Trajectory Optimization and Orbital Stabilization of Underactuated Euler-Lagrange Systems with Impacts

Christian Fredrik Sætre, Anton Shiriaev and Torleif Anstensrud

***Abstract*— A numerical framework for finding and stabilizing periodic trajectories of underactuated mechanical systems with impacts is presented. By parameterizing a trajectory by a set of synchronization functions, whose parameters we search for, the dynamical constraints arising due to underactuation can be reduced to a single equation on integral form. This allows for the discretization of the planning problem into a parametric nonlinear programming problem by Gauss-Legendre quadratures. A convenient set of candidates for transverse coordinates are then introduced. The origin of these coordinates correspond to the target motion, along which their dynamics can be analytically linearized. This allows for the design of an orbitally stabilizing feedback controller, which is also applicable for degrees of underactuation higher than one.**

***Index Terms*— Trajectory optimization, underactuated mechanical systems, transverse linearization, orbital stabilization.**

I. Introduction

Underactuation in mechanical systems, i.e. systems possessing fewer actuators than degrees of freedom, poses several challenges in terms of both planning and stabilizing feasible trajectories [1]. These tasks are further complicated when the continuous dynamics are complemented by impact events causing discrete (instantaneous) jumps in the system states. One approach for both planning and stabilizing periodic trajectories of such systems in the case of one degree of underactuation was outlined in [2]. There, under the assumption of the invariance of a set of kinematic relations, it is shown that the dimensionality of the planning problem can be reduced to a single second-order differential equation in a scalar variable. Moreover, the method naturally gives rise to a set of transverse coordinates, whose dynamics can be linearized along the target motion, allowing for the design of an orbitally stabilizing feedback controller using mainly tools from linear control theory (see also [3], [4]).

Although [2] presents a general method and necessary steps for finding feasible trajectories for such systems in the case of underactuation one, no efficient numerical scheme for searching for such trajectories was presented. In this paper, we present such a method. We state the planning problem as an optimal control problem (see Sec. II) which is then transcribed by Gauss-Legendre quadratures into a parametric nonlinear programming problem (see Sec. III). Furthermore, we present a set of transverse coordinates, different to those in [2], and a velocity-independent feedforward-like controller (see Sec. IV), allowing for straightforward computation of the linearized transverse dynamics for a class of trajectories. The method is then illustrated on a three-link biped walking robot with only one actuator (see Sec. V).

More specifically, our method allows for: 1) Writing the dynamical constraints arising from underactuation on integral form, in a multiple-shooting fashion, allowing for their direct computation through Gauss-Legendre quadratures; 2) Finding feasible trajectories without constraining the search space to a particular motion generator; 3) Not having to make any requirements on the stability of the system in the planning phase; 4) Deriving analytical expression for computing gradients and Hessians of the objective and constraints; 5) The handling of higher degrees of underactuation directly through constraints; and lastly 6) The design of orbitally stabilizing feedback controllers, whose structure is not highly dependent on the degree of actuation.

II. Problem Formulation

As stated in the introduction, we consider systems consisting of a continuous phase and a discrete impact (jump) event. We will consider systems in which the continuous time dynamics are given by controlled Euler-Lagrange equations[1]

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{B}\mathbf{u} \tag{1}$$

where $\mathbf{q} \in \mathbf{Q}$ are the n_q generalized coordinates, $\mathbf{Q} \subseteq \mathbb{R}^{n_q}$ the configuration manifold, $\dot{\mathbf{q}} \in \mathbb{R}^{n_q}$ the corresponding vector of generalized velocities, $\mathbf{u} \in \mathcal{U}$ is a vector of n_u control inputs, $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ the space of admissible controls, and $\mathbf{B}$ is an n_q by n_u matrix of full rank. The system is *underactuated*, i.e. $n_u < n_q$, with degree of underactuation $m = n_q - n_u$. Furthermore, we assume the matrix functions in (1) to be continuously differentiable, with $\mathbf{M}(\mathbf{q})$ denoting the symmetric, positive definite inertia matrix, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{M}}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{1}{2}\frac{\partial}{\partial \mathbf{q}}[\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}]$ contains Coriolis and centrifugal terms, and $\mathbf{G}(\mathbf{q})$ is a vector of potential forces.

Letting $\mathbf{x}(t) := [\mathbf{q}^\mathsf{T}(t)\ \dot{\mathbf{q}}^\mathsf{T}(t)]^\mathsf{T} \in \mathbf{TQ}$ denote the system states and $\mathbf{TQ} = \mathbf{Q} \times \mathbb{R}^{n_q}$ the tangent bundle, the discrete impact dynamics is given by the set of triples

$$\{\mathbf{\Gamma}_-, \mathbf{\Gamma}_+, \mathbf{P}(\cdot)\}, \quad \mathbf{P}(\cdot): \mathbf{\Gamma}_- \to \mathbf{\Gamma}_+, \tag{2}$$

such that $\mathbf{\Gamma}_+ \ni \mathbf{x}_+ = \mathbf{P}(\mathbf{x}_-)$. Here the notation $\mathbf{x}_- = \mathbf{x}(t_-)$ and $\mathbf{x}_+ = \mathbf{x}(t_+)$ denotes the system's states immediately prior to and after an impact, respectively.

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[1] In general, the equations of motion, even the number of coordinates, can vary between different sub-arcs of a trajectory. However, due to space limitations, we will here only consider trajectories given as a single arc.

We will consider the following task of finding pseudo-periodic trajectories of the system (1)-(2).

Problem 1: Given an initial configuration $\mathbf{q}_0$, find an initial state $\mathbf{x}_0$ and control input $\mathbf{u}(t) \in \mathcal{U}$ such that, after some time $T > 0$, we have $\mathbf{x}_T \in \mathbf{\Gamma}_-$ and $\mathbf{x}_0 = \mathbf{P}\big(\mathbf{x}_T\big)$.

This problem can be restated as the following optimal control problem (OCP). Determine the state $\mathbf{x}(t) \in \mathbf{TQ}$ and control input $\mathbf{u}(t) \in \mathcal{U}$ for all $t \in \mathcal{T} := [0, T]$ (with T possibly unknown) which minimizes the functional

$$J = \int_0^T \mathbf{F}\left(\mathbf{x}(t), \mathbf{u}(t)\right) dt, \tag{3}$$

for some $\mathcal{C}^2$-smooth, positive definite function $\mathbf{F} : \mathbf{TQ} \times \mathbb{R}^{n_u} \to \mathbb{R}_+$, subject to the continuous-time dynamics (1), the discrete periodic impact

$$\mathbf{\Gamma}_+ \ni \mathbf{x}_0 = \mathbf{P}\big(\mathbf{x}_T\big), \quad \mathbf{x}_T \in \mathbf{\Gamma}_-, \tag{4}$$

and the boundary configuration conditions

$$\mathbf{q}(0) = \mathbf{q}_+, \quad \mathbf{q}(T) = \mathbf{q}_-. \tag{5}$$

For brevity, we leave out additional constraints such as those related to reaction forces or other path constraints, which can additionally be added as desired.

III. Motion Planning Scheme

A. Reparameterization of a trajectory

Let a feasible trajectory of the hybrid system (1)-(2) driven by the control $\mathbf{u}_* : \mathcal{T} \to \mathbb{R}^{n_u}$ be denoted by

$$\eta_* := \{(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbf{TQ} : \ \mathbf{q}(t) = \mathbf{q}_*(t), \ \dot{\mathbf{q}}(t) = \dot{\mathbf{q}}_*(t), \ t \in \mathcal{T}\}.$$

Suppose it admits a reparametrization of the form

$$\mathbf{q}_*(t) = \mathbf{\Phi}(\mathbf{p}, s(t)), \ \dot{\mathbf{q}}_*(t) = \mathbf{\Phi}'(\mathbf{p}, s(t))\dot{s}_*(t), \ t \in \mathcal{T}, \tag{6}$$

where $s : \mathcal{T} \to \mathcal{S} := [0, 1]$ is a monotonically increasing, scalar variable which we will refer to as the *motion generator* (MG), while the vector function $\mathbf{\Phi}(s, \mathbf{p}) = [\phi_1(s, \mathbf{p_1}), \ldots, \phi_{n_q}(s, \mathbf{p}_{n_q})]^T$ consists of $\mathcal{C}^2$-smooth *synchronization functions*, $\phi_j : \mathcal{S} \to \mathbb{R}$, and a set of constant parameters $\mathbf{p} = \{\mathbf{p}_1, \ldots, \mathbf{p}_{n_q}\}$. We consider the MG to be monotonically increasing on the interval $\mathcal{S} = [0, 1]$, i.e. $\dot{s}(t) > 0$ for all $t \in \mathcal{T}$, with $s(0) = 0$ and $s(T) = 1$.

Under the assumption of the invariance of the relations (6), the the dynamics (1) can be written as

$$\mathcal{A}(s)\ddot{s} + \mathcal{B}(s)\dot{s}^2 + \mathcal{G}(s) = \mathbf{B}\mathbf{u} \tag{7}$$

where[2] $\mathcal{A}(s) := \mathbf{M}(\mathbf{\Phi})\mathbf{\Phi}'$, $\mathcal{G}(s) := \mathbf{G}(\mathbf{\Phi})$ and $\mathcal{B}(s) := \mathcal{A}'(s) + \mathcal{D}(s)$ with $\mathcal{D}(s) := \mathbf{C}(\mathbf{\Phi}, \mathbf{\Phi}')\mathbf{\Phi}' - \dot{\mathbf{M}}(\mathbf{\Phi}, \mathbf{\Phi}')\mathbf{\Phi}'$. Therefore, due to the full rank property of the matrix $\mathbf{B} \in \mathbb{R}^{n_q \times n_u}$, the nominal control input along the trajectory (6) can thus be found using its left inverse $\mathbf{B}^\dagger \in \mathbb{R}^{n_u \times n_q}$:

$$\mathbf{u} = \mathbf{B}^\dagger \left[\mathcal{A}(s)\ddot{s} + \mathcal{B}(s)\dot{s}^2 + \mathcal{G}(s)\right]. \tag{8}$$

[2] For notational simplicity, we will sometimes omit the s-arguments whenever no confusion may arise.

Similarly, as $\mathbf{B}$ must have a left annihilator $\mathbf{B}^\perp \in \mathbb{R}^{m \times n_q}$, i.e. $\mathbf{B}^\perp\mathbf{B}\mathbf{u} \equiv \mathbf{0}_{m\times 1}$ (recall that $m = n_q - n_u$ is the degree of underactuation), we have that

$$\underbrace{\mathbf{B}^\perp\mathcal{A}(s)}_{:=\mathcal{A}_r(s)}\ddot{s} + \underbrace{\mathbf{B}^\perp\mathcal{B}(s)}_{:=\mathcal{B}_r(s)}\dot{s}^2 + \underbrace{\mathbf{B}^\perp\mathcal{G}(s)}_{:=\mathcal{G}_r(s)} = \mathbf{0}_{m\times 1}. \tag{9}$$

These are the so-called *reduced dynamics* arising due to the underactuation of the system. They can be thought of as dynamical constraints on the evolution of the MG, which must hold at all time moments along any feasible trajectory of (1) given a particular parameterization of the form (6).

Now, using the relation $2\ddot{s} \equiv \frac{d\dot{s}^2}{ds}$, we can write (9) as m first-order differential equations in the variable $S := \dot{s}^2$:

$$\frac{1}{2}\alpha_i(s)S' + \beta_i(s)S + \gamma_i(s) = 0, \quad i = 1, \ldots, m, \tag{10}$$

where $\beta_i(s) = \alpha_i'(s) + \delta_i(s)$ and $S' = \frac{d}{ds}S$. It can be shown that these equations are all integrable, with integrating factor

$$\mu_i(s_0, s) := \alpha_i(s)\exp\left\{2\int_{s_0}^{s}\frac{\delta_i(\tau)}{\alpha_i(\tau)}d\tau\right\} =: \alpha_i(s)\psi_i(s_0, s);$$

hence, any two points (s_0, S_0) and (s, S), with $S_0 = S(s_0)$, correspond to the same trajectory if and only if [3]

$$\frac{1}{2}\psi_i(s_0, s)\alpha_i^2(s)S - \frac{1}{2}\alpha_i^2(s_0)S_0 + \int_{s_0}^{s}\psi_i(s_0, \tau)\alpha_i(\tau)\gamma_i(\tau)d\tau = 0. \tag{11}$$

Consequently, this, together with (8), allows finding the nominal control as a function only in terms of the MG: $\mathbf{u}_* = \mathbf{u}_*(s)$. That is, we can find S from (11) given the initial velocity S_0 if the following assumption is satisfied.

A1: For some $i \in \{1, \ldots, m\}$, $\alpha_i(s) \neq 0$ for all $s \in \mathcal{S}$. □
Suppose the following assumption holds as well.

A2: For some $i \in \{1, \ldots, m\}$, $\delta_i(s) \equiv 0$ for all $s \in \mathcal{S}$. □
As then $\psi_i(\cdot) \equiv 1$, it allows us to write (11) as

$$\frac{1}{2}\alpha_i^2(s)S - \frac{1}{2}\alpha_i^2(s_0)S_0 + \int_{s_0}^{s}\alpha_i(\tau)\gamma_i(\tau)d\tau = 0, \tag{12}$$

thus avoiding the nested integrals appearing when integrating $\psi_i(\cdot)$. The occurrence of the property A2 in one of the equations in (10) is not uncommon; for instance, it is always satisfied whenever a passive (unactuated) degree of freedom acts as a pivot in an open kinematic chain (see e.g. the arguments in [5]), which is the case for bipedal walkers with passive ankles (see Sec. V). Thus, to best clarify the method presented, we will assume this property to hold for, say, the i^{th} equation hereinafter. It should however be noted that it is not strictly necessary for our method, although it does somewhat simplify both the procedure and its numerical evaluation, possibly also increasing speed and convergence.

B. Restating and discretizing the optimal control problem

Let $\hat{\mathbf{x}}(s, \dot{s}, \mathbf{p}) = [\mathbf{\Phi}(s, \mathbf{p})^\mathsf{T} \ \mathbf{\Phi}'(s, \mathbf{p})^\mathsf{T}\dot{s}]^\mathsf{T}$ denote a trajectory parameterized on the form (6). Utilizing this parameterization and noticing that $\dot{s} := \dot{s}(s)$ is readily available from (12) ((11) in general), we can restate the OCP as the following trajectory optimization problem (TOP).

With respect to the decision variables $\{\dot{s}_+, \dot{s}_-, \mathbf{p}\}$, minimize the objective function

$$\int_0^1 \frac{1}{\dot{s}} \mathbf{F}\left(\hat{\mathbf{x}}(s, \dot{s}, \mathbf{p}), \hat{\mathbf{u}}(s, \dot{s}, \mathbf{p})\right) ds, \tag{13}$$

subject to $\hat{\mathbf{x}}(s, \dot{s}, \mathbf{p}) \in \mathbf{TQ}$, $\hat{\mathbf{u}}(s, \dot{s}, \mathbf{p}) \in \mathcal{U}$ (with $\hat{\mathbf{u}}$ found from (8) using $\ddot{s}$ from (10)), the continuous dynamics integral constraint (12), the instantaneous impact update

$$\hat{\mathbf{x}}(0, \dot{s}_+, \mathbf{p}) = \mathbf{P}\left(\hat{\mathbf{x}}(1, \dot{s}_-, \mathbf{p})\right), \tag{14}$$

and boundary conditions

$$\mathbf{\Phi}(0, \mathbf{p}) = \mathbf{q}_+, \quad \mathbf{\Phi}(1, \mathbf{p}) = \mathbf{q}_-. \tag{15}$$

Since the objective (13) is evaluated over a constant interval, it is natural to discretize (transcribe) the problem using a Gauss-Legendre quadrature of order n_J (although, in principle, any quadrature rule may be used). Given the set of ordered nodes $\{\hat{s}_j\}_{j=1}^{n_J}$ and corresponding weights $\{w_j\}_{j=1}^{n_J}$, the discretized objective function can then be written as

$$\min_{(\mathbf{S}, \mathbf{p})} \frac{1}{2} \sum_{j=1}^{n_J} w_j \frac{1}{\sqrt{S_j}} \mathbf{F}\left(\hat{s}_j, S_j, \mathbf{p}\right), \tag{16}$$

where $\mathbf{S} := \{S_j\}_{j=0}^{n_J+1}$, with $S_0 := \dot{s}_+^2$, $S_j := \dot{s}^2(\hat{s}_j)$ and $S_{n_J+1} := \dot{s}_-^2$, are new decision variables. By their definition they must satisfy the additional inequality constraints

$$S_j \geq 0, \quad \forall j \in \{0, \ldots, n_j + 1\}. \tag{17}$$

Note that, due to the requirement of $s(t)$ being monotonically increasing on $\mathcal{T}$, i.e. $\dot{s}_*(t) > 0$, none of the inequality constraints (17) can be active in the nonlinear program for any feasible solution. Moreover, the introduction of these new decision variables also lets us divide the integral dynamics constraint (12) in a multiple shooting fashion into $n_J + 1$ collocation-like constraints of the form:

$$\alpha_0(\hat{s}_j)^2 S_j - \alpha_0(\hat{s}_{j-1})^2 S_{j-1} + \Delta\hat{s}_j \sum_{k=1}^{n_I} \tilde{w}_k \alpha(\tilde{s}_k)\gamma(\tilde{s}_k) = 0 \tag{18}$$

for all $j \in \{1, 2, \ldots, n_J + 1\}$, where $\Delta\hat{s}_j := \hat{s}_j - \hat{s}_{j-1}$. Here we have discretized the integral using a Gauss-Legendre quadrature of order n_I with node-weight pairing $\{(\tilde{s}_k, \tilde{w}_k)\}_{k=1}^{n_I}$.

This particular structure of the TOP makes the gradients of the constraints and objective in terms of the decision variables $\{\mathbf{S}, \mathbf{p}\}$ easily attainable, although we leave out the explicit expressions due to space limitations. Furthermore, the Hessian of the Lagrangian is also available if desired.

Note that both the Hessian of the Lagrangian and gradients of the constraints will result in quite sparse matrices in general, with the level of sparsity generally depending on the choice of synchronization functions.

IV. Orbitally stabilizing controller design

A. Determining the motion generator

Suppose that a feasible trajectory $s \mapsto \mathbf{x}_*(s)$ of the hybrid system (1)-(2) parameterized by (6) is found by, e.g., the method presented in Section III. Since the MG $s : \mathcal{T} \to \mathcal{S}$ is then not known as an explicit function of the system states, we can not use it directly for feedback purposes. However, suppose that on a given subarc of the trajectory, denoted $\mathcal{S}_k \subseteq \mathcal{S}$, there exists a function $h_k : \mathbf{TQ} \times \mathcal{S} \to \mathbb{R}$ which, in some non-zero neighbourhood $\mathcal{X} \subset \mathbf{TQ}$ of the nominal trajectory (6), satisfies

$$h_k(\mathbf{x}_*(s), s) \equiv 0 \quad \text{and} \quad \frac{\partial h_k}{\partial s}(\mathbf{x}_*(s), s) \neq 0, \tag{19}$$

for all $s \in \mathcal{S}_k$. Then, by the implicit function theorem, there exists a function $P_k : \mathbf{TQ} \to \mathcal{S}_k$ such that for all $\mathbf{x} \in \mathcal{X}$, we have $h_k(\mathbf{x}, P_k(\mathbf{x})) = 0$ as well as

$$\frac{\partial P_k}{\partial \mathbf{x}}(\mathbf{x}) = -\left(\frac{\partial h_k}{\partial s}(\mathbf{x}, P_k(\mathbf{x}))\right)^{-1} \frac{\partial h_k}{\partial \mathbf{x}}(\mathbf{x}, P_k(\mathbf{x})).$$

Hence, for $\mathbf{x}(t) \in \mathcal{X}$, one can take

$$s = P_k(\mathbf{x}(t)) \tag{20}$$

as the projection of the states at time t onto the $\mathcal{S}_k$ subarc of the orbit (6).

In some cases, $h(\cdot)$ can be taken as one of the generalized coordinates; for example, suppose $|\phi_j'(s)| > 0$ for all $s \in \mathcal{S}$. Then, given a measurement $q_j(t)$, one can find an approximation of the corresponding $s(t)$ by iterating through

$$s_{k+1} = s_k - \frac{(\phi_j(s_k) - q_j(t))}{\phi_j'(s_k)}, \tag{21}$$

i.e. Newton's method, until $|s_{k+1} - s_k|$ is less than a desired accuracy. Thus the nominal trajectory can then be completely re-parameterized in terms of q_j:

$$\frac{\partial \phi_i}{\partial q_j}(s) = \frac{\phi_i'(s)}{\phi_j'(s)} \tag{22}$$

$$\frac{\partial^2 \phi_i}{\partial q_j^2}(s) = \frac{\phi_i''(s)\phi_j'(s) - \phi_i'(s)\phi_j''(s)}{(\phi_j'(s))^3} \tag{23}$$

In the following, we will, due to space limitations, only consider the case of $|\phi_j'(s)| > 0$, thus letting us utilize (21). Moreover, without loss of generality, we will consider $j = 1$ such that $q_1 \equiv \phi_1(s)$ is assumed to always be satisfied.

B. Linearized transverse dynamics and orbital stabilization

Let us introduce the following coordinates:

$$\mathbf{q} = \mathbf{\Phi}(s) + \mathbf{L}\mathbf{y} := \mathbf{\Phi}(s) + \begin{bmatrix} \mathbf{0}_{1 \times n_q - 1} \\ \mathbf{I}_{n_q - 1} \end{bmatrix} \mathbf{y} \tag{24}$$

where $\mathbf{y} = [y_2, \ldots, y_{n_q}]^\mathsf{T}$ ($y_1 = q_1 - \phi_1(s) \equiv 0$), as well as

$$\dot{\mathbf{q}} = \mathbf{\Phi}'(s)\zeta(s) + \mathbf{z}. \tag{25}$$

Here $\zeta : [0, 1] \to \mathbb{R}_+$ represents the velocity of the MG on the nominal orbit, i.e. $\zeta(s(t)) := \dot{s}_*(s)$, which is readily available from (12) (or (11) in general). The coordinates $\mathbf{y}$:

$\mathbf{Q} \times \mathcal{S} \to \mathbb{R}^{n_q-1}$ thus measures the deviation of positions away from the nominal trajectory, whereas $\mathbf{z} : \mathbb{R}^{n_q} \times \mathcal{S} \to \mathbb{R}^{n_q}$ is a measure of the deviation of the velocities. It is not difficult to show that they are related by

$$\dot{\mathbf{y}} = \begin{bmatrix} -\mathbf{L}_y^{\mathsf{T}} \mathbf{\Phi}'(s)/\phi_1'(s) & \mathbf{I}_{n_q-1} \end{bmatrix} \mathbf{z} =: \mathbf{L}_z^{\dot{y}}(s)\mathbf{z}, \tag{26}$$

where we have used the fact that $\dot{s} = (z_1 + \phi_1'(s)\zeta(s))/\phi_1'(s)$. Furthermore, the dynamics of the $\mathbf{z}$-coordinate can be found from (25) to be

$$\dot{\mathbf{z}} = \ddot{\mathbf{q}} - \mathcal{F}(s)\dot{s} = \mathbf{M}^{-1}(\mathbf{q})\left[\mathbf{B}\mathbf{u} - \mathbf{U}(\mathbf{q}, \dot{\mathbf{q}}, s)\right] - \mathcal{F}\frac{\partial P}{\partial \mathbf{q}}\mathbf{z}$$

where $\partial P(\mathbf{q})/\partial \mathbf{q} = [1/\phi_1'(s), \mathbf{0}_{1\times n_q-1}]$ and with

$$\begin{aligned} \mathcal{F}(s) &:= \mathbf{\Phi}''(s)\zeta(s) + \mathbf{\Phi}'(s)\zeta'(s), \\ \mathbf{U}(\mathbf{q}, \dot{\mathbf{q}}, s) &:= \mathbf{M}(\mathbf{q})\mathcal{F}(s)\zeta(s) + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}). \end{aligned}$$

Then, if we take[3]

$$\mathbf{u} = \mathbf{B}^{\dagger}\mathbf{U}(\mathbf{q}, \mathbf{\Phi}'\zeta, s) + \mathbf{v}, \tag{27}$$

for some $\mathbf{v} \in \mathbb{R}^{n_u}$ to be defined, we can write

$$\dot{\mathbf{z}} = \mathbf{\Lambda}_y(\mathbf{y}, s) + \mathbf{\Lambda}_z(\mathbf{y}, \mathbf{z}, s)\mathbf{z} + \mathbf{g}_{\perp}(\mathbf{y}, s)\mathbf{v} \tag{28}$$

where

$$\mathbf{\Lambda}_y(\mathbf{y}, s) := \mathbf{M}^{-1}(\mathbf{q})\left(\mathbf{B}\mathbf{B}^{\dagger} - \mathbf{I}\right)\mathbf{U}(\mathbf{q}, \mathbf{\Phi}'\zeta, s), \tag{29a}$$

$$\begin{aligned} \mathbf{\Lambda}_z(\mathbf{y}, \mathbf{z}, s) := &- \mathbf{M}^{-1}(\mathbf{q})\big[\mathbf{C}(\mathbf{q}, \mathbf{z}) + 2\mathbf{C}(\mathbf{q}, \mathbf{\Phi}'(s)\zeta(s)) \\ &+ \mathbf{M}(\mathbf{q})\mathcal{F}(s)\frac{\partial P}{\partial \mathbf{q}}(\mathbf{q})\big], \end{aligned} \tag{29b}$$

$$\mathbf{g}_{\perp}(\mathbf{y}, s) := \mathbf{M}^{-1}(\mathbf{q})\mathbf{B}. \tag{29c}$$

Due to (7) it is clear that we have $\mathbf{\Lambda}_y(\mathbf{0}, s) \equiv \mathbf{0}$; thus the first order Taylor expansion of $\mathbf{\Lambda}_y(\cdot)$ about $\mathbf{y} = \mathbf{0}$ is simply

$$\begin{aligned} \mathbf{\Lambda}_y(\mathbf{y}, s) &\approx \mathbf{\Lambda}_y(\mathbf{0}, s) + \frac{\partial \mathbf{\Lambda}_y}{\partial \mathbf{y}}(\mathbf{0}, s)\mathbf{y} \\ &= \mathbf{M}^{-1}(\mathbf{\Phi})\left(\mathbf{B}\mathbf{B}^{\dagger} - \mathbf{I}\right)\frac{\partial \mathbf{U}}{\partial \mathbf{y}}(\mathbf{0}, s)\mathbf{y} =: \delta\mathbf{\Lambda}_y(s)\mathbf{y}. \end{aligned}$$

Therefore, if we define the vector of *transverse coordinates*, $\mathbf{x}_{\perp} := [\mathbf{y}^{\mathsf{T}}, \mathbf{z}^{\mathsf{T}}]^{\mathsf{T}}$, and linearize its dynamics given by (26) and (28) along the the target motion, we obtain a linear, s-dependent system:

$$\frac{d}{ds}\delta\mathbf{x}_{\perp} = \mathbf{A}_{\perp}(s)\delta\mathbf{x}_{\perp} + \mathbf{B}_{\perp}(s)\mathbf{v} \tag{30}$$

where $\mathbf{A}_{\perp}(s) := \bar{\mathbf{A}}_{\perp}(s)/\zeta(s)$ and $\mathbf{B}_{\perp}(s) := \bar{\mathbf{B}}_{\perp}(s)/\zeta(s)$ with

$$\bar{\mathbf{A}}_{\perp}(s) = \begin{bmatrix} \mathbf{0}_{n_q-1} & \mathbf{L}_z^{\dot{y}}(s) \\ \delta\mathbf{\Lambda}_y(s) & \mathbf{\Lambda}_z(\mathbf{0}, \mathbf{0}, s) \end{bmatrix}, \bar{\mathbf{B}}_{\perp}(s) = \begin{bmatrix} \mathbf{0}_{(n_q-1\times n_u)} \\ \mathbf{g}_{\perp}(\mathbf{0}, s) \end{bmatrix}.$$

Thus, in a neighbourhood of the continuous part of a nominal trajectory parameterized by the MG, this system describes the evolution of the transverse coordinates upon the linearization of a *moving Poincaré section* when traversing the trajectory (see e.g. [6], [7], [8]).

[3]If accurate velocity measurement, $\dot{\mathbf{q}}$, are available, one can instead here take the partial feedback-linearizing controller $\mathbf{u} = \mathbf{B}^{\dagger}\mathbf{U}(\mathbf{q}, \dot{\mathbf{q}}, s) + \mathbf{v}$, which simplifies the following expressions somewhat.

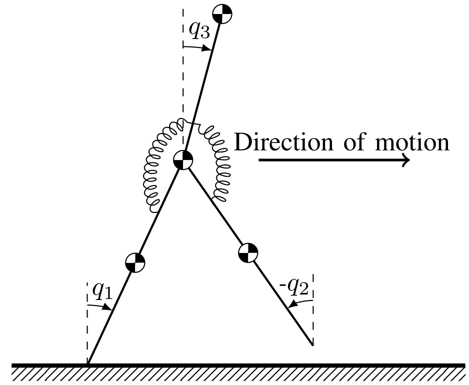

Fig. 1. Schematic of the biped system.

In order to account for how the impact (2) affects these coordinates, we form, using the procedure outlined in [2], the discrete mapping

$$\mathbf{x}_{\perp}(t_+) = \mathbf{F}_{\perp}\mathbf{x}_{\perp}(t_-). \tag{31}$$

Then a controller which stabilizes the origin of (30), and consequently the nominal trajectory of the hybrid system (1)-(2) in the orbital sense (at least locally), can be found as

$$\mathbf{v} = -\mathbf{\Gamma}\mathbf{B}_{\perp}^T(s)\mathbf{R}(s)\mathbf{x}_{\perp}. \tag{32}$$

Here the matrix function $\mathbf{R}(\cdot)$ is the symmetric, positive semi-definite solution of the differential Riccati equation

$$\begin{aligned} &\frac{d}{ds}\mathbf{R}(s) + \mathbf{A}_{\perp}^T(s)\mathbf{R}(s) + \mathbf{R}(s)\mathbf{A}_{\perp}(s) + \mathbf{Q} \\ &\quad + 2\kappa\mathbf{R}(s) - \mathbf{R}(s)\mathbf{B}_{\perp}(s)\mathbf{\Gamma}^{-1}\mathbf{B}_{\perp}^T(s)\mathbf{R}(S) = 0 \end{aligned} \tag{33}$$

for $\mathbf{\Gamma} = \mathbf{\Gamma}^T \succ 0$, $\mathbf{Q} = \mathbf{Q}^T \succeq 0$, $\kappa \geq 0$, which, in addition, should satisfy

$$\mathbf{F}_{\perp}^T\mathbf{R}(0)\mathbf{F}_{\perp} \leq \mathbf{R}(1). \tag{34}$$

This is to ensure that the Lyapunov function candidate $V(\mathbf{x}_{\perp}, s) = \mathbf{x}_{\perp}{}^T\mathbf{R}(s)\mathbf{x}_{\perp}$ is decrescent in a neighbourhood of the nominal motion over the hybrid cycle.

Note that determining the existence of such an $\mathbf{R}(\cdot)$ is still an open problem.

V. Example: Three-link Biped with one actuator

To illustrate our scheme, we will consider the task of creating symmetric gaits for a three-link biped robot with springs attached between its torso and legs. The simple structure, consisting of three links connected together a central hip joint, as seen in Figure 1, is a common testbed for trajectory planning and stabilization methods within the scope of bipedal walking [9], [5], [10]. For simplicity, we will assume that the initial configuration is given, such that the final configuration can be found directly from the assumption of a symmetric gait:

$$q_2(T) = q_1(0), \quad q_1(T) = q_2(0), \quad q_3(T) = q_3(0). \tag{35}$$

Here $q_1(0)$ can be computed from the desired step length L_s by $q_0(0) = -\arcsin L_s/(2r)$ given the length r of the stance leg. The velocity impact map is taken from [9] and is of the form $\dot{\mathbf{q}}_+ = \mathbf{P}_{\dot{\mathbf{q}}}(\mathbf{q}_-)\dot{\mathbf{q}}_-$, whereas the system matrices $\mathbf{M}(\cdot)$, $\mathbf{C}(\cdot)$ and $\mathbf{G}(\cdot)$ are taken according to [10] (which are identical to those in [9] with the addition of linear springs). The physical parameters of the system are taken as in [9], with the spring stiffness K left as a possible additional decision variable in our search.

A. Results from numerical optimization

We considered the task of finding symmetric gaits of length $L_s = 0.5\,\mathrm{m}$ and with initial lean angle $q_3(0) = 0.2\,\mathrm{rad}$. As an objective function to minimize, we considered

$$J = \frac{1}{g m_T L_s} \int_0^T \sum_{i=1}^{n_u} \left| u_i \mathbf{B}_i^T \dot{\mathbf{q}} \right| dt \tag{36}$$

corresponding to the energetic cost of transport (CoT) of the system over one step. Here m_T denotes the total mass of the system and $\mathbf{B}_i^T$ denotes the i^{th} row of the transpose of the input mapping matrix $\mathbf{B}$.

The order of the quadratures for computing the objective (16) and integral dynamics constraints (18) were taken as $n_J = 50$ and $n_I = 5$, respectively. The synchronization functions in (6) were taken as Bézier polynomials of order n_b, whose first and last parameters are taken according to (35), with the remaining initialized as zero. The total number of decision variables, $\{\mathbf{S}, \mathbf{p}, K\}$, thus equaled $3n_b + n_j$, where the decision variables $\mathbf{S}$ were all initialized as one.

1) One degree of underactuation: We first considered the system with two actuators, $\mathbf{u} = [u_1, u_2]^T$, and

$$\mathbf{B} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{\mathsf{T}}. \tag{37}$$

In order to compare to [9], [5], we set $K = 0$ and omitted it as an decision variable, and took $n_b = 3$. The search[4] converged after 119 iterations and took approximately $18.5\,\mathrm{s}$ on average. The resulting gait, having a CoT of approximately $3.16 \cdot 10^{-2}$, can be seen in Figure 2 .

2) Two degrees of underactuation: To complicate our search, we considered the system with only a single actuator, here denoted by u_2, with the new input mapping matrix taken as $\mathbf{B} = [1, -1, 0]^{\mathsf{T}}$. This was done by adding the constraint $u_1(\hat{s}_i) = 0$ for all $i \in \{0, 1, \ldots, n_j + 1\}$. However, as that constraint is quite demanding, we relaxed it to $|u_1(\hat{s})| < \epsilon$, such that for $\epsilon \geq 0$ sufficiently small, a found solution corresponds to an "almost feasible" gait in which $u_1(\cdot)$ can be seen as a disturbance that can be later handled by appropriate feedback. By adding K as an optimization variable with initial value $20\,\mathrm{N\,m^{-1}}$, as well as taking $n_b = 9$ and $\epsilon = 10^{-5}$, the gait shown in Figure 3 was found, whose CoT was approximately $7.19 \cdot 10^{-2}$. The search converged after 954 iterations and took approximately $74\,\mathrm{s}$ on average.

[4]The optimization problems were solved using the *fmincon* command in MATLAB running the interior-point algorithm solver on a 64bit operating system with an Intel Core I7 2.8 GHz processor. Gradients of the constraints and objective were provided to the solver, whilst the Hessian of the Lagrangian was estimated using the BFGS algorithm.

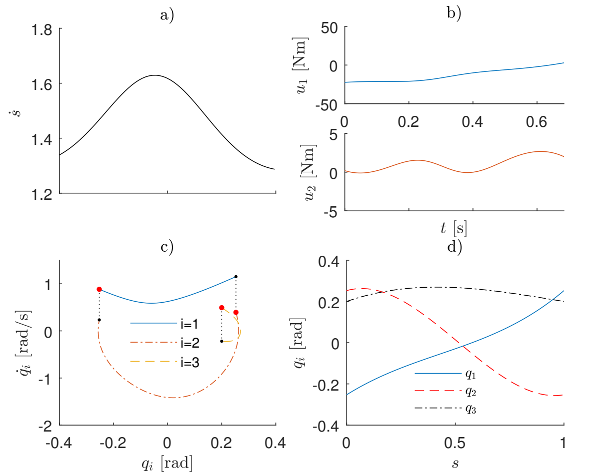

Fig. 2. Gait found with $n_b = 3$ and two actuators. a) phase portrait of the MG; b) control inputs; c) phase portraits of the system coordinates (initial points in red); and c) generalized coordinates as functions of the MG.

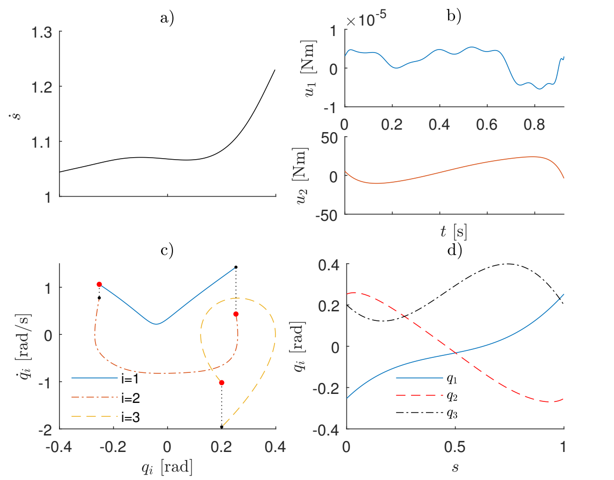

Fig. 3. Gait found with $n_b = 9$ and one actuator. a) phase portrait of the MG; b) control inputs; c) phase portraits of the system coordinates (initial points in red); and c) generalized coordinates as functions of the MG.

B. Orbital stabilization and numerical simulation

As $\dot{q}_1(t) > 0$ for all $t \in [0, T]$ for the gait in Figure 3, we could utilize the control strategy outlined in Section IV. We solved (33) subject to (34) with $Q = \mathbf{I}_5$ and $\Gamma = \kappa = 1$ using the SDP approach in [11], representing $\mathbf{R}(s)$ by Beziér curves of order 10.[5] The evolution of the transverse coordinates and the control input from simulating the system in closed-loop with one actuator saturated at $\pm 50\,\mathrm{N\,m}$ and initiated with perturbed initial conditions can be seen in Figure 4. After 10 steps, the norm of the transverse coordinates became bounded below 10^{-5}.

[5]The found solution to the SDP was not a solution of the DRE (33) but to the corresponding differential Riccati inequality, i.e. the equality in (33) is substituted with ≥ 0. The found solution was however still stabilizing, with the characteristic multipliers of the state transition matrix over the hybrid cycle (see Eq. (36) in [10]) lying well within the unit circle.

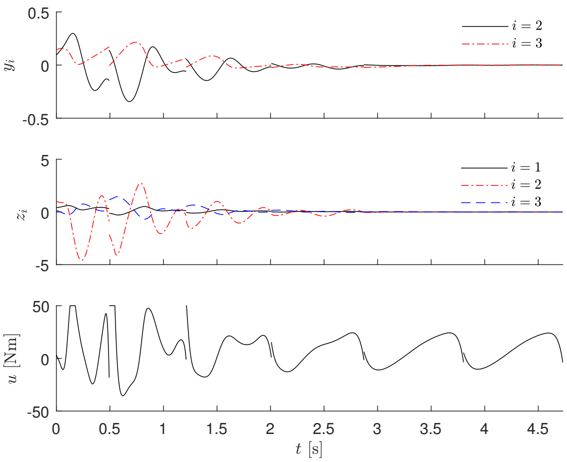

Fig. 4. Evolution of the transverse coordinates and the control input from simulating the closed-loop system with perturbed initial conditions and a single actuator saturated at $\pm 50\,\mathrm{N\,m}$. The perturbations vanish almost fully after approximately four steps.

VI. Discussion and Future work

We have presented a scheme for both finding and stabilizing trajectories of a class of underactuated mechanical systems with impacts. In what follows, we give some brief comments regarding our method, results and future work, as well as some very brief comparisons to related methods.

Choice of motion generator: Commonly, the motion generation (MG) is chosen as a known function of the generalized coordinates. Such a choice can often be justified, as in the case of bipedal robots where the angle of stance leg is commonly chosen [9], [5], [10]. For more complicated tasks, however, in which picking a specific MG is non-trivial, only requiring its monotonicity as in our approach has clear advantages as knowledge of the trajectory (e.g. the monotonicity of a coordinate) regardless can be utilized through appropriate constraints on the synchronization functions.

Handling higher degrees of underactuation: As illustrated, our approach allows one to find "almost feasible" trajectories for degrees of underactuation greater than one by treating them as (in-)equality (collocation) constraints. However, due to the sensitivity of such constraints, this will often require sufficiently many discretization points (nodes) and the synchronization functions to be sufficiently flexible (e.g. have high polynomial degree), resulting in slow convergence. The approach in [10] avoids this be finding these functions as solutions to a differential equation, and seems to surpass our method in terms of convergence rate. On the other hand, it lacks the flexibility in terms of differing actuation and choice of MG, as well as the ease of adding additional constraints.

Generalizing the numerical scheme: With small modifications, the method presented in Sec. III can be extended to become a more generalized trajectory planner for systems of the form (1). This requires removing assumption **A1**, thus allowing for the occurrence of singularities in (10), i.e. points s_s such that $\alpha_i(s_s) = 0$. For many trajectories, such points are natural, even necessary; for example they will occur at a system's equilibrium states. The main difficulty with removing **A1** is, however, that $\ddot{s}$ is no longer guaranteed to be found from (10), which we require in order to find the control input from (8). This can be handled by interpolating $\mathbf{S} := \{S_j\}_{j=0}^{n_J+1}$ using, say, a Lagrange polynomial, such that S_j' can be found. Then one can add collocation constraints corresponding to (9) evaluated at the discretization points, and likewise find the control input directly from (8).

Transverse coordinates and orbital stabilization: In our scheme, we find an implicit function letting us retrieve the MG for a given trajectory. For the example shown, this allowed for straightforward computation of a set of transverse coordinates. Similar coordinates and approaches have previously been considered (see e.g. [12], [13]). Our approach, however, differs both in the way that we parameterize the trajectory by a MG implicitly defined, as well as by the velocity independent feedforward-like control input, allowing for a transverse linearization only dependent on the MG. It also differs from the method of [4] in that it can handle singularities in (10), and that a change in actuation only requires updating the input mapping matrix. We plan to present the generalization of this approach using excessive transverse coordinates in a future publication.

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