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Stability Margins of $\mathcal{L}_1$ Adaptive Controller: Part II

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***Abstract*— In [1], we present a novel $\mathcal{L}_1$ adaptive control architecture that enables fast adaptation and leads to uniformly bounded transient and asymptotic tracking for system's both signals, input and output, simultaneously. In this paper, we derive the stability margins of it and verify those in simulations.**

I. Introduction

In [1], we have introduced novel $\mathcal{L}_1$ adaptive control architecture that has guaranteed transient performance in the presence of unknown time-varying parameters and bounded disturbances. In this paper, we derive the time-delay and the gain margins of it in the presence of unknown constant parameters and bounded time-varying disturbances.We notice that characterization of the time-delay margin is extremely difficult as compared to the gain-margin analysis for closed-loop nonlinear systems. To the best of our knowledge there are no such results in adaptive control theory. On the other hand, this is not surprising since the time-delay margin cannot be characterized if the transient is not guaranteed.

The paper is organized as follows. Section II gives the problem formulation and $\mathcal{L}_1$ adaptive controller. Stability margins are derived in Sections III, IV. Results are generalized in Section V. In section VI, simulation results are presented, while Section VII concludes the paper. The proof of the main theorem is in Appendix.

II. Problem Formulation

Consider the following single-input single-output system:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\left(\omega u(t) + \theta^\top x(t) + \sigma(t)\right), \\ y(t) &= c^\top x(t)\,, x(0) = x_0 = 0 \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is control signal, $y \in \mathbb{R}$ is the regulated output, $b, c \in \mathbb{R}^n$ are known constant vectors, $A_m \in \mathbb{R}^{n\times n}$ is given Hurwitz matrix, $\omega \in \mathbb{R}$ is unknown constant with given sign, $\theta \in \mathbb{R}^n$ is unknown constant vector, and $\sigma(t) \in \mathbb{R}$ is a uniformly bounded time-varying disturbance with a uniformly bounded derivative. Without loss of generality, we assume

$$\omega \in \Omega_0 = [\omega_{l_0},\ \omega_{u_0}]\,,\ \theta \in \Theta\,,\ |\sigma(t)| \le \Delta_0\,,\ \forall\ t \ge 0\,, \quad (2)$$

where $\omega_{u_0} > \omega_{l_0} > 0$ are known (conservative) upper and lower bounds, Θ is a known (conservative) compact set and $\Delta_0 \in \mathbb{R}^+$ is a known (conservative) $\mathcal{L}_\infty$ bound of $\sigma(t)$. We further assume that $\sigma(t)$ is continuously differentiable and its derivative is uniformly bounded, i.e. $|\dot{\sigma}(t)| \le d_\sigma < \infty$ for any $t \ge 0$, where finite d_σ can be arbitrarily large.

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We repeat the $\mathcal{L}_1$ adaptive control architecture from [1].

State Predictor: The state predictor model is:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m\hat{x}(t) + b(\hat{\omega}(t)u(t) + \hat{\theta}^\top(t)x(t) + \hat{\sigma}(t))\,, \\ \hat{y}(t) &= c^\top \hat{x}(t)\,, \quad \hat{x}(0) = x_0\,, \end{aligned} \quad (3)$$

which has the same dynamic structure as the system in (1). Only the unknown parameters and the disturbance $\omega, \theta, \sigma(t)$ are replaced by their adaptive estimates $\hat{\omega}(t), \hat{\theta}(t), \hat{\sigma}(t)$.

Adaptive Laws: Adaptive estimates are governed by the following laws:

$$\dot{\hat{\theta}}(t) = \Gamma_\theta \mathrm{Proj}(\hat{\theta}(t), -x(t)\tilde{x}^\top(t)Pb),\ \hat{\theta}(0) = \hat{\theta}_0\,, \quad (4)$$

$$\dot{\hat{\sigma}}(t) = \Gamma_\sigma \mathrm{Proj}(\hat{\sigma}(t), -\tilde{x}^\top(t)Pb), \qquad \hat{\sigma}(0) = \hat{\sigma}_0\,, \quad (5)$$

$$\dot{\hat{\omega}}(t) = \Gamma_\omega \mathrm{Proj}(\hat{\omega}(t), -\tilde{x}^\top(t)Pbu(t)), \hat{\omega}(0) = \hat{\omega}_0\,, \quad (6)$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$, $\Gamma_\theta = \Gamma_c I_{n\times n} \in \mathbb{R}^{n\times n}$, $\Gamma_\sigma = \Gamma_\omega = \Gamma_c > 0$ are the adaptation rates, and $P = P^\top > 0$ is the solution of the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$, $Q > 0$. In the implementation of the projection operator we use the compact set Θ as given in (2), while we replace Δ_0, Ω_0 by larger sets Δ and $\Omega = [\omega_l,\ \omega_u]$ such that

$$\Delta_0 < \Delta, \qquad 0 < \omega_l < \omega_{l_0} < \omega_{u_0} < \omega_u\,, \quad (7)$$

the purpose of which will be clarified in the analysis of the time-delay and gain margins.

Control Law: The control signal is defined as:

$$\chi(s) = D(s)r_u(s)\,, \quad u(s) = -k\chi(s)\,, \quad (8)$$

where $r_u(t) = \hat{\omega}(t)u(t) + \bar{r}(t)$, $k > 0$ is a feedback gain, $\bar{r}(t) = \hat{\theta}^\top(t)x(t) + \hat{\sigma}(t) - k_g r(t)$, $k_g = -\frac{1}{c^\top A_m^{-1} b}$, and $D(s)$ is a LTI system that needs to be chosen to ensure

$$C(s) = \omega k D(s)/(1 + \omega k D(s)) \quad (9)$$

is stable and strictly proper with $C(0) = 1$. We now give the $\mathcal{L}_1$ performance requirement that ensures desired transient performance, [1].

$\mathcal{L}_1$-gain stability requirement: Design $D(s)$ to ensure that $C(s)$ in (9) satisfies

$$\|G(s)\|_{\mathcal{L}_1} L < 1\,, \quad (10)$$

where $G(s) = H(s)(1 - C(s))$, and $H(s) = (sI - A_m)^{-1}b$.

The complete $\mathcal{L}_1$ adaptive controller consists of (3), (4)-(6), (8) subject to (10). We notice that the $\mathcal{L}_1$-gain stability requirement depends only upon the choice of Θ and is independent of the choice of Δ_0, Ω_0 or Δ, Ω.

Consider the following closed-loop reference system, the stability of which follows from (10), [1]:

$$\begin{aligned} &\dot{x}_{ref} = A_m x_{ref} + b(\omega u_{ref} + \theta^\top x_{ref} + \sigma), \\ &u_{ref}(s) = (C(s)/\omega)\bar{r}_{ref}(s)\,, y_{ref}(t) = c^\top x_{ref}(t), \end{aligned} \quad (11)$$

with $x_{ref}(0) = x_0$, where $\bar{r}_{ref}(t) = -\theta^\top x_{ref}(t) - \sigma(t) + k_g r(t)$. The main result of [1] implies that by increasing the adaptation gain, $x(t)$ and $u(t)$ can track $x_{ref}(t)$ and $u_{ref}(t)$ arbitrarily closely both in transient and asymptotically.

III. Time-delay Margin Analysis

A. $\mathcal{L}_1$ adaptive controller in the presence of time-delay

In this section, we develop the time-delay margin analysis for the system in (1). We rewrite the open-loop system

$$x(s) = \bar{H}(s)(\omega u(s) + \sigma(s)), \quad x(0) = 0, \qquad (12)$$

where $\bar{H}(s) = (sI - A_m - b\theta^\top)^{-1} b$. We further consider the following three systems.

System 1. Let $x_d(t)$ be the delayed signal of the open-loop state $x(t)$ of (12) by a constant time interval τ, i.e

$$x_d(t) = \begin{cases} x(t-\tau) & t \geq \tau, \\ 0 & t < \tau. \end{cases} \qquad (13)$$

We close the loop of (12) with $\mathcal{L}_1$ adaptive controller (3), (4)-(6), (8), using $x_d(t)$ from (13) instead of $x(t)$ everywhere in the definition of (3), (4)-(6), (8). We denote the resulting control and the state trajectory of this closed-loop system by $u(t)$ and $x_d(t)$. We further notice that this closed-loop adaptive system has a unique solution. It is the stability of this closed-loop system that we are trying to determine dependent upon τ. It is important to point out that while applying the $\mathcal{L}_1$ adaptive controller to the system in (12) using $x_d(t)$ from (13), stability and analysis results in [1] are invalid.

System 2. Next, we consider the following closed-loop system with the same zero initial conditions:

$$\dot{x}_q(t) = A_m x_q(t) + b\left(\omega u_q(t) + \theta^\top x_q(t) + \sigma(t) + \eta(t)\right), \qquad (14)$$

where $x_q(0) = x(0)$, $u_q(t)$ is defined via (3), (4)-(6) and (8) with $x(t)$ being replaced by $x_q(t)$, while $\eta(t)$ is a continuously differentiable bounded signal with uniformly bounded derivative. As compared to (1) or (12), the system in (14) has one more additional disturbance signal $\eta(t)$. If

$$|\sigma(t) + \eta(t)| \leq \Delta, \qquad (15)$$

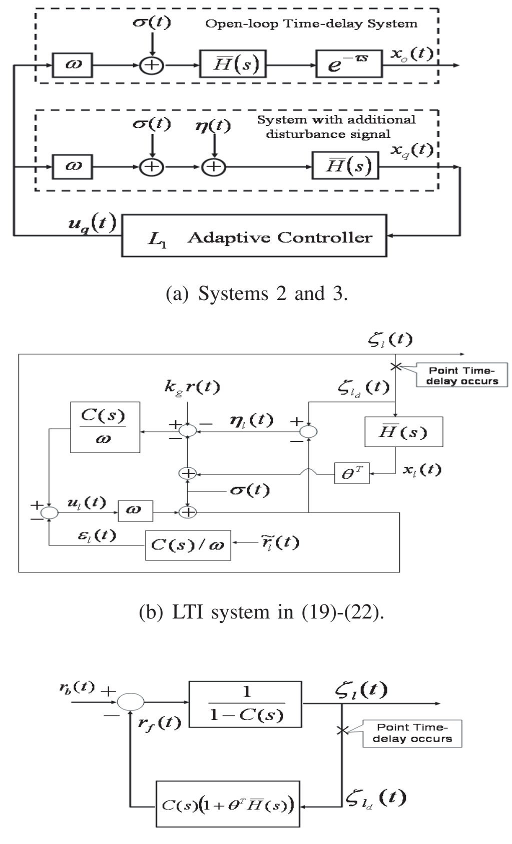
where Δ has been defined in (7), then application of $\mathcal{L}_1$ adaptive controller to the system in (14) is well defined, and hence the results in [1] are valid. We denote by $u_q(t)$ the time trajectory of the $\mathcal{L}_1$ adaptive controller, resulting from its application to (14).

System 3. Finally, we consider the open-loop system in (12)-(13) and apply $u_q(t)$ to it and look at its delayed output $x_o(t)$, where the subindex o is added to indicate the open-loop nature of this signal. It is important to notice that at this point we view $u_q(t)$ as a time-varying input signal for (12), and not as a feedback signal, so that (12) remains an open-loop system in this context.

Illustration of these last two systems is given in Fig. 1(a).

Lemma 1: If the time-delayed output of the open-loop System 3 has the same time history as the closed-loop output of System 2, i.e.

$$x_o(t) = x_q(t), \; \forall\, t \geq 0, \qquad (16)$$



(a) Systems 2 and 3.

(b) LTI system in (19)-(22).

(c) LTI system in (26).

Fig. 1.

then $u(t) = u_q(t)$, $x_d(t) = x_q(t)$, $\forall\, t \geq 0$, where $u(t)$ and $x_d(t)$ denote the control and state trajectories of the closed-loop System 1 in (12)-(13) with $\mathcal{L}_1$ adaptive controller.

Proof. Eq. (16) implies that the open-loop time-delayed System 3 in (12)-(13) generates $x_q(t)$ in response to the input $u_q(t)$. When applied to (14), $u_q(t)$ leads to $x_q(t)$. Hence, $u_q(t)$ and $x_q(t)$ are also solutions of the closed-loop adaptive System 1 in (12)-(13) with (3), (4)-(6), (8). □

This Lemma consequently implies that to ensure stability of the System 1 in the presence of a given time-delay τ, it is sufficient to prove existence of $\eta(t)$ in System 2, satisfying (15) and verifying (16). We notice, however, that the closed-loop System 2 is a nonlinear system due to the nonlinear adaptive laws, so that the proof on existence of such $\eta(t)$ for this system and explicit construction of the set Δ is not straightforward. Moreover, we note that the condition in (16) relates the time-delay τ of System 1 (or System 3) to the signal $\eta(t)$ implicitly. In the next section we introduce an equivalent LTI system that helps to prove existence of such $\eta(t)$ and leads to explicit construction of Δ. Definition of this LTI system is the key step in the overall time-delay margin analysis. It has an exogenous input that lumps the time trajectories of the nonlinear elements of the closed-loop System 2. For this LTI system, the time delay margin can be computed via its open-loop transfer function, which consequently defines a conservative, but guaranteed, lower bound for the time-delay margin of the adaptive system.

B. LTI System in the Presence of Time-delay in its Output

Consider the following closed-loop LTI system:

$$x_l(s) = \bar{H}(s)\zeta_l(s), \quad \epsilon_l(s) = (C(s)/\omega)\tilde{r}_l(s),$$
$$u_l(s) = (1/\omega)C(s)(k_g r(s) - \theta^\top x_l(s) - \sigma(s) - \eta_l(s)) - \epsilon_l(s),$$

where $\zeta_l(s) = \omega u_l(s) + \sigma(s)$, $\eta_l(s) = \zeta_l(s) - \omega u_l(s) - \sigma(s)$, $x_l(t)$, $u_l(t)$ and $\epsilon_l(t)$ are the states, $\zeta_l(t)$ is its output signal, and $\tilde{r}_l(t)$ is an exogenous signal. We note that the system trajectories are uniquely defined once $\tilde{r}_l(t)$ is given. Since $x_l(s) = \bar{H}(s)\zeta_l(s)$, we have

$$x_l(s)/r(s) = (\bar{H}(s)C(s))/(1 + C(s)\theta^\top \bar{H}(s)), \quad (17)$$
$$x_l(s)/\sigma(s) = (\bar{H}(s)(1 - C(s)))/(1 + C(s)\theta^\top \bar{H}(s)). \quad (18)$$

We notice that for the reference system in (11), in case of constant θ, the signals $x_{ref}(s)/r(s)$ and $x_{ref}(s)/\sigma(s)$ are equivalent to those in (17) and (18). We also notice that the LTI system in the absence of time-delay ensures stable transfer functions from the inputs $r(t)$, $\sigma(t)$ and $\tilde{r}_l(t)$ to the output $\zeta_l(t)$.

Assume the system output $\zeta_l(t)$ experiences time-delay τ, so that in the presence of the time-delay we have:

$$x_l(s) = \bar{H}(s)\zeta_{l_d}(s) \quad (19)$$
$$u_l(s) = (C(s)/\omega)(k_g r(s) - \theta^\top x_l(s) - \sigma(s) - \eta_l(s)) - \epsilon_l(s) \quad (20)$$
$$\epsilon_l(s) = (C(s)/\omega)\tilde{r}_l(s) \quad (21)$$
$$\zeta_l(s) = \omega u_l(s) + \sigma(s), \quad (22)$$

where $\zeta_{l_d}(t)$ is the time-delayed signal of $\zeta_l(t)$, i.e

$$\zeta_{l_d}(t) = \begin{cases} 0 & t < \tau, \\ \zeta_l(t-\tau) & t \geq \tau, \end{cases} \quad (23)$$

consequently leading to redefined $\eta_l(s)$:

$$\eta_l(s) = \zeta_{l_d}(s) - \omega u_l(s) - \sigma(s). \quad (24)$$

Let

$$x_l(0) = 0, \quad u_l(0) = 0, \quad \epsilon_l(0) = 0. \quad (25)$$

We notice that the system in (19)-(22) is highly coupled. Its diagram is plotted in Figure 1(b).

C. Time-Delay Margin of the LTI System

We notice that the phase margin of this LTI system can be determined by its open-loop transfer function from $\zeta_{l_d}(t)$ to $\zeta_l(t)$. It follows from (19), (20), and (24) that

$$\omega u_l(s) = \frac{C(s)\left(k_g r(s) - \zeta_{l_d}(s) - \theta^\top \bar{H}(s)\zeta_{l_d}(s)\right) - \omega\epsilon_l(s)}{1 - C(s)},$$

and hence the relationship in (22) implies that $\zeta_l(s) = \frac{C(s)\left(k_g r(s) - \zeta_{l_d}(s) - \theta^\top \bar{H}(s)\zeta_{l_d}(s)\right) - \omega\epsilon_l(s)}{1 - C(s)} + \sigma(s)$. Therefore, it can be equivalently written as:

$$\zeta_l(s) = (1/(1 - C(s)))(r_b(s) - r_f(s)),$$
$$r_f(s) = C(s)(1 + \theta^\top \bar{H}(s))\zeta_{l_d}(s), \quad (26)$$
$$r_b(s) = C(s)k_g r(s) + (1 - C(s))\sigma(s) - \omega\epsilon_l(s).$$

Assume that $\tilde{r}_l(t)$ is such that $\epsilon_l(t)$ is bounded. Since $\sigma(t)$ and $r(t)$ are bounded, $C(s)$ is strictly proper and stable, then $r_b(t)$ is also bounded. The block-diagram of the closed-loop system in (26) is shown in Figure 1(c).

The open-loop transfer function of the system in (26) is:

$$H_o(s) = (C(s)/1 - C(s))(1 + \theta^\top \bar{H}(s)), \quad (27)$$

the phase margin $\mathcal{P}(H_o(s))$ of which can be derived from its Bode plot easily. Its time-delay margin is given by:

$$\mathcal{T}(H_o(s)) = \mathcal{P}(H_o(s))/\omega_c, \quad (28)$$

where $\mathcal{P}(H_o(s))$ is the phase margin of the open-loop system $H_o(s)$, and ω_c is the cross-over frequency of $H_o(s)$. The next lemma states a sufficient condition for boundedness of all the states in the system (19)-(22), including the internal states.

Lemma 2: Let

$$\tau < \mathcal{T}(H_o(s)), \quad (29)$$

and ϵ_b be any positive number such that $\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b$. Then the signals $\zeta_l(t)$, $x_l(t)$, $u_l(t)$, $\eta_l(t)$ are bounded.

Proof: Since $\epsilon_l(t)$ is bounded and $\tau < \mathcal{T}(H_o(s))$, the boundedness of $\zeta_l(t)$ follows from the definition of $\mathcal{T}(H_o(s))$. The boundedness of $\zeta_{l_d}(t)$ follows from its definition in (23). Since $\zeta_l(t)$ and $\sigma(t)$ are bounded, it follows from (22) that $u_l(t)$ is bounded, and (24) implies that $\eta_l(t)$ is bounded. Notice that since $u_l(t)$ and $\epsilon_l(t)$ are bounded, it follows from (20) that $\theta^\top x_l(t)$ is bounded. Finally, we notice that $x_l(s)$ in (19) can be written as $x_l(s) = H(s)(\theta^\top x_l(s) + \zeta_{l_d}(s))$, which leads to boundedness of $x_l(t)$. □

For any $\tau < \mathcal{T}(H_o(s))$ and any $\epsilon_b > 0$, Lemma 2 guarantees that the map $\Delta_n : \mathbb{R}^+ \times [0, \mathcal{T}(H_o(s))) \to \mathbb{R}^+$

$$\Delta_n(\epsilon_b, \tau) \quad = \max_{\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b} \|\sigma + \eta_l\|_{\mathcal{L}_\infty} \quad (30)$$

is well defined. We note that strictly speaking $\eta_l(t)$ depends not only on $\epsilon_l(t)$ and τ, but also upon other arguments, like $\sigma(t)$ and other variables of the system that are used in the definition of $\eta_l(t)$. These are dropped due to their non-crucial role in the subsequent analysis.

Lemma 3: Let τ comply with (29), and ϵ_b be any positive number. If $\tilde{r}_l(t)$ is such that the resulting $\epsilon_l(t)$ is bounded

$$\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b, \quad (31)$$

and

$$2\omega\|u_l\|_{\mathcal{L}_\infty} + 2L\|x_l\|_{\mathcal{L}_\infty} + 2\Delta \geq \|\tilde{r}_l\|_{\mathcal{L}_\infty}, \quad (32)$$

where

$$\Delta = \Delta_n(\epsilon_b, \tau) + \delta_1, \quad (33)$$

with $\delta_1 > 0$ being arbitrary constant, then $\eta_l(t)$ has a uniformly bounded derivative.

Proof: Using Lemma 2, we immediately conclude that $x_l(t)$, $u_l(t)$, $\Delta_n(\epsilon_b, \tau)$ are bounded. Hence, it follows from (32) that $\tilde{r}_l(t)$ is also bounded. Since $C(s)$ is strictly proper and stable, bounded $\tilde{r}_l(t)$ ensures that $\epsilon_l(t)$ is differentiable with bounded derivative. Using similar methods, we prove that both $u_l(t)$ and $\zeta_{l_d}(t)$ have bounded derivatives. Since $\dot{\sigma}(t)$ is bounded, it follows from (24) that $\dot{\eta}_l(t)$ is bounded. □

For any $\tau < \mathcal{T}(H_o(s))$ and any $\epsilon_b > 0$, Lemma 3 guarantees that the following map $\Delta_d : \mathbb{R}^+ \times [0, \mathcal{T}(H_o(s))) \to \mathbb{R}^+$

$$\Delta_d(\epsilon_b, \tau) = \max_{\tilde{r}_l(t)} \|\dot{\sigma} + \dot{\eta}_l\|_{\mathcal{L}_\infty} \tag{34}$$

is well defined, where $\tilde{r}_l(t)$ complies with (31) and (32). Further, let

$$\begin{aligned} \theta_m(\epsilon_b, \tau) &\triangleq \max_{\theta \in \Theta} \sum_{i=1}^{n} 4\theta_i^2 + 4\Delta^2 + 4\left(\omega_u - \omega_l\right)^2 \\ &\quad + 4\lambda_{\max}(P)\Delta_d(\epsilon_b, \tau)\Delta/\lambda_{\min}(Q)\,, \quad (35) \\ \epsilon_c(\epsilon_b, \tau) &= \left\|C(s)(c_o^\top H(s))^{-1} c_o^\top\right\|_{\mathcal{L}_1} \\ &\quad \sqrt{\theta_m(\epsilon_b, \tau)/(\lambda_{\max}(P)\epsilon_b^2)}\,. \quad (36) \end{aligned}$$

We notice that for any finite $\epsilon_b \in \mathbb{R}^+$ and any τ verifying (29), we have finite $\Delta_n(\epsilon_b, \tau)$ and $\Delta_d(\epsilon_b, \tau)$, and hence finite $\epsilon_c(\epsilon_b, \tau)$, if $\tilde{r}_l(t)$ complies with (31) and (32).

D. Time-delay Margin of the Closed-loop Adaptive System

In this section we formulate the main result for the time-delay margin of $\mathcal{L}_1$ adaptive controller.

Theorem 1: Consider the closed-loop adaptive system, comprised of System 1 in (12)-(13) with (3), (4)-(6), (8) and the LTI system in (19)-(22) in the presence of the same time delay τ. For any $\epsilon_b \in \mathbb{R}^+$ choose the set Δ as in (33) and let

$$\Gamma_c \geq \sqrt{\epsilon_c(\epsilon_b, \tau)} + \delta_2\,, \tag{37}$$

where δ_2 is arbitrary positive constant. Then for every τ satisfying $\tau < \mathcal{T}(H_o(s))$, there exists an exogenous signal $\tilde{r}_l(t)$ ensuring that $\|\epsilon_l\|_{\mathcal{L}_\infty} < \epsilon_b$, and $x_l(t) = x_d(t)\,, u_l(t) = u(t)\,,\ \forall\ t \geq 0\,.$

The proof is in Appendix.

Theorem 1 establishes the equivalence of state and control trajectories of the closed-loop adaptive system and the LTI system in (19)-(22) in the presence of the same time-delay. Therefore the time-delay margin of the system in (19)-(22) can be used as a conservative lower bound for the time-delay margin of the closed-loop adaptive system.

Corollary 1: Given the system in (1) with constant θ and the $\mathcal{L}_1$ adaptive controller defined via (3), (4)-(6) and (8) subject to (10), where Γ_c and Δ are selected appropriately large, the closed-loop adaptive system is stable in the presence of time delay τ in its output if $\tau < \mathcal{T}(H_o(s))$, where $\mathcal{T}(H_o(s))$ is defined in (28).

The proof follows from Lemma 2 and Theorem 1 directly.

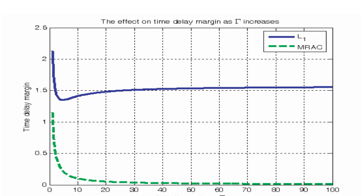


Fig. 2. Effect of adaptive gain on time-delay margins of MRAC (slashed) and $\mathcal{L}_1$ adaptive controller (solid)

Remark 1: If we omit the uncertainties due to θ and ω in (1) and retain only constant $\sigma(t) = const$, both MRAC and $\mathcal{L}_1$ adaptive controller degenerate into LTI systems. We can verify the time-delay margins using frequency domain tools. Letting $\dot{x}(t) = -x(t) + u(t) + \sigma$, Fig. 2 shows the time-delay margin of both architectures with respect to Γ_c. As $\Gamma_c \to \infty$, the time-delay margin of MRAC decreases, while $\mathcal{L}_1$ adaptive controller has a time-delay margin equal to $\pi/2$. We verify from (28) that in this case $\mathcal{T}(H_o(s)) = \pi/2\,.$

IV. Gain Margin Analysis

We now analyze the gain margin of the system in (1) with $\mathcal{L}_1$ adaptive controller. By inserting a gain module g into the control loop, the system in (1) can be formulated as:

$$\dot{x}(t) = A_m x(t) + b\left(\omega_g u(t) + \theta^\top(t)x(t) + \sigma(t)\right)\,, \tag{38}$$

where $\omega_g = g\omega$. We note that this transformation implies that the set Ω in the application of the Projection operator for adaptive laws needs to increase accordingly. However, increased Ω will not violate the condition in (10). Thus, it follows from (7) that the gain margin of the $\mathcal{L}_1$ adaptive controller is determined by:

$$\mathcal{G}_m = [\omega_l/\omega_{l_0},\ \ \omega_u/\omega_{u_0}]. \tag{39}$$

If $g \in \mathcal{G}_m$, then the closed-loop system in (38) satisfies the $\mathcal{L}_1$ stability criterion in (10), implying that the entire closed-loop system is stable. We note that the lower-bound of $\mathcal{G}_m$ is greater than zero. Eq. (39) implies that arbitrary gain margin can be obtained through appropriate choice of Ω.

V. Main Results

Theorem 2: Given the system in (1) with constant unknown parameters θ and the $\mathcal{L}_1$ adaptive controller defined via (3), (4)-(6) and (8) subject to (10), we have:

$$\begin{aligned} &\lim_{\Gamma_c \to \infty} \mathcal{T} \geq \mathcal{T}(H_o(s))\,, \quad \mathcal{G} \supseteq \mathcal{G}_m\,, \quad (40) \\ &\lim_{\Gamma_c \to \infty} (x(t) - x_{ref}(t)) = 0\,, \lim_{\Gamma_c \to \infty} (u(t) - u_{ref}(t)) = 0\,, \end{aligned}$$

for any $t \geq 0$, where $\mathcal{T}$ and $\mathcal{G}$ are the time-delay and the gain margins of the $\mathcal{L}_1$ adaptive controller, while $\mathcal{T}(H_o(s))$, $\mathcal{G}_m$ are defined in (28) and (39).

VI. Simulations

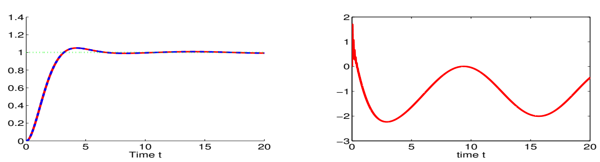
We consider the same system from [1], in which a single-link robot arm is rotating on a vertical plane. Assuming constant $\theta(t)$, it can be cast into the form in (1) with $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $\theta = [2\ \ 2]^\top$, $\omega = 1$, $\sigma(t) = \sin(\pi t)$, so that the compact sets can be conservatively chosen as $\Omega_0 = [0.2,\ 5]$, $\Theta = [-10,\ 10]$, $\Delta_0 = [-10,\ 10]$, respectively. Next, we analyze the stability margins of the $\mathcal{L}_1$ adaptive controller for this system numerically.

For $\theta = [2\ \ 2]^\top$, $\omega = 1$ we can derive $H_o(s)$ in (27). Its Bode plot indicates phase margin $88.1°(1.54\mathrm{rad})$ at cross frequency $9.55\mathrm{Hz}(60\mathrm{rad/s})$. Hence, the time-delay margin can be derived from (28) as: $\mathcal{T}(H_o(s)) = \frac{1.54\mathrm{rad}}{60\mathrm{rad/s}} = 0.0256$.

We set $\Delta = [-1000 \quad 1000]^\top$, $\Gamma_c = 500000$, and run the $\mathcal{L}_1$ adaptive controller with time-delay $\tau = 0.02$. The simulations in Figs. 3(a)-3(b) verify Corollary 1. As stated in Theorem 2, the time-delay margin of the LTI system in (27) provides only a conservative lower bound for the time-delay margin of the closed-loop adaptive system. So, we simulate the $\mathcal{L}_1$ adaptive controller in the presence of larger time-delay, like $\tau = 0.1$ sec., and observe that the system is not losing its stability. Since θ and ω are unknown, we derive the $\mathcal{T}(H_o(s))$ for all possible $\theta \in \Theta$ and $\omega \in \Omega$ and use the most conservative value. It gives $\mathcal{T}(H_o(s)) = 0.005s$. The gain margin can be arbitrarily large as stated in (40).

VII. Conclusion

In this paper, we derive the stability margins of $\mathcal{L}_1$ adaptive controller presented in [1]. To the best of our knowledge, this is the first attempt to quantify the time-delay margin for a closed-loop adaptive system. With this particular architecture, we prove that increasing the adaptive gain improves the time-delay margin. This presents a significant improvement over conventional adaptive control schemes, in which increasing the adaptive gain leads to reduced tolerance to time-delay in input/output channels.



(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

(b) Time-history of $u(t)$

Fig. 3. Performance of $\mathcal{L}_1$ adaptive controller with time-delay $0.02s$

References

[1] C. Cao, N. Hovakimyan. Guaranteed transient performance with $\mathcal{L}_1$ adaptive controller for systems with unknown time-varying parameters and bounded disturbances: Part I. *American Control Conference*, 2007.

Appendix

Let $x_h(t)$ be the state variable of the LTI system $H_x(s)$, while $x_i(t)$ and $x_s(t)$ be the input and the output signals of it. We note that for any time instant t_1 and any fixed time-interval $[t_1,\ t_2]$, where $t_2 > t_1$, given $x_h(t_1)$ and a continuous input signal $x_i(t)$ over $[t_1, t_2)$, $x_s(t)$ is uniquely defined for $t \in [t_1, t_2]$. Let $\mathcal{S}$ be the map $x_s(t)|_{t\in[t_1,\ t_2]} = \mathcal{S}(H_x(s), x_h(t_1), x_i(t)|_{t\in[t_1,\ t_2)})$. We note that $x_s(t)$ is continuous, if $x_i(t)$ has no δ function. Also, $x_s(t)$ is defined in the space of continuous functions over the closed interval $[t_1,\ \ t_2]$, i.e $\mathcal{C}_{[t_1,\ t_2]}$, although $x_i(t)$ is defined in the $\mathcal{L}_\infty$ space over the open set $[t_1,\ \ t_2)$. Let $x_{o_1}|_{t\in[t_1,\ t_2]} = \mathcal{S}(H_x(s), x_{h_1}, x_{i_1}(t)|_{t\in[t_1,\ t_2)})$, $x_{o_2}|_{t\in[t_1,\ t_2]} = \mathcal{S}(H_x(s), x_{h_2}, x_{i_2}(t)|_{t\in[t_1,\ t_2)})$. Definition of $\mathcal{S}$ implies that if $x_{h_1} = x_{h_2}$ and $x_{i_1}(t) = x_{i_2}(t)$ over $[t_1\,, t_2)$, then $x_{o_1}(t) = x_{o_2}(t)$ for any $t \in [t_1\,, t_2]$.

Proof of Theorem 1: In the closed-loop adaptive system in (14) for any $t^* \geq 0$, we notice that if $\|(\sigma+\eta)_{t^*}\|_{\mathcal{L}_\infty} \leq \Delta$, and $\sigma(t),\ \eta(t)$ have finite derivatives over $[0,\ t^*]$, then application of $\mathcal{L}_1$ adaptive controller is well-defined. Let

$$d_{t^*} = \|(\dot{\sigma}+\dot{\eta})_{t^*}\|_{\mathcal{L}_\infty}\,. \tag{41}$$

It follows from (3) and (14) that $\tilde{x}_q(s) = H(s)\tilde{r}(s)$, where $\tilde{x}_q(t) = \hat{x}(t) - x_q(t)$ and

$$\tilde{r}(t) = \tilde{\omega}(t)u_q(t) + \tilde{\theta}^\top(t)x_q(t) + \tilde{\sigma}(t)\,. \tag{42}$$

This along with Eq. (25) in [1] implies that $u_q(t)_{t\in[0,t^*]} = \mathcal{S}(C(s)/\omega, u_q(0), (k_g r(t) - \theta^\top x_q(t) - \sigma(t) - \eta(t) - \tilde{r}(t))_{t\in[0,t^*)})$, $\tilde{x}_q(t)_{t\in[0,t^*]} = \mathcal{S}(H(s), \tilde{x}_q(0), \tilde{r}(t)_{t\in[0,t^*)})$, where $\tilde{\sigma}(t) = \hat{\sigma}(t) - (\sigma(t) + \eta(t))$. Note that

$$\begin{aligned} u_q(t)_{t\in[0,t^*]} &= \mathcal{S}(C(s)/\omega, u_q(0), (k_g r(t) - \theta^\top x_q(t) - \sigma(t) \\ &\quad -\eta(t))_{t\in[0,t^*)}) - \epsilon(t)_{t\in[0,t^*]}\,, \end{aligned} \tag{43}$$

where

$$\epsilon(t)_{t\in[0,t^*]} = \mathcal{S}(C(s)/\omega, 0, \tilde{r}(t)_{t\in[0,t^*)})\,. \tag{44}$$

We further define

$$\begin{aligned} \theta_{t^*} \quad \triangleq \quad & \max_{\theta\in\Theta}\sum_{i=1}^{n} 4\theta_i^2 + \max_{\sigma\in\Delta} 4\sigma^2 + 4\left(\omega_u - \omega_l\right)^2 \\ & +4\lambda_{\max}(P)d_{t^*}\Delta/\lambda_{\min}(Q)\,. \end{aligned} \tag{45}$$

It can be verified that Lemma 7 in [1] holds for truncated norms as well so that $\|\tilde{x}_{q_{t^*}}\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_{t^*}}{\lambda_{\min}(P)\Gamma_c}}$. Since $\epsilon(s) = \frac{C(s)}{\omega c_o^\top H(s)} c_o^\top H(s)\tilde{r}(s) = \frac{C(s)}{\omega c_o^\top H(s)} c_o^\top \tilde{x}_q(s)$, then

$$\|\epsilon_{t^*}\|_{\mathcal{L}_\infty} \leq \left\| C(s)(\omega c_o^\top H(s))^{-1} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\theta_{t^*}/(\lambda_{\min}(P)\Gamma_c)}\,. \tag{46}$$

In the three steps below, we prove the existence of a continuously differentiable $\eta(t)$ with uniformly bounded derivative in the closed-loop adaptive system (14), (3), (4)-(6), (8) and the existence of $r_l(t)$ in the time-delayed LTI system such that for any $t \geq 0$

$$|\sigma(t) + \eta(t)| < \Delta\,, \quad x_o(t) = x_q(t)\,, \tag{47}$$

$$\|\epsilon_{l_t}\|_{\mathcal{L}_\infty} < \epsilon_b\,, x_l(t) = x_q(t)\,, u_l(t) = u_q(t)\,, \epsilon_l(t) = \epsilon(t)\,. \tag{48}$$

With (47), Lemma 1 implies that $x_d(t) = x_q(t), u(t) = u_q(t)$ for any $t \geq 0$, while (48) proves Theorem 1.

Step 1: Let

$$\zeta(t) = \omega u_q(t) + \sigma(t)\,. \tag{49}$$

We further define

$$\zeta_d = \left\{ \begin{array}{cc} 0\,, & t \in [0,\tau) \\ \zeta(t-\tau)\,, & t \geq \tau \end{array} \right.\,. \tag{50}$$

Since (12) and (13) imply that $x_o(t) = 0$ for any $t \in [0,\tau]$, it follows from (50) and the definition of the map $\mathcal{S}$ that $x_o(t)|_{t\in[0,\tau]} = \mathcal{S}(\bar{H}(s), x_o(0), \zeta_d(t)_{t\in[0,\tau)})$. For $i \geq 1$, it follows from the definition of the time-delayed open-loop System 3 that

$$x_o(t)|_{t\in[i\tau,(i+1)\tau]} = \mathcal{S}(\bar{H}(s), x_o(i\tau), \zeta_d(t)_{t\in[i\tau,(i+1)\tau)})\,. \tag{51}$$

We note that (51) holds for any i. Also, it follows from (44) that $\epsilon(0) = 0$. Taking into consideration the initial conditions and definitions in (12), (13), (23), (25), we have that for $i = 0$,

$$\begin{aligned} & u_q(i\tau) = u_l(i\tau)\,, \epsilon(i\tau) = \epsilon_l(i\tau)\,, x_o(i\tau) = x_q(i\tau) = x_l(i\tau)\,, \\ & \quad \zeta_d(t) = \zeta_{l_d}(t)\,, \quad t < (i+1)\tau\,, \quad |\epsilon(t)| < \epsilon_b\,, \quad t \leq i\tau\,. \end{aligned}$$

Step 2: Assume that for any i the following conditions hold:

$$u_q(t) = u_l(t)\,, \quad t \leq i\tau\,, \tag{52}$$

$$\epsilon(t) = \epsilon_l(t)\,, \quad t = i\tau\,, \tag{53}$$

$$x_o(t) = x_q(t) = x_l(t)\,, \quad t \leq i\tau\,, \tag{54}$$

$$\zeta_d(t) = \zeta_{ld}(t)\,, \ \forall\ t \in [i\tau, (i+1)\tau)\,, \tag{55}$$

$$|\epsilon(t)| < \epsilon_b\,, \ \forall\ t \leq i\tau\,. \tag{56}$$

For $i \geq 1$, further assume that there exist bounded $\tilde{r}_l(t)$ and continuously differentiable $\eta(t)$ with bounded derivative over $t \in [0, i\tau)$ such that $\forall\ t < i\tau$

$$\eta(t) = \eta_l(t)\,, \quad |\sigma(t) + \eta(t)| < \Delta\,. \tag{57}$$

We prove below that there exist bounded $\tilde{r}_l(t)$ and continuously differentiable $\eta(t)$ with bounded derivative over $t \in [0, (i+1)\tau)$ such that (52)-(57) hold for $i+1$, too. We note that (19) implies

$$x_l(t)|_{t\in[i\tau,(i+1)\tau]} = \mathcal{S}(\bar{H}(s), x_l(i\tau), \zeta_{l_d}(t)_{t\in[i\tau,(i+1)\tau)}). \tag{58}$$

Using (54)-(55), it follows from (51) and (58) that

$$x_o(t) = x_l(t), \quad \forall\ t \in [i\tau,\, (i+1)\tau]\,. \tag{59}$$

We assumed in (57) that if $i \geq 1$, then there exists continuous $\eta(t)$ over $[0, i\tau)$ with uniformly bounded derivative. We now define $\eta(t)$ over $[i\tau, (i+1)\tau)$ as:

$$\eta(t) = \zeta_d(t) - \omega u_q(t) - \sigma(t)\,, \quad t \in [i\tau, (i+1)\tau)\,. \tag{60}$$

Since (14) implies that $x_q(t)|_{t\in[i\tau,(i+1)\tau]} = \mathcal{S}(\bar{H}(s), x_q(i\tau)(\omega u_q(t)+\sigma(t)+\eta(t))_{t\in[i\tau,(i+1)\tau)})$, it follows from (60) that $x_q(t)|_{t\in[i\tau,(i+1)\tau]} = \mathcal{S}(\bar{H}(s), x_q(i\tau), \zeta_d(t)_{t\in[i\tau,(i+1)\tau)})$. Along with (51) and (54) this ensures that

$$x_q(t) = x_o(t), \ \ \forall\ t \in [i\tau, (i+1)\tau]\,. \tag{61}$$

However, the definition in (60) does not guarantee

$$|\sigma(t) + \eta(t)| < \Delta\,, \ \ t \in [i\tau, (i+1)\tau)\,, \tag{62}$$

which is required for application of the $\mathcal{L}_1$ adaptive controller.

We prove (62) by contradiction. Since $\eta(t)$ is continuous over $[i\tau, (i+1)\tau)$, if (62) is not true, there must exist $t' \in [i\tau, (i+1)\tau)$ such that $|\sigma(t) + \eta(t)| < \Delta$ for any $t < t'$ and

$$|\sigma(t') + \eta(t')| = \Delta\,. \tag{63}$$

It follows from (51) and (60) that $x_o(t)_{t\in[i\tau,t']} = \mathcal{S}(\bar{H}(s), x_o(i\tau), (\omega u_q(t) + \sigma(t) + \eta(t))_{t\in[i\tau,t')})$. It follows from (43) and (44) that

$$u_q(t)|_{t\in[i\tau,\,t']} = \mathcal{S}(C(s)/\omega, u_q(i\tau) + \epsilon(i\tau), (k_g r(t) \\ -\theta^\top x_q(t) - \sigma(t) - \eta(t))_{t\in[i\tau,\,t')}) - \epsilon(t)|_{t\in[i\tau,\,t']} \tag{64}$$

where

$$\epsilon(t)|_{t\in[i\tau,\,t']} = \mathcal{S}(C(s)/\omega, \epsilon(i\tau), \tilde{r}(t)_{t\in[i\tau,\,t')})\,. \tag{65}$$

We notice that if $i \geq 1$, then $\tilde{r}_l(t)$ is well defined on $[0, i\tau)$. Let

$$\tilde{r}_l(t) = \tilde{r}(t)\,, \quad t \in [i\tau,\, t')\,. \tag{66}$$

We have $\epsilon_l|_{t\in[i\tau,\,t']} = \mathcal{S}(C(s)/\omega, \epsilon_l(i\tau), \tilde{r}(t)_{t\in[i\tau,\,t')})$, which along with (53) and (65) implies

$$\epsilon_l(t) = \epsilon(t), \quad \forall\ t \in [i\tau, t']\,. \tag{67}$$

Hence, (52), (59), (61), (64) yield

$$u_q(t)|_{t\in[i\tau,\,t']} = \mathcal{S}(C(s)/\omega, u_l(i\tau) + \epsilon(i\tau), (k_g r(t) \\ -\theta^\top x_l(t) - \sigma(t) - \eta(t))_{t\in[i\tau,\,t')}) - \epsilon(t)|_{t\in[i\tau,\,t']}\,. \tag{68}$$

It follows from (67) and (68) that

$$u_q(t)|_{t\in[i\tau,\,t']} = \mathcal{S}(C(s)/\omega, u_l(i\tau) + \epsilon_l(i\tau), (k_g r(t) \\ -\theta^\top x_l(t) - \sigma(t) - \eta(t))_{t\in[i\tau,\,t')}) - \epsilon_l(t)|_{t\in[i\tau,\,t']}\,. \tag{69}$$

The relationships in (24) and (55) imply that

$$\eta_l(t) = \zeta_d(t) - \omega u_l(t) - \sigma(t)\,, \quad t \in [i\tau, t']\,, \tag{70}$$

which along with (20) yields

$$u_l(t)|_{t\in[i\tau,\,t']} = \mathcal{S}(C(s)/\omega, u_l(i\tau) + \epsilon_l(i\tau), (k_g r(t) \\ -\theta^\top x_l(t) - \sigma(t) - \eta_l(t))_{t\in[i\tau,\,t')}) - \epsilon_l(t)|_{t\in[i\tau,\,t']}\,. \tag{71}$$

From (60), (69), (70) and (71), we have

$$u_q(t) \ = \ u_l(t), \quad \forall t \in [i\tau,\, t'] \tag{72}$$

$$\eta(t) \ = \ \eta_l(t), \quad \forall t \in [i\tau,\, t')\,. \tag{73}$$

It follows from (57) and (73) that

$$\eta(t) \ = \ \eta_l(t), \quad \forall\ t \in [0,\, t')\,. \tag{74}$$

We now prove by contradiction that

$$|\epsilon(t)| < \epsilon_b\,, \quad \forall\ t\ \in [i\tau, t']\,. \tag{75}$$

If (75) is not true, then since $\epsilon(t)$ is continuous, there exists $\bar{t} \in [i\tau, t']$ such that $|\epsilon(t)| < \epsilon_b, \forall t \in [i\tau, \bar{t})$, and

$$|\epsilon(\bar{t})| = \epsilon_b\,. \tag{76}$$

It follows from (56) that

$$|\epsilon(t)| \leq \epsilon_b\,, \qquad \forall\ [0, \bar{t}]\,. \tag{77}$$

The relationships in (52), (54), (59), (61) and (72) imply that $u_q(t) = u_l(t)\,,\ x_q(t) = x_l(t)$ for any $t \in [0, \bar{t}]$. Therefore, (42) and (66) imply that $\tilde{r}_l(t) = \tilde{\omega}(t)u_l(t) + \tilde{\theta}^\top(t)x_l(t) + \tilde{\sigma}(t)$, and

$$\|\tilde{r}_{l_{\bar{t}}}\|_{\mathcal{L}_\infty} \leq 2\omega\|u_{l_{\bar{t}}}\|_{\mathcal{L}_\infty} + L\|x_{l_{\bar{t}}}\|_{\mathcal{L}_\infty} + 2\Delta\,. \tag{78}$$

Using (77) and (78), Lemmas 2 and 3 imply that $\eta_l(t)$ is bounded and differentiable with bounded derivative. Further, it follows from (30) and (34) that $|\sigma(t) + \eta_l(t)| \leq \Delta_n(\epsilon_b, \tau)$, $|\dot{\sigma}(t) + \dot{\eta}_l(t)| \leq \Delta_d(\epsilon_b, \tau)$ for any $t \in [0, \bar{t}]$. Since (74) holds $\forall\ t \in [0,\, t')$, $\eta(t)$ is also bounded and differentiable with bounded derivative and further

$$|\sigma(t) + \eta(t)| \leq \Delta_n(\epsilon_b, \tau), |\dot{\sigma}(t) + \dot{\eta}(t)| \leq \Delta_d(\epsilon_b, \tau), \tag{79}$$

for any $t \in [0, \bar{t}]$. It follows from (46) that $\|\epsilon_{\bar{t}}\|_{\mathcal{L}_\infty} \leq \left\|C(s)(\omega c_o^\top H(s))^{-1}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\theta_{\bar{t}}/(\lambda_{\min}(P)\Gamma_c)}$. The relationships in (35), (45) and (79) imply that $\theta_{\bar{t}} \leq \theta_m(\epsilon_b, \tau)$, and using the upper bound from (46) we have $\|\epsilon_{\bar{t}}\|_{\mathcal{L}_\infty} \leq \left\|C(s)\frac{1}{\omega c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m(\epsilon_b,\tau)}{\lambda_{\min}(P)\Gamma_c}}$. From (36) and (37) we have $\|\epsilon_{\bar{t}}\|_{\mathcal{L}_\infty} < \epsilon_b$, which contradicts (76). Therefore, (75) holds.

If (75) is true, it follows from (56) that $|\epsilon(t)| < \epsilon_b\,,\ \forall\ t \in [0, t']\,.$ Hence, it follows from (30) and (74) that $|\sigma(t)+\eta(t)| \leq \Delta_n < \Delta\,,$ which contradicts (63). Hence, we have

$$|\sigma(t) + \eta(t)| < \Delta, \ \ \forall\ t \in [i\tau, (i+1)\tau]. \tag{80}$$

Therefore, from (59), (61), (67), (72), (73), (75), (80) it follows that there exist $\tilde{r}_l(t)$ and continuously differentiable $\eta(t)$ in $[0, (i+1)\tau)$, which ensure

$$x_o(t) \ = \ x_q(t) = x_l(t), \forall\ t \in [i\tau, (i+1)\tau], \tag{81}$$

$$\epsilon(t) \ = \ \epsilon_l(t), \ \ \forall\ t \in [i\tau, (i+1)\tau], \tag{82}$$

$$u_q(t) \ = \ u_l(t), \ \ \forall\ t \in [i\tau, (i+1)\tau], \tag{83}$$

$$\eta(t) \ = \ \eta_l(t), \ \ \forall\ t \in [i\tau, (i+1)\tau), \tag{84}$$

$$|\epsilon(t)| \ < \ \epsilon_b, \ \ \forall\ t \in [0, (i+1)\tau], \tag{85}$$

$$|\sigma(t) + \eta(t)| \ < \ \Delta, \ \ \forall\ t \in [0, (i+1)\tau]. \tag{86}$$

It follows from (22), (49) and (83) that $\zeta(t) = \zeta_l(t), \quad \forall\ t \in [i\tau, (i+1)\tau)$. Therefore (23) and (50) imply that

$$\zeta_d(t) = \zeta_{l_d}(t), \quad \forall\ t \in [(i+1)\tau, (i+2)\tau)\,. \tag{87}$$

We note that the relationships in (81)-(87) prove the Step 2.

Step 3: Step 1 implies that the relationships (52)-(56) hold for $i = 0$. By iterating the results from Step 2, we prove (47)-(48), which conclude proof of the Theorem. □