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$\mathcal{L}_1$ Adaptive Controller for a Class of Systems with Unknown Nonlinearities: Part I

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***Abstract*— This paper presents a novel adaptive control methodology for a class of uncertain systems in the presence of time-varying unknown nonlinearities. The adaptive controller ensures uniformly bounded transient and asymptotic tracking for system's both input and output signals simultaneously. The performance bounds can be systematically improved by increasing the adaptation rate. Part II extends the results to a class of systems in the presence of unmodeled dynamics.**

I. Introduction

This paper extends the results of [1], [2] to a class of uncertain systems in the presence of time-varying and state dependent unknown nonlinearities. We prove that subject to a set of mild assumptions the system can be transformed into an equivalent linear system with time-varying unknown parameters and disturbances. For the latter, we extend the methodology from [1], [2], which ensures uniformly bounded transient response for system's both input and output signals simultaneously, in addition to stable tracking. The $\mathcal{L}_\infty$ norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by increasing the adaptation rate.

The paper is organized as follows. Section II gives the problem formulation. In Section III, the $\mathcal{L}_1$ adaptive control architecture is presented. Stability and uniform performance bounds are presented in Section IV. In Section V, simulation results are presented, while Section VI concludes the paper.

Throughout this paper, $\mathbb{I}$ indicates the identity matrix of appropriate dimension, $||H(s)||_{\mathcal{L}_1}$ denotes the $\mathcal{L}_1$ gain of $H(s)$, $||x||_{\mathcal{L}_\infty}$ denotes the $\mathcal{L}_\infty$ norm of $x(t)$, $||x_t||_{\mathcal{L}_\infty}$ denotes the truncated $\mathcal{L}_\infty$ norm of $x(t)$ at the time instant t, and $\|x\|_2$ and $\|x\|_\infty$ indicate the 2- and ∞- norms of the vector x respectively. Some of the proofs are in included in the Appendix.

II. Problem Formulation

Consider the following system dynamics:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\Big(\omega u(t) + f(x(t),t)\Big)\,, \\ y(t) &= c^\top x(t)\,, \quad x(0) = x_0\,, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the regulated output, $b, c \in \mathbb{R}^n$ are known constant vectors, A_m is a known $n \times n$ Hurwitz matrix, ω is an unknown constant, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is an unknown nonlinear function.

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Assumption 1: [Semiglobal Lipschitz condition] For any $\delta > 0$, there exist $L_\delta > 0$ and $B > 0$ such that $|f(x,t) - f(\bar{x},t)| \le L_\delta \|x - \bar{x}\|_\infty\,,\quad |f(0,t)| \le B\,,$ for all $\|x\|_\infty \le \delta$ and $\|\bar{x}\|_\infty \le \delta$ uniformly in t.

Assumption 2: [Known sign for control effectiveness] There exist upper and lower bounds $\omega_u > \omega_l > 0$ such that $\omega_l \le \omega \le \omega_u$.

Assumption 3: [Semiglobal uniform boundedness of partial derivatives] For any $\delta > 0$, there exist $d_{f_x}(\delta) > 0$, and $d_{f_t}(\delta) > 0$ such that for any $\|x\|_\infty \le \delta$, the partial derivatives of $f(x,t)$ are piece-wise continuous and bounded $\left\|\dfrac{\partial f(x,t)}{\partial x}\right\| \le d_{f_x}(\delta)\,,\quad \left|\dfrac{\partial f(x,t)}{\partial t}\right| \le d_{f_t}(\delta)\,.$

The control objective is to design a full-state feedback adaptive controller to ensure that $y(t)$ tracks a given bounded reference signal $r(t)$ *both in transient and steady state*, while all other error signals remain bounded.

III. $\mathcal{L}_1$ Adaptive Controller

In this section we develop an adaptive control architecture for the system in (1) that permits complete transient characterization for both $u(t)$ and $x(t)$. The design of $\mathcal{L}_1$ adaptive controller involves a strictly proper transfer function $D(s)$ and a gain $k \in \mathbb{R}^+$, which leads to a strictly proper stable

$$C(s) = \frac{\omega k D(s)}{1 + \omega k D(s)} \tag{2}$$

with DC gain $C(0) = 1$. The simplest choice of $D(s)$ is $D(s) = \frac{1}{s}$, which yields a first order strictly proper $C(s)$ in the following form $C(s) = \frac{\omega k}{s + \omega k}$. Let $H(s) = (s\mathbb{I} - A_m)^{-1} b$, and $r_0(t)$ be the signal with its Laplace transform $(s\mathbb{I} - A_m)^{-1} x_0$. Since A_m is Hurwitz and x_0 is finite, $\|r_0\|_{\mathcal{L}_\infty}$ is finite.

For the proof of stability and uniform performance bounds the choice of $D(s)$ and k needs to ensure that there exists ρ_r such that

$$\begin{aligned} \|G(s)\|_{\mathcal{L}_1} < \Big(&\rho_r - \|k_g C(s) H(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} \\ &- \|r_0\|_{\mathcal{L}_\infty}\Big) / (\rho_r L_{\rho_r} + B)\,, \end{aligned} \tag{3}$$

where $G(s) = H(s)(1 - C(s))$, and

$$k_g = -\frac{1}{c^\top A_m^{-1} b}\,. \tag{4}$$

Remark 1: We notice that the upper bound in (3) is a consequence of the *semiglobal* Lipschitz property of $f(x,t)$, stated in Assumption 1. If $f(x,t)$ is globally Lipschitz with uniform Lipschitz constant L, then $\lim\limits_{\rho_r \to \infty} \Big(\rho_r -$

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$\|k_g C(s)H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} - \|r_0\|_{\mathcal{L}_\infty}\Big)/(\rho_r L + B) = \dfrac{1}{L}$, and the upper bound in (3) degenerates into $\|G(s)\|_{\mathcal{L}_1} < 1/L$, which is the same as the one derived in [3] for systems with constant unknown parameters.

We consider the following state predictor (or passive identifier) for generation of the adaptive laws:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m\hat{x}(t) + b\Big(\hat{\omega}(t)u(t) + \hat{\theta}(t)\|x(t)\|_\infty + \hat{\sigma}(t)\Big)\\ \hat{y}(t) &= c^\top \hat{x}(t)\,, \quad \hat{x}(0) = x_0\,. \end{aligned} \tag{5}$$

The adaptive estimates $\hat{\omega}(t), \hat{\theta}(t), \hat{\sigma}(t)$ are defined as:

$$\begin{aligned}\dot{\hat{\theta}}(t) &= \Gamma \mathrm{Proj}(\hat{\theta}(t), -\|x(t)\|_\infty \tilde{x}^\top(t)Pb),\ \hat{\theta}(0) = \hat{\theta}_0\\ \dot{\hat{\sigma}}(t) &= \Gamma \mathrm{Proj}(\hat{\sigma}(t), -\tilde{x}^\top(t)Pb), \qquad \hat{\sigma}(0) = \hat{\sigma}_0 \\ \dot{\hat{\omega}}(t) &= \Gamma \mathrm{Proj}(\hat{\omega}(t), -\tilde{x}^\top(t)Pbu(t)),\ \hat{\omega}(0) = \hat{\omega}_0\end{aligned} \tag{6}$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$, $\Gamma \in \mathbb{R}^+$ is the adaptation gain, P is the solution of the algebraic equation $A_m^\top P + PA_m = -Q$, $Q > 0$, and the projection operator ensures that the adaptive estimates $\hat{\omega}(t)$, $\hat{\theta}(t)$, $\hat{\sigma}(t)$ remain inside the compact sets $[\omega_l,\ \omega_u]$, $[-\theta_b,\ \theta_b]$, $[-\sigma_b,\ \sigma_b]$, respectively, with θ_b, σ_b defined as follows

$$\theta_b = L_\rho, \qquad \sigma_b = B + \epsilon\,, \tag{7}$$

where ϵ is an arbitrary positive constant and

$$\rho = \rho_r + \beta\,, \tag{8}$$

with β being an arbitrary positive constant that satisfies $\|G(s)\|_{\mathcal{L}_1} L_\rho < 1$. We notice that (3) implies that $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$. Since L_ρ depends continuously on ρ, then $\|G(s)\|_{\mathcal{L}_1} L_\rho < 1$ can always be satisfied if β is small enough.

Remark 2: In the following analysis we demonstrate that ρ_r and ρ characterize the domain of attraction of the closed loop reference system (yet to be defined) and the system in (1) respectively. We notice that since β can be set arbitrarily small, ρ can approximate ρ_r arbitrarily closely.

The control signal is generated through gain feedback of the following system:

$$\chi(s) = D(s)\bar{r}(s)\,, \quad u(s) = -k\chi(s)\,, \tag{9}$$

where $k \in \mathbb{R}^+$ is introduced in (2), and $\bar{r}(s)$ is the Laplace transform of

$$\bar{r}(t) = \hat{\omega}(t)u(t) + \hat{\theta}(t)\|x(t)\|_\infty + \hat{\sigma}(t) - k_g r(t). \tag{10}$$

The complete $\mathcal{L}_1$ adaptive controller consists of (5), (6) and (9) subject to the $\mathcal{L}_1$-gain upper bound in (3).

IV. Analysis of $\mathcal{L}_1$ Adaptive Controller

A. Closed-loop Reference System

We now consider the following closed-loop reference system with its control signal and system response being defined as follows:

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + b\Big(\omega u_{ref}(t) + f(x_{ref}(t), t)\Big) \tag{11}$$

$$u_{ref}(s) = (C(s)/\omega)(k_g r(s) - \bar{r}_{ref}(s)) \tag{12}$$

$$y_{ref}(t) = c^\top x_{ref}(t)\,, \quad x_{ref}(0) = x_0\,, \tag{13}$$

where $\bar{r}_{ref}(s)$ is the Laplace transformation of the signal $\bar{r}_{ref}(t) = f(x_{ref}(t), t)$, and k_g is introduced in (4). The next Lemma establishes the stability of the closed-loop reference system in (11)-(13).

Lemma 1: For the closed-loop reference system in (11)-(13) subject to the $\mathcal{L}_1$-gain upper bound in (3), if

$$\|x_0\|_\infty \leq \rho_r\,, \tag{14}$$

then

$$\|x_{ref}\|_{\mathcal{L}_\infty} < \rho_r\,, \tag{15}$$

$$\|u_{ref}\|_{\mathcal{L}_\infty} < \rho_{u_r}\,, \tag{16}$$

where ρ_r is introduced in (3) and $\rho_{u_r} = \|C(s)/\omega\|_{\mathcal{L}_1}(\rho_r L_{\rho_r} + B + k_g\|r\|_{\mathcal{L}_\infty})$.

B. Equivalent Linear Time-Varying System

In this section we demonstrate that the nonlinear system in (1) can be transformed into an equivalent linear system with unknown time-varying parameters. To streamline the subsequent analysis, we need to introduce several notations. Let γ_0 be the desired performance bound for $||\tilde{x}||_{\mathcal{L}_\infty}$, and β_1 be an arbitrary positive constant verifying the following upper bound

$$\gamma_1 \triangleq \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L_\rho}\gamma_0 + \beta_1 < \beta\,, \tag{17}$$

where β was introduced in (8). Further, let

$$\rho_u = \rho_{u_r} + \gamma_2\,, \tag{18}$$

$$\gamma_2 = \left\|\frac{C(s)}{\omega}\right\|_{\mathcal{L}_1} L_\rho \gamma_1 + \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H_o(s)} c_o^\top\right\|_{\mathcal{L}_1} \gamma_0\,. \tag{19}$$

It follows from Lemma 4 in [1] that there exists $c_o \in \mathbb{R}^n$ such that

$$c_o^\top H(s) = \frac{N_n(s)}{N_d(s)}\,, \tag{20}$$

where the order of $N_d(s)$ is one more than the order of $N_n(s)$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials. We will prove that by increasing the adaptation rate Γ, one can arbitrarily reduce γ_0.

Lemma 2: If the truncated $\mathcal{L}_\infty$ norms of $x(t)$ and $u(t)$ verify

$$\|x_t\|_{\mathcal{L}_\infty} \leq \rho\,,\ \|u_t\|_{\mathcal{L}_\infty} \leq \rho_u\,, \tag{21}$$

then there exist differentiable $\theta(\tau)$ and $\sigma(\tau)$ with uniformly bounded derivatives over $\tau \in [0,\ t]$ such that

$$|\theta(\tau)| < \theta_b\,, \tag{22}$$

$$|\sigma(\tau)| < \sigma_b\,, \tag{23}$$

$$f(x(\tau), \tau) = \theta(\tau)\|x(\tau)\|_\infty + \sigma(\tau)\,. \tag{24}$$

Proof. Using the definitions in (7), we have

$$f(x(0), 0) \leq L_\rho\|x(0)\|_\infty + B < \theta_b\|x(0)\|_\infty + \sigma_b\,, \tag{25}$$

which implies that there exist $\theta(0)$ and $\sigma(0)$ such that

$$|\theta(0)| < \theta_b\,, \quad |\sigma(0)| < \sigma_b \tag{26}$$

and

$$f(x(0),0)=\theta(0)\|x(0)\|_\infty+\sigma(0)\,. \tag{27}$$

We construct the trajectories of $\theta(\tau)$ and $\sigma(\tau)$ according to the following dynamics:

$$\begin{bmatrix}\dot\theta(\tau)\\ \dot\sigma(\tau)\end{bmatrix}=A_\eta^{-1}\begin{bmatrix}\frac{df(x(\tau),\tau)}{d\tau}-\theta(\tau)\frac{d\|x(\tau)\|_\infty}{d\tau}\\ 0\end{bmatrix}, \tag{28}$$

where

$$A_\eta=\begin{bmatrix}\|x(\tau)\|_\infty & 1\\ -(\sigma_b-|\sigma(\tau)|) & \theta_b-|\theta(\tau)|\end{bmatrix}, \tag{29}$$

with the initial values being bounded according to (26). The determinant of A_η is:

$$\det(A_\eta)=\|x(\tau)\|_\infty(\theta_b-|\theta(\tau)|)+\sigma_b-|\sigma(\tau)|\,. \tag{30}$$

For any $\tau\in[0,\bar t)$, where $\bar t$ is an arbitrary constant or ∞, if

$$\begin{aligned}|\theta(\tau)| &< \theta_b\,,\\ |\sigma(\tau)| &< \sigma_b\,,\end{aligned} \tag{31}$$

it follows from (30) that $\det(A_\eta(\tau))\neq 0$ over $[0,\bar t)$. Hence, it follows from (28), (29) that

$$\frac{d\Big(\theta(\tau)\|x(\tau)\|_\infty+\sigma(\tau)\Big)}{d\tau}=\frac{df(x(\tau),\tau)}{d\tau}\,, \tag{32}$$

$$\frac{\dot\sigma(\tau)}{\sigma_b-|\sigma(\tau)|}=\frac{\dot\theta(\tau)}{\theta_b-|\theta(\tau)|} \tag{33}$$

for any $\tau\in[0,\bar t)$. Using the initial condition from (27), we can integrate to obtain

$$\theta(\tau)\|x(\tau)\|_\infty+\sigma(\tau)=f(x(\tau),\tau)\,,\forall\tau\in[0,\ \bar t) \tag{34}$$

$$\int_0^{\bar t_-}\frac{\dot\sigma(\tau)}{\sigma_b-|\sigma(\tau)|}d\tau=\int_0^{\bar t_-}\frac{\dot\theta(\tau)}{\theta_b-|\theta(\tau)|}d\tau\,, \tag{35}$$

where $\int_0^{\bar t_-}(\cdot)d\tau\triangleq\lim_{\bar\tau\to\bar t}\int_0^{\bar\tau}(\cdot)d\tau$ with $\bar\tau$ approaches $\bar t$ from the left.

Next we calculate $\int_0^{\bar t_-}\frac{\dot\sigma(\tau)}{\sigma_b-|\sigma(\tau)|}d\tau$, assuming that $|\sigma(\tau)|<\sigma_b$. Since $\sigma(\tau)$ may cross zero, we divide $[0,\bar t)$ into countable subsets: $[0,t_1)$, $[t_1,t_2)$,..., $[t_I,\bar t)$ where $\sigma(t_1)=\cdots=\sigma(t_I)=0$, and write

$$\begin{aligned}&\int_0^{\bar t_-}\frac{\dot\sigma(\tau)}{\sigma_b-|\sigma(\tau)|}d\tau=\sum_{i=1,..,I}\int_{t_{i-1}}^{t_i}\frac{\dot\sigma(\tau)}{\sigma_b-|\sigma(\tau)|}d\tau\\ &+\int_{t_I}^{\bar t_-}\frac{\dot\sigma(\tau)}{\sigma_b-|\sigma(\tau)|}d\tau=\lim_{\tau\to\bar t}\mathrm{sign}(\sigma(\tau))\ln(\sigma_b-|\sigma(\tau)|)\\ &-\mathrm{sign}(\sigma(0))\ln(\sigma_b-|\sigma(0)|)\,.\end{aligned} \tag{36}$$

From (35) and (36), we have

$$\begin{aligned}&\lim_{\tau\to\bar t}\mathrm{sign}(\sigma(\tau))\ln(\sigma_b-|\sigma(\tau)|)\\ &-\mathrm{sign}(\sigma(0))\ln(\sigma_b-|\sigma(0)|)=\\ &\lim_{\tau\to\bar t}\mathrm{sign}(\theta(\tau))\ln(\theta_b-|\theta(\tau)|)\\ &-\mathrm{sign}(\theta(0))\ln(\theta_b-|\theta(0)|)\,.\end{aligned} \tag{37}$$

In what follows, we prove (22)-(23) by contradiction. If (22)-(23) is not true, since $\theta(\tau)$ and $\sigma(\tau)$ are continuous, it follows from (26) that there exists $\bar t\in[0,\ t]$ such that either

$$(i)\qquad \lim_{\tau\to\bar t}|\theta(\tau)|=\theta_b\,,\ \text{or} \tag{38}$$

$$(ii)\qquad \lim_{\tau\to\bar t}|\sigma(\tau)|=\sigma_b\,, \tag{39}$$

while

$$|\theta(\tau)|<\theta_b\,,\ |\sigma(\tau)|<\sigma_b\,,\ \forall\ \tau\in[0,\ \bar t)\,. \tag{40}$$

$i)$ In this case we have

$$|\lim_{\tau\to\bar t}\mathrm{sign}(\theta(\tau))\ln(\theta_b-|\theta(\tau)|)|=\infty. \tag{41}$$

Since it is obvious that $\mathrm{sign}(\sigma(0))\ln(\sigma_b-|\sigma(0)|)$ and $\mathrm{sign}(\theta(0))\ln(\sigma_b-|\theta(0)|)$ are bounded, it follows from (37) that

$$|\lim_{\tau\to\bar t}\mathrm{sign}(\sigma(\tau))\ln(\sigma_b-|\sigma(\tau)|)|=\infty\,,$$

and hence

$$\lim_{\tau\to\bar t}|\sigma(\tau)|=\sigma_b\,. \tag{42}$$

It follows from (34) that $\lim_{\tau\to\bar t}(\theta(\tau)\|x(\tau)\|_\infty+\sigma(\tau))=\lim_{\tau\to\bar t}f(x(\tau),\tau)$, which along with (38) and (42) implies that

$$|\lim_{\tau\to\bar t}f(x(\tau),\tau)|=|f(x(\bar t),\bar t)|=\theta_b\|x(\bar t)\|_\infty+\sigma_b\,. \tag{43}$$

From Assumption 1 it follows that $|f(x(\bar t),\bar t)|\leq L_\rho\|x(\bar t)\|_\infty+B=\theta_b\|x(\bar t)\|_\infty+\sigma_b-\epsilon$, which contradicts (43), and therefore (38) is not true.

$ii)$ Following the same steps as above, one can derive a contradicting argument to (39).

Since (38), (39) are not true, then the relationships in (22), (23) hold. Eq. (24) follows from (22), (23) and (34) directly.

Further, notice from (1) that bounded $x(\tau)$ and $u(\tau)$ imply bounded right hand side for the system dynamics, and hence bounded $\dot x(\tau)$ over $[0,t]$. In the light of Assumption 3, $\frac{df(x(\tau),\tau)}{d\tau}$ and $\frac{d\|x(\tau)\|_\infty}{d\tau}$ are bounded, although the derivative $\frac{d\|x(\tau)\|_\infty}{d\tau}$ may not be continuous. Since $\theta(\tau)$ is bounded, then $\frac{df(x(\tau),\tau)}{d\tau}-\theta(\tau)\frac{d\|x(\tau)\|_\infty}{d\tau}$ is bounded. From (22), (23) it follows that $\det A_\eta(\tau)\neq 0$, and therefore we conclude from (28) that $\dot\theta(\tau)$ and $\dot\sigma(\tau)$ are bounded. This concludes the proof. □

If (21) holds, Lemma 2 implies that the system in (1) can be rewritten over $\tau\in[0,\ t]$ as:

$$\begin{aligned}\dot x(\tau)&=A_m x(\tau)+b\left(\omega u(\tau)+\theta(\tau)\|x(\tau)\|_\infty+\sigma(\tau)\right),\\ y(\tau)&=c^\top x(\tau),\quad x(0)=x_0\,,\end{aligned} \tag{44}$$

where $\theta(\tau)$, $\sigma(\tau)$ are unknown time-varying signals subject to the upper bounds (22), (23) for all $\forall\ \tau\in[0,\ t]$, while their derivatives for all $\tau\in[0,\ t]$ are subject to

$$|\dot\theta(\tau)|\leq d_\theta(\rho,\rho_u)<\infty\,,\quad |\dot\sigma(\tau)|\leq d_\sigma(\rho,\rho_u)<\infty\,. \tag{45}$$

Remark 3: We notice that though Lemma 2 proves the existence and boundedness of $\dot\theta(\tau)$, $\dot\sigma(t)$, their continuity is not guaranteed. The reason is that $\frac{d\|x(\tau)\|_\infty}{d\tau}$ can be piece-wise continuous due to the definition of the ∞ norm. Thus, in (28) the right-hand sides can be piece-wise continuous functions of t.

C. *Tracking error signal*

Lemma 3: For the system in (1) and the $\mathcal{L}_1$ adaptive controller in (5), (6) and (9), for any t such that (21) holds, we have

$$\|\tilde{x}_t\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_m(\rho,\rho_u)}{\lambda_{\min}(P)\Gamma}}\,, \tag{46}$$

where

$$\theta_m(\rho,\rho_u) \triangleq 4\theta_b^2 + 4\sigma_b^2 + (\omega_u-\omega_l)^2 + 4\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\left(\theta_b d_\theta(\rho,\rho_u) + \sigma_b d_\sigma(\rho,\rho_u)\right)\,. \tag{47}$$

D. *Transient and Steady-State Performance*

Theorem 1: Consider the reference system in (11)-(13) and the closed-loop $\mathcal{L}_1$ adaptive controller in (5), (6), (9) subject to (3). If

$$\|x_0\|_\infty \leq \rho_r\,, \tag{48}$$

and the adaptive gain is chosen to verify the lower bound:

$$\Gamma > \frac{\theta_m(\rho,\rho_u)}{\lambda_{\min}(P)\gamma_0^2}\,, \tag{49}$$

we have:

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \gamma_0\,, \tag{50}$$

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} < \gamma_1\,, \tag{51}$$

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} < \gamma_2\,, \tag{52}$$

where γ_1 and γ_2 are defined in (17) and (19).
It follows from (49) that arbitrarily small γ_0 can be obtained by increasing the adaptive gain.

V. Simulations

Consider the dynamics of a single-link robot arm rotating on a vertical plane:

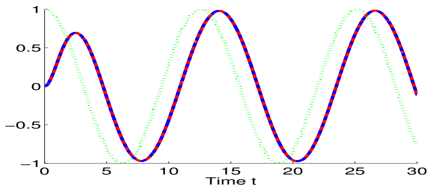
$$I\ddot{q}(t) + F(q(t),\dot{q}(t),t) = u(t)\,, \tag{53}$$

where $q(t)$ and $\dot{q}(t)$ are the measured angular position and velocity, respectively, $u(t)$ is the input torque, I is the unknown moment of inertia, $F(q(t),\dot{q}(t),t)$ is an unknown nonlinear function that lumps the forces and torques due to gravity, friction, disturbance and other external sources. The control objective is to design $u(t)$ to achieve tracking of a bounded reference input $r(t)$ by $q(t)$, where $\|r\|_{\mathcal{L}_\infty} \leq 1$. Let $x = [x_1\ x_2]^\top = [q\ \dot{q}]^\top$. The system in (53) can be presented in the canonical form:

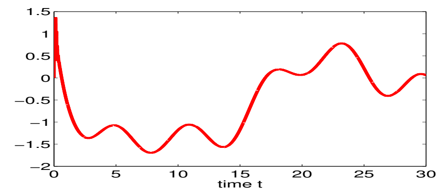
$$\dot{x}(t) = A_m x(t) + b(\omega u(t) + f(x(t),t))\,,\ y(t) = c^\top x(t),$$

where $b = [0 \quad 1]^\top$, $c = [1 \quad 0]^\top$, $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$, $\omega = 1/I$, and $f(x(t),t)) = [1 \ \ 1.4]x(t) - F(x_2(t),x_1(t),t)$. Let $\omega = 1/I = 1$ and $F(x_2(t),x_1(t),t) = x_2^2(t) + x_1^2(t) + \sin(0.2t)$, so that the compact sets can be conservatively chosen as $\omega_l = 0.5$, $\omega_u = 2$, $\theta_b = 20$, $\sigma_b = 10$.

In the implementation of the $\mathcal{L}_1$ adaptive controller, we set $Q = 2\mathbb{I}$, and hence $P = \begin{bmatrix} 1.4143 & 0.5000 \\ 0.5000 & 0.71430 \end{bmatrix}$.

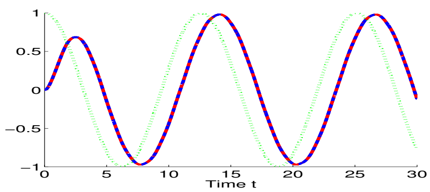


(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

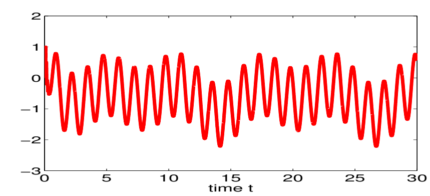


(b) Time-history of $u(t)$

Fig. 1. Performance of $\mathcal{L}_1$ adaptive controller for $F(x_2(t),x_1(t),t) = x_2^2(t) + x_1^2(t) + \sin(0.2t)$

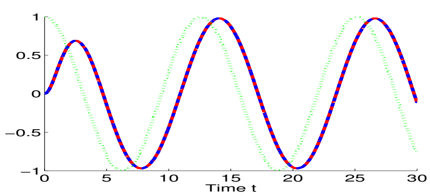


(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

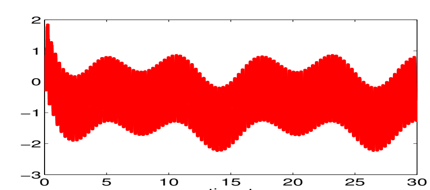


(b) Time-history of $u(t)$

Fig. 2. Performance of $\mathcal{L}_1$ adaptive controller for $F(x_2(t),x_1(t),t) = x_2^2(t) + x_1^2(t) + \sin(5t)$



(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)



(b) Time-history of $u(t)$

Fig. 3. Performance of $\mathcal{L}_1$ adaptive controller for $F(x_2(t),x_1(t),t) = x_2^2(t) + x_1^2(t) + \sin(20t)$

We need to verify the condition in (3). Letting $D(s) = 1/s$, we have $G(s) = \frac{s}{s+\omega k}H(s)$, $H(s) = [\frac{1}{s^2+1.4s+1} \ \frac{s}{s^2+1.4s+1}]^\top$, and we choose conservative $L_\rho = 20$. One can numerically verify that for $\omega k > 30$ the upper bound $\|G(s)\|_{\mathcal{L}_1} L_\rho < 1$ holds. Since $\omega > 0.5$, we set $k = 60$. We set the adaptive gain $\Gamma_c = 10000$.

The simulation results of the $\mathcal{L}_1$ adaptive controller are shown in Figures 1(a)-1(b) for reference input $r = \cos(0.5t)$. Next, without any retuning of the controller we consider different nonlinearity $F(x_2(t), x_1(t), t) = x_2^2(t) + x_1^2(t) + \sin(5t)$. The simulation results are shown in 2(a)-2(b). Finally, we consider much higher frequencies in the nonlinearity: $F(x_2(t), x_1(t), t) = x_2^2(t) + x_1^2(t) + \sin(20t)$. The simulation results are shown in 3(a)-3(b). We note that the $\mathcal{L}_1$ adaptive controller guarantees smooth and uniform transient performance in the presence of different unknown nonlinearities without requiring any retuning. We also notice that $x_1(t)$ and $\hat{x}_1(t)$ are almost the same in Figs. 1(a), 2(a) and 3(a).

VI. Conclusion

A novel $\mathcal{L}_1$ adaptive control architecture is presented that has guaranteed transient response in addition to stable tracking for uncertain systems in the presence of unknown state and time-dependent nonlinearities. The control signal and the system response approximate the same signals of a closed-loop reference system, which can be designed to achieve desired specifications. In Part II, we present an extension to a class of systems in the presence of unmodeled dynamics, [5]. The results of these papers question the need for the neural network based adaptive control paradigm.

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Appendix

Proof of Lemma 1. It follows from (11)-(13) that

$$x_{ref}(s) = G(s)\bar{r}_{ref}(s) + H(s)C(s)k_g r(s) + (s\mathbb{I} - A_m)^{-1}x_0 \,. \tag{54}$$

Example 5.2 in [6] (page 199) implies that

$$\begin{aligned}&\|x_{ref_t}\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1}\|\bar{r}_{ref_t}\|_{\mathcal{L}_\infty} + \\ &\|k_g C(s)H(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty} + \|r_0\|_{\mathcal{L}_\infty} \,.\end{aligned} \tag{55}$$

If (15) is not true, since $\|x_{ref}(0)\|_\infty = \|x_0\|_\infty < \rho_r$ and $x_{ref}(t)$ is continuous, there exists t such that

$$\|x_{ref_t}\|_{\mathcal{L}_\infty} \le \rho_r \,, \tag{56}$$

$$x_{ref}(t) = \rho_r \,. \tag{57}$$

Using Assumption 1 and the upper bound in (56), we arrive at the following upper bound

$$\|\bar{r}_{ref_t}\|_{\mathcal{L}_\infty} \le L_{\rho_r}\|x_{ref_t}\|_{\mathcal{L}_\infty} + B \,. \tag{58}$$

Substituting (58) into (55), and noticing that $\|r_t\|_{\mathcal{L}_\infty} \le \|r\|_{\mathcal{L}_\infty}$, we obtain

$$\begin{aligned}&\|x_{ref_t}\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1}L_{\rho_r}\rho_r + \|k_g C(s)H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \\ &+\|r_0\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}B \,.\end{aligned} \tag{59}$$

The condition in (3) can be solved for ρ_r to obtain the following upper bound

$$\begin{aligned}&\|G(s)\|_{\mathcal{L}_1}L_{\rho_r}\rho_r + \|k_g C(s)H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \\ &+\|r_0\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}B < \rho_r \,,\end{aligned} \tag{60}$$

which implies that $\|x_{ref_t}\|_{\mathcal{L}_\infty} < \rho_r$, and contradicts (57). This proves (15).

Using (15), it follows from Assumption 1 that

$$\|\bar{r}_{ref}\|_{\mathcal{L}_\infty} < \rho_r L_{\rho_r} + B \,. \tag{61}$$

Example 5.2 in [6] (page 199) further implies that

$$\|u_{ref}\|_{\mathcal{L}_\infty} < \|C(s)/\omega\|_{\mathcal{L}_1}(\rho_r L_{\rho_r} + B + k_g\|r\|_{\mathcal{L}_\infty}) \,, \tag{62}$$

which proves (16). □

Proof of Lemma 3. It follows from (21) that (44) and (45) hold for any $\tau \in [0, t]$. Consider the following Lyapunov function candidate:

$$\begin{aligned}&V(\tilde{x}(\tau), \tilde{\omega}(\tau), \tilde{\theta}(\tau), \tilde{\sigma}(\tau)) = \tilde{x}^\top(\tau)P\tilde{x}(\tau) + \\ &\Gamma^{-1}\left(\tilde{\omega}^2(\tau) + \tilde{\theta}^2(\tau) + \tilde{\sigma}^2(\tau)\right) \,,\end{aligned} \tag{63}$$

where

$$\tilde{\omega}(\tau) \triangleq \hat{\omega}(\tau) - \omega, \tilde{\theta}(\tau) \triangleq \hat{\theta}(\tau) - \theta(\tau), \tilde{\sigma}(\tau) \triangleq \hat{\sigma}(\tau) - \sigma(\tau) \,. \tag{64}$$

It follows from (5) and (44) that over $[0, t]$

$$\dot{\tilde{x}}(\tau) = A_m\tilde{x}(\tau) + b\left(\tilde{\omega}u(t) + \tilde{\theta}(\tau)\|x(\tau)\|_\infty + \tilde{\sigma}(\tau)\right) \,, \tag{65}$$

where $\tilde{x}(0) = 0$. We can verify straightforwardly that

$$V(0) \le \left(4\theta_b^2 + 4\sigma_b^2 + (\omega_u - \omega_l)^2\right)/\Gamma \le \frac{\theta_m(\rho, \rho_u)}{\Gamma} \,.$$

Let $t_1 \in (0, t]$ be the first time-instant of the discontinuity of either of the derivatives of $\dot{\hat{\theta}}(t)$, $\dot{\hat{\sigma}}(t)$, $\dot{\theta}(t)$, $\dot{\sigma}(t)$. Next we prove that

$$V(\tau) \le \frac{\theta_m(\rho, \rho_u)}{\Gamma} \,, \quad \forall\, \tau \in [0, t_1] \,. \tag{66}$$

Using the projection based adaptation laws from (6), one has the following upper bound for $\dot{V}(\tau)$:

$$\dot{V}(\tau) \le -\tilde{x}^\top(\tau)Q\tilde{x}(\tau) + 2\Gamma^{-1}\left|\tilde{\theta}(\tau)\dot{\theta}(\tau) + \tilde{\sigma}(\tau)\dot{\sigma}(\tau)\right| \tag{67}$$

for any $\tau \in [0, t_1)$. The projection algorithm ensures that for all $\tau \in [0, t_1)$

$$\omega_l \le \hat{\omega}(t) \le \omega_u \,, \quad |\hat{\theta}(\tau)| \le \theta_b, \quad |\hat{\sigma}(\tau)| \le \sigma_b \,, \tag{68}$$

and therefore

$$\begin{aligned}&\max_{\tau \in [0, t_1)}\left(\Gamma^{-1}(\tilde{\omega}^2 + \tilde{\theta}^2(\tau) + \tilde{\sigma}^2(\tau)\right) \le \\ &\left((\omega_u - \omega_l)^2 + 4\theta_b^2 + 4\sigma_b^2\right)/\Gamma\end{aligned} \tag{69}$$

for any $\tau \in [0, t_1)$. If at any $\tau \in [0, t_1)$

$$V(\tau) \ge \frac{\theta_m(\rho, \rho_u)}{\Gamma} \,, \tag{70}$$

where $\theta_m(\rho,\rho_u)$ is defined in (47), then it follows from (69) that

$$\tilde{x}^\top(\tau)P\tilde{x}(\tau) \geq \frac{4\lambda_{\max}(P)}{\Gamma\lambda_{\min}(Q)}\left(\theta_b d_\theta(\rho,\rho_u)+\sigma_b d_\sigma(\rho,\rho_u)\right), \quad (71)$$

and hence

$$\begin{aligned}\tilde{x}^\top(\tau)Q\tilde{x}(\tau) &\geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\tilde{x}^\top(\tau)P\tilde{x}(\tau)\\ &\geq 4\frac{\theta_b d_\theta(\rho,\rho_u)+\sigma_b d_\sigma(\rho,\rho_u)}{\Gamma}. \quad (72)\end{aligned}$$

It follows from (22), (23) and (68) that

$$|\tilde{\theta}(\tau)| \leq 2\theta_b, \quad |\tilde{\sigma}(\tau)| \leq 2\sigma_b \quad (73)$$

for all $\tau \in [0,t_1)$. Since $\dot{\theta}(\tau)$ and $\dot{\sigma}(\tau)$ are continuous over $[0,t_1)$, the upper bounds in (45) and (73) lead to the following upper bound:

$$\frac{|\tilde{\theta}(\tau)\dot{\theta}(\tau)+\tilde{\sigma}(\tau)\dot{\sigma}(\tau)|}{\Gamma} \leq 2\frac{\theta_b d_\theta(\rho,\rho_u)+\sigma_b d_\sigma(\rho,\rho_u)}{\Gamma}. \quad (74)$$

Hence, if $V(\tau) \geq \dfrac{\theta_m(\rho,\rho_u)}{\Gamma}$, then from (67) and (72) we have

$$\dot{V}(\tau) \leq 0. \quad (75)$$

It follows from (75) that $V(\tau) \leq \dfrac{\theta_m(\rho,\rho_u)}{\Gamma}$ for any $\tau \in [0,t_1)$.

Since $\lambda_{\min}(P)\|\tilde{x}(\tau)\|^2 \leq \tilde{x}^\top(\tau)P\tilde{x}(\tau) \leq V(\tau)$, then for any $\tau \in [0,t_1)$

$$||\tilde{x}(\tau)||^2 \leq \frac{\theta_m(\rho,\rho_u)}{\lambda_{\min}(P)\Gamma}.$$

Since $V(\tau)$ is continuous, we further have

$$\|\tilde{x}(\tau)\|_\infty \leq \sqrt{\frac{\theta_m(\rho,\rho_u)}{\lambda_{\min}(P)\Gamma}}, \quad \tau \in [0,t_1]. \quad (76)$$

Given $t_1 \in [0,t]$ such that

$$V(t_1) \leq \frac{\theta_m(\rho,\rho_u)}{\Gamma},$$

let $t_2 \in (t_1,t]$ be the next time-instant such that discontinuity of any of the derivatives $\dot{\tilde{\theta}}(t)$, $\dot{\tilde{\sigma}}(t)$, $\dot{\theta}(t)$, and $\dot{\sigma}(t)$. Using similar derivations as above, we can prove that

$$\|\tilde{x}(\tau)\|_\infty \leq \sqrt{\frac{\theta_m(\rho,\rho_u)}{\lambda_{\min}(P)\Gamma}}, \quad \tau \in [t_1,t_2]. \quad (77)$$

Iterating the process until the time instant t, we get

$$\|\tilde{x}_t\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_m(\rho,\rho_u)}{\lambda_{\min}(P)\Gamma}}, \quad (78)$$

which concludes the proof. □

Proof of Theorem 1. The proof will be done by contradiction. Assume that (51)-(52) are not true. Then, since $\|x(0)-x_{ref}(0)\|_\infty = 0 \leq \gamma_1$, $u(0)-u_{ref}(0)=0$, and $x(\tau)$, $x_{ref}(\tau)$, $u(\tau)$, $u_{ref}(\tau)$ are continuous, there exists $t \geq 0$ such that

$$\begin{aligned}\|x(t)-x_{ref}(t)\|_\infty &= \gamma_1, \text{ or} \quad (79)\\ \|u(t)-u_{ref}(t)\|_\infty &= \gamma_2, \quad (80)\end{aligned}$$

while

$$\|(x-x_{ref})_t\|_{\mathcal{L}_\infty} \leq \gamma_1,\ \|(u-u_{ref})_t\|_{\mathcal{L}_\infty} \leq \gamma_2. \quad (81)$$

Since (48) holds, (15)-(16) follow from Lemma 1 directly. Taking into consideration the relationships in (8), (18) and (81), we have $\|x_t\|_{\mathcal{L}_\infty} \leq \rho, \quad \|u_t\|_{\mathcal{L}_\infty} \leq \rho_u$. Hence, it follows from (49) and Lemma 3 that

$$\|\tilde{x}_t\|_{\mathcal{L}_\infty} \leq \gamma_0. \quad (82)$$

Let $\tilde{r}(\tau) = \tilde{\omega}(\tau)u(\tau) + \tilde{\theta}(\tau)\|x(\tau)\|_\infty + \tilde{\sigma}(\tau)$, $r_1(\tau) = \theta(\tau)\|x(\tau)\|_\infty + \sigma(\tau)$. It follows from (9) that $\chi(s) = D(s)(\omega u(s)+r_1(s)-k_g r(s)+\tilde{r}(s))$, where $\tilde{r}(s)$ and $r_1(s)$ are the Laplace transformations of signals $\tilde{r}(\tau)$ and $r_1(\tau)$. Consequently

$$\begin{aligned}\chi(s) &= \frac{D(s)}{1+k\omega D(s)}(r_1(s)-k_g r(s)+\tilde{r}(s)),\\ u(s) &= -\frac{kD(s)}{1+k\omega D(s)}(r_1(s)-k_g r(s)+\tilde{r}(s)). \quad (83)\end{aligned}$$

Using the definition of $C(s)$ from (2), we can write

$$\omega u(s) = -C(s)(r_1(s)-k_g r(s)+\tilde{r}(s)), \quad (84)$$

and the system in (1) consequently takes the form:

$$\begin{aligned}x(s) &= H(s)\Big((1-C(s))r_1(s)+C(s)k_g r(s) -\\ &\quad C(s)\tilde{r}(s)\Big) + (s\mathbb{I}-A_m)^{-1}x_0. \quad (85)\end{aligned}$$

Let $e(\tau) = x(\tau)-x_{ref}(\tau)$. It follows from (54) that $e(s) = H(s)\left((1-C(s))r_2(s)-C(s)\tilde{r}(s)\right), \quad e(0)=0$, where $r_2(s)$ is the Laplace transformation of the signal

$$r_2(\tau) = \theta(\tau)(\|x(\tau)\|_\infty - \|x_{ref}(\tau)\|_\infty). \quad (86)$$

Example 5.2 in [6] (page 199) gives the following upper bound:

$$\|e_t\|_{\mathcal{L}_\infty} \leq \|G(s)\|_{\mathcal{L}_1}\|r_{2_t}\|_{\mathcal{L}_\infty} + \|r_{3_t}\|_{\mathcal{L}_\infty}, \quad (87)$$

where $r_3(\tau)$ is the signal with its Laplace transformation being $r_3(s) = C(s)H(s)\tilde{r}(s)$. From the relationship in (65) we have $\tilde{x}(s) = H(s)\tilde{r}(s)$, which leads to $r_3(s) = C(s)\tilde{x}(s)$, and hence $\|r_{3_t}\|_{\mathcal{L}_\infty} \leq \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. Since $\|x(\tau)\|_\infty - \|x_{ref}(\tau)\|_\infty \leq \|x(\tau)-x_{ref}(\tau)\|_\infty \leq \|e_t\|_{\mathcal{L}_\infty}$ for any $\tau \in [0,t]$, it follows from (7), (22) and (86) that $\|r_{2_t}\|_{\mathcal{L}_\infty} \leq L_\rho\|e_t\|_{\mathcal{L}_\infty}$. From (87) we have $\|e_t\|_{\mathcal{L}_\infty} \leq \|G(s)\|_{\mathcal{L}_1}L_\rho\|e_t\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. Eq. (82) and the $\mathcal{L}_1$-gain upper bound from (3) lead to the following upper bound $\|e_t\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1}L_\rho}\gamma_0$, which along with (17) leads to

$$\|e_t\|_{\mathcal{L}_\infty} \leq \gamma_1 - \beta_1 < \gamma_1. \quad (88)$$

We notice that from (12) and (84) one can derive $u(s) - u_{ref}(s) = -\frac{C(s)}{\omega}\theta^\top(\tau)(x(s)-x_{ref}(s)) - r_4(s)$, where $r_4(s) = \frac{C(s)}{\omega}\tilde{r}(s)$. Therefore, it follows from Example 5.2 in [6] (page 199)

$$\|(u-u_{ref})_t\|_{\mathcal{L}_\infty} \leq \frac{L_\rho}{\omega}\|C(s)\|_{\mathcal{L}_1}\|(x-x_{ref})_t\|_{\mathcal{L}_\infty} + \|r_{4_t}\|_{\mathcal{L}_\infty}. \quad (89)$$

We have $r_4(s) = \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top H(s)\tilde{r}(s) = \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\tilde{x}(s)$, where c_o was introduced in (20). Using the polynomials from (20), we can write that $\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} = \frac{C(s)}{\omega}\frac{N_d(s)}{N_n(s)}$. Since $C(s)$ is stable and strictly proper, the complete system $C(s)\frac{1}{c_o^\top H(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$ gain exists and is finite. Hence, we have $\|r_{4_t}\|_{\mathcal{L}_\infty} \leq \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}$. The upper bound in (82) leads to the following upper bound:

$$\|r_{4_t}\|_{\mathcal{L}_\infty} \leq \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\gamma_0. \quad (90)$$

It follows from (88), (89), and (90) and the definition of γ_2 in (19) that

$$\begin{aligned}&\|(u-u_{ref})_t\|_{\mathcal{L}_\infty} \leq (L_\rho/\omega)\|C(s)\|_{\mathcal{L}_1}(\gamma_1-\beta_1) +\\ &\left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\gamma_0 < \gamma_2. \quad (91)\end{aligned}$$

We note that the upper bounds in (88) and (91) contradict the equality in (80), which proves (51)-(52). The upper bound in (50) follows from (51)-(52) and (82) directly. □