Control of a 1–D Time-Variant Linear Hyperbolic PDE using Infinite-Dimensional Backstepping

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***Abstract*— We derive a state-feedback controller for a scalar 1–D linear hyperbolic partial differential equation (PDE) with a spatially- and time-varying interior-domain parameter. The resulting controller ensures convergence to zero in a finite time d_1, corresponding to the propagation time from one boundary to the other. The control law requires predictions of the in-domain parameter a time d_1 into the future. The state-feedback controller is also combined with a boundary observer into an output-feedback control law. Lastly, under the assumption that the interior-domain parameter can be decoupled into a time-varying and a spatially-varying part, a stabilizing adaptive output-feedback control law is derived for an uncertain spatially varying parameter, stabilizing the system in the L_2-sense from a single boundary measurement only. All derived controllers are implemented and demonstrated in simulations.**

I. Introduction

Systems of hyperbolic partial differential equations (PDEs) describe flow and transport phenomena. Typical examples range from traffic [1], and oil wells to time-delays [2] and predator-prey systems [3]. Several approaches have been used for design of estimators and controllers for such systems, ranging from control Lyapunov functions [4] and Riemann invariants [5] to frequency domain approaches [6], to mention a few.

Infinite-dimensional backstepping has in the last decade and a half decade proven itself to be a powerful tool in the design of controllers and observers for linear PDEs. The key strength of infinite-dimensional backstepping for controller (and observer) design of PDEs, is the introduction of an invertible Volterra transform - the backstepping transform - and a control law that map the system of interest into a target system designed with some desirable stability properties. The analysis is hence done on the infinite-dimensional system directly, avoiding any discretization before an eventual implementation on a computer.

Starting in the early 2000s with non-adaptive stabilization of the heat equation [7], the backstepping method quickly found its application in adaptive control problems for parabolic PDEs [8]. Several results on adaptive control of more general parabolic PDEs using the backstepping method followed in the later years [9] [10] [11], and even a book [12] was published on the topic.

The first use of backstepping for control of linear *hyperbolic* PDEs, was in 2008 in the paper [2] for a scalar 1-D system. Extensions to more complicated systems of hyperbolic PDEs were derived a few years later in [13], for two coupled linear hyperbolic PDEs, and in [14] and [15] for more complicated systems of PDEs. Several adaptive solutions have also been proposed, where hyperbolic PDEs with uncertain system parameters have been stabilized both when assuming full-state measurements [16], [17] and boundary measurements [18], [19], [20] are available. However, all the above mentioned results on control of hyperbolic PDEs using backstepping considered systems with time-invariant system parameters.

The amount of material regarding the use of backstepping for stabilization of hyperbolic PDEs with time-varying parameters, however, is very limited. To the best of the authors' knowledge no such control result exists in the literature. However, an observer based on backstepping was derived in [21] for a hyperbolic partial differential integro-differential equation (PIDE) with time-varying parameters.

We will in this paper consider a control problem for a scalar 1–D linear hyperbolic PDE with an in-domain parameter that is allowed to vary with both space and time. The problem is formally stated in Section II. A state-feedback controller is derived in Section III, assuming full-state measurements are available. The controller achieves convergence to zero in a finite time corresponding to the propagation time from one boundary to the other. We believe this is the first such results, where a linear hyperbolic PDE with a time-varying parameter is stabilized using infinite-dimensional backstepping. The resulting controller is also in Section IV combined with a boundary observer into an output-feedback controller. Additionally, in Section V, we assume the in-domain parameter can be decoupled into a spatially varying and a time-varying part, and derive an adaptive output-feedback controller stabilizing the system in the L_2-sense from a single boundary measurement only. All derived controllers require predictions of the time-varying parameter a time into the future corresponding to the total propagation time in the PDE. All derived controllers are implemented and simulated in VI, while some concluding remarks are offered in Section VII.

II. Problem statement

We consider a 1–D linear hyperbolic partial differential equation in the form

$$u_t(x,t) - \mu u_x(x,t) = \varpi(x,t)u(0,t) \tag{1a}$$

$$u(1,t) = U(t) \tag{1b}$$

$$u(x,0) = u_0(x) \tag{1c}$$

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where $u(x,t)$ is the system state defined for over $\mathcal{S}$, where

$$\mathcal{S} = \{(x,t) \mid x \in [0,1], t \geq 0\}. \tag{2}$$

The system parameters and initial condition are assumed to satisfy

$$\mu \in \mathbb{R},\ \mu > 0, \qquad \varpi \in C^0([0,1]\times[0,\infty)) \tag{3a}$$
$$u_0 \in L_2([0,1]). \tag{3b}$$

We will derive a backstepping-based state feedback control law $U(t)$ in the form

$$U(t) = \int_0^1 k(\xi,t)u(\xi,t)d\xi \tag{4}$$

that stabilizes system (1), and specifically achieves $u \equiv 0$ in a finite time d_1, defined as

$$d_1 = \mu^{-1}. \tag{5}$$

In order to achieve this, we assume the following.

Assumption 1: The parameter $\varpi(x,t)$ is known for all for $x \in [0,1]$ and for all time t, and is predictable a time d_1 into the future. Moreover, there exists a constant $\bar{\varpi}$ so that

$$|\varpi(x,t)| \leq \bar{\varpi},\ \forall (x,t) \in \mathcal{S} \tag{6}$$

where $\mathcal{S}$ is defined in (2).

We will also show how a (trivial) output-feedback solution can be implemented, requiring boundary sensing

$$y(t) = u(0,t) \tag{7}$$

only.

Lastly, we show how a previously derived adaptive controller can be slightly altered to solve an adaptive output-feedback stabilization problem for system (1), assuming $\varpi(x,t)$ can be separated in its spatially-varying and time-varying parts, that is: ϖ is on the form

$$\varpi(x,t) = \theta(x)g(t), \tag{8}$$

where

$$\theta \in C^0([0,1]), \qquad g \in C^0([0,\infty)). \tag{9}$$

This adaptive control law is derived subject to the following assumption.

Assumption 2: The parameter g is known for all t and a time d_1 into the future. Moreover, we are in knowledge of some positive constants $\bar{\theta}$ and $\bar{g}$ so that

$$|\theta(x)| \leq \bar{\theta},\ \forall x \in [0,1], \tag{10a}$$
$$|g(t)| \leq \bar{g},\ \forall t \geq 0. \tag{10b}$$

III. Non-adaptive state-feedback controller

Consider the control law (4), which we for the reader's convenience restate here

$$U(t) = \int_0^1 k(\xi,t)u(\xi,t)d\xi, \tag{11}$$

and let k be taken as the solution to the Volterra integral equation

$$\mu k(x,t) = \int_x^1 k(1+x-\xi,t)\varpi(1-\xi,t+d_1x)d\xi - \varpi(1-x,t+d_1x) \tag{12}$$

where d_1 is defined in (5). The kernel is bounded for all $t \geq 0$, following Assumption 1.

Theorem 3: Consider system (1). Assume Assumption 1 holds.- Then the control law (11) with k given as the solution to the Volterra integral equation (12) ensures

$$u \equiv 0 \tag{13}$$

for all $t \geq d_1$, where d_1 is defined in (5).

Proof: We will use a backstepping technique similar to the one used for stabilizing a time-invariant system in [2]. Consider the following backstepping transformation

$$\alpha(x,t) = u(x,t) - \int_0^x K(x,\xi,t)u(\xi,t)d\xi. \tag{14}$$

for some kernel $K(x,\xi,t)$ defined over $\mathcal{T}_1$, where

$$\mathcal{T}_1 = \mathcal{T} \times \{t \geq 0\} \tag{15a}$$
$$\mathcal{T} = \{(x,\xi) \mid 0 \leq \xi \leq x \leq 1\} \tag{15b}$$

satisfying the PDE

$$K_t(x,\xi,t) = \mu K_x(x,\xi,t) + \mu K_\xi(x,\xi,t) \tag{16a}$$
$$\mu K(x,0,t) = \int_0^x K(x,\xi,t)\varpi(\xi,t)d\xi - \varpi(x,t) \tag{16b}$$
$$K(x,\xi,0) = K_0(x,\xi) \tag{16c}$$

for some bounded initial condition K_0 defined over $\mathcal{T}$. The kernel K is bounded for all $t \geq 0$, following Assumption 1.

We will show that the transformation (14) and control law (11) map system (1) into the target system

$$\alpha_t(x,t) - \mu\alpha_x(x,t) = 0 \tag{17a}$$
$$\alpha(1,t) = 0 \tag{17b}$$
$$\alpha(x,0) = \alpha_0(x) \tag{17c}$$

for some initial condition $\alpha_0 \in L_2([0,1])$.

Differentiating (14) with respect to time and space, respectively, inserting the dynamics (1a), and integration by parts, we obtain

$$\begin{aligned} u_t(x,t) &= \alpha_t(x,t) + \mu K(x,x,t)u(x,t) \\ &\quad - \mu K(x,0,t)u(0,t) - \int_0^x \mu K_\xi(x,\xi,t)u(\xi,t)d\xi \\ &\quad + \int_0^x K(x,\xi,t)\varpi(\xi,t)d\xi u(0,t) \\ &\quad + \int_0^x K_t(x,\xi,t)u(\xi,t)d\xi \end{aligned} \tag{18}$$

and

$$\begin{aligned} u_x(x,t) &= \alpha_x(x,t) + K(x,x,t)u(x,t) \\ &\quad + \int_0^x K_x(x,\xi,t)u(\xi,t)d\xi, \end{aligned} \tag{19}$$

respectively. Inserting (18) and (19) into the dynamics (1a), we obtain

$$
\begin{aligned}
& u_t(x,t) - \mu u_x(x,t) - \varpi(x,t)u(0,t) \\
&= \alpha_t(x,t) - \mu\alpha_x(x,t) \\
&\quad - \bigg[\mu K(x,0,t) + \varpi(x,t) \\
&\qquad\qquad - \int_0^x K(x,\xi,t)\varpi(\xi,t)d\xi\bigg]u(0,t) \\
&\quad + \int_0^x \bigg[K_t(x,\xi,t) - \mu K_x(x,\xi,t) \\
&\qquad\qquad - \mu K_\xi(x,\xi,t)\bigg]u(\xi,t)d\xi = 0 \qquad (20)
\end{aligned}
$$

using the equations (16a)–(16b) gives the target system dynamics (17a).

Evaluating (14) at $x = 1$ and inserting the boundary condition (1b)

$$
\alpha(1,t) = U(t) - \int_0^1 K(1,\xi,t)u(\xi,t)d\xi \qquad (21)
$$

Choosing

$$
U(t) = \int_0^1 K(1,\xi,t)u(\xi,t)d\xi \qquad (22)
$$

gives (17b). Lastly, the initial condition (17c) is found from evaluating (14) at $t = 0$ to yield

$$
w_0(x) = u_0(x) - \int_0^x K_0(x,\xi)u_0(\xi)d\xi. \qquad (23)
$$

Next, we analyze the kernel equations (16). Using the method of characteristics, we can obtain

$$
\frac{d}{ds}K(x+\mu s, \xi+\mu s, t-s) = 0 \qquad (24)
$$

Integrating in s from $s = 0$ to $s = d_1(1-x)$, and noting from comparing (11) and (22) that

$$
K(1,\xi,t) = k(\xi,t), \qquad (25)
$$

we obtain

$$
K(x,\xi,t) = k(1+\xi-x, t-d_1(1-x)), \qquad (26)
$$

where we have assumed that the initial condition K_0 given in (16c) is compatible with the equations (16a)–(16b) for past values of t. From (26), we specifically have

$$
K(x,0,t) = k(1-x, t-d_1(1-x)). \qquad (27)
$$

Inserting (26) and (27) into (16b) gives

$$
\begin{aligned}
\mu k(1-x, t-d_1(1-x)) &= -\varpi(x,t) \\
&\quad + \int_0^x k(\xi+1-x, t-d_1(1-x))\varpi(\xi,t)d\xi. \qquad (28)
\end{aligned}
$$

A substitution $x \to 1-x$ followed by a time-shift $(t - d_1x) \to t$ give

$$
\begin{aligned}
\mu k(x,t) &= \int_0^{1-x} k(\xi+x,t)\varpi(\xi,t+d_1x)d\xi \\
&\quad - \varpi(1-x,t+d_1x) \qquad (29)
\end{aligned}
$$

and appropriate substitution $\gamma = 1-\xi$ in the integral gives

$$
\begin{aligned}
\mu k(x,t) &= \int_x^1 k(1+x-\gamma,t)\varpi(1-\gamma,t+d_1x)d\gamma \\
&\quad - \varpi(1-x,t+d_1x) \qquad (30)
\end{aligned}
$$

which is the same as (12).

It is clear from the simple structure of the target system (17) that $\alpha \equiv 0$ for $t \geq d_1$, and due to the invertibility of the backstepping transformation (14), the result follows. ■

IV. Non-adaptive output-feedback controller

Designing an observer for system (1) and hence an output-feedback controller is almost trivial. Consider the observer

$$
\begin{aligned}
\check{u}_t(x,t) - \mu\check{u}_x(x,t) &= \varpi(x,t)u(0,t) && (31a) \\
\check{u}(1,t) &= U(t) && (31b) \\
\check{u}(x,0) &= \check{u}_0(x) && (31c)
\end{aligned}
$$

for some initial condition $\check{u}_0 \in L_2([0,1])$. Consider also the control law

$$
U(t) = \int_0^1 k(x,t)\check{u}(x,t)d\xi, \qquad (32)
$$

where k is the solution to the Volterra integral equation (12).

Theorem 4: Consider system (1) and the observer (31). Assume Assumption 1 holds. Then the control law (32) with k given as the solution to the Volterra integral equation (12) ensures

$$
\check{u} \equiv u \qquad (33)
$$

for $t \geq d_1$, and

$$
u \equiv 0 \qquad (34)
$$

for $t \geq 2d_1$, where d_1 is defined in (5).

Proof: The observer error $\tilde{u} = u - \check{u}$ can straightforwardly, using (1) and (31) be shown to have the dynamics

$$
\begin{aligned}
\tilde{u}_t(x,t) - \mu\tilde{u}_x(x,t) &= 0 && (35a) \\
\tilde{u}(1,t) &= 0 && (35b) \\
\tilde{u}(x,0) &= \tilde{u}_0(x) && (35c)
\end{aligned}
$$

where $\tilde{u}_0 = u_0 - \check{u}_0$, from which it is clear that $\tilde{u} \equiv 0$ and hence $\check{u} \equiv u$ for $t \geq d_1$. The control law (32) is therefore for $t \geq d_1$ equivalent with the control law (11), for which $u \equiv 0$ for $t \geq 2d_1$ follows from Theorem 3. ■

V. Adaptive output-feedback controller

We now assume (8), that is, ϖ is on the form

$$
\varpi(x,t) = \theta(x)g(t). \qquad (36)
$$

That is, we investigate the system

$$
\begin{aligned}
u_t(x,t) - \mu u_x(x,t) &= \theta(x)g(t)u(0,t) && (37a) \\
u(1,t) &= U(t) && (37b) \\
u(x,0) &= u_0(x) && (37c)
\end{aligned}
$$

$$y(t) = u(0,t) \tag{37d}$$

where we also have added the measurement (7). Moreover, we assume Assumption 2 holds.

The control strategy we will use, is heavily based on a similar problem originally solved in [18], and involves expressing the system state u as a linear combination of a set of filters, and the uncertain parameter θ. However, the stability proof will, due to the time-varying parameter g be significantly altered.

A. Filter design

We introduce the filters

$$\psi_t(x,t) - \mu\psi_x(x,t) = 0, \qquad \psi(1,t) = U(t)$$
$$\psi(x,0) = \psi_0(x) \tag{38a}$$
$$\phi_t(x,t) - \mu\phi_x(x,t) = 0, \qquad \phi(1,t) = g(t)u(0,t)$$
$$\phi(x,0) = \phi_0(x) \tag{38b}$$

where ψ and ϕ are defined over $\mathcal{S}$ defined in (2), with initial conditions

$$\psi_0, \phi_0 \in L_2([0,1]). \tag{39}$$

Consider the non-adaptive state estimate of u generated as

$$\bar{u}(x,t) = \psi(x,t) + d_1 \int_x^1 \theta(\xi)\phi(1-(\xi-x),t)d\xi \tag{40}$$

Lemma 5: Consider the system (37) and the non-adaptive state estimate generated from (40). Then

$$\bar{u} \equiv u \tag{41}$$

for $t \geq d_1$.

Proof: Define the non-adaptive state estimation error e as

$$e(x,t) = u(x,t) - \bar{u}(x,t) \tag{42}$$

We will show that e satisfies the dynamics

$$e_t(x,t) - e_x(x,t) = 0 \tag{43a}$$
$$e(1,t) = 0 \tag{43b}$$
$$e(x,0) = e_0(x) \tag{43c}$$

for some $e_0 \in L_2([0,1])$. From differentiating (42) with respect to time and space, and inserting the dynamics (37a) and (38), we obtain

$$e_t(x,t) = \mu u_x(x,t) + \theta(x)g(t)u(0,t) - \mu\psi_x(x,t) - \int_x^1 \theta(\xi)\phi_x(1-(\xi-x),t)d\xi \tag{44}$$

and

$$e_x(x,t) = u_x(x,t) - \psi_x(x,t) + d_1\theta(x)\phi(1,t) - d_1\int_x^1 \theta(\xi)\phi_x(1-(\xi-x))d\xi, \tag{45}$$

respectively, which immediately gives (43a) when inserting the boundary condition (38b). Evaluating (42) at $x=1$ and using the boundary conditions (37b) and (38a) gives (43b). The initial condition (43c) is given as

$$e(x,0) = u(x,0) - \bar{u}(x,0) = u_0(x) - \psi_0(x) - d_1\int_x^1 \theta(\xi)\phi_0(1-(\xi-x))d\xi \tag{46}$$

From the dynamics (43), it is evident that $e \equiv 0$ and hence $\bar{u} \equiv u$ for $t \geq d_1$. ■

B. Adaptive laws

From the relationship (40) and Lemma 5, we have

$$y(t) = u(0,t) = \psi(0,t) + d_1\int_0^1 \theta(\xi)\phi(1-\xi,t)d\xi \tag{47}$$

from which we propose the adaptive law

$$\hat{\theta}_t(x,t) = \text{proj}_{\bar{\theta}}\left\{\gamma(x)\frac{\hat{e}(0,t)\phi(1-x,t)}{1+||\phi(t)||^2}, \hat{\theta}(x,t)\right\} \tag{48a}$$
$$\hat{\theta}(x,0) = \hat{\theta}_0(x) \tag{48b}$$

where the initial condition $\hat{\theta}_0$ is chosen in inside the feasible domain

$$|\hat{\theta}_0(x)| \leq \bar{\theta}, \ \forall x \in [0,1], \tag{49}$$

and $\gamma(x) > 0, \ \forall x \in [0,1]$ is a design gain, the projection operator is given as

$$\text{proj}_a(\tau,\omega) = \begin{cases} 0 & \text{if } \omega = -a \text{ and } \tau \leq 0 \\ 0 & \text{if } \omega = a \text{ and } \tau \geq 0 \\ \tau & \text{otherwise} \end{cases} \tag{50}$$

and

$$\hat{e}(x,t) = u(x,t) - \hat{u}(x,t) \tag{51}$$

is the prediction error, computed from the adaptive state estimate $\hat{u}$ generated from

$$\hat{u}(x,t) = \psi(x,t) + d_1\int_x^1 \hat{\theta}(\xi,t)\phi(1-(\xi-x),t)d\xi. \tag{52}$$

Lemma 6: The adaptive law (48) with initial condition satisfying (49) provide the following signal properties

$$|\hat{\theta}(x,t)| \leq \bar{\theta}, \forall x \in [0,1], t \geq 0 \tag{53a}$$
$$||\tilde{\theta}_t|| \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{53b}$$
$$\sigma \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{53c}$$

where $\tilde{\theta} = \theta - \hat{\theta}$, and

$$\sigma(t) = \frac{\hat{e}(0,t)}{\sqrt{1+||\phi(t)||^2}}. \tag{54}$$

Proof: Similar proofs like this have been stated many times before, e.g. in [18], [19], [22], but we include a proof here for completeness. The property (53a) follows from the projection operator in (48) and the initial conditions (49). Consider the Lyapunov function candidate

$$V(t) = \frac{1}{2}\int_0^1 \gamma^{-1}(x)\tilde{\theta}^2(x,t)dx. \tag{55}$$

Differentiating with respect to time, inserting the adaptive law (53) and using the property $-\tilde{\theta}\text{proj}_{\bar{\theta}}(\tau,\hat{\theta}) \leq -\tilde{\theta}\tau$ ([23, Lemma E.1]), we find

$$\dot{V}(t) \leq -\frac{\hat{e}(0,t)}{1+||\phi(t)||^2}\int_0^1 \tilde{\theta}(x,t)\phi(1-x,t)dx. \tag{56}$$

From (42), (40), (52) and (51), we can derive the relationship

$$\hat{e}(0,t) = d_1\int_0^1 \tilde{\theta}(x,t)\phi(1-x,t)dx + e(0,t) \tag{57}$$

where $e(0,t) = 0$ for $t \geq d_1$. Inserting (57) into (56), we obtain

$$\dot{V}(t) \leq -\sigma^2(t) \tag{58}$$

for $t \geq d_1$, with σ defined in (54). This proves that V is bounded and non-increasing, and hence has a limit V_∞ as $t \to \infty$. Integrating (58) in time from zero to infinity, we find

$$\int_0^\infty \sigma^2(t)dt \leq V(0) - V_\infty \leq V(0) < \infty, \tag{59}$$

which proves that $\sigma \in \mathcal{L}_2$. Using (57), we obtain, for $t \geq d_1$, using Cauchy-Schwarz' inequality

$$\begin{aligned}\frac{|\hat{e}(0,t)|}{\sqrt{1+||\phi(t)||^2}} &= \frac{|\int_0^1 \tilde{\theta}(\xi,t)\phi(1-\xi,t)d\xi|}{\sqrt{1+||\phi(t)||^2}}\\ &\leq ||\hat{\theta}(t)||\frac{||\phi(t)||}{\sqrt{1+||\phi(t)||^2}} \leq ||\hat{\theta}(t)|| \end{aligned}\tag{60}$$

which proves that $\sigma \in \mathcal{L}_\infty$. From the adaptation law (48), we have

$$\begin{aligned}||\hat{\theta}_t(t)|| &\leq ||\gamma||\frac{|\hat{e}(0,t)|}{\sqrt{1+||\phi(t)||^2}}\frac{||\phi(t)||}{\sqrt{1+||\phi(t)||^2}}\\ &\leq ||\gamma|||\sigma(t)|\end{aligned}\tag{61}$$

which, along with (53c) gives (53b). ■

C. Adaptive state estimate dynamics

Using the filter dynamics (38), and the definition of $\hat{u}$ in (52), it is straightforwardly possible to derive the dynamics for $\hat{u}$ as

$$\begin{aligned}\hat{u}_t(x,t) - \mu\hat{u}_x(x,t) &= \hat{\theta}(x,t)g(t)u(0,t)\\ &\quad + d_1\int_x^1 \hat{\theta}_t(\xi,t)\phi(1-(\xi-x),t)d\xi \quad (62a)\\ \hat{u}(1,t) &= U(t) \quad (62b)\\ \hat{u}(x,0) &= \hat{u}_0(x) \quad (62c)\end{aligned}$$

for some initial condition $\hat{u}_0 \in L_2([0,1])$.

D. Adaptive control law

We propose the following control law

$$U(t) = \int_0^1 \hat{k}(\xi,t)\hat{u}(\xi,t)d\xi \tag{63}$$

where $\hat{k}$ is given as the solution to the Volterra integral equation

$$\begin{aligned}\mu\hat{k}(x,t) &= \int_x^1 \hat{k}(1+x-\gamma,t)\hat{\theta}(1-\gamma,t)g(t+d_1x)d\gamma\\ &\quad - \hat{\theta}(1-x,t)g(t+d_1x)\end{aligned}\tag{64}$$

where $\hat{\theta}$ is generated using (48).

Theorem 7: Consider the system (37), the adaptive state estimate (52) and the adaptive law (48). Assume Assumption 2 holds. Then the control law (63) ensures

$$||u||, ||\hat{u}||, ||\phi||, ||\psi|| \in \mathcal{L}_2 \cap \mathcal{L}_\infty \tag{65a}$$
$$||u||, ||\hat{u}||, ||\phi||, ||\psi|| \to 0. \tag{65b}$$

Proof: We now consider the same type of backstepping as in the non-adaptive case, and consider

$$\begin{aligned}w(x,t) &= \hat{u}(x,t) - \int_0^x \hat{K}(x,\xi,t)\hat{u}(\xi,t)d\xi\\ &= T[\hat{u}](x,t)\end{aligned}\tag{66}$$

where $\hat{K}(x,\xi,t)$ is defined over $\mathcal{T}_1$ given in (15a), and given from $\hat{k}$ as

$$\hat{K}(x,\xi,t) = \hat{k}(1+\xi-x, t-d_1(1-x)). \tag{67}$$

We note that, $\hat{\theta}$ and g are uniformly bounded (the former by projection, the latter by Assumption 2), $\hat{k}$ as the solution to (64) and hence also $\hat{K}$ will be uniformly bounded. That is; there exists a constant $\bar{k} \geq 0$ (depending on $\bar{\theta}$ and $\bar{g}$) so that

$$|\hat{k}(x,t)| \leq \bar{k},\ \forall x \in [0,1]\times\{t \geq 0\} \tag{68}$$
$$|\hat{K}(x,\xi,t)| \leq \bar{k},\ \forall (x,\xi,t) \in \mathcal{T}_1. \tag{69}$$

Since the kernel $\hat{K}(x,\xi,t)$ is uniformly bounded, the invertibility of (66) follows, and there exists a constant $G_1 > 0$ (depending on $\bar{k}$) so that

$$||w(t)|| = ||T[\hat{u}](t)|| \leq G_1||\hat{u}(t)||, \quad \forall t \geq 0. \tag{70}$$

Next, we will show that the backstepping transformation (66) and the control law (63) map (62) into the following target system

$$\begin{aligned}w_t(x,t) - \mu w_x(x,t) &= \int_{t-d_1(1-x)}^t \hat{\theta}_t(x,\tau)d\tau g(t)w(0,t)\\ &\quad + d_1T\left[\int_x^1 \hat{\theta}_t(\xi,t)\phi(1-(\xi-x),t)d\xi\right](x,t)\\ &\quad + T[\hat{\theta}](x,t)g(t)\hat{e}(0,t) \quad (71a)\\ w(1,t) &= 0 \quad (71b)\\ w(0,t) &= w_0(x). \quad (71c)\end{aligned}$$

Performing the same steps as in the non-adaptive case, by differentiating (66) with respect to time and space, inserting the dynamics (62a) and integration by parts, yields

$$\begin{aligned}\hat{u}_t(x,t) &= w_t(x,t) + \mu\hat{K}(x,x,t)\hat{u}(x,t)\\ &\quad - \mu\hat{K}(x,0,t)\hat{u}(0,t) - \int_0^x \mu\hat{K}_\xi(x,\xi,t)\hat{u}(\xi,t)d\xi\end{aligned}$$

$$+\int_0^x \hat{K}(x,\xi,t)\theta(\xi,t)d\xi\hat{u}(0,t)$$
$$+\int_0^x \hat{K}(x,\xi,t)\theta(\xi,t)d\xi\hat{e}(0,t)$$
$$+d_1\int_0^x \hat{K}(x,\xi,t)\int_\xi^1 \hat{\theta}_t(s,t)\phi(1-(s-\xi),t)dsd\xi$$
$$+\int_0^x \hat{K}_t(x,\xi,t)\hat{u}(\xi,t)d\xi \tag{72}$$

and

$$\hat{u}_x(x,t) = w_x(x,t) + \hat{K}(x,x,t)\hat{u}(x,t) + \int_0^x \hat{K}_x(x,\xi,t)\hat{u}(\xi,t)d\xi, \tag{73}$$

where we have used that $u(0) = \hat{u}(0) + \hat{e}(0)$. Inserting (72) and (73) into (62a) and using $u(0) = \hat{u}(0) + \hat{e}(0)$ again, results in

$$\begin{aligned}0 &= \hat{u}_t(x,t) - \mu\hat{u}_x(x,t) - \hat{\theta}(x,t)g(t)\hat{u}(0,t) \\ &\quad - \hat{\theta}(x,t)g(t)\hat{e}(0,t) - d_1\int_x^1 \hat{\theta}_t(\xi,t)\phi(1-(\xi-x),t)d\xi \\ &= w_t(x,t) - \mu w_x(x,t) - f(x,t)\hat{u}(0,t) \\ &\quad - T[\hat{\theta}](x,t)g(t)\hat{e}(0,t) \\ &\quad - d_1 T\left[\int_x^1 \hat{\theta}_t(\xi,t)\phi(1-(\xi-x),t)d\xi\right](x,t)\end{aligned} \tag{74}$$

where we used the fact that

$$\hat{K}_t(x,\xi,t) = \mu\hat{K}(x,\xi,t) + \mu\hat{K}(x,\xi,t) \tag{75}$$

which is easily verified from (67), and where

$$f(x,t) = \mu\hat{K}(x,0,t) + \hat{\theta}(x,t)g(t) - \int_0^x \hat{K}(x,\xi,t)\hat{\theta}(\xi,t)d\xi g(t). \tag{76}$$

Inserting (67) into (76), we have

$$f(x,t) = \mu\hat{k}(1-x,t-d_1(1-x)) + \hat{\theta}(x,t)g(t) - \int_0^x \hat{k}(1+\xi-x,t-d_1(1-x))\hat{\theta}(\xi,t)d\xi g(t). \tag{77}$$

From (64), we have

$$\begin{aligned}&\mu\hat{k}(1-x,t-d_1(1-x)) \\ &= \int_0^x \hat{k}(1-x+\xi,t-d_1(1-x))\hat{\theta}(\xi,t)g(t)d\xi \\ &\quad - \hat{\theta}(x,t-d_1(1-x))g(t)\end{aligned} \tag{78}$$

and inserting this, we obtain

$$\begin{aligned}f(x,t) &= \left[\hat{\theta}(x,t) - \hat{\theta}(x,t-d_1(1-x))\right]g(t) \\ &= \int_{t-d_1(1-x)}^t \hat{\theta}_t(x,\tau)d\tau g(t)\end{aligned} \tag{79}$$

which, when inserted into (74) gives the dynamics (71a) when noting that $\hat{u}(0,t) = w(0,t)$.

The boundary condition (71b) follows from evaluating (66) at $x = 1$, inserting the boundary condition (62b) and the control law (63) and noting from (67) that $\hat{K}(1,\xi,t) = \hat{k}(\xi,t)$. Lastly, the boundary condition (71c) is given from $\hat{u}_0$ as $w_0(x) = T[\hat{u}_0](x)$, found from evaluating (66) at $t = 0$.

We now prove stability of the closed loop system. Consider the functions

$$V_1(t) = \int_0^1 (1+x)w^2(x,t)dx \tag{80a}$$
$$V_2(t) = \int_0^1 (1+x)\phi^2(x,t)dx. \tag{80b}$$

Differentiating (80a) with respect to time, inserting the dynamics (71a), integration by parts and inserting the boundary condition (71b), one find

$$\begin{aligned}&\dot{V}_1(t) = -\mu w^2(0,t) - \mu||w(t)||^2 \\ &+ 2\int_0^1 (1+x)w(x,t)\int_{t-d_1(1-x)}^t \hat{\theta}_t(x,\tau)d\tau g(t)w(0,t)dx \\ &+ 2d_1\int_0^1 (1+x)w(x,t) \\ &\qquad \times T\left[\int_x^1 \hat{\theta}_t(\xi,t)\phi(1-(\xi-x),t)d\xi\right](x,t)dx \\ &+ 2\int_0^1 (1+x)w(x,t)T[\hat{\theta}](x,t)dxg(t)\hat{e}(0,t)\end{aligned} \tag{81}$$

Using Young's inequality on the cross terms, this can be bounded as

$$\begin{aligned}&\dot{V}_1(t) = -\mu w^2(0,t) - \mu||w(t)||^2 \\ &+ \rho_1\int_0^1 (1+x)w^2(x,t)dx \\ &+ \frac{1}{\rho_1}\int_0^1 (1+x)\left(\int_{t-d_1(1-x)}^t \hat{\theta}_t(x,\tau)d\tau\right)^2 dx\bar{g}^2w^2(0,t) \\ &+ \rho_2 d_1\int_0^1 (1+x)w^2(x,t)dx \\ &+ \frac{d_1}{\rho_2}\int_0^1 (1+x) \\ &\qquad \times T^2\left[\int_x^1 \hat{\theta}_t(\xi,t)\phi(1-(\xi-x),t)d\xi\right](x,t)dx \\ &+ \rho_3 d_1\int_0^1 (1+x)w^2(x,t)dx \\ &+ \frac{1}{\rho_3}\int_0^1 (1+x)w(x,t)T[\hat{\theta}](x,t)dxg(t)\hat{e}(0,t)\end{aligned} \tag{82}$$

for some arbitrary positive constants ρ_1, ρ_2, ρ_3. Using the bounds (10) and (70), Cauchy-Schwarz' inequality and choosing $\rho_1 = \rho_2 = \rho_3 = \frac{\mu}{12}$ we further bound $\dot{V}_1$ as

$$\begin{aligned}\dot{V}_1(t) \leq &-\left(\mu - \zeta^2(t)\right)w^2(0,t) \\ &- \frac{\mu}{4}V_1(t) + 24d_1^2G_1^2||\hat{\theta}_t(t)||^2||\phi(t)||^2 \\ &+ 24\bar{g}^2d_1G_1^2\bar{\theta}^2\hat{e}^2(0,t)\end{aligned} \tag{83}$$

where

$$\zeta^2(t) = 24\bar{g}^2d_1^2\int_{t-d_1(1-x)}^t ||\hat{\theta}_t(\tau)||^2d\tau. \tag{84}$$

Using σ as defined in (54), we can expanding $\hat{e}^2(0,t)$ as

$$\hat{e}^2(0,t) = \sigma^2(t)(1 + ||\phi(t)||^2) \tag{85}$$

and write (83) as

$$\begin{aligned}\dot{V}_1(t) \leq &- \left(\mu - \zeta^2(t)\right) w^2(0,t) - \frac{\mu}{4} V_1(t) \\ &+ l_1(t) V_2(t) + l_2(t)\end{aligned} \tag{86}$$

where $l_1(t)$ and $l_2(t)$, defined as

$$l_1(t) = 24 d_1^2 G_1^2 ||\hat{\theta}_t(t)||^2 + l_2(t) \tag{87a}$$

$$l_2(t) = 24 \bar{g}^2 d_1 G_1^2 \bar{\theta}^2 \sigma^2(t) \tag{87b}$$

are nonnegative, integrable functions (Lemma 6).

Consider now (80b). By differentiating with respect to time, inserting the dynamics (38b), integration by parts and inserting the boundary condition (38b), we obtain

$$\dot{V}_2(t) = 2\mu g^2(t) u^2(0,t) - \mu\phi^2(0,t) - \mu ||\phi(t)||^2 \tag{88}$$

Using $u(0) = \hat{u}(0) + \hat{e}(0) = w(0) + \hat{e}(0)$ and the expansion (85) of $\hat{e}^2(0)$, we can bound (88) as

$$\begin{aligned}\dot{V}_2(t) \leq &\, 4\mu \bar{g}^2 w^2(0,t) - \frac{\mu}{2} V_2(t) \\ &+ l_3(t) V_2(t) + l_3(t)\end{aligned} \tag{89}$$

where

$$l_3(t) = 4\mu \bar{g}^2 \sigma^2(t) \tag{90}$$

is a nonnegative, integrable function (Lemma 6).

Now forming the Lyapunov function candidate

$$V_3(t) = 8\bar{g}^2 V_1(t) + V_2(t) \tag{91}$$

we find, using (86) and (89)

$$\begin{aligned}\dot{V}_3(t) \leq &-8\bar{g}^2 \left(\frac{\mu}{2} - \zeta^2(t)\right) w^2(0,t) \\ &- cV_3(t) + l_4(t) V_2(t) + l_5(t)\end{aligned} \tag{92}$$

where $c = \frac{\mu}{4}$ is a positive constant, and

$$l_4(t) = 8\bar{g}^2 l_1(t) + l_3(t), \quad l_5(t) = 8\bar{g}^2 l_2(t) + l_3(t) \tag{93}$$

are nonnegative, integrable functions.

We now prove that

$$V_3 \in \mathcal{L}_1 \cap \mathcal{L}_\infty, \qquad V_3 \to 0. \tag{94}$$

We consider two cases. If $\zeta^2(t) \leq \frac{\mu}{2}$ for $t \geq 0$, then (94) immediately follows from Lemma 8 in Appendix A. If, however, $\zeta^2(t) \not\leq \frac{\mu}{2}$ for $t \geq 0$, we note from Lemma 6 that $||\hat{\theta}_t|| \in \mathcal{L}_2$, which means that $\lim_{t\to\infty} \int_{t-d_1(1-x)}^t ||\hat{\theta}_t(\tau)||^2 d\tau = 0$. Specifically, this implies that for every $\epsilon_0 > 0$, there must exist a $T_0 \geq 0$ so that

$$\int_{t-d_1(1-x)}^t ||\hat{\theta}_t(\tau)||^2 d\tau < \epsilon_0 \tag{95}$$

for all $t \geq T_0$. Let ϵ_0 be taken as $\epsilon_0 = \frac{\mu^3}{48\bar{g}^2}$ which, from the definition of ζ^2 in (84) implies that $\zeta^2(t) < \frac{\mu}{2}$ for all $t \geq T_0$, and Lemma 8 in Appendix A gives (94).

From (94), $||w||, ||\phi|| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $||w||, ||\phi|| \to 0$ follow. From the invertibility of transform (66), we have $||\hat{u}|| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $||\hat{u}|| \to 0$. The relationship (52) then gives $||\psi|| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $||\psi|| \to 0$, while (40) and Lemma 5 finally gives

$$||u|| \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \quad ||u|| \to 0. \tag{96}$$

■

VI. SIMULATIONS

A. Non-adaptive controllers

System (1) along with the controllers of Theorems 3 and 4 were implemented in MATLAB, using the system parameters

$$\mu = 1 \qquad \varpi(x,t) = \frac{1}{2}(2 + \sin(\pi t)) e^{\frac{1}{2}x} \tag{97}$$

The system's initial condition was in both cases set to

$$u_0(x) = x \tag{98}$$

while the initial condition for the observer was set identically zero. The kernel equation (12) was solved at each time step using successive approximations. In Figure 1, the parameter ϖ is depicted, and also the system norm in the open loop case, showing that when left uncontrolled, the system diverges. In the closed loop case, the system is stabilized in finite time, as seen in Figure 2, the state estimation error norm, and state norms in the state-feedback and output-feedback cases converge in the finite time as predicted by theory. The actuation signals are also seen to converge to zero.

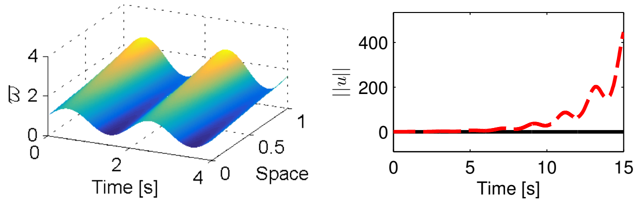

Fig. 1: *Left*: System parameter $\varpi(x,t)$. *Right*: State norm in the open loop case ($U = 0$).

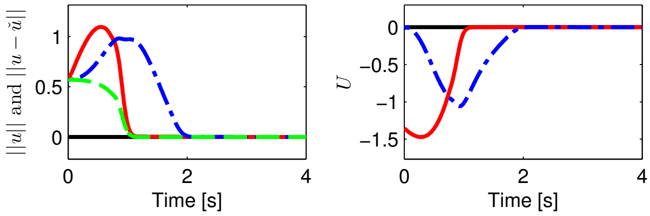

Fig. 2: *Left*: State norm during non-adaptive state feedback (solid red) and output feedback (dashed-dotted blue) and the state estimation error norm (dashed green). *Right*: Actuation signal during non-adaptive state feedback (solid red) and output feedback (dashed-dotted blue).

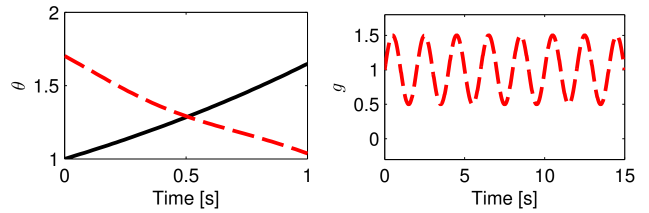

Fig. 3: *Left*: Actual (solid black) and estimated (dashed red) parameter θ. *Right*: Parameter g.

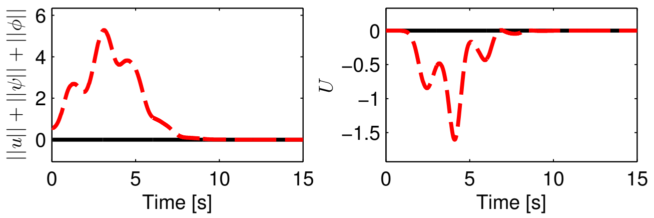

Fig. 4: *Left*: State norms in during adaptive output-feedback. *Right*: Actuation signal during adaptive output-feedback.

B. Adaptive controller

System (37) was here implemented with the controller of Theorem 7 using the same system parameters as in the non-adaptive case, by noting that ϖ defined in (97) can be written in the form (36), with

$$\theta(x) = e^{\frac{1}{2}x}, \qquad g(t) = \frac{1}{2}(2 + \sin(\pi t)). \qquad (99)$$

The design parameters were set to

$$\gamma = 1, \qquad \bar{\theta} = 10^3. \qquad (100)$$

Figure 3 shows the parameters θ and g, and the final estimate $\hat{\theta}$. It can be noted that the estimated θ is very different from the actual θ, even though the state and filter norms and the actuation signal all converge to zero, as seen from Figure 4.

VII. Concluding remarks

We have considered a scalar 1–D linear hyperbolic PDE with an interior-domain parameter that is a function of time and space. A state-feedback control law was derived stabilizing the system in finite time, subject to the requirement that the in-domain parameter can be predicted a time into the future corresponding to the propagation time between the boundaries. The control law was also combined with an observer into an output-feedback control law. Lastly, when assuming the interior-domain parameter can be decoupled into a time-varying and spatially varying part, the latter was allowed to be uncertain, and an adaptive output feedback control law was derived stabilizing the system from a single boundary sensing only. All derived controllers were implemented and demonstrated in simulations.

A natural next step is to consider systems with more involved time-varying in-domain parameters, and also systems of coupled linear hyperbolic PDEs with time-varying parameters.

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APPENDIX

A. Stability and convergence lemma

Lemma 8: Let $v(t)$, $l_1(t)$, $l_2(t)$, be real-valued functions defined for $t \geq 0$. Suppose

$$v(t), l_1(t), l_2(t) \geq 0, \ \forall t \geq 0 \tag{101a}$$

$$l_1, l_2 \in \mathcal{L}_1 \tag{101b}$$

$$\dot{v}(t) \leq -cv(t) + l_1(t)v(t) + l_2(t) \tag{101c}$$

where c is a positive constant. Then

$$v \in \mathcal{L}_1 \cap \mathcal{L}_\infty \tag{102}$$

and

$$\lim_{t \to \infty} v(t) = 0. \tag{103}$$

Proof: Proof of (102) is given in [24, Lemma B.6], while proof of (103) is given in [25, Lemma 3]. ■