# Constructive Tool for Orbital Stabilization of Underactuated Nonlinear Systems: Virtual Constraints Approach

Anton Shiriaev, John W. Perram, and Carlos Canudas-de-Wit

***Abstract*—We present a constructive tool for generation and orbital stabilization of periodic solutions for underactuated nonlinear systems. Our method can be applied to *any* mechanical system with a number of independent actuators smaller than the number of degrees of freedom by one. The synthesized feedback control law is nonlinear and time-dependent. It is derived from a feedback structure that explicitly uses the *general* or *full integral* of the systems zero dynamics. The control law generates a periodic solution and makes it exponentially orbitally stable.**

***Index Terms*—Constructive control procedures, nonlinear and time-varying controllers, orbital stabilization.**

## I. Introduction

THE problem of orbital stabilization arises from applications, where the desired operation mode is oscillatory. In many cases, the oscillations, however, are not present in the open-loop dynamics. Therefore, it is relevant to study new control design methods forcing the system dynamics to exhibit a limit cycle. Walking mechanisms, inverted pendulums (the full system state should behave periodically), rotating machines (the internal states, i.e., current and flux, are oscillatory if a torque output is kept constant), vertically landing aircraft on an oscillating platform (e.g., an aircraft carrier) are examples of such systems. The goal of the feedback design is either to render an existing periodic motion orbitally stable or to generate a new periodic motion and to ensure its orbital stability.

### A. Literature Review

Forcing orbitally stable oscillations via feedback and analysis of such oscillations in nonlinear fully actuated mechanical systems is an old area of research; several well-known methods are available in the literature; see, for example, [1] and [3]. The problem of forcing oscillations via feedback in underactuated systems has a more recent origin. Hauser and Choo (see [10]) have developed an analysis framework for the computation of Lyapunov functions, allowing to determine if the present limit cycle is exponentially stable. They have applied their approach to the cart-pendulum system, however, no procedure to ensure the existence of the limit cycle has been proposed. In [2], the authors also investigate induced oscillations for the cart-pendulum system as well as for the Furuta pendulum. They propose to match a desired closed-loop oscillatory structure by a suitable control design precedent by partial feedback linearization. Although the method ensures the existence of an orbit in a subset of the state variables, it does not, however, assess the stability of the other state variables. The approach of [19] is to compute suboptimal controllers for orbital stabilization. A drawback of this method, as well as for the one of [2], is that for the needed attenuation properties to hold, it is necessary to make *a priori* hypothesis on $L_2$ bounds of the internal states of the system. Feedback linearization with warranted stability of the zero-dynamics is another approach that can be applied for a restricted class of underactuated systems (systems with cyclic underactuated joint) (see [7] for details). Recently, the classical methods of absolute stability [23], [25] have been applied in [11], [12], [14] to generate oscillations. The problem of periodic stabilization of walking mechanism has been studied in [5], [6], and [22]. A class of particular orbits for underactuated mechanical systems has been recently investigated in [16] and [17]. An original method to match a particular oscillatory exo-system or a given closed-curve has been proposed in [4]. The solution presented below extends this idea. However, unlike [4], we present a *constructive* method for control design to stabilize a *prespecified* orbit.

### B. Main Idea

We consider an underactuated controlled Euler–Lagrange system

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = B(q)u. \tag{1}$$

Here, $q$, $\dot{q}$ are vectors of generalized coordinates and velocities; $u$ is a vector of independent control inputs, the function

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q) \tag{2}$$

is a Lagrangian of (1), $M(q)$ is a positive–definite matrix of inertia, $V(q)$ is a potential energy of the systems, and $B(q)$ is a full rank matrix function of appropriate dimensions.

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The underactuation in (1) means that

$$\dim\, u < \dim\, q \tag{3}$$

i.e., the number of actuators in (1) is less than the number of its degrees of freedom.

Our goal is to derive a constructive method to generate an exponentially orbitally stable periodic solution of (1).

We considers only the simplest case of underactuation

$$\dim\, u = \dim\, q - 1.$$

Under this simplifying assumption, we show that if feedback stabilization of an arbitrarily chosen set, defined by a system of $(n-1)$ independent geometric relations [1] (or by the sliding surface determined by them)

$$q_1 = \phi_1(\theta) \quad q_2 = \phi_2(\theta), \quad \ldots, \quad q_n = \theta$$

is achieved, then zero-dynamics[2] of the mechanical system (1) can be written as

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0. \tag{4}$$

Moreover, it turns out that (4) has a general integral of motion $I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0)$. The function $I$ preserves its values along the solutions of (4) $[\theta(t), \dot{\theta}(t)]$, provided the initial conditions $[\theta_0, \dot{\theta}_0]$ are chosen appropriately. We derive the explicit form of $I$ and analyze some of its properties. Existence of the conserved quantity implies that the zero-dynamics (4) has no asymptotically stable solutions. We present some examples where (4) has unbounded solutions and solutions with finite escape time.

In the case when (4) has at least one periodic solution,[3] and under certain technical assumptions the dynamics of the system (1) can be transformed into an almost linear form of a new kind. This transformation allows us to reduce the complexity of the stability analysis for a high-dimension nonlinear system to stability analysis for an *auxiliary linear periodic in time controlled system of lower dimension*.[4] We show that if this auxiliary linear system is controllable over a period, then one can proceed with a modified LQR design in order to derive a nonlinear time-varying feedback control law that locally exponentially orbitally stabilizes the chosen periodic motion.

The rest of the paper is organized as follows. In Section II, we introduce the concept of *virtual holonomic constraint* and explore some properties of the resulting zero dynamics, i.e., of the *virtual limit system*. In Section III, we describe a control design for orbital stabilization of a chosen periodic motion. An illustrative example is presented in Section IV. We conclude with some remarks in Section V.

[1]These geometrical relations are named later in the text as *virtual holonomic constraints*. Indeed, the adjective *holonomic* refers from one side to the class of nonlinear systems (1) considered, and from other side to the fact that these are only constraints on generalized coordinates. The adjective *virtual* refers to the fact that these constraint are not physical, and are to be reproduced by feedback control action.

[2]This dynamical system (4) is named later in the text as a *virtual limit system*.

[3]Sufficient (and almost necessary) conditions for this are given in [18].

[4]It is worth to mention that this linear system is not derived via linearization of (1) around the periodic solution.

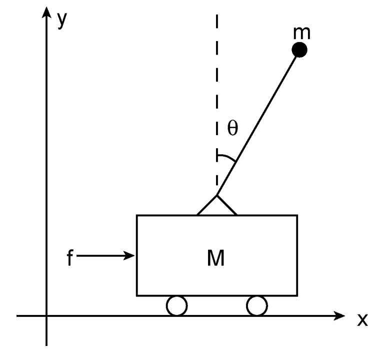


Fig. 1. Cart pendulum system.

## II. Virtual Constraints as a Tool for Orbital Stabilization

Let us start with the cart-pendulum motivating example. For this system, we show the main idea of our design and illustrate how to use a virtual constraint to generate a motion. In addition, we discuss differences between *virtually* and *physically* constrained systems and describe some general properties of the virtually constrained Lagrangian systems (1) with one-degree of underactuation.

### A. Illustrating Example

In order to illustrate some elements of the proposed control design and motivate the further theoretical development we investigate the following control problem. Consider the cart-pendulum system shown on Fig. 1.

Our goal is *to design a control law that ensures the presence of the pendulum oscillations around the upright equilibrium*.[5]

For simplicity, we assume that the mass $m$ of the pendulum, the mass $M$ of the cart, and the distance $l$ between the center of mass of the pendulum and the suspension point are all equal to 1. The dynamics of the system can be described by

$$2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = u_f \tag{5}$$

$$\cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0 \tag{6}$$

where $x$ is the horizontal displacement of the cart; $\theta$ is the angle between the pendulum rod and the vertical, which is zero at the upright position; and $u_f$ is the force applied to the cart in the horizontal direction, which is generated by the controller.

In order to solve the control problem posted above, we consider a *virtual constraint*. Suppose the following geometrical relation between the position of the cart and the angle of the pendulum is imposed:

$$x(t) + L \cdot \sin\theta(t) = a \tag{7}$$

where $L$ and $a$ are given constants, the parameters of the constraint. The relation (7) means that a particular point of the pendulum's rod (on the distance of $L$ from the suspension point) is preserved on the vertical line $x = a$.

Suppose there exists a static feedback control law, which ensures that the constraint (7) is invariant. Then, the closed-loop system under this control law can be rewritten as

$$\ddot{x} + L\cos\theta \cdot \ddot{\theta} - L\sin\theta \cdot \dot{\theta}^2 = 0 \tag{8}$$

$$\cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0. \tag{9}$$

[5]In Section IV, we show how to render this newly generated oscillation orbitally stable.

Here, (8) is derived by differentiating (7) twice. One can eliminate $\ddot{x}$ from the system (8) and (9) and obtain the equation

$$(1 - L \cdot \cos^2\theta)\ddot{\theta} + L \cdot \cos\theta \cdot \sin\theta \cdot \dot{\theta}^2 - g \cdot \sin\theta = 0. \quad (10)$$

We observe the following.

- If a controller stabilizes the relation (7), then solutions of the closed-loop system converge to some solutions of (10).
- For any value $L$, the function
$$U(\theta, \dot{\theta}) = [1 - L\cos^2\theta]\dot{\theta}^2 + 2g \cdot \cos\theta \quad (11)$$
is constant along each solution of (10), provided it is well-defined.[6]
- Existence of the conserved quantity (11) qualitatively defines behavior of the solutions of (7). Clearly, there are no asymptotically stable motions and, moreover, some (or almost all) solutions are unbounded and some escape to infinity in finite time dependent on a value of the parameter $L$.
- Equation (10) describes, in fact, not one system but rather a family of systems parameterized by the constraint parameter $L$.
- In order to generate oscillations, we need to identify the values of $L$, such that the system (10) has a limit cycle around its equilibrium $\theta = 0$.
- It could be shown that for $L < 1$, the linearization of (10) around $\theta = 0$ has a saddle. Hence, existence of the conserved quantity (11) does not guarantee a periodic orbit in a vicinity of the equilibrium $\theta = 0$.

Here, we describe those values of the parameter $L$, for which (10) has a family of periodic solutions around the upright equilibrium.

*Proposition 1:* If $L > 1$, then *the virtual limit system* (10) has a center at the upright equilibrium

$$\theta = 0 \quad \dot{\theta} = 0. \quad (12)$$

*Proof:* is based on the Lagrange–Dirichlet stability criterion. We check whether the Hessian

$$H = \begin{bmatrix} \frac{\partial^2 U}{\partial \dot{\theta}^2} & \frac{\partial^2 U}{\partial \theta \partial \dot{\theta}} \\ \frac{\partial^2 U}{\partial \theta \partial \dot{\theta}} & \frac{\partial^2 U}{\partial \theta^2} \end{bmatrix}$$

of the first integral $U(\theta, \dot{\theta})$; see (11), at the equilibrium (12) is sign definite. The direct calculations

$$H = \left.\begin{bmatrix} 2(1 - L\cos^2\theta) & 4\dot{\theta}\cos\theta\sin\theta \\ 4\dot{\theta}\cos\theta\sin\theta & 2\dot{\theta}^2\cos 2\theta - 2g\cos\theta \end{bmatrix}\right|_{\theta=\dot{\theta}=0} = \begin{bmatrix} 2(1-L) & 0 \\ 0 & -2g \end{bmatrix}$$

show that the Hessian is negative definite when $L > 1$. Hence, the equilibrium (12) is stable in the sense of Lyapunov. In turn, the level sets of the function $U$ are closed curves around (12). They define the periodic orbits of the system (10). ■

[6]The solution may fail to exist; for example, if $L = 1$, then for any initial conditions $[\theta_0, \dot{\theta}_0]$ with $\theta_0 = 0$ the solution is not well defined.

Conclusions from Proposition 1 are as follows. The constraint (7) can be used for generating oscillation of the pendulum around the upright equilibrium. All possible oscillations, based on the constraint (7), are solutions of the system (10). Finally, the constraint (7) ensures that the position of the cart remains bounded and periodic with the same period independent of a chosen oscillation of $\theta$.

### B. Physical Constraints Versus Virtual Ones

It might be thought that an existence of the first integral (11) of (10) is either the consequence of particular choice of the constraint (7) and something exceptional or the consequence of the well-known reduction procedure of classical mechanics due to D'Alembert. In the next section, we show that the observed integrability in the example is a general fact, valid for any controlled $n$ degrees of freedom Lagrangian system (1) subject to $(n-1)$ holonomic *virtual* constraints. Note, however, that this property cannot deducted from the classical reduction principle due to D'Alembert. To illustrate this point, let us show that the derived dynamical system (10) differs from the system (5), (6) when the constraint (7) is present physically.

As well known from classical mechanics, the two-degrees of freedom Lagrangian system (5) and (6) with the *holonomic* constraint

$$F(x, \theta) = x + L\sin\theta - a = 0$$

is equivalent to a one-degree-of-freedom unconstrained Lagrangian system, which can be found via D'Alembert reduction principle. According to this principle, the dynamics of the physically constrained system with $u_f = 0$ is

$$2\ddot{x} + \cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2 = \lambda\frac{\partial}{\partial x}F(x, \theta) = \lambda \quad (13)$$

$$\cos\theta\ddot{x} + \ddot{\theta} - g\sin\theta = \lambda\frac{\partial}{\partial \theta}F(x, \theta) = \lambda L\cos\theta \quad (14)$$

where $\lambda$ is the Lagrangian multiplier. Eliminating $\lambda$ and $\ddot{x}$ from (13), (14), and (8), we obtain the following 1-degree of freedom Lagrangian system:

$$\left(1 - 2L(1-L)\cos^2\theta\right)\ddot{\theta} + L(1-L)\sin(2\theta)\dot{\theta}^2 - g\sin\theta = 0. \quad (15)$$

To compute the energy of the reduced system (15), one could use the energy of the original cart-pendulum system (5) and (6). Namely, the energy $E(\theta, \dot{\theta})$ of (15) is

$$E(\theta, \dot{\theta}) = \left(1 - 2L(1-L)\cos^2\theta\right)\dot{\theta}^2 + 2g\cos\theta. \quad (16)$$

It is readily seen that the systems (10) and (15) are different since their energies (11) and (16) are different.

### C. Properties of Virtual Limit System

Given a controlled Lagrangian system (1) of $n$-degrees of freedom and with $(n-1)$ actuators (i.e., $\dim u = \dim q - 1$), consider the following geometrical relations:

$$q_1 = \phi_1(\theta, c) \quad q_2 = \phi_2(\theta, c), \quad \ldots, \quad q_n = \phi_n(\theta, c) \quad (17)$$

imposed among the generalized coordinates $q_1, \ldots, q_n$, and the new independent scalar variable $\theta$. Here, $\phi_1, \ldots, \phi_n$ are smooth functions of $\theta$, parametrized by a constant vector[7] $c$.

*Proposition 2:* Suppose that there exists a control law $u^*$ for (1) that makes the relations (17) invariant along solutions of the closed-loop system. Then, $\theta$ is a solution of the system

$$\alpha(\theta,c)\ddot{\theta} + \beta(\theta,c)\dot{\theta}^2 + \gamma(\theta,c) = 0 \tag{18}$$

where $\alpha(\theta,c)$, $\beta(\theta,c)$ and $\gamma(\theta,c)$ are scalar functions.[8] ■

*Proof:* The invariance of the relations (17) implies that

$$\begin{aligned} \frac{d}{dt}q_i &= \phi_i'(\theta,c)\frac{d}{dt}\theta \\ \frac{d^2}{dt^2}q_i &= \phi_i''(\theta,c)\left(\frac{d}{dt}\theta\right)^2 + \phi_i'(\theta,c)\frac{d^2}{dt^2}\theta \end{aligned} \tag{19}$$

where $i = 1, \ldots, n$. The controlled Lagrangian system (1) can be rewritten as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)u \tag{20}$$

where the matrix $C(q,\dot{q})$ depends on $\dot{q}$ linearly. Substituting (17) and (19) into (20) with $u = u^*$, we obtain the following system of $n$ second-order differential equations:

$$M(\Phi(\theta))\left[\Phi'(\theta)\ddot{\theta} + \Phi''\dot{\theta}^2\right] + C\left(\Phi(\theta), \Phi'(\theta)\dot{\theta}\right)\Phi'(\theta)\dot{\theta} + G(\Phi(\theta)) = B(\Phi(\theta))\,u^* \tag{21}$$

where $\Phi(\theta)$, $\Phi'(\theta)$ and $\Phi''(\theta)$ are the vectors

$$\begin{aligned} \Phi(\theta) &= [\phi_1(\theta,c),\ \phi_2(\theta,c),\ \ldots,\ \phi_n(\theta,c)]^T \\ \Phi'(\theta) &= [\phi_1'(\theta,c),\ \phi_2'(\theta,c),\ \ldots,\ \phi_n'(\theta,c)]^T \\ \Phi''(\theta) &= [\phi_1''(\theta,c),\ \phi_2''(\theta,c),\ \ldots,\ \phi_n''(\theta,c)]^T. \end{aligned} \tag{22}$$

Since the rank of the smooth $n \times (n-1)$ matrix function $B(q)$ is equal to $(n-1)$, there exists a $1 \times n$ raw function $B^{\perp}(q)$ such that $B^{\perp}(q)B(q)u^* = 0 \forall q$. It is not hard to check that the functions $\alpha$, $\beta$, and $\gamma$ in (18) can be computed as follows[9]:

$$\begin{aligned} \alpha(\theta) &= B^{\perp}(\Phi(\theta))\,M(\Phi(\theta))\,\Phi'(\theta) \\ \beta(\theta) &= B^{\perp}(\Phi(\theta))[C(\Phi(\theta), \Phi'(\theta))\,\Phi'(\theta) + M(\Phi(\theta))\,\Phi''(\theta)] \\ \gamma(\theta) &= B^{\perp}(\Phi(\theta))\,G(\Phi(\theta)). \end{aligned}$$

This completes the proof. ■

It is important to investigate properties of (18) because it is clear that if a feedback controller renders the relations (17) asymptotically invariant, then the solutions of the system with such a controller in the loop asymptotically tend to solutions of (18). The next properties of (18), derived in [13], are of special interest.

*Theorem 1 [13]:* Suppose the function $\alpha(\theta)$ has only isolated zeros. If the solution $[\theta(t), \dot{\theta}(t)]$ of (18) with initial conditions $\theta(0) = \theta_0$, $\dot{\theta}(0) = \dot{\theta}_0$ exists and is continuously differentiable, then along this solution the function

$$I(\theta,\dot{\theta},\theta_0,\dot{\theta}_0) = \dot{\theta}^2 - \psi(\theta_0,\theta)\left[\dot{\theta}_0^2 - \int_{\theta_0}^{\theta} \psi(s,\theta_0)\frac{2\gamma(s)}{\alpha(s)}ds\right] \tag{23}$$

with

$$\psi(\theta_0,\theta_1) = \exp\left\{-2\int_{\theta_0}^{\theta_1} \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\} \tag{24}$$

preserves its zero value. ■

*Proof:* See Appendix I-A. It is worth to notice that even the solution $[\theta(t), \dot{\theta}(t)]$ is unbounded the property stated in Theorem 1 holds.

*Theorem 2 [13]:* With $x$ and $y$ being some constants, the time derivative of the function $I(\theta, \dot{\theta}, x, y)$ defined by (23), calculated along a solution $[\theta(t), \dot{\theta}(t)]$ of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = u \tag{25}$$

can be computed as

$$\frac{d}{dt}I = \dot{\theta}\left\{\frac{2}{\alpha(\theta)}u - \frac{2\beta(\theta)}{\alpha(\theta)}I\right\}. \tag{26}$$

*Proof:* See Appendix I-B.

## III. Controller Design

In this section, we present main results of this paper: A family of feedback control laws and conditions ensuring exponential orbital stabilization of periodic solutions of the *virtual limit system* (18).

### A. Choice of Periodic Solution

Given the *virtual constraints* (17), suppose that there exists a vector of parameters $c$ such that the resulted *virtual limit system* (18) has a $T$-periodic solution, i.e.,

$$\theta_\gamma(t) = \theta_\gamma(t+T) \qquad \forall t. \tag{27}$$

Here, existence of a periodic orbit of the dynamical system (18) for a certain value of $c$ is postulated.[10]

The problem is *to determine a feedback controller that guarantees invariance of the chosen virtual constraints* (17) *and an orbital asymptotic stability of the chosen periodic solution* (27) *for the closed-loop system.*

### B. Partial Feedback Linearization

Given the *virtual constraints* (17), introduce new coordinates for (1) as follows:

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_n = q_n - \phi_n(\theta). \tag{28}$$

[7]For the cart-pendulum example these relations correspond to $q_1 = a - L \cdot \sin\theta$, $q_2 = \theta$, and the vector $c$ has two components $L$ and $a$.

[8]The explicit expressions are given in the proof.

[9]Here and later the dependence on $c$ is omitted.

[10]In general, it is a difficult problem since, as is well known, existence of periodic solutions in nonlinear systems does not follow from stability of the linearization. The problem is solved in [18].

The $(n+1)$ scalar quantities $y_1, y_2, \ldots, y_n$ and $\theta$ are excessive coordinates for the controlled $n$-degrees of freedom Euler–Lagrange system (1). Therefore, one of these $(n+1)$ coordinates could be always locally expressed as a function of the others and excluded from the consideration. Let us assume that this is the case for $y_n$, and new independent coordinates are

$$y = (y_1, \ldots, y_{n-1})^T \quad \text{and} \quad \theta.$$

Thus, the last equality in (28) could be rewritten as

$$q_n = \phi_n(\theta) + h(y_1, \ldots, y_{n-1}, \theta) \tag{29}$$

where $h$ is a scalar smooth function of its arguments. It is readily seen that the first and second time derivatives of $y$ and $\theta$ are related to the original coordinates $q$ and their time derivatives as follows:

$$\dot{q} = L(\theta, y)\begin{bmatrix}\dot{y}\\ \dot{\theta}\end{bmatrix},\ \ddot{q} = L(\theta, y)\begin{bmatrix}\ddot{y}\\ \ddot{\theta}\end{bmatrix} + N(\theta, \dot{\theta}, y, \dot{y}) \tag{30}$$

where $N$ is $n \times 1$ matrix function and

$$L(\theta, y) = \begin{bmatrix} I_{n-1},\ 0_{(n-1)\times 1} \\ \operatorname{grad} h \end{bmatrix} + \left[0_{n\times(n-1)},\ \Phi'\right] \tag{31}$$

where $\operatorname{grad} h = [(\partial h/\partial y_1), \ldots, (\partial h/\partial y_{n-1}), (\partial h/\partial \theta)]$ and the vector function $\Phi'$ is defined in (22).

In the newly introduced coordinates, the main problem of this paper is to make the following periodic solution:

$$\theta_\gamma(t) = \theta_\gamma(t+T) \quad y(t) = 0_{(n-1)\times 1} \tag{32}$$

of the closed-loop system orbitally asymptotically stable.

*Proposition 3:* Suppose the $n \times n$ matrix function $L(\theta, y)$, introduced in (31), and the $(n-1) \times (n-1)$ matrix function $K(\theta, y)$ defined as follows:

$$K = [I_{n-1}, \mathbf{0}]L(\theta, y)^{-1}\left[M(q)^{-1}B(q)\right]\Big|_{\substack{q_1 = y_1 + \phi(\theta)\\ \cdots \\ q_{n-1} = y_{n-1} + \phi_{n-1}(\theta)\\ q_n = \phi(\theta) + h(y, \theta)}} \tag{33}$$

with $\mathbf{0} = 0_{(n-1)\times 1}$, are both nonsingular in a vicinity of the orbit of (32). Then, the feedback transformation[11]

$$u = K(y, \theta)^{-1}\left[v - R(y, \theta, \dot{y}, \dot{\theta})\right] \tag{34}$$

well defined in this vicinity, brings the dynamics of the controlled Euler–Lagrange system (1) into the partly linear form

$$\begin{aligned}\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y}\\ &\quad + g_v(\theta, \dot{\theta}, y, \dot{y})v \end{aligned}\tag{35}$$

$$\ddot{y} = v \tag{36}$$

where the left-hand side of (35) matches the structure of the *virtual limit system* (18) and $g_y$, $g_{\dot{y}}$, $g_v$ are smooth functions of appropriate dimensions. ■

*Proof:* See Appendix I-C.

Based on Theorem 2, one can introduce new differential relations

$$\dot{I} = \frac{2\dot{\theta}}{\alpha(\theta)}\left\{g_y(\cdot)y + g_{\dot{y}}(\cdot)\dot{y} + g_v(\cdot)v - \beta(\theta)I\right\} \tag{37}$$

$$\ddot{y} = v. \tag{38}$$

It is important to notice that the systems (35), (36) and (37), (38) are of different orders and are not equivalent. Moreover, the system (37), (38) cannot be solved unless the function of time $\theta(t)$ is given.[12] Nevertheless, it will be shown here that this incomplete nonlinear system (37), (38) plays an important role in developing a stabilizing controller.

### *C. Feedback Stabilization of Auxiliary Linear Periodic in Time System*

As a step for controller design, let us consider the incomplete nonlinear system (37), (38), where the functions

$$\frac{\dot{\theta}g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})}{\alpha(\theta)}, \frac{\dot{\theta}g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})}{\alpha(\theta)}, \frac{\dot{\theta}g_v(\theta, \dot{\theta}, y, \dot{y})}{\alpha(\theta)}, \frac{\dot{\theta}\beta(\theta)}{\alpha(\theta)}$$

from the right-hand side of the (37) are evaluated along the chosen periodic solution (32). This means that we consider the following auxiliary linear system:

$$\dot{I} = \frac{2\dot{\theta}_\gamma(t)\{g_y(t)y + g_{\dot{y}}(t)\dot{y} + g_v(t)v - \beta(\theta_\gamma(t))I\}}{\alpha(\theta_\gamma(t))} \tag{39}$$

$$\ddot{y} = v \tag{40}$$

where the functions

$$\begin{aligned} g_y(t) &= g_y\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), \ddot{\theta}_\gamma(t), 0_{(n-1)\times 1}, 0_{(n-1)\times 1}\right)\\ g_{\dot{y}}(t) &= g_{\dot{y}}\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), \ddot{\theta}_\gamma(t), 0_{(n-1)\times 1}, 0_{(n-1)\times 1}\right)\\ g_v(t) &= g_v\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), 0_{(n-1)\times 1}, 0_{(n-1)\times 1}\right)\end{aligned}$$

are periodic in time. Let $\zeta \in R^{(2n-1)}$ be defined as

$$\zeta = [I, y, \dot{y}]^T.$$

The state–space representation of (39), (40) is

$$\dot{\zeta} = A(t)\zeta + b(t)v \tag{41}$$

where $A(t) \in R^{(2n-1)\times(2n-1)}$ is

$$A(t) = \begin{bmatrix} \kappa_1(t) & \kappa_2(t) & \kappa_3(t)\\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} & I_{(n-1)}\\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)}\end{bmatrix}$$

$0_{m\times l}$ is a $m \times l$ matrix with zero elements, $I_l$ is an $l \times l$ identity matrix, $b(t) \in R^{(2n-1)\times(n-1)}$ is

$$b(t)^T = \left[\rho(t),\ 0_{(n-1)\times(n-1)},\ I_{n-1}\right]$$

and the $T$-periodic functions $\kappa_i(t)$ and $\rho(t)$ are given in Appendix II-A, so that $A(t) = A(t+T)$ and $b(t) = b(t+T)$.

In the further development, we will be interested in situations when the auxiliary linear system (41) is completely controllable on the time interval $[0, T]$. This can be verified by the well-known tests (see, for example, [15, Th. 9.2, p. 143]), while the analytical computations for arbitrary functions $\kappa_i(t)$, $i = 1, 2, 3$, might be tedious.

Controllability of the linear periodic system (39), (40) allows to achieve its exponential stabilization (see, for example, [15, Th. 14.7, p. 245]). The next statement, taken from [24, Th. 3], describes one of the stabilizing controllers.

*Proposition 4:* Given an $(n-1)\times(n-1)$ matrix $\Gamma = \Gamma^T > 0$ and a $(2n-1) \times (2n-1)$ matrix $G = G^T > 0$, suppose that the system (39), (40) is completely controllable on $[0, T]$, then there exists a $(2n-1) \times (2n-1)$ matrix function $R(t)$,

[11]For the explicit form of the matrix function $R$, see the proof.

[12]$\theta(t)$ may be seen as an external signal to the system (37), (38).

$R(t) = R(t+T)$ and $R(t) = R(t)^T$ for all $t \in [0, T]$, that satisfies the Riccati equation

$$\dot{R}(t) + A(t)^T R(t) + R(t)A(t) + G = R(t)b(t)\Gamma^{-1}b(t)^T R(t) \tag{42}$$

$\forall t \in [0, T]$ with $A(t)$ and $b(t)$ defined in (41), and such that the feedback controller

$$v = -\Gamma^{-1}b(t)^T R(t)\zeta \tag{43}$$

renders the linear periodic system (41) exponentially stable. Moreover, along any solution $\zeta(t)$ of the closed-loop system (39), (40), (43), the following differential equality holds:

$$\frac{d}{dt}\left\{\zeta(t)^T R(t)\zeta(t)\right\} = -\zeta(t)^T G\zeta(t) - v(t)^T \Gamma v(t). \tag{44}$$

### D. Constructive Procedure for Control Design

It is not hard to show that, in general, the original nonlinear system (35), (36) cannot be stabilized by (43). However, the following *ad-hoc* modification:

$$v(t) = -\Gamma^{-1}b(\theta, \dot{\theta}, y, \dot{y})^T R(t)\zeta \tag{45}$$

where $R(t)$ is the stabilizing solution of the Riccati equation (42) and

$$b(\theta, \dot{\theta}, y, \dot{y})^T = \left[\frac{2\dot{\theta}g_v(\theta, \dot{\theta}, y, \dot{y})}{\alpha(\theta)},\ 0_{(n-1)},\ I_{(n-1)}\right] \tag{46}$$

exponentially orbitally stabilizing for the periodic solution (32). This is the main result of this paper.

*Theorem 3:* Given an underactuated controlled Euler–Lagrange system (1) with $n$ degrees of freedom $(\dim q = n)$ and with $n-1$ independent control inputs $(\dim u = n-1)$; assume that

1) there are a set of constraints $q_i = \phi_i(\theta, c)$, $i = 1, \ldots, n$, and a value of the vector $c$ such that the resulting virtual limit system (18) has a nontrivial $T$-periodic solution $\theta_\gamma(t)$;
2) the matrix functions $L(\cdot)$ and $K(\cdot)$, defined by (31) and (33), are nonsingular in some neighborhood of the orbit (32);
3) the linear periodic in time system (41) is completely controllable over the period $[0, T]$.

Take the control law (34), (45), where $R(t)$ and $\Gamma$ are from Proposition 4, $\zeta^T = [I(\theta, \dot{\theta}, \theta_\gamma(0), \dot{\theta}_\gamma(0)), y^T, \dot{y}^T]$ with $y$ and $\theta$ defined as explained after (28), and the scalar function $I$ defined by [see (23)]

$$\begin{aligned} I\left(\theta, \dot{\theta}, \theta_\gamma(0), \dot{\theta}_\gamma(0)\right) &= \dot{\theta}^2 - \psi\left(\theta_\gamma(0), \theta\right)\dot{\theta}_\gamma^2(0) \\ &\quad + \psi\left(\theta_\gamma(0), \theta\right)\int_{\theta_\gamma(0)}^{\theta} \psi\left(s, \theta_\gamma(0)\right)\frac{2\gamma(s)}{\alpha(s)}ds \end{aligned} \tag{47}$$

where $\psi(\theta_\gamma(0), \theta)$ is defined in (24) and the functions $\alpha(\theta)$, $\beta(\theta)$, $\gamma(\theta)$, and $g_v(\theta, \dot{\theta}, y, \dot{y})$ are defined in (35).

Then, the chosen periodic solution (32) is exponentially orbitally stable for the closed-loop system (1), (34), (45). ■

*Proof:* By construction, the target periodic solution (32) is one of solutions of the closed-loop system (35), (36), (45). Note that the closed-loop system (35), (36), (45) is a smooth time-varying nonlinear dynamical system, where the time dependence is only due to the $T$-periodic matrix factor $R(t)$ in the expression for the controller.

Since the closed-loop system is smooth, $T$-periodic, and has $T$-periodic solution (32), the following consequence of the *theorem of continuous dependence on initial conditions* [9] holds true.

*Property 1:* For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if two vectors

$$w_{01} = [\theta_{01}, \dot{\theta}_{01}, y_{01}, \dot{y}_{01}] \quad w_{02} = [\theta_{02}, \dot{\theta}_{02}, 0, 0]$$

are such that

$$\|w_{01} - w_{02}\| < \delta \quad \text{and} \quad w_{02} \in \mathcal{O} \tag{48}$$

where the set $\mathcal{O}$ is the orbit of the chosen solution (32), i.e.,

$$\mathcal{O} := \left\{[\theta, \dot{\theta}, y, \dot{y}] : \theta = \theta_\gamma(t), \dot{\theta} = \dot{\theta}_\gamma(t), y = \dot{y} = 0\right\} \tag{49}$$

then the solutions $w_1(t) = w_1(t, w_{01})$, $w_2(t) = w_2(t, w_{02})$ of the closed-loop system (35), (36), (45) initiated at $w_{01}$ and $w_{02}$ at $t = t_0$, correspondingly, satisfy the inequality

$$\|w_1(t) - w_2(t)\| < \varepsilon, \qquad t \in [t_0, t_0 + T]. \tag{50}$$

The inequality (50) holds irrespective of stability of the orbit (32). Indeed, it holds only on the final time interval. ■

This property is instrumental for showing orbital stability of the solution (32). Let us introduce a Lyapunov function candidate

$$V(t, \theta, \dot{\theta}, y, \dot{y}) = \zeta^T R(t)\zeta \tag{51}$$

where $\zeta = [I, y, \dot{y}]^T$, $I$ is the function defined in (47), and $R(t)$ is the stabilizing solution of the Riccati equation defined in Statement 4. Choose a vector

$$w_0 = [\theta_0, \dot{\theta}_0, y_0, \dot{y}_0] \tag{52}$$

of initial conditions and consider the solution $w(t) = [\theta(t), \dot{\theta}(t), y(t), \dot{y}(t)]$ of the closed loop system (35), (36), (45) initiated at this point. Let us introduce the following notation:

$$\bar{A}(\cdot) = \begin{bmatrix} -\frac{2\dot{\theta}\beta(\theta)}{\alpha(\theta)} & \frac{2\dot{\theta}g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})}{\alpha(\theta)} & \frac{2\dot{\theta}g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})}{\alpha(\theta)} \\ 0_{m\times 1} & 0_{m\times m} & I_m \\ 0_{m\times 1} & 0_{m\times m} & 0_{m\times m} \end{bmatrix}$$

$$\bar{b}(\cdot)^T = \left[\frac{2\dot{\theta}g_v(\theta, \dot{\theta}, y, \dot{y})}{\alpha(\theta)},\ 0_{m\times m},\ I_m\right]$$

where $m = n - 1$. The time derivative of $V$ along the solution takes the form [see also (37), (38)]

$$\dot{V} = 2\zeta^T R(t)\bar{b}(\cdot)v + \zeta^T \left\{\dot{R}(t) + \bar{A}(\cdot)^T R(t) + R(t)\bar{A}(\cdot)\right\}\zeta = \zeta^T \left\{-2R(t)\bar{b}\Gamma^{-1}\bar{b}^T R(t) + \dot{R}(t) + \bar{A}^T R(t) + R(t)\bar{A}\right\}\zeta.$$

Let us add and subtract to this equation the expression

$$\zeta^T \left\{A(t)^T R(t) + R(t)A(t) - 2R(t)b(t)\Gamma^{-1}b(t)^T R(t)\right\}\zeta$$

where $A(t)$, $b(t)$ are as defined in (41). Then, one obtains

$$\begin{aligned}\dot{V} &= \zeta^T \Big\{\dot{R}(t) + A(t)^T R(t) + R(t)A(t) \\ &\qquad -2R(t)b(t)\Gamma^{-1}b(t)^T R(t) + \Delta(t)\Big\}\zeta \\ &= \zeta^T \left\{-G - R(t)b(t)\Gamma^{-1}b(t)^T R(t) + \Delta(t)\right\}\zeta. \quad (53)\end{aligned}$$

Here, we have used the differential Riccati equation (42), while the matrix function $\Delta(t) = \Delta(t, \theta(t), \dot{\theta}(t), \theta_\gamma(t), \dot{\theta}_\gamma(t), y(t), \dot{y}(t))$ is

$$\begin{aligned}\Delta(t) = &\left(\bar{A}(\cdot) - A(t)\right)^T R(t) + R(t)\left(\bar{A}(\cdot) - A(t)\right) \\ &+ R(t)\left(\bar{b}(\cdot) - b(t)\right)\Gamma^{-1}\left(\bar{b}(\cdot) + b(t)\right)^T R(t) \quad (54)\end{aligned}$$

where

$$\bar{A}(\cdot) - A(t) = \begin{bmatrix} \Delta_1(t) & \Delta_2(t) & \Delta_3(t) \\ 0_{m\times 1} & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times 1} & 0_{m\times m} & 0_{m\times m}\end{bmatrix}$$

$$\bar{b}(\cdot) - b(t) = \begin{bmatrix} \Delta_4(t) \\ 0_{m\times m} \\ 0_{m\times m}\end{bmatrix} \quad \bar{b}(\cdot) + b(t) = \begin{bmatrix} \Delta_5(t) \\ 0_{m\times m} \\ 2I_m\end{bmatrix}.$$

The expression for the $\Delta_i(t)$ are given in Appendix II-B.

In the previous computations, no assumption about the initial conditions $w_0$ [see (52)] of the closed-loop system solution has been made and the initial point $w_0$ has been chosen arbitrarily. Let us now consider only those initial conditions that belong to some vicinity of the orbit $\mathcal{O}$, defined in (49). More precisely, choose $\varepsilon_*$ as half of the smallest eigenvalue of $G$, i.e.,

$$\varepsilon_* = \frac{1}{2}\min\left\{\lambda(G)\right\}. \quad (55)$$

Property 1 implies that for such a positive constant $\varepsilon_*$ there exists a positive constant $\delta_1 = \delta_1(\varepsilon_*)$ such that if the vector $w_0$ of the initial conditions (52) is chosen on a distance less than $\delta_1$ from the orbit $\mathcal{O}$

$$\mathrm{dist}\{w_0, \mathcal{O}\} < \delta_1 \quad (56)$$

then the corresponding solution $w(t) = w(t, w_0)$ initiated at $w_0$ satisfies the inequality

$$\mathrm{dist}\left\{w(t), \mathcal{O}\right\} < \varepsilon_*, \qquad t \in [t_0, t_0 + T]. \quad (57)$$

In other words, the solution of the closed-loop system on the period belongs to the open tube around the orbit $\mathcal{O}$ of the radius $\varepsilon_*$.

This, together with the observation that the functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$, $g_v(\cdot)$, $\beta(\cdot)$ and $\alpha(\cdot)$ are smooth and, therefore, bounded around the orbit $\mathcal{O}$ [see (49)] allows us to make the values of functions $\Delta_i(t)$, $i = 1, 2, 3, 4$, as close to zero as we like, uniformly on the time interval $[t_0, t_0 + T]$, provided that the vector $w_0$ of the initial conditions is chosen sufficiently close to $\mathcal{O}$.

The boundedness of the matrix function $R(t)$ and the fact that the function $\Delta(t)$ in (54) is linear in $\Delta_i(t)$, $i = 1, 2, 3, 4$ imply that $|\Delta(t)|$ could be made as small as we like, uniformly on the time interval $[t_0, t_0 + T]$, provided that the vector $w_0$ of the initial conditions is chosen sufficiently close to $\mathcal{O}$. That is, if necessary, we can find $\delta_2$ such that $0 < \delta_2 \le \delta_1(\varepsilon_*)$ and along the solution $w(t) = w(t, w_0)$ the closed-loop system with

$$\mathrm{dist}\{w_0, \mathcal{O}\} < \delta_2$$

in addition to (57), the inequality

$$\left\|\Delta\left(t, \theta(t), \dot{\theta}(t), \theta_\gamma(t), \dot{\theta}_\gamma(t)\right)\right\| \le \varepsilon_* \quad (58)$$

holds for $t \in [t_0, t_0 + T]$.

Inequality (58) allows us to bound from above the time derivative of $V$ [see (53)] as follows:

$$\begin{aligned}\dot{V}(t) &\le -\zeta(t)^T G\zeta(t) + \varepsilon_* \left\|\zeta(t)\right\|^2 \\ &\le \left(\varepsilon_* - \min\left\{\lambda(G)\right\}\right)\left\|\zeta(t)\right\|^2 \\ &= -\varepsilon_* \left\|\zeta(t)\right\|^2. \end{aligned} \quad (59)$$

Here, we have used the choice of $\varepsilon_*$ in (55). The differential relation (59) holds only on the time interval $[t_0, t_0 + T]$. Integrating (59) over this time interval, we obtain

$$V(t_0 + T) \le V(t_0) - \varepsilon_* \int_{t_0}^{t_0+T} \left\|\zeta(t)\right\|^2 dt. \quad (60)$$

The last inequality suggests that if we choose only those initial conditions $w_0$ such that the implication

$$V(t_0) = \zeta_0^T R(t_0)\zeta_0 < c \Rightarrow \mathrm{dist}\{w_0, \mathcal{O}\} < \delta_2 \quad (61)$$

holds for some $c > 0$, then the set

$$V_c(t_0) = \{w_0 : \text{the implication (61) holds}\} \quad (62)$$

around $\mathcal{O}$ belongs to this orbit's region of attraction[13] and any solution of the closed-loop system initiated inside this set[14] at $t = t_0$ exponentially converges to $\mathcal{O}$.

[13]Note that the initial time $t_0$ is fixed.

[14]Employing the arguments presented in the proof of Theorem 1, see Appendix I-A, it could be shown that $V_c(t_0)$ is a nontrivial open set.

It has been shown that if $w(t)$ satisfies $\operatorname{dist}\{w(t_0), \mathcal{O}\} < \delta_2$, then $\operatorname{dist}\{w(t_0+T), \mathcal{O}\} < \varepsilon_*$. Furthermore, the equality (60) together with (61) imply that

$$\operatorname{dist}\{w(t_0+T), \mathcal{O}\} < \delta_2 \tag{63}$$

that is, over the period $T$ the solution $w(t)$ of the closed-loop system comes back to the same neighborhood of the orbit $\mathcal{O}$ and the value of the Lyapunov function candidate $V$ satisfies the relation (60) over the period.

Inequality (63) allows us to repeat the same arguments for the solution $w(t)$ over the next period, i.e., on the time interval $[t_0+T, t_0+2T]$. Namely, if we choose new vector $w_{0new}$ of initial conditions with components

$$\theta_0 = \theta(t_0+T),\ \dot\theta_0 = \dot\theta(t_0+T),\ y_0 = y(t_0+T),\ \dot y_0 = \dot y(t_0+T)$$

and repeat all the derivations done previously, then we obtain

$$V(2T+t_0) \le V(t_0) - \varepsilon_* \int_{t_0}^{t_0+2T} \|\zeta(t)\|^2\, dt \quad \text{and} \quad \operatorname{dist}\{w(t_0+2T), \mathcal{O}\} < \delta_2.$$

Following inductive arguments, we obtain that for any $n \ge 1$

$$\begin{aligned} V(nT+t_0) &\le V(t_0) - \varepsilon_* \int_{t_0}^{t_0+nT} \|\zeta(t)\|^2\, dt \\ &\le V(t_0) - \frac{\varepsilon_*}{\max\limits_{t\in[0,T]} \|R(t)\|} \int_{t_0}^{t_0+nT} V(\tau) d\tau. \end{aligned}$$

This implies that $w(t)$ converges to the target orbit (32) exponentially.

Finally, we need to show that the constant $c$ in the implication (61) can be chosen independent of $t_0$ provided that $\delta_2$ is sufficiently small.

Let

$$V_* = \bigcap_{t_0\in[0,T]} V_c(t_0).$$

Since each of the sets $V_c(t_0)$ is open and contains the orbit $\mathcal{O}$, therefore the set $V_*$ contains $\mathcal{O}$. Using the fact that $R(t) > \varepsilon_0 I_{n-1}$, it can be shown that $V_*$ is open. ■

## IV. Example

We have shown in Section II that if the *virtual holonomic constraint* (7) made invariant for the dynamics of the cart-pendulum system (5), (6), then the *virtual limit system* (10) that has the centre at the upright equilibrium $\theta = 0$ of the pendulum, provided that $L > 1$. In Fig. 2 the phase portrait of (10) is shown for the case when $L = 1.5$. It is clear that the system has the center at the equilibrium $\theta = \dot\theta = 0$ and its neighborhood is filled with cycles. In Fig. 3, the corresponding solutions are shown in the time domain. The target orbit is given in bold. Let us denote

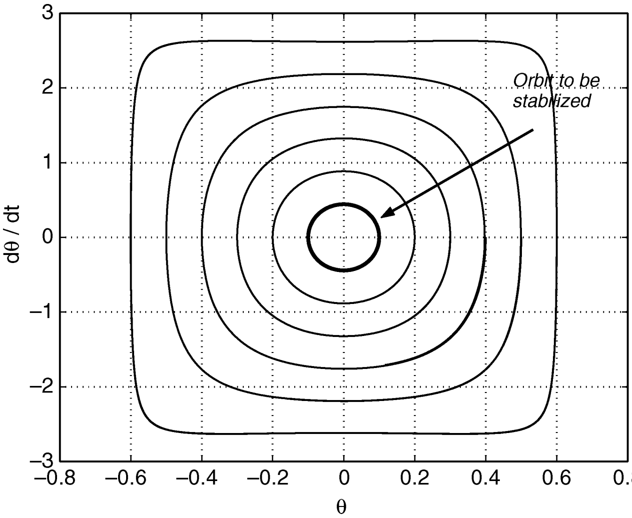


Fig. 2. Phase portrait of the *virtual limit system* (10) with $L = 1.5$. Here, the solution chosen to be stabilized is shown in bold.

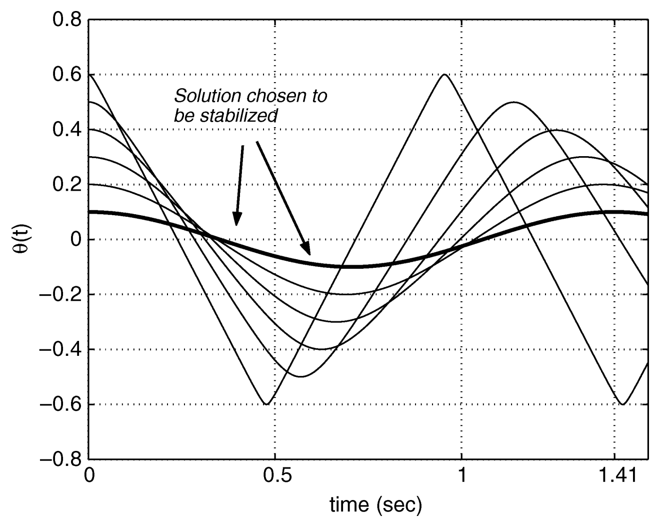


Fig. 3. Solutions of the *virtual limit system* (10) with $L = 1.5$ as a time varying functions shown previously on the phase portrait of (10); see Fig. 2. Here, the solution chosen to be stabilized is shown in bold.

the periodic solution of the differential equation (10), that corresponds to this orbit, as

$$\theta_\gamma(t) = \theta_\gamma(t+T).$$

Simulation shows that it is of a period $T$, which is approximately equal to 1.41 s.

The matrices $L(y,\theta)$ and $K(y,\theta)$ [see (31) and (33)] for this example are

$$L(y,\theta) = \begin{bmatrix} 1 & -L\cos\theta \\ 0 & 1 \end{bmatrix} \quad K(y,\theta) = \frac{1 - L\cos^2\theta}{1+\sin^2\theta}$$

and are nonsingular $\forall\theta$. Then, Proposition 3 is applicable and one can check that the transformed system (35), (36) is now

$$\alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta) = -\cos\theta \cdot v \tag{64}$$

$$\ddot y = v \tag{65}$$

with $\alpha(\theta) = (1 - L\cos^2\theta)$, $\beta(\theta) = L\cos\theta\sin\theta$, and $\gamma(\theta) = g\sin\theta$. The feedback transformation is

$$u_f = \frac{1}{K(y,\theta)}\left[v - R(\theta,\dot\theta)\right] \tag{66}$$

where

$$R(\theta,\dot\theta) = \frac{\sin\theta(\dot\theta^2 - g\cos\theta)}{1+\sin^2\theta} + \frac{L\cos\theta(2g\sin\theta - \sin\theta\cos\theta\dot\theta^2)}{1+\sin^2\theta} - L\sin\theta\dot\theta^2$$

and the control variable $v$ to be defined. The auxiliary linear system (41) now looks as

$$\begin{aligned}\dot I &= \dot\theta_\gamma(t)\left\{\frac{2g_v(\theta_\gamma(t))}{\alpha(\theta_\gamma(t))}v - \frac{2\beta(\theta_\gamma(t))}{\alpha(\theta_\gamma(t))}I\right\}\\ &= \dot\theta_\gamma(t)\left\{\frac{-2\cos(\theta_\gamma(t))}{1 - L\cdot\cos^2(\theta_\gamma(t))}v - \frac{2L\cos(\theta_\gamma(t))\sin(\theta_\gamma(t))}{1 - L\cdot\cos^2(\theta_\gamma(t))}I\right\} \end{aligned} \tag{67}$$

$$\ddot y = v \tag{68}$$

Controllability of this system is verified in Appendix I-D. Hence, we can find a stabilizing controller in the form (45). The key step is finding an stabilizing solution $R(t)$ of the periodic Riccati equation (42). This equation has been solved numerically for the following value of the weighting matrices: $G = I_3$ and $\Gamma = 1$.

Finally, the stabilizing controller for the cart-pendulum system (8), (9) is given by (66) and

$$v = -\left[\frac{-2\dot\theta\cos(\theta)}{1 - L\cos^2(\theta)},\ 0,\ 1\right]^T R(t)\begin{bmatrix} I \\ y \\ \dot y\end{bmatrix}$$

where

$$I = \dot\theta^2(t) - \frac{1 - L\cos^2\theta_\gamma(0)}{1 - L\cos^2\theta(t)}\dot\theta_\gamma^2(0) + 2g\frac{\cos\theta(t) - \cos\theta_\gamma(0)}{1 - L\cos^2\theta(t)}$$

$y = x + L\sin\theta$, $\dot y = \dot x + L\dot\theta\cos\theta$, and $R(t)$ is the periodic $3\times3$ matrix function, computed numerically. In Figs. 4 and 5, we show the behavior of the closed-loop system variables when initial conditions are

$$\theta_0 = 0.4 \quad \dot\theta_0 = -0.2 \quad x_0 = 0.1 \quad \dot x_0 = -0.1.$$

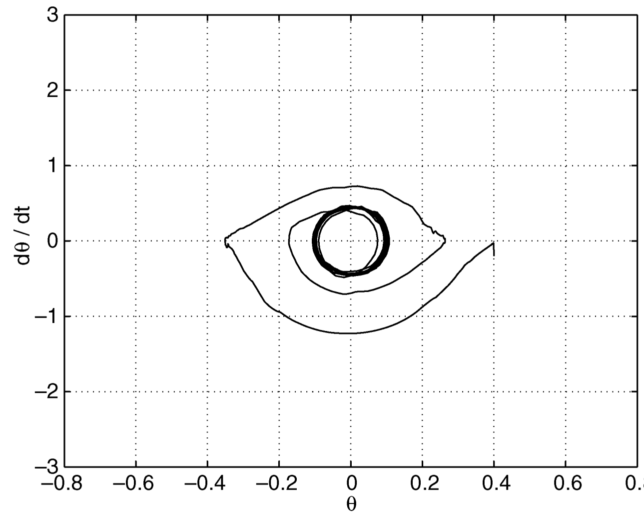


Fig. 4.Behavior of $[\theta(t), \dot\theta(t)]$ in the closed loop system with an added white noise in measurements. The orbit chosen to be stabilized is shown in bold on Fig. 2.

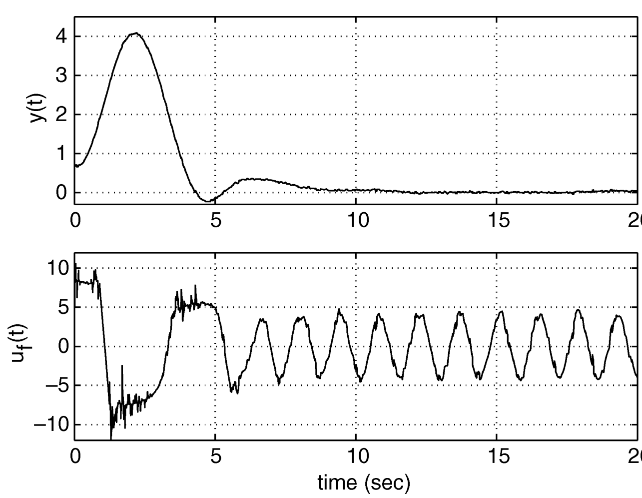


Fig. 5. Behavior of regulated output $y(t)$ and control signal $u_f(t)$ in the closed loop system with a white noise in measurements.

## V. Conclusion

In this paper, we have introduced a constructive method for control design to produce stable oscillations for underactuated Euler–Lagrange systems with the number of independent actuators smaller than the number of degrees of freedom by one.

The method is based on the idea of introducing virtual holonomic constraints that define a set of possible target periodic motions for the closed-loop system.

The main contribution of this paper is a step-by-step procedure for design of a state feedback control law that ensures exponential orbital stabilization of each feasible periodic target motion. The first step is a nonstandard partial feedback linearization, where the remaining nonlinear dynamics is integrable. On the next step, we construct an auxiliary linear periodic control system of reduced order. Finally, the feedback control law for the original nonlinear system is obtained by modification of the LQR-based control law, designed for the auxiliary system.

The resulting nonlinear control law contains time-varying periodic components, which are fundamental for the stability of the orbit in the closed-loop system. The proof of exponential orbital stability is Lyapunov based. The Lyapunov function is time-dependent and quadratic in special variables, that measure a distance to the target orbit.

The proposed methodology is applied to the cart-pendulum system. We have achieved stable oscillations of the pendulum around its upright equilibrium while the position of the cart remains bounded.

## APPENDIX I
## APPENDICES: PROOFS OF THE MAIN RESULTS

### *A. Proof of Theorem 1*

Let us introduce the variable

$$Y = \dot\theta^2(t).$$

It is easy to see that

$$\frac{dY}{dt} = \frac{d}{dt}\left(\dot\theta^2(t)\right) = 2\dot\theta\ddot\theta$$

and

$$\frac{dY}{dt} = \frac{dY}{d\theta}\frac{d\theta}{dt} = \frac{dY}{d\theta}\dot\theta.$$

Therefore, along any solution of the dynamical system (18) the identity

$$\ddot\theta = \frac{1}{2}\frac{dY}{d\theta}$$

holds. Then, one can rewrite (18) in the equivalent form

$$\alpha(\theta)\frac{1}{2}\frac{d}{d\theta}Y + \beta(\theta)Y + \gamma(\theta) = 0. \qquad (69)$$

This differential equation for $Y$ is linear with $\theta$ being (instead of $t$) an independent variable.

Let us first consider the case where along the solution $[\theta(t), \dot\theta(t)]$ the function $\alpha(\theta(t))$ is different from zero. Under this assumption, one can rewrite (69) as

$$\frac{d}{d\theta}Y + \frac{2\beta(\theta)}{\alpha(\theta)}Y + \frac{2\gamma(\theta)}{\alpha(\theta)} = 0 \qquad (70)$$

Its general solution has the following form:

$$Y(\theta) = \psi(\theta_0,\theta)Y(\theta_0) - \psi(\theta_0,\theta)\int_{\theta_0}^{\theta}\psi(s,\theta_0)\frac{2\gamma(s)}{\alpha(s)}ds \qquad (71)$$

with $\psi(\cdot,\cdot)$ defined in (24). It follows that along any solution of (18) the function

$$I\left(\theta(t), \dot\theta(t), \theta_0, \dot\theta_0\right) = \dot\theta(t)^2 - Y\left(\theta(t)\right)$$

is identically equal to zero.

Suppose now that there exists finite moment of time $t = T_*$ such that

a) $\exists\varepsilon > 0$: the solution $[\theta(t), \dot\theta(t)]$ of (18) is well defined and remains continuous on $[T_* - \varepsilon, T_* + \varepsilon]$;

b) the function $\alpha(\theta)$ equals to zero at $\theta = \theta(T_*)$.

Since the function $I(\cdot)$ continuously depends on its arguments and the solution is bounded on the time interval $[T_* - \varepsilon, T + \varepsilon]$ the following limit relations hold:

$$\begin{aligned}
&\lim_{t\to T_*^-} I\left(\theta(t), \dot\theta(t), \theta(T_*-\varepsilon), \dot\theta(T_*-\varepsilon)\right) \\
&\quad = \lim_{\substack{x\to\theta(T_*)^- \\ y\to\dot\theta(T_*)^-}} I\left(x, y, \theta(T_*-\varepsilon), \dot\theta(T_*-\varepsilon)\right) = 0 \\
&\lim_{t\to T_*^+} I\left(\theta(T_*+\varepsilon), \dot\theta(T_*+\varepsilon), \theta(t), \dot\theta(t)\right) \\
&\quad = \lim_{\substack{x\to\theta(T_*)^+ \\ y\to\dot\theta(T_*)^+}} I\left(\theta(T_*+\varepsilon), \dot\theta(T_*+\varepsilon), x, y\right) = 0.
\end{aligned}$$

Therefore

$$I\left(\theta(t), \dot\theta(t), \theta(T_*-\varepsilon), \dot\theta(T_*-\varepsilon)\right) = 0$$

for all $t \in [T_* - \varepsilon, T_* + \varepsilon]$. ■

### *B. Proof of Theorem 2*

The time derivative of the function $I = I(\theta(t), \dot\theta(t), x, y)$ along a solution of (25) is

$$\frac{d}{dt}I = \dot\theta\frac{\partial}{\partial\theta}I + \ddot\theta\frac{\partial}{\partial\dot\theta}I \qquad (72)$$

where

$$\begin{aligned}
\frac{\partial}{\partial\dot\theta}I &= 2\dot\theta \\
\frac{\partial}{\partial\theta}I &= \frac{2\gamma(\theta)}{\alpha(\theta)} - \frac{2\beta(\theta)}{\alpha(\theta)}[I - \dot\theta^2]
\end{aligned}$$

and $\ddot\theta$ is defined by (25). Therefore

$$\begin{aligned}
\frac{d}{dt}I &= \dot\theta\left\{\frac{2\gamma(\theta)}{\alpha(\theta)} - \frac{2\beta(\theta)}{\alpha(\theta)}[I-\dot\theta^2]\right\} + \frac{\left(u - \beta(\theta)\dot\theta^2 - \gamma(\theta)\right)}{\alpha(\theta)}2\dot\theta \\
&= \dot\theta\left\{\frac{2}{\alpha(\theta)}u - \frac{2\beta(\theta)}{\alpha(\theta)}I\right\}. \qquad (73)
\end{aligned}$$

### *C. Proof of Statement 3*

System (1) can be equivalently rewritten as

$$\ddot q = M(q)^{-1}\left[B(q)u - C(q,\dot q)\dot q - G(q)\right] \qquad (74)$$

and in the new coordinates $y$, $\theta$ as

$$\begin{aligned}
&L(y,\theta)\begin{bmatrix}\ddot y \\ \ddot\theta\end{bmatrix} + N(\theta,\dot\theta,y,\dot y) \\
&= \left(M(q)^{-1}\left[B(q)u - C(q,\dot q)\dot q - G(q)\right]\right)\Big|_{\substack{q_1 = y_1 + \phi(\theta) \\ \cdots \\ q_{n-1} = y_{n-1} + \phi_{n-1}(\theta) \\ q_n = \phi(\theta) + h(y,\theta).}}
\end{aligned}$$

Therefore

$$\ddot y = K(y,\theta)u + R(y,\theta,\dot y,\dot\theta) \qquad (75)$$

where a function $R(\cdot)$ is independent of the control input $u$ and $K(\cdot)$ is defined in (33). By assumption, the $(n-1)\times(n-1)$ matrix function $K(y,\theta)$ is nonsingular around the orbit (32). Then, one can introduce the following feedback transformation:

$$u = K(y,\theta)^{-1}\left[v - R(y,\theta,\dot{y},\dot{\theta})\right] \tag{76}$$

to brings (75) into

$$\ddot{y} = v. \tag{77}$$

One way to find the remaining dynamics for $\theta$ variable is to consider the scalar equation

$$\begin{aligned} B^{\perp}(q)\Bigg\{M(q)\bigg(L(y,\theta)\begin{bmatrix}\ddot{y}\\ \ddot{\theta}\end{bmatrix} + N(\theta,\dot{\theta},y,\dot{y})\bigg) + C(q,\dot{q})\dot{q} + G(q)\Bigg\} \\ = B^{\perp}(q)B(q)u \\ = 0. \end{aligned} \tag{78}$$

Here, $B^{\perp}(q)$ is an $1\times n$ annihilator for the $n\times(n-1)$ matrix function $B(q)$ and $q$ is defined as follows:

$$\begin{aligned} q_1 &= y_1 + \phi(\theta),\ \ldots,\ q_{n-1} = y_{n-1} + \phi_{n-1}(\theta) \\ q_n &= \phi(\theta) + h(y_1,\ldots,y_{n-1},\theta) \end{aligned}$$

while $\dot{q}$ is as in (30). It is easy to see from the derivations in the proof of Proposition 2 that the (78) coincides with the virtual limit system (18) provided that $y = \dot{y} = \ddot{y} = 0_{(n-1)\times 1}$. Using the H'Adamard Lemma (see [9, Lemma 3.1, p. 122]), one can rewrite (78) as (35). ■

### D. Controllability of Auxiliary System for the Cart-Pendulum Example

To verify controllability of the linear periodic system (67), (68), we check the necessary and sufficient condition given in the next statement.

*Proposition 5:* Suppose $\kappa_2(t) = 0$, $\kappa_3(t) = 0$ and the function $g_v(\cdot)$ is independent on $\dot{\theta}$. The linear $T$-periodic system (41) is completely controllable over the period $[0,T]$ if and only if the inequality

$$\int_0^T \frac{\rho^2(t)}{\tilde{\kappa}^2(t)}dt > \frac{12}{T^3}\left|\int_0^T \frac{t}{\tilde{\kappa}(t)}\rho(t)dt\right|^2 \tag{79}$$

holds. Here

$$\tilde{\kappa}(t) = \exp\left\{\int_0^t \kappa_1(\tau)d\tau\right\} \tag{80}$$

the functions $\rho(t)$ and $\kappa_1(t)$ are defined in Appendix II-A. ■

*Proof:* The system (41) is completely controllable (see [15, Th. 9.2, p. 143]) if and only if the matrix

$$K = \int_0^T [X_0(t)]^{-1} b(t)b(t)^T \left[X_0(t)^T\right]^{-1} dt \tag{81}$$

is positive definite. Here, $X_0(t)$ is defined by

$$\frac{d}{dt}X_0 = A(t)X_0 \quad X_0(0) = I_{2n+1}.$$

The straightforward calculation shows that

$$X_0(t) = \begin{bmatrix} \tilde{\kappa}(t) & 0_{1\times n} & 0_{1\times n} \\ 0_{n\times 1} & I_n & t\cdot I_n \\ 0_{n\times 1} & 0_{n\times n} & I_n \end{bmatrix}$$

where $\tilde{\kappa}(t)$ is defined in (80).

It follows from the inherent boundedness of the periodic desired orbit that $\tilde{\kappa}(t) > \varepsilon > 0$ for $t\in[0,T]$. Therefore

$$[X_0(t)]^{-1} = \begin{bmatrix} \frac{1}{\tilde{\kappa}(t)} & 0_{1\times n} & 0_{1\times n} \\ 0_{n\times 1} & I_n & -t\cdot I_n \\ 0_{n\times 1} & 0_{n\times n} & I_n \end{bmatrix}$$

and

$$b(t)b(t)^T = \begin{bmatrix} \rho^2(t) & 0_{1\times n} & \rho(t) \\ 0_{n\times 1} & 0_{n\times n} & 0_{n\times n} \\ \rho(t)^T & 0_{n\times n} & I_n \end{bmatrix}.$$

Substituting these formulas into (81), one obtains

$$K = \begin{bmatrix} \int_0^T \frac{\rho^2(t)}{\tilde{\kappa}^2(t)}dt & -\int_0^T \frac{t\rho(t)}{\tilde{\kappa}(t)}dt & \int_0^T \frac{\rho(t)}{\tilde{\kappa}(t)}dt \\ -\int_0^T \frac{t\rho(t)^T}{\tilde{\kappa}(t)}dt & \frac{T^3}{3}\cdot I_n & -\frac{T^2}{2}\cdot I_n \\ \int_0^T \frac{\rho(t)^T}{\tilde{\kappa}(t)}dt & -\frac{T^2}{2}\cdot I_n & T\cdot I_n \end{bmatrix}. \tag{82}$$

Let us check that

$$\int_0^T \frac{1}{\tilde{\kappa}(t)}\rho(t)dt = 0. \tag{83}$$

The function $\rho(t)/\tilde{\kappa}(t)$ is of particular form,[15] so that

$$\begin{aligned} \int_0^T \frac{\rho(t)}{\tilde{\kappa}(t)}dt = \int_{\theta_\gamma(0)}^{\theta_\gamma(T)} \frac{2g_v(\theta_\gamma(t))}{\alpha(\theta_\gamma(t))} \\ \times \exp\left\{2\int_{\theta_\gamma(0)}^{\theta_\gamma(t)} \frac{\beta(\theta_\gamma(\tau))}{\alpha(\theta_\gamma(\tau))}d\theta_\gamma(\tau)\right\} d\theta_\gamma(t). \end{aligned}$$

This integral is equal to zero since the upper and lower limits of integration, $\theta_\gamma(0)$ and $\theta_\gamma(T)$, are the same due to periodicity of $\theta_\gamma(t)$.

Hence

$$K = \begin{bmatrix} \int_0^T \frac{\rho^2(t)}{\tilde{\kappa}^2(t)}dt & -\int_0^T \frac{t\rho(t)}{\tilde{\kappa}(t)}dt & 0_{1\times n} \\ -\int_0^T \frac{t\rho(t)^T}{\tilde{\kappa}(t)}dt & \frac{T^3}{3}\cdot I_n & -\frac{T^2}{2}\cdot I_n \\ 0_{n\times 1} & -\frac{T^2}{2}\cdot I_n & T\cdot I_n \end{bmatrix}$$

[15]See Appendix II-A.

and $K$ is positive definite if and only if its determinant is positive, since the other main minors

$$\begin{bmatrix} \frac{T^3}{3}\cdot I_n & -\frac{T^2}{2}\cdot I_n \\ -\frac{T^2}{2}\cdot I_n & T\cdot I_n \end{bmatrix} \quad T\cdot I_n$$

are positive definite. From the straightforward calculations (Schur lemma), we see that

$$\det K = \frac{T^4}{12}\left[\int_0^T \frac{\rho^2(t)}{\tilde{\kappa}^2(t)}dt - \frac{12}{T^3}\left(\int_0^T \frac{t}{\tilde{\kappa}(t)}\rho(t)dt\right)^2\right]. \quad (84)$$

It is positive iff the inequality (79) is valid. ■

To verify controllability of the linear periodic system (67), (68) for the cart-pendulum example, we check the necessary and sufficient conditions (79) derived above.

We have $T \approx 1.41$s and

$$\rho(t) = -\dot{\theta}_\gamma(t)\frac{2\cos(\theta_\gamma(t))}{1 - L\cdot\cos^2(\theta_\gamma(t))}$$

$$\tilde{\kappa}(t) = \exp\left\{-\int_0^t \dot{\theta}_\gamma(\tau)\frac{2L\cos(\theta_\gamma(\tau))\sin(\theta_\gamma(\tau))}{1 - L\cdot\cos^2(\theta_\gamma(\tau))}d\tau\right\}.$$

From the numerical integration

$$\int_0^T \frac{\rho^2(t)}{\tilde{\kappa}^2(t)}dt \approx 2.1610 > 1.4407 \approx \frac{12}{T^3}\left|\int_0^T \frac{t}{\tilde{\kappa}(t)}\rho(t)dt\right|^2.$$

Thus, the inequality (79) is valid, and the system (67), (68) is completely controllable over the period.

## Appendix II
## Appendix: Definitions

### A. Expression for the Components of $A(t)$ and $b(t)$ in (41)

$$\rho(t) = \dot{\theta}_\gamma(t)\frac{2g_v\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), 0, 0\right)}{\alpha(\theta_\gamma(t))}$$

$$\kappa_1(t) = -\dot{\theta}_\gamma(t)\frac{2\beta(\theta_\gamma(t))}{\alpha(\theta_\gamma(t))}$$

$$\kappa_2(t) = \dot{\theta}_\gamma(t)\frac{2g_y\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), \ddot{\theta}_\gamma(t), 0, 0\right)}{\alpha(\theta_\gamma(t))}$$

$$\kappa_3(t) = \dot{\theta}_\gamma(t)\frac{2g_{\dot{y}}\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), \ddot{\theta}_\gamma(t), 0, 0\right)}{\alpha(\theta_\gamma(t))}.$$

### B. Expression for Functions $\Delta_i(t)$ in (54)

$$\Delta_1(t) = \frac{2\dot{\theta}_\gamma(t)\beta(\theta_\gamma(t))}{\alpha(\theta_\gamma(t))} - \frac{2\dot{\theta}(t)\beta(\theta(t))}{\alpha(\theta(t))}$$

$$\Delta_2(t) = \frac{2\dot{\theta}(t)g_y\left(\theta(t), \dot{\theta}(t), \ddot{\theta}(t), y(t), \dot{y}(t)\right)}{\alpha(\theta(t))} - \frac{2\dot{\theta}_\gamma(t)g_y\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), \ddot{\theta}_\gamma(t), 0, 0\right)}{\alpha(\theta_\gamma(t))}$$

$$\Delta_3(t) = \frac{2\dot{\theta}(t)g_{\dot{y}}\left(\theta(t), \dot{\theta}(t), \ddot{\theta}(t), y(t), \dot{y}(t)\right)}{\alpha(\theta(t))} - \frac{2\dot{\theta}_\gamma(t)g_{\dot{y}}\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), \ddot{\theta}_\gamma(t), 0, 0\right)}{\alpha(\theta_\gamma(t))}$$

$$\Delta_4(t) = \frac{2\dot{\theta}(t)g_v\left(\theta(t), \dot{\theta}(t), \ddot{\theta}(t), y(t), \dot{y}(t)\right)}{\alpha(\theta(t))} - \frac{2\dot{\theta}_\gamma(t)g_v\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), \ddot{\theta}_\gamma(t), 0, 0\right)}{\alpha(\theta_\gamma(t))}$$

$$\Delta_5(t) = \frac{2\dot{\theta}(t)g_v\left(\theta(t), \dot{\theta}(t), \ddot{\theta}(t), y(t), \dot{y}(t)\right)}{\alpha(\theta(t))} + \frac{2\dot{\theta}_\gamma(t)g_v\left(\theta_\gamma(t), \dot{\theta}_\gamma(t), \ddot{\theta}_\gamma(t), 0, 0\right)}{\alpha(\theta_\gamma(t))}.$$

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