Annual Reviews in Control 32 (2008) 200–211

Annual Reviews in Control

www.elsevier.com/locate/arcontrol

Can we make a robot ballerina perform a pirouette? Orbital stabilization of periodic motions of underactuated mechanical systems☆

A.S. Shiriaev [a,b,*], L.B. Freidovich [a], I.R. Manchester [a]

[a] Department of Applied Physics and Electronics, Umeå University, SE-901 87 Umeå, Sweden

[b] Department of Engineering Cybernetics, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

Received 7 December 2007; accepted 13 July 2008

Abstract

This paper provides an introduction to several problems and techniques related to controlling periodic motions of dynamical systems. In particular, we consider planning periodic motions and designing feedback controllers for orbital stabilization. We review classical and recent design methods based on the Poincaré first-return map and the transverse linearization. We begin with general nonlinear systems and then specialize to a class of underactuated mechanical systems for which a particularly rich structure allows many of the problems to be solved analytically.

Keywords: Periodic motion planning; Poincaré first-return map; Orbital stabilization; Transverse linearization; Virtual holonomic constraints

1. Introduction

Most people will agree that performing a pirouette is intrinsically challenging: for humans it takes both natural talent and years of training. Looking at it from the perspective of a control-systems scientist does not necessarily make it any easier, but does allow us to be more specific about what the problem is. For one, the motion is periodic, and it is well known that stabilization of periodic motions provides many challenges over and above those found when stabilizing an equilibrium. A second difficulty is that, standing on tip-toes, the dancer cannot directly maintain their upright position, that is, the system is underactuated: there are less independent control inputs than dynamical degrees of freedom.

The classical and the most frequently used tool for analysis of existence and stability of periodic trajectories of dynamical systems is the Poincaré first-return map, defined on a hyper-surface (Poincaré section) transversal to dynamics at a point of the cycle, see Poincaré (1916–1954), Leonov (2006) and many others. Calculation of the Poincaré map of a nonlinear system typically cannot be done analytically and requires numerical solution of the system dynamics for a large number of initial conditions. It is often computationally expensive and of limited use if we look for periodic solutions which are open-loop unstable, or if we look for design of a stabilizing controller. This motivates investigation of alternative strategies.

Given a periodic trajectory, one can approach the problem of synthesis of a stabilizing controller using the concept of a moving Poincaré section, see Leonov (2006). This can be done by introducing not one surface transverse to the cycle but a family of transverse surfaces parameterized by the points on the cyclic trajectory. The linearization of the dynamics on the foliation of these surfaces is a linear time-periodic system, the dimension of which is less than the dimension of the nonlinear system by one. Stability (stabilization) of this auxiliary linear system is equivalent to exponential orbital stability (stabilization) of the corresponding periodic motion of the original nonlinear dynamical system. This linear system is called a transverse linearization of the dynamics around the cycle.

In this paper, we consider a large class of controlled mechanical systems that includes many popular research set-ups such as the Furuta pendulum, the Pendubot, the Acrobot, a pendulum on a cart, a spherical pendulum and applications such as bipeds. Remarkably, for this class of nonlinear systems both

☆ A preliminary version of this paper was presented as a plenary talk at *the 3rd IFAC Workshop on Periodic Control Systems*, St. Petersburg, August 29–31, 2007. This work has been partly supported by the Swedish Research Council (the grants: 2005–4182, 2006–5243) and the Kempe Foundation.

* Corresponding author at: Department of Applied Physics and Electronics, Umeå University, SE-901 87 Umeå, Sweden.

E-mail addresses: Anton.Shiriaev@tfe.umu.se (A.S. Shiriaev), Leonid.Freidovich@tfe.umu.se (L.B. Freidovich), Ian.Manchester@tfe.umu.se (I.R. Manchester).

1367-5788/$ – see front matter
doi:10.1016/j.arcontrol.2008.07.001

the search of nontrivial periodic orbits and the computation for the transverse linearization around any feasible orbit can be done analytically.

As discussed in the paper, existence of a nontrivial periodic solution for a system in this class is equivalent to existence of a periodic solution of a particular second order differential equation, which can be computed by imposing an appropriate virtual holonomic constraint and is proved to be always integrable. The last property significantly simplifies the analysis and allows derivation of easy-to-check analytical conditions for presence of cycles in the original nonlinear system. The analysis of this auxiliary second order integrable differential equation allows one to proceed further and to introduce a set of new coordinates for a controlled mechanical system in a vicinity of its periodic solution. The main feature of these coordinates is that the transversal linearization of the dynamics of the nonlinear system around the cycle becomes transparent and can be computed analytically, provided that the dynamics are written in these coordinates.

The possibility of analytical construction of a transverse linearization and of periodic motion planning opens up a wide range of opportunities for using linear control theory for creating exponentially stable limit cycles in nonlinear systems.

The most prominent challenges which remain – and restrict wider application of this technique – are numerical. To design an orbitally stabilizing controller, one must ultimately find a stabilizing controller for a periodic linear system. For example, the theory behind the LQR approach for linear systems with periodic coefficients is well established, but requires finding the stabilizing solution of a matrix Riccati differential equation with periodic coefficients.[1] Similar comments are appropriate if the pole-placement technique is chosen for synthesis of a controller. In certain cases numerical solutions are achievable; however, there is a strong need for more reliable numerical methods. We refer the reader to Colaneri (2005) and references therein for a review of the theory of linear periodic systems and Varga and Van Dooren (2001), Varga (2007) and references therein for a review of appropriate numerical methods.

The rest of the paper is organized as follows. In Section 2, we describe a number of mathematical tools that are useful for the analysis and stabilization of periodic motions of general nonlinear systems. In Section 3, we specialize to the class of mechanical systems of underactuation degree one. In Section 4, we provide references to a number of recent applications of the techniques described. Finally, we give some brief conclusions in Section 5.

2. Tools for analyzing periodic motions of dynamical systems

In this section, we provide an introductory overview of a number of techniques that can be used to study periodic motions of dynamical systems, in particular: the Poincaré first-return map, transverse dynamics, and the controlled transverse linearization.

2.1. Problem formulation

Consider a general nonlinear control system, the dynamics of which can be described by

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathbb{R}^n$ is a state vector, $f(\cdot,\cdot)$ is a continuously differentiable vector function, and $u \in \mathbb{R}^m$ is a vector of control inputs.

For this system one can formulate the following task.

Problem 1 (*Periodic motion planning*).
Find two vector functions of time $u_\star(t)$ and $x_\star(t)$, such that $x_\star(t)$ is a solution of (1) with $u = u_\star(t)$,

$$x_\star(t) = x_\star(t+T) \quad \forall\, t \geq 0 \tag{2}$$

for some $T > 0$, and satisfies certain pre-defined specifications.

Sometimes, the specifications include a particular period T and certain desired ranges for the components of $x_\star(t)$.

If the desired motion is successfully planned, as assumed, e.g. in Byrnes, Isidori, and Willems (1991), Spong (1997), Bloch, Leonard, and Marsden (2000), Ortega, Spong, Gomez, and Blankenstein (2002), the problem of feedback stabilization can be formulated.

However, in some situations it is more natural to force the systems trajectories not to track a particular periodic motion (2) but to stay as close as possible or to approach an orbit defined by this motion:

$$\mathcal{M} = \{x \in \mathbb{R}^n : x = x_\star(t), t \in [0, T]\} \tag{3}$$

So that the following control task can be defined.

Problem 2 (*Exponential orbital stabilization*).
Find a function $k(x)$ such that the solutions of (1) with $u = k(x)$ initiated in a neighborhood of the desired orbit $\mathcal{M}$, defined by (3), exponentially approach the compact set $\mathcal{M}$, i.e. there exist $c_1 > 0$ and $c_2 > 0$ such that

$$d(x(t), \mathcal{M}) \leq c_1 d(x(t_0), \mathcal{M}) \exp\{-c_2(t - t_0)\}, \quad \forall\, t \geq t_0, \tag{4}$$

where $d(x, \mathcal{M})$ is the distance function from the vector x to the set $\mathcal{M}$.

Let us briefly discuss tools that are instrumental in analysis of orbital exponential (in-)stability of a periodic solution for an autonomous system:

$$\dot{x} = \mathcal{F}(x), \tag{5}$$

which in the present context can be defined by

$$\mathcal{F}(x) = f(x, k(x)).$$

[1] To the best of our knowledge, the solution for the LQR problem for linear control systems with periodic coefficients was first reported in Yakubovich (1986), see also Bittanti, Colaneri, and de Nicolao (1991).

2.2. Poincaré first-return map

One of the classical tools for verifying existence and stability of nontrivial periodic orbits is Poincaré first-return map analysis, see Poincaré (1916–1954).

Let a surface S be transversal to the flow of a periodic orbit $\mathcal{M}$ of (5) and consider a sufficiently small region $S_0 \subset S$ which is open relative to S and contains the point of intersection of S and $\mathcal{M}$. The Poincaré map $P : S_0 \to S$ is defined by the first hit rule, i.e. it maps the initial points of the solutions of (5) belonging to S_0 into the points where these solutions hit S again for the first time, see Fig. 1.

Under certain natural regularity conditions, this map is well-defined because of continuous dependence of solutions on initial conditions (see, e.g. Khalil, 2002). The trajectory $x^\star(t)$ corresponds to the fixed point of $P(\cdot)$. If the Poincaré map is contracting, then the orbit is asymptotically stable. Exponential orbital stability can be verified using linearization of the Poincaré map $dP : TS \to TS$, which acts on the tangent to S space at the point of intersection of S and $\mathcal{M}$.

It is well-known, see, e.g. Andronov and Vitt (1933), Urabe (1967), Hale (1980), Yoshizawa (1966), Rouche and Mawhin (1980), Leonov (2006), that (4) is satisfied if and only if all the eigenvalues of the linear operator dP are strictly inside the unit circle. Moreover, the rate of contraction toward the orbit over the period, which is estimated by $\exp\{-c_2 T\}$, is defined by the absolute value of the eigenvalue of dP closest to the unit circle.

Control design exploiting Poincaré map analysis is hard and only a few successful applications are known, see, e.g. the discussion in Fradkov and Evans (2005) and the references therein for a general case. Such technique was used in Grizzle, Abba, and Plestan (1999, 2001), Westervelt, Grizzle, and Koditschek (2003), Chevallereau, Formal'sky, and Djoudi (2004), Chevallereau, Westervelt, and Grizzle (2005) in the context of bipedal walking robots and in Nakaura, Kawaida, Matsumoto, and Sampei (2004) for controlling enduring rotations of a devil stick, where with the help of particular structure of the feedback control law it was possible to reduce stability analysis to that of a one-dimensional Poincaré map.

2.3. Transverse dynamics

In order to introduce a Poincaré map, it is sufficient to have a surface S that is transverse to the orbit at a single point. However, sometimes it is useful to introduce moving Poincaré sections, see Leonov (2006): a family $\{S(t)\}_{t\in[0,T]}$ of surfaces each of which is transversal to the orbit (3) and intersects it at $x_\star(t)$, see Fig. 2.

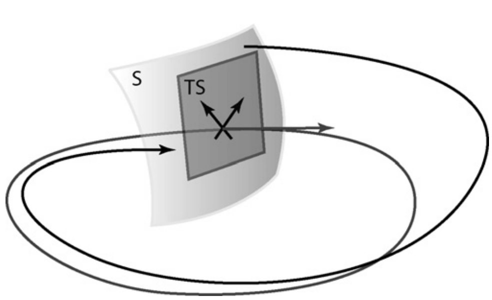

Fig. 1. Poincaré map $P : S \to S$ for the periodic trajectory $x_\star(t)$ (grey) is defined by the first hit rule.

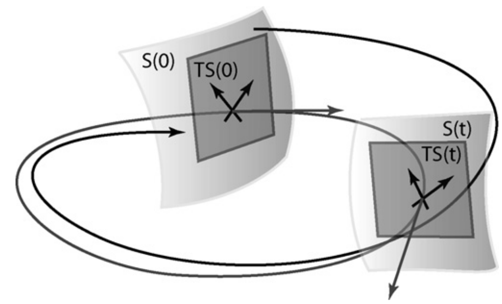

Fig. 2. Moving Poincaré section for the periodic trajectory $x_\star(t)$ (grey) of (5), $\mathcal{F}(x_\star(t)) \notin TS(t)$. In a vicinity of $\mathcal{M}$, defined by (3), we have $\mathcal{M} \bigcap S(t) = \{x_\star(t)\}$.

Suppose $S(T) = S(0)$ and the union of all the surfaces in the family covers a neighborhood of the orbit. Then, one can define a new set of coordinates $x_\perp(\cdot)$ and $\varphi(\cdot)$ in a vicinity of each point of the orbit, where $x_\perp(t)$ are also coordinates on $S(t)$ with the origin in $x_\star(t)$ and $\varphi(t)$ is a scalar coordinate that travels along the orbit, see Urabe (1967), Hale (1980), Hauser and Chung (1994).

The original and new coordinates are related by the transform:

$$x(t) = U(x_\star(t)) \begin{bmatrix} \varphi(t) \\ x_\perp(t) \end{bmatrix}$$

Assuming that the matrix function $U(\cdot)$ is continuously differentiable together with its inverse, it is easy to rewrite (5) in terms of the new coordinates.

Linearizing the dynamics for $x_\perp(t)$ around the desired trajectory:

$$x_\star(t) = U(x_\star(t)) \begin{bmatrix} \varphi_\star(t) \\ 0 \end{bmatrix}$$

one can define a linear comparison (first approximation) system:

$$\dot{z}(t) = A(t)\, z(t), \quad A(t) = A(t+T) \tag{6}$$

where $z(t)$ is the vector from the tangent space $TS(t)$. For another equivalent way of describing these dynamics see Leonov (2006).

Exponential stability of the zero solution for the linear system (6) is equivalent to exponential orbital stability of the solution $x_\star(t)$ of (5). This system is called transverse linearization, short for linearization of the transverse dynamics. The concept has been used for feedback control of various classes of systems, see, e.g. Nam and Arapostathis (1992), Samson (1995), Gillespie, Colgate, and Peshkin (2001), Altafini (2002), Coelho and Nunes (2003), Banaszuk and Hauser (1995), Chung and Hauser (1997).

It is worth noting that

$$\dim\{z(t)\} = \dim\{x(t)\} - 1.$$

Correspondingly, the comparison system (6) is different from the classical linearization of (5) around the trajectory $x_\star(t)$:

$$\dot{Z}(t) = \mathcal{A}(t)\,Z(t), \quad \mathcal{A}(t) = \left(\frac{\partial \mathcal{F}(x)}{\partial x}\right)\Bigg|_{x=x_\star(t)}$$

with $\dim\{Z(t)\} = \dim\{x(t)\}$. Moreover, the later system cannot be exponentially stable unless the target periodic solution $x_\star(t)$ is trivial, i.e. an equilibrium of (5). To prove this it is enough to notice that $z = \dot{x}_\star(t)$ is a non-vanishing solution (Andronov & Vitt, 1933; Yoshizawa, 1966).

In summary: if one can find appropriate transverse coordinates at all points of the orbit, then proving exponential orbital stability (or instability) is reduced to analysis of a particular time-periodic linear system of dimension of transverse dynamics.

2.4. Controlled transverse linearization

The transverse-linearization-based approach, as described above, can be used for the analysis of (in-)stability of periodic solutions. Here, however, we are interested in exploiting this technique for control design.

The orbital exponential stabilization task (see Problem 2 above) in the case when the target periodic solution is trivial, $x_\star(t) = \text{const}$, $u_\star(t) = \text{const}$, can be approached via analysis of the standard controlled linearization:

$$\dot{Z}(t) = \mathcal{A}\,Z(t) + \mathcal{B}\,W(t),$$

with $\dim\{Z(t)\} = \dim\{x(t)\}$ and

$$\mathcal{A} = \frac{\partial f(x,u)}{\partial x}\Bigg|_{\substack{x=x_\star\\ u=u_\star}}, \quad \mathcal{B} = \frac{\partial f(x,u)}{\partial u}\Bigg|_{\substack{x=x_\star\\ u=u_\star}}$$

In the case when $x_\star(t)$ is a nontrivial periodic solution, such an approach might be of limited use. However, by defining a family of Poincaré sections $\{S(t)\}_{t\in[0,T]}$ as above, and by linearizing the transverse dynamics corresponding to $x_\perp(t)$ for appropriately rewritten dynamics of the controlled nonlinear system (1), one obtains a controlled transverse linearization in the following form (Hauser & Chung, 1994; Nielsen & Maggiore, 2006):

$$\dot{z}(t) = A(t)\,z(t) + B(t)w(t), \tag{7}$$

where $A(t) = A(t+T)$, $B(t) = B(t+T)$, and $z(t)$ is the vector of the transversal coordinates, which belongs to the tangent space $TS(t)$ with $\dim\{z(t)\} = \dim\{x(t)\} - 1$.

The main step is then to design a feedback controller $w = K(t)z$ for the periodic linear control system (7) and to transform it into an orbitally exponentially stabilizing time-invariant state feedback controller for the nonlinear system. This step has a clear intuitive meaning and involves introduction of a particular projection operator $\mathcal{T}$, defined in a neighborhood of the orbit $\mathcal{M}$, mapping points onto an appropriate point on the orbit $\mathcal{M}$.

Such an operator $\mathcal{T}$ possibly introduces another moving Poincaré section to the target orbit defining a particular time stamp for points of each transversal section. This allows to modify the feedback gain $K(t) = K(t+T) = K(x_\star(t))$ from being the function of $x_\star(t)$ into the function of a state vector of nonlinear system, i.e. $K = K(\mathcal{T}(x))$, well-defined in a vicinity of the orbit. Detailed discussions of this procedure can be found in Banaszuk and Hauser (1995), Shiriaev, Freidovich, and Manchester (submitted for publication).

3. Theory for a class of under-actuated mechanical systems

In this section, we elaborate further on the tools presented above for a class of underactuated nonlinear mechanical systems. Controlling mechanical systems with a limited number of actuators is a challenging task (Bloch et al., 2000; Byrnes et al., 1991; Ortega et al., 2002; Spong, 1997). When the target behavior is more complicated than a simple equilibrium, e.g. a periodic trajectory, the challenges become greater still (Fradkov & Pogromsky, 1998, Chapter 6), and even establishing existence of periodic motions in a nonlinear system is often difficult, see, e.g. Rouche and Mawhin (1980), Yoshizawa (1966).

In particular, we consider orbital stabilization for systems that have one fewer independent control inputs than mechanical degrees of freedom, i.e. systems of under-actuation degree one. Even with this restricted focus, the problems are sufficiently challenging, and the class of motivating examples is sufficiently rich, to warrant a detailed study. This class includes popular "control challenge" systems such as the inverted pendulum on a cart, the Furuta pendulum, the Pendubot, and the Acrobot, and can also represent the dynamics of practical systems such as humanoid robots, surface vessels, helicopters and many others. Examples of controller designs for underactuated mechanical systems can be found in Aracil, Gordillo, and Acosta (2002), Anami, Nakaura, and Sampei (2007), Miossec and Aoustin (2005), Bittanti and Moiraghi (1994), Bittanti, Lorito, and Strada (1996), Chevallereau et al. (2003, 2004, 2005), Duindam and Stramigioli (2005), Das and Mukherjee (2001), Grizzle et al. (2001), Grizzle, Moog, and Chevallereau (2005), Fossen and Strand (2001), Fossen (2002), Kuo (2002), Manchester, Shiriaev, and Savkin (2007), Mukherjee and Chen (1993), Nakaura et al. (2004), Nair and Leonard (2008), Skjetne, Fossen, and Kokotović (2004), Shimizu, Nakaura, and Sampei (2006), Mazenc and Bowong (2003), Canudas-de-Wit, Espiau, and Urrea (2002), Orlov, Aguilar, Acho, and Ortiz (2008a, 2008b), Santiesteban, Floquet, Orlov, Riachy, and Richard (2008), van Oort and Stramigioli (2007), Wisse, Feliksdal, van Frankenhuyzen, and Moyer (2007) and others.

Consider an n-degree-of-freedom controlled Euler-Lagrange system (Ortega, Loria, Nicklasson, & Sira-Ramirez, 1998):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = B(q)u. \tag{8}$$

Here $q \in \mathbb{R}^n$ is a vector of generalized coordinates, $u \in \mathbb{R}^{n-1}$ is a vector of independent control inputs, the function:

$$\mathcal{L}(q,\dot q) = \frac{1}{2}\dot q^T M(q)\dot q - V(q) \qquad (9)$$

is the Lagrangian of the system (8), $M(q)$ is a positive definite matrix of inertia, $V(q)$ is the potential energy of the system, and $B(q)$ is a full-rank $n \times (n-1)$ matrix function, which defines applications of generalized controlled forces and is often constant.

The system (8) and (9) can be rewritten (Ortega et al., 1998; Spong, Hutchinson, & Vidyasagar, 2006) as

$$M(q)\ddot q + C(q,\dot q)\dot q + G(q) = B(q)u, \qquad (10)$$

where $G(q) = [(\partial V(q)/\partial q_1), \ldots, (\partial V(q)/\partial q_n)]^{\mathrm{T}}$ and $C(q,\dot q)$ is a matrix of Coriolis and generalized centrifugal forces.

For the class of systems described by (10) it is possible to suggest approaches for solving the problems introduced above. We start with the periodic motion planning task (Problem 1).

3.1. Planning periodic orbits

Before suggesting a procedure for planning an orbit, it is useful to make some general observations about properties of mechanical systems for which such a problem is already solved. So, suppose someone managed to find a nontrivial periodic trajectory $q_\star(t) = q_\star(t+T)$ with components:

$$q_1 = q_{1\star}(t), q_2 = q_{2\star}(t), \ldots, q_n = q_{n\star}(t), \quad t \in [0,T] \qquad (11)$$

of the nonlinear mechanical system (10) shaped by an appropriate choice of the control variable $u_\star(t) = u_\star(t+T)$.

It is not hard to see that the found periodic solution (11), being parameterized by time, can be reparametrized by another scalar variable[2] $\theta_\star$:

$$q_1 = \underbrace{q_{1\star}(t(\theta_\star))}_{\phi_1(\theta_\star)}, q_2 = \underbrace{q_{2\star}(t(\theta_\star))}_{\phi_2(\theta_\star)}, \ldots, q_n = \underbrace{q_{n\star}(t(\theta_\star))}_{\phi_n(\theta_\star)},$$
$$\theta_\star \in [\Theta_b, \Theta_e] \qquad (12)$$

The variable $\theta_\star(\cdot)$ can be viewed as a candidate for one of the new generalized coordinates for the mechanical system (10), defined in the beginning only on the cycle itself and later on to be defined in a vicinity of the cycle.

In some cases the $\theta_\star$-variable can be chosen as one of the generalized coordinates of the mechanical system, i.e. $\theta_\star(\cdot) = q_{i\star}(\cdot)$. In all situations it can be the arc-length along the target periodic orbit in the state space $[q,\dot q]$ of the mechanical system (10).

It is worth mentioning that the relations (12) define uniquely n-functions $\phi_1(\cdot), \ldots, \phi_n(\cdot)$, which, in turn, are well-defined provided someone knows the cycle and chooses the way that $\theta_\star$ parametrizes points on it.

[2] Along the desired trajectory, time is a function of this scalar variable $t = t(\theta_\star)$, $\theta_\star \in [\Theta_b, \Theta_e]$.

The relations (12) motivate the introduction of the following concept, see Grizzle et al. (2001), Shiriaev, Perram, and Canudas-de-Wit (2005): Given C^2-smooth scalar function $\phi_1(\cdot), \ldots, \phi_n(\cdot)$, the relations

$$q_1 = \phi_1(\theta), q_2 = \phi_2(\theta), \ldots, q_n = \phi_n(\theta) \qquad (13)$$

are called *virtual holonomic constraints*, if the control $u(\cdot)$ ensures invariance of these relations for one or several solutions of the mechanical system (10). Here θ is a scalar variable.

The behavior of θ is not explicitly determined by the choice of functions $\phi_i(\cdot)$ in (13). It might be chosen arbitrarily if the mechanical system (10) is fully actuated, and might not if some of degrees of freedom are not directly actuated. As shown in Perram, Shiriaev, Canudas-de-Wit, and Grognard (2003), Shiriaev, Perram, et al. (2005), Shiriaev, Robertsson, Perram, and Sandberg (2006), for the case when only one degree of freedom of (10) is not actuated, i.e. rank $B(q) = (n-1)$, the variable θ is not free or directly controllable, but is a solution of the following second order differential equation:

$$\alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta) = 0, \qquad (14)$$

where

$$\alpha(\theta) = B^\perp(q)M(\Phi(\theta))\Phi'(\theta),$$
$$\beta(\theta) = B^\perp(q)[C(\Phi(\theta),\Phi'(\theta))\Phi'(\theta) + M(\Phi(\theta))\Phi''(\theta)],$$
$$\gamma(\theta) = B^\perp(q)G(\Phi(\theta)),$$
$$\Phi(\theta) = [\phi_1(\theta), \ldots, \phi_n(\theta)]^{\mathrm{T}}, \quad \Phi'(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(\theta),$$
$$\Phi''(\theta) = \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Phi(\theta),$$

and $B^\perp(q)$ is a full rank matrix such that $B^\perp(q)B(q) = 0$.

The system (14) has a number of important properties that allow detailed qualitative and quantitative analysis. As shown in Perram et al. (2003), Shiriaev, Robertsson, Perram, et al. (2006), if (14) admits the solution $\theta = \theta_\star(t)$, then the following function:

$$I(x,y,a,b) = y^2 - \psi(a,x)b^2 + 2\int_a^x \psi(s,x)\frac{\gamma(s)}{\alpha(s)}\,\mathrm{d}s \qquad (15)$$

with

$$\psi(z_1,z_2) = \exp\left\{-2\int_{z_1}^{z_2}\frac{\beta(\tau)}{\alpha(\tau)}\mathrm{d}\tau\right\}, \qquad (16)$$

keeps a value of zero for $x = \theta_\star(t)$, $y = \dot\theta_\star(t)$, $a = \theta_\star(0)$, $b = \dot\theta_\star(0)$, i.e. it is zero along every solution as long as it exists:

$$I(\theta_\star(t), \dot\theta_\star(t), \theta_\star(0), \dot\theta_\star(0)) \equiv 0.$$

The existence of a conserved quantity (15) and (16) for the reduced dynamics (14) and the fact that every periodic motion can be defined by an appropriate choice of virtual holonomic constraints (13), describing synchronizations among the generalized coordinates, inspires the following generic procedure for planning a periodic motion of the controlled mechanical system (10):

Step 1 Define a set of C^2-functions $\phi_1(P,\theta),\ldots,\phi_n(P,\theta)$ parametrized by a vector of parameters $P=(p_1,\ldots,p_k)$.

Step 2 Use the parametric set of functions from previous step for defining the set of virtual holonomic constraints (13), compute the corresponding family of systems:

$$\alpha(P,\theta)\ddot{\theta}+\beta(P,\theta)\dot{\theta}^2+\gamma(P,\theta)=0. \tag{17}$$

Step 3 Find an appropriate value for $P=P_\star$ such that there exists a periodic solution $\theta_\star(t)=\theta_\star(t+T)$ for (17) that generates the periodic motion:

$$q_1(t)=\phi_1(P_\star,\theta_\star(t)),\ q_2(t)=\phi_2(P_\star,\theta_\star(t)),\ldots,q_n(t)=\phi_n(P_\star,\theta_\star(t)) \tag{18}$$

for the controlled mechanical system (10) with the required properties.

The proposed procedure largely depends on a choice of the set of functions $\phi_1(P,\theta)$, ..., $\phi_n(P,\theta)$ in Step 1 and a criterion in Step 3 used for detecting periodic solutions of (17). Integrability of (17) for any choice of virtual holonomic constraints helps in detecting periodic solutions. We do not bring the complete analysis of the phase portrait of the system (17), but formulate the criterion for detecting a center at an equilibrium of the nonlinear system (17).

Theorem 1 (*Existence of a center, Shiriaev, Robertsson, Perram, et al. (2006)*).
Let θ_0 be an equilibrium of the system (17), i.e. $\gamma(P,\theta_0)=0$. Suppose that:

(1) *There is a vicinity $\mathcal{V}$ of θ_0 such that the scalar functions $\alpha(P,\cdot)$, $\beta(P,\cdot)$ and $\gamma(P,\cdot)$ are continuous on $\mathcal{V}$*, i.e. $\alpha(P,\theta)$, $\beta(P,\theta)$, $\gamma(P,\theta)\in C^0(\mathcal{V})$;
(2) *The function $(\gamma(P,\theta)/\alpha(P,\theta))$ is continuously differentiable at $\theta=\theta_0$.*
(3) *For any $\theta_i\in\mathcal{V}$, there exists $\delta>0$ such that for any $\dot{\theta}_i$ with $|\dot{\theta}_i|<\delta$, the solution of the nonlinear system (17) initiated at $(\theta(0),\dot{\theta}(0))=(\theta_i,\dot{\theta}_i)$ exists for all $t\geq 0$ and is unique.*

If the linear system:

$$\frac{d^2}{dt^2}z+\left[\frac{d}{d\theta}\frac{\gamma(P,\theta)}{\alpha(P,\theta)}\right]\bigg|_{\theta=\theta_0}\cdot z=0.$$

has a center at $z=0$, that is, when the constant $\omega=[(d/d\theta)(\gamma(P,\theta)/\alpha(P,\theta))]|_{\theta=\theta_0}$ is positive, then the nonlinear system (17) has a center at the equilibrium θ_0.

The statement above is a generalization of a theorem by Lyapunov (1892, pp. 662–675) related to the famous center-focus problem posted by Poincaré.

3.2. *Generic choice for transverse coordinates in a vicinity of a cycle*

Here we discuss one of possible choices for transverse coordinates (a moving Poincaré section) for the orbit:

$$\mathcal{M}=\{[q,\dot{q}]:q=q_\star(t),\dot{q}=\dot{q}_\star(t),t\in[0,T]\}. \tag{19}$$

defined by a given periodic motion $q_\star(t)=q_\star(t+T)$ of the controlled mechanical system (8).

As explained above, knowing $q_\star(t)$, one can construct n-scalar functions $\phi_1(\cdot),\ldots,\phi_n(\cdot)$ that constitute a parametrization of the same periodic solution $q_\star(t)$ by a scalar variable other than time, see (11) and (12).

Given functions $\phi_1(\cdot),\ldots,\phi_n(\cdot)$, let us introduce the error variables:

$$y_1=q_1-\phi_1(\theta),\ldots,y_n=q_n-\phi_n(\theta). \tag{20}$$

In an open subset of $\mathbb{R}^n$ one can consider the $n+1$ scalar quantities $y_1,y_2,\ldots,y_n$, and θ as excessive coordinates for the controlled n-degree-of-freedom Euler-Lagrange system (8).

Therefore, one coordinate can be expressed as a function of the other coordinates. Without loss of generality, let us assume that this is the case for y_n, so the new independent coordinates are

$$y=(y_1,\ldots,y_{n-1})^{\mathrm{T}}\in\mathbb{R}^{n-1}\quad\text{and}\quad\theta\in\mathbb{R} \tag{21}$$

and the last equality in (20) can be rewritten as

$$q_n=\phi_n(\theta)+h(y,\theta),$$

where $h(\cdot)$ is a scalar smooth function, so that

$$q=\Phi(\theta)+\left[y^{\mathrm{T}},h(y,\theta)\right]^{\mathrm{T}},\qquad \Phi(\theta):=[\phi_1(\theta),\phi_2(\theta)\ldots,\phi_n(\theta)]^{\mathrm{T}}. \tag{22}$$

In the new coordinates, the motion $q_\star(t)=q_\star(t+T)$ is

$$y=y_\star(t)=0_{(n-1)\times 1},\quad \theta=\theta_\star(t)=\theta_\star(t+T) \tag{23}$$

and the orbit (19) is

$$\mathcal{M}=\{[y,\theta,\dot{y},\dot{\theta}]:y=y_\star(t)=0,\dot{y}=\dot{y}_\star(t)=0,\theta=\theta_\star(t),\dot{\theta}=\dot{\theta}_\star(t)\}. \tag{24}$$

Proposition 1 (*Feedback transformation*).
Denote

$$L(q)=\begin{bmatrix}I_{n-1},0_{(n-1)\times 1}\\ \mathrm{grad}\,h(q)\end{bmatrix}+[0_{n\times(n-1)},\Phi'(\theta)],$$

where

$$\mathrm{grad}\,h(q)=\left[\frac{\partial h(\cdot)}{\partial y_1},\ldots,\frac{\partial h(\cdot)}{\partial y_{n-1}},\frac{\partial h(\cdot)}{\partial\theta}\right]$$

with y and θ substituted in terms of q using the inverse transform to (22). Suppose the matrix function $L(q)$ and the matrix function:

$$N(q)=[I_{n-1},0_{(n-1)\times 1}]L(q)^{-1}\left[M(q)^{-1}B(q)\right]$$

are both non-singular in a vicinity of the orbit (19). *Then, there exists the feedback transformation in the form*:

$$u = N(q)^{-1}[v - R(q, \dot{q})], \tag{25}$$

well-defined in this vicinity, so that the dynamics of the system (8) *can be rewritten in the new coordinates* (21) *as follows*

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = g_y(\cdot)\, y + g_{\dot{y}}(\cdot)\, \dot{y} + g_v(\cdot)\, v, \tag{26}$$

$$\ddot{y} = v, \tag{27}$$

where the left hand side of (26) *matches the structure of the reduced system* (14) *and*

$$g_y = g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}), \quad g_{\dot{y}} = g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}),$$
$$g_v = g_v(\theta, \dot{\theta}, y, \dot{y})$$

are smooth functions of appropriate dimensions.

This is a relaxed version of the result that can be found in Shiriaev et al. (submitted for publication).

Eq. (26) is not resolved with respect to $\ddot{\theta}$, where functions $g_y(\cdot)$ and $g_{\dot{y}}(\cdot)$ are left dependent on $\ddot{\theta}$. The reason for doing this is that the left hand side of Eq. (26) is integrable. This motivates the following choice of the transverse coordinates $x_\perp$ for the orbit (24):

$$x_\perp = [I(\theta, \dot{\theta}, \theta_\star(0), \dot{\theta}_\star(0)), y_1, \ldots, y_{n-1}, \dot{y}_1, \ldots, \dot{y}_{n-1}]^{\mathrm{T}}. \tag{28}$$

where the scalar variable $I(\theta, \dot{\theta}, \theta_\star(0), \dot{\theta}_\star(0))$ is defined by (15) and (16).

3.3. Analytical construction of a transverse linearization in the case of underactuation degree one

To linearize the dynamics of $x_\perp$, introduced by (28), in a vicinity of the periodic motion (23) and to show that this is indeed a transverse linearization for the system (26), (27), the following properties of the integral function $I(\cdot)$, defined by (15) and (16), are essential.

Property 1 (*Independence on initial point, Shiriaev et al. (submitted for publication)*).
For any x_1 and x_2 the function (15) *satisfies the identity* (*see* Fig. 3)

$$I(x_1, x_2, \theta_\star(0), \dot{\theta}_\star(0)) \equiv I(x_1, x_2, \theta_\star(\rho_0), \dot{\theta}_\star(\rho_0)) \tag{29}$$

for all $\rho_0 \in [0, T]$.

Property 2 ($I \sim$ *distance to the orbit, Shiriaev et al. (submitted for publication)*).

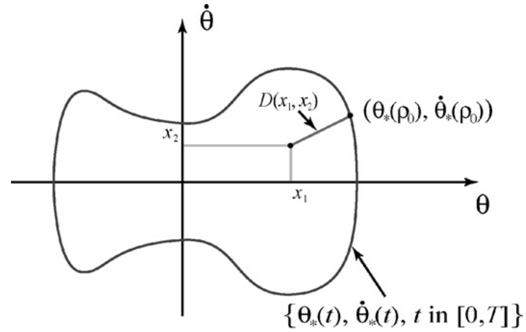

Fig. 3. An illustration for Properties 1 and 2 of (15).

In a vicinity of the orbit, (*see* Fig. 3)

$$\begin{aligned} I^2&(x_1, x_2, \theta_\star(0), \dot{\theta}_\star(0)) \\ &= 4[\dot{\theta}_\star(\rho_0)^2 + \ddot{\theta}_\star^2(\rho_0)] \times D^2(x_1, x_2) + O(|x_1 - \theta_\star(\rho_0)|^3) \\ &\quad + O(|x_2 - \dot{\theta}_\star(\rho_0)|^3), \end{aligned} \tag{30}$$

where

$$D(x_1, x_2) = \min_{0 \le \rho < T} \left\{ \sqrt{|x_1 - \theta_\star(\rho)|^2 + |x_2 - \dot{\theta}_\star(\rho)|^2} \right\}$$

is the Euclidean distance to the orbit from the point with coordinates (x_1, x_2),

$$\rho_0 = \arg \min_{0 \le \rho < T} \{|x_1 - \theta_\star(\rho)|^2 + |x_2 - \dot{\theta}_\star(\rho)|^2\}.$$

defines the point on the orbit, which is the closest to (x_1, x_2).

Now, it is clear that the new coordinates (28) can be locally used for computing the distance to the orbit defined by the desired trajectory $\theta_\star(t)$. To linearize the dynamics of $x_\perp$, one needs to linearize the dynamics of the integral $I(\cdot)$. The following property shows how to compute its time derivative.

Property 3 ($(d/dt)I(\cdot)$ *away from the cycle, Shiriaev, Perram, et al. (2005)*).
With θ_0 and $\dot{\theta}_0$ being constants, the time derivative of the function $I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0)$ defined by (15), *calculated along a solution $[\theta(t), \dot{\theta}(t)]$ of the system*

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = W, \tag{31}$$

can be computed as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}& I(\theta(t), \dot{\theta}(t), \theta_0, \dot{\theta}_0) \\ &= \frac{2\,\dot{\theta}(t)}{\alpha(\theta(t))}\{W - \beta(\theta(t)) \cdot I(\theta(t), \dot{\theta}(t), \theta_0, \dot{\theta}_0)\}. \end{aligned} \tag{32}$$

Now, exploiting the structure of (26) and with the help of Property 3, one can readily compute the linearization of the dynamics in the transverse coordinates $x_\perp$ (28) in a vicinity of the periodic motion (23) in the form (7):

$$\dot{z} = A(t)z + B(t)w, \tag{33}$$

where

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} & I_{(n-1)\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \end{bmatrix}, \quad B(t) = \begin{bmatrix} b_1(t) \\ 0_{(n-1)\times(n-1)} \\ I_{(n-1)\times(n-1)} \end{bmatrix} \tag{34}$$

with the coefficients

$$b_1(t) = \dot{\theta}_\star(t)\frac{2g_v(\theta_\star(t), \dot{\theta}_\star(t), 0, 0)}{\alpha(\theta_\star(t))},$$

$$a_{12}(t) = \dot{\theta}_\star(t)\frac{2g_y(\theta_\star(t), \dot{\theta}_\star(t), \ddot{\theta}_\star(t), 0, 0)}{\alpha(\theta_\star(t))},$$

$$a_{11}(t) = -\dot{\theta}_\star(t)\frac{2\beta(\theta_\star(t))}{\alpha(\theta_\star(t))},$$

$$a_{13}(t) = \dot{\theta}_\star(t)\frac{2g_{\dot{y}}(\theta_\star(t), \dot{\theta}_\star(t), \ddot{\theta}_\star(t), 0, 0)}{\alpha(\theta_\star(t))}.$$

3.4. Exponential orbital stabilization in the case of underactuation degree one

The following statement provides a method for constructing an orbitally exponentially stabilizing feedback controller based on a controller developed for the comparison linear periodic system.

Theorem 2. *The following statements are equivalent*:

Statement 1: *There exists a periodic matrix gain $K(t) = K(t+T)$ such that the feedback controller*

$$w = K(t)z \tag{35}$$

exponentially stabilizes the origin $z = 0$ of the linear control system (33) and (34).

Statement 2: *There exists a state feedback controller of the form*

$$v = f(\theta, \dot{\theta}, y, \dot{y}) \tag{36}$$

that makes the target periodic motion (23) orbitally exponentially stable in the transformed system (26) and (27).

Furthermore, the feedback controllers can be constructed as follows:

- *Given (35), a possible choice for (36)* is

$$u = K(\mathcal{T}(\theta, \dot{\theta}))\,x_\perp, \tag{37}$$

where $x_\perp$ is given by (28) with $I(\cdot)$ defined by (15) and (16), and *$\mathcal{T}(\cdot)$is a smooth projector operator to the orbit $\{[\theta_\star(t), \dot{\theta}_\star(t)]\}$ that recovers the time stamp of the point on the orbit*;

- *Given (36), a possible choice for (35) is*

$$K(t) = \left.\begin{bmatrix} \dfrac{\partial f(\theta, \dot{\theta}, y, \dot{y})}{\partial y} \\ \dfrac{\partial f(\theta, \dot{\theta}, y, \dot{y})}{\partial \dot{y}} \\ \dfrac{(\partial f(\theta, \dot{\theta}, y, \dot{y})/\partial\dot{\theta})\dot{\theta}_\star(t) - (\partial f(\theta, \dot{\theta}, y, \dot{y})/\partial\theta)\ddot{\theta}_\star(t)}{2(\dot{\theta}_\star(t))^2 + 2\,(\ddot{\theta}_\star(t))^2} \end{bmatrix}\right|_{\substack{y=\dot{y}=0 \\ \theta=\theta_\star(t) \\ \dot{\theta}=\dot{\theta}_\star(t)}} \tag{38}$$

.

This result is a particular case of a more general theorem, proved in Shiriaev et al. (submitted for publication), where the assumption of Proposition 1, which are needed to conclude exponential orbital stability of the desired motion of the original system (10) (not only the transformed one) are relaxed. Summing up, to solve the problem of orbital exponential stabilizations it is necessary and sufficient to design a controller for the linear time-periodic system (33) and (34).

4. Application examples

To validate the methodology described in the previous sections for motion planning, constructing a transverse linearization, and checking exponential orbital stability, besides computer simulations, we have organized experiments with the systems which we have in our laboratory: the Furuta pendulum, the inertia wheel pendulum and the Pendubot.

As emphasized in the introduction, numerical methods for analysis and stabilization of linear control systems with periodic coefficients are the main challenges in this approach especially for organizing experimental studies. The proposed theoretical results are based on linearized models and allow only local analysis. To succeed with an experiment, one can use various reasonings for choice of poles of the closed-loop system and weights for the LQR design, but this will require versatile software to find coefficients of stabilizing controllers for these choices numerically.

To this end it is worth mentioning the result of Varga (2005a), where a multiple-shooting-based method is proposed to solve continuous-time-periodic Riccati equations (and thus improving the original method of Yakubovich, 1986). The above algorithm of Varga (2005a) has been examined and tested in Johansson, Kågström, Shiriaev, and Varga (2007). The numerical comparisons in Johansson et al. (2007) were performed using both general purpose and symplectic integration methods for solving the associated Hamiltonian differential systems. In the multi-shot method a stable subspace is determined using recent algorithms by Granat and Kågström (2006), Granat, Kågström, and Kressner (2006, 2007a) for computing a reordered periodic real Schur form. Another algorithm for solving continuous-time-periodic Riccati equations was recently reported in Gusev, Shiriaev, and Freidovich (2007).

Unfortunately, at present software packages for periodic linear systems are not available, and for all reported simulations and experiments quite conservative settings were used for which we were able to find stabilizing controllers by programming our own algorithms based on the original method of Yakubovich and the method of Gusev et al. (2007). It is hoped that this theoretical investigation will be of interest for the research community focused on numerical methods for linear periodic systems, bringing a new and large class of examples, which may stimulate further developments. To this end it is worth mentioning continuing efforts for development of such software, see Varga (2005b, in press), and the recent development of Granat, Kågström, and Kressner (2007b) towards Matlab tools for solving periodic eigenvalue and subspace problems.

Below, we list the mechanical systems and related control problems, we have considered, and briefly comment the results.

- *The Furuta pendulum* is an unactuated pendulum mounted on the end of an actuated horizontal-plane rotary arm. The problems of shaping oscillations around its upright and downward equilibriums and various swing-up strategies were considered, solved and verified through simulations and experiments. The results were partly reported in Shiriaev, Freidovich, Robertsson, Johansson, and Sandberg (2007), Freidovich, Shiriaev, and Manchester (2007), LaHera, Freidovich, Shiriaev, and Mettin (in press).
- *The inertial wheel pendulum* is a planar pendulum with an actuated inertia wheel attached to its end. For this system algorithms for organization stable oscillations around upright unstable equilibrium and swing-up are proposed. Furthermore, for this system the difficult problem of choosing a virtual holonomic constraint to achieve oscillations of a particular period, shape and amplitude is solved analytically. The last means that each of imposed specifications on a target cycle is translated into a particular algebraic equation on parameters of virtual holonomic constraint function. Details of these investigations are reported in Freidovich, Robertsson, Shiriaev, and Johansson (2007); the experimental studies are reported in Freidovich, LaHera, et al. (submitted for publication).
- *The Pendubot* is a double link planar pendulum with an actuator attached to the first link. For this set-up the problems of swinging-up to vicinities of its unstable equilibriums have no satisfactory solution, and the reported methods are typically based on dedicated open-loop strategy elaborated for particular initial conditions, see Graichen and Zeits (2005). These problems and the problems of shaping stable oscillations around unstable equilibriums of the Pendubot were approached and solved by the proposed method. Steps in motions planning and controller design as well as results of simulations and experimental studies are partly reported in Freidovich, Robertsson, Shiriaev, and Johansson (2008).
- *The pendulum on a cart* is a freely moving planar pendulum attached to a cart moving on the horizontal plane. This was the first system approached by the presented methodology for planning a periodic motion of the pendulum around its upright equilibrium. Steps were reported in Shiriaev, Perram, et al. (2005). Furthermore, we were able to extend this method and suggested the modification of the controller to include the specification on average forward (backward) speed of the cart, using an invariance of the cart-pendulum dynamics with respect to shifts of the cart position, see Shiriaev, Robertsson, Perram, et al. (2006). It is worth mentioning that the problem of shaping stable oscillations of the pendulum around stable downward position through stabilization of natural modes of the pendulum (which is a particular case of the virtual holonomic constraint imposed only on a position of the cart) were considered and solved in Chung and Hauser (1995).
- *The devil stick* is a juggling device, and can be modeled as a mechanical system consisting of two parts: a hand stick and a center stick which is floating in the air and rolls along the hand stick without sliding. The goal is to push the center stick by the completely controlled hand stick to propel. The mechanical set-up resembles the behavior of the system built at Tokyo Institute of Technology by Professor Sampei and his students, who was also among the first researchers suggested analytical solutions for motion planning and feedback design for this system following the approach close to the method of this paper, see Nakaura et al. (2004). The results of Shiriaev, Freidovich, Robertsson, and Johansson (2006) elaborate further and modify the motion planning arguments and the controller design steps of Nakaura et al. (2004) to improve the rate of convergence and robustness of the *hybrid* closed-loop system.
- *The 3-DoF underactuated ship* model is commonly used for motion planning and control or dynamical positioning of ships, see Fossen and Strand (2001), Fossen (2002). The presented technique was applied and extended for motion planning to a model of an underactuated ship in order to analyze the feasibility of certain motion-planning and control tasks, see Shiriaev, Robertsson, Pacull, and Fossen (2005), Shiriaev, Robertsson, Freidovich, and Johansson (2006), Manchester et al. (2007).
- *Walking gaits* analysis, motion planning and orbital stabilization are reported in Mettin, LaHera, Freidovich, Shiriaev, and Helbo (2008), Freidovich, Shiriaev, and Manchester (2008), Freidovich, Mettin, Shiriaev, and Spong (submitted for publication). Here it is shown how to use virtual holonomic constraints for parameterizing natural gaits of a human, for planning and stabilizing a gait for underactuated biped, and how to use this concept for searching walking cycles of a passive biped.

5. Conclusions

Stabilization of periodic motions is a challenging task, considerably more difficult than stabilization of an equilibrium point. In this paper, we have given an overview of some classical and some more recent mathematical tools that can be brought to bear on the problem.

The notion of a transverse linearization has been defined and developed in detail for the class of mechanical systems with the number of actuators one less than the number of degrees of freedom.

Remarkably, for this class of nonlinear controlled systems, the difficult problem of planning a periodic motion can be solved analytically. Furthermore, for any nontrivial periodic motion of such systems there is a particular set of coordinates, which can be readily partitioned into a transverse part and a scalar variable that changes along the cycle. Such partition of coordinates defines a moving Poincaré section for this cycle prior to the choice of a control signal in a vicinity of the cycle.

As shown, the linearization of these transverse coordinates along the periodic motion, results in a particular periodic linear control system (transverse linearization), whose coefficients can be computed analytically. This opens up a number of possibilities. For instance,

- One can design an orbitally exponentially stabilizing controller for the desired periodic motion of the nonlinear system if and only if one can stabilize the auxiliary periodic linear control system.
- In order to analyze robustness, rate of convergence, sensitivity to parameters variations, *etc*., of the closed-loop system with a given nonlinear feedback law, one can compute the transverse linearization of the system dynamics around its cycle with an already fixed controller and explore the corresponding properties.
- One can find a quadratic Lyapunov function for the closed-loop system, written in terms of a distance function to the cycle, as commonly used for analysis of stability of a nonlinear system around an equilibrium.
- One can compute analytically the linearization of the first-return Poincaré map of the system dynamics around the cycle without explicit introduction of Poincaré section and computing of the Poincaré map itself.

The proposed arguments are supported by many examples in simulations and successful experimental studies. However, application of this technique requires reliable numerical methods for analysis and control for linear systems with periodic coefficients and presents an important opportunity for future development.

References

Altafini, C. (2002). Following a path of varying curvature as an output regulation problem. *IEEE Transactions on Automatic Control, 47*(9), 1551–1556.

Anami, R., Nakaura, S., & Sampei, M. (2007). Swing up control for the Acrobot considering compliance of high bar and energy interaction with each component.. In *Proceedings of the 46th IEEE Conference on Decision and Control*.

Andronov, A., & Vitt, A. (1933). On Lyapunov stability.. *Experimental Theoretical Physics, 3*, 373–374 (In Russian).

Aracil, J., Gordillo, F., & Acosta, J. (2002). Stabilization of oscillations in the inverted pendulum. In *Proceedings of the 15th IFAC World Congress*.

Banaszuk, A., & Hauser, J. (1995). Feedback linearization of transverse dynamics for periodic orbits. *Systems and Control Letters, 26*, 95–105.

Bittanti, S., Colaneri, P., & de Nicolao, G. (1991). The periodic Riccati equations. In S. Bittanti, A. Laub, & J. Willems (Eds.), *The Riccati Equation* (pp. 127–162). Berlin, Germany: Springer.

Bittanti, S., Lorito, F., & Strada, S. (1996). An LQ approach to active control of vibrations in helicopters. *ASME Transactions on Dynamic Systems, Measurement and Control, 118*, 482–488.

Bittanti, S., & Moiraghi, L. (1994). Active control of vibrations in helicopters via pole assignment techniques. *IEEE Transactions on Control Systems Technology, 2*, 343–351.

Bloch, A., Leonard, N., & Marsden, J. E. (2000). Controlled Lagrangians and the stabilization of mechanical systems. I. The first matching theorem. *IEEE Transactions on Automatic Control, 45*(12), 2253–2270.

Byrnes, C., Isidori, A., & Willems, J. (1991). Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Transactions on Automatic Control, 36*, 1228–1240.

Canudas-de-Wit, C., Espiau, B., & Urrea, C. (2002). Orbital stabilization of underactuated mechanical systems. In *Proceedings of the 15th IFAC World Congress*.

Chevallereau, C., Abba, G., Aoustin, Y., Plestan, F., Westervelt, E., Canudas-de-Wit, C., & Grizzle, J. (2003). Rabbit: A testbed for advanced control theory. *IEEE Control Systems Magazine, 23*(5), 57–79.

Chevallereau, C., Formal'sky, A., & Djoudi, D. (2004). Tracking a joint path for the walk of an underactuated biped. *Robotica, 22*(1), 15–28.

Chevallereau, C., Westervelt, E., & Grizzle, J. (2005). Asymptotically stable running for a five-link, four-actuator, planar, bipedal robot. *International Journal of Robotics Research, 24*(6), 431–464.

Chung, C., & Hauser, J. (1995). Nonlinear control of a swinging pendulum. *Automatica, 31*, 851–862.

Chung, C., & Hauser, J. (1997). Nonlinear H_∞ control around periodic orbits. *Systems and Control Letters, 30*(2), 127–137.

Coelho, P., & Nunes, U. (2003). Lie algebra application to mobile robot control: A tutorial. *Robotica, 21*, 483–493.

Colaneri, P. (2005). Theoretical aspects of continuous-time periodic systems. *Annual Reviews in Control, 29*(1), 205–215.

Das, T., & Mukherjee, R. (2001). Dynamic analysis of rectilinear motion of a self-propelling disk with unbalance masses. *ASME Journal of Applied Mechanics, 68*(1), 58–66.

Duindam, V., & Stramigioli, S. (2005). Port-based control of a compass-gait bipedal robot. In *Proceedings of the 16th IFAC World Congress*.

Fossen, T. (2002). *Marine Control Systems*. Marine Cybernetics.

Fossen, T., & Strand, J. (2001). Nonlinear passive weather optimal positioning control for ships and rigs: Experimental results. *Automatica, 37*, 701–715.

Fradkov, A., & Evans, R. (2005). Control of chaos: Methods and applications in engineering. *Annual Reviews in Control, 29*, 33–56.

Fradkov, A., & Pogromsky, A. (1998). *Introduction to control of oscillations and chaos*. World Scientific, Singapore.

Freidovich, L., LaHera, P., Mettin, U., Robertsson, A., Shiriaev, A., & Johansson, R. (submitted for publication-a). Stable periodic motions of the Inertia Wheel Pendulum via virtual holonomic constraints.

Freidovich, L., Mettin, U., Shiriaev, A., & Spong, M. (submitted for publication-b). A passive 2d of walker: Hunting for gaits using virtual holonomic constraints.

Freidovich, L., Robertsson, A., Shiriaev, A., & Johansson, R. (2007a). Stable periodic motions of inertia wheel pendulum via virtual holonomic constraints. In *Proceedings of the European Control Conference*.

Freidovich, L., Robertsson, A., Shiriaev, A., & Johansson, R. (2008a). Periodic motions of the Pendubot via virtual holonomic constraints: Theory and experiment. *Automatica, 44*(3), 785–791.

Freidovich, L., Shiriaev, A., & Manchester, I. (2007b). Experimental implementation of stable oscillations of the Furuta pendulum around the upward equilibrium. In *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems*.

Freidovich, L., Shiriaev, A., & Manchester, I. (2008b). Stability analysis and control design for an underactuated walking robot via computation of a transverse linearization. In *Proceedings of the 17th IFAC World Congress*.

Gillespie, R., Colgate, J., & Peshkin, M. (2001). A general framework for cobot control. *IEEE Transactions on Robotics and Automation, 17*(4), 391–401.

Graichen, K., & Zeits, M. (2005). Nonlinear feedforward feedback tracking control with input constraints solving the pendubot swing-up problem.. In *Proceedings of the 15th IFAC World Congress*.

Granat, R., & Kågström, B. (2006). Direct eigenvalue reordering in a product of matrices in extended periodic schur form. *SIAM Journal on Matrix Analysis and Applications, 28*(1), 285–300.

Granat, R., Kågström, B., & Kressner, D. (2006). Reordering the eigenvalues of a periodic matrix pair with applications in control. In *Proceedings of the 2006 IEEE Conference on Computer Aided Control Systems Design (CACSD)*.

Granat, R., Kågström, B., & Kressner, D. (2007a). Computing periodic deflating subspaces associated with a specified set of eigenvalues. *BIT Numerical Mathematics, 47*, 763–791.

Granat, R., Kågström, B., & Kressner, D. (2007b). Matlab tools for solving periodic eigenvalue problems. In *Proceedings of the 3rd IFAC Workshop PSYCO'07*.

Grizzle, J., Abba, G., & Plestan, F. (1999). Proving asymptotic stability of a walking cycle for a five dof biped robot model. In *Proceedings of the 2nd International Conference on Climbing and Walking Robots*.

Grizzle, J., Abba, G., & Plestan, F. (2001). Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE Transactions on Automatic Control, 46*(1), 51–64.

Grizzle, J., Moog, C., & Chevallereau, C. (2005). Nonlinear control of mechanical systems with an unactuated cyclic variable. *IEEE Transactions on Automatic Control, 50*(5), 559–576.

Gusev, S., Shiriaev, A., & Freidovich, L. (2007). LMI approach for solving periodic matrix Riccati equation. In *Proceedings of the 3rd IFAC Workshop Periodic Control Systems*.

Hale, J. (1980). *Ordinary Differential Equations*. Malabar, FL, USA: Krieger.

Hauser, J., & Chung, C. (1994). Converse Lyapunov function for exponential stable periodic orbits. *Systems and Control Letters, 23*, 27–34.

Johansson, S., Kågström, B., Shiriaev, A., & Varga, A. (2007). Comparing one-shot and multi-shot methods for solving periodic Riccati differential equations. In *Proceedings of the 3rd IFAC Workshop Periodic Control Systems*.

Khalil, H. (2002). *Nonlinear systems* (3rd Edition). Prentice Hall.

Kuo, A. (2002). Energetics of actively powered locomotion using the simplest walking model. *ASME Journal of Biomechanical Engineering, 124*(2), 113–120.

LaHera, P., Freidovich, L., Shiriaev, A., & Mettin, U. (in press). Swinging up the Furuta pendulum via stabilization of planned trajectory: Theory and experiments. *IFAC Journal of Mechatronics*.

Leonov, G. (2006). Generalization of the Andronov-Vitt theorem. *Regular and chaotic dynamics, 11*(2), 281–289.

Lyapunov, A. (1892). The general problem of the stability of motion. The Mathematical Society of Kharkov. *International Journal of Control, 55*(3), 531–773 (reprinted in 1992).

Manchester, I., Shiriaev, A., & Savkin, A. (2007). On motion planning for an underactuated ship: Fundamental limitations and a bearings-only navigation strategy. In *Proceedings of the 46th Conference on Decision and Control*.

Mazenc, F., & Bowong, S. (2003). Tracking trajectories of the cart-pendulum system. *Automatica, 39*(4), 677–684.

Mettin, U., LaHera, P., Freidovich, L., Shiriaev, A., & Helbo, J. (2008). Motion planning for humanoid robots based on virtual constraints extracted from recorded human movements. *Journal of Intelligent Service Robotics, 1*(4), 289–301.

Miossec, S., & Aoustin, Y. (2005). A simplified stability study for a biped walk with underactuated and overactuated phases. *International Journal of Robotics Research, 24*(7), 537–551.

Mukherjee, R., & Chen, D. (1993). Control of free-flying under-actuated manipulators to equilibrium manifolds. *IEEE Transactions on Robotics and Automation, 9*(5), 561–570.

Nair, S., & Leonard, N. (2008). Stable synchronization of mechanical system networks. *SIAM Journal of Control and Optimization, 47*(2), 661–683.

Nakaura, S., Kawaida, Y., Matsumoto, T., & Sampei, M. (2004). Enduring rotatory motion control of devil stick. In *Proceedings of the 6th IFAC Symposium on Nonlinear Control Systems*.

Nam, K., & Arapostathis, A. (1992). A sufficient condition for local controllability of nonlinear systems along closed orbits. *IEEE Transactions on Automatic Control, 37*(3), 378–380.

Nielsen, C., & Maggiore, M. (2006). Further results on transverse feedback linearization of multi-input systems. In *Proceedings of the 45th Conference on Decision and Control*.

Orlov, Y., Aguilar, L., Acho, L., & Ortiz, A. (2008a). Asymptotic harmonic generator and its application to finite time orbital stabilization of a friction pendulum with experimental verification. *International Journal of Control, 81*(2), 227–234.

Orlov, Y., Aguilar, L., Acho, L., & Ortiz, A. (2008b). Robust orbital stabilization of pendubot: Algorithm synthesis, experimental verification, and application to swing up and balancing control. In G. Bartolini, L. Fridman, A. Pisano, & E. Usai (Eds.), *Modern Sliding Mode Control Theory - New Perspectives and Applications. Vol. 375 of Lecture Notes in Control and Inf. Sci*. Springer–Verlag.

Ortega, R., Loria, A., Nicklasson, P., & Sira-Ramirez, H. (1998). *Passivity-based control of Euler-Lagrange systems. Communications and Control Engineering*. Berlin, Germany: Springer–Verlag.

Ortega, R., Spong, M., Gomez, F., & Blankenstein, G. (2002). Stabilization of underactuated mechanical systems via interconnection and damping assignment. *IEEE Transactions on Automatic Control, 47*(8), 1218–1233.

Perram, J., Shiriaev, A., Canudas-de-Wit, C., & Grognard, F. (2003). Explicit formula for a general integral of motion for a class of mechanical systems subject to holonomic constraint. In *Proceedings of the 2nd IFAC Workshop Control Methods for Lagrangian and Hamiltonian Systems*.

Poincaré, H. (1916–1954). *Oeuvres comleés, Vol. 11*. Paris, France: Gauthier-Villars.

Rouche, N., & Mawhin, J. (1980). *Ordinary differential equations. Stability and periodic solutions*. Pitman Publishing Ltd.

Samson, C. (1995). Control of chained systems: Application to path following and time-varying point-stabilization of mobile robots. *IEEE Transactions on Automatic Control, 40*(1), 64–77.

Santiesteban, R., Floquet, T., Orlov, Y., Riachy, S., & Richard, J. (2008). Second order sliding mode control of underactuated mechanical systems II: Orbital stabilization with application to swing up/balancing control of an inverted pendulum. *International Journal of Robust and Nonlinear Control, 18*, 544–556.

Shimizu, T., Nakaura, S., & Sampei, M. (2006). The control of bipedal running robot based on output zeroing considered the rotation of the ankle joint.. In *Proceedings of the 45th Conference on Decision and Control*.

Shiriaev, A., Freidovich, L., & Manchester, I. (submitted for publication). Transverse linearization for mechanical systems with underactuation degree one. *SIAM Journal on Control and Optimization*.

Shiriaev, A., Freidovich, L., Robertsson, A., & Johansson, R. (2006a). Generating stable propeller motions for devil stick. In *Proceedings of the 3rd IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*.

Shiriaev, A., Freidovich, L., Robertsson, A., Johansson, R., & Sandberg, A. (2007). Virtual-constraints-based design of stable oscillations of Furuta pendulum: Theory and experiments. *IEEE Transactions on Robotics, 23*(4), 827–832.

Shiriaev, A., Perram, J., & Canudas-de-Wit, C. (2005a). Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. *IEEE Transactions on Automatic Control, 50*(8), 1164–1176.

Shiriaev, A., Robertsson, A., Freidovich, L., & Johansson, R. (2006b). Coordinating control for a fleet of underactuated ships. In K. Pettersen, J. Gravdahl, & H. Nijmeijer (Eds.), *Group Coordination and Cooperative Control. Vol. 336 of Lecture Notes in Control and Inf. Sci.* (pp. 233–250). Springer–Verlag

Shiriaev, A., Robertsson, A., Pacull, P., & Fossen, T. (2005b). Motion planning and feedback stabilization for under-actuated ships: Virtual constraints approach.. In *Proceedings of the 16th IFAC World Congress*.

Shiriaev, A., Robertsson, A., Perram, J., & Sandberg, A. (2006c). Periodic motion planning for virtually constrained mechanical system. *Systems and Control Letters, 55*(11), 900–907.

Skjetne, R., Fossen, T., & Kokotović, P. (2004). Robust output maneuvering for a class of nonlinear systems. *Automatica, 40*, 373–383.

Spong, M. (1997). Underactuated mechanical systems. In B. Siciliano & K. Valavanis (Eds.), *Control Problems in Robotics and Automation. Vol. 230 of Lecture Notes in Control and Inf. Sci.* London, UK: Springer–Verlag.

Spong, M., Hutchinson, S., & Vidyasagar, M. (2006). *Robot modeling and control*. John Wiley and Sons.
Urabe, M. (1967). *Nonlinear autonomous oscillations*. NY, USA: Academic Press.
van Oort, G., & Stramigioli, S. (2007). Using time-reversal symmetry for stabilizing a simple 3d walker model. In *Proceedings of the IEEE International Conference on Robotics and Automation*.
Varga, A. (2005a). On solving periodic differential matrix equations with applications to periodic system norms computation. In *Proceedings of the CDC'05*.
Varga, A. (2005b). A periodic systems toolbox for matlab. In *Proceedings of the IFAC'05 World Congress*.
Varga, A. (2007). An overview of recent developments in computational methods for periodic systems. In *Proceedings of the IFAC Workshop on Periodic Control Systems*.
Varga, A. (in press). On solving periodic riccati equations. *Journal of Numerical Linear Algebra with Applications*.
Varga, A., & Van Dooren, P. (2001). Computational methods for periodic systems-An overview. .
Westervelt, E., Grizzle, J., & Koditschek, D. (2003). Hybrid zero dynamics of planar biped walkers. *IEEE Transactions on Automatic Control, 48*(1), 42–56.
Wisse, M., Feliksdal, G., van Frankenhuyzen, J., & Moyer, B. (2007). Passive-based walking robot: Denise, a simple, efficient, and lightweight biped. *IEEE Robotics and Automation Magazine, 14*(2), 52–62.
Yakubovich, V. (1986). Linear quadratic optimization problem and frequency theorem for periodic systems. *Siberian Mathematical Journal, 27*(4), 181–200 (In Russian).
Yoshizawa, T. (1966). *Stability theory by Liapunov's second method*. Tokyo, Japan: The mathematical society of Japan.

Anton S. Shiriaev received the M.Sc. and the Ph.D. degrees both in Applied Mathematics from St. Petersburg State University, Russia in 1993 and 1997, respectively. In 2003 he became Professor in Automatic Control at the Department of Applied Physics and Electronics, Umeå University, Sweden. Since 2006 he holds a part-time Professor position at Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, Norway. Professor Shiriaev held visiting research positions at the Lund Institute of Technology (Sweden), the University of Seville (Spain), the Centre National de la Recherche Scientifique (France), the Mittag-Leffler Institute of the Royal Swedish Academy of Sciences (Sweden), the University of Aalborg (Denmark) and the Sungkyunkwan University, Suwon (Republic of Korea). His research interests in systems and control theory, as well as in robotics and mechatronics subjects, are related to motion planning, stability and stabilization of nonlinear systems.

Leonid B. Freidovich received the M.Sc. in Mechanics and Engineering and the Kandidat of Physical and Mathematical Sciences degrees from Saint-Petersburg State Technical University, Russia, and the Ph.D. degree in Mathematics from Michigan State University, East Lansing, in 1996, 1999, and 2005, respectively. He is currently an Assistant Professor in Control Systems at the Department of Applied Physics and Electronics, Umeå University, Sweden. His research interests include nonlinear control systems and robotics.

Ian R. Manchester was born in Sydney, Australia, in 1979. He completed the B.E. (Hons 1) degree in 2001 and the Ph.D. degree in 2005, both in electrical engineering at the University of New South Wales (UNSW), Sydney, Australia. Since then he has been at the Department of Applied Physics and Electronics, Ume\aa University, Sweden, and has held visiting positions at UNSW, Sungkyunkwan University (Republic of Korea) and Massachusetts Institute of Technology (USA). His current research interests include the control of underactuated and nonholonomic systems, walking robots, and system-identification approaches to medical diagnosis, with particular attention to disorders of the cerebrospinal fluid system.