LINEAR FUNCTIONS AND DUALITY ON THE INFINITE POLYTORUS

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Abstract. We consider the following question: Are there exponents $2 < p < q$ such that the Riesz projection is bounded from L^q to L^p on the infinite polytorus? We are unable to answer the question, but our counter-example improves a result of Marzo and Seip by demonstrating that the Riesz projection is unbounded from L^∞ to L^p if $p \geq 3.31138$. A similar result can be extracted for any $q > 2$. Our approach is based on duality arguments and a detailed study of linear functions. Some related results are also presented.

1. Introduction

Let $\mathbb{T}^\infty = \mathbb{T} \times \mathbb{T} \times \cdots$ denote the countably infinite cartesian product of the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We equip the $\mathbb{T}^\infty$ with its Haar measure μ_∞, which is equal to the infinite product of the normalized Lebesgue arc measure on $\mathbb{T}$ in each variable. Let $1 \leq p \leq \infty$. Every f in $L^p(\mathbb{T}^\infty)$ has a Fourier series expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}_0^\infty} c_\alpha z^\alpha$$

where the Fourier coefficients are defined in the standard way and $\alpha \in \mathbb{Z}_0^\infty$ means that the multi-index α contains only a finite number of non-zero components. The Riesz projection on $\mathbb{T}^\infty$ is defined by

$$Pf(z) = \sum_{\alpha \in \mathbb{N}_0^\infty} c_\alpha z^\alpha. \tag{1}$$

The initial motivation for the present paper is the following.

Question. What is the largest $p = p_\infty$ such that the Riesz projection (1) is bounded from $L^\infty(\mathbb{T}^\infty)$ to $L^p(\mathbb{T}^\infty)$?

The Riesz projection is certainly a contraction on the Hilbert space $L^2(\mathbb{T}^\infty)$ and since $\|f\|_{L^2(\mathbb{T}^\infty)} \leq \|f\|_{L^\infty(\mathbb{T}^\infty)}$, we get that $p_\infty \geq 2$. This question has previously been investigated by Marzo and Seip [8] who demonstrated that $p_\infty \leq 3.67632$. We will obtain the following improvement.

Theorem 1. *$p_\infty \leq p = 3.31138\ldots$, where p denotes the unique positive solution of the equation*

$$\Gamma\left(1 + \frac{p}{2}\right)^{\frac{1}{p}} = \frac{2}{\sqrt{\pi}}.$$

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For $2 \leq p \leq q \leq \infty$, let $\|P\|_{q,p}$ denote the norm of the Riesz projection from $L^q(\mathbb{T}^\infty)$ to $L^p(\mathbb{T}^\infty)$. In the case that the Riesz projection is unbounded, we use the convention $\|P\|_{q,p} = \infty$. As explained in [8], for each fixed $2 \leq q \leq \infty$ there is a number $2 \leq p_q \leq q$, called the critical exponent, with the property that

$$\|P\|_{p,q} = \begin{cases} 1 & \text{if } p \leq p_q, \\ \infty & \text{if } p > p_q. \end{cases} \tag{2}$$

The dichotomy (2) is a direct consequence of the fact that we are on the infinite polytorus. Let f be a function in the unit ball of $L^q(\mathbb{T}^\infty)$ such that $\|Pf\|_{L^p(\mathbb{T}^\infty)} > 1$. Consider the function

$$f_2(z) = f(z_1, z_3, z_5, \ldots) \cdot f(z_2, z_4, z_6, \ldots)$$

which is also in the unit ball of $L^q(\mathbb{T}^\infty)$. The Riesz projection (1) acts independently on the variables, so we find that

$$Pf_2(z) = Pf(z_1, z_3, z_5, \ldots) \cdot Pf(z_2, z_4, z_6, \ldots)$$

which implies that $\|Pf_2\|_{L^p(\mathbb{T}^\infty)} = \|Pf\|^2_{L^p(\mathbb{T}^\infty)} > \|Pf\|_{L^p(\mathbb{T}^\infty)}$. This procedure can be repeated and so we obtain (2). The example from [8] producing $p_\infty \leq 3.67632$ is a function of only two variables.

The present paper is inspired by [3], where linear functions are used as building blocks in an similar way to what was just described to construct a counter-example related to Nehari's theorem for Hankel forms on $\mathbb{T}^\infty$. The example from [3] improves on an earlier example from [9] by replacing a function of two variables by a linear function in an infinite number of variables.

Our approach differs from that of [8] (and [2]) in that we do not attempt to directly construct a counter-example, but instead use duality arguments to infer its existence. This approach leads us to consider the Hardy spaces $H^p(\mathbb{T}^\infty)$, which are the subspaces of $L^p(\mathbb{T}^\infty)$ consisting of elements such that $Pf = f$. A standard argument involving the Hahn–Banach theorem (see e.g. [5, Sec. 7.2]) yields that

$$\inf_{P\psi=\varphi} \|\psi\|_{L^q(\mathbb{T}^\infty)} = \|\varphi\|_{(H^r(\mathbb{T}^\infty))^*} = \sup_{f \in H^r(\mathbb{T}^\infty)} \frac{|\langle f, \varphi\rangle_{L^2(\mathbb{T}^\infty)}|}{\|f\|_{H^r(\mathbb{T}^\infty)}} \tag{3}$$

for $1 \leq r < \infty$ and $q^{-1} + r^{-1} = 1$. We will choose φ and try to find the optimal f in $H^r(\mathbb{T}^\infty)$ attaining the supremum. This will ensure the existence of ψ in $L^q(\mathbb{T}^\infty)$ attaining the infimum, which be our counter-example through (2).

We shall see in Section 3 that if we know the optimal f in the supremum on the right hand side of (3), we can use Hölder's inequality to construct the element ψ in $L^q(\mathbb{T}^\infty)$ of minimal norm such that $P\psi = \varphi$, thereby attaining the infimum on the left hand side of (3).

As in [3] we will primarily be working with linear functions, which are of the form

$$f(z) = \sum_{j=1}^{\infty} c_j z_j. \tag{4}$$

Clearly, $\|f\|^2_{H^2(\mathbb{T}^\infty)} = \sum_{j\geq 1} |c_j|^2$ and we easily check that $\|f\|_{H^\infty(\mathbb{T}^\infty)} = \sum_{j\geq 1} |c_j|$. For $1 \leq p < \infty$, optimal norm estimates are given by Khintchine's inequality.

Define

$$a_p = \min\left(1,\, \Gamma\left(1+\frac{p}{2}\right)^{\frac{1}{p}}\right) \qquad \text{and} \qquad b_p = \max\left(1,\, \Gamma\left(1+\frac{p}{2}\right)^{\frac{1}{p}}\right).$$

If f is a linear function (4) and $1 \leq p < \infty$, then we restate a result from [7] as

$$a_p\|f\|_{H^2(\mathbb{T}^\infty)} \leq \|f\|_{H^p(\mathbb{T}^\infty)} \leq b_p\|f\|_{H^2(\mathbb{T}^\infty)} \tag{5}$$

and the constants in (5) are optimal. We shall obtain the following companion inequality for dual norms, which might be of independent interest.

Theorem 2. *Let $1 \leq p < \infty$. If f is a linear function* (4)*, then*

$$b_p^{-1}\|f\|_{H^2(\mathbb{T}^\infty)} \leq \|f\|_{(H^p(\mathbb{T}^\infty))^*} \leq a_p^{-1}\|f\|_{H^2(\mathbb{T}^\infty)}. \tag{6}$$

The constants are optimal.

Remark. In the case $p = \infty$, it is easy to deduce by similar considerations (Lemma 4) that $\|f\|_{(H^\infty(\mathbb{T}^\infty))^*} = \sup_{j\geq 1}|c_j|$ if f is a linear function (4).

Optimality of the constants containing the Gamma function in (5) and (6) both arise from the function

$$f(z) = \frac{z_1 + z_2 + \cdots + z_d}{\sqrt{d}}$$

as $d \to \infty$ through the central limit theorem. In view of (2) and (3), we can therefore obtain the following general result. Note that Theorem 1 corresponds to the particular case $q = \infty$, since $\Gamma(3/2) = \sqrt{\pi}/2$.

Theorem 3. *Let $2 \leq p \leq q \leq \infty$ and set $q^{-1} + r^{-1} = 1$. If*

$$\Gamma\left(1+\frac{p}{2}\right)^{\frac{1}{p}}\Gamma\left(1+\frac{r}{2}\right)^{\frac{1}{r}} > 1,$$

then the Riesz projection is unbounded from $L^q(\mathbb{T}^\infty)$ to $L^p(\mathbb{T}^\infty)$.

Remark. Theorem 3 is an improvement on the same statement with requirement $p/2 \cdot r/2 > 1$, which can be deduced from a one-variable example found in [2, Sec. 4]. Here is an alternative example to that of [2] obtained by our approach using the Hahn–Banach theorem. For $w \in \mathbb{D}$, the functional of point evaluation $f \mapsto f(w)$ has norm $(1-|w|^2)^{-1/r}$ on $H^r(\mathbb{T})$ and the analytic symbol is $\varphi_w(z) = (1-\overline{w}z)^{-1}$. Hence, if $w = \varepsilon > 0$ then $\|\varphi_\varepsilon\|_{(H^r(\mathbb{T}))^*} = 1 + r^{-1}\varepsilon^2 + O(\varepsilon^4)$ as $\varepsilon \to 0$. Furthermore,

$$\|\varphi_\varepsilon\|_{H^p(\mathbb{T})} = \left\|(1-\varepsilon z)^{-p/2}\right\|_{H^2(\mathbb{T})}^{2/p} = 1 + \frac{p}{4}\varepsilon^2 + O(\varepsilon^4),$$

so we obtain the desired counter-example as soon as $r^{-1} > p/4$ in view of (2). The optimal ψ_w in $L^q(\mathbb{T})$ for this functional can be found in [4, Thm. 6.1], and we note that it is similar (but not equal to) the counter-example constructed in [2].

The present paper is organised into two additional sections. In Section 2 we prove Theorem 2 and Theorem 3. Section 3 is devoted to constructing the element ψ in $L^q(\mathbb{T}^\infty)$ for $1 < q \leq \infty$ of minimal norm such that $P\psi(z) = z_1 + z_2 + \cdots + z_d$, thereby realising the infimum (3) in this special case, which is of particular interest due to the crucial role it plays in the proof of Theorem 2 and Theorem 3.

2. Linear functions on $\mathbb{T}^\infty$

In preparation for the proof of Theorem 2 and Theorem 3, let us recall some basic facts about linear functions and projections on $\mathbb{T}^\infty$. The projection A_d obtained by formally setting $z_j = 0$ for $j > d$ has the representation

$$A_d f(z_1, z_2, \ldots) = \int_{\mathbb{T}^\infty} f(z_1, z_2, \ldots, z_d, z_{d+1}, z_{d+2}, \ldots)\, d\mu_\infty(z_{d+1}, z_{d+2}, \ldots).$$

Since $A_d f$ is a function the first d variables, we take L^p norm with respect to these variables and use the triangle inequality to obtain

$$\|A_d f\|_{L^p(\mathbb{T}^\infty)} \leq \|f\|_{L^p(\mathbb{T}^\infty)}. \tag{7}$$

Let $k \in \mathbb{Z}$. We say that f is k-homogeneous if

$$f(e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3, \ldots) = e^{ki\theta} f(z_1, z_2, z_3, \ldots).$$

Clearly every f in $L^p(\mathbb{T}^\infty)$ can be decomposed in k-homogeneous parts, say

$$f(z) = \sum_{k \in \mathbb{Z}} f_k(z), \tag{8}$$

where f_k is k-homogeneous. The following simple lemma is well-known, but we include a short proof for the readers convenience.

Lemma 4. *Let $1 \leq p \leq \infty$ and suppose that f in $L^p(\mathbb{T}^\infty)$ is decomposed as in* (8)*. Then $\|f_k\|_{L^p(\mathbb{T}^\infty)} \leq \|f\|_{L^p(\mathbb{T}^\infty)}$ for every $k \in \mathbb{Z}$.*

Proof. By the decomposition (8), we find that

$$f_k(z) = \int_{-\pi}^{\pi} f(e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3, \ldots)\, e^{-ki\theta}\, \frac{d\theta}{2\pi}.$$

By the triangle inequality and interchanging the order of integration, we obtain

$$\|f_k\|_{L^p(\mathbb{T}^\infty)}^p \leq \int_{-\pi}^{\pi} \int_{\mathbb{T}^\infty} \left| f(e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3, \ldots) \right|^p \, d\mu_\infty(z)\, \frac{d\theta}{2\pi} = \|f\|_{L^p(\mathbb{T}^\infty)}^p,$$

since for each θ the rotation $z_j \mapsto e^{i\theta} z_j$ does not change the $L^p(\mathbb{T}^\infty)$ norm of f. □

Let $\mathrm{Lin}(\mathbb{T}^\infty)$ denote the space of linear functions (4). Lemma 4 states that the projection from $H^p(\mathbb{T}^\infty)$ to $\mathrm{Lin}(\mathbb{T}^\infty) \cap H^p(\mathbb{T}^\infty)$ is contractive. This fact is crucial to the proof of Theorem 2 and Theorem 3 since it allows us to compute the $(H^p(\mathbb{T}^\infty))^*$ norm of a linear function φ by testing only against functions f from $\mathrm{Lin}(\mathbb{T}^\infty) \cap H^p(\mathbb{T}^\infty)$.

In view of Khintchine's inequality (5), the space $\mathrm{Lin}(\mathbb{T}^\infty) \cap H^p(\mathbb{T}^\infty)$ consists of linear functions (4) with square summable coefficients for each $1 \leq p < \infty$, although the norms are generally different.

Armed with these preliminaries, we will now obtain the key new ingredient needed in the proofs of Theorem 2 and Theorem 3.

Lemma 5. *Let $1 \leq p < \infty$ and set $\varphi_d(z) = (z_1 + \cdots + z_d)/\sqrt{d}$. Then*

$$\|\varphi_d\|_{(H^p(\mathbb{T}^\infty))^*} = \|\varphi_d\|_{H^p(\mathbb{T}^\infty)}^{-1}.$$

Proof. For the lower bound, we simply note that since φ_d is in $H^p(\mathbb{T}^\infty)$ we obtain

$$\|\varphi_d\|_{(H^p(\mathbb{T}^\infty))^*} = \sup_{f \in H^p(\mathbb{T}^\infty)} \frac{|\langle f, \varphi_d \rangle_{H^2(\mathbb{T}^\infty)}|}{\|f\|_{H^p(\mathbb{T}^\infty)}} \geq \frac{\|\varphi_d\|_{H^2(\mathbb{T}^\infty)}^2}{\|\varphi_d\|_{H^p(\mathbb{T}^\infty)}} = \|\varphi_d\|_{H^p(\mathbb{T}^\infty)}^{-1}. \tag{9}$$

For the upper bound, we first use (7) and Lemma 4 to the effect that

$$\|\varphi_d\|_{(H^p(\mathbb{T}^\infty))^*} = \sup_{f\in H^p(\mathbb{T}^\infty)} \frac{|\langle f,\varphi_d\rangle_{H^2(\mathbb{T}^\infty)}|}{\|f\|_{H^p(\mathbb{T}^\infty)}} = \sup_{f\in \mathrm{Lin}(\mathbb{T}^d)} \frac{|\langle f,\varphi_d\rangle_{H^2(\mathbb{T}^\infty)}|}{\|f\|_{H^p(\mathbb{T}^\infty)}}. \tag{10}$$

Any non-trivial element f in $\mathrm{Lin}(\mathbb{T}^d)$ is of the form

$$f(z) = \sum_{j=1}^d c_j z_j$$

with at least one non-zero coefficient. Define

$$\lambda = \langle f,\varphi_d\rangle_{H^2(\mathbb{T}^\infty)} = \frac{c_1+\cdots+c_d}{\sqrt{d}}. \tag{11}$$

After rotating each of the variables if necessary, we may assume that $c_j \geq 0$ for $1\leq j\leq d$ so that $\lambda>0$ whenever f is a non-trivial element in $\mathrm{Lin}(\mathbb{T}^d)$.

For $1\leq k\leq d$, let f_k denote the polynomial obtained by replacing the coefficient sequence $(c_1,\ldots,c_d)$ of f with the shifted sequence

$$(c_k, c_{k+1},\ldots,c_d,c_1,\ldots,c_{k-1}).$$

By symmetry, we find that $\|f_k\|_{H^p(\mathbb{T}^\infty)} = \|f\|_{H^p(\mathbb{T}^\infty)}$. Note also that

$$\frac{1}{d}\sum_{k=1}^d f_k(z) = \frac{c_1+\cdots+c_d}{d}\sum_{j=1}^d z_j = \lambda\varphi_d(z).$$

The triangle inequality therefore allows us to conclude that

$$\lambda\|\varphi_d\|_{H^p(\mathbb{T}^\infty)} \leq \frac{1}{d}\sum_{k=1}^d \|f_k\|_{H^p(\mathbb{T}^\infty)} = \|f\|_{H^p(\mathbb{T}^\infty)}. \tag{12}$$

Using (10) with (11) and (12), we obtain the upper bound

$$\|\varphi_d\|_{(H^p(\mathbb{T}^\infty))^*} = \sup_{f\in H^p(\mathbb{T}^\infty)} \frac{|\langle f,\varphi_d\rangle_{H^2(\mathbb{T}^\infty)}|}{\|f\|_{H^p(\mathbb{T}^\infty)}} \leq \frac{\lambda}{\lambda\|\varphi_d\|_{H^p(\mathbb{T}^\infty)}} = \|\varphi_d\|^{-1}_{H^p(\mathbb{T}^\infty)}$$

which, when combined with the lower bound (9), completes the proof. □

Another viewpoint is to consider $(z_j)_{j\geq1}$ a sequence of independently distributed random variables on the torus and $f(z)=\sum_{j\geq1} c_j z_j$ as a weighted random walk in the plane. The norms $\|f\|_{H^p(\mathbb{T}^\infty)}$ can now be interpreted as moments of this random walk. A simple computation (see Section 3) gives that $\|z_1+z_2\|_{H^1(\mathbb{T}^\infty)} = 4/\pi$ and it is demonstrated in [1] that

$$\|z_1+z_2+z_3\|_{H^1(\mathbb{T}^\infty)} = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right) = 1.57459\ldots$$

In general it is difficult to compute $\|f\|_{H^p(\mathbb{T}^\infty)}$ even for simple linear polynomials f (when p is not an even integer). However, the central limit theorem gives that

$$\lim_{d\to\infty}\left\|\frac{z_1+z_2+\cdots+z_d}{\sqrt{d}}\right\|^p_{H^p(\mathbb{T}^\infty)} = \int_{\mathbb{C}} |Z|^p e^{-|Z|^2}\,\frac{dZ}{\pi} = \Gamma\left(1+\frac{p}{2}\right), \tag{13}$$

since $(z_1+z_2+\cdots+z_d)/\sqrt{d}$ has a limiting complex normal distribution.

We are now ready to prove Theorem 2. To conform with the notations of the present section and to make the proof clearer, we consider now φ in $(H^p(\mathbb{T}^\infty))^*$ and f in $H^p(\mathbb{T}^\infty)$, so φ plays the role of f in the statement of the theorem.

Proof of Theorem 2. Let φ be a linear function in $(H^p(\mathbb{T}^\infty))^*$. By Lemma 4, the Cauchy–Schwarz inequality and Khintchine's inequality (5), we find that

$$\|\varphi\|_{(H^p(\mathbb{T}^\infty))^*} = \sup_{f\in\mathrm{Lin}(\mathbb{T}^\infty)} \frac{|\langle f,\varphi\rangle_{H^2(\mathbb{T}^\infty)}|}{\|f\|_{H^p(\mathbb{T}^\infty)}} \leq \sup_{f\in\mathrm{Lin}(\mathbb{T}^\infty)} \frac{\|f\|_{H^2(\mathbb{T}^\infty)}\|\varphi\|_{H^2(\mathbb{T}^\infty)}}{\|f\|_{H^p(\mathbb{T}^\infty)}} \leq \frac{\|\varphi\|_{H^2(\mathbb{T}^\infty)}}{a_p}.$$

Conversely, Khintchine's inequality (5) also gives that

$$\|\varphi\|_{(H^p(\mathbb{T}^\infty))^*} = \sup_{f\in H^p(\mathbb{T}^\infty)} \frac{|\langle f,\varphi\rangle_{H^2(\mathbb{T}^\infty)}|}{\|f\|_{H^p(\mathbb{T}^\infty)}} \geq \frac{\|\varphi\|^2_{H^2(\mathbb{T}^\infty)}}{\|\varphi\|_{H^p(\mathbb{T}^\infty)}} \geq \frac{\|\varphi\|_{H^2(\mathbb{T}^\infty)}}{b_p},$$

since φ is in $H^p(\mathbb{T}^\infty)$. To prove optimality of the constants, we appeal to Lemma 5 and consider $\varphi_d(z) = (z_1 + \cdots + z_d)/\sqrt{d}$ for $d = 1$ and as $d \to \infty$. □

Theorem 3 also follows easily from Lemma 5 and (13).

Proof of Theorem 3. Let $2 \leq p \leq q \leq \infty$ and set $q^{-1} + r^{-1} = 1$. Suppose that

$$\Gamma\left(1+\frac{p}{2}\right)^{\frac{1}{p}}\Gamma\left(1+\frac{r}{2}\right)^{\frac{1}{r}} > 1. \tag{14}$$

We want to to prove that the Riesz projection is unbounded from $L^q(\mathbb{T}^\infty)$ to $L^p(\mathbb{T}^\infty)$. In view of (2), it is sufficient to find ψ in $L^q(\mathbb{T}^\infty)$ such that

$$\frac{\|P\psi\|_{L^p(\mathbb{T}^\infty)}}{\|\psi\|_{L^q(\mathbb{T}^\infty)}} > 1.$$

We pick ψ_d in $L^q(\mathbb{T}^\infty)$ of minimal norm such that $P\psi_d = \varphi_d$, where φ_d denotes the function from Lemma 5. By (3) and Lemma 5, we obtain

$$\frac{\|P\psi_d\|_{L^p(\mathbb{T}^\infty)}}{\|\psi_d\|_{L^q(\mathbb{T}^\infty)}} = \|\varphi_d\|_{L^p(\mathbb{T}^\infty)}\|\varphi_d\|_{L^r(\mathbb{T}^\infty)}.$$

By (13) and our assumption (14), the right hand side is strictly larger than 1 for some sufficiently large d. □

3. Minimal $L^q(\mathbb{T}^\infty)$ norm

We will now solve the following problem: For $1 < q \leq \infty$, find the element ψ in $L^q(\mathbb{T}^\infty)$ of minimal norm such that

$$P\psi(z) = z_1 + z_2 + \cdots + z_d = \varphi(z).$$

The strict convexity of $L^q(\mathbb{T}^\infty)$ when $1 < q < \infty$ means that the minimizer is unique. Uniqueness of the minimizer holds also for $q = \infty$, but in this case it is a consequence of the continuity of φ on the polytorus (see e.g. [5, Sec. 8.2]).

In view of (3) and (the proof of) Lemma 5, we know that ψ satisfies

$$\|\psi\|_{L^q(\mathbb{T}^\infty)} = \frac{\langle\varphi,\psi\rangle_{L^2(\mathbb{T}^\infty)}}{\|\varphi\|_{L^p(\mathbb{T}^\infty)}} = \frac{d}{\|\varphi\|_{L^p(\mathbb{T}^\infty)}} \tag{15}$$

with $p^{-1} + q^{-1} = 1$. On the left hand side of (15) we have attained equality in Hölder's inequality, which implies that $|\psi| = C|\varphi|^{p-1}$ almost everywhere. Inserting this into the norm expression $\|\psi\|_{L^q(\mathbb{T}^\infty)}$ in (15) and using that $(p-1)q = p$, we find that $C = d\|\varphi\|^{-p}_{L^p(\mathbb{T}^\infty)}$. From Hölder's inequality and (15) we also see that

$$\langle|\varphi|, |\psi|\rangle_{L^2(\mathbb{T}^\infty)} \leq \|\varphi\|_{L^p(\mathbb{T}^\infty)}\|\psi\|_{L^q(\mathbb{T}^\infty)} = \langle\varphi, \psi\rangle_{L^2(\mathbb{T}^\infty)},$$

which is only possible if $\varphi\overline{\psi} \geq 0$ almost everywhere. Combining these observations yields that

$$\psi(z) = \frac{d}{\|\varphi\|^p_{L^p(\mathbb{T}^\infty)}}|\varphi(z)|^{p-2}\varphi(z)$$

is the element in $L^q(\mathbb{T}^\infty)$ of minimal norm such that $P\psi(z) = z_1+z_2+\cdots+z_d = \varphi(z)$ for $1 < p \leq \infty$ and $p^{-1} + q^{-1} = 1$. Note that ψ is 1-homogeneous, which we knew in advance by Lemma 4. We can also directly verify that

$$\int_{\mathbb{T}^\infty} \psi(z)\,\overline{z_j}\,dm_\infty(z) = \int_{\mathbb{T}^\infty} \psi(z)\,\frac{\overline{z_1}+\overline{z_2}+\cdots+\overline{z_d}}{d}\,d\mu_\infty(z) = 1,$$

since ψ inherits the symmetry of φ.

When $d = 2$, we can actually compute the Fourier series explicitly. We begin by using the trick $z_1 + z_2 = z_2(1 + z_1\overline{z_2})$ to write $\psi(z) = z_2\Psi(z_1\overline{z_2})$, where

$$\Psi(z) = \frac{2}{\|1+z\|^p_{L^p(\mathbb{T})}}|1+z|^{p-2}(1+z).$$

Then we get that

$$\frac{\|1+z\|^p_{L^p(\mathbb{T})}}{2} = \frac{1}{2}\int_{-\pi}^{\pi}|1+e^{i\theta}|^p\,\frac{d\theta}{2\pi} = 2^{p-1}\int_{-\pi}^{\pi}\cos^p\left(\frac{\theta}{2}\right)\frac{d\theta}{2\pi} = \frac{2^p}{\pi}\int_0^{\pi/2}\cos^p(\vartheta)\,d\vartheta.$$

Similarly, we compute:

$$\begin{aligned}\int_{-\pi}^{\pi}|1+e^{i\theta}|^{p-2}(1+e^{i\theta})\,e^{-ik\theta}\,\frac{d\theta}{2\pi} &= 2^{p-1}\int_{-\pi}^{\pi}\cos^{p-1}\left(\frac{\theta}{2}\right)\,e^{-i(k-1/2)\theta}\,\frac{d\theta}{2\pi} \\ &= 2^{p-1}\int_{-\pi/2}^{\pi/2}\cos^{p-1}(\vartheta)\,e^{-i(2k-1)\vartheta}\,\frac{d\vartheta}{\pi} \\ &= \frac{2^p}{\pi}\int_0^{\pi/2}\cos^{p-1}(\vartheta)\cos((1-2k)\vartheta)\,d\vartheta\end{aligned}$$

The latter integral, which contains the former as the special case $k = 0, 1$ is known (see e.g. [6, p. 399]) and we obtain that

$$\int_{-\pi}^{\pi}|1+e^{i\theta}|^{p-2}(1+e^{i\theta})\,e^{-ik\theta}\,\frac{d\theta}{2\pi} = \frac{1}{p\,\mathrm{Beta}\left(\frac{p+1-2k+1}{2}, \frac{p-1+2k+1}{2}\right)}$$

for $\mathrm{Beta}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Combining everything, we find that

$$\psi(e^{i\theta_1}, e^{i\theta_2}) = \sum_{k\in\mathbb{Z}}\frac{\Gamma(1+p/2)\Gamma(p/2)}{\Gamma(1+p/2-k)\Gamma(p/2+k)}\,e^{ik\theta_1}e^{i(1-k)\theta_2}.$$

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References

1. J. M. Borwein, D. Nuyens, A. Straub, and J. Wan, *Some arithmetic properties of short random walk integrals*, Ramanujan J. **26** (2011), no. 1, 109–132.
2. O. F. Brevig, J. Ortega-Cerdà, K. Seip, and J. Zhao, *Contractive inequalities for Hardy spaces*, Funct. Approx. Comment. Math. **59** (2018), no. 1, 41–56.
3. O. F. Brevig and K.-M. Perfekt, *Failure of Nehari's theorem for multiplicative Hankel forms in Schatten classes*, Studia Math. **228** (2015), no. 2, 101–108.
4. B. J. Cole and T. W. Gamelin, *Representing measures and Hardy spaces for the infinite polydisk algebra*, Proc. London Math. Soc. (3) **53** (1986), no. 1, 112–142.
5. P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970.
6. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, eighth ed., Elsevier/Academic Press, Amsterdam, 2015.
7. H. König and S. Kwapień, *Best Khintchine type inequalities for sums of independent, rotationally invariant random vectors*, Positivity **5** (2001), no. 2, 115–152.
8. J. Marzo and K. Seip, *L^∞ to L^p constants for Riesz projections*, Bull. Sci. Math. **135** (2011), no. 3, 324–331.
9. J. Ortega-Cerdà and K. Seip, *A lower bound in Nehari's theorem on the polydisc*, J. Anal. Math. **118** (2012), no. 1, 339–342.

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