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$\mathcal{L}_1$ Adaptive Controller for Nonlinear Systems in the Presence of Unmodelled Dynamics: Part II

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***Abstract*— This paper presents a novel adaptive control methodology for a class of uncertain nonlinear systems in the presence of unmodelled dynamics. The adaptive controller ensures uniformly bounded transient response for system's both input and output signals simultaneously. The performance bounds can be systematically improved by increasing the adaptation gain.**

I. INTRODUCTION

Reference [1] considers a class of uncertain systems in the presence of time and state dependent unknown nonlinearities and develops an adaptive control methodology that ensures uniformly bounded transient response for system's input/output signals simultaneously. This paper extends the results of [1] to a class of uncertain nonlinear systems in the presence of unmodeled dynamics. The $\mathcal{L}_\infty$ norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by increasing the adaptation gain.

The paper is organized as follows. Section II gives the problem formulation. In Section III, the novel $\mathcal{L}_1$ adaptive control architecture is presented. Stability and uniform performance bounds are presented in Section IV. In Section V, simulation results are presented, while Section VI concludes the paper.

II. PROBLEM FORMULATION

Consider the following system dynamics:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\left(\omega u(t) + f(x(t), z(t), t)\right) \\ z(t) &= g_o(x_z, t)\,, \quad \dot{x}_z(t) = g(x_z(t), x(t), t)\,, \\ y(t) &= c^\top x(t), \quad x(0) = x_0\,, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the regulated output, $b, c \in \mathbb{R}^n$ are known constant vectors, A_m is a known $n \times n$ Hurwitz matrix, $\omega \in \mathbb{R}$ is the unknown control effectiveness, z and x_z are the output and the state vector of the unmodelled dynamics, while f, g_o, g are unknown nonlinear functions.

Assumption 1: [Known sign for control effectiveness] There exist $\omega_u > \omega_l > 0$ such that $\omega_l \le \omega \le \omega_u$.

Assumption 2: [Stability of internal dynamics] The z-dynamics are bounded–input–bounded–output (BIBO) stable, i.e. there exist $L_1 > 0$ and $L_2 > 0$ such that

$$\|z_t\|_{\mathcal{L}_\infty} \le L_1 \|x_t\|_{\mathcal{L}_\infty} + L_2\,. \tag{2}$$

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Further, let

$$X \triangleq [x^\top \;\; z^\top]^\top\,.$$

Assumption 3: [Semiglobal Lipschitz condition] For any $\delta > 0$, there exist positive K_δ and B such that

$$|f(X_1, t) - f(X_2, t)| \le K_\delta \|X_1 - X_2\|_\infty\,, \tag{3}$$

$$|f(0, t)| \le B \tag{4}$$

for all $\|X_i\|_\infty \le \delta$, $i = 1, 2$, uniformly in t.

Assumption 4: [Semiglobal uniform boundedness of partial derivatives] For any $\delta > 0$, there exist $d_{f_x}(\delta) > 0$, and $d_{f_t}(\delta) > 0$ such that for any $\|x\|_\infty \le \delta$, the partial derivatives of $f(X, t)$ are piece-wise continuous and bounded

$$\left\|\frac{\partial f(X,t)}{\partial X}\right\| \le d_{f_x}(\delta)\,, \quad \left|\frac{\partial f(X,t)}{\partial t}\right| \le d_{f_t}(\delta)\,. \tag{5}$$

The control objective is to design a full-state feedback adaptive controller to ensure that $y(t)$ tracks a given bounded reference signal $r(t)$ *both in transient and steady state*, while all other error signals remain bounded.

III. $\mathcal{L}_1$ ADAPTIVE CONTROLLER

The design of $\mathcal{L}_1$ adaptive controller involves a strictly proper transfer function $D(s)$ and a gain $k \in \mathbb{R}^+$, which leads to a strictly proper stable

$$C(s) = \frac{\omega k D(s)}{1 + \omega k D(s)} \tag{6}$$

with DC gain $C(0) = 1$. The simplest choice of $D(s)$ is

$$D(s) = \frac{1}{s}\,, \tag{7}$$

which yields a first order strictly proper $C(s)$ in the following form:

$$C(s) = \frac{\omega k}{s + \omega k}\,. \tag{8}$$

Let

$$H(s) = (s\mathbb{I} - A_m)^{-1} b\,, \tag{9}$$

and $r_0(t)$ be the signal with its Laplace transformation $(s\mathbb{I} - A_m)^{-1} x_0$. Since A_m is Hurwitz and x_0 is finite, $\|r_0\|_{\mathcal{L}_\infty}$ is finite. Further, for every $\delta > 0$ let

$$L_\delta \;\triangleq\; \frac{\bar{\delta}}{\delta} K_{\bar{\delta}}\,, \tag{10}$$

where

$$\bar{\delta} \triangleq \max\{\delta + \gamma_1,\, L_1(\delta + \gamma_1) + L_2\}, \tag{11}$$

and γ_1 is an arbitrary positive constant.

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For the proofs of stability and performance bounds, the choice of $D(s)$ and k further needs to ensure that there exists ρ_r such that:

$$\|G(s)\|_{\mathcal{L}_1} < \Big(\rho_r - \|k_g C(s)H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} - \|r_0\|_{\mathcal{L}_\infty}\Big)/(\rho_r L_{\rho_r} + B)\,, \quad (12)$$

where

$$G(s) = H(s)(1 - C(s))\,,$$

and

$$k_g = -\frac{1}{c^\top A_m^{-1} b}\,. \quad (13)$$

We consider the following state predictor (or passive identifier) for generation of the adaptive laws:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m\hat{x}(t) + b\Big(\hat{\omega}(t)u(t) + \hat{\theta}(t)\|x_t\|_{\mathcal{L}_\infty} + \hat{\sigma}(t)\Big)\\ \hat{y}(t) &= c^\top\hat{x}(t)\,, \quad \hat{x}(0) = x_0\,. \end{aligned} \quad (14)$$

The adaptive estimates $\hat{\omega}(t), \hat{\theta}(t), \hat{\sigma}(t)$ are defined as:

$$\begin{aligned}\dot{\hat{\theta}}(t) &= \Gamma\mathrm{Proj}(\hat{\theta}(t), -\|x_t\|_{\mathcal{L}_\infty}\tilde{x}^\top(t)Pb),\ \hat{\theta}(0) = \hat{\theta}_0\\ \dot{\hat{\sigma}}(t) &= \Gamma\mathrm{Proj}(\hat{\sigma}(t), -\tilde{x}^\top(t)Pb), \qquad \hat{\sigma}(0) = \hat{\sigma}_0 \quad (15)\\ \dot{\hat{\omega}}(t) &= \Gamma\mathrm{Proj}(\hat{\omega}(t), -\tilde{x}^\top(t)Pbu(t)),\ \hat{\omega}(0) = \hat{\omega}_0\,,\end{aligned}$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$, $\Gamma \in \mathbb{R}^+$ is the adaptation gain, P is the solution of the algebraic equation $A_m^\top P + PA_m = -Q$, $Q > 0$, and the projection operator ensures that the adaptive estimates $\hat{\omega}(t)$, $\hat{\theta}(t)$, $\hat{\sigma}(t)$ remain inside the compact sets $[\omega_l,\ \omega_u]$, $[-\theta_b,\ \theta_b]$, $[-\sigma_b,\ \sigma_b]$, respectively, with θ_b, σ_b defined as follows:

$$\theta_b = L_\rho,\ \sigma_b = B + \epsilon\,, \quad (16)$$

where ϵ is arbitrary positive constant, and

$$\rho = \rho_r + \beta\,, \quad (17)$$

with arbitrary $\beta > \gamma_1$.

Remark 1: In the following analysis, we demonstrate that ρ_r and ρ characterize the domain of attraction of the closed loop reference system (yet to be defined) and the system in (1) respectively. Since γ_1 and β can be set arbitrarily small, ρ can approximate ρ_r arbitrarily closely.

The control signal is generated through gain feedback of the following system:

$$\chi(s) = D(s)\bar{r}(s)\,, \quad u(s) = -k\chi(s)\,, \quad (18)$$

where $k \in \mathbb{R}^+$ was introduced in (6), while $\bar{r}(s)$ is the Laplace transformation of the signal

$$\bar{r}(t) = \hat{\omega}(t)u(t) + \hat{\theta}(t)\|x_t\|_{\mathcal{L}_\infty} + \hat{\sigma}(t) - k_g r(t). \quad (19)$$

The complete $\mathcal{L}_1$ adaptive controller consists of (14), (15) and (18), subject to the $\mathcal{L}_1$-gain upper bound in (12). As compared to the corresponding $\mathcal{L}_1$ adaptive control architecture for systems without unmodeled dynamics in [1], the only difference is that here we use $\|x_t\|_{\mathcal{L}_\infty}$ in (14) and (15) instead of $\|x(t)\|_\infty$. Consequently, the subsequent analysis is also principally different from the one in [1].

IV. Analysis of $\mathcal{L}_1$ Adaptive Controller

A. Reference System

We now consider the following closed-loop reference system with its control signal and system response being defined as follows:

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + b\Big(f(x_{ref}(t), z(t), t) + \omega u_{ref}(t)\Big), \quad x_{ref}(0) = x_0\,, \quad (20)$$

$$u_{ref}(s) = (C(s)/\omega)(k_g r(s) - \bar{r}_{ref}(s))\,, \quad (21)$$

$$y_{ref}(t) = c^\top x_{ref}(t)\,, \quad (22)$$

where $\bar{r}_{ref}(s)$ is the Laplace transformation of the signal $\bar{r}_{ref}(t) = f(x_{ref}(t), z(t), t)$, and k_g is introduced in (13). The next Lemma establishes the stability of the closed-loop system in (20)-(22).

Lemma 1: For the closed-loop reference system in (20)-(22), subject to the $\mathcal{L}_1$-gain upper bound in (12), if $\|x_0\|_\infty < \rho_r$ and

$$\|z_t\|_{\mathcal{L}_\infty} \le L_1(\|x_{ref_t}\|_{\mathcal{L}_\infty} + \gamma_1) + L_2\,, \quad (23)$$

then

$$\|x_{ref_t}\|_{\mathcal{L}_\infty} < \rho_r\,, \quad (24)$$

$$\|u_{ref_t}\|_{\mathcal{L}_\infty} < \rho_{u_r}\,, \quad (25)$$

where ρ_r is introduced in (12) and

$$\rho_{u_r} = \|C(s)/\omega\|_{\mathcal{L}_1}(\rho_r L_{\rho_r} + B + k_g\|r\|_{\mathcal{L}_\infty}).$$

Proof. It follows from (20)-(22) that

$$x_{ref}(s) = G(s)\bar{r}_{ref}(s) + H(s)C(s)k_g r(s) + (s\mathbb{I} - A_m)^{-1}x_0\,. \quad (26)$$

Example 5.2 in [2] (page 199) implies that

$$\|x_{ref_\tau}\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1}\|\bar{r}_{ref_\tau}\|_{\mathcal{L}_\infty} + \|k_g C(s)H(s)\|_{\mathcal{L}_1}\|r_\tau\|_{\mathcal{L}_\infty} + \|r_0\|_{\mathcal{L}_\infty}\,. \quad (27)$$

If (24) is not true, since $\|x_{ref}(0)\|_\infty = \|x_0\|_\infty < \rho_r$ and $x_{ref}(t)$ is continuous, there exists $\tau \in [0,\ t]$ such that

$$\|x_{ref_\tau}\|_{\mathcal{L}_\infty} \le \rho_r\,, \quad (28)$$

$$x_{ref}(\tau) = \rho_r\,. \quad (29)$$

It follows from (23) and (28) that

$$\|z_\tau\|_{\mathcal{L}_\infty} \le L_1(\rho_r + \gamma_1) + L_2\,,$$

and hence

$$\|X_\tau\|_{\mathcal{L}_\infty} \le \bar{\rho}_r \triangleq \max\{\rho_r + \gamma_1, L_1(\rho_r + \gamma_1) + L_2\}\,. \quad (30)$$

Further, it follows from Assumption 3 that

$$\|\bar{r}_{ref_\tau}\|_{\mathcal{L}_\infty} \le K_{\bar{\rho}_r}\bar{\rho}_r + B\,,$$

and the redefinition in (10) leads to the following upper bound

$$\|\bar{r}_{ref_\tau}\|_{\mathcal{L}_\infty} \le L_{\rho_r}\rho_r + B\,. \quad (31)$$

Since $\|r_\tau\|_{\mathcal{L}_\infty} \le \|r\|_{\mathcal{L}_\infty}$, it follows from (27) that

$$\|x_{ref_\tau}\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1}L_{\rho_r}\rho_r + \|r_0\|_{\mathcal{L}_\infty} + \|k_g C(s)H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}B\,. \quad (32)$$

The condition in (12) can be solved for ρ_r to obtain the following upper bound

$$\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} \rho_r + \|k_g C(s) H(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} + \|r_0\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} B < \rho_r \,, \quad (33)$$

leading to

$$\|x_{ref_\tau}\|_{\mathcal{L}_\infty} < \rho_r \,, \quad (34)$$

which contradicts (29) and proves the upper bound in (24). This further implies that the upper bound in (31) holds for any τ, i.e.

$$\|\bar{r}_{ref}\|_{\mathcal{L}_\infty} < \rho_r L_{\rho_r} + B \,. \quad (35)$$

Example 5.2 in [2] (page 199) leads to the following bound

$$\|u_{ref}\|_{\mathcal{L}_\infty} < \|C(s)/\omega\|_{\mathcal{L}_1} (\rho_r L_{\rho_r} + B + k_g \|r\|_{\mathcal{L}_\infty}) \,, \quad (36)$$

which proves (25). □

B. Equivalent Linear Time-Varying System

In this section we demonstrate that the system with unmodelled dynamics in (1) can be transformed into a linear system with unknown time-varying parameters.

To streamline the subsequent analysis, we need to introduce several notations. Let γ_0 be the desired performance bound for $||\tilde{x}||_{\mathcal{L}_\infty}$, and β_1 be an arbitrary positive constant verifying the following equality

$$\frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} K_{\bar{\rho}_r}} \gamma_0 + \beta_1 = \gamma_1 \,, \quad (37)$$

where $\bar{\rho}_r$ was defined in (30), and γ_1 was introduced in (11). It follows from (12) that $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$. Since $K_{\bar{\rho}_r} < L_{\rho_r}$, we have

$$1 - \|G(s)\|_{\mathcal{L}_1} K_{\bar{\rho}_r} > 0 \,,$$

which implies that the condition in (37) can be always satisfied by selecting γ_0 and β_1 sufficiently small.

Further, let

$$\rho_u = \rho_{u_r} + \gamma_2 \,, \quad (38)$$

$$\gamma_2 = K_{\bar{\rho}_r} \gamma_1 \left\| \frac{C(s)}{\omega} \right\|_{\mathcal{L}_1} + \gamma_0 \left\| \frac{C(s)}{\omega} \frac{1}{c_o^\top H_o(s)} c_o^\top \right\|_{\mathcal{L}_1} \quad (39)$$

It follows from Lemma 4 in [3] that there exists $c_o \in \mathbb{R}^n$ such that

$$c_o^\top H(s) = \frac{N_n(s)}{N_d(s)} \,, \quad (40)$$

where the order of $N_d(s)$ is one more than the order of $N_n(s)$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials.

Lemma 2: For the system in (1), if

$$\|x_t\|_{\mathcal{L}_\infty} \leq \rho \,, \quad \|u_t\|_{\mathcal{L}_\infty} \leq \rho_u \,, \quad (41)$$

there exist differentiable $\theta(\tau)$ and $\sigma(\tau)$ with bounded $\dot{\theta}(\tau)$ and $\dot{\sigma}(\tau)$ over $\tau \in [0,\ t]$ such that

$$\begin{aligned} |\theta(\tau)| &< \theta_b \,, & (42)\\ |\sigma(\tau)| &< \sigma_b \,, & (43)\\ f(x(\tau), z(\tau), \tau) &= \theta(\tau) \|x_\tau\|_{\mathcal{L}_\infty} + \sigma(\tau) \,. & (44) \end{aligned}$$

The proof is similar to the proof of Lemma 2 in [1] and is therefore omitted.

If (41) holds, Lemma 2 implies that the system in (1) can be rewritten over $\tau \in [0,\ t]$ as:

$$\begin{aligned} \dot{x}(\tau) &= A_m x(\tau) + b\left(\theta(\tau)\|x_\tau\|_{\mathcal{L}_\infty} + \omega u(\tau) + \sigma(\tau)\right) \,, \\ y(\tau) &= c^\top x(\tau), \quad x(0) = x_0 \,, \end{aligned} \quad (45)$$

where $\theta(\tau)$, $\sigma(\tau)$ are unknown time-varying signals subject to (42)-(43) for all $\forall\, \tau \in [0,\ t]$, and their derivatives are also uniformly bounded for all $\tau \in [0,\ t]$:

$$|\dot{\theta}(\tau)| \leq d_\theta(\rho, \rho_u) < \infty \,, \quad |\dot{\sigma}(\tau)| \leq d_\sigma(\rho, \rho_u) < \infty . \quad (46)$$

Let

$$\tilde{\theta}(\tau) \triangleq \hat{\theta}(\tau) - \theta(\tau),\ \tilde{\omega}(\tau) \triangleq \hat{\omega}(\tau) - \omega, \tilde{\sigma}(\tau) \triangleq \hat{\sigma}(\tau) - \sigma(\tau) \,. \quad (47)$$

It follows from (14) and (45) that for all $\tau \in [0, t]$

$$\dot{\tilde{x}}(\tau) = A_m \tilde{x}(\tau) + b\left(\tilde{\omega}(\tau) u(\tau) + \tilde{\theta}(\tau)\|x_\tau\|_{\mathcal{L}_\infty} + \tilde{\sigma}(\tau)\right) \quad (48)$$

and $\tilde{x}(0) = 0$.

C. Tracking error signal

Lemma 3: For the system in (1) and the $\mathcal{L}_1$ adaptive controller in (14), (15) and (18), if

$$\|x_t\|_{\mathcal{L}_\infty} \leq \rho \,, \quad \|u_t\|_\infty \leq \rho_u \,, \quad (49)$$

then

$$\|\tilde{x}_t\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_m(\rho, \rho_u)}{\lambda_{\min}(P)\Gamma}} \,, \quad (50)$$

where

$$\begin{aligned} \theta_m(\rho, \rho_u) &\triangleq 4\theta_b^2 + 4\sigma_b^2 + (\omega_u - \omega_l)^2 \\ &+ 4\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \left(\theta_b d_\theta(\rho, \rho_u) + \sigma_b d_\sigma(\rho, \rho_u)\right) . \end{aligned} \quad (51)$$

The proof is similar to Lemma 3 in [1].

D. Transient and Steady-State Performance

The block diagram of the closed-loop system with $\mathcal{L}_1$ adaptive controller and the reference system is illustrated in Figure 1. We note that the reference system is not implementable since it uses the unknown signal $z(t)$ and the unknown function f. This closed-loop system is only used for analysis purposes and does not affect the implementation of $\mathcal{L}_1$ adaptive controller. In the following Theorem, we prove the stability and the transient performance of the closed-loop system with $\mathcal{L}_1$ adaptive controller.

Theorem 1: Consider the closed-loop system with $\mathcal{L}_1$ adaptive controller defined via (14), (15), (18), subject to (12), and the reference system in (20)-(22). If

$$\|x_0\|_\infty \leq \rho_r \,, \quad (52)$$

and the adaptive gain verifies the lower bound

$$\Gamma > \frac{\theta_m(\rho, \rho_u)}{\lambda_{\min}(P)\gamma_0^2} \,, \quad (53)$$

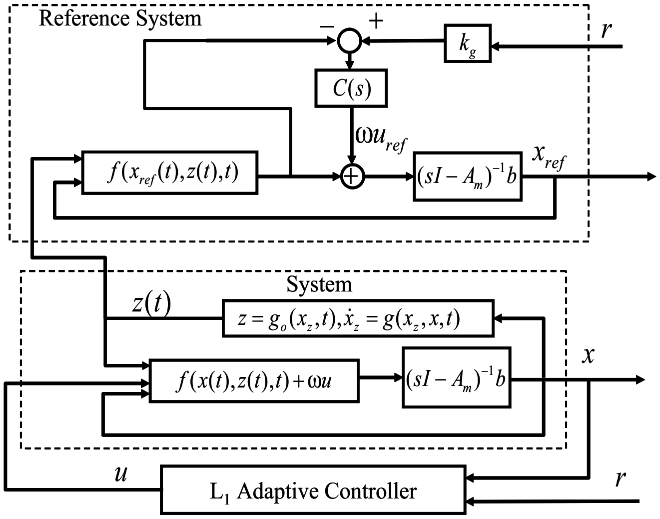


Fig. 1. Closed-loop system with $\mathcal{L}_1$ adaptive controller and reference system.

we have:

$$\begin{aligned} \|x_{ref}\|_{\mathcal{L}_\infty} &\le \rho_r\,, &(54)\\ \|u_{ref}\|_{\mathcal{L}_\infty} &\le \rho_{u_r}\,, &(55)\\ \|\tilde{x}\|_{\mathcal{L}_\infty} &\le \gamma_0\,, &(56)\\ \|x-x_{ref}\|_{\mathcal{L}_\infty} &< \gamma_1\,, &(57)\\ \|u-u_{ref}\|_{\mathcal{L}_\infty} &< \gamma_2\,, &(58)\end{aligned}$$

where γ_2 is defined in (39).

Proof. The proof will be done by contradiction. Assume that (57)-(58) are not true. Then, since $\|x(0)-x_{ref}(0)\|_\infty = 0 \le \gamma_1$, $u(0)-u_{ref}(0)=0$, and $x(\tau)$, $x_{ref}(\tau)$, $u(\tau)$, $u_{ref}(\tau)$ are continuous, there exists $t\ge 0$ such that

$$\begin{aligned} \|x(t)-x_{ref}(t)\|_\infty &= \gamma_1\,, \text{ or}\\ \|u(t)-u_{ref}(t)\|_\infty &= \gamma_2\,, &(59)\end{aligned}$$

while

$$\|(x-x_{ref})_t\|_{\mathcal{L}_\infty} \le \gamma_1, \quad \|(u-u_{ref})_t\|_{\mathcal{L}_\infty} \le \gamma_2\,. \quad (60)$$

On the other hand, it follows from Assumption 2 that

$$\|z_t\|_{\mathcal{L}_\infty} \le L_1(\|x_{ref_t}\|_{\mathcal{L}_\infty} + \gamma_1) + L_2\,. \quad (61)$$

Consequently, Lemma 1 implies that

$$\|x_{ref_t}\|_{\mathcal{L}_\infty} \le \rho_r\,, \quad \|u_{ref_t}\|_{\mathcal{L}_\infty} \le \rho_{u_r}\,. \quad (62)$$

Taking into consideration the definitions from (17) and (38), it follows from (60) that

$$\|x_t\|_{\mathcal{L}_\infty} \le \rho_r+\gamma_1 \le \rho\,, \quad \|u_t\|_{\mathcal{L}_\infty} \le \rho_{u_r}+\gamma_2 = \rho_u\,. \quad (63)$$

Choosing the adaptive gain according to (53), Lemma 3 implies that

$$\|\tilde{x}_t\|_{\mathcal{L}_\infty} \le \gamma_0\,. \quad (64)$$

Let $\tilde{r}(\tau) = \tilde{\omega}(\tau)u(\tau) + \tilde{\theta}(\tau)\|x_\tau\|_{\mathcal{L}_\infty} + \tilde{\sigma}(\tau)\,,\ r_1(\tau) = \theta(\tau)\|x_\tau\|_{\mathcal{L}_\infty} + \sigma(\tau)\,.$ It follows from (18) that $\chi(s) = D(s)(\omega u(s) + r_1(s) - k_g r(s) + \tilde{r}(s))$, where $\tilde{r}(s)$ and $r_1(s)$ are the Laplace transformations of signals $\tilde{r}(\tau)$ and $r_1(\tau)$. Consequently

$$\begin{aligned} \chi(s) &= \frac{D(s)}{1+k\omega D(s)}(r_1(s) - k_g r(s) + \tilde{r}(s))\,,\\ u(s) &= -\frac{kD(s)}{1+k\omega D(s)}(r_1(s) - k_g r(s) + \tilde{r}(s))\,. &(65)\end{aligned}$$

Using the definition of $C(s)$ from (6), we can write

$$\omega u(s) = -C(s)(r_1(s) - k_g r(s) + \tilde{r}(s))\,, \quad (66)$$

and the system in (1) consequently takes the form:

$$\begin{aligned} x(s) &= H(s)\Big((1-C(s))r_1(s) + C(s)k_g r(s) -\\ &\quad C(s)\tilde{r}(s)\Big) + (s\mathbb{I}-A_m)^{-1}x_0\,. &(67)\end{aligned}$$

Let $e(\tau) = x(\tau) - x_{ref}(\tau)$. Then from (26) we have

$$e(s) = H(s)\left((1-C(s))r_2(s) - C(s)\tilde{r}(s)\right), \quad e(0)=0\,, \quad (68)$$

where $r_2(s)$ is the Laplace transformation of the signal

$$r_2(\tau) = \theta(\tau)(\|x_\tau\|_{\mathcal{L}_\infty} - \|x_{ref_\tau}\|_{\mathcal{L}_\infty})\,. \quad (69)$$

Since (44) implies that

$$\begin{aligned} f(x(\tau),z(\tau),\tau) &= \theta(\tau)\|x_\tau\|_{\mathcal{L}_\infty} + \sigma(\tau)\,,\\ f(x_{ref}(\tau),z(\tau),\tau) &= \theta(\tau)\|x_{ref_\tau}\|_{\mathcal{L}_\infty} + \sigma(\tau)\,,\end{aligned}$$

it follows from (69) that

$$r_2(\tau) = f(x(\tau),z(\tau),\tau) - f(x_{ref}(\tau),z(\tau),\tau)\,. \quad (70)$$

Example 5.2 in [2] (page 199) gives the following upper bound:

$$\|e_t\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1}\|r_{2_t}\|_{\mathcal{L}_\infty} + \|r_{3_t}\|_{\mathcal{L}_\infty}\,, \quad (71)$$

where $r_3(\tau)$ is the signal with its Laplace transformation being $r_3(s) = C(s)H(s)\tilde{r}(s)$. From the error dynamics in (48) we have $\tilde{x}(s) = H(s)\tilde{r}(s)$, which leads to $r_3(s) = C(s)\tilde{x}(s)$, and hence $\|r_{3_t}\|_{\mathcal{L}_\infty} \le \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. Substituting (62) into (61), we obtain

$$\|z_t\|_{\mathcal{L}_\infty} \le L_1(\rho_r+\gamma_1) + L_2\,,$$

and hence

$$\|X_t\|_{\mathcal{L}_\infty} \le \max\{\rho_r+\gamma_1, L_1(\rho_r+\gamma_1)+L_2\}\,. \quad (72)$$

Similarly, we have

$$\|X_{ref_t}\|_{\mathcal{L}_\infty} \le \|X_t\|_{\mathcal{L}_\infty} \le \max\{\rho_r+\gamma_1, L_1(\rho_r+\gamma_1)+L_2\}\,, \quad (73)$$

where $X_{ref} = [x_{ref}^\top \ \ z^\top]^\top$. It follows from (72), (73) that $\|X(\tau)\|_\infty \le \bar{\rho}_r, \|X_{ref}(\tau)\|_\infty \le \bar{\rho}_r$, over $\tau\in[0,\,t]$, and hence Assumption 3 implies that $|r_2(\tau)| \le K_{\bar{\rho}_r}\|e(\tau)\|_\infty$ over $\tau\in[0,\,t]$, which implies that

$$\|r_{2_t}\|_{\mathcal{L}_\infty} \le K_{\bar{\rho}_r}\|e_t\|_{\mathcal{L}_\infty}\,. \quad (74)$$

From (71) we have $\|e_t\|_{\mathcal{L}_\infty} \le \|G(s)\|_{\mathcal{L}_1}K_{\bar{\rho}_r}\|e_t\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. The upper bound in (64) and the $\mathcal{L}_1$-gain upper bound in (12) lead to the following upper bound

$\|e_t\|_{\mathcal{L}_\infty} \leq \gamma_0 \|C(s)\|_{\mathcal{L}_1}/(1 - \|G(s)\|_{\mathcal{L}_1} K_{\bar{\rho}_r})$, which along with (37) leads to

$$\|e_t\|_{\mathcal{L}_\infty} \leq \gamma_1 - \beta_1 < \gamma_1 \,. \tag{75}$$

We notice that from (21) and (66) that one can derive $u(s) - u_{ref}(s) = -\frac{C(s)}{\omega} r_2(s) - r_4(s)$, where $r_4(s) = \frac{C(s)}{\omega}\tilde{r}(s)$. Therefore, it follows from Example 5.2 in [2] (page 199) that

$$\|(u - u_{ref})_t\|_{\mathcal{L}_\infty} \leq \|C(s)/\omega\|_{\mathcal{L}_1} \|r_{2_t}\|_{\mathcal{L}_\infty} + \|r_{4_t}\|_{\mathcal{L}_\infty} \,,$$

and hence

$$\|(u - u_{ref})_t\|_{\mathcal{L}_\infty} \leq \|C(s)/\omega\|_{\mathcal{L}_1} K_{\bar{\rho}_r} \|e_t\|_{\mathcal{L}_\infty} + \|r_{4_t}\|_{\mathcal{L}_\infty} \,. \tag{76}$$

We have $r_4(s) = \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} c_o^\top H(s)\tilde{r}(s) = \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} c_o^\top \tilde{x}(s)$, where c_o is introduced in (40). Using the polynomials from (40), we can write that $\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} = \frac{C(s)}{\omega}\frac{N_d(s)}{N_n(s)}$. Since $C(s)$ is stable and strictly proper, the complete system $C(s)\frac{1}{c_o^\top H(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$ gain exists and is finite. Hence, we have $\|r_{4_t}\|_{\mathcal{L}_\infty} \leq \left\| \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \|\tilde{x}_t\|_{\mathcal{L}_\infty}$. The upper bound in (63) leads to

$$\|r_{4_t}\|_{\mathcal{L}_\infty} \leq \left\| \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \gamma_0 \,. \tag{77}$$

It follows from (75), (76) and (77) and the definition of γ_2 in (39) that

$$\|(u - u_{ref})_t\|_{\mathcal{L}_\infty} \leq K_{\bar{\rho}_r} \|C(s)/\omega\|_{\mathcal{L}_1} (\gamma_1 - \beta_1) + \left\| \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \gamma_0 < \gamma_2 \,. \tag{78}$$

We note that the upper bounds in (75) and (78) contradict the equality in (59), which proves (57)-(58). Since (57)-(58) are uniform bounds for all $t \geq 0$, the upper bound in (56) follows from (64) directly, while the upper bounds in (54)-(55) follow from (62) correspondingly. □

It follows from (53) that we can achieve arbitrarily small γ_0 by increasing the adaptive gain.

V. Simulations

Consider the dynamics of a single-link robot arm rotating on a vertical plane:

$$I\ddot{q}(t) + F(q(t), \dot{q}(t), z(t), t) = u(t) \,, \tag{79}$$

where $q(t)$ and $\dot{q}(t)$ are measured angular position and velocity, respectively, $u(t)$ is the input torque, I is the unknown moment of inertia, $z(t)$ represents the output of unmodelled dynamics, $F(q(t), \dot{q}(t), z(t), t)$ is an unknown nonlinearity that lumps the forces and torques due to gravity, friction, disturbance, other external sources and unmodelled dynamics. The control objective is to design $u(t)$ to achieve tracking of bounded reference input $r(t)$ by $q(t)$, where $\|r\|_{\mathcal{L}_\infty} \leq 1$. Let $x = [x_1\ x_2]^\top = [q\ \dot{q}]^\top$. The system in (79) can be presented in the canonical form:

$$\dot{x}(t) = A_m x(t) + b(\omega u(t) + f(x(t), t)) \,,\ y(t) = c^\top x(t),$$

where $b = [0\ \ 1]^\top$, $c = [1\ \ 0]^\top$, $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$, $\omega = 1/I$ is the unknown control effectiveness, and the unknown function f is given by:

$$f(x(t), t) = [1\ \ 1.4]x(t) - F(x_2(t), x_1(t), z(t), t) \,.$$

Let the unknown control effectiveness, the unknown nonlinearity be given by $\omega = 1/I = 1$ and $F(x_1(t), x_2(t), z(t), t) = x_2^2(t) + x_1^2(t) + z^2(t)$, while $z(t)$ is the output of the unmodelled dynamics, given by:

$$\begin{aligned} z(s) &= \frac{s-1}{s^2+3s+2}\bar{\sigma}(s) \,, \\ \bar{\sigma}(t) &= \sin(0.2t)x_1(t) + x_2(t) \,, \end{aligned}$$

so that the compact sets can be conservatively chosen according to the following upper and lower bounds $\omega_l = 0.5$, $\omega_u = 2$, $\theta_b = 20$, $\sigma_b = 10$.

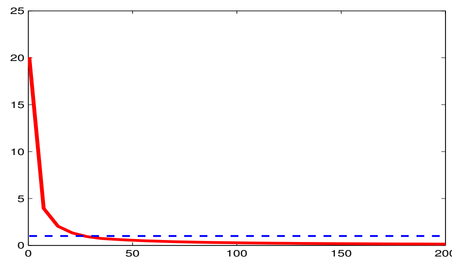
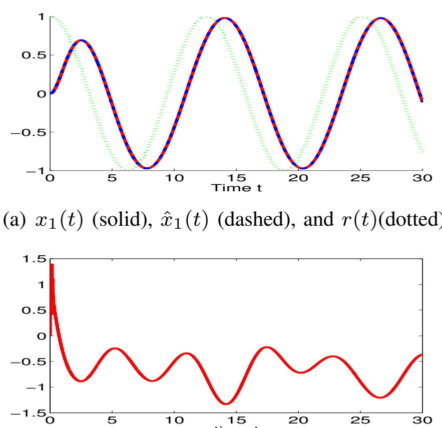


Fig. 2. $\|G(s)\|_{\mathcal{L}_1} L_\rho$ with respect to ωk.



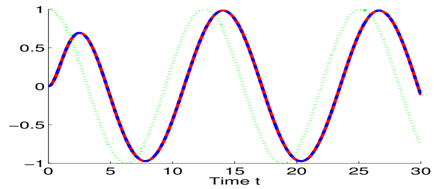
(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

(b) Time-history of $u(t)$

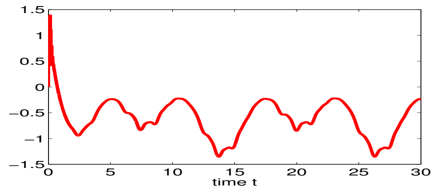
Fig. 3. Performance of $\mathcal{L}_1$ adaptive controller for $\bar{\sigma}(t) = \sin(0.2t)x_1(t) + x_2(t)$.

In the implementation of the $\mathcal{L}_1$ adaptive controller, we set $Q = 2\mathbb{I}$ and hence $P = \begin{bmatrix} 1.4143 & 0.5000 \\ 0.5000 & 0.71430 \end{bmatrix}$.

First, we need to verify the condition in (12). Letting $D(s) = 1/s$, we have $G(s) = \frac{s}{s+\omega k}H(s)$, $H(s) = [\frac{1}{s^2+1.4s+1}\ \ \frac{s}{s^2+1.4s+1}]^\top$. We choose conservative $L_\rho = 20$. In Fig. 2, we plot $\|G(s)\|_{\mathcal{L}_1} L_\rho$ as a function of ωk and compare it to 1. We notice that for $\omega k > 30$, we have

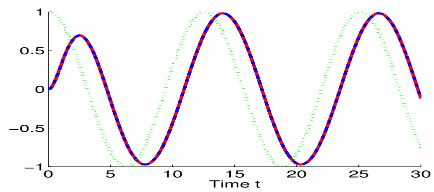


(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

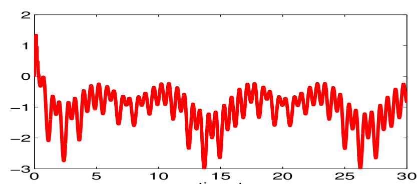


(b) Time-history of $u(t)$

Fig. 4. Performance of $\mathcal{L}_1$ adaptive controller for $\bar{\sigma}(t) = \sin(5t)x_1(t) + x_2(t)$.



(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)



(b) Time-history of $u(t)$

Fig. 5. Performance of $\mathcal{L}_1$ adaptive controller for $\bar{\sigma}(t) = \sin(5t)x_1(t) + x_2(t) + 5\sin(5t)$.

$\|G(s)\|_{\mathcal{L}_1} L_\rho < 1$. Since $\omega > 0.5$, we set $k = 60$. We set the adaptive gain $\Gamma_c = 10000$.

The simulation results of the $\mathcal{L}_1$ adaptive controller are shown in Figures 3(a)-3(b) for the reference input $r = \cos(0.5t)$. Next, we consider the same system in the presence of $\bar{\sigma}(t) = \sin(5t)x_1(t) + x_2(t)$. The simulation results, without any retuning of the controller, are shown in 4(a)-4(b). Finally, we change $\bar{\sigma}(t) = \sin(5t)x_1(t) + x_2(t) + 5\sin(5t)$. The simulation results are shown in 5(a)-5(b). We note that the $\mathcal{L}_1$ adaptive controller guarantees smooth and uniform transient performance in the presence of different unknown nonlinearities and does not require any retuning. We also notice that $x_1(t)$ and $\hat{x}_1(t)$ are almost the same in Figs. 3(a), 4(a) and 5(a).

VI. Conclusion

A novel $\mathcal{L}_1$ adaptive control architecture is presented that has guaranteed transient response in addition to stable tracking for uncertain nonlinear systems in the presence of unmodelled dynamics. The control signal and the system response approximate the same signals of a closed-loop reference system, which can be designed to achieve desired specifications.

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