

# Periodic Motion Planning for Virtually Constrained (Hybrid) Mechanical Systems

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**Abstract**—The paper presents sufficient and almost necessary conditions for the presence of periodic solutions for zero dynamics of virtually constrained under-actuated Euler-Lagrange system. This result is further extended to detect periodic solutions for a class of hybrid systems in the plane and analyze their orbital stability and instability. Illustrative examples are given.

**Index Terms**—Motion planning, Hybrid systems, Orbital feedback stabilization, Virtual holonomic constraints, Lyapunov lemma

## I. INTRODUCTION

This note addresses the question of description of the qualitative behaviour of a particular class of second order dynamical systems. The following systems

$$\alpha(q) \frac{d^2 q}{dt^2} + \beta(q) \left( \frac{dq}{dt} \right)^2 + \gamma(q) = 0, \quad (1)$$

are investigated. Here  $q \in R^1$ ;  $\alpha(q)$ ,  $\beta(q)$  and  $\gamma(q)$  are continuous scalar functions.

It turns out that the dynamical systems (1) naturally appear following the control design method based on an idea of *virtual holonomic constraints* recently elaborated in [7], [8] and others.

To give an insight how the system (1) gets into consideration, let us briefly repeat the motivation example from [8]: Consider the cart-pendulum system

$$\begin{aligned} (M + m)\ddot{x} + ml \cos \theta \ddot{\theta} - ml \sin \theta \dot{\theta}^2 &= f \\ ml \cos \theta \ddot{x} + ml^2 \ddot{\theta} - mgl \sin \theta &= 0 \end{aligned} \quad (2)$$

where  $x \in R^1$  is the horizontal displacement of the cart,  $\theta \in S^1$  is the angle between the pendulum rod and the vertical axis, which is zero at the upright position;  $m$ ,  $M$  are the masses of the pendulum and the cart respectively;  $l$  is the distance from the pendulum's suspension to the center of mass of the rod;  $f$  is the control variable, see Fig. 1.

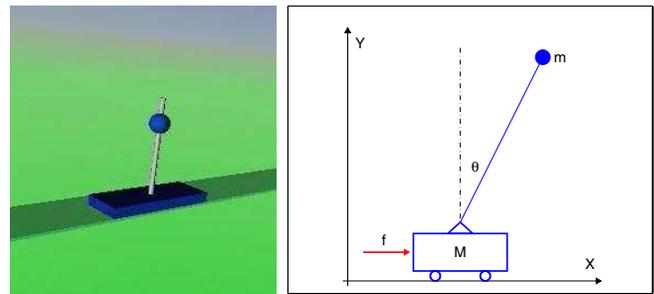


Fig. 1. The cart pendulum system

The problem is then how to generate and further orbitally stabilize a periodic motion (with approximately given frequency  $\omega_c$ ) of the pendulum around the upright equilibrium when the cart moves forwards with a prescribed average velocity ( $= V_x$ ) over the oscillation period.

To simplify the calculations, we will assume that  $m = M = l = 1$ . Then the equations (2) are

$$2\ddot{x} + \cos \theta \ddot{\theta} - \sin \theta \dot{\theta}^2 = f \quad (3)$$

$$\cos \theta \ddot{x} + \ddot{\theta} - g \sin \theta = 0 \quad (4)$$

Consider the following *virtual constraint* that relates the position of the cart  $x$  and the angle of the pendulum  $\theta$

$$x = V_x(t - t_0) - \left[ 1 + \frac{g}{\omega_c^2} \right] \ln \left( \frac{1 + \sin \theta}{\cos \theta} \right) \quad (5)$$

where  $\omega_c$  is a desired frequency of the pendulum oscillations;  $V_x$  is a desired speed of the cart;  $t$  is time;  $t_0$  is an initial time instant.

Supposing that there is a controller<sup>1</sup> that ensures that (5) holds along the solutions of the closed-loop system, then we

<sup>1</sup>It is constructed in [8].

can rewrite (3)–(4) into the new form

$$\ddot{x} + \left[1 + \frac{g}{\omega_c^2}\right] \frac{1}{\cos \theta} \ddot{\theta} + \left[1 + \frac{g}{\omega_c^2}\right] \frac{\sin \theta}{\cos^2 \theta} \dot{\theta}^2 = 0 \quad (6)$$

$$\cos \theta \ddot{x} + \ddot{\theta} - g \sin \theta = 0 \quad (7)$$

where (6) comes from double differentiation of (5). Solving and substituting the expression for  $\ddot{x}$  from (6)–(7), one gets the following equation for  $\theta$

$$-\frac{g}{\omega_c^2} \ddot{\theta} - \left[1 + \frac{g}{\omega_c^2}\right] \frac{\sin \theta}{\cos \theta} \dot{\theta}^2 - g \sin \theta = 0$$

or equivalently

$$\ddot{\theta} + \left[1 + \frac{\omega_c^2}{g}\right] \frac{\sin \theta}{\cos \theta} \dot{\theta}^2 + \omega_c^2 \sin \theta = 0. \quad (8)$$

The system (8) has the same structure as the general equation (1) with

$$\alpha(\theta) = 1, \quad \beta(\theta) = \left[1 + \frac{\omega_c^2}{g}\right] \frac{\sin \theta}{\cos \theta}, \quad \gamma(\theta) = \omega_c^2 \sin \theta$$

Only solutions of (8) could be observed in the closed loop (after transition) if a controller stabilizes the geometrical relations (5).

The problem of providing a qualitative description of the dynamics of (8) or in general (1) becomes then of interest<sup>2</sup>. It happens that the system (1) cannot have asymptotically stable solutions, and search of conditions when it has bounded motions is quite nontrivial.

The situation gets even more complicated, if one includes and takes into consideration a discontinuous update law

$$[q(t_+), \dot{q}(t_+)] = F(q(t_-), \dot{q}(t_-), t_-) \quad (9)$$

acting from time to time on solutions of (1). Introduction of the map (9) makes the combined system (1), (9) hybrid in nature. Such systems have gained a lot of attention during the last decade. Recently the application of hybrid control strategies has been used for generating periodic patterns for locomotion of walking robots, see [1], [6] and references therein.

The main contribution of this paper is the derivation of new sufficient conditions for the presence of periodic solutions in dynamics of the continuous-time system (1) and the hybrid system (1), (9). Both test are rigorously proven and extend some of the known results reported in [3], [4] and others.

## II. IMPORTANT PRELIMINARY RESULT

The next statement extracted from [8] shows that any dynamical system of the form (1) has a general integral of motion, that is a function of 4 variables:

$$q_0, \quad \dot{q}_0, \quad q, \quad \dot{q}$$

<sup>2</sup>It can be shown that any  $n$ -degree of freedom under-actuated controlled Euler-Lagrange system with the degree of under-actuation equal to one, always results in the system (1), provided a controller is chosen to stabilize any  $(n - 1)$  constraints imposed on generalized coordinates [7].

which being evaluated along a solution

$$q = q(t), \quad \dot{q} = \dot{q}(t)$$

of the system (1) with the origin in  $[q_0, \dot{q}_0]$ , remains equal to a constant.

*Theorem 1:* Given initial conditions  $[q_0, \dot{q}_0]$ , if the solution

$$[q(t), \dot{q}(t)] = [q(t, q_0, \dot{q}_0), \dot{q}(t, q_0, \dot{q}_0)]$$

of the system (1) exists for these initial conditions, then the function

$$I_0(q, \dot{q}, q_0, \dot{q}_0) = \dot{q}^2 - \exp \left\{ -2 \int_{q_0}^q \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \times \quad (10)$$

$$\times \left( \dot{q}_0^2 + \int_{q_0}^q \exp \left\{ 2 \int_{q_0}^s \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \frac{2\gamma(s)}{\alpha(s)} ds \right)$$

preserves a zero value along this solution irrespective of the boundedness of the solution  $[q(t), \dot{q}(t)]$ . ■

## III. MAIN RESULTS

### A. Sufficient Conditions for Existence of Periodic Solution of Virtually Constraint Euler-Lagrange System (1)

*Theorem 2:* Let  $q_0$  be an equilibrium of the system (1), i.e., the solution of the equation

$$\gamma(q_0) = 0. \quad (11)$$

Suppose that the scalar functions  $\alpha(\cdot)$ ,  $\beta(\cdot)$  and  $\gamma(\cdot)$  are  $C^1$ -smooth. Consider a linearization of the nonlinear system (1) around this equilibrium

$$\frac{d^2}{dt^2} z + \left[ \frac{d}{dq} \frac{\gamma(q)}{\alpha(q)} \right] \Big|_{q=q_0} \cdot z = 0 \quad (12)$$

If the linear system (12) has a center at  $z = 0$ , then the nonlinear system (1) has a center at the equilibrium  $q_0$ . ■

*Proof.* It is readily seen that if the constant

$$\omega = \left[ \frac{d}{dq} \frac{\gamma(q)}{\alpha(q)} \right] \Big|_{q=q_0}$$

is negative, then the linear system (12) has a saddle point. By the Hartman-Grobman theorem [2] the nonlinear system (1) has a saddle point too. If  $\omega = 0$ , then (12) is a double integrator that is again unstable, but this might not be related to the behaviour of the nonlinear system (1).

If  $\omega$  is positive, then the linear system (12) has a center, its solutions are oscillations of frequency  $\sqrt{\omega}$ . This fact does not directly imply that the nonlinear system (1) has a center at  $q_0$ , but it implies that the nonlinear system has either a stable or an unstable focus at  $q_0$ , or a center.

Introduce the polar coordinates

$$q - q_0 = r \cos \theta, \quad \dot{q} = r \sin \theta \quad (13)$$

and consider a solution  $[r(t), \theta(t)]$  of (1), (13) with initial condition at  $r_0 > 0$ ,  $\theta_0 = 0$ .

Due to the fact that the nonlinear system (1) has a focus at  $q_0$ , one can conclude that there exists a time moment  $T > 0$  so that  $\theta(T) = 0$  and  $r(T)$  is positive<sup>3</sup>. It is seen that if  $r(T) < r_0$  then the focus is stable, while if  $r(T) > r_0$  then the focus is unstable.

To prove that  $r(T) = r_0$ , consider the integral expression (10) that remains zero on the solution  $[r(t), \theta(t)]$

$$\begin{aligned}
 0 &= I(q(T), \dot{q}(T), q_0, \dot{q}_0) = \\
 &= [r(T) \sin \theta(T)]^2 - \exp \left\{ -2 \int_{r_0 \cos \theta_0}^{r(T) \cos \theta(T)} \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \times \\
 &\times \left( [r_0 \sin \theta_0]^2 - \int_{r_0 \cos \theta_0}^{r(T) \cos \theta(T)} \exp \left\{ 2 \int_{r_0 \cos \theta_0}^s \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \frac{2\gamma(s)}{\alpha(s)} ds \right) \\
 &= \exp \left\{ -2 \int_{r_0}^{r(T)} \frac{\beta_1(\tau)}{\alpha(\tau)} d\tau \right\} \times \\
 &\times \int_{r_0}^{r(T)} \exp \left\{ 2 \int_{r_0}^s \frac{\beta_1(\tau)}{\alpha(\tau)} d\tau \right\} \frac{2\gamma(s)}{\alpha(s)} ds
 \end{aligned} \tag{14}$$

The exponential factor in the product (14) is positive, therefore, the equality (14) implies that

$$\int_{r_0}^{r(T)} 2 \exp \left\{ 2 \int_{r_0}^s \frac{\beta_1(\tau)}{\alpha(\tau)} d\tau \right\} \frac{\gamma(s)}{\alpha(s)} ds = 0 \tag{15}$$

By the mean value theorem, there exists  $r_* \in [r_0, r(T)]$  (or  $r_* \in [r(T), r_0]$  in the case if  $r(T) \leq r_0$ ) such that

$$\begin{aligned}
 0 &= \int_{r_0}^{r(T)} 2 \exp \left\{ 2 \int_{r_0}^s \frac{\beta_1(\tau)}{\alpha(\tau)} d\tau \right\} \frac{\gamma(s)}{\alpha(s)} ds \\
 &= (r(T) - r_0) 2 \exp \left\{ 2 \int_{r_0}^{r_*} \frac{\beta_1(\tau)}{\alpha(\tau)} d\tau \right\} \frac{\gamma(r_*)}{\alpha(r_*)}
 \end{aligned}$$

If one supposes that  $r(T) \neq r_0$  then by necessity the next equality

$$2 \exp \left\{ 2 \int_{r_0}^{r_*} \frac{\beta_1(\tau)}{\alpha(\tau)} d\tau \right\} \frac{\gamma(r_*)}{\alpha(r_*)} = 0$$

holds. However, this is not possible, because the exponential function is positive irrespective of the argument value, while the other factor satisfies the next equality

$$\frac{\gamma(r)}{\alpha(r)} = \omega r + o(|r|) \quad \text{with} \quad \lim_{r \rightarrow 0+} \frac{o(|r|)}{r} = 0$$

when the radius  $r$  remains sufficiently small.

<sup>3</sup>This is true provided that  $r_0$  is chosen as a sufficiently small value.

It is readily seen that the last approximation implies that

$$\frac{\gamma(r)}{\alpha(r)}$$

remains positive on the interval  $[r_0, r(T)]$  provided that  $r_0$  is chosen sufficiently small. Therefore the identity (15) is only valid when  $r(T) = r_0$ . In turn, this fact implies that the nonlinear system (1) has a center at  $q_0$ . ■

*Remark 1:* As well known, the dynamical system having a center at its equilibrium is usually not structurally stable, so that only a few mathematical tools are available for establishing presence of the center. Most known are the Lagrange-Dirichlet stability test and Lyapunov lemma.

The first one is applied to any Lagrangian system checking sign-definiteness of the Hessian of its energy, which is its first integral. If the Hessian is sign-definite (might be positive definite or negative definite), then one can conclude that this equilibrium is stable in Lyapunov sense. For a Lagrangian system with one degree of freedom this conclusion implies that the system has a center at the equilibrium.

The second one - the Lyapunov lemma - is applied only to second order system, which might not be of mechanical origin, but which have a first integral. The lemma says that if a linearization of such a dynamical system has eigenvalues on the imaginary axis, and if the first integral and the right-hand side of the dynamical system are analytic functions, then the nonlinear system has a center at the equilibrium.

It is readily seen that neither the Lagrange-Dirichlet stability test nor the Lyapunov lemma can be directly applied to the system (1). Indeed, we have not been able to compute the first integral of the system (1), and we have not assumed any smoothness conditions for the functions  $\alpha(q)$ ,  $\beta(q)$  and  $\gamma(q)$  in Theorem 2. ■

*Remark 2:* Coming to the motivating example, it is seen that the linearization of the nonlinear system (8)

$$\ddot{\theta} + \left[ 1 + \frac{\omega_c^2}{g} \right] \frac{\sin \theta}{\cos \theta} \dot{\theta}^2 + \omega_c^2 \sin \theta = 0.$$

around its equilibrium  $\theta = 0$ , which corresponds to upright position of the pendulum, is

$$\ddot{\theta} + \omega_c^2 \theta = 0; \tag{16}$$

and it is marginally stable. Hence, Theorem 2 ensures that the nonlinear system (8) has a center at the equilibrium  $\theta = 0$ . ■

## B. Sufficient Conditions for Existence of Periodic Solution of Virtually Constraint Hybrid Lagrangian System (1),(9)

Combination of the instantaneous update law (9) with the continuous dynamics (1) gives a possibility to have a hybrid cycle in the hybrid system even if the conditions of Theorem 2 are not satisfied. Having in mind examples of controlled mechanical system, especially bipeds, see e.g. [1], [3], [4], [5], [6], for which the controller design reduces after a number of steps to analysis of hybrid cycles in the system (1), (9), let us impose assumption on the map  $F$ .

Suppose that the updating law (9) is defined and acts on a switching surface  $\Gamma_s$ , which is represented on the phase plane of the continuous-time system (1) by some  $C^0$ -curve. More precisely, suppose that the curve  $\Gamma_s$  consists of two components

$$\Gamma_s = \Gamma_- \cup \Gamma_+, \quad (17)$$

where the partition is uniquely defined by the  $F$ -operator in (9): When a solution of the continuous-time system (1) arrives to some point  $p_- \in \Gamma_-$ , this solution exhibits an instantaneous jump to a new point  $p_+ \in \Gamma_+$  of the phase plane, which, in turn, serves as an initial condition for reinitialization of the solution of the continuous-time system (1) after update.

The search of periodic solutions in the hybrid system (1), (9) can be organized based on the next statement.

*Lemma 1:* Suppose the hybrid system (1)-(9) has a periodic solution

$$q(t) = q(t + T)$$

of the period  $T$  with one nontrivial continuous-time sub-arc and one nontrivial update jump, then it is by necessity a stationary point of the next equations

$$\begin{aligned} I(q_-, \dot{q}_-, q_+, \dot{q}_+) &= 0, \\ [q_+, \dot{q}_+] &= F(q_-, \dot{q}_-) \end{aligned} \quad (18)$$

where the function  $I$  is defined in (10); and  $(q_-, \dot{q}_-)$  is uniquely defined point of the solution that belongs to  $\Gamma_-$ . ■

*Proof* of the statement follows from the fact that the function  $I$  preserves its zero value along the continuous-time arc of the solution, see Theorem 1. ■

Of course, the hybrid system (1), (9) might have a periodic solution with multiple continuous-time sub-arcs, but if this number of sub-arcs is finite, Lemma 1 can be easily extended to cover the case.

An important aspect of periodic solutions in the hybrid system (1), (9), is their orbital stability/instability. For the continuous-time system (1) asymptotic orbital stability of any bounded motion cannot take place. Indeed, the general integral preserving its value shows that any two periodic solutions of the system (1) should remain separated and of non-vanishing distance over all the time. In hybrid system, this is not the case anymore and depending on the update law (9) we can observe either orbitally asymptotically stable or unstable periodic solutions.

It is clear that such analysis can be reduced to analysis of the behaviour on the switching surface  $\Gamma$ .

*Theorem 3:* Consider the hybrid system (1), (9), suppose that it has a periodic solution

$$q(t) = q(t + T), \quad T > 0$$

of period  $T$  having one nontrivial continuous-time arc and one non-trivial update. Then this periodic solution of the hybrid system is orbitally asymptotically stable (unstable) if

the corresponding stationary point of the difference equation

$$\begin{aligned} I(q_{k+1}, \dot{q}_{k+1}, q_k, \dot{q}_k) &= 0, \\ [q_{k+2}, \dot{q}_{k+2}] &= F(q_{k+1}, \dot{q}_{k+1}) \end{aligned} \quad (19)$$

evolving on the switching surface  $\Gamma$  is asymptotically stable (unstable). ■

*Proof* of Theorem 3 follows from the standard Poincare map arguments. ■

*Remark 3:* The main advantage of Theorem 3 is that we know how to compute the integral function explicitly, that is, we can avoid the difficult tasks of solving the differential equations and find the analytical expression for the solution, when it arrives at switching surface  $\Gamma_-$ . These results applied to a class of particular virtually constrained systems - 2D bipeds - have been the key point for proving stability of newly generated cycles in hybrid dynamics of walking robot, [3], [4]. ■

## IV. EXAMPLES

### A. Example 1

Consider the case when the system (1) is a linear oscillator

$$\ddot{q} + q = 0 \quad (20)$$

defined only for  $q \geq 0$  and the nonlinear updating law (9) is of the form

$$F = \begin{cases} q_+ = 0, \quad \dot{q}_+ = \dot{q}_- \sin(\dot{q}_-), & \text{if } q = 0 \\ q_+ = q_-, \quad \dot{q}_+ = \dot{q}_-, & \text{otherwise} \end{cases} \quad (21)$$

To find periodic motions for the hybrid system (20), (21) following the Lemma, one needs to solve the system algebraic equations (18), which in this case are

$$\begin{aligned} (\dot{q}_+)^2 + (q_+)^2 - (\dot{q}_-)^2 - (q_-)^2 &= 0 \\ \dot{q}_+ &= \dot{q}_- \sin(\dot{q}_-) \\ q_- &= 0 \\ q_+ &= 0 \end{aligned} \quad (22)$$

where the last two equations are the equations of the switching curve  $\Gamma$ .

In the case of  $\Gamma_- = \{q|q < 0\}$  and  $\Gamma_+ = \{q|q \geq 0\}$  straightforward calculations show that the solutions are those trajectories of the system, whose velocity at  $q = 0$  equals to

$$\dot{q}_+ = 2\pi \cdot n + \frac{\pi}{2}, \quad n = 0, 1, \dots \quad (23)$$

*Remark:* In fact, without the distinction of  $\Gamma_-$  and  $\Gamma_+$  the equations (22) have another set of solutions

$$\dot{q}_- = -(\pi \cdot n + \frac{\pi}{2}), \quad n = 1, 2, \dots \quad (24)$$

which correspond to a set of stationary points of the hybrid system on  $\Gamma_-$ .

Coming back to the example (20), (21), let us prove that any found periodic solution is not orbitally stable<sup>4</sup>.

<sup>4</sup>the trivial periodic solution - the equilibrium at  $q = \dot{q} = 0$  is excluded from consideration.

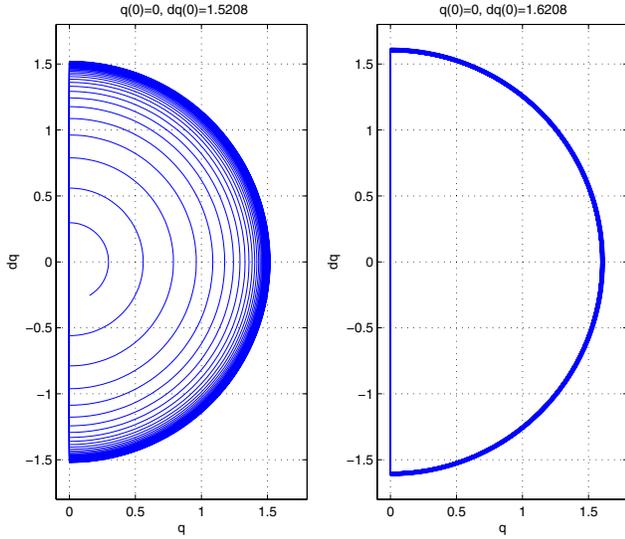


Fig. 2. Simulation of hybrid system (20),(21) with different initial conditions. (left):  $(q_0, \dot{q}_0) = (0, \pi/2 - 0.05)$ , (right):  $(q_0, \dot{q}_0) = (0, \pi/2 + 0.05)$ ,

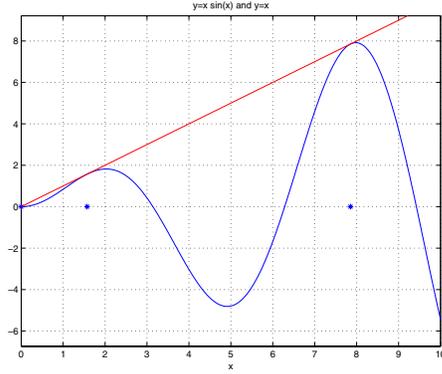


Fig. 3. Plot of switch law (21) with intersections corresponding to (23).

Indeed, the subset of the switching curve  $\Gamma_+$  in this example coincides with the semi-axis  $\dot{q} > 0, q = 0$ . Select any equilibrium (23), let say  $\dot{q}^* = \pi/2$  and consider the solution of the difference system (19). For the example, it can be equivalently rewritten as

$$x_{k+1} = \text{abs}(x_k \sin(x_k)), \quad x_0 = \dot{q}$$

If one initializes this equation by value  $x_0 = \dot{q}^* - \varepsilon$ , then it is clear that  $x_k$  will decrease in value and will never come back to the initial value  $x_0$ . See Fig. 2 for simulations with varying initial condition around  $(\theta_0, \dot{\theta}_0) = (0, \pi/2 + 2\pi n)$ .

### B. Example 2

Consider the hybrid system defined by the dynamic equation

$$\ddot{q} + q \cdot \dot{q}^2 + q = 0 \quad (25)$$

valid for  $q \geq 0$  and the update law

$$F = \begin{cases} q_+ = 0, \dot{q}_+ = (\dot{q}_-)^2, & \text{if } q = 0 \\ q_+ = q_-, \dot{q}_+ = \dot{q}_-, & \text{otherwise} \end{cases} \quad (26)$$

We readily obtain

$$\alpha(q) = 1, \quad \beta(q) = q, \quad \gamma(q) = q \quad (27)$$

and

$$\begin{aligned} I(q_-, \dot{q}_-, q_+, \dot{q}_+) &= \dot{q}_-^2 - \exp \left\{ -2 \int_{q_+}^{q_-} \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \times \\ &\left[ \dot{q}_+^2 - \int_{q_+}^{q_-} \exp \left\{ 2 \int_{q_+}^s \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \cdot 2 \frac{\gamma(s)}{\alpha(s)} ds \right] \\ &= \dot{q}_-^2 - e^{-(q_-^2 - q_+^2)} \left[ \dot{q}_+^2 - \left( e^{(q_-^2 - q_+^2)} - 1 \right) \right] \end{aligned}$$

From Lemma 1 we know that any cycles, if existing, must satisfy the relations

$$I(q_-, \dot{q}_-, q_+, \dot{q}_+) = 0, \quad q_+ = q_- = 0, \quad \dot{q}_+ = \dot{q}_-$$

Therefore

$$I(0, \dot{q}_-, 0, \dot{q}_+) = \dot{q}_-^2 - \dot{q}_+^2 = \dot{q}_+(1 - \dot{q}_+) = 0, \quad (28)$$

which implies that  $\dot{q}_+ = 1$  or  $\dot{q}_+ = 0$ . Hence, the only nontrivial cycle is initiated at  $q(0) = q_+ = 0$  and  $\dot{q}(0) = \dot{q}_+ = 1$ . Existence of solutions follows from analysis of the system without updates. To check the stability we use Theorem 3.

$$\theta_+ \rightarrow \theta_- \quad (I = 0) :$$

$$x_k \rightarrow x_{k+1} \quad \text{with } (x_{k+1})^2 - (x_k)^2 = 0$$

$$\theta_- \rightarrow \theta_+ \quad (\text{update}) :$$

$$x_{k+1} \rightarrow x_{k+2} \quad \text{with } (x_{k+2}) = (x_{k+1})^2$$

The Poincare first return map  $x_k \rightarrow x_{k+2}$  is given by

$$x_{k+2} = (x_k)^2 \quad (29)$$

and the cycle is thus *unstable* (i.e.,  $x_0 = 1 \Rightarrow x_k = 1$ ,  $x_0 < 1 \Rightarrow x_k \rightarrow 0$ ,  $x_0 > 1 \Rightarrow x_k \rightarrow \infty$ ). For numerical simulations of system (25),(26), see Fig. 4.

### C. Example 3

Consider again the continuous dynamics as in the Example 2, see (25), being valid for  $q \geq 0$ , but introduce a new update law according to

$$F = \begin{cases} q_+ = 0, \dot{q}_+ = \frac{1}{\ln 2} \ln(1 - \dot{q}_-), & \text{if } q = 0 \\ q_+ = q_-, \dot{q}_+ = \dot{q}_-, & \text{otherwise} \end{cases} \quad (30)$$

For this hybrid system any cycles, if existing, must satisfy the new relations

$$I(q_-, \dot{q}_-, q_+, \dot{q}_+) = 0, \quad q_+ = q_- = 0, \quad \dot{q}_+ = \frac{\ln(1 - \dot{q}_-)}{\ln 2}$$

which are equivalent to the following one

$$\dot{q}_+ = \frac{\ln(1 + \dot{q}_+)}{\ln 2} \quad (31)$$

The reader is asked to check that its solutions again, as in Example 2, are  $\dot{q}_+ = 1$  or  $\dot{q}_+ = 0$ , while the only nontrivial cycle is initiated at  $q(0) = q_+ = 0$  and  $\dot{q}(0) = \dot{q}_+ = 1$ .

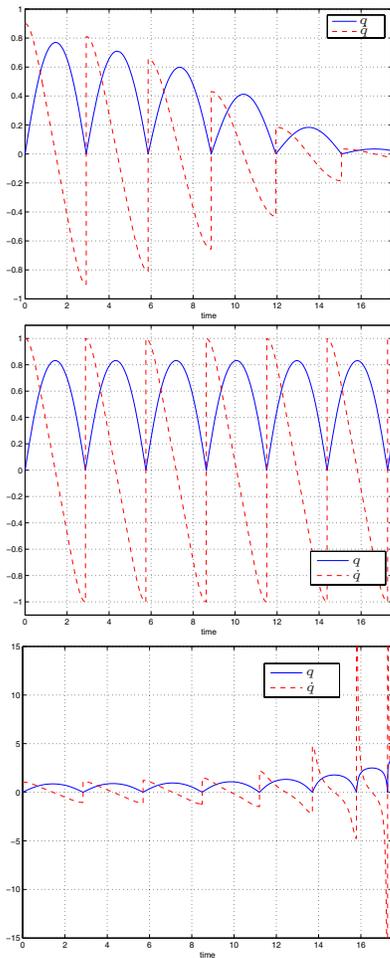


Fig. 4. Simulation results for system (25),(26) with initial conditions (upper):  $(q(0), \dot{q}(0)) = (0, 0.9)$ , (middle):  $(q(0), \dot{q}(0)) = (0, 1)$ , (lower):  $(q(0), \dot{q}(0)) = (0, 1.05)$ .

This hybrid system generates the Poincare map on the switching surface  $\Gamma = \{q = 0\}$  and looks like

$$x_{k+2} = f(x_k) = \frac{\ln(1 + x_k)}{\ln 2}, \quad x_k \geq 0 \quad (32)$$

The stability of the found cycle with  $x = 1$ , can be accessed by computing  $f'$  at  $x = 1$ . The straightforward calculations show that

$$\left. \frac{d}{dx} \left( \frac{\ln(1 + x_k)}{\ln 2} \right) \right|_{x=1} = \frac{1}{2 \ln 2} < 1$$

Hence the found hybrid cycle is asymptotically stable.

## V. CONCLUSIONS

The paper presents several analytical results related to the motion planning task, following the controller design methodology based on the virtual holonomic constraint approach for feedback stabilization of under-actuated Euler-Lagrange systems. Sufficient and almost necessary conditions for the presence of periodic solutions for zero dynamics of virtually constrained under-actuated Euler-Lagrange system are given. This result is further extended to detect

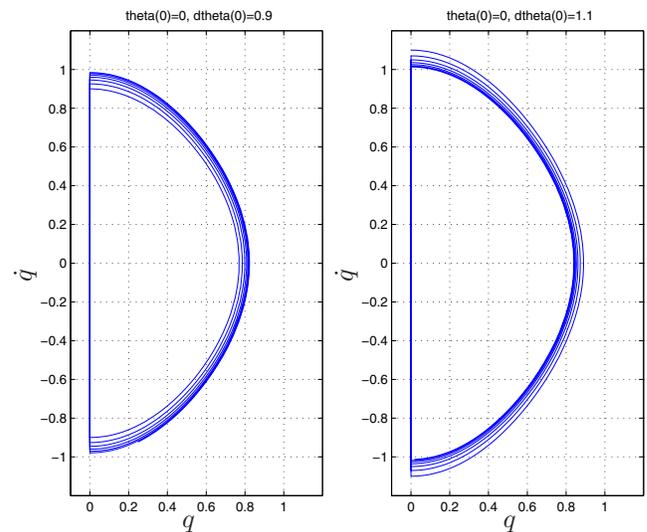


Fig. 5. Simulations of stable cycle for the hybrid system (25),(30) in Example 3 (different initial conditions). (left):  $(q_0, \dot{q}_0) = (0, 0.9)$ , (right):  $(q_0, \dot{q}_0) = (0, 1.1)$ .

periodic solutions for a class of hybrid systems in the plane and to analyze the orbital stability/instability of the cycles.

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## REFERENCES

- [1] Grizzle J.W., G. Abba, and F. Plestan. "Asymptotically stable walking for biped robots: Analysis via systems with impulse effects." *IEEE Transactions on Automatic Control*, 46:51–64, 2001.
- [2] Hartman Ph. *Ordinary Differential Equations*, John Wiley & Sons, 1964.
- [3] Westervelt E., J.W. Grizzle, and D.E. Koditschek. "Zero dynamics of planar biped walkers with one degree of under actuation." In *IFAC 2002*, Barcelona, Spain, 2002.
- [4] Westervelt E., J.W. Grizzle, and D.E. Koditschek. "Hybrid zero dynamics of planar biped walkers." *IEEE Transactions on Automatic Control*, December 2001.
- [5] Westervelt E., J.W. Grizzle, and C. Canudas-de Wit. "Switching and PI control of walking motions of planar biped walkers." *IEEE Transactions on Automatic Control*, 2003.
- [6] Chevallereau C., G. Abba, Y. Aoustin, F. Plestan, E.R. Westervelt, C. Canudas-de Wit, and J.W. Grizzle. "A Testbed for Advanced Control Theory". *IEEE Control Systems Magazine*, December, 2003.
- [7] Shiriaev A.S., J. Perram and C. Canudas-de-Wit. "Constructive tool for orbital stabilization of underactuated nonlinear systems: virtual constraints approach," *IEEE Transactions on Automatic Control*, Vol. 50, No. 8, pp. 1164-1176, 2005.
- [8] Shiriaev A.S., J. Perram, A. Robertsson and A. Sandberg. "Explicit Formulas for General Integral of Motion for a Class of Mechanical Systems subject to Virtual Constraints," *Proceedings of the 43rd IEEE Conference on Decision and Control (CDC2004)*, pp. 1158–1163, Paradise Island, Bahamas, Dec 2004.