morphism of DG modules, and Cf is the *mapping cone* of f, in the sense that $Cf = M \oplus L[1]$ as a graded k-module, with

$$d_{Cf} = \begin{bmatrix} d_M & f \\ 0 & d_{L[1]} \end{bmatrix}, \mu_{Cf}\left(\begin{bmatrix} m \\ l \end{bmatrix}, a\right) = \begin{bmatrix} ma \\ la \end{bmatrix},$$

for $m \in M$ and $l \in L^p$. The morphism g in the triangle is the inclusion of M into Cf, and $-h$ is the canonical projection $Cf \to L[1]$.

Endowed with these structures $\mathbf{K}(\Lambda)$ becomes a triangulated category, but before we can prove that we will need the a few new definitions and a lemma. An *exact category* (in the sense of Quillen, see [Qui73]) is a pair $(\mathscr{B}, \mathscr{S})$, where $\mathscr{B}$ is a full subcategory of an abelian category $\mathscr{A}$, and $\mathscr{S}$ is the set of exact sequences in $\mathscr{A}$ with terms in $\mathscr{B}$. Also $\mathscr{B}$ must be closed under extensions, that is, if the first and last term of a short exact sequence is in $\mathscr{B}$, then the middle term is also in $\mathscr{B}$. A *Frobenius category* is an exact category with enough projectives and injectives, where all injective modules are projective and vice versa. The *stable category* $\underline{\mathscr{B}}$ of a Frobenius category $\mathscr{B}$, is the category whose objects are the objects of $\mathscr{B}$, and whose morphisms are the morphisms of $\mathscr{B}$ modulo morphisms that factor through projective-injective objects.

Lemma 3.2.3. *The category of dg Λ-modules, together with the exact structure given by sequences that split as graded Λ-modules, is a Frobenius category whose stable category is $\boldsymbol{K}(\Lambda)$.*

Proof. We start by proving that the category is Frobenius. First notice that the given exact structure is equivalent to sequences of complexes that are degree-wise split, but which don't necessarily commute with the differentials. We want to show that the category has enough projectives and injectives. To do so, we will take an arbitrary differential graded Λ-module (A, d), where A is a graded module and d is its differential, and find a monomorphism to an injective, and an epimorphism from a projective. Given any dg Λ-module (A, d), we have the canonical morphisms

$$A \hookrightarrow A \oplus A[1] \qquad \text{and} \qquad A \oplus A[1] \twoheadrightarrow A[1],$$

which are clearly a monomorphism and an epimorphism, respectively. So if we can show that the DG module $(A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right])$ is both injective and projective, we are done. We show that it is projective, injectivity is shown with a dual argument. We look at the set of morphisms of dg modules from $A \oplus A[1]$, in other words, morphisms of graded Λ-modules starting in $A \oplus A[1]$, which respect the grading.

$$\begin{aligned} \mathrm{Hom}_{dg\,\Lambda}\big((A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right]), (M, e)\big) &= \big\{[f \quad g] \mid [f \quad g]\left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right] = e[f \quad g]\big\} \\ &= \{[f \quad g] \mid [fd \quad f - gd] = [ef \quad eg]\} \\ &= \{[f \quad g] \mid fd = ef \text{ and } f - gd = eg\} \end{aligned}$$

Notice that the second condition gives that $f = gd + eg$, which implies that the first condition also is satisfied, since $fd = (gd + eg)d = egd = e(gd+eg) = ef$. So for a given morphism of graded Λ-modules $g \colon A[1] \to M$, we get a morphism of dg Λ-modules $[f \quad g] \colon (A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right]) \to (M, e)$ by setting $f = gd + eg$. This means that for any DG Λ-module (M, e), we have a bijection $\mathrm{Hom}_{dg\,\Lambda}\big((A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right]), (M, e)\big) \leftrightarrow \mathrm{Hom}_{gr\,\Lambda}(A[1], M)$, where $\mathrm{Hom}_{gr\,\Lambda}$ denotes morphisms of graded Λ-modules. Now, to see that $(A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right])$ is projective, we show that given the solid part of this diagram, the dashed arrow is induced

$$\begin{array}{ccccc} & & (A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right]) & & \\ & \swarrow & \downarrow{\scriptstyle [f \quad g]} & & \\ (N, c) & \xrightarrow{\varphi} & (M, e) & \longrightarrow & 0. \end{array}$$

To do so, we observe that the bijection of Hom-sets allows us to instead look at the following diagram of graded Λ-modules

$$\begin{array}{ccccc} & & A[1] & & \\ & \swarrow{\scriptstyle \psi g} & \downarrow{\scriptstyle g} & & \\ N & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & M & \longrightarrow & 0. \end{array}$$

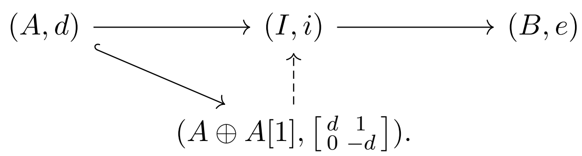
By construction of the exact structure, all epimorphisms of graded modules are split. Thus we know that there exists a $\psi\colon M \to N$ such that $\varphi\psi = id_M$. Composing this with g gives a morphism $A[1] \to N$ (the dashed arrow above) such that the triangle commutes. But then using the bijection again, we get a morpism $(A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right]) \to (N, c)$ making the top triangle commute, which is what we wanted. Thus we have shown that $(A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right])$ is projective. The argument that it is injective is entirely dual, and we will skip it.

Now, to conclude that the category is Frobenius, the only thing we need to show is that any projective object is injective, and vice versa. To see this, take a projective dg Λ-module (P, p) and an injective dg Λ-module (I, i). We know that we have morphisms

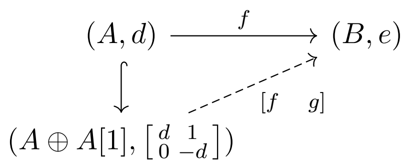
$$(P \oplus P[1], \left[\begin{smallmatrix} p & 1 \\ 0 & -p \end{smallmatrix}\right]) \twoheadrightarrow (P, p) \qquad\qquad (I, i) \hookrightarrow (I \oplus I[1], \left[\begin{smallmatrix} i & 1 \\ 0 & -i \end{smallmatrix}\right]).$$

Any epimorphism to a projective is split, so (P, p) is a direct summand of $(P \oplus P[1], \left[\begin{smallmatrix} p & 1 \\ 0 & -p \end{smallmatrix}\right]$, which as we know is injective. Thus (P, p) is injective, since direct summands of injectives are injective. Similarly, monomorphisms from injectives are split, so (I, i) is a direct summand of a projective, and is thus projective.

The only thing left to show now is that the stable category is equal to the homotopy category $\mathbf{K}(\Lambda)$. To do this, we will show that a morphism factors through an injective module if and only if it is null-homotopic. Because assume that we have a morphism between two dg Λ-modules (A, d) and (B, e) which factors through an injective module (I, i). We know that (A, d) has an injective envelope $(A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right])$, which means that ι is a monomorphism. Thus, since (I, i) is injective, we have the dashed arrow such that the triangle in the following diagram commutes.



This means that a morphism factors through an injective if and only if it factors through a module on the form $(A \oplus A[1], \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right])$. But this is equivalent to, given the morphism f in the following diagram, finding a graded morphism $g\colon A[1] \to B$ such that $[f \quad g]$ makes the diagram commute.



Writing out what this actually means, we want g such that

$$\begin{aligned} & e[f \quad g] = [f \quad g] \left[\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right] \\ \Leftrightarrow \quad & [ef \quad eg] = [fd \quad f - gd] \\ \Leftrightarrow \quad & ef = fd \quad \text{and} \quad f = eg + gd \end{aligned}$$

The first condition is satisfied by definition, since f is a morphism of dg Λ-modules. Notice that g can be viewed as a graded morphism of degree -1 from (A, d) to (B, e). So the second condition is exactly the definition of f being a null-homotopic morphism. So the morphism f factors through $A \oplus A[1]$ if and only if it is null-homotopic. Combining this with the previous result, we see that a morphism of dg Λ-modules factoring through an injective is equvalent with it being null-homotopic. So both conditions define the same equivalence relation on the category of dg Λ-modules, and we conclude that its stable category is $\mathbf{K}(\Lambda)$. □

Proposition 3.2.4. *The category $\boldsymbol{K}(\Lambda)$, with the functor* [1] *and triangles isomorphic to standard triangles, is a triangulated category.*

Proof. We refer to a theorem due to Happel in [Hap87], which states that the stable category of a Frobenius category is triangulated. In lemma 3.2.3, we showed that the category of dg Λ-modules, with the exact structure given by sequences that split as graded Λ-modules, is a Frobenius category. We also showed that the associated stable category is $\mathbf{K}(\Lambda)$. Thus, we conclude that $\mathbf{K}(\Lambda)$ is a triangulated category. □

Chapter 4

Proof using differential graded algebras

In this chapter we will present an alternate proof of Rickard's theorem, due to Bernhard Keller [Kel98]. It employs a different approach than the previous proof, one based on the theory of differential graded algebras. The techniques used in this proof are more general, in that they can be applied to different problems as well. In fact, this is even explicitly shown in Keller's article. He spends most of the article developing theory, and then he presents three different applications of that theory, one of them being a proof of Rickard's theorem. We will not present the other applications here, but focus on the general theory and Rickard's theorem.

4.1 Strategy of the proof

This proof is structured as follows:

- We define so-called homotopically projective and homotopically injective modules, and develop some theory surrounding them.
- Next, we present the principle of infinite dévissage, which gives a criterion for when a triangulated subcategory of $\mathbf{D}(\Lambda)$ is equal to $\mathbf{D}(\Lambda)$.
- With the help of homotopically projective and injective modules we define total left and right derived functors of tensor and hom. We then use these to state and prove a theorem which essentially is Ricksrd's theorem with an extra assumption, namely the existence of a bimodule complex.
- The rest of the chapter is devoted to showing how the assumptions in Rickard's theorem allows us to construct a bimodule complex. The proof then follows from the previous theorem. This is where we need the theory of differential graded algebras.

Remark. In this proof we use a slightly different statement of Rickard's theorem than what we used in the previous proof. There are two main differences. The first is that, because of how the bimodule complex is constructed, we get an explicit description of the functor which acts as the triangle equivalence in Rickard's theorem. The second difference is that this statement doesn't require the derived categories to be bounded. See theorem 4.7.1 for the actual statement we use in this chapter.

4.2 Unbounded resolutions

Definition 4.2.1. A complex K is *homotopically projective* if

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(K, N) = 0$$

for all acyclic complexes N. Dually, the complex is *homotopically injective* if

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(N, K) = 0$$

for all acyclic complexes N.

We write $\mathbf{K}_p(\Lambda)$ for the category of homotopically projective complexes as a full subcategory of $\mathbf{K}(\Lambda)$ (and similarly $\mathbf{K}_i(\Lambda)$ for the full subcategory of homotopically injective complexes). If K is any complex with projective components and *vanishing* differential, then K is homotopically projective. To see this, observe that for a given degree n of a map $f \in \operatorname{Hom}(K, N)$, we get the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{0} & P^{n-1} & \xrightarrow{0} & P^{n} & \xrightarrow{0} & P^{n+1} & \xrightarrow{0} & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^{n}} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \xrightarrow{d} & N^{n-1} & \xrightarrow{d^{n-1}} & N^{n} & \xrightarrow{d^{n}} & N^{n+1} & \xrightarrow{d} & \cdots
\end{array}$$

where P^i is projective for all i, and $\operatorname{Im} d^{n-1} = \operatorname{Ker} d^n$. By commutativity $d^n f^n = 0$, which implies that $\operatorname{Im} f^n \subseteq \operatorname{Ker} d^n = \operatorname{Im} d^{n-1}$. This means that we have an epimorphism $N^{n-1} \to \operatorname{Im} f^n$, and since P^n is projective we thus get a map $h^n \colon P^n \to N^{n-1}$ such that $f^n = d^{n-1} h^n$. Because this is true for all degrees of the chain map f, we have that f is null-homotopic, which is the zero map in $\mathbf{K}(\Lambda)$. It is easy to see that direct sums of homotopically projective complexes are homotopically projective, since $\operatorname{Hom}(-, N)$ takes coproducts to products, and products of zeros are zero. Another important observation regarding homotopically projective complexes is the following proposition.

Proposition 4.2.2. *The collection of all homotopically projective complexes is a full triangulated subcategory of* $\boldsymbol{K}(\Lambda)$

Proof. By simply taking the collection of all homotopically projective complexes, and all morphisms between them, we get that $\mathbf{K}_p(\Lambda)$ forms a full subcategory of $\mathbf{K}(\Lambda)$. To see that it is also triangulated, we must check that both the shift functor and taking cones respect homotopical projectivity. For the shift functor we have that $\mathrm{Hom}(K, N) = 0$ implies $\mathrm{Hom}(K[n], N) \simeq \mathrm{Hom}(K, N[-n]) = 0$ for all n, since the shift of an acyclic complex is still acyclic. To see that $\mathbf{K}_p(\Lambda)$ is closed under cones, take a morphism of complexes $f\colon X \to Y$, with X and Y homotopically projective. Because of the triangulated structure on $\mathbf{K}(\Lambda)$, we can complete this to a triangle with the cone of f:

$$X \xrightarrow{f} Y \longrightarrow \mathrm{Cone}(f) \longrightarrow X[1] \ .$$

By applying the functor $\mathrm{Hom}(-, N)$ with N acyclic, we get a long exact sequence. Since X, $X[1]$ and Y are homotopically projective, all terms coming from them are zero, so $\mathrm{Hom}(\mathrm{Cone}(f), N)$ must be zero too. □

Now take the following directed system in the category of complexes

$$\begin{array}{c} P_0 \xrightarrow{i_0} P_1 \xrightarrow{i_1} P_2 \longrightarrow \cdots \longrightarrow P_q \xrightarrow{i_q} P_{q+1} \longrightarrow \cdots \\ \| \\ 0 \end{array} \tag{4.1}$$

where for all $q \in \mathbb{N}$, the chain map i_q has split monomorphisms in each degree, and for each q the degreewise complex of quotients P_{q+1}/P_q has vanishing differentials and is projective in each degree. As we have seen above, this means that P_{q+1}/P_q is homotopically projective for all q. We will now use induction to prove that P_q is homotopically projective for all q. To do so, we will use lemma A.2.1, which states that any sequence of complexes which is split exact in each degree, gives rise to a distinguished triangle in $\mathbf{K}(\Lambda)$. We start by observing that $P_0 = 0$, which is trivially homotopically projective, thus proving the base case. Next, we assume that P_q is homotopically projective, and look at the following sequence of complexes

$$P_q \xrightarrow{i_q} P_{q+1} \longrightarrow P_{q+1}/P_q \ ,$$

which looks like this in each degree

$$0 \longrightarrow P_q^j \xrightarrow{i_q^j} P_{q+1}^j \longrightarrow P_{q+1}^j/P_q^j \longrightarrow 0 \ .$$

It is split exact because i_q^j is a split monomorphism for all j. Thus, by lemma A.2.1 we have the following distinguished triangle in $\mathbf{K}(\Lambda)$:

$$P_q \xrightarrow{i_q} P_{q+1} \longrightarrow P_{q+1}/P_q \longrightarrow P_q[1] \ .$$

If we apply the functor $\operatorname{Hom}(-, N)$ to the triangle, with N acyclic, we get a long exact sequence. Now, if we assume that P_q is homotopically projective, then the terms coming from P_q and P_{q+1}/P_q are all zero. This means that $\operatorname{Hom}(P_{q+1}, N) = 0$, and thus P_{q+1} is homotopically projective. Consequently, all terms of our directed system are homotopically projective. Next, we want to show that this holds for $K = \varinjlim P_q$ as well. In order to do so, we will use the so-called *Milnor's Triangle*, which is constructed as follows: Define the morphism

$$\Phi \colon \bigoplus_{p \in \mathbb{N}} P_p \to \bigoplus_{q \in \mathbb{N}} P_q$$

whose components are the compositions

$$P_q \xrightarrow{\binom{1}{-i_p}} P_p \bigoplus P_{p+1} \longrightarrow \bigoplus_{q \in \mathbb{N}} P_q \ , \tag{4.2}$$

where the last map is the canonical incusion. The map Φ gives the following sequence

$$0 \longrightarrow \bigoplus_{p \in \mathbb{N}} P_p \xrightarrow{\Phi} \bigoplus_{q \in \mathbb{N}} P_q \xrightarrow{\Theta} \varinjlim P_q \longrightarrow 0 \ , \tag{4.3}$$

where Θ consists of the canonical morphisms $f_q \colon P_q \to \varinjlim P_q$.

Proposition 4.2.3. *The sequence in eq.* (4.3) *is split exact.*

Proof. First, notice that the following is true for all q

$$P^j_{q+1} \simeq \operatorname{Im} i^j_q \oplus P^j_{q+1}/\operatorname{Im} i^j_q \simeq P^j_q \oplus P^j_{q+1}/P^j_q,$$

where the last isomorphism holds since i_q is split mono in each degree j. This means that we can write the directed sequence in each degree, excluding zero, as the following (remember that since $P^j_0 = 0$ for all j, we have $P^j_1 \simeq 0 \oplus P^j_1/0 \simeq P^j_0 \oplus P^j_1/P^j_0$)

$$\begin{array}{ccccccc}
P^j_1 & \xrightarrow{i^j_1} & P^j_2 & \xrightarrow{i^j_2} & P_3 & \longrightarrow & \cdots \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
P_0 \oplus P^j_1/P^j_0 & \longrightarrow & P^j_1 \oplus P^j_2/P^j_1 & \longrightarrow & P_2 \oplus P^j_3/P^j_2 & \longrightarrow & \cdots \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
P^j_1/P^j_0 & \xrightarrow{\binom{1}{0}} & P^j_1/P^j_0 \oplus P^j_2/P^j_1 & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} & \bigoplus_{i=1}^{3} P^j_i/P^j_{i-1} & \longrightarrow & \cdots
\end{array} \tag{4.4}$$

By induction, we see that we can write $P_q^j \simeq \bigoplus_{i=1}^{q} P_i^j/P_{i-1}^j$ for each $q \in \mathbb{N}$. Combining this with the map i_q^j, we get the following commutative diagram

$$\begin{array}{ccc} P_q^j & \xrightarrow{\quad i_q^j \quad} & P_{q+1}^j \\ \downarrow\wr & & \downarrow\wr \\ \bigoplus_{i=1}^{q} P_i^j/P_{i-1}^j & \hookrightarrow & \bigoplus_{i=1}^{q+1} P_i^j/P_{i-1}^j \\ \cup & & \cup \\ x_q & \longmapsto & (x_q, 0) \end{array}.$$

So up to isomorphism, we can view i_q^j as the canonical inclusion which sends $x_q \in \bigoplus_{i=1}^{q} P_i^j/P_{i-1}^j$ to itself in $\bigoplus_{i=1}^{q+1} P_i^j/P_{i-1}^j$. We then see that we can write the maps from eq. (4.2) (the component maps of Φ) as

$$\begin{array}{ccccc} P_p^j & \xrightarrow{\begin{pmatrix} 1 \\ -i_p^j \end{pmatrix}} & P_p^j \bigoplus P_{p+1}^j & \longrightarrow & \bigoplus_{q\in\mathbb{N}} P_q^j \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \bigoplus_{i=1}^{p} P_i^j/P_{i-1}^j & \longrightarrow & \bigoplus_{i=1}^{p} P_i^j/P_{i-1}^j \oplus \bigoplus_{i=1}^{p+1} P_i^j/P_{i-1}^j & \longrightarrow & \bigoplus_{q\in\mathbb{N}} \bigoplus_{i=1}^{q} P_i^j/P_{i-1}^j \end{array}.$$

For any given $p \in \mathbb{N}$, each element $x_p \in \bigoplus_{i=1}^{p} P_i^j/P_{i-1}^j$ is sent to the same element in the component sums corresponding to $q = p$ and $q = p+1$. Now notice that we can view Φ as a morphism

$$\bigoplus_{p\in\mathbb{N}} \bigoplus_{i=1}^{p} P_i^j/P_{i-1}^j \longrightarrow \bigoplus_{q\in\mathbb{N}} \bigoplus_{i=1}^{q} P_i^j/P_{i-1}^j.$$

We can rearrange this expression into

$$\bigoplus_{i\in\mathbb{N}} \bigoplus_{p\geq i} P_i^j/P_{i-1}^j \longrightarrow \bigoplus_{i\in\mathbb{N}} \bigoplus_{q\geq i} P_i^j/P_{i-1}^j,$$

which then can be written as

$$\bigoplus_{i\in\mathbb{N}} \left[\bigoplus_{p\geq i} P_i^j/P_{i-1}^j \longrightarrow \bigoplus_{q\geq i} P_i^j/P_{i-1}^j \right].$$

□

By lemma A.2.1, we see that the degreewise split exact sequence above gives rise to the following distinguished triangle in $\mathbf{K}(\Lambda)$, which is the so-called *Milnor's Triangle*:

$$\bigoplus_{p\in\mathbb{N}} P_p \xrightarrow{\overline{\Phi}} \bigoplus_{q\in\mathbb{N}} P_q \xrightarrow{\overline{\Theta}} \varinjlim P_q \longrightarrow (\bigoplus_{p\in\mathbb{N}} P_p)[1] \ .$$

Remember that our goal is to prove that $\varinjlim P_q$ is homotopically projective, and to do so we apply the functor $\mathrm{Hom}_{\mathbf{K}(\Lambda)}(-, N)$ to the triangle, where N is an acyclic complex. This is a cohomological functor, which means that applying it to the distinguished triangle gives the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}((\bigoplus_{p\in\mathbb{N}} P_p)[1], N) \longrightarrow \mathrm{Hom}(\varinjlim P_q, N) \longrightarrow \mathrm{Hom}(\bigoplus_{q\in\mathbb{N}} P_q, N) \longrightarrow \cdots \ .$$

We now use the fact that

$$\mathrm{Hom}(\bigoplus_{p\in\mathbb{N}} P_p, N) \simeq \prod_{p\in\mathbb{N}} \mathrm{Hom}(P_p, N) \ ,$$

which is zero because P_p is homotopically projective for all $p \in \mathbb{N}$. Thus, the term $\mathrm{Hom}_{\mathbf{K}(\Lambda)}(\varinjlim P_q, N)$ is between two zeros in the long exact sequence, so it must be zero. Since this is true for any acyclic complex N, we conclude that $\varinjlim P_q$ is homotopically projective.

Theorem 4.2.4. *1. A complex K is homotopically projective if and only if it is the colimit of a directed, ascending system in the category of complexes,*

$$P_0 \xrightarrow{i_0} P_1 \longrightarrow \cdots \longrightarrow P_j \xrightarrow{i_j} P_{j+1} \longrightarrow \cdots \ , \qquad j \in \mathbb{N}$$

where the chain maps i_j are split mono in each degree and P_{j+1}/P_j has projective components and vanishing differential for each j.

2. For any complex K, there exists a homotopically projective complex $\boldsymbol{p}K$ and an acyclic complex $\boldsymbol{a}K$ which fit in a triangle

$$\boldsymbol{p}K \to K \to \boldsymbol{a}K \to \boldsymbol{p}K[1].$$

This triangle is unique in the sense that any triangle (P, K, A) with P homotopically projective and A acyclic is isomorphic to $(\boldsymbol{p}K, K, \boldsymbol{a}K)$ and there is a unique such isomorphism extending the identity of K.

3. A complex K is homotopically injective if and only if it is the colimit of a directed, descending system in the category of complexes,

$$I_0 \xleftarrow{p_0} I_1 \longleftarrow \cdots \longleftarrow I_j \xleftarrow{p_j} I_{j+1} \longleftarrow \cdots, \qquad j \in \mathbb{N}$$

where the chain maps p_j are split epi in each degree and $Ker p_j$ has injective components and vanishing differential for each j.

4. *For any complex K, there exists a homotopically injective complex $\boldsymbol{i}K$ and an acyclic complex $\boldsymbol{a}'K$ which fit in a triangle*

$$\boldsymbol{a}'K \to K \to \boldsymbol{i}K \to \boldsymbol{a}K[1].$$

This triangle is unique in the sense that any triangle (A', K, I) with I homotopically injective and A' acyclic is isomorphic to $(\boldsymbol{a}'K, K, \boldsymbol{i}K)$ and there is a unique such isomorphism extending the identity of K.

We skip the proof here, because theorem 4.6.1 is a more general version of this theorem, with a similar proof. An immediate corollary of the second part of the theorem is the following

Corollary 4.2.5. *Any complex is quasi-isomorphic to a homotopically projective complex.*

Proof. Part 2 of the theorem states that for any complex K, the complex $\boldsymbol{a}K$ is isomorphic to the cone of the map $\boldsymbol{p}K \to K$ (since that is how distinguished triangles are defined in $\mathbf{K}(\Lambda)$). This shows that $\boldsymbol{p}K \to K$ is a quasi-isomorphism, by the fact that a chain map f is a quasi-isomorphism if and only if Cone(f) is acyclic. Thus, any complex is quasi-isomorphic to a homotopically projective complex. □

We call $\boldsymbol{p}K$ a *homotopically projective resolution* of K. In fact, we get that $\boldsymbol{p}$ defines a functor which is right adjoint to the inclusion $\iota_p \colon \mathbf{K}_p(\Lambda) \hookrightarrow \mathbf{K}(\Lambda)$. Likewise $\boldsymbol{a}$ defines a left adjoint to the inclusion $\iota_a \colon \mathbf{K}_a(\Lambda) \hookrightarrow \mathbf{K}(\Lambda)$, where $\mathbf{K}_a(\Lambda)$ is the full triangulated subcategory of $\mathbf{K}(\Lambda)$ consisting of acyclic complexes. The adjoint properties give us the these isomorphisms

$$\mathrm{Hom}_{K(\Lambda)}(\iota_p X, Y) \simeq \mathrm{Hom}_{K_p(\Lambda)}(X, \boldsymbol{p}Y)$$

$$\mathrm{Hom}_{K(\Lambda)}(X', \iota_a Y') \simeq \mathrm{Hom}_{K_a(\Lambda)}(\boldsymbol{a}X', Y').$$

Notice that for any homotopically projective complex X, if Y is acyclic then $\mathrm{Hom}_{\mathbf{K}_p(\Lambda)}(X, \boldsymbol{p}Y) = 0$. This means that $\boldsymbol{p}Y = 0$ in $\mathbf{K}_p(\Lambda)$. Similarly, for any acyclic complex Y', we see that X' homotopically projective implies $\mathrm{Hom}_{\mathbf{K}_a(\Lambda)}(\boldsymbol{a}X', Y) = 0$. So $\boldsymbol{a}X' = 0$ in $\mathbf{K}_a(\Lambda)$.

Proposition 4.2.6. *The functors $\mathbf{p}$ and $\boldsymbol{a}$ commute with infinite direct sums.*

Proof. Assume that we have complexes K_q with corresponding triangles

$$\boldsymbol{p}K_q \to K_q \to \boldsymbol{a}K_q \to \boldsymbol{p}K[1], \qquad q \in \mathbb{N}.$$

By taking the direct sum of all these triangles, we get a new triangle

$$\bigoplus_{q\in\mathbb{N}} \boldsymbol{p}K_q \to \bigoplus_{q\in\mathbb{N}} K_q \to \bigoplus_{q\in\mathbb{N}} \boldsymbol{a}K_q \to \bigoplus_{q\in\mathbb{N}} \boldsymbol{p}K[1],$$

and since both $\mathbf{K}_p(\Lambda)$ and $\mathbf{K}_a(\Lambda)$ are closed under infinite direct sums, we see that $\bigoplus_{q\in\mathbb{N}} \boldsymbol{p}K_q$ is homotopically projective and $\bigoplus_{q\in\mathbb{N}} \boldsymbol{a}K_q$ is acyclic. So we have a triangle on the form (P, K, A) in part 2 of theorem 4.2.4, and thus it is isomorphic to

$$\boldsymbol{p}\bigoplus_{q\in\mathbb{N}} K_q \to \bigoplus_{q\in\mathbb{N}} K_q \to \boldsymbol{a}\bigoplus_{q\in\mathbb{N}} K_q \to \boldsymbol{p}\bigoplus_{q\in\mathbb{N}} K[1],$$

the triangle given by applying $\boldsymbol{p}$ and $\boldsymbol{a}$ to $\bigoplus_{q\in\mathbb{N}} K_q$. The triangles being isomorphic shows that both $\boldsymbol{p}$ and $\boldsymbol{a}$ must commute with infinite direct sums. □

A similar argument holds for the functor $\boldsymbol{i}$.

4.3 Unbounded derived categories

Let $\mathcal{S}$ be the class of quasi-isomorphisms (up to homotopy) in $\mathbf{K}(\Lambda)$.

Definition 4.3.1. The *derived category of* Λ is the localization of the homotopy category by the class of quasi-isomorphisms, that is

$$\mathbf{D}(\Lambda) = \mathcal{S}^{-1}\mathbf{K}(\Lambda)$$

The next theorem is analogus to a similar theorem for projective and injective resolutions. Before we can prove it, we need the following two lemmas

Lemma 4.3.2. *If P is a homotopically projective complex, then any quasi-isomorphism $q\colon \tilde{P} \to P$ from any complex $\tilde{P}$ to P is a split epimorphism in $\boldsymbol{K}(\Lambda)$.*

Proof. First, recall that for any quasi-isomorphism $q\colon \tilde{P} \to P$, the complex $\operatorname{Cone}(q)$ is exact. Since P is homotopically projective, this means that $\operatorname{Hom}(P, \operatorname{Cone}(q)) = 0$. We then have the following diagram

$$\begin{array}{ccccccc}
\tilde{P} & \xrightarrow{q} & P & \xrightarrow{0} & \operatorname{Cone}(q) & \longrightarrow & \tilde{P}[1] \\
 & & {\scriptstyle 1}\uparrow & & \uparrow & & \\
P & \xrightarrow{1} & P & \longrightarrow & 0 & \longrightarrow & P[1],
\end{array}$$

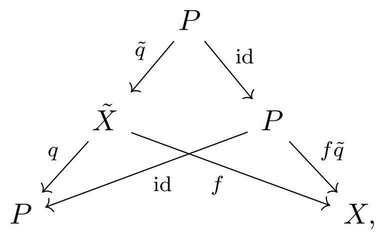
which commutes. By the "2 out of 3"-property of triangulated categories, we know that there exists $q'\colon P \to \tilde{P}$ such that $qq' = \mathrm{id}_P$. In other words, q is a split epimorphism. □

Lemma 4.3.3. *Let P be a homotopically projective complex, and let X be an arbitrary complex. Then the map*

$$Hom_{\boldsymbol{K}(\Lambda)}(P, X) \to Hom_{\boldsymbol{D}(\Lambda)}(P, X),$$

given by the projection functor $\boldsymbol{K}(\Lambda) \to \boldsymbol{D}(\Lambda)$, is an isomorphism.

Proof. For any roof $f \cdot q^{-1} \in \mathrm{Hom}_{\mathbf{D}(\Lambda)}(P, X)$, lemma 4.3.2 implies that the quasi-isomorphism q is a split epimorphism. We then find $\tilde{q}$ such that $q\tilde{q} = \mathrm{id}_P$. By the commutativity of the following diagram



we have that $f \cdot q^{-1} = (f\tilde{q}) \cdot \mathrm{id}^{-1}$ in $\mathbf{D}(\Lambda)$. This means that $f \cdot q^{-1}$ lies in the image of the projection functor $\mathbf{K}(\Lambda) \to \mathbf{D}(\Lambda)$, consisting of (roofs equivalent to) roofs on the form $\varphi \cdot \mathrm{id}^{-1}$. This shows surjectivity. To show injectivity, we use the fact that a map f in $\mathbf{K}(\Lambda)$ vanishes in $\mathbf{D}(\Lambda)$ if and only if there exists a quasi-isomorphism $q\colon \tilde{P} \to P$, from some complex $\tilde{P}$, such that $fq = 0$. By lemma 4.3.2 this quasi-isomorphism is split epi, so there exists a map $\tilde{q}$ such that $f = fq\tilde{q} = 0 \circ \tilde{q} = 0$. So only the zero map is sent to zero, which shows injectivity. This concludes the proof that $\mathrm{Hom}_{\mathbf{K}(\Lambda)}(P, X) \simeq \mathrm{Hom}_{\mathbf{D}(\Lambda)}(P, X)$ □

Theorem 4.3.4. *The projection functor $\boldsymbol{K}(\Lambda) \to \boldsymbol{D}(\Lambda)$ induces equivalences $\boldsymbol{K}_p(\Lambda) \xrightarrow{\sim} \boldsymbol{D}(\Lambda)$ and $\boldsymbol{K}_i(\Lambda) \xrightarrow{\sim} \boldsymbol{D}(\Lambda)$. The quasi-inverse functors are induced by $\boldsymbol{p}\colon \boldsymbol{D}(\Lambda) \xrightarrow{\sim} \boldsymbol{K}_p(\Lambda)$ and $\boldsymbol{i}\colon \boldsymbol{D}(\Lambda) \xrightarrow{\sim} \boldsymbol{K}_i(\Lambda)$. More precisely, $\boldsymbol{p}$ induces a fully faithful left adjoint to the projection functor, and $\boldsymbol{i}$ induces a fully faithful right adjoint.*

Proof. To see that $\boldsymbol{p}$ and $\boldsymbol{i}$ induce well-defined functors $\mathbf{D}(\Lambda) \to \mathbf{K}(\Lambda)$, it is sufficient to observe that both vanish on acyclic complexes. For $\boldsymbol{p}$ being right adjoint to the projection functor, note that for complexes L and M,

$$\mathrm{Hom}_{KL}(\boldsymbol{p}L, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\Lambda)}(\boldsymbol{p}L, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\Lambda)}(L, M). \tag{4.5}$$

The first isomorphism is given by lemma 4.3.3. The second comes from corollary 4.2.5, and the fact that quasi-isomorphisms give isomorphisms in $\mathbf{D}(\Lambda)$.

A similar argument holds for $\boldsymbol{i}$, which gives

$$\mathrm{Hom}_{KL}(L, \boldsymbol{i}M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\Lambda)}(L, \boldsymbol{i}M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\Lambda)}(L, M).$$

□

4.4 Infinite dévissage

The principle of infinite dévissage is a very useful tool, originally used in algebraic geometry (see for example [Gro61]). For our use, we don't really need to know why it's called infinite dévissage or how it's motivated. Actually, whenever we mention infinite dévissage in this thesis, it is basically just to have a name to refer to when applying the following theorem. Thorughout this section, we keep the assumptions from section 4.2.

Proposition 4.4.1. *A full triangulated subcategory of $\boldsymbol{D}(\Lambda)$ is equal to $\boldsymbol{D}(\Lambda)$ if and only if it contains Λ_Λ and is closed under forming infinite direct sums.*

Instead of proving the proposition directly, we prove the corresponding statement for $\mathbf{K}_p(\Lambda)$, which is sufficient because of the equivalence $\mathbf{K}_p(\Lambda) \simeq \mathbf{D}(\Lambda)$. Before we give the proof of the proposition, we need the following lemma:

Lemma 4.4.2. *If $\mathcal{U}$ is the smallest full triangulated subcategory of $\boldsymbol{K}_p(\Lambda)$ which contains Λ_Λ and is closed under infinite direct sums, then $\mathcal{U}$ contains all projective Λ-modules.*

Proof. First, note that since $\mathcal{U}$ contains Λ and is closed under direct sums, it contains all free Λ-modules. By induction on length, it can also be shown that $\mathcal{U}$ contains all finite complexes of free Λ-modules. Any projective module can be written as the image of an idempotent endomorphism on a free module, by the following argument:

$$P \text{ projective} \iff P \oplus Q \xrightarrow{\sim} F$$

for some module Q and some free module F. So for a given projective P, the following composition gives an endomorphism on F which is clearly idempotent, and whose image is equal to P:

$$e\colon F \xrightarrow{\binom{1}{0}} P \oplus Q \xrightarrow{\sim} F.$$

Now, let e be an idempotent endomorphism of a free module F, with $e(F)$ being the image of e. We can take a resolution of $e(F)$ with a copy of F in each degree, given by the following exact complex

$$\cdots \longrightarrow F \xrightarrow{1-e} F \xrightarrow{e} F \xrightarrow{1-e} F \xrightarrow{e} e(F) \longrightarrow 0.$$

From this we see that $e(F)$ is homotopy equivalent to this complex

$$F^\bullet = \cdots \longrightarrow F \xrightarrow{1-e} F \xrightarrow{e} F \xrightarrow{1-e} F \longrightarrow 0,$$

in other words $e(F)$ is isomorphic to the complex $F^\bullet$ in $\mathbf{K}(\Lambda)$. We now need to show that $F^\bullet$ is contained in $\mathcal{U}$. Consider the sequence of truncated subcomplexes $F^\bullet(p)$ with zero to the left of degree p for all $p \in \mathbb{N}$. Then $F^\bullet$ is the colimit $\varinjlim F^\bullet(p)$. Notice that all the assumptions we needed to invoke Milnor's triangle in section 4.2 are satisfied:

1. $\mathcal{U}$ is a full triangulated subcategory of $\mathbf{K}(\Lambda)$ closed under infinite direct sums.

2. Each morphism $i_p \colon F^\bullet(p) \to F^\bullet(p+1)$ is an isomorphism in all degrees except $p+1$, where it is zero. Consequently, each i_p has split monomorphisms in each degree.

3. For a given p, we have that $F^\bullet(p+1)/F^\bullet(p)$ is isomorphic to a shifted copy of F. Since it is zero in all degrees except one, where it is free, it has vanishing differential and is projective in each degree.

Because the same assumptions are satisfied, we can use exactly the same construction as in section 4.2 to make Milnor's triangle in this case. So we get that

$$\bigoplus_{p\in\mathbb{N}} F^\bullet(p) \longrightarrow \bigoplus_{q\in\mathbb{N}} F^\bullet(q) \longrightarrow \varinjlim F^\bullet(q) \longrightarrow \bigoplus_{p\in\mathbb{N}} F^\bullet(p)[1]$$

is a distinguished triangle in $\mathbf{K}(\Lambda)$. The two first terms are contained in $\mathcal{U}$ since it is closed under infinite direct sum. And then $\varinjlim F^\bullet(q)$ is contained in $\mathcal{U}$ as well, because $\mathcal{U}$ is assumed to be a triangulated subcategory. Since we know that $\varinjlim F^\bullet(p) = F^\bullet$, and that $F^\bullet \simeq e(F)$ in $\mathbf{K}(\Lambda)$, this shows that $e(F)$ is contained in $\mathcal{U}$. And as we have shown, all projective Λ-modules can be written as $e(F)$ for some idempotent e and some free module F. Thus, all projective modules are contained in $\mathcal{U}$. □

Proof of proposition 4.4.1. We need to show that the smallest full triangulated subcategory of $\mathbf{K}(\Lambda)$ which contains all projective modules and is closed under direct sums is $\mathbf{K}_p(\Lambda)$. Because then, by lemma 4.4.2, the subcategory $\mathcal{U}$ is equal to $\mathbf{K}_p(\Lambda)$. By theorem 4.2.4 we know that a complex is homotopically projective if and only if it is the limit of a directed system satisfying some conditions. But these conditions are satisfied by $\mathcal{U}$. Since $\mathcal{U}$ contains all projective modules, and is closed under direct sum, we can create all directed systems on the form eq. (4.1). We then apply Milnor's triangle, and use the fact that $\mathcal{U}$ is triangulated to conclude that all limits of such systems also lie in $\mathcal{U}$. In other words, we have shown that $\mathbf{K}_p(\Lambda) \subseteq \mathcal{U}$, and since $\mathcal{U}$ is minimal, we get equality. This concludes the proof of the proposition. □

4.5 Derived equivalences

Let X be a complex of $\Gamma-\Lambda$-bimodules, where Λ and Γ are rings. We can use X to define two functors between complexes of Λ-modules and complexes of Γ-modules, namely the tensor product and the covariant Hom-functor. Given L a complex of Λ-modules, we have that $L \otimes_\Lambda X$ is a complex of Γ-modules defined by:

$$(L \otimes_\Lambda X)^n = \bigoplus_{p+q=n} L^p \otimes X^p$$

$$d(l \otimes x) = (dl) \otimes x + (-1)^p, l \in L^p, x \in X^q.$$

Similarly, if M is a complex of Γ-modules, then $\mathcal{H}om_\Gamma(X, M)$ is a complex of Λ-modules defined by

$$\mathcal{H}om_\Gamma(X, M)^n = \prod_{-p+q=n} \operatorname{Hom}_\Gamma(X^p, M^q)$$

$$(df)(x) = d(f(x)) - (-1)^n f(dx), f \in \mathcal{H}om_\Gamma(X, M)^n.$$

This gives the functors between complex categories $F := - \otimes_\Lambda X$ and $G := \mathcal{H}om_\Gamma(X, -)$, which in turn induce functors between $\mathbf{K}(\Lambda)$ and $\mathbf{K}(\Gamma)$. For simplicity, we keep the same notation, and we have

$$\mathbf{K}(\Lambda) \underset{G=\mathcal{H}om_\Gamma(X,-)}{\overset{F=-\otimes_\Lambda X}{\rightleftarrows}} \mathbf{K}(\Gamma) .$$

It is a known result that the functors F and G form an adjoint pair. Now we can construct the total derived functors between $\mathbf{D}(\Lambda)$ and $\mathbf{D}(\Gamma)$. In order to do so, we use the functors $\boldsymbol{p}(-)$ and $\boldsymbol{i}(-)$, as well as the following diagram

$$\begin{array}{ccccc} & & \mathbf{K}_p(\Lambda) & & \\ & \overset{\boldsymbol{p}}{\nearrow} \sim & & \hookrightarrow\searrow & \\ \mathbf{D}(\Lambda) & & & & \mathbf{K}(\Lambda) \\ & \underset{\sim}{\searrow} \boldsymbol{i} & & \hookrightarrow\nearrow & \\ & & \mathbf{K}_i(\Lambda) & & \end{array} .$$

The total left derived functor $\mathbb{L}F \colon \mathbf{D}(\Lambda) \to \mathbf{D}(\Gamma)$ is given as the following composition

$$\begin{CD} \mathbf{K}_p(\Lambda) @>F>> \mathbf{K}(\Gamma) \\ @A{\boldsymbol{p}}AA @VV{\pi_\Gamma}V \\ \mathbf{D}(\Lambda) @. \mathbf{D}(\Gamma) \end{CD} ,$$

where π_Γ is the projection functor $\mathbf{K}(\Gamma) \to \mathbf{D}(\Gamma)$. In other words, $\mathbb{L}F = \boldsymbol{p}(-) \otimes_\Lambda X$. Similarly, the total right derived functor $\mathbb{R}G \colon \mathbf{D}(\Gamma) \to \mathbf{D}(\Lambda)$ is given by the composition

$$\begin{array}{ccc} \mathbf{K}_i(\Gamma) & \xrightarrow{G} & \mathbf{K}(\Lambda) \\ \boldsymbol{i}\uparrow & & \downarrow \pi_\Lambda \\ \mathbf{D}(\Gamma) & & \mathbf{D}(\Lambda) \end{array},$$

so we get that $\mathbb{R}G = \mathcal{H}om_\Gamma(X, \boldsymbol{i}(-))$. Now we will show that $\mathbb{L}F$ and $\mathbb{R}G$ form an adjoint pair:

$$\begin{aligned} \operatorname{Hom}_{\mathbf{D}(\Gamma)}(\mathbb{L}F\,L, M) &\xrightarrow{\sim} \operatorname{Hom}_{\mathbf{K}(\Gamma)}(F(\boldsymbol{p}L), \boldsymbol{i}M) \\ &\xrightarrow{\sim} \operatorname{Hom}_{\mathbf{K}(\Lambda)}(\boldsymbol{p}L, G(\boldsymbol{i}M)) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}(\Lambda)}(L, \mathbb{R}G\,M) \end{aligned}$$

The first and last isomorphisms are given by properties of morphisms in derived categories, and the middle one comes from the fact that F and G are adjoints. Now, recall that a prefect complex of Λ-modules is a bounded complex of finitely generated projective Λ-modules, and $\operatorname{per}\Lambda$ is the full subcategory of $\mathbf{D}(\Lambda)$ consisting of complexes that are quasi-isomorphic to perfect complexes. In other words $\operatorname{per}\Lambda = \mathbf{K}^b(proj - \Lambda)$. The next proposition will be used in our proof for Rickard's Morita theorem.

Theorem 4.5.1. *The following are equivalent*

i) The functor $\mathbb{L}F\colon \boldsymbol{D}(\Lambda) \to \boldsymbol{D}(\Gamma)$ *is an equivalence.*

ii) The functor $\mathbb{L}F$ *induces an equivalence* $per\,\Lambda \to per\,\Gamma$.

iii) The object $T = \mathbb{L}F\,A$ *satisfies the following conditions*

(a) The map

$$A \to Hom_{\boldsymbol{D}(\Gamma)}(T, T)$$

is bijective and $Hom_{\boldsymbol{D}(\Gamma)}(T, T[n]) = 0$ *for* $n \neq 0$.

(b) $T \in per\,\Gamma$.

(c) The smallest full triangulated subcategory of $\boldsymbol{D}(\Gamma)$ *containing* T *and closed under forming direct summands equals* $per\,\Gamma$.

Proof. $i) \implies ii)$

Proposition 2.5.3 states that a complex $K \in \mathbf{K}^b(Proj - \Lambda)$ lies in $\operatorname{per}\Lambda$ if and only if the functor $\operatorname{Hom}_{\mathbf{D}(\Lambda)}(K, -)$ commutes with arbitrary direct sums.[1] This fact is then used to show that an equvalence in $\boldsymbol{K}^b$ restricts down to an equivalence between perfect complexes. We will now prove that the statement in proposition 2.5.3 holds for any complex $K \in \mathbf{D}(\Lambda)$, not

[1]Such a complex is called a compact object in $\mathbf{K}^b(Proj - \Lambda)$

just in $\mathbf{K}^b(Proj-\Lambda)$. Thus, an equivalence $\mathbf{D}(\Lambda) \simeq \mathbf{D}(\Gamma)$ will restrict down to $\operatorname{per}\Lambda \simeq \operatorname{per}\Gamma$ in a similar way.

In other words, we want to prove that for any $K \in \mathbf{D}(\Lambda)$, we have that

$$K \in \operatorname{per}\Lambda \iff \operatorname{Hom}_{\mathbf{D}(\Lambda)}(K,-) \text{ commutes with arbitrary direct sums.}$$

The $\implies$ direction is easy. Since $\operatorname{per}\Lambda \subseteq \mathbf{K}^b(Proj-\Lambda)$, we clearly still have from proposition 2.5.3 that

$$K \in \operatorname{per}\Lambda \implies \operatorname{Hom}_{\mathbf{D}(\Lambda)}(K,-) \text{ commutes with arbitrary direct sums.}$$

To show the other direction, we take a complex $K \in \boldsymbol{K}(Proj-\Lambda)$ instead of $\mathbf{D}(\Lambda)$ (by using the fact that any complex is quasi-isomorphic to a complex of projectives). If we can show that this is homotopy equivalent to a bounded complex, we are done, because then we can apply proposition 2.5.3. We start by looking at the chain map $\varphi \in \operatorname{Hom}(K, \bigoplus\limits_{i\in\mathbb{Z}} \operatorname{Cok} d^i[-i-1])$, which corresponds to the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{-2}} & K^{-1} & \xrightarrow{d^{-1}} & K^0 & \xrightarrow{d^0} & K^1 & \xrightarrow{d^1} & \cdots \\
 & & \big\downarrow{\scriptstyle\varphi^{-1}} & & \big\downarrow{\scriptstyle\varphi^{0}} & & \big\downarrow{\scriptstyle\varphi^{1}} & & \\
\cdots & \xrightarrow{0} & \operatorname{Cok} d^{-2} & \xrightarrow{0} & \operatorname{Cok} d^{-1} & \xrightarrow{0} & \operatorname{Cok} d^{0} & \xrightarrow{0} & \cdots
\end{array}$$

We now assume that $\operatorname{Hom}(K,-)$ commutes with arbitrary direct sums, which in particular means that we have an isomorphism:

$$\begin{array}{ccc}
\operatorname{Hom}(K, \bigoplus_{i\in\mathbb{Z}} \operatorname{Cok} d^i[-i-1]) & \xrightarrow{\sim} & \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}(K, \operatorname{Cok} d^i[-i-1]) \\
\cup & & \cup \\
\varphi & \longmapsto & (\varphi^i)_i
\end{array}$$

From the definition of the direct sum, we see that only finitely many components in the image of φ are nonzero, so we get that $\varphi^i = 0$ for $i \ll 0$ and $i \gg 0$. Since each φ^i is epi, this also implies that $\operatorname{Cok} d^i = 0$ for all sufficiently large and sufficiently small values of i, which means that d^i is epi for those same i's. recall that we want to show that the complex K is bounded (up to homotopy), that is, both left bounded and right bounded. We'll consider the two cases separately:

Right bounded: Pick a sufficiently large i, such that d^j is epi for all $j \geq i$. Since the map $d^i\colon K^i \to K^{i+1}$ is epi, its image is equal to K^{i+1}. But $\operatorname{Im} d^i \subseteq \operatorname{Ker} d^{i+1} \subseteq K^{i+1}$, which means that $K^{i+1} = \operatorname{Ker} d^{i+1}$. So d^{i+1} sends everything to zero, and since it is also epi, K^{i+2} must be zero. Then, $d^{i+2}\colon K^{i+2} \to K^{i+3}$ is an epimorphism from $K^{i+2} = 0$, so K^{i+3} is zero as well. The same argument shows that $K^j = 0$ for all $j > i+1$. Thus, K is right bounded.

Left bounded: Pick a sufficiently small i, such that d^j is epi for all $j \leq i$. The fact that d^i is an epimorphism implies that the sequence

$$0 \to \operatorname{Ker} d^i \to K^i \xrightarrow{d^i} K^{i+1} \to 0$$

is exact. Since it ends in K^{i+1}, which is projective, it is in fact split exact, which means that $K^i \simeq \operatorname{Ker} d^i \oplus K^{i+1}$. Thus, $K^{i+1} \xrightarrow{1} K^{i+1}$ is a direct summand in K, and by theorem A.1.1 we can remove it. That leaves us with the following complex, which is equal to K up to homotopy:

$$\cdots \to K^{i-3} \xrightarrow{d^{i-3}} K^{i-2} \xrightarrow{d^{i-2}} K^{i-1} \xrightarrow{f^{i-1}} \operatorname{Ker} d^i \to 0 \to K^{i+2} \to \cdots$$

Note that this complex is equal to K in all degrees except i and $i+1$, and that f^{i-1} is the component map to $\operatorname{Ker} d^i$ in d^{i-1}. Because d^{i-1} is epi, f^{i-1} is also epi. Thus, we get a short exact sequence

$$0 \to \operatorname{Ker} f^{i-1} \to K^{i-1} \to \operatorname{Ker} d^i \to 0,$$

which again is split since $\operatorname{Ker} d^i$ is a direct summand of a projective module, and thus projective. So we have that $K^{i-1} \simeq \operatorname{Ker} f^{i-1} \oplus \operatorname{Ker} d^i$, and we get a direct summand $\operatorname{Ker} d^i \xrightarrow{1} \operatorname{Ker} d^i$, which we can remove. This gives us the following complex, which still is homotopy equivalent to K:

$$\cdots \to K^{i-3} \xrightarrow{d^{i-3}} K^{i-2} \xrightarrow{f^{i-2}} \operatorname{Ker} f^{i-1} \to 0 \to 0 \to K^{i+2} \to \cdots$$

This process of showing that K^j splits, and then using a trivial summand to remove whatever nonzero term is left in degree $j+1$, works indefinitely, since d^j is epi for all $j \leq i$. Thus, up to homotopy, $K^i = 0$ for $i \ll 0$, and K is left bounded.

All in all, this means that for any complex $K \in \mathbf{D}(\Lambda)$, we have that if $\operatorname{Hom}(K, -)$ commutes with arbitrary direct sums, then K is homotopy equivalent to a complex in $\mathbf{K}^b(Proj - \Lambda)$. From proposition 2.5.3 we know that if K' is a complex in $\mathbf{K}^b(Proj - \Lambda)$ such that $\operatorname{Hom}(K', -)$ commutes with arbitrary direct sums, then $K' \in \operatorname{per} \Lambda$. Thus, we see that K must be in $\operatorname{per} \Lambda$, which is what we wanted to show. This concludes the proof of the if-and-only-if statement. That this statement implies that the equivalence restricts as needed, is given by the exact same arument as in proposition 2.5.3.

ii) $\implies$ *iii)*

For condition *a*), notice that $\mathbb{L}F$ is an equivalence, and therefore fully faithful. This means that both arrows in this diagram are bijections

$$\Lambda \to \operatorname{Hom}_{\mathbf{D}(\Lambda)}(\Lambda, \Lambda) \to \operatorname{Hom}_{\mathbf{D}(\Gamma)}(T, T).$$

We also have the bijection $\mathrm{Hom}_{\mathbf{D}(\Gamma)}(T, T[n]) \leftrightarrow \mathrm{Hom}_{\mathbf{D}(\Lambda)}(\Lambda, \Lambda[n])$, and we know that there are no nonzero maps from Λ to $\Lambda[n]$ in $\mathbf{D}(\Lambda)$ for $n \neq 0$. Thus, we get that $\mathrm{Hom}_{\mathbf{D}(\Gamma)}(T, T[n]) = \mathrm{Hom}_{\mathbf{D}(\Lambda)}(\Lambda, \Lambda[n]) = 0$ for $n \neq 0$. Condition $b)$ holds because $\mathbb{L}F\, K$ is perfect in $\mathbf{D}(\Gamma)$ if and only if K is perfect in $\mathbf{D}(\Lambda)$, since equivalences preserve perfectness. For $c)$, we observe that any full triangulated subcategory of $\mathbf{D}(\Lambda)$ which contains Λ and is closed under forming direct summands, must contain $\mathrm{per}\,\Lambda$. Thus, $\mathrm{per}\,\Lambda$ is the smallest such subcategory, and because each of the properties are preserved by the equivalence, applying $\mathbb{L}F$ shows that the condition is satisfied.

iii) $\implies$ *i)*

We start by showing that $\mathbb{L}F$ is fully faithful. Because $\mathbb{R}G$ is right adjoint to $\mathbb{L}F$, we have the isomorphism

$$\mathrm{Hom}_{\mathbf{D}(\Gamma)}(\mathbb{L}F\, M, \mathbb{L}F\, M) \simeq \mathrm{Hom}_{\mathbf{D}(\Lambda)}(M, \mathbb{R}G\, \mathbb{L}F\, M).$$

This means that we only need to show that we have a bijection

$$\mathrm{Hom}_{\mathbf{D}(\Lambda)}(M, M) \leftrightarrow \mathrm{Hom}_{\mathbf{D}(\Lambda)}(M, \mathbb{R}G\, \mathbb{L}F\, M),$$

in other words that the adjunction morphism

$$\varphi M \colon M \to \mathbb{R}G\, \mathbb{L}F\, M$$

is invertible for each $M \in \mathbf{D}(\Lambda)$. To do this, we will define $\mathcal{U}$ to be the smallest full subcategory of objects M in $\mathbf{D}(\Lambda)$ such that φM is invertible, and then use infinite dévissage to show that $\mathcal{U}$ in fact is equal to $\mathbf{D}(\Lambda)$. That is, we must show that $\mathcal{U}$ contains Λ_Λ, is triangulated and commutes with infinite direct sums. To see that $\mathcal{U}$ contains Λ_Λ, notice that we have the following

$$T = \mathbb{L}F\, \Lambda = \boldsymbol{p}(\Lambda) \otimes_\Lambda X = \Lambda \otimes_\Lambda X \simeq X,$$

where the first and second equality hold by definition, the third equality is true since Λ is homotopically projective, and the last isomorphism is a property of the tensor product. Thus, when we use the adjunction morphism on Λ, we get

$$\varphi\Lambda \colon \Lambda \to \mathbb{R}\mathcal{H}om_\Gamma(X, \Lambda \otimes^{\mathbb{L}}_\Lambda X) = \mathbb{R}\mathcal{H}om_\Gamma(T, T).$$

Now, by taking the n-th homology and using known properties of derived functors, we see that

$$\mathtt{H}^n \mathbb{R}\mathcal{H}om_\Gamma(T, T) = \mathrm{Ext}^n(T, T) = \mathrm{Hom}_{\mathbf{D}(\Gamma)}(T, T[n]).$$

From condition $c)$ we know that $\Lambda \to \mathrm{Hom}_{\mathbf{D}(\Gamma)}(T, T)$ is bijective, and that $\mathrm{Hom}_{\mathbf{D}(\Gamma)}(T, T[n]) = 0$ for $n \neq 0$, so we conclude that $\varphi\Lambda$ is bijective.

This means that $\Lambda \in \mathcal{U}$. Since both $\mathbb{L}F$ and $\mathbb{R}G$ are morphisms of triangulated categories, and $\mathcal{U}$ consists of the objects for which $\mathbb{R}G\,\mathbb{L}F$ is invertible, we see that $\mathcal{U}$ is a triangulated subcategory. The only thing remaining to show is that $\mathcal{U}$ is closed under infinite direct sums. In fact, we only need to check that $\mathbb{R}G$ commutes with infinite direct sums, since $\mathbb{L}F$ is a left adjoint, and thus commutes with all coproducts. For $\mathbb{R}G = \mathbb{R}\mathcal{H}om_\Gamma(T,-)$ it is enough to consider $\mathtt{H}^n\mathbb{R}G = \mathrm{Hom}_{\mathbf{D}(\Gamma)}(T,-[n])$. If we prove that this commutes with direct sums for all n, we're done, since H^n is an additive functor. But the fact that $\mathrm{Hom}_{\mathbf{D}(\Gamma)}(T,-[n])$ commutes with direct sums follows the fact that T, by condition $b)$ in the lemma, has projective and finitely generated terms. This shows that $\mathbb{R}G$ commutes with infinite direct sums, which means that $\mathbb{R}G\mathbb{L}F$ does so too. Thus, $\mathcal{U}$ is closed under infinite direct sums, so $\mathcal{U} = \mathbf{D}(\Lambda)$. This means that $\mathbb{L}F$ is fully faithful. To show that it is an equivalence, all we need now is to show that it is dense. But that follows easily from infinite dévissage, so we are done. □

Note that by proving theorem 4.5.1, we are very close to having a proof of Rickard's Morita theorem. Actually, the only difference between the theorems is the assumption that we have a complex X of Λ-Γ-bimodules. Such a complex ensures that the tensor product gives Γ-modules and not just abelian groups, thus making $- \otimes X$ a functor $\mathbf{K}(\Lambda) \to \mathbf{K}(\Gamma)$. So we need to find a way to construct a complex of Λ-Γ-bimodules, using the assumptions in Rickard's theorem. It will turn out that the existence of a tilting complex T is precisely what we need, because we will see that applying a certain functor to T will yield a complex of bimodules. But before we can do that, we need to develop some concepts of homological algebra for dg modules.

4.6 Resolutions of dg modules

In chapter 3 we gave the main definitions surrounding differential graded algebras, or dg albegras. In this section we will define some more concepts, as well as present a little more of the theory we need. This includes homotopically projective dg modules, the derived category of dg modules, dg bimodules, and tensor products of dg modules. Our goal is to use the theory we present here to make a functor which, when applied to a tilting complex, gives a complex of bimodules. We will then use this complex, together with theorem 4.5.1, to prove Rickard's Morita theorem.

As defined in chapter 3, dg Λ-modules are graded Λ-modules equipped with a differential, i.e. a graded map of degree 1 satisfying some commutativity properties. Note that chain complexes are one example of dg modules, but there are others as well. Like for regular chain complexes, we defined the homotopy category for dg Λ-modules $\mathbf{K}(\Lambda)$, and showed that it was triangulated.

Homotopically projective dg modules

We define *homotopically projective* dg modules similarly to chain complexes, namely a dg Λ-module K is homotopically projective if

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(K, N) = 0 \quad \forall N \text{ acyclic.}$$

Homotopically injective dg modules are defined in a similar way. We will now show that $\mathbf{K}_p(\Lambda)$, the full subcategory of $\mathbf{K}(\Lambda)$ consisting of the homotopically projective dg Λ-modules, is triangulated and closed under arbitrary direct sums, and contains Λ_Λ. The same is true for $\mathbf{K}_i(\Lambda)$, homotopically injective dg Λ-modules, by a similar proof.

We start by showing that Λ_Λ is homotopically projective, when viewed as a dg module over itself. First, observe that $\operatorname{Hom}_{dg-\Lambda}(\Lambda_\Lambda, M) \simeq \mathrm{Z}^0 M$ by the canonical map $f \mapsto f(1)$, which is an isomorphism to $\mathrm{Z}^0 M$ (for graded Λ-modules we have that $\operatorname{Hom}(\Lambda_\Lambda, M) \simeq M$, and since f must commute with the differential we have $\operatorname{Im} f \subseteq \operatorname{Ker} d^0 = \mathrm{Z}^0 M$). Now, the null-homotopic maps in $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(\Lambda_\Lambda, M)$ are precisely the ones where $f\colon \Lambda \to M^0$ factors through M^{-1}, which means that $\operatorname{Im} f \subseteq \operatorname{Im} d^{-1} = \mathrm{B}^0 M$. Since maps in $\mathbf{K}(\Lambda)$ are precisely dg Λ-morphisms modulo null-homotopic maps, we have that

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(\Lambda_\Lambda, M) \simeq \mathrm{Z}^0 M / \mathrm{B}^0 M \simeq \mathrm{H}^0 M.$$

If we now let N be an acyclic dg Λ-module, then obviously

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(\Lambda_\Lambda, N) \simeq \mathrm{H}^0 N = 0,$$

so Λ_Λ is homotopically projective. We easily see that $\mathbf{K}_p(\Lambda)$ is closed under arbitrary direct sums, because $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(-, N)$ takes coproducts to products, and products of zero are still zero. It remains to show that $\mathbf{K}_p(\Lambda)$ is triangulated, and to do so we must check that it is closed under shift and taking cones. Shift is straight forward, just notice that

$$\operatorname{Hom}_{\mathbf{K}(\Lambda)}(P[n], N) \simeq \operatorname{Hom}_{\mathbf{K}(\Lambda)}(P, N[-n]),$$

and since the shift of an acyclic dg module still is acyclic, we get that $P \in \mathbf{K}_p(\Lambda) \implies P[n] \in \mathbf{K}_p(\Lambda)$ for all $n \in \mathbb{Z}$. Now assume that we have two homotopically projective modules X and Y, and a map $X \xrightarrow{f} Y$. Using the triangulated structure on $\mathbf{K}(\Lambda)$, we can complete this to a distinguished triangle as follows

$$X \xrightarrow{\ f\ } Y \longrightarrow \operatorname{Cone}(f) \longrightarrow X[1].$$

By applying the functor $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(-, N)$ we get a long exact sequence. If N is acyclic, the terms coming from both X and Y are zero, since they are homotopically projective. This means that $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(\operatorname{Cone}(f))$ is between two zeros in a long exact sequence, and thus it must be zero (likewise for all shifts). So $\operatorname{Cone}(f)$ is homotopically projective, and we conclude that $\mathbf{K}_p(\Lambda)$ is a triangulated subcategory.

In our discussion of homotopically projective complexes we showed that complexes with projective components and zero differential were homotopically projective. It is not immediately obvious what the corresponding statement should be for dg Λ-modules. Notice that any chain complex with projective components and zero differential can be written as a direct sum of direct summands of shifted copies of Λ, in other words they lie in the subcategory $\operatorname{Add}(\Lambda[n] \mid n \in \mathbb{Z})$ (which from now on will be refered to simply as $\operatorname{Add}(\Lambda[n])$). This description is what we will apply to dg Λ-modules.

It is easy to see that $\operatorname{Add}(\Lambda[n])$ is a full subcategory of $\mathbf{K}_p(\Lambda)$. We have already seen that $\Lambda_\Lambda \in \mathbf{K}_p(\Lambda)$, and we will now show that all direct summands of Λ are in $\mathbf{K}_p(\Lambda)$. If $\Lambda \simeq \Lambda_1 \oplus \Lambda_2$, we can form the distinguished triangle

$$\Lambda_1 \xrightarrow{\ \iota\ } \Lambda \longrightarrow \operatorname{Cone}(\iota) \longrightarrow \Lambda_1[1],$$

where by definition $\operatorname{Cone}(\iota) = \Lambda \oplus \Lambda_1[1] \simeq \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1[1] \simeq \Lambda_2$. The last isomorphism follows because $\Lambda_1 \oplus \Lambda_1[1]$ is a trivial summand up to homotopy, and by theorem A.1.1 it can be removed. From this we can see that $\mathbf{K}_p(\Lambda)$ contains the direct summands of Λ. From before we know that $\mathbf{K}_p(\Lambda)$ is closed under shift and direct sums. Thus, we see that $\operatorname{Add}(\Lambda[n])$, which again consists of direct sums of direct summands of shifted copies of Λ, is contained in $\mathbf{K}_p(\Lambda)$. The following theorem corresponds to theorem 4.2.4, for homotopically projective dg Λ-modules.

Note: the statement for homotopically injective Λ-modules is also carried over to the differential graded case. We will not state or prove the version of the following theorem for homotopically injective dg Λ-modules. We will just say that the proof is similar, with some slight adjustments since homology in general doesn't commute with limits (the way it does with colimits).

Theorem 4.6.1. *For any dg Λ-module $X \in \boldsymbol{K}(\Lambda)$ there is a triangle*

$$\boldsymbol{p}X \longrightarrow X \longrightarrow \boldsymbol{a}X \longrightarrow \boldsymbol{p}X[1], \tag{4.6}$$

where $\boldsymbol{p}X$ is homotopically projective and $\boldsymbol{a}X$ is acyclic.

Proof. First, observe that $\boldsymbol{a}X$ is acyclic if $\mathrm{H}^n(\boldsymbol{a}X) \simeq \mathrm{H}^n(\operatorname{Hom}(\Lambda, \boldsymbol{a}X)) \simeq \operatorname{Hom}_{\mathbf{K}(\Lambda)}(\Lambda, \boldsymbol{a}X[n]) = 0$ for all $n \in \mathbb{Z}$. If we know that eq. (4.6) is a distinguished triangle, applying the functor $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(\Lambda, -)$ gives a long exact sequence. Then $\boldsymbol{a}X$ is acyclic if all terms in the sequence on the form $\operatorname{Hom}(\Lambda, \boldsymbol{a}X[n])$ are zero, which is equivalent with having an isomorphism $\operatorname{Hom}_{\mathbf{K}(\Lambda)}(\Lambda, \boldsymbol{p}X[n]) \simeq \operatorname{Hom}_{\mathbf{K}(\Lambda)}(\Lambda, X[n])$ for each $n \in \mathbb{Z}$. So if we can find a homotopically projective dg Λ-module $\boldsymbol{p}X$ with a map $\boldsymbol{p}X \to X$, such that

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{K}(\Lambda)}(\Lambda, \boldsymbol{p}X) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{K}(\Lambda)}(\Lambda, X) \\ \| & & \| \\ \mathrm{H}^n(\boldsymbol{p}) & & \mathrm{H}^n(X), \end{array}$$

then setting $\boldsymbol{a}X$ to be the cone of that map will give us the triangle we are looking for. We will find $\boldsymbol{p}X$ as the colimit of a certain directed system, which we will construct inductively. More precisely, we will create a directed system

$$P_1 \to P_2 \to \cdots \to P_i \to P_{i+1} \to \cdots \tag{4.7}$$

where for all $i \in \mathbb{Z}$, we choose $P_i \in \mathrm{Add}(\Lambda[n])$ such that $\mathrm{H}^*(P_i) \to \mathrm{H}^*(X)$ is an epimorphism. We can always find a P_1 which satisfies this, for example take a basis for $H^i(X)$ and let P_1 be a direct sum of as many copies of $\Lambda[-i]$ as there are elements in that basis. If we now let X_i be the cone of $P_i \to X$, so that we have a triangle

$$P_i \to X \xrightarrow{\alpha} X_i \to P_i[1],$$

then $\mathrm{H}^*(P_i) \to \mathrm{H}^*(X)$ being an epimorphism is equivalent to $H^*(\alpha) = 0$. Let's assume that we have a P_i which satisfies the conditions. We must find a $P_{i+1} \in \mathrm{Add}(\Lambda[n])$ and a map $P_i \to P_{i+1}$, such that $\mathrm{H}^*(P_{i+1}) \xrightarrow{\sim} \mathrm{H}^*(X)$. To do so, we use the following diagram

$$\begin{array}{ccccccc} P_i & \dashrightarrow & P_{i+1} & \dashrightarrow & P'_i & \longrightarrow & P_i[1] \\ \| & & \vdots & & \downarrow & & \| \\ P_i & \longrightarrow & X & \xrightarrow{\alpha} & X_i & \longrightarrow & P_i[1] \\ & & \downarrow{\scriptstyle \beta\alpha} & & \downarrow{\scriptstyle \beta} & & \\ & & X_{i+1} & = & X_{i+1} & & \\ & & \vdots & & \downarrow & & \\ & & P_{i+1}[1] & = & P_{i+1}[1] & & \end{array} \tag{4.8}$$

We assume that we have the second row, where X_i is the cone of the map $P_i \to X$, so it is a distinguished triangle. We also assume that $\mathrm{H}^*(\alpha) = 0$, to have that $\mathrm{H}^*(P_i) \to \mathrm{H}^*(X)$ is epi. By the same construction as above, we can find $P'_i \to X_i$ such that the rightmost column is a distinguished triangle with $\mathrm{H}^*(\beta) = 0$. Because $\mathrm{Add}(\Lambda[n])$ is triangulated, we can find a $P_{i+1} \in \mathrm{Add}(\Lambda[n])$, as well as the dashed arrows, so that the top row in the diagram becomes a distinguished triangle. We then use the octahedral axiom to find the dotted maps which makes the first column into a distinguished triangle. Notice that $\mathrm{H}^*(\beta\alpha) = 0$, since it is the composition of two zero maps, which means that $\mathrm{H}^*(P_{i+1}) \to \mathrm{H}^*(X)$ is epi. Also notice the square we get in the top left corner being commutative ensures that we actually get a directed system of P_i. Since all P_i have morphisms to X, we can write the system as

$$\begin{array}{ccccccccccc} P_1 & \longrightarrow & P_2 & \longrightarrow & \cdots & \longrightarrow & P_i & \longrightarrow & P_{i+1} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ X & = & X & = & \cdots & = & X & = & X & = & \cdots \end{array}$$

If we apply H^* to the top two triangles in diagram 4.8, we get the solid part of the following

$$\begin{array}{ccccccc} \mathrm{H}^*(P_i'[-1]) & \longrightarrow & \mathrm{H}^*(P_i) & & \xrightarrow{\quad f_i \quad} & & \mathrm{H}^*(P_2) \\ & & & \searrow & & \nearrow & \\ \big\downarrow & & \big\| & & \operatorname{Im} f_i & & \big\downarrow \\ & & & & & \searrow & \\ \mathrm{H}^*(X_i[-1]) & \hookrightarrow & \mathrm{H}^*(P_1) & & \twoheadrightarrow & & \mathrm{H}^*(X), \end{array}$$

and the maps to and from $\operatorname{Im} f_i$ are given since a map always factors through its image. Composition gives the dotted map from $\operatorname{Im} f_i$ to $\mathrm{H}^*(X)$. We will now show that this map is an isomorphism. To see this, observe that we have the following commutative diagram, where the rows are exact

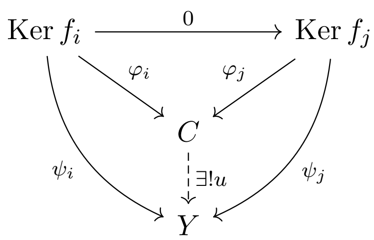
$$\begin{array}{ccccccccc} \operatorname{Ker} f_i & \hookrightarrow & \mathrm{H}^*(P_i) & \twoheadrightarrow & \operatorname{Im} f_i & \longrightarrow & 0 & \longrightarrow & 0 \\ \big\downarrow & & \big\| & & \big\downarrow & & \big\| & & \big\| \\ \mathrm{H}^*(X_i[-1]) & \hookrightarrow & \mathrm{H}^*(P_i) & \twoheadrightarrow & \mathrm{H}^*(X) & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Now, the second, fourth and fifth vertical maps are equalities, and the first is an epimorphism. Thus, the five lemma states that the middle map is an isomorphism. Note that this is true for all $i \in \mathbb{Z}$, and since $\mathrm{H}^*(X)$ is independent of i, this means that up to isomorphism, all f_i have the same image. This means that we can construct the following system

$$\begin{array}{ccccccccccc} \operatorname{Ker} f_1 & \xrightarrow{k_1} & \operatorname{Ker} f_2 & \xrightarrow{k_2} & \cdots & \longrightarrow & \operatorname{Ker} f_i & \xrightarrow{k_i} & \operatorname{Ker} f_{i+1} & \xrightarrow{k_{i+1}} & \cdots \\ \big\downarrow \iota_1 & & \big\downarrow \iota_2 & & & & \big\downarrow \iota_i & & \big\downarrow \iota_{i+1} & & \\ \mathrm{H}^*(P_1) & \xrightarrow{f_1} & \mathrm{H}^*(P_2) & \xrightarrow{f_2} & \cdots & \longrightarrow & \mathrm{H}^*(P_i) & \xrightarrow{f_i} & \mathrm{H}^*(P_{i+1}) & \xrightarrow{f_{i+1}} & \cdots \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow & & \\ \mathrm{H}^*(X) & = & \mathrm{H}^*(X) & = & \cdots & = & \mathrm{H}^*(X) & = & \mathrm{H}^*(X) & = & \cdots \end{array}$$

We will now show that $k_i = 0$ for all i, that is all maps between the kernels are zero. To see this, notice that because $\mathrm{H}^*(X) \simeq \operatorname{Im} f_i$, we have that f_i factors through $\mathrm{H}^*(X)$ for all i. This means that the composition $f_i\iota_i\colon \operatorname{Ker} f_i \to \mathrm{H}^*(P_{i+1})$ factors through $\mathrm{H}^*(X)$, and is thus 0. By the commutativity of the upper squares, we get that $\iota_{i+1}k_i = f_i\iota_i = 0$, and since ι_{i+1} is a monomorphism, this means that $k_i = 0$ for all i. To finish the proof we will now take the colimit of each of the directed systems in the previous diagram, and look at the induced sequence of colimits.

We start with the system of kernels. By definition, the colimit is a pair $(C, \{\varphi_i\})$, where $\varphi_i\colon \operatorname{Ker} f_i \to C$ satisfies $\varphi_i = \varphi_j \circ 0 = 0$ for all $j > i$ (since all maps between the $\operatorname{Ker} f_i$'s are 0). The colimit satisfies the universal property that for any other pair $(Y, \{\psi_i\})$ with $\psi_i\colon \operatorname{Ker} f_i \to Y$ such that $\psi_i = \psi_j \circ 0 = 0$ for all $j > i$, there exists a unique map $u\colon C \to Y$ such that $\psi_i = u \circ \varphi_i$ for all i. In terms of diagrams, this statement becomes



As we have seen $\varphi_i = \psi_i = 0$ for all i, which means that any map $u'\colon C \to Y$ will satisfy $\psi_i = u' \circ \varphi_i$. Thus, since u is the *unique* such map, there can be only one map $C \to Y$, which means that C must be 0. So we have showed that $\varinjlim \operatorname{Ker} f_i = 0$.

Now, consider the colimit of $\mathrm{H}^*(P_i)$. Note that since $\varinjlim$ is an exact functor, and thus commutes with homology, we have that $\varinjlim \mathrm{H}^*(P_i) = H^*(\varinjlim P_i)$. We also obviously have that $\varinjlim H^*(X) = H^*(X)$. So we end up with the sequence

$$0 \longrightarrow \mathrm{H}^*(\varinjlim P_i) \longrightarrow\!\!\!\!\to \mathrm{H}^*(X),$$

which is exact, since $\varinjlim$ preserves exactness. Thus the map $\mathrm{H}^*(\varinjlim P_i) \to H^*(X)$ is an isomorphism. All we need now is to show that $\varinjlim P_i$ is homotopically projective. An analogous result to proposition 4.2.3 is true for the system (4.7) (the proof is similar). Recall that $P_i \in \operatorname{Add}(\Lambda[n])$ for all i, and that $\operatorname{Add}(\Lambda[n])$ is closed under direct sums and taking cones. This means that $\varinjlim P_i \in \operatorname{Add}(\Lambda[n]) \subseteq \mathbf{K}_p(\Lambda)$, and we are done. Because then we can set $\boldsymbol{p}X \coloneqq \varinjlim P_i$ and set $\boldsymbol{a}X$ to be the cone of the induced map $\varinjlim P_i \to X$, and thus get the wanted triangle. $\square$

The derived category of a dg algebra

We define *quasi-isomorphisms* of dg Λ-modules similar to how we define them for regular chain complexes, which is to say a map $f\colon M \to N$ is a quasi-isomorphism if it induces an isomorphism $\mathrm{H}^*(f)\colon \mathrm{H}^*(M) \xrightarrow{\sim} \mathrm{H}^*(N)$ in homology. We also define the *derived category of* Λ like before, as

$$\mathbf{D}(\Lambda) \coloneqq \mathcal{S}^{-1}\mathbf{K}(\Lambda).$$

In other words, $\mathbf{D}(\Lambda)$ is the localization of $\mathbf{K}(\Lambda)$ by $\mathcal{S}$, the equivalence class of all quasi-isomorphisms up to homotopy. Now we can take advantage of the work we already have done. Corollary 4.2.5, lemma 4.3.2, lemma 4.3.3, theorem 4.3.4, eq. (4.5), and the principle of infinite dévissage are all valid in this setup as well, with the proofs transferring directly. In addition, we have the following formula

$$\mathrm{Hom}_{\mathbf{D}(\Lambda)}(\Lambda, M) \simeq \mathrm{Hom}_{\mathbf{K}(\Lambda)}(\boldsymbol{p}\Lambda, M) \simeq \mathrm{Hom}_{\mathbf{K}(\Lambda)}(\Lambda, M) \simeq \mathrm{H}^0 M.$$

The first isomorphism is given by eq. (4.5), and the second comes from corollary 4.2.5.

Derived equivalences

Total derived functors are defined by the same formulas for dg algebras as for ordinary algebras (see section 4.5). Given two dg algebras Λ and Γ, we define a new dg algebra $\Lambda^{\mathrm{op}} \otimes \Gamma$ by

$$\begin{aligned}(\Lambda^{\mathrm{op}} \otimes \Gamma)^n &= \bigoplus_{p+q=n} \Lambda^p \otimes \Gamma^q \\ d(a \otimes b) &= (da) \otimes b + (-1)^p a \otimes (db) \\ (a \otimes b)(a' \otimes b') &= (-1)^{pq'} aa' \otimes bb',\end{aligned}$$

for all $a \in \Lambda^p$, $b \in \Gamma^q$, $a' \in \Lambda^{p'}$ and $b' \in \Gamma^{q'}$. Now let ${}_\Lambda X_\Gamma$ be a dg Λ-Γ-bimodule, which means that it can be written as

$$X = \bigoplus_{p \in \mathbb{Z}} X^p,$$

being both a graded left Λ-module and a graded right Γ-module. The two actions must commute and coincide on k, and X must have a k-linear differential d, which is graded of degree 1 and satisfies

$$d(axb) = (da)xb + (-1)^p a(dx)b + (-1)^{p+q} ax(db)$$

for all $a \in \Lambda$, $x \in X^q$ and $b \in \Gamma$. We may view this X as a right dg $\Lambda^{\mathrm{op}} \otimes \Gamma$-module by defining multiplication as

$$x(a \otimes b) = (-1)^{rp} axb$$

for all $x \in X^r$, $a \in \Lambda^p$ and $b \in \Gamma$. Now, given a dg Λ-module M, we can use X to create a corresponding dg Γ-module. We define $M \otimes_k X$ to be the dg Γ-module with multiplication given as

$$(m \otimes x)b = m \otimes (xb), \quad m \in M,\ x \in X,\ b \in \Gamma,$$

and with graded structure and differential given by

$$\begin{aligned}(M \otimes_k X)^n &= \bigoplus_{p+q=n} M^p \otimes_k X^q \\ d(m \otimes x) &= (dm) \otimes x + (-1)^p m \otimes (dx),\end{aligned}$$

for all $m \in M^p$ and $x \in X$. What we actually want here is the tensor product over Λ, not just over k. To show this, we use the fact that $M \otimes_\Lambda X$ is the quotient group $(M \otimes_k X)/S$, where S is the k-submodule generated by all differences $ma \otimes x - m \otimes ax$. If we can show that S is stable under d and multiplication by elements of Γ, then we get that $(M \otimes_\Lambda X)$ is a well defined dg Γ-module. This is straight forward to check. For $s \in S$, we have

$$\begin{aligned}d(s) &= d(ma \otimes x - m \otimes ax) \\ &= d(ma \otimes x) - d(m \otimes ax) \\ &= d(ma) \otimes x + (-1)^{p+q} ma \otimes dx - (dm \otimes ax - (-1)^p m \otimes d(ax)) \\ &= ((dm)a + (-1)^p mda) \otimes x + (-1)^{p+q} ma \otimes dx \\ &\qquad\qquad - dm \otimes ax - ((-1)^p m \otimes ((da)x + (-1)^q adx) \\ &= ((dm)a \otimes x - dm \otimes ax) + ((-1)^p mda \otimes x - (-1)^p m \otimes dax) \\ &\qquad\qquad + ((-1)^{p+q} ma \otimes dx - (-1)^p m \otimes (-1)^q adx))\end{aligned}$$

We see that $d(s)$ is still on the form $m'a' \otimes x' - m' \otimes a'x'$, for some $m' \in M$, $a' \in \Lambda$ and $x' \in X$, since it is the sum of three terms on that form. As for multiplication by elements of Γ, let $s \in S$ and $b \in \Gamma$. Then

$$\begin{aligned}sb &= (ma \otimes x - m \otimes ax)b \\ &= (ma \otimes x)b - (m \otimes ax)b \\ &= ma \otimes xb - m \otimes axb,\end{aligned}$$

so $sb \in S$, which means that S is closed under multiplication by elements of Γ. Thus we conclude that $M \otimes_\Lambda X = (M \otimes_k X)/S$ is a dg Γ-module. Furthermore, $M \otimes_\Lambda X$ is functorial in both M and X. Now let N be a dg Γ-module. Like before, we define the dg Λ-module $\mathcal{H}om_\Gamma(X, N)$. Respectively, the graded structure, differential, and Λ-action are given by

$$\begin{aligned}\mathcal{H}om_\Gamma(X, N)^n &= \prod_{-p+q=n} \mathrm{Hom}_\Gamma(X^p, N^q) \\ (df)(x) &= d(f(x)) - (-1)^n f(dx) \\ (fa)(x) &= f(ax) \qquad ,\end{aligned}$$

where $x \in X$, $a \in \Lambda$, and f is a graded map of degree n.

We now have the two functors $F := - \otimes_\Lambda X$ and $G := \mathcal{H}om_\Gamma(X, -)$ between the categories of dg Λ-modules and dg Γ-modules, which in turn induce functors between $\mathbf{K}(\Lambda)$ and $\mathbf{K}(\Gamma)$. For simplicity, we keep the same notation, and we have

$$\mathbf{K}(\Lambda) \underset{G = \mathcal{H}om_\Gamma(X,-)}{\overset{F = - \otimes_\Lambda X}{\rightleftarrows}} \mathbf{K}(\Gamma) \; .$$

We now define the total left derived functor in the same way as for ordinary algebras, by applying the functor F to homotopically projective resolutions. That is, the total left derived functor $\mathbb{L}F \colon \mathbf{D}(\Lambda) \to \mathbf{D}(\Gamma)$ is given as the following composition:

$$\begin{array}{ccc} \mathbf{K}_p(\Lambda) & \xrightarrow{F} & \mathbf{K}(\Gamma) \\ {\scriptstyle \boldsymbol{p}}\uparrow & & \downarrow{\scriptstyle \pi_\Gamma} \\ \mathbf{D}(\Lambda) & & \mathbf{D}(\Gamma) \end{array}$$

The total right derived functor $\mathbb{R}G \colon \mathbf{D}(\Gamma) \to \mathbf{D}(\Lambda)$ is defined similarly, but using G and homotopically injective resolutions, and is given by the composition:

$$\begin{array}{ccc} \mathbf{K}_i(\Gamma) & \xrightarrow{G} & \mathbf{K}(\Lambda) \\ {\scriptstyle \boldsymbol{i}}\uparrow & & \downarrow{\scriptstyle \pi_\Lambda} \\ \mathbf{D}(\Gamma) & & \mathbf{D}(\Lambda) \end{array}$$

4.7 Rickard's Morita theorem

Now we use the theory we have developed so far to give an alternate proof of Rickard's Morita theorem.

Construction of bimodule complexes

In this section we assume that Γ is a flat k-algebra, that T is a homotopically projective complex over Γ, and that Γ satisfies the so-called *Toda condition*

$$\mathrm{Hom}_{\mathbf{K}(\Gamma)}(T, T[n]) = 0$$

for all $n < 0$. Let

$$\Lambda := \mathrm{Hom}_{\mathbf{K}(\Gamma)}(T, T).$$

What we want now is a functor $\mathbf{K}(\Lambda) \to \mathbf{K}(\Gamma)$ which sends Λ_Λ to T, much like $- \otimes^{\mathbb{L}}_\Lambda T$ does in theorem 4.5.1. The problem is that T isn't a complex of Λ-Γ-bimodules, and $- \otimes^{\mathbb{L}}_\Lambda T$ only gives a complex of abelian groups, not of Γ-modules. So we need to find a complex X of Λ-Γ-bimodules, such that

X_Γ is quasi-isomorphic to T, and such that we can define up to homotopy a left action of Λ on T given such the following diagram

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & X_\Gamma \\ \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle \lambda(a)} \\ T & \xrightarrow{\varphi} & X_\Gamma \end{array}$$

commutes for all $a \in \Lambda$, where φ is a quasi-isomorphism and $\lambda(a)$ is left multiplication by a. We will now construct such a complex of Λ-Γ-bimodules X, as well as the quasi-isomorphism $\varphi\colon T \to X_\Gamma$. Let $C = \mathcal{H}om_\Gamma(T,T)$, which is a differential graded k-algebra, as seen in example 3.1.7. We now take the differential graded subalgebra of C whose underlying complex is

$$\cdots \to C^{-2} \to C^{-1} \to \mathtt{Z}^0 C \to 0 \to 0 \to \cdots,$$

and call this dg subalgebra C_-. Notice that the 0-component of C_- is $\mathtt{Z}^0 C$, which is canonically isomorphic to $\mathrm{Hom}_{\boldsymbol{C}(\Lambda)}(T,T)$, the group of chain complex homomorphisms $T \to T$ (see lemma A.3.1 for details). Since we can view Λ as a dg algebra concentrated in degree 0, this means that we can view Λ as a quotient of C_-. Specifically, we have that Λ is equal to C_- modulo the complex

$$\cdots \to C^{-2} \to C^{-1} \to \mathtt{B}^0 C \to 0 \to 0 \to \cdots.$$

Thus, we have morphisms of dg algebras $\Lambda \leftarrow C_- \subseteq C$ Since T is a complex of right Γ-modules, it can be viewed as a right dg Γ-module, by viewing Γ as a dg algebra concentrated in degree zero (see example 3.1.6). From example 3.1.7, we get that T is also a left dg C-module, so T is a dg C-Γ-bimodule. By restriction, T also becomes a dg C_--Γ-bimodule. We are now ready to define the complex of Λ-Γ-bimodules we need. We set

$$X \coloneqq \Lambda \otimes_{C_-} \boldsymbol{p}T,$$

where the homotopically projective resolution $\boldsymbol{p}T$ is taken by viewing T as a $C_- \otimes \Gamma$-module. Note that X is indeed a complex of Λ-Γ-bimodules. Now we must construct a quasi-isomorphism $\varphi \coloneqq T \to X$, and to do so we will use the (still unused) conditions we placed on Γ and T. From the Toda condititon we get that the map $C_- \to \Lambda$ is a quasi-isomorphism. To see this, note that $\mathrm{H}^n(C_-) = \mathrm{Hom}_{\mathbf{K}(\Gamma)}(T, T[n])$ for $n \geq 0$ (and $\mathrm{H}^n(C_-) = 0$ for $n > 0$). Thus, the Toda condition means precisely that $\mathrm{H}^n(C_-) = 0$ for all $n \neq 0$, and since $\mathrm{H}^0(C_-) = \mathrm{Hom}_{\mathbf{K}(\Gamma)}(T,T) = \Lambda$, the map $C_- \to \Lambda$ is a quasi-isomorphism. Next, we want to show that the map

$$C_- \otimes_{C_-} \boldsymbol{p}T \to \Lambda \otimes_{C_-} \boldsymbol{p}T$$

is still a quasi-isomorphism. By the long exact sequence of homology, we see that an equivalent statement is that the mapping cone is acyclic, which is what we will show. To do this we can use the principle of infinite dévissage, much like we did in the proof of theorem 4.5.1. Let $\mathcal{U}$ be the full subcategory in the category of dg C_--Γ-bimodules consisting of all M such that $-\otimes_{C_-} M$ preserves acyclicity. To see that $\mathcal{U}$ is a triangulated subcategory, notice that shifts don't change acyclicity, so $\mathcal{U}$ is closed under shifts. Also, if $M, N \in \mathcal{U}$ we can take the long exact sequence of cohomology of the triangle

$$M \xrightarrow{f} N \to \operatorname{Cone}(f) \to M[1]$$

and then apply $- \otimes_{C_-} -$. Since tensor products commute with cohomology, applying this to an acyclic module A gives the following sequence, where by definition $\mathrm{H}^*(A \otimes_{C_-} M) = \mathrm{H}^*(A \otimes_{C_-} N) = 0$

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{H}^n(A \otimes_{C_-} \operatorname{Cone}(f)) \longrightarrow 0 \longrightarrow \cdots$$

This sequence is still exact, so $\mathrm{H}^*(A \otimes_{C_-} \operatorname{Cone}(f)) = 0$, which means that $- \otimes_{C_-} \operatorname{Cone}(f)$ preserves acyclicity, and thus $\mathcal{U}$ is closed under taking cones. This shows that $\mathcal{U}$ is a triangulated subcategory. Next, observe that $\mathcal{U}$ is closed under direct sums, since tensor products commute with coproducts. The final thing we need to check is that $(C_- \otimes \Gamma) \in \mathcal{U}$, in other words that $- \otimes_{C_-} (C_- \otimes_k \Gamma)$ preserves acyclicity. But we have that $- \otimes_{C_-} (C_- \otimes_k \Gamma) \simeq (- \otimes_{C_-} C_-) \otimes_k \Gamma \simeq - \otimes_k \Gamma$, which preserves acyclicity by assumtion, since Γ is assumed to be flat. So all the conditions for infinite dévissage are satisfied, and we conclude that $\mathcal{U}$ is equal to the category of C_--Γ-bimodules. Thus, the functor $- \otimes_{C_-} \boldsymbol{p}T$ preserves quasi-isomorphisms. Now, using the natural isomorphism

$$C_- \otimes_{C_-} \boldsymbol{p}T \to \boldsymbol{p}T,$$

we set φ to be the unique morphism in $\mathbf{D}(\Gamma)$ making the following diagram commutative

$$\begin{array}{ccc} C_- \otimes_{C_-} \boldsymbol{p}T & \xrightarrow{\sim} & \boldsymbol{p}T \\ \downarrow & & \downarrow \\ \Lambda \otimes_{C_-} \boldsymbol{p}T & \xleftarrow[\varphi]{} & T \end{array}$$

To see that we actually get a morphism of complexes, recall that we have the isomorphism $\operatorname{Hom}_{\mathbf{D}(\Gamma)}(T, \Lambda \otimes_{C_-} \boldsymbol{p}T) \simeq \operatorname{Hom}_{\mathbf{K}(\Gamma)}(\boldsymbol{p}T, \Lambda \otimes_{C_-} \boldsymbol{p}T) \simeq \operatorname{Hom}_{\mathbf{K}(\Gamma)}(T, \Lambda \otimes_{C_-} \boldsymbol{p}T)$. The last isomorphism comes from the fact that T is assumed to be homotopically projective. Thus, we have constructed a complex of Λ-Γ-bimodules $X = \Lambda \otimes_{C_-} \boldsymbol{p}T$, and a quasi-isomorphism $\varphi \colon T \to X_\Gamma$. Now we are ready to prove Rickard's Morita theorem.

Rickard's Morita theorem

Theorem 4.7.1. *Let k be a commutative ring and Λ, Γ two k-algebras such that Γ is flat over k. The following are equivalent.*

i) There is a complex of Λ-Γ-bimodules such that the functor $-\otimes^{\mathbb{L}}_{\Lambda} X \colon \boldsymbol{D}(\Lambda) \to \boldsymbol{D}(\Gamma)$ is an equivalence.

ii) There is a triangle equivalence $F \colon \boldsymbol{D}(\Lambda) \to \boldsymbol{D}(\Gamma)$.

iii) There is a triangle equivalence $per\,\Lambda \to per\,\Gamma$.

iv) There is an object $T \in \boldsymbol{D}(\Gamma)$ such that

(a) There is an isomorphism

$$\Lambda \simeq Hom_{\boldsymbol{D}(\Gamma)}(T, T)$$

and $Hom_{\boldsymbol{D}(\Gamma)}(T, T[n]) = 0$ for $n \neq 0$.

(b) $T \in per\,\Gamma$.

(c) The smallest full triangulated subcategory of $\boldsymbol{D}(\Gamma)$ which contains T and is closed under direct summands is equal to $per\,B$

Proof. It is obvious that *i*) implies *ii*). Simply set $F := - \otimes^{\mathbb{L}}_{\Lambda} X$, which is an equivalence by assumption, and a triangle functor since tensor products preserve triangles. Observe that the implications from *ii*) to *iii*) and from *iii*) to *iv*) are true by theorem 4.5.1. So all we need to prove is *iv*) implies *i*). Since we have an equivalence $\mathbf{D}(\Gamma) \simeq \mathbf{K}_p(\Gamma)$, we can assume that T is homotopically projective. Then, all the assumptions of section 4.7 are satisfied, so we can apply the construction to T. This gives us a complex of Λ-Γ-bimodules $X = \Lambda \otimes_{C_-} \boldsymbol{p}T$, such that the functor $- \otimes^{\mathbb{L}}_{\Lambda} X$ sends Λ_Λ to T. The claim now follows from theorem 4.5.1. □

Chapter 5

An example

We will now look at a concrete example of Rickard's Morita theorem being used to show that two rings have equivalent derived categories.

Example 5.0.1. Let Q be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. We define the path algebras $A_1 = kQ$ and $A_2 = kQ/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the relation $\beta\alpha$. Clearly, A_1 and A_2 are not isomorphic as rings, but they might be derived equivalent. To find out, we try to find a tilting complex over A_1 such that A_2 is isomorphic to the opposite endomorphism ring of that tilting complex. The indecomposable modules over A_1, up to isomorphism, are

$$P_1 = I_3 = [k \xrightarrow{1} k \xrightarrow{1} k], \quad P_2 = [0 \to k \xrightarrow{1} k], \quad P_3 = [0 \to 0 \to k]$$

$$I_2 = [k \xrightarrow{1} k \to 0], \quad I_1 = [k \to 0 \to 0], \quad S_2 = [0 \to k \to 0].$$

The P_i's are the indecomposable projectives, and the I_i's are indecomposable injectives. Now consider the complex

$$T = \cdots \to 0 \to P_2 \xrightarrow{\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}} P_1 \oplus P_1 \oplus P_3 \to 0 \to \cdots$$

where all terms to the left and right of this are zero, and i is the inclusion $P_2 \to P_1$. We want to show that this is a tilting complex. First, we note that T clearly is a bounded complex of finitely generated projective A_1-modules, so it is in $\operatorname{per} A_1$. What we must check now is that

1. for all $i \neq 0$, the set $\operatorname{Hom}_{\boldsymbol{D}^b(A_1)}(T, T[i])$ of shifted endomorphisms of T in $\boldsymbol{D}^b(A_1)$ vanishes,
2. the category $add(T)$ generates $\operatorname{per} A_1$ as a triangulated category.

To show that condition 1 is satisfied, we notice that shifting in either direction by more that 1 makes all morphisms zero, since there will be no overlap in the nonzero degrees. So we just need to show that $\mathrm{Hom}_{\boldsymbol{D}^b(A_1)}(T, T[i]) = 0$ for $i \in \{-1, 1\}$. We start with the case $i = 1$. A map from T to $T[1]$ will only have one nonzero component, namely

$$P_2 \xrightarrow{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} P_1 \oplus P_1 \oplus P_3.$$

So a chain map $T \to T[1]$ is completely determined by the maps a, b and c. We note that c must be zero, because there are no nonzero maps from P_2 to P_3. Next, observe that we can find maps b' and φ making the following diagram commutative

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \xrightarrow{\;0\;} & P_2 & \xrightarrow{\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}} & P_1 \oplus P_1 \oplus P_3 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle 0} & \overset{b'}{\swarrow} & \downarrow{\scriptstyle \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}} & \overset{\varphi}{\swarrow} & \downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & P_2 & \xrightarrow[\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}]{} & P_1 \oplus P_1 \oplus P_3 & \xrightarrow{\;0\;} & 0 & \longrightarrow & \cdots
\end{array}$$

More preciesly, we can choose b' such that $ib' = b$, and take $\varphi = \begin{pmatrix} 0 & a' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, with a' such that $a'i = a$. This works for any a and b, and thus for any map from T to $T[1]$. In other words, any map $T \to T[1]$ is null-homotopic, which means that $\mathrm{Hom}_{\boldsymbol{D}^b(A_1)}(T, T[1]) = 0$.

For the case $i = -1$, we also get that a chain map from T to $T[-1]$ has only one possible nonzero component. Such a chain map is given by the following diagram, where each square commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}} & P_1 \oplus P_1 \oplus P_3 & \xrightarrow{\;0\;} & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & 0 & \xrightarrow{\;0\;} & P_2 & \xrightarrow[\begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}]{} & P_1 \oplus P_1 \oplus P_3 & \longrightarrow & \cdots
\end{array}$$

The fact that each square commutes means that $ih = 0$, and since i is an inclusion (and thus mono), we see that h must be zero. This means that $\mathrm{Hom}_{\boldsymbol{D}^b(A_1)}(T, T[-1]) = 0$, which concludes the proof that T has no self-extensions.

Let's consider the second condition. Notice that P_1 and P_3 appear as direct summands of T, which means that they are contained in $add(T)$. If we can find a distingusihed triangle consisting of P_2 and two direct summands of T, then P_2 will be contained in the triangulated category generated by $add(T)$. Thus, we will have a full, triangulated subcategory of per A_1 which is closed under direct sums and summands, and which contains all indecomposable projective modules. Since any object in per A_1 can be constructed from the indecomposable projectives using direct sums, direct summands, shifts and cones, this must generate all of per A_1.

So we just need to find a distinguished triangle where one term is P_2, and the other terms are direct summands of T. In addition to P_1 and P_3, the last direct summand of T is $P_2 \xrightarrow{i} P_1$. Note that the cokernel of i is I_1, and that the short exact sequence $P_2 \xrightarrow{i} P_1 \xrightarrow{p} I_1$ defines a chain map

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P_2 & \xrightarrow{i} & P_1 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_1 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

This is a quasi-isomorphism, hence an isomorphism in $\boldsymbol{D}^b(A_1)$. So up to isomorphism, I_3 is a direct summand of T, which means that we get a distinguished triangle on the form we want from the short exact sequence above, namely $P_2 \xrightarrow{i} P_3 \xrightarrow{p} I_1 \rightsquigarrow$. We conclude that $add(T)$ generates per A_1 as a triangulated category, which means that both conditions are satisfied. Thus, T is a tilting complex over A_1.

Now, let's look at the endomorphism ring of T. Since the direct summand $P_2 \to P_1$ is isomorphic to I_1, we have that $T \simeq I_1 \oplus P_1 \oplus P_3$, which is easier to work with. Since Hom commutes with direct sums, it is enough to consider the homomorphisms between the direct summands. Each of the direct summands has endomorphism ring isomorphic to k (I_1 and P_3 are simple. A map $P_1 \to P_1$ consists of 3 scalar multiplications, but commutativity ensures that they must be equal). The only nonzero homomorphism between different direct summands are $P_3 \xrightarrow{a} P_1$ and $P_1 \xrightarrow{b} I_1$, and they compose to zero. Altogether we can look at this as a path algebra, and write it as

$$\text{End}(P_3) \xrightarrow{a\circ -} \text{End}(P_1) \xrightarrow{b\circ -} \text{End}(I_1), \quad ba = 0$$

We observe that this is isomorphic as an algebra to A_2^{op}, so we get that $A_2 \simeq \text{End}_{\boldsymbol{D}^b(A_1)}(T)^{op}$. This means that T satisfies the conditions in Rickard's Mortia theorem, and we conclude that A_1 and A_2 have equivalent derived categories.

In this case, there are techniques we could use to calculate $\boldsymbol{D}^b(A_1)$ and $\boldsymbol{D}^b(A_2)$ directly, and see that they are equivalent. But for more complicated examples, this may not be possible. In that case, Rickard's Morita theorem is probably our best bet to check if two rings are derived equivalent.

Appendix A

Some useful results

A.1 Trivial summands in complexes

Since we are working with right bound complexes of projective modules up to chain homotopy, there are some useful techniques we can employ. The first is called to add a trivial summand, and actually holds for any complex in the homotopy category $\boldsymbol{K}(\Lambda)$. The idea is that if you take a chain complex of modules and add a term of the form $M \xrightarrow{1} M$, then the resulting complex will be homotopy equivalent to the one you started with (that is, isomorphic in $\boldsymbol{K}(\Lambda)$).

Theorem A.1.1. *Let X be any complex in $\boldsymbol{K}(\Lambda)$, and let M be any Λ-module. Denote by X' a complex which is equal to X, but where you have added a copy of M to degree i and $i+1$, with the identity map between them. Then X' is homotopy equivalent to X, hence isomorphic in $\boldsymbol{K}(\Lambda)$.*

To prove this take a complex in $\boldsymbol{K}(\Lambda)$

$$\cdots \xrightarrow{d} X^{i-1} \xrightarrow{d} X^{i} \xrightarrow{d} X^{i+1} \xrightarrow{d} X^{i+2} \xrightarrow{d} \cdots$$

and by adding the summand in degrees i and $i+1$, we get

$$\cdots \xrightarrow{d} X^{i-1} \xrightarrow{\binom{d}{0}} X^{i} \bigoplus M \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)} X^{i+1} \bigoplus M \xrightarrow{(d\ 0)} X^{i+2} \xrightarrow{d} \cdots$$

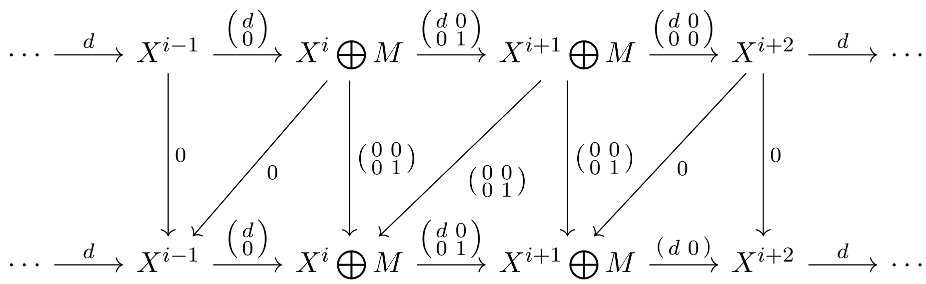
To show that they are homotopy equivalent, we need to find two maps between them such that both compositions are homotopic to the identity. Now consider the following diagram

$$
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d} & X^{i-1} & \xrightarrow{\binom{d}{0}} & X^{i}\bigoplus M & \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)} & X^{i+1}\bigoplus M & \xrightarrow{(d\ 0)} & X^{i+2} & \xrightarrow{d} & \cdots \\
 & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle (1\ 0)} & & \Big\downarrow{\scriptstyle (1\ 0)} & & \Big\downarrow{\scriptstyle 1} & & \\
\cdots & \xrightarrow{d} & X^{i-1} & \xrightarrow{d} & X^{i} & \xrightarrow{d} & X^{i+1} & \xrightarrow{d} & X^{i+2} & \xrightarrow{d} & \cdots \\
 & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \binom{1}{0}} & & \Big\downarrow{\scriptstyle \binom{1}{0}} & & \Big\downarrow{\scriptstyle 1} & & \\
\cdots & \xrightarrow{d} & X^{i-1} & \xrightarrow{\binom{d}{0}} & X^{i}\bigoplus M & \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)} & X^{i+1}\bigoplus M & \xrightarrow{(d\ 0)} & X^{i+2} & \xrightarrow{d} & \cdots \\
 & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle (1\ 0)} & & \Big\downarrow{\scriptstyle (1\ 0)} & & \Big\downarrow{\scriptstyle 1} & & \\
\cdots & \xrightarrow{d} & X^{i-1} & \xrightarrow{d} & X^{i} & \xrightarrow{d} & X^{i+1} & \xrightarrow{d} & X^{i+2} & \xrightarrow{d} & \cdots
\end{array}
$$

The last two vertical chain maps compose to the identity on each X^i, so all we need is to show that the composition of the first two vertical chain maps is homotopic to the identity. Composing the first two chain maps gives the following chain map

$$
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d} & X^{i-1} & \xrightarrow{\binom{d}{0}} & X^{i}\bigoplus M & \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)} & X^{i+1}\bigoplus M & \xrightarrow{(d\ 0)} & X^{i+2} & \xrightarrow{d} & \cdots \\
 & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle 1} & & \\
\cdots & \xrightarrow{d} & X^{i-1} & \xrightarrow{\binom{d}{0}} & X^{i}\bigoplus M & \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)} & X^{i+1}\bigoplus M & \xrightarrow{(d\ 0)} & X^{i+2} & \xrightarrow{d} & \cdots
\end{array}
$$

To see that this is homotopic to the identity, simply observe that in the difference between the identity and the composition, the maps in degrees i and $i+1$ are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and in all other degrees the maps are zero. It is then easy to see, by the following diagram, that this difference is null-homotopic, since every vertical map factors as wanted.



Thus we have shown that the two complexes are homotopy equivalent. This means that if you take a complex and add $M \xrightarrow{1} M$ as a direct summand in some degree, then the the complex you end up with is isomorphic in the homotopy category to the one you started with. In other words, adding a trivial direct summand doesn't change a complex in $\boldsymbol{K}(\Lambda)$.

A.2 Degreewise split exact sequences of complexes

Lemma A.2.1. *Any degreewise split exact sequence of complexes induces a distinguished triangle in the homotopy category.*

Proof. Take a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \ ,$$

where $B^k \simeq A^k \bigoplus C^k$, with maps s^k and π^k for each k such that $s^k\alpha^k = id_{A^k}$ and $\beta^k\pi^k = id_{C^k}$. We then get a map $\delta\colon C \to A[1]$, given by $s^{k+1}d_B^k\pi^k\colon C^k \to A^{k+1}$. This gives us a triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\delta} A[1] \ ,$$

which is distinguished in the homotopy category because it is quasi-isomorphic to the triangle

$$A \xrightarrow{\alpha} B \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow A[1] \ .$$

□

A.3 Total complex of the Hom-functor

Let X be a complex of Γ-Λ-bimodules, and let M be a complex of Γ-modules. We obtain a new complex by forming the double Hom complex given by X and M, and then forming the product total complex of that. This will be a complex of Λ-modules, which we will denote as $\mathcal{H}om_\Gamma(X, M)$, and whose components and differentials are given as follows

$$\mathcal{H}om_\Gamma(X, M)^n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_\Gamma(X^i, M^{i+n})$$

$$\partial\colon \mathcal{H}om_\Gamma(X, M)^n \longrightarrow \mathcal{H}om_\Gamma(X, M)^{n+1}$$

$$(f_i)_{i\in\mathbb{Z}} \longmapsto (d_M \circ f_i - (-1)^n f_{i+1} \circ d_X).$$

Notice that the elements of $\mathcal{H}om_\Gamma(X, M)^n$ are collections $(f_i)_{i\in\mathbb{Z}}$ of morphisms in each degree from X to $M[n]$, but these morphisms don't necessarily commute with the differentials. Now we have the following lemma concerning the homologies of this complex

Lemma A.3.1. *If X is a complex of Γ-Λ-bimodules and M is a complex of Γ-modules, then*

$$\mathtt{H}^n\mathcal{H}om_\Gamma(X,M) = Hom_{\boldsymbol{K}(\Lambda)}(X, M[n])$$

Proof. First, we see that the n-cycles are given by

$$\begin{aligned}\mathtt{Z}^n\mathcal{H}om_\Gamma(X,M) &= \operatorname{Ker}(\partial^n)\\ &= \big\{(f_i)_{i\in\mathbb{Z}} \in \mathcal{H}om_\Gamma(X,M)^n \mid d_M \circ f_i = (-1)^n f_{i+1}\circ d_X, \forall i\in\mathbb{Z}\big\},\end{aligned}$$

which is precisely the condition we require for (f_i) to be a chain map from X to $M[n]$. In other words, $\mathtt{Z}^n\mathcal{H}om_\Gamma(X,M) = \operatorname{Hom}_{\boldsymbol{C}(\Lambda)}(X, M[n])$. For the n-boundaries we have

$$\begin{aligned}\mathtt{B}^n\mathcal{H}om_\Gamma(X,M) &= \operatorname{Im}(\partial^{n-1})\\ &= \big\{(g_i)_{i\in\mathbb{Z}} \in \mathcal{H}om_\Gamma(X,M)^n \mid g_i = d_M \circ f_i + (-1)^n f_{i+1}\circ d_X, \forall i\in\mathbb{Z}\big\},\end{aligned}$$

where (f_i) are maps from X^i to M^{i+n-1}. This is the definition of (g_i) being null-homotopic, which means that $\mathtt{B}^n\mathcal{H}om_\Gamma(X,M) = \{$null-homotopic maps in $\operatorname{Hom}_{\boldsymbol{C}(\Lambda)}(X, M[n])\}$. So the n-th homology of the complex is

$$\mathtt{H}^n\mathcal{H}om_\Gamma(X,M) = \mathtt{Z}^n/\mathtt{B}^n = \operatorname{Hom}_{\boldsymbol{C}(\Lambda)}(X, M[n])/\{\text{null-homotopic maps}\},$$

which is precisely how morphisms are defined in the homotopy category. So $\mathtt{H}^n\mathcal{H}om_\Gamma(X,M) = \operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, M[n])$, and in particular $\mathtt{H}^0\mathcal{H}om_\Gamma(X,M) = \operatorname{Hom}_{\mathbf{K}(\Lambda)}(X, M)$. $\square$

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