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$\mathcal{L}_1$ Adaptive Output Feedback Controller for Systems with Time-varying Unknown Parameters and Bounded Disturbances

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Abstract— **This paper presents an extension of the $\mathcal{L}_1$ adaptive controller to output feedback for systems with time-varying unknown parameters and time-varying bounded disturbances. The adaptive controller ensures uniformly bounded transient and asymptotic tracking for system's both signals, input and output, simultaneously. The performance bounds can be systematically improved by increasing the adaptation rate. Simulations verify the theoretical findings.**

I. Introduction

This paper extends the results of [1], [2] to an output feedback framework. The methodology ensures uniformly bounded transient response for system's both signals, input and output, simultaneously, in addition to asymptotic tracking. The $\mathcal{L}_\infty$ norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by increasing the adaptation gain.

The paper is organized as follows. Section II gives the problem formulation. In Section III, the novel $\mathcal{L}_1$ adaptive control architecture is presented. Stability and uniform transient tracking bounds of the $\mathcal{L}_1$ adaptive controller are presented in Section IV. In section V, simulation results are presented, while Section VI concludes the paper. The proofs are in Appendix.

II. Problem Formulation

Consider the following system dynamics:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\left(u(t) + \theta^\top(t)x(t) + \sigma(t)\right), \; x(0) = x_0 \\ y(t) &= c^\top x(t), \quad y(0) = y_0 \,, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state vector, which is not measured, $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the only measured output, $b, c \in \mathbb{R}^n$ are known constant vectors, A is a known Hurwitz $n \times n$ matrix, $\theta(t) \in \mathbb{R}^n$ is a vector of unknown time-varying bounded parameters, $\sigma(t) \in \mathbb{R}$ is a time-varying bounded disturbance, and $H(s) = c^\top(s\mathbb{I} - A)^{-1}b$ is a stable minimum phase system with relative degree 1.

Assumption 1: Assume $\theta(t) \in \Theta$ and $|\sigma(t)| \leq \Delta$ for any $t \geq 0$, where Θ is a known compact set, and $\Delta \in \mathbb{R}^+$ is a known (conservative) bound of $\sigma(t)$.

Assumption 2: $\theta(t)$ and $\sigma(t)$ are continuously differentiable and their derivatives are uniformly bounded: $\|\dot{\theta}(t)\| \leq d_\theta < \infty$, $|\dot{\sigma}(t)| \leq d_\sigma < \infty$, where the numbers d_θ, d_σ can be arbitrarily large.

The control objective is to design an adaptive output feedback controller to ensure that $y(t)$ tracks a given bounded reference signal $r(t)$ *both in transient and steady state*, while all other error signals remain bounded.

III. $\mathcal{L}_1$ Adaptive Controller

At first, we introduce the following notations that will be used throughout the paper. For a signal $\xi(t),\ t \geq 0,\ \xi \in \mathbb{R}^n$, its truncated $\mathcal{L}_\infty$ norm and $\mathcal{L}_\infty$ norm are defined as $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{0\leq\tau\leq t}|\xi_i(\tau)|)$, $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{\tau\geq 0}|\xi_i(\tau)|)$, where ξ_i is the i^{th} component of ξ. For a stable proper LTI system $H(s)$, its $\mathcal{L}_1$ gain will be denoted as $||H(s)||_{\mathcal{L}_1}$.

Lemma 1: For a stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have $\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}, \quad \forall\ t \geq 0$, and $\|x\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}$.

In this section, we develop a novel adaptive output feedback control architecture for the system in (1) that permits complete transient characterization for both $u(t)$ and $y(t)$. Towards that end, we first transform the system in (1) to another input-output equivalent system with Strictly Positive Real (SPR) transfer function.

A. System transformation

Let

$$H_x(s) = (s\mathbb{I} - A)^{-1}b = \frac{A_T\left[s^{n-1}\ s^{n-2}\ ..\ 1\right]^\top}{s^n + a_1 s^{n-1} + ... + a_n} \tag{2}$$

$$H(s) = c^\top H_x(s) \triangleq \frac{H_n(s)}{H_d(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + ... + b_n}{s^n + a_1 s^{n-1} + ... + a_n}\,. \tag{3}$$

Since $H(s)$ is a stable minimum-phase system with relative degree 1, both $H_n(s)$ and $H_d(s)$ are stable polynomials and the order of $H_d(s)$ and $H_n(s)$ are n and $n-1$, respectively. Let

$$A_m = \left[\begin{array}{ccccc} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & \cdots & ... & -a_2 & -a_1 \end{array}\right] \tag{4}$$

and $b_m = [0\ \cdots\ 0\ 1]^\top$. Since A_m is Hurwitz, for any $Q > 0$ there exists a $P = P^\top > 0$ that solves the algebraic Lyapunov equation

$$A_m^\top P + PA_m = -Q\,. \tag{5}$$

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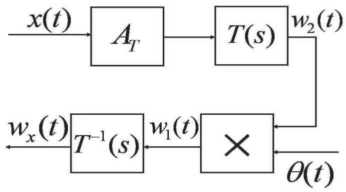


Fig. 1. System $\mathcal{W}$

Let

$$c_m = Pb_m\,. \tag{6}$$

Then, it follows from Kalman-Yakubovich-Popov lemma that

$$H_m(s) = c_m^\top(s\mathbb{I} - A_m)^{-1}b_m \triangleq H_p(s)/H_d(s) \tag{7}$$

is strictly positive real (SPR). Let

$$u(s) = T(s)v(s)\,, \tag{8}$$

where $v(t)$ will be defined later and

$$T(s) = H_p(s)/H_n(s)\,. \tag{9}$$

We notice that $H_m(s) = H(s)T(s)$.

We further let $w_x(t)$ be the output of the following system $\mathcal{W}$, driven by the input $x(t)$:

$$w_x(s) = T^{-1}(s)w_1(s)\,,\ w_1(t) = \theta^\top(t)w_2(t)\,, \tag{10}$$
$$w_2(s) = T(s)A_T x(s)\,,\ x(0) = 0\,. \tag{11}$$

Fig. 1 presents a sketch of this system. We note that for constant θ the output of the system (10) simply reduces to $w_x(t) = \theta^\top A_T x(t)$. Let $\theta_L = \max_{\theta\in\Theta}\|\theta\|$. It follows from Lemma 1 that

$$\|w_x\|_{\mathcal{L}_\infty} \le L\|x\|_{\mathcal{L}_\infty}\,, \tag{12}$$

where

$$L = \theta_L\|T^{-1}(s)\|_{\mathcal{L}_1}\|T(s)A_T\|_{\mathcal{L}_1}\,. \tag{13}$$

Lemma 2: Given $v(t)$, $\theta(t)$ and $\sigma(t)$, there exists a bounded signal $\sigma_m(t)$, whose derivative $\dot\sigma_m(t)$ is also bounded, such that the output $y(t)$ of the system in (1) is equal to the output $y_m(t)$ of the following system:

$$\begin{aligned}\dot x_m(t) &= A_m x_m(t) + b_m\left(v(t) + w_{x_m}(t) + \sigma_m(t)\right)\,,\\ y_m(t) &= c_m^\top x_m(t)\,, \qquad x_m(0) = \hat x_0\,,\end{aligned} \tag{14}$$

where A_m, b_m, c_m are defined in (4)-(6), $w_{x_m}(t)$ is computed following (10) with $x(s)$ being replaced by $x_m(s)$ in (11), and $\hat x_0$ is any point satisfying

$$c_m^\top \hat x_0 = y_0, \tag{15}$$

where $y_0 \triangleq c^\top x_0$ is the only available initial condition of the system output in (1).

Remark 1: Condition $c_m^\top\hat x_0 = c^\top x_0$ was crucial for proving the existence of a bounded $\sigma_{m2}(t)$ with bounded derivative.

Remark 2: It follows from (62) that

$$|\sigma_m(t)| \le \Delta_m < \infty\,, \quad |\dot\sigma_m(t)| \le d_{\sigma_m} < \infty\,, \quad \forall\ t \ge 0\,,$$

and the bounds Δ_m and d_{σ_m} can be derived explicitly from the original bounds using the intermediate filtering constructions.

B. Closed-loop Reference System

We now consider the following closed-loop reference system with its control signal and system response being defined as follows:

$$\begin{aligned}\dot x_{ref}(t) &= A_m x_{ref}(t) + \\ &\quad b_m\left(v_{ref}(t) + w_{x_{ref}}(t) + \sigma_m(t)\right)\,, &(16)\\ v_{ref}(s) &= C(s)\bar r_{ref}(s)\,, \quad x_{ref}(0) = \hat x_0\,, &(17)\\ y_{ref}(t) &= c_m^\top x_{ref}(t)\,, &(18)\end{aligned}$$

where $w_{x_{ref}}(t)$ is the output of the system $\mathcal{W}$ in (10) for $x_{ref}(t)$, $\bar r_{ref}(s)$ is the Laplace transformation of the signal $\bar r_{ref}(t) = -w_{x_{ref}}(t) - \sigma_m(t) + k_g r(t)\,,$

$$k_g = -1/(c_m^\top A_m^{-1}b_m)\,, \tag{19}$$

and $C(s)$ is a strictly proper stable transfer function with $C(0) = 1$, subject to the following $\mathcal{L}_1$ gain restriction:

$\mathcal{L}_1$-gain stability requirement: The $\mathcal{L}_1$ gain of $C(s)$ needs to verify

$$\|G(s)\|_{\mathcal{L}_1}L < 1\,, \tag{20}$$

where

$$G(s) = (sI - A_m)^{-1}b_m(1 - C(s))\,, \tag{21}$$

and L is defined in (13).

We notice that the reference system in (16)-(18) depends upon the unknown parameters and the disturbance of the original system. The next Lemma establishes its stability.

Lemma 3: If $C(s)$ verifies the condition in (20), the closed-loop reference system in (16)-(18) is stable.

Remark 3: We note that in the absence of $C(s)$, i.e. when $C(s) = 1$, the reference system in (16)-(18) reduces to

$$\dot x_{ref}(t) = A_m x_{ref}(t) + b_m k_g r(t), \quad y_{ref}(t) = c_m^\top x_{ref}(t),$$

with $x_{ref}(0) = \hat x_0$.

C. $\mathcal{L}_1$ adaptive controller

Since for any $v(t)$ the output of the system in (14) is equivalent to the output of the system in (1) with (8), we will design an adaptive output feedback controller $v(t)$ for the system in (14) and, using (8), we will implement it for the system in (1).

1) Notations: Let

$$\bar\Delta = \Delta_m + L\left(\rho + \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}\bar\gamma\right)\,, \tag{22}$$

where $\bar\gamma > 0$ is an arbitrary constant. Let

$$\beta_1 = \beta_{01}\frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}, \quad \beta_2 = \beta_{02} + \beta_{01}\rho\,, \tag{23}$$

$$\begin{aligned}\beta_{01} &= 4\bar\Delta L\left(\frac{d_\theta}{\theta_L} + \|A_m\|_{\mathcal{L}_1} + \|b_m\|_{\mathcal{L}_1}L\right)\\ \beta_{02} &= 4\bar\Delta\Big(d_\Delta + L\|b_m\|_{\mathcal{L}_1}\\ &\quad (\|C(s)\|_{\mathcal{L}_1}(k_g\|r\|_{\mathcal{L}_\infty} + \bar\Delta) + \Delta_m)\Big)\,, \end{aligned} \tag{24}$$

$$\beta_3 = \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\beta_1, \quad \beta_4 = 4\bar{\Delta}^2 + \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\beta_2 \,. \tag{25}$$

2) $\mathcal{L}_1$ adaptive controller: The elements of $\mathcal{L}_1$ adaptive controller are introduced below.

State Predictor: We consider the following state predictor:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m\hat{x}(t) + b_m\left(v(t) + \hat{\sigma}(t)\right) , \\ \hat{y}(t) &= c_m^\top \hat{x}(t) , \quad \hat{x}(0) = \hat{x}_0 , \end{aligned} \tag{26}$$

where $\hat{x}_0$ is defined in (15), and the adaptive estimate $\hat{\sigma}(t)$ is governed by the following adaptation law.

Adaptive Law: The adaptation of $\hat{\sigma}(t)$ is defined as:

$$\dot{\hat{\sigma}}(t) = \Gamma_c \mathrm{Proj}(\hat{\sigma}(t), -\tilde{y}(t)), \qquad \hat{\sigma}(0) = 0, \tag{27}$$

where $\tilde{y}(t) = \hat{y}(t) - y_m(t)$ is the error signal between the outputs of the system in (14) and the state predictor in (26), $\Gamma_c \in \mathbb{R}^+$ is the adaptation rate subject to the following lower bound:

$$\Gamma_c > \max\left\{\frac{\alpha\beta_3}{(\alpha-1)^2\beta_4\lambda_{\min}(P)}, \frac{\alpha\beta_4}{\lambda_{\min}(P)\bar{\gamma}^2}\right\} \tag{28}$$

with $\alpha > 1$ being an arbitrary constant, while projection is performed on the set $\hat{\sigma} \in \bar{\Delta}$, the latter being defined in (22). Letting

$$\gamma_0 = \sqrt{\alpha\beta_4/(\Gamma_c\lambda_{\min}(P))} \,, \tag{29}$$

it follows from (28) that $\bar{\gamma} > \gamma_0$, and hence

$$\bar{\Delta} \geq \Delta_m + L\left(\rho + \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L}\gamma_0\right) \tag{30}$$

and

$$|\hat{\sigma}(t)| \leq \bar{\Delta} \,, \quad \forall\ t \geq 0. \tag{31}$$

Control Law: The control signal is generated as:

$$v(s) = C(s)\left(k_g r(s) - \hat{\sigma}(s)\right) , \tag{32}$$

where k_g is introduced in (19).

The complete $\mathcal{L}_1$ adaptive controller consists of (26), (27), (8), (32) subject to the lower bound in (28) for the adaptive gain, and the $\mathcal{L}_1$-gain restriction in (20) for the filter choice.

Remark 4: We note that the lower bound for Γ_c in (28) depends upon Δ_m, which in its turn depends upon $||\sigma_{m2}(t)||_{\mathcal{L}_\infty}$. The latter can be reduced only via minimization of $||\hat{x}_0 - x_0||$ over the set to which x_0 might belong. In that sense, the approach in this paper establishes only semiglobal results.

IV. Analysis of $\mathcal{L}_1$ Adaptive Controller

In this section, we analyze stability and performance of $\mathcal{L}_1$ adaptive controller. Same as in [1], we have the following Lemma.

Lemma 4: If (A, b) is controllable and $(s\mathbb{I} - A)^{-1}b$ is strictly proper and stable, there exists $c_o \in \mathbb{R}^n$ such that $c_o^\top(s\mathbb{I} - A)^{-1}b$ is minimum phase with relative degree one. It follows from Lemma 4 that there exists $c_o \in \mathbb{R}^n$ such that

$$c_o^\top H_{xm}(s) = N_n(s)/N_d(s) \,, \tag{33}$$

where the order of $N_d(s)$ is one more than the order of $N_n(s)$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials.

Theorem 1: Given the system in (14) and the $\mathcal{L}_1$ adaptive controller defined via (26), (27) and (32) subject to (20), we have:

$$\begin{aligned} \|\tilde{x}\|_{\mathcal{L}_\infty} &< \gamma_0 \,, & (34) \\ \|x_m - x_{ref}\|_{\mathcal{L}_\infty} &\leq \gamma_1 \,, & (35) \\ \|y_m - y_{ref}\|_{\mathcal{L}_\infty} &\leq \|c_m^\top\|_{\mathcal{L}_1}\gamma_1 \,, & (36) \\ \|v - v_{ref}\|_{\mathcal{L}_\infty} &\leq \gamma_2 \,, & (37) \end{aligned}$$

where $\tilde{x}(t) = \hat{x}(t) - x_m(t)$, γ_0 is defined in (29), and $\gamma_1 = \gamma_0\|C(s)\|_{\mathcal{L}_1}/(1-\|G(s)\|_{\mathcal{L}_1}L)$, $\gamma_2 = L\,\|C(s)\|_{\mathcal{L}_1}\gamma_1 + \left\|(C(s)/(c_o^\top H_{xm}(s)))c_o^\top\right\|_{\mathcal{L}_1}\gamma_0$.

The following corollary follows from Theorem 1 directly.

Corollary 1: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller defined via (26), (27) subject to (28), and (8), (32) subject to $\mathcal{L}_1$-gain restriction in (20), we have:

$$\begin{aligned} \lim_{\Gamma_c\to\infty}\left(y(t) - y_{ref}(t)\right) &= 0\,, \qquad \forall t \geq 0, & (38) \\ \lim_{\Gamma_c\to\infty}\left(u(t) - u_{ref}(t)\right) &= 0\,, \qquad \forall t \geq 0\,, & (39) \end{aligned}$$

where $u_{ref}(s) = T(s)v_{ref}(s)$.

Thus, the tracking error between $y(t)$ and $y_{ref}(t)$, as well between $v(t)$ and $v_{ref}(t)$, is uniformly bounded by a constant inverse proportional to Γ_c. This implies that during the transient one can achieve arbitrarily close tracking performance for both signals simultaneously by increasing Γ_c.

We note that the control law $u_{ref}(t)$ in the closed-loop reference system, which is used in the analysis of $\mathcal{L}_\infty$ norm bounds, is not implementable since its definition involves the unknown parameters. Theorem 1 ensures that the $\mathcal{L}_1$ adaptive controller approximates $u_{ref}(t)$ both in transient and steady state. So, it is important to understand how these bounds can be used for ensuring uniform transient response with *desired* specifications. We notice that the following *ideal* control signal for system in (16)

$$v_{ideal}(t) = k_g r(t) - w_{x_{ideal}}(t) - \sigma(t) \tag{40}$$

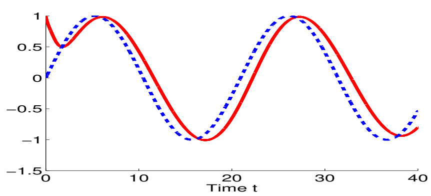
is the one that leads to desired system response:

$$\begin{aligned} \dot{x}_{ideal}(t) &= A_m x_{ideal}(t) + b_m k_g r(t) & (41) \\ y_{ideal}(t) &= c_m^\top x_{ideal}(t), \quad x_{ideal} = \hat{x}_0 \,, & (42) \end{aligned}$$

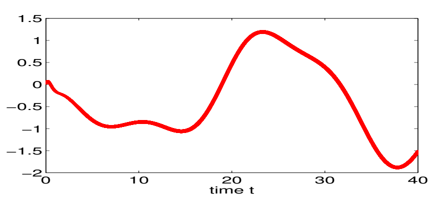
by cancelling the uncertainties exactly. It follows from (41)-(42) that $y_{ideal}(s) = H_m(s)r(s)$. In the closed-loop reference system (16)-(18), $v_{ideal}(t)$ is further low-pass filtered by $C(s)$ in (17) to have guaranteed low-frequency range. Thus, the reference system in (16)-(18) has a different response as compared to (41), (42) with (40). In [3], specific design guidelines are suggested for selection of $C(s)$ if the unknown parameters θ are constants. In case of fast varying $\theta(t)$, it is obvious that the bandwidth of the controller needs to be matched correspondingly.

V. Simulations

As an illustrative example, consider the system in (1) where $A = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are known, $\theta(t) = [0.5 + 0.5\sin(0.3t) \quad 0.5 + 0.2\sin(0.3t) + 0.1\cos(0.2t)]^\top$, $\sigma(t) = \sin(0.2t)$, $x_0 = [1 \quad 1]^\top$ are unknown time-varying parameters, disturbances and initial conditions, respectively. Let $\Theta_i = [-1.5, 1.5]$, $i = 1, 2$, and $\Delta = 2$, $Q = \mathbb{I}_{2\times 2}$. It follows from (5), (6), and (9) that $P = \begin{bmatrix} 1.1667 & 0.1667 \\ 0.1667 & 0.1667 \end{bmatrix}$, $c_m = \begin{bmatrix} 2.5 \\ 0.5 \end{bmatrix}$, $T(s) = \frac{0.5s+2.5}{2s+3}$. Let $C(s) = \frac{8}{s+8}$, it can be verified numerically that $\|G(s)\|_{\mathcal{L}_1} L \approx 0.5$ and, hence, the condition in (20) is satisfied. For implementation of the $\mathcal{L}_1$ adaptive controller in (26), (27), (8), and (32), we choose $\bar{\Delta} = 10$ and $\Gamma_c = 500000$. The simulation results are shown in Figures 2(a)-2(b) for the reference input $r = \sin(0.3t)$. Next, we consider

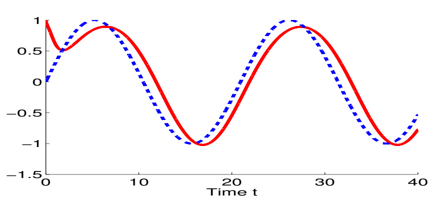


(a) $y(t)$ (solid) and $r(t)$ (dashed)

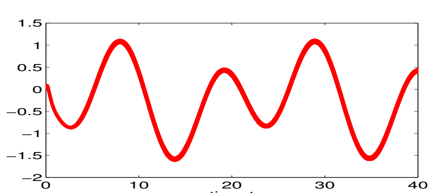


(b) Time-history of $u(t)$

Fig. 2. Performance for $\sigma(t) = \sin(0.2t)$



(a) $y(t)$ (solid) and $r(t)$ (dashed)



(b) Time-history of $u(t)$

Fig. 3. Performance for $\sigma(t) = \sin(0.6t)$

the same controller for a faster time-varying disturbance $\sigma(t) = \sin(0.6t)$ without any retuning. The system response and the control signal are plotted in Figs. 3(a)-3(b). We notice that theoretically we can always increase the bandwidth of $C(s)$ to compensate for such uncertainties, however, it will require to further increase the adaptive gain.

VI. Conclusion

A novel $\mathcal{L}_1$ adaptive output feedback control architecture is presented that has guaranteed transient response in addition to stable tracking for systems with time-varying unknown parameters and bounded disturbances. The control signal and the system response approximate the same signals of a closed-loop reference system, which can be designed to achieve desired specifications.

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Appendix

Proof of Lemma 2. Consider the following system:

$$\dot{x}_1(t) = Ax_1(t) + b\left(u(t) + \theta^\top(t)(x_1(t) + x_2(t)) + \sigma(t)\right)$$
$$y_1(t) = c^\top x_1(t)\,, \quad x_1(0) = 0\,, \tag{43}$$

where $x_2(t)$ is the solution of:

$$\begin{aligned} \dot{x}_2(t) &= Ax_2(t)\,, \\ y_2(t) &= c^\top x_2(t)\,, \quad x_2(0) = x_0\,. \end{aligned} \tag{44}$$

It can be verified immediately that

$$y(t) = y_1(t) + y_2(t)\,. \tag{45}$$

It follows from (2) and (43) that

$$x_1(s) = H_x(s)T(s)(v(s) + T^{-1}(s)z(s))\,, \tag{46}$$

where $z(s)$ is the Laplace transformation of $z(t) = \theta^\top(t)(x_1(t) + x_2(t)) + \sigma(t)$. Define

$$H_{xm}(s) = (s\mathbb{I} - A_m)^{-1}b_m = \frac{\left[s^{n-1} \ s^{n-2} \ .. \ 1\right]^\top}{H_d(s)}\,. \tag{47}$$

It follows from (2) that $H_x(s) = A_T H_{xm}(s)$. Letting

$$x_{m1}(s) \quad = \quad H_{xm}(s)(v(s) + T^{-1}(s)z(s))\,, \tag{48}$$

subject to $x_{m1}(0) = 0$, it follows from (46) that

$$x_1(s) = T(s)A_T x_{m1}(s)\,. \tag{49}$$

Letting $y_{m1}(t) = c_m^\top x_{m1}(t)$, it follows from (7) and (48) that

$$y_{m1}(s) = H_m(s)(v(s) + T^{-1}(s)z(s))\,. \tag{50}$$

Eqs. (43) and (46) lead to

$$y_1(s) = H(s)T(s)(v(s) + T^{-1}(s)z(s))\,. \tag{51}$$

Since (9) implies that $H_m(s) = H(s)T(s)$, and both $x_1(0) = 0$ and $x_{m1}(0) = 0$, it follows from (50) and (51) that

$$y_{m1}(t) = y_1(t)\,. \tag{52}$$

Hence, using the expression for $x_1(s)$ via $x_{m1}(s)$ from (49), we notice that $y_{m1}(t)$ is the output of the following well-defined system:

$$\begin{aligned} \dot{x}_{m1}(t) &= A_m x_{m1}(t) \\ &+ b_m(v(t) + w_{x_{m1}}(t) + z_1(t) + \sigma_{m1}(t)), \\ y_{m1}(t) &= c_m^\top x_{m1}(t), \quad x_{m1}(0) = 0, \end{aligned} \tag{53}$$

where $w_{x_{m1}}(t)$ is the output of the system $\mathcal{W}$ in (10) for $x_{m1}(t)$, while $z_1(t)$ is defined as $z_1(s) = T^{-1}(s)z_2(s)$, $z_2(t) = \theta^\top(t)x_2(t)$, and

$$\sigma_{m1}(s) = T^{-1}(s)\sigma(s). \tag{54}$$

It follows from (44) that the Laplace transformation of $y_2(t)$ is: $y_2(s) = c^\top(s\mathbb{I} - A)^{-1}x_0$. Define

$$y_{m2}(s) = c_m^\top(s\mathbb{I} - A_m)^{-1}\hat{x}_0. \tag{55}$$

Since A and A_m have the same characteristic polynomial, it is straightforward to verify that the Laplace transformation of $y_2(t) - y_{m2}(t)$ can be rewritten as:

$$y_2(s) - y_{m2}(s) = H_2(s)/H_d(s), \tag{56}$$

where $H_d(s)$ is defined in (3) and $H_2(s)$ is a polynomial of order $< n$. Since from definition of $\hat{x}_0$ it follows that $y_2(0) - y_m(0) = c^\top x_0 - c_m^\top \hat{x}_0 = 0$, the initial value theorem implies $\lim_{s\to\infty} s\dfrac{H_2(s)}{H_d(s)} = y_2(0) - y_{m2}(0) = 0$, and, hence, the order of $H_2(s)$ is $\leq n-2$. Let $\sigma_{m2}(t)$ be

$$y_2(s) - y_{m2}(s) = H_m(s)\sigma_{m2}(s). \tag{57}$$

It follows from (56) that $\sigma_{m2}(s) = \frac{y_2(s)-y_{m2}(s)}{H_m(s)} = \frac{H_2(s)}{H_p(s)}$. Since the order of $H_2(s)$ is less and equal than $n-2$ and the order of $H_p(s)$ is $n-1$, we note that $\sigma_{m2}(s)$ is a strictly proper and stable transfer function, which implies that $\sigma_{m2}(t)$ is a continuous and bounded signal, and its derivative $\dot{\sigma}_{m2}(t)$ is also bounded. Denoting

$$y_{m3}(t) = y_2(t) - y_{m2}(t), \tag{58}$$

it follows from (55) and (57) that:

$$\begin{aligned} \dot{x}_{m2}(t) &= A_m x_{m2}(t), \\ y_{m2}(t) &= c_m x_{m2}(t), \quad x_{m2}(0) = \hat{x}_0, \\ \dot{x}_{m3}(t) &= A_m x_{m3}(t) + b_m\sigma_{m2}(t), \\ y_{m3}(t) &= c_m x_{m3}(t), \quad x_{m3}(0) = 0, \end{aligned} \tag{59}$$

and (45), (52), (58) imply that

$$y(t) = y_{m1}(t) + y_{m2}(t) + y_{m3}(t). \tag{60}$$

Letting $x_m(t) = x_{m1}(t) + x_{m2}(t) + x_{m3}(t)$, it follows from (53), (59) and (60) that:

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + b_m\Big(v(t) + w_{x_m}(t) + z_1(t) \\ &\quad -(w_{x_{m2}}(t) + w_{x_{m3}}(t)) + \sigma_{m1}(t) + \sigma_{m2}(t)\Big), \\ y_m(t) &= c_m x_m(t), \qquad x_m(0) = \hat{x}_0, \end{aligned} \tag{61}$$

where $w_{x_m}(t), w_{x_{m2}}(t), w_{x_{m3}}(t)$ are the outputs of the system (10) for the inputs $x_m(t)$, $x_{m2}(t)$ and $x_{m3}(t)$ respectively. Letting $\sigma_m(t) = z_1(t) - (w_{x_{m2}}(t) + w_{x_{m3}}(t)) + \sigma_{m1}(t) + \sigma_{m2}(t)$, the system dynamics in (61) leads to (14) directly. It follows from (54), (12) and Lemma 1 that

$$\begin{aligned} \|\sigma_m\|_{\mathcal{L}_\infty} &\leq \theta_L\|T^{-1}\|_{\mathcal{L}_1}\|x_2\|_{\mathcal{L}_\infty} + L\|x_{m2} + x_{m3}\|_{\mathcal{L}_\infty} + \\ &\quad \|T^{-1}(s)\|_{\mathcal{L}_1}\Delta + \|\sigma_{m2}\|_{\mathcal{L}_\infty}. \end{aligned} \tag{62}$$

Since $x_2(t)$, $x_{m2}(t)$, $x_{m3}(t)$ and $\sigma_{m2}(t)$ are bounded, then $\sigma_m(t)$ is also bounded. Since $x_2(t)$, $x_{m2}(t)$, $x_{m3}(t)$ $\sigma_{m2}(t)$, $\sigma(t)$ and $\theta(t)$ are differentiable with bounded derivative, it follows that $\dot{\sigma}_m(t)$ exists and is bounded. Since (61) implies that $y_m(t) = y_{m1}(t) + y_{m2}(t) + y_{m3}(t)$, it follows from (60) that $y(t) = y_m(t)$. The proof is complete. □

Proof of Lemma 3. The closed loop system in (16)-(17) can be equivalently represented as:

$$\begin{aligned} \dot{x}_{ref_1}(t) &= A_m x_{ref_1}(t) + b_m\Big(v_{ref}(t) + w_{x_{ref1}}(t) + \\ &\quad w_{x_{ref2}}(t) + \sigma_m(t)\Big),\ x_{ref_1}(0) = 0, \end{aligned} \tag{63}$$

$$\begin{aligned} v_{ref}(s) &= C(s)(\bar{r}_{ref_1}(s) + \bar{r}_{ref_2}(s)), \\ \dot{x}_{ref_2}(t) &= A_m x_{ref_2}(t), \quad x_{ref_2}(0) = \hat{x}_0, \end{aligned} \tag{64}$$

where $\bar{r}_{ref_1}(t) = -w_{x_{ref1}}(t) - \sigma_m(t) + k_g r(t)$, $\bar{r}_{ref_2}(t) = -w_{x_{ref2}}(t)$, and $w_{x_{ref1}}(t)$ and $w_{x_{ref2}}(t)$ are the outputs of the system $\mathcal{W}$ in (10) for $x_{ref_1}(t)$ and $x_{ref_2}(t)$ respectively. It follows that

$$x_{ref}(t) = x_{ref_1}(t) + x_{ref_2}(t), \qquad \forall\, t \geq 0. \tag{65}$$

For any given bounded $\hat{x}_0$, it follows from (64) that $x_{ref_2}(t)$ is uniquely defined and bounded. Let

$$\rho_2 = \|x_{ref_2}\|_{\mathcal{L}_\infty}. \tag{66}$$

It follows from (47), (21) and (63) that

$$x_{ref_1}(s) = G(s)r_1(s) + H_{xm}(s)C(s)k_g r(s), \tag{67}$$

where $r_1(t) = w_{x_{ref1}}(t) + w_{x_{ref2}}(t) + \sigma_m(t)$. The following bound that can be derived from (12) and (66):

$$\|r_1\|_{\mathcal{L}_\infty} \leq L\|x_{ref_1}\|_{\mathcal{L}_\infty} + L\rho_2 + \|\sigma_m\|_{\mathcal{L}_\infty}. \tag{68}$$

Since $C(s)$ verifies the condition in (20), then application of Theorem 1 in [1] to (67) ensures that the closed-loop system in (16)-(18) is stable. It follows from (67) and (68) that $\|x_{ref_1}\|_{\mathcal{L}_\infty} \leq \rho_1 \triangleq (k_g\|H_{xm}(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}(\|\sigma_m\|_{\mathcal{L}_\infty} + L\rho_2))/(1 - \|G(s)\|_{\mathcal{L}_1}L)$, and hence (65) implies that

$$\|x_{ref}\|_{\mathcal{L}_\infty} \leq \|x_{ref_1}\|_{\mathcal{L}_\infty} + \|x_{ref_2}\|_{\mathcal{L}_\infty} \leq \rho, \tag{69}$$

where $\rho = \rho_1 + \rho_2$. The proof is complete. □

Proof of Theorem 1. Let

$$\begin{aligned} \tilde{\sigma}(t) &= \hat{\sigma}(t) - w_{x_m}(t) - \sigma_m(t), \\ r_2(t) &= w_{x_m}(t) + \sigma_m(t), \end{aligned} \tag{70}$$

where $w_{x_m}(t)$ is the output of the system $\mathcal{W}$ in (10) for $x_m(t)$. It follows from (32) that

$$v(s) = C(s)(k_g r(s) - r_2(s) - \tilde{\sigma}(s)), \tag{71}$$

and the system in (14) consequently takes the form:

$$\begin{aligned} x_m(s) &= H_{xm}(s)\Big((1 - C(s))r_2(s) \\ &\quad + C(s)k_g r(s) - C(s)\tilde{\sigma}(s)\Big) + (s\mathbb{I} - A_m)^{-1}x_m(0). \end{aligned} \tag{72}$$

It follows from (16)-(17) that

$$\begin{aligned} x_{ref}(s) &= H_{xm}(s)\Big((1 - C(s))(w_{x_{ref}}(s) + \sigma_m(s)) \\ &+ C(s)k_g r(s)\Big) + (s\mathbb{I} - A_m)^{-1}x_{ref}(0), \end{aligned} \tag{73}$$

where $w_{x_{ref}}(s)$ is the Laplace transformation of the signal $w_{x_{ref}}(t)$, which is the output of the system $\mathcal{W}$ in (10) for $x_{ref}(t)$. Let $e(t) = x_m(t) - x_{ref}(t)$. Then, using (72), (73), and taking into consideration that $x_m(0) = x_{ref}(0) = \hat{x}(0)$, one gets $e(s) = H_{xm}(s)\left((1 - C(s))w_e(s) - C(s)\tilde{\sigma}(s)\right)$, where $w_e(t)$ is the output of the system $\mathcal{W}$ in (10) for $e(t)$. Lemma 1 gives the following upper bound:

$$\|e_t\|_{\mathcal{L}_\infty} \leq \|H_{xm}(s)(1 - C(s))\|_{\mathcal{L}_1}\|w_{e_t}\|_{\mathcal{L}_\infty} + \|r_{3_t}\|_{\mathcal{L}_\infty}, \tag{74}$$

where $r_3(t)$ is the signal with its Laplace transformation $r_3(s) = C(s)H_{xm}(s)\tilde{\sigma}(s)$. It follows from (14) and (26) that

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + b_m\tilde{\sigma}(t), \quad \tilde{x}(0) = 0\,, \tag{75}$$

which leads to $\tilde{x}(s) = H_{xm}(s)\tilde{\sigma}(s)$, and therefore $r_3(s) = C(s)\tilde{x}(s)$. Hence $\|r_{3_t}\|_{\mathcal{L}_\infty} \le \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. Using the definition of L in (13), one can verify easily that $\|w_{e_t}\|_{\mathcal{L}_\infty} \le L\|e_t\|_{\mathcal{L}_\infty}$. From (74) we have $\|e_t\|_{\mathcal{L}_\infty} \le \|H_{xm}(s)(1-C(s))\|_{\mathcal{L}_1}L\|e_t\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$, and hence

$$\|e_t\|_{\mathcal{L}_\infty} \le \frac{\|C(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1}L}\|\tilde{x}_t\|_{\mathcal{L}_\infty}\,. \tag{76}$$

First we prove the bound in (34) by a contradiction argument. Since $\tilde{x}(0) = 0$ and $\tilde{x}(t)$ is continuous, then assuming the opposite implies that there exists t' such that

$$\begin{aligned} \|\tilde{x}(t)\| &< \gamma_0, \quad \forall\, 0 \le t < t'\,, & (77)\\ \|\tilde{x}(t')\| &= \gamma_0\,, & (78)\end{aligned}$$

which leads to

$$\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty} \le \gamma_0\,. \tag{79}$$

Since $x_m(t) = x_{ref}(t) + e(t)$, it follows from (69), (76) and (79) that

$$\begin{aligned}\|x_{m_{t'}}\|_{\mathcal{L}_\infty} &\le \|x_{ref_{t'}}\|_{\mathcal{L}_\infty} + \|e_{t'}\|_{\mathcal{L}_\infty}\\ &\le \rho + \frac{\|C(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1}L}\gamma_0\,. \end{aligned}\tag{80}$$

It follows from (12) and (30) that $|\sigma_m(t)+w_{x_m}(t)| \le \bar{\Delta}$ and hence $|\tilde{\sigma}(t)| \le 2\bar{\Delta}$ for any $0 \le t \le t'$. Consider the following candidate Lyapunov function: $V(\tilde{x}(t), \tilde{\sigma}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \Gamma_c^{-1}\tilde{\sigma}^2(t)$, where $\tilde{\sigma}(t)$ is defined in (70). It follows from (6) that

$$\dot{V}(t) = -\tilde{x}^\top(t)Q\tilde{x}(t) + 2\tilde{y}(t)\tilde{\sigma}(t) + 2\Gamma_c^{-1}\tilde{\sigma}(t)\dot{\tilde{\sigma}}(t)\,. \tag{81}$$

The adaptive law (27) ensures the following inequality for all $0 \le t \le t'$:

$$\dot{V}(t) \le -\tilde{x}^\top(t)Q\tilde{x}(t) - 2\Gamma_c^{-1}\tilde{\sigma}(t)(\dot{\sigma}_m(t) + \dot{w}_{x_m}(t)). \tag{82}$$

It follows from (32) and (31) that $\|v_{t'}\|_{\mathcal{L}_\infty} \le \|C(s)\|_{\mathcal{L}_1}(k_g\|r\|_{\mathcal{L}_\infty} + \bar{\Delta})$, and hence (14) implies that

$$\begin{aligned}\|\dot{x}_{m_{t'}}\|_{\mathcal{L}_\infty} &\le \|A_m\|_{\mathcal{L}_1}\|x_{m_{t'}}\|_{\mathcal{L}_\infty} + \|b_m\|_{\mathcal{L}_1}\\ &\quad \Big(\|v_{t'}\|_{\mathcal{L}_\infty} + L\|x_{m_{t'}}\|_{\mathcal{L}_\infty} + \Delta_m\Big)\\ &\le (\|A_m\|_{\mathcal{L}_1} + L\|b_m\|_{\mathcal{L}_1})\|x_{m_{t'}}\|_{\mathcal{L}_\infty} + \|b_m\|_{\mathcal{L}_1}\\ &\quad \Big(\|C(s)\|_{\mathcal{L}_1}(k_g\|r\|_{\mathcal{L}_\infty} + \bar{\Delta}) + \Delta_m\Big)\,. \end{aligned}\tag{83}$$

Let

$$w_{m1}(t) = \theta^\top(t)w_{m2}(t),\ w_{m2}(s) = T(s)A_Tx_m(s)\,. \tag{84}$$

It can be verified easily that

$$\begin{aligned}\|\dot{w}_{x_{m_{t'}}}\|_{\mathcal{L}_\infty} &\le \|T^{-1}(s)\|_{\mathcal{L}_1}\|\dot{w}_{m1_{t'}}\|_{\mathcal{L}_\infty}\,,\\ \|\dot{w}_{m2_{t'}}\|_{\mathcal{L}_\infty} &\le \|T(s)A_T\|_{\mathcal{L}_1}\|\dot{x}_{m_{t'}}\|_{\mathcal{L}_\infty}\,.\end{aligned}$$

Eq. (84) imply that

$$\begin{aligned}&\|\dot{w}_{m1_{t'}}\|_{\mathcal{L}_\infty} \le \theta_L\|\dot{w}_{m2_{t'}}\|_{\mathcal{L}_\infty} + d_\theta\|w_{m2_{t'}}\|_{\mathcal{L}_\infty}\,,\\ &\|w_{m2_{t'}}\|_{\mathcal{L}_\infty} \le \|T(s)A_T\|_{\mathcal{L}_1}\|x_{m_{t'}}\|_{\mathcal{L}_\infty}\,,\end{aligned}$$

and hence

$$\Big\|\dot{w}_{x_{m_{t'}}}\Big\|_{\mathcal{L}_\infty} \le L\|\dot{x}_{m_{t'}}\|_{\mathcal{L}_\infty} + \frac{d_\theta L}{\theta_L}\|x_{m_{t'}}\|_{\mathcal{L}_\infty}\,. \tag{85}$$

It follows from (82), (83), (85) and the definitions of β_{01}, β_{02} in (24) that for all $0 \le t \le t'$

$$\begin{aligned}\dot{V}(t) &\le -\tilde{x}^\top(t)Q\tilde{x}(t) + 4\bar{\Delta}\Gamma_c^{-1}(d_{\sigma_m} + \|\dot{w}_{x_{m_{t'}}}\|_{\mathcal{L}_\infty})\\ &\le -\tilde{x}^\top(t)Q\tilde{x}(t) + (\beta_{01}\|x_{m_{t'}}\|_{\mathcal{L}_\infty} + \beta_{02})/\Gamma_c\,.\end{aligned}$$

Using the definitions of β_1 and β_2 in (23), it follows from (80) that for all $0 \le t \le t'$

$$\dot{V}(t) \le -\tilde{x}^\top(t)Q\tilde{x}(t) + \Gamma_c^{-1}\Big(\beta_1\gamma_0 + \beta_2\Big)\,. \tag{86}$$

The projection algorithm ensures that $|\hat{\sigma}(t)| \le \bar{\Delta}$ for all $t \ge 0$, and therefore

$$\max_{t' \ge t \ge 0} \Gamma_c^{-1}\tilde{\sigma}^2(t) \le 4\bar{\Delta}^2/\Gamma_c\,. \tag{87}$$

Let $\theta_{\max} \triangleq \beta_3\gamma_0 + \beta_4$, where β_3 and β_4 are defined in (25). If $V(t) > \theta_{\max}/\Gamma_c$ at any $t \in [0, t']$, then it follows from (81) and (87) that $\tilde{x}^\top(t)P\tilde{x}(t) > \lambda_{\max}(P)\left(\beta_1\gamma_0 + \beta_2\right)/(\Gamma_c\lambda_{\min}(Q))$, and hence

$$\tilde{x}^\top(t)Q\tilde{x}(t) > \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\tilde{x}^\top(t)P\tilde{x}(t) > \frac{\beta_1\gamma_0 + \beta_2}{\Gamma_c}\,. \tag{88}$$

From (86) and (88) it follows that if for some $t \in [0, t']$ $V(t) > \theta_{\max}/\Gamma_c$, then $\dot{V}(t) < 0$. Since $\tilde{x}(0) = 0$, we can verify that $V(0) \le \Big(\beta_3\gamma_0 + \beta_4\Big)/\Gamma_c$. It follows from $\dot{V}(t) < 0$ that

$$V(t) \le \theta_{\max}/\Gamma_c\,, \quad 0 \le t \le t'. \tag{89}$$

Since $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \le \tilde{x}^\top(t)P\tilde{x}(t) \le V(t)$, then it follows from (89) that

$$||\tilde{x}(t)||^2 \le (\beta_3\gamma_0 + \beta_4)/(\Gamma_c\lambda_{\min}(P))\,, \quad 0 \le t \le t'\,. \tag{90}$$

It follows from (79) and (90) that $\gamma_0^2 \le \dfrac{\beta_3\gamma_0 + \beta_4}{\Gamma_c\lambda_{\min}(P)}$, which along with (29) leads to $\alpha\beta_4 \le \beta_3\gamma_0 + \beta_4$. This further implies that $(\alpha-1)^2\beta_4 \le \dfrac{\alpha\beta_3}{\Gamma_c\lambda_{\min}(P)}$, which limits the adaptive gain

$$\Gamma_c \le \alpha\beta_3/\Big((\alpha-1)^2\beta_4\lambda_{\min}(P)\Big) \tag{91}$$

and hence contradicts (28). Hence, (91) is not true which further implies that (78) does not hold. Therefore, (34) is true.

It follows from the $\mathcal{L}_1$-gain requirement in (20), (34) and (76) that $\|e_t\|_{\mathcal{L}_\infty} \le \frac{\|C(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1}L}\gamma_0$, which holds uniformly for all $t \ge 0$ and therefore leads to (35). Since $y - y_{ref} = c_m^\top(x_m - x_{ref})$, then (36) follows from (35) directly.

To prove the bound in (37), we notice that from (17) and (71) one can derive $v(s) - v_{ref}(s) = -C(s)r_5(s) - r_4(s)$, where $r_4(s) = C(s)\tilde{\sigma}(s)$ and $r_5(t) = w_{x_m}(t) - w_{x_{ref}}(t)$. Therefore, it follows from Lemma 1 and the relationship in (12) that

$$\|v - v_{ref}\|_{\mathcal{L}_\infty} \le L\|C(s)\|_{\mathcal{L}_1}\|x - x_{ref}\|_{\mathcal{L}_\infty} + \|r_4\|_{\mathcal{L}_\infty}\,. \tag{92}$$

We have $r_4(s) = (C(s)/(c_o^\top H_{xm}(s)))c_o^\top H_{xm}(s)\tilde{\sigma}(s) = (C(s)/(c_o^\top H_{xm}(s)))c_o^\top\tilde{x}(s)$, where c_o is introduced in (33). Using the polynomials from (33), we can write that $C(s)/(c_o^\top H_{xm}(s)) = C(s)N_d(s)/N_n(s)$, where $N_d(s)$, $N_n(s)$ are stable polynomials and the order of $N_n(s)$ is one less than the order of $N_d(s)$. Since $C(s)$ is stable and strictly proper, the complete system $C(s)\frac{1}{c_o^\top H_{xm}(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$ gain exists and is finite. Hence, we have $\|r_4\|_{\mathcal{L}_\infty} \le \Big\|C(s)\dfrac{1}{c_o^\top H_{xm}(s)}c_o^\top\Big\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}$. Lemma 1 consequently leads to the upper bound: $\|r_4\|_{\mathcal{L}_\infty} \le \Big\|C(s)\frac{1}{c_o^\top H_{xm}(s)}c_o^\top\Big\|_{\mathcal{L}_1}\gamma_0$, which, when substituted into (92), leads to (37). □