Transverse Linearization for Impulsive Mechanical Systems With One Passive Link

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***Abstract*—A general method for planning and orbitally stabilizing periodic motions for impulsive mechanical systems with underactuation one is proposed. For each such trajectory, we suggest a constructive procedure for defining a sufficient number of nontrivial quantities that vanish on the orbit. After that, we prove that these quantities constitute a possible set of transverse coordinates. Finally, we present analytical steps for computing linearization of dynamics of these coordinates along the motion. As a result, for each such planned periodic trajectory, a hybrid transverse linearization for dynamics of the system is computed in closed form. The derived impulsive linear system can be used for stability analysis and for design of exponentially orbitally stabilizing feedback controllers. A geometrical interpretation of the method is given in terms of a novel concept of a moving Poincaré section. The technique is illustrated on a devil stick example.**

***Index Terms*—Impulsive mechanical systems, moving Poincaré section, orbital stability, transverse linearization, underactuation one, virtual holonomic constraints.**

I. Introduction

Finding feasible motions in nonlinear dynamical systems with switchings, exploring their properties, and designing for them orbitally stabilizing feedback controllers are challenging important in applications problems, see e.g. [1]–[3] and references therein. However, despite the complexity, it is clear that any nontrivial periodic solution of an impulsive dynamical system [4] should consist of interchanged sub-arcs of continuous-in-time parts and jumps due to instantaneous updates of the states. Here we consider controlled impulsive mechanical systems, i.e. we assume that the continuous dynamics can be described by Euler-Lagrange equations

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q,\dot{q})u \tag{1}$$

where $q \in \mathbb{R}^n$ and $\dot{q} \in \mathbb{R}^n$ are vectors of generalized coordinates and velocities, $u \in \mathbb{R}^{n-1}$ is a vector of control inputs, $B(q,\dot{q})$ is a matrix function of constant rank $(n-1)$. Instantaneous jumps along a solution are defined by a discrete-in-time part of the system dynamics. It consists of a collection of pairs of hypersurfaces $\{\Gamma_-^{(i)}, \Gamma_+^{(i)}\}_{i=1}^{N_d}$ in the state space of the mechanical system (1) and instantaneous mappings $F^{(i)}$, so that the update law is defined by the set of triples

$$\left\{\Gamma_-^{(i)}, \Gamma_+^{(i)}, F^{(i)}(\cdot)\right\}_{i=1}^{N_d}, \qquad F^{(i)} : \Gamma_-^{(i)} \to \Gamma_+^{(i)} \tag{2}$$

where N_d denotes the number of possible jumps and can be arbitrarily large but finite. Finding a periodic solution of the hybrid system (1), (2) is not easy. The common approach is to use one of the switching surfaces, e.g. $\Gamma_+^{(1)}$, as a hypersurface, where the Poincaré first-return map can be defined and computed [3], [5]. Parameterizing somehow a set of control inputs, a cycle is typically found via an extensive numerical search. If a periodic motion and an associated control input are obtained, then the orbital (in-)stability can be also verified numerically.

We propose below a new approach for planning periodic motions and for analyzing their orbital stability as well as stabilizability. In particular, we derive new necessary conditions for presence of a cycle in the dynamics of (1), (2) and show how to use them. Furthermore, we argue that for each hybrid periodic motion there is a natural candidate for a Poincaré section different from the standard choice, which is one of the hypersurfaces of (2) [5]. As another important contribution, we compute analytically a linearization of transverse dynamics of the impulsive system prior to design of a controller. As a result, the challenging task of feedback stabilization of a cycle is reformulated as a simpler and tractable problem of stabilization of a hybrid linear control system with a linear update law that is regularly activated over a constant time, which is equal to the period of the planned motion. The ideas are illustrated on a devil stick example; an application to a walking robot example is presented in [6].

Due to lack of space, we elaborate in detail only the simplest case when the hybrid cycle of (1), (2) consists of one continuous-in-time sub-arc and one jump. The constructive procedure for planning such a cycle is described in Section II. The method for computing a transverse linearization for a periodic motion of (1), (2) is given in Section III; the key here is a constructive choice of the change of coordinates in a vicinity of the motion. The example and conclusions are put in Sections IV and V respectively.

II. Planning a Cycle for Impulsive Mechanical System (1), (2) With One Jump

The procedure for planning periodic motions for the impulsive mechanical system (1), (2) is based on the important observation: The continuous-in-time sub-arc of any hybrid cycle $q_\star(t) = [q_{1\star}(t), \ldots, q_{n\star}(t)] \in \mathbb{R}^n$, $t \in [0, T_h]$ can be always re-parameterized using virtual holonomic constraints [3], [7], [8] as

$$q_{1\star} = \phi_1(\theta), \ldots, q_{n\star} = \phi_n(\theta), \theta = \theta_\star(t), \quad t \in [0, T_h] \tag{3}$$

where θ is a scalar variable representing one of the (old or new) generalized coordinates for (1). There are many choices for such re-parametrization; the obvious one is using the distance of travel along the orbit of the motion

$$\mathcal{O}(q_\star) := \left\{[q;\dot{q}] \in \mathbb{R}^{2n} : q = q_\star(t), \dot{q} = \dot{q}_\star(t), t \in [0, T_h]\right\}. \tag{4}$$

Due to the presence of one passive link, the evolution of $\theta = \theta_\star(t)$ cannot be any, but should comply with the constraints imposed by the

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dynamics of (1). As shown in [9], see also [7], [8], $\theta_\star(t)$ must be a solution of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0 \tag{5}$$

where the functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ are computed from (1) and (3). So that (5) naturally appears knowing just one solution of (1)—the continuous-in-time sub-arc of the hybrid cycle. If by some reasons the geometrical relations (3) are valid not only for the choice $\theta = \theta_\star(t)$, but for any $\theta(t)$, then these functions are solutions of (5), and their phase curves $[\theta(t); \dot{\theta}(t)]$ fill out a 2-D manifold Z in the $2n$-dimensional state-space of (1)

$$Z = \left\{ [q;\dot{q}] : q = \varphi(\theta), \dot{q} = \varphi'(\theta)\dot{\theta}, \theta(t) \text{ is a solution of } (5) \right\}. \tag{6}$$

The closure of this manifold can be invariant with respect to the update laws (2) as assumed in [3] or not. The system (5) is integrable [9]: The function $I = I(\theta(t), \dot{\theta}(t), \theta(0), \dot{\theta}(0))$ defined by

$$I = \dot{\theta}^2(t) - e^{\left\{ -\int_{\theta(0)}^{\theta(t)} \frac{2\beta(\tau)}{\alpha(\tau)} d\tau \right\}} \dot{\theta}^2(0) + \int_{\theta(0)}^{\theta(t)} e^{\left\{ \int_{\theta(t)}^{s} \frac{2\beta(\tau)}{\alpha(\tau)} d\tau \right\}} \frac{2\gamma(s)}{\alpha(s)} ds \tag{7}$$

keeps its zero value for any solution $\theta(t)$ of (5) for all $t \geq 0$, for which $\theta(t)$ is well-defined. Correspondingly, the phase curves of (5) can be found from (7) avoiding numerical solutions of the differential equation. Altogether these facts allow us to formulate necessary conditions for existence of a hybrid cycle.

Theorem 1: Given the controlled impulsive mechanical system, where the continuous-time dynamics (1) is with n degrees of freedom and is of underactuation one, and where the hypersurfaces Γ_- and Γ_+ and the mapping $F : \Gamma_- \to \Gamma_+$ define the discrete dynamics. Suppose that for some control signal $u = u_\star(t) = u_\star(t + T_h)$, the impulsive mechanical system has a periodic solution

$$q = q_\star(t) = q_\star(t + T_h), \quad \forall t, \quad T_h > 0 \tag{8}$$

with only one jump, i.e. $[q_\star(0+); \dot{q}_\star(0+)] \in \Gamma_+$, $[q_\star(T_h-); \dot{q}_\star(T_h-)] \in \Gamma_-$, $F([q_\star(T_h-); \dot{q}_\star(T_h-)]) = [q_\star(0+); \dot{q}_\star(0+)]$. Suppose the continuous-in-time arc of (8) admits a re-parametrization $q_\star(t) = \phi(\theta_\star(t))$ defined by (3) with C^2-functions $\phi_1(\cdot), \ldots, \phi_n(\cdot)$. Compute the dynamics of (1), when these relations are kept invariant, i.e. compute the coefficients of the second order system (5). Then, by necessity, the algebraic equations

$$\begin{aligned} & I\left(\theta_\star(0), \dot{\theta}_\star(0), \theta_\star(T_h), \dot{\theta}_\star(T_h)\right) = 0 \\ & F\left(\begin{bmatrix} q_- \\ \dot{q}_- \end{bmatrix}\right)\Bigg|_{\substack{q_- = \phi(\theta_\star(T_h)) \\ \dot{q}_- = \phi'(\theta_\star(T_h))\dot{\theta}_\star(T_h)}} = \begin{bmatrix} q_+ \\ \dot{q}_+ \end{bmatrix}\Bigg|_{\substack{q_+ = \phi(\theta_\star(0)) \\ \dot{q}_+ = \phi'(\theta_\star(0))\dot{\theta}_\star(0)}} \end{aligned} \tag{9}$$

hold. Here $I(\cdot)$ is an integral of (5) and can be taken as (7). ■

Proof: The function $\theta_\star(t)$ by construction is the solution of the system (5) and the first relation of (9) is valid because the function $I(\cdot)$ keeps the zero value on it [9]. The second relation of (9) is the mapping $F(\cdot)$ on the cycle written in terms of the variable θ. ■

Planning cycles of an impulsive mechanical systems can be based on Theorem 1 following the next steps.[1]

[1]A more restrictive motion planning procedure for a class of models for planar biped robots is described in [3]; we do not require $\mathcal{F}(\gamma_-) \subset \gamma_+$.

1) Let $P = (p_1, \ldots, p_k)$ be a vector of parameters and θ be a scalar variable; choose a set of C^2-smooth functions: $\phi(\theta, P) = \{\phi_1(\theta, P), \phi_2(\theta, P), \ldots, \phi_n(\theta, P)\}$;
2) Simplify the dynamics of (1) under the assumption that the relations $q_1 = \phi_1(\theta, P), \ldots, q_n = \phi_n(\theta, P)$ are all kept invariant. This results in the family of 2-D manifolds $Z(P) \subset \mathbb{R}^n \times \mathbb{R}^n$, defined by (6), and systems

$$\alpha(\theta, P)\ddot{\theta} + \beta(\theta, P)\dot{\theta}^2 + \gamma(\theta, P) = 0 \tag{10}$$

phase curves of which fill out $Z(P)$. For each choice of P, (10) is integrable; i.e. for any solution $\theta = \theta(t, P)$ of (10) well-defined for $t = \tau_1 > 0$, the function $I(\cdot)$ computed as in (7), satisfies

$$I\left(\theta(\tau_1, P), \dot{\theta}(\tau_1, P), \theta(0, P), \dot{\theta}(0, P)\right) = 0$$

3) Define the curves γ_-, γ_+ and the mapping $\mathcal{F} : \gamma_- \to \Gamma_+$

$$\begin{aligned} \gamma_+ = \Gamma_+ \cap Z(P), \quad \gamma_- = \Gamma_- \cap Z(P), \\ \mathcal{F}\left([\theta; \dot{\theta}]\right)\Big|_{[\theta;\dot{\theta}] \in \gamma_-} = F\left([q; \dot{q}]\right)\big|_{q = \phi(\theta, P),\ \dot{q} = \phi'(\theta, P)\dot{\theta}} \end{aligned} \tag{11}$$

4) Search for parameters $P = P_\star$ such that the following *algebraic* equations have a non-trivial solution

$$\begin{aligned} I(a, b, x, y) = 0, \quad \mathcal{F}([x; y]) = [a; b], \\ [a; b] \in \gamma_+, \quad [x; y] \in \gamma_-, \quad a, b, x, y \in \mathbb{R}^1. \end{aligned} \tag{12}$$

If the search is successful and $\theta = \theta_\star(t, P_\star)$ is the solution of (10) with $P = P_\star$ initiated at $\theta_\star(0, P_\star) = a, \dot{\theta}_\star(0, P_\star) = b$ such that $\theta_\star(T_h, P_\star) = x\ \dot{\theta}_\star(T_h, P_\star) = y$ for some $T_h > 0$, then the hybrid mechanical system has the hybrid cycle defined by

$$q_1 = \phi_1\left(\theta_\star(t, P_\star), P_\star\right), \quad \ldots, \quad q_n = \phi_n\left(\theta_\star(t, P_\star), P_\star\right).$$

III. Transverse Linearization of (1), (2) Around Its Periodic Solution With One Jump

Given a one-jump cycle $q = q_\star(t) = q_\star(t + T_h)$ of the impulsive mechanical system (1), (2) with n degrees of freedom and under-actuation one, computing *a transverse linearization*, an impulsive linear control system of dimension $(2n - 1)$, for the system dynamics along this motion consists of three steps:

Step 1) Linearizing the update law $\{\Gamma_-, \Gamma_+, F(\cdot)\}$ around $[q_\star(T_{h-}); \dot{q}_\star(T_{h-})] \in \Gamma_-$;

Step 2) Linearizing the transverse dynamics of (1) along the continuous-in-time sub-arc of $q_\star(t)$;

Step 3) Merging the linearizations of the continuous and discrete parts of the dynamics.

A. Step 1: Linearizing the Discrete-in-Time Dynamics

Linearization of $\{\Gamma_-, \Gamma_+, F(\cdot)\}$ is the Jacobian of $F(q)$ calculated at $[q_\star(T_{h-}); \dot{q}_\star(T_{h-})] \in \Gamma_-$

$$dF := \frac{\partial F}{\partial [q; \dot{q}]}\Bigg|_{\substack{q = q_\star(T_{h-}) \\ \dot{q} = \dot{q}_\star(T_{h-})}} : T\Gamma_-\big|_{\substack{q = q_\star(T_{h-}) \\ \dot{q} = \dot{q}_\star(T_{h-})}} \to T\Gamma_+\big|_{\substack{q = q_\star(0_+) \\ \dot{q} = \dot{q}_\star(0_+)}} \tag{13}$$

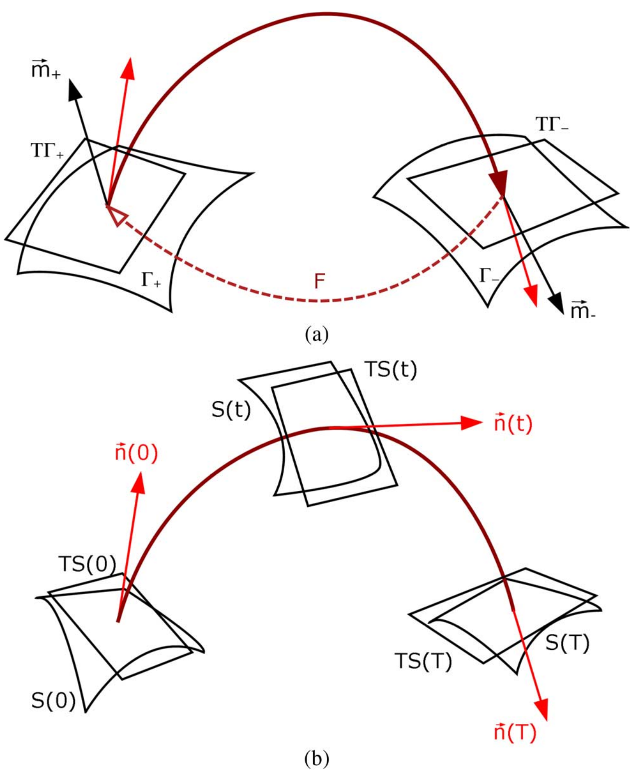


Fig. 1. (a) Tangent planes $T\Gamma_-$ and $T\Gamma_+$ to the switching surfaces Γ_- and Γ_+ at two points, where the periodic trajectory $q_\star(t)$ hits and originates from the switching surfaces. The linearization of $F(\cdot)$ in a vicinity of the hybrid cycle is the linear mapping $dF : T\Gamma_- \to T\Gamma_+$, see (13). $\vec{m}_-$ and $\vec{m}_+$ denote vectors normal to $T\Gamma_-$ and $T\Gamma_+$ respectively; (b) the moving Poincaré section is a family of $(2n-1)$-dimensional surfaces $S(t)$ transversal to the continuous-in-time sub-arc of the hybrid cycle. The linearization of transverse dynamics is a linear control system defined on $TS(t)$, the tangent planes to $S(t)$. $\vec{n}(t)$ denote vectors normal to $TS(t)$.

here $T\Gamma_-$ and $T\Gamma_+$ are tangent planes to the C^1-smooth hypersurfaces Γ_- and Γ_+ at two points, where the periodic trajectory $q_\star(t)$ hits and originates from the switching surfaces, see Fig. 1(a).

B. Step 2: Linearizing the Continuous-in-Time Dynamics

Linearizing the transverse dynamics of (1) along the continuous-in-time sub-arc of $q_\star(t)$ is based on the concepts of moving Poincaré sections [10] and transverse dynamics [11]–[13] defined next.

Definition 1: Let $q_s(t), t \in [0, T_h]$, be a solution of the n-degree-of-freedom mechanical system (1) with the initial conditions at $q_s(0) = q_0, \dot{q}_s(0) = \dot{q}_0$, driven by the control signal $u_s(t) \in C^1([0, T_h])$ such that $(|\dot{q}_s(t)|^2 + |\ddot{q}_s(t)|^2) > 0$ for all $t \in [0, T_h]$. Let us define an ε-tube around its orbit $\mathcal{O}(q_s)$, see (4), as

$$\mathcal{O}_\varepsilon(q_s) = \left\{ x = [q; \dot{q}] \in \mathbb{R}^{2n} : \operatorname{dist}(x, \mathcal{O}(q_s)) \le \varepsilon \right\},$$
$$\operatorname{dist}(x, \mathcal{O}(q_s)) = \min_{\tau \in [0,T]} \| x - [q_s(\tau); \dot{q}_s(\tau)] \| . \tag{14}$$

1) A family of $(2n-1)$-dimensional C^1-smooth surfaces $\{S(t), t \in [0, T_h]\}$ is called a *moving Poincaré section* associated with the solution $q_s(t), t \in [0, T_h]$, if there exists $\varepsilon > 0$ such that:
   - The surfaces $S(t)$ are disjoint, i.e. $S(\tau_1) \cap S(\tau_2) \cap \mathcal{O}_\varepsilon(q_s) = \emptyset$, $\forall \tau_1, \tau_2 \in [0, T_h), \tau_1 \neq \tau_2$;
   - $S(\tau) \cap \{[q_s(t); \dot{q}_s(t)], |t - \tau| < \varepsilon\} \cap \mathcal{O}_\varepsilon(q_s) = \{[q_s(\tau); \dot{q}_s(\tau)]\}$ for each $\tau \in [0, T_h]$;
   - The surfaces $S(t)$ are locally smoothly parametrized by evolution along the trajectory, i.e. $\exists f_s \in C^1(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}) : S(t) \cap \mathcal{O}_\varepsilon(q_s) = \{[q; \dot{q}] \in \mathbb{R}^n \times \mathbb{R}^n : f_s(q, \dot{q}, t) = 0\} \cap \mathcal{O}_\varepsilon(q_s)$.

2) Given a moving Poincaré section $\{S(t), t \in [0, T_h]\}$, in a tube $\mathcal{O}_\varepsilon(q_s)$ around the trajectory $[q_s(t); \dot{q}_s(t)], t \in [0, T_h]$, the state vector $[q; \dot{q}]$ of (1) can be changed into: a scalar variable $\psi(t)$ that parameterizes a position along the trajectory and a $(2n-1)$-dimensional vector $x_\perp(t)$ that defines location on the surface $S(t)$. $x_\perp$ is known as a vector of *transverse coordinates*, while $x_\perp$-dynamics are called *transverse*.

3) The dynamics of (1) rewritten in $[\psi; x_\perp]$-coordinates and linearized along the solution $q_s(t), t \in [0, T_h]$ give rise to a linear time-varying control system of dimension $2n$ defined on $(0, T_h)$. Its subsystem corresponding to a linearization of the $x_\perp$-dynamics is called a *transverse linearization*. ■

Fig. 1(b) illustrates the concept of a moving Poincaré section for the continuous-in-time sub-arc of a hybrid cycle. Note that $S(0)$ and $S(T_h)$ might not coincide with the switching surfaces Γ_+ and Γ_-.

The transverse linearization is not uniquely defined and depends on the choice of a moving Poincaré section $\{S(t)\}_{t \in [0, T_h]}$ or, in other words, on the choice of transverse coordinates $x_\perp$. Let us show that for any non-trivial motion of (1), there is one generic choice of transverse coordinates. Indeed, given the scalar functions $\phi_1(\cdot), \ldots, \phi_n(\cdot)$ defined by the relations (3), let us consider the quantities

$$\theta, \quad y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_n = q_n - \phi_n(\theta) \tag{15}$$

as excessive coordinates for the n-DOF mechanical system (1). Locally, one of them can be always expressed as a function of the others. Assuming that this can be done globally for y_n, we can take

$$y = (y_1, \ldots, y_{n-1})^T \quad \text{and} \quad \theta \tag{16}$$

as the new generalized coordinates, see [7], [8] for details.

In these coordinates, a possible choices for the transverse coordinates for the dynamics of (1) along $q_\star(t), t \in [0, T_h]$ is

$$x_\perp = \left[I\left(\theta, \dot{\theta}, \theta_\star(0), \dot{\theta}_\star(0)\right); y; \dot{y} \right] \tag{17}$$

where $I(\cdot)$ and $y(\cdot)$ are defined by (7), (15), and (16). To define a moving Poincaré section $\{S(t)\}_{t \in [0, T_h]}$ associated with the continuous-in-time sub-arc of solution $q_\star(t)$ of (1), theoretically one can proceed as follows.

1) Change in a vicinity of the target motion the state vector $[q; \dot{q}]$ of the system (1) into $[\theta; \dot{\theta}; y; \dot{y}]$.
2) Make another change from $[\theta; \dot{\theta}; y; \dot{y}]$ into $[\psi; I; y; \dot{y}]$, where I is defined by (7). This step introduces the scalar variable $\psi = \psi(q, \dot{q})$ such that $\psi_\star(t) := \psi(q_\star(t), \dot{q}_\star(t))$ monotonically changes with time along the target trajectory[2] that can be written as $\{\psi = \psi_\star(t), I = 0, y = 0, \dot{y} = 0\}$. These coordinates are the ones that were known to exist [11] and that were assumed to be somehow given for analysis and stabilization of smooth cycles of smooth nonlinear systems in [12] and other papers. The key novelty here is the *systematic and explicit procedure for computation of transversal states* (17) and *their linearization for controlled mechanical systems*. These steps were impossible to accomplish for general nonlinear systems considered e.g. in [10], [14].

[2]This is due to the assumption that $|\dot{q}_\star(t)|^2 + |\ddot{q}_\star(t)|^2 > 0$ for $t \in [0, T_h]$.

3) After that, the moving Poincaré section is defined by

$$S(t) := \{[q;\dot{q}]:\ \psi(q,\dot{q}) - \psi_\star(t) = 0\}, \quad t \in [0, T_h]. \tag{18}$$

Rewriting dynamics of (1) in terms of the new variables $[\psi; I; y; \dot{y}]$ and computing $S(t)$ in (18) is often a nontrivial task. Fortunately, the tangent planes $TS(t)$ can be readily found without introducing the variable ψ. Indeed, if the dynamics of (1) are expressed in $[\psi; I; y; \dot{y}]$-coordinates, then at any point of the cycle the vector $\vec{n}(t)$ normal to $TS(t)$ is: $\vec{n}(t) = [1, 0, \ldots, 0]^T \in \mathbb{R}^{2n}$. At the same time, the direction of the vector field of (1) at this point is proportional to the same vector. Therefore, in the original coordinates

$$\vec{n}(t) = [\dot{q}_\star(t); \ddot{q}_\star(t)] \Big/ \sqrt{|\dot{q}_\star(t)|^2 + |\ddot{q}_\star(t)|^2}. \tag{19}$$

So, by construction, surfaces $S(t)$ and hyperplanes $TS(t)$ are orthogonal to the orbit of the target motion, and hence are transversal to it. This makes the choice of transverse coordinates (17) to be the most natural.

With the transverse coordinates (17) and the associated moving Poincaré section (18), the transverse linearization of the n-DOF controlled mechanical system (1) with $(n-1)$ independent control variables $u = (u_1, \ldots, u_{n-1})^T$ around the motion $q = q_\star(t)$ can be found analytically. Expressed in the particular coordinates on $TS(\tau)$, this linear control system has the form [8], [13]

$$\begin{aligned}
\frac{d}{d\tau}X(\tau) &= A(\tau)X(\tau) + B(\tau)V_\bullet(\tau),\\
X(\tau) &= \begin{bmatrix} I_{\bullet(\tau)} \\ Y_{1\bullet}(\tau) \\ Y_{2\bullet}(\tau) \end{bmatrix}, \quad B(\tau) = \begin{bmatrix} b_1(\tau) \\ \mathbf{0}_{(n-1)\times(n-1)} \\ \mathbf{1}_{(n-1)\times(n-1)} \end{bmatrix},\\
A(\tau) &= \begin{bmatrix} a_{11}(\tau) & a_{12}(\tau) & a_{13}(\tau) \\ \mathbf{0}_{(n-1)\times 1} & \mathbf{0}_{(n-1)\times(n-1)} & \mathbf{1}_{(n-1)\times(n-1)} \\ \mathbf{0}_{(n-1)\times 1} & \mathbf{0}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times(n-1)} \end{bmatrix}.
\end{aligned} \tag{20}$$

Here $X(\tau) \in \mathbb{R}^{2n-1}$ is a vector approximating a behavior of transverse coordinates $x_\perp(\cdot)$ defined by (17); $V_\bullet \in \mathbb{R}^{n-1}$ is a control input, which approximates new control variable $v(\cdot)$ for (1) introduced by a feedback transformation in the form [8]

$$u = \varepsilon(q,\dot{q}) + \delta(q,\dot{q})v \tag{21}$$

linearizing the dynamics of y-variables (16) to obtain $\ddot{y} = v$; $a_{11}(\cdot)$, $a_{12}(\cdot)$, $a_{13}(\cdot)$, and $b_1(\cdot)$ are functions on $[0, T_h]$ of appropriate dimensions. Explicit formulas for coefficients of (20) can be found in [8, Eqns. (33), (34)]; they are computed knowing (1) and the solution $q_\star(t)$ as a function of time.

C. Step 3: Merging the Two Parts of Linearized Dynamics

Combining the linear mapping (13), i.e. the linearization of the update law, with the transverse linearization (20) requires certain care. Indeed, the linear mapping (13) acts between the hyperplanes $T\Gamma_-$ and $T\Gamma_+$, defined by the switching surfaces, while the linear differential equation (20) maps a vector on $TS(0)$ into a vector on $TS(T_h)$. Clearly, the hyperplanes $T\Gamma_-$ and $TS(T_h)$, as well as $T\Gamma_+$ and $TS(0)$, might not coincide. Therefore, to define a transverse linearization for our impulsive system around its hybrid periodic motion $q_\star(t)$, we have either to introduce a transverse linearization corresponding to another choice of a moving Poincaré section such that $TS(0) = T\Gamma_+$ and $TS(T_h) = T\Gamma_-$ or to transform the linearization of the update law $dF(\cdot)$ in such a way that it acts from $TS(T_h)$ onto $TS(0)$ with the choice of $S(t)_{t\in[0,T_h]}$ given on **Step 2**. Difficulties associated with computing alternative to (20) transverse linearizations analytically[3] motivate the second choice.

Definition 2: Suppose the following are given:

1) The hyperplanes $TS(0)$, $TS(T_h)$, the normal vectors $\vec{n}(0)$, $\vec{n}(T_h)$, see Fig. 1(b), defined by a moving Poincaré section $\{S(t)\}_{t\in[0,T_h]}$ associated with the continuous-in-time sub-arc of $q_\star(t)$.
2) The tangent planes $T\Gamma_+$ and $T\Gamma_-$ to the switching surfaces Γ_+ and Γ_-, see Fig. 1(a), that are transversal to the hybrid periodic motion defined at the end-points of the continuous-in-time sub-arc

$$[\dot{q}_\star(0_+); \ddot{q}_\star(0_+)] \notin T\Gamma_+ \text{ and } [\dot{q}_\star(T_{h-}); \ddot{q}_\star(T_{h-})] \notin T\Gamma_-. \tag{22}$$

3) The linear mapping $dF : T\Gamma_- \to T\Gamma_+$ defined by (13).

Let us denote by $P^+_{\vec{n}(0)} : T\Gamma_+ \to TS(0)$ the projection along $\vec{n}(0)$. This operator can be introduced by the following geometric rule. For a given $z_0 \in T\Gamma_+$ consider the line l_0 parallel to $\vec{n}(0)$ that passes through z_0. Denote by y_0 the point of intersection of $TS(0)$ and this line l_0. Then, y_0 is the image of z_0 under the map $P^+_{\vec{n}(0)}$. Similarly, let $P^-_{\vec{n}(T_h)} : TS(T_h) \to T\Gamma_-$ be the projection along $\vec{n}(T_h)$. From the conditions (22) both projection operators are well-defined and linear.[4] The linear operator

$$[d^{TS}F] : TS(T_h) \ni \xi \mapsto \eta := \left[P^+_{\vec{n}(0)}\right] dF \left[P^-_{\vec{n}(T_h)}\right] \xi \in TS(0) \tag{23}$$

is called *a linearization of $F(\cdot)$ associated with the moving Poincaré section $\{S(t)\}_{t\in[0,T_h]}$*. ■

The next statement relates behaviors of the linear system (20), (23) and the nonlinear system (1), (21), (2).

Theorem 2: Given the impulsive mechanical system (1), (2) and its T_h-periodic solution (8) with one jump such that the relations (22) hold. Consider the impulsive linear control system, a solution of which $X = X(\tau) \in \mathbb{R}^{2n-1}$ is defined by the next inductive rule:

- On the time intervals $((k-1)T_h, kT_h)$, $k \in \mathbb{N}$ the solution is defined by the linear control system

$$\frac{d}{d\tau}X(\tau) = A(\tau \bmod T_h)X(\tau) + B(\tau \bmod T_h)V_\bullet(\tau) \tag{24}$$

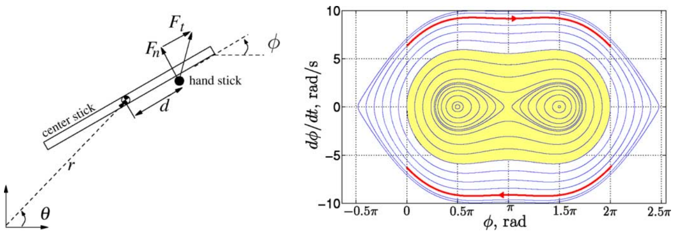
where matrices $A(\tau)$ and $B(\tau)$ are from (20), and $V_\bullet(\tau) \in \mathbb{R}^{n-1}$ is a vector of control inputs.
- At each of the time moments $\tau_s = kT_h$, $k \in \mathbb{N}$ the state $X(\tau_s)$ of the linear system (24) is instantaneously changed to the new vector, defined by the linear transformation

$$X(\tau_{s-}) \longmapsto X(\tau_{s+}) := \left[d^{TS}F\right] X(\tau_{s-}) \tag{25}$$

where the operator $[d^{TS}F]$ is from (23). After the update, the solution is defined by (24) until $\tau = \tau_s + T_h$, where the next instantaneous update (25) occurs and so on.

[3]A computational procedure, based on a concept of orthogonalizing transform, has been proposed in [15].

[4]The explicit formulae for the projection operators are given in [6].



(a) (b)

Fig. 2. (a) Model of the devil stick adopted from [16]. The system consists of one (rotating) center stick (CS) driven by the force from the hand stick (HS), which rolls along the CS without sliding. (b) Phase portrait of system (33). The shaded area of the phase portrait possesses no solution that satisfies $\phi(T) = \phi(0) + 2\pi$. Meanwhile, the trajectories indicated by bold red arrows satisfy this and $\dot\phi(T) = \dot\phi(0)$, so they can be sub-arcs of the desired hybrid cycle.

Then, the origin $X = 0$ of the linear system (24), (25) is exponentially stabilized by

$$V_\bullet(\tau) = K(\tau \bmod T_h)X(\tau), \quad K(\cdot) \in C^1[0, T_h] \tag{26}$$

if and only if the feedback controller (21) with

$$v(t) = K(\tau)x_\perp(t) \tag{27}$$

and τ defined at every t by one of the following two relations

$$\{[q(t); \dot q(t)] \in TS(\tau) \cap \mathcal{O}_\varepsilon(q_\star)\} \text{ or } \{[q(t); \dot q(t)] \in S(\tau) \cap \mathcal{O}_\varepsilon(q_\star)\} \tag{28}$$

makes the hybrid periodic motion of the impulsive mechanical system (1), (2) orbitally exponentially stable. Here $x_\perp(\cdot)$ and $\mathcal{O}_\varepsilon(q_\star) = \mathcal{O}_\varepsilon(q_s)$ are defined in (17) and (14), and $\varepsilon > 0$ is small. ■

Proof of Theorem 2 is given in Appendix A.

IV. Example: Stable Rotations for a Devil Stick

To illustrate our technique, consider the task of planning stable perpetual rotations of the mechanical system resembling the behavior of a 'devil stick'—an entertainment juggling device. It consists of two parts: a hand stick (HS) and a center stick (CS) which is floating in the air and rolls along the HS without sliding, see Fig. 2(a). The dynamics[5] of the CS, in polar coordinates, is [16]

$$\begin{aligned} m\ddot r &= mr\dot\theta^2 - mg\sin\theta + \cos(\theta - \phi)F_t + \sin(\theta - \phi)F_n \\ rm\ddot\theta &= -2m\dot r\dot\theta - mg\cos\theta - \sin(\theta - \phi)F_t + \cos(\theta - \phi)F_n \\ J\ddot\phi &= d(t)F_n = (d_0 - \rho\phi)F_n. \end{aligned} \tag{29}$$

Here $q = [r; \theta; \phi]$ with r, θ being the polar coordinates for the center of mass of the CS, ϕ being the angle that the CS makes with the horizontal; m, J are the mass and the moment of inertia of the CS; F_t, F_n are the tangential and normal components of the force applied to the CS by the HS at the point of contact; $d(t)$ is the *resettable* instantaneous position at which the CS and the HS are in contact. Let us make the following assumptions: The normal force F_n can change sign without breaking the contact between the sticks, so that dragging is possible. Furthermore, assume that there is no sliding even if the normal and tangential projections of the force violate a certain cone-like friction condition. The 'no-sliding' assumption results in a relation valid at the contact point: $\dot d = -\rho\dot\phi$ with $d(t_0) = d_0$, where d_0 is the contact position when $\phi(t_0) = 0$ and ρ is the radius of the HS. Integrating this relation over time, we obtain the right-hand side of the third equation in (29).

If someone manage to achieve a counterclockwise propeller motion, $(d/dt)\phi(t)$ must be kept positive. Since the length of the CS is finite, one cannot keep $(d/dt)d(t)$ negative forever and must instantaneously change $d(t)$ via an infinitely fast motion of the CS. This makes any cycle of (29) *hybrid*.

The update law is a part of a controller, so there is a certain freedom in its definition. However, the following choices for the switching surfaces and the update mapping

$$\Gamma_- \triangleq \left\{[q; \dot q] : \dot\phi = \dot\phi_\bullet(0), \frac{3\pi}{2} < \phi < \frac{5\pi}{2}\right\},$$

$$\Gamma_+ \triangleq \left\{[q; \dot q] : \dot\phi = \dot\phi_\bullet(0), -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right\}, \tag{30}$$

$$F : (r, \theta, \phi, \dot r, \dot\theta, \dot\phi) \ni \Gamma_- \mapsto (r, \theta, \phi - 2\pi, \dot r, \dot\theta, \dot\phi) \in \Gamma_+ \tag{31}$$

can be motivated from the analysis of motions performed by human artists and are simple for implementation. Here $\dot\phi_\bullet(0)$ is some chosen value of the angular velocity. *Planning a hybrid cycle* for (29), (30), (31) can be done using arguments of Section II. Two relations from [16]

$$r = p_0, \qquad \theta = \phi - p_1 \tag{32}$$

parameterize behavior of the polar coordinates of the center of mass of the CS as functions of the other coordinate ϕ. The class of systems (10) parameterized by $P = (p_0, p_1)$ looks as

$$[J - mp_0\cos p_1 d(\phi)]\,\ddot\phi - mp_0\sin p_1 d(\phi)\dot\phi^2 - mgd(\phi)\cos\phi = 0 \tag{33}$$

and the 2-D manifolds (6) are $Z(P) = \{[q; \dot q] : r = p_1, \dot r = 0, \theta = \phi - p_1, \dot\theta = \dot\phi\}$. In turn, the curves γ_- and γ_+ defined in (11) become intervals of straight lines, e.g.

$$\gamma_- = \left\{r = p_1, \dot r = 0, \theta = \phi - p_1, \dot\theta = \dot\phi = \dot\phi_\bullet(0), \frac{3\pi}{2} < \phi < \frac{5\pi}{2}\right\}.$$

The four equations in (12) can be rewritten as one with respect to $a = \phi_\bullet(0)$, the angle in the beginning of the cycle

$$\left[\exp\left\{\int_a^{a+2\pi} \frac{2mp_0\sin p_1 d(\tau)d\tau}{J - mp_0\cos p_1 d(\tau)}\right\} - 1\right]\dot\phi_\bullet^2(0) = \int_a^{a+2\pi} \exp\left\{\int_{a+2\pi}^{s} \frac{-2mp_0\sin p_1 d(\tau)d\tau}{J - mp_0\cos p_1 d(\tau)}\right\} \frac{2mgd(s)\cos s ds}{J - mp_0\cos p_1 d(s)}.$$

Depending on physical parameters and the initial velocity $\dot\phi_\bullet(0)$, this equation might have solutions or not. The phase portrait of (33) with $p_0 = 0.05$, $p_1 = \pi/2$ and $m = 0.2$ [kg], $J = 0.01$ [kgm^2], $\rho =$

[5]The hand-stick dynamics is neglected and several other simplifying assumptions are made. It is assumed that r can be only positive so that there is no problem with singularity of dynamics at $r = 0$.

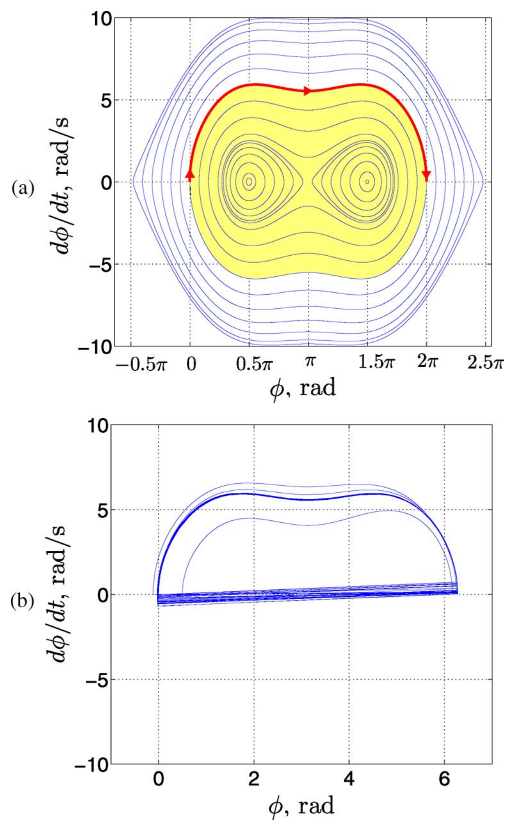


Fig. 3. (a) Phase portrait of system (33). The particular trajectory representing the boundary of shaded area has the zero velocity at the beginning and the end of sub-arc: $\dot{\phi}_\bullet(T) = \dot{\phi}_\bullet(0) = 0$. (b) Evolution of ϕ versus $\dot{\phi}$ along the solution of the closed-loop system with initial conditions outside the desired cycle and added measurement noise.

0.03 [m], $g = 9.81$ [kg/s^2], $d_0 = \rho\pi$ is depicted in Figs. 2(b) and 3(a). The shaded area of the phase portrait possesses no solution that satisfies $\phi(T) = \phi(0) + 2\pi$. Meanwhile, the trajectories indicated by bold red arrows satisfy also $\dot{\phi}(T) = \dot{\phi}(0)$, so they can be sub-arcs of a desired hybrid cycle.

Orbital stabilization of the hybrid cycle generated by the relations (32) and the motion of $\phi_\bullet(\cdot)$ depicted on Fig. 3(a), can be achieved via stabilization of a transverse linearization and the modification of the controller proposed[6] in Theorem 2. This motion is of period $T_h \approx 2.854$ s, coefficients of (20) for (29) along this motion are $b_1(\tau) = [-md(\phi_\bullet(\tau))\dot{\phi}_\bullet(\tau)/J, 0]$, $a_{11}(\tau) = p_0 \cdot m \cdot d(\phi_\bullet(\tau)) \cdot \dot{\phi}_\bullet(\tau)/J$, $a_{12}(\tau) = (m \cdot d(\phi_\bullet(\tau)) \cdot \dot{\phi}_\bullet(\tau)/J)[\dot{\phi}_\bullet^2(\tau), p_0 \cdot \ddot{\phi}_\bullet(t)]$, and $a_{13}(\tau) = [0, 2p_0 \cdot m \cdot d(\phi_\bullet(\tau)) \cdot \dot{\phi}_\bullet^2(\tau)/J]$. To compute $d^{TS}F$ for this cycle, we observe that the linearization of $F(\cdot)$ in (31) is the identity, and the hyperplanes $T\Gamma_-$ and $TS(T_h)$ as well as $T\Gamma_+$ and $TS(0)$ are different, their normals are

$$\vec{n}(0) = [0,0,0,0,1,1]^T, \quad \vec{m}_+ = [0,0,0,0,0,1]^T,$$
$$\vec{n}(T_h) = -\vec{n}(0), \quad \vec{m}_- = -\vec{m}_+$$

However, the operators $P^+_{\vec{n}(0)}$ and $P^-_{\vec{n}(T_h)}$ are inverse of each other, so that $d^{TS}F$ is the identity as well, see (23). The stabilizing controller for this linear hybrid system has been found and modified as in Theorem 2. A representative solution of the closed-loop system with added measurement noise is depicted in Fig. 3(b).

[6]The restriction of the dynamics (29) and (31) to Z does not have this cycle orbitally asymptotically stable.

V. Conclusion

We have considered the class of impulsive mechanical system with underactuation degree one. Motion planning problem for systems in the class that are not feedback linearizable and non-minimum phase is challenging since the lack of control inputs does not allow assigning trajectories for each degree of freedom independently. Necessary conditions for existence of a hybrid periodic motions with one jump in the form of a set of algebraic equations are obtained. As shown, they provide basis for a new algorithm for motion planning.

For a nontrivial hybrid periodic trajectory, a linearization of controlled dynamics transversal to its orbit is computed in closed form. The obtained linear control system is defined on the most natural choice of a moving Poincaré section and allows not only analyzing stability for the closed-loop system with a given control law but also designing an exponential orbitally stabilizing controller.

The results are illustrated on a problem of stabilizing a permanent rotation for a simplified model of the devil stick. Another example of applying the presented ideas can be found in [6], where an impulsive model for a three-link walking robot is successfully treated.

Appendix

A. *Proof of Theorem 2*

In a vicinity of the orbit (4) the dynamics (1) are [7], [8]

$$\begin{aligned}\alpha(\theta)\ddot{\theta}+\beta(\theta)\dot{\theta}^2+\gamma(\theta) &= g_y(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})y+g_{\dot{y}}(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})\dot{y}\\ &\quad + g_v(\theta,\dot{\theta},y,\dot{y})v,\\ \ddot{y} &= v\end{aligned} \tag{34}$$

where the left-hand side of the first equation in (34) coincides with (5); $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$, $g_v(\cdot)$ are smooth vector functions of appropriate dimensions. The system (34) makes an introduction of transverse coordinates for (1) transparent. Indeed, the transverse coordinates (17) are a part of the state vector $[\psi; x_\perp] = [\psi; I; y; \dot{y}]$ for (34) introduced instead of $[\theta; \dot{\theta}; y; \dot{y}]$, see the arguments after (17). Rewriting the dynamics of (1) in such coordinates might be difficult, however a linearization of the dynamics for $[I; y; \dot{y}]$ along the motion $q_\star(t)$ can be analytically found.

If, for instance, the vector of the initial conditions $[\psi(0); x_\perp(0)]$ belongs to the hypersurface $S(0)$, which is the first one from the moving Poincaré section, defined as a family $\{S(t)\}_{t\in[0,T_h]}$ associated with the continuous-in-time arc of the cycle $q_\star(t)$, then the solution of the closed-loop system (1) with the controller (21), (27) can be approximated for the time interval from 0 to the first instant it reaches Γ_-, as follows:

$$\begin{aligned}&\psi(t) = \psi_\star(\tau) + o(\cdot), \quad I(t) = I_\bullet(\tau) + o(\cdot)\\ &y(t) = Y_{1\bullet}(\tau) + o(\cdot), \quad \dot{y}(t) = Y_{2\bullet}(\tau) + o(\cdot)\\ &o(\cdot) = o\left(|I(0)| + |y(0)| + |\dot{y}(0)|\right).\end{aligned} \tag{35}$$

Here, $I_\bullet(\tau)$, $Y_{1\bullet}(\tau)$, and $Y_{2\bullet}(\tau)$ are components of the state $X(\tau)$ at $\tau = t + o(\cdot)$ of the linear system (20) with the linear feedback controller (26) and with the initial conditions on the hyperplane $TS(0)$ defined as: $I_\bullet(0) = I(0)$, $Y_{1\bullet}(0) = y(0)$, $Y_{2\bullet}(0) = \dot{y}(0)$.

The time moment T when the solution of the nonlinear closed-loop system (1), (21), (27) hits the switching surface Γ_- for the first time, i.e. $[\psi(T); I(T); y(T); \dot{y}(T)] \in \Gamma_-$, is approximated by $T = T_h + o(\cdot)$ and can be larger or smaller than the period T_h. However, the relations (35) imply that if we consider the solution at time moment $t = T_h$ (making if necessary the switching rule non-active), then the approximations

$$\begin{aligned} \psi(T_h) &= \psi_\star(T_h) + o(\cdot), \quad I(T_h) = I_\bullet(T_h) + o(\cdot) \\ y(T_h) &= Y_{1\bullet}(T_h) + o(\cdot), \quad \dot{y}(T_h) = Y_{2\bullet}(T_h) + o(\cdot) \end{aligned} \tag{36}$$

hold. Here the vector $X(T_h) = [I_\bullet(T_h); Y_{1\bullet}(T_h); Y_{2\bullet}(T_h)]$ belongs to the hyperplane $TS(T_h)$ and the state vector $[\psi(T_h); I(T_h); y(T_h); \dot{y}(t_h)]$ belongs to the hypersurface $S(T_h)$.

To find a point, where the solution $[\psi(\cdot); I(\cdot); y(\cdot); \dot{y}(\cdot)]$ of the closed-loop system hits Γ_-, we can use the approximation for $[\psi(T_h); I(T_h); y(T_h); \dot{y}(t_h)]$ in (36) translated along the direction of $\vec{n}(T_h)$ defined in (19) until it intersects Γ_- (note that such a translation is the first-order approximation for the trajectories of the nonlinear system). Here the update law is activated and further maps this point into the one on Γ_+. To find where the solution of the closed-loop system, originated from the point on Γ_+, hits the hypersurface $S(0)$, we can translate it along $\vec{n}(0)$ until it intersects $S(0)$. The same procedure can be used for approximating the point on the hyperplane $T\Gamma_-$, mapping it by the linear operator dF to a point on $T\Gamma_+$, and translating the image along $\vec{n}(0)$ until hitting $TS(0)$, see [6].

These steps repeated for $(2n-1)$ independent sets of initial conditions on $S(0)$ allow us computing a linear approximation, the monodromy matrix, for the first-return Poincaré map defined on this hypersurface, see [13]. As a result, asymptotic stability of the origin for the linear impulsive system is equivalent to exponential orbital stability of the periodic motion $q_\star(t)$ for the nonlinear one. ∎

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REFERENCES

[1] R. Goebel, R. Sanfelice, and A. Teel, "Hybrid dynamical systems," *IEEE Control Syst. Mag.*, vol. 4, pp. 28–93, 2009.

[2] D. Liberzon, *Switching in Systems and Control*. Cambridge, MA: Birkhaüser, 2003.

[3] E. Westervelt, J. Grizzle, C. Chevallereau, J. Choi, and B. Morris, *Feedback Control of Dynamic Bipedal Robot Locomotion*. Orlando, FL: CRC Press, 2007.

[4] D. Bainov and P. Simeonov, *Systems With Impulse Effects: Stability, Theory and Applications*, ser. Ellis Horwood Series: Mathematics and its Applications. Chichester, U.K.: Ellis Horwood, 1989.

[5] S. Nersesov, V. Chellaboina, and W. Haddad, "A generalization of Poincaré's theorem to hybrid and impulsive dynamical systems," *Int. J. Hybrid Syst.*, vol. 2, no. 1, pp. 39–55, 2002.

[6] L. Freidovich, A. Shiriaev, and I. Manchester, "Stability analysis and control design for an underactuated walking robot via computation of a transverse linearization," in *Proc. 17th IFAC World Congress*, Seoul, Korea, 2008, pp. 10166–10171.

[7] A. Shiriaev, J. Perram, and C. Canudas-de-Wit, "Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach," *IEEE Trans. Automat. Control*, vol. 50, no. 8, pp. 1164–1176, Aug. 2005.

[8] A. Shiriaev, L. Freidovich, and I. Manchester, "Can we make a robot ballerina perform a pirouette? Orbital stabilization of periodic motions of underactuated mechanical systems," *Annu. Rev. Control*, vol. 32, no. 2, pp. 200–211, 2008.

[9] J. Perram, A. Shiriaev, C. Canudas-de-Wit, and F. Grognard, "Explicit formula for a general integral of motion for a class of mechanical systems subject to holonomic constraint," in *Proc. 2nd IFAC Workshop Control Methods Lagrangian Hamiltonian Syst.*, Seville, Spain, 2003, pp. 99–104.

[10] G. Leonov, "Generalization of the Andronov-Vitt theorem," *Regular and Chaotic Dynamics*, vol. 11, no. 2, pp. 281–289, 2006.

[11] M. Urabe, *Nonlinear Autonomous Oscillations*. New York: Academic Press, 1967.

[12] J. Hauser and C. Chung, "Converse Lyapunov functions for exponentially stable periodic orbits," *Syst. Control Lett.*, vol. 23, no. 1, pp. 27–34, 1994.

[13] A. Shiriaev, L. Freidovich, and S. Gusev, "Transverse linearization for controlled mechanical systems with several passive degrees of freedom," *IEEE Trans. Autom. Control*, to be published.

[14] C. Nielsen and M. Maggiore, "On local transverse feedback linearization," *SIAM J. Control Optim.*, vol. 7, no. 5, pp. 2227–2250, 2008.

[15] G. Song and M. Zefran, "Stabilization of hybrid periodic orbits with application to bipedal walking," in *Proc. Amer. Control Conf.*, Minneapolis, MN, 2006, pp. 2504–2509.

[16] S. Nakaura, Y. Kawaida, T. Matsumoto, and M. Sampei, "Enduring rotatory motion control of devil stick," in *Proc 6th IFAC Symp. Nonlin. Control Syst.*, Stuttgart, Germany, 2004, pp. 1073–1078.