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$\mathcal{L}_1$ Adaptive Output Feedback Controller to Systems of Unknown Dimension

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Abstract— **This paper presents an extension of the $\mathcal{L}_1$ adaptive output feedback controller to systems of unknown dimension in the presence of unmodeled dynamics and time-varying uncertainties. The adaptive output feedback controller ensures uniformly bounded transient and asymptotic tracking for system's both signals, input and output, simultaneously. The performance bounds can be systematically improved by increasing the adaptation rate. Simulations of an unstable non-minimum phase system verify the theoretical findings.**

I. Introduction

This paper extends the results of [1]–[3] to an output feedback framework for a single input signal output (SISO) system of unknown dimension in the presence of unmodeled dynamics and time-varying disturbances. The methodology ensures uniformly bounded transient response for system's both signals, input and output, simultaneously, in addition to asymptotic tracking. The $\mathcal{L}_\infty$ norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference LTI system can be systematically reduced by increasing the adaptation gain.

The paper is organized as follows. Section II states some preliminary definitions, and Section III gives the problem formulation. In Section IV, the novel $\mathcal{L}_1$ adaptive control architecture is presented. Stability and uniform transient tracking bounds of the $\mathcal{L}_1$ adaptive controller are presented in Section V. In section VI, simulation results are presented, while Section VII concludes the paper.

II. Preliminaries

In this Section, we recall basic definitions and facts from linear systems theory.

Definition 1: For a signal $\xi(t)$, $t \geq 0$, $\xi \in \mathbb{R}^n$, its truncated $\mathcal{L}_\infty$ and $\mathcal{L}_\infty$ norms are $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{0\leq\tau\leq t}|\xi_i(\tau)|)$, $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{\tau\geq 0}|\xi_i(\tau)|)$, where ξ_i is the i^{th} component of ξ.

Definition 2: The $\mathcal{L}_1$ gain of a stable proper SISO system is defined $||H(s)||_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$, where $h(t)$ is the impulse response of $H(s)$.

Definition 3: For a stable proper m input n output system $H(s)$ its $\mathcal{L}_1$ gain is defined as $\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,..,n}(\sum_{j=1}^m \|H_{ij}(s)\|_{\mathcal{L}_1})$, where $H_{ij}(s)$ is the corresponding entry of $H(s)$.

Lemma 1: For a stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have $\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}, \quad \forall\ t \geq 0$.

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III. Problem Formulation

Consider the following SISO system:

$$y(s) = A(s)(u(s) + d(s))\,, \quad y(0) = 0 \tag{1}$$

where $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the system output, $A(s)$ is a strictly proper unknown transfer function, $d(s)$ is the Laplace transform of the time-varying uncertainties and disturbances $d(t) = f(t, y(t))$, while f is an unknown map, subject to the following assumptions.

Assumption 1: There exist constants $L > 0$ and $L_0 > 0$ such that the following inequalities hold uniformly in t:

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|\,, \quad |f(t, y)| \leq L|y| + L_0.$$

Assumption 2: There exist constants $L_1 > 0$, $L_2 > 0$ and $L_3 > 0$ such that for all $t \geq 0$:

$$|\dot{d}(t)| \leq L_1|\dot{y}(t)| + L_2|y(t)| + L_3\,. \tag{2}$$

We note that the numbers L, L_0, L_1, L_2, L_3 can be arbitrarily large. Let $r(t)$ be a given bounded continuous reference input signal. The control objective is to design an adaptive output feedback controller $u(t)$ such that the system output $y(t)$ tracks the reference input following a desired reference model, i.e. $y(s) \approx M(s)r(s)$. In this paper, we consider a first order system, i.e.

$$M(s) = m/(s + m)\,, \quad m > 0. \tag{3}$$

We note that the system in (1) can be simply rewritten as:

$$\begin{aligned} y(s) &= M(s)\,(u(s) + \sigma(s))\,, & (4)\\ \sigma(s) &= ((A(s) - M(s))u(s) + A(s)d(s))/M(s)\,. & (5) \end{aligned}$$

IV. $\mathcal{L}_1$ Adaptive Controller

A. Closed-loop Reference System

Consider the following closed-loop reference system:

$$\begin{aligned} y_{ref}(s) &= M(s)(u_{ref}(s) + \sigma_{ref}(s)) & (6)\\ \sigma_{ref}(s) &= \frac{(A(s) - M(s))u_{ref}(s) + A(s)d_{ref}(s)}{M(s)} & (7)\\ u_{ref}(s) &= C(s)(r(s) - \sigma_{ref}(s))\,, & (8) \end{aligned}$$

where $d_{ref}(t) = f(t, y_{ref}(t))$, and $C(s)$ is a strictly proper system with $C(0) = 1$. One simple choice would be

$$C(s) = \omega/(s + \omega)\,. \tag{9}$$

We note that there is no algebraic loop involved in the definition of $\sigma(s)$, $u(s)$ and $\sigma_{ref}(s)$, $u_{ref}(s)$.

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$\mathcal{L}_1$**-gain stability requirement:** $C(s)$ and $M(s)$ need to ensure that

$$H(s) = A(s)M(s)/(C(s)A(s) + (1 - C(s))M(s)) \quad (10)$$

is stable and

$$\|G(s)\|_{\mathcal{L}_1} L < 1\,, \quad (11)$$

where $G(s) = H(s)(1 - C(s))$.

The condition in (11) restricts the class of systems $A(s)$ in (1) that can be stabilized by the controller architecture in this paper. However, as discussed in section V-A, the class of such systems is not empty. Letting

$$A(s) = \frac{A_n(s)}{A_d(s)}\,, \quad C(s) = \frac{C_n(s)}{C_d(s)}\,, \quad M(s) = \frac{M_n(s)}{M_d(s)}\,, \quad (12)$$

it follows from (10) that

$$H(s) = \frac{C_d(s)M_n(s)A_n(s)}{M_d(s)C_n(s)A_n(s) + (C_d(s) - C_n(s))M_n(s)A_d(s)} \quad (13)$$

We note that a strictly proper $C(s)$ implies that the order of $C_d(s) - C_n(s)$ and $C_d(s)$ is the same. Since the order of $A_d(s)$ is higher than that of $A_n(s)$, we note that the transfer function $H(s)$ is strictly proper.

The next Lemma establishes the stability of the closed-loop system in (6)-(8).

Lemma 2: If $C(s)$ and $M(s)$ verify the condition in (11), the closed-loop reference system in (6)-(8) is stable.

Proof. It follows from (7)-(8) that $u_{ref}(s) = C(s)r(s) - C(s)((A(s) - M(s))u_{ref}(s) + A(s)d_{ref}(s))/M(s)$, and hence

$$u_{ref}(s) \quad = \quad \frac{C(s)M(s)r(s) - C(s)A(s)d_{ref}(s)}{C(s)A(s) + (1 - C(s))M(s)}\,. \quad (14)$$

It follows from (6)-(7) that

$$y_{ref}(s) = A(s)(u_{ref}(s) + d_{ref}(s))\,. \quad (15)$$

Substituting (14) into (15), it follows from (10) that

$$\begin{aligned} y_{ref}(s) &= A(s)\Big(\frac{C(s)M(s)r(s) - C(s)A(s)d_{ref}(s)}{C(s)A(s) + (1 - C(s))M(s)} \\ &+ d_{ref}(s)\Big) \\ &= A(s)M(s)\left(\frac{C(s)r(s) + (1 - C(s))d_{ref}(s)}{C(s)A(s) + (1 - C(s))M(s)}\right) \\ &= H(s)\left(C(s)r(s) + (1 - C(s))d_{ref}(s)\right)\,. \quad (16) \end{aligned}$$

Since $H(s)$ is strictly proper and stable, $G(s)$ is also strictly proper and stable and further

$$\begin{aligned} \|y_{ref}\|_{\mathcal{L}_\infty} \leq\ & \|H(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \\ & + \|G(s)\|_{\mathcal{L}_1}(L\|y_{ref}\|_{\mathcal{L}_\infty} + L_0)\,. \quad (17) \end{aligned}$$

It follows from (11) and (17) that

$$\|y_{ref}\|_{\mathcal{L}_\infty} \leq \rho\,, \quad (18)$$

where

$$\rho = \frac{\|H(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}L_0}{1 - \|G(s)\|_{\mathcal{L}_1}L}\,, \quad (19)$$

and hence $\|y_{ref}\|_{\mathcal{L}_\infty}$ is finite, which implies that the closed-loop reference system in (6)-(8) is stable. □

1) Notations: Choose arbitrary $P > 0$ and let $Q = 2mP$. Define

$$H_0(s) = A(s)/(C(s)A(s) + (1 - C(s))M(s)), \quad (20)$$

$$H_1(s) = \frac{(A(s) - M(s))C(s)}{C(s)A(s) + (1 - C(s))M(s)}\,. \quad (21)$$

Using (12) in (20)-(21), we have $H_0(s) = C_d(s)A_n(s)M_d(s)/H_d(s)$, and

$$H_1(s) = (C_n(s)A_n(s)M_d(s) - C_n(s)A_d(s)M_n(s))/H_d(s), \quad (22)$$

where $H_d(s) = C_n(s)A_n(s)M_d(s) + M_n(s)A_d(s)(C_d(s) - C_n(s))$. Since the relative order between $C_d(s) - C_n(s)$ and $C_n(s)$ is greater than zero, the order of $M_n(s)A_d(s)(C_d(s) - C_n(s))$ is higher than $C_n(s)A_d(s)M_n(s)$. Since the relative order between $A_d(s)$ and $A_n(s)$ is greater than zero, while the relative order between $M_n(s)$ and $M_d(s)$ is -1, we note that the order of $M_n(s)A_d(s)(C_d(s) - C_n(s))$ is higher than that of $C_n(s)A_n(s)M_d(s)$. Therefore, $H_1(s)$ is strictly proper. We note from (13) and (22) that $H_1(s)$ has the same denominator as $H(s)$ and it follows from the $\mathcal{L}_1$-gain stability requirement that $H_1(s)$ is stable. Using similar arguments, it can be verified that $H_0(s)$ is proper and stable.

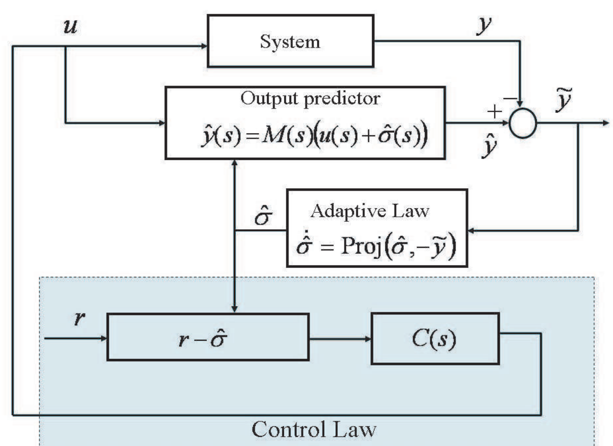


Fig. 1. Closed-loop system with $\mathcal{L}_1$ adaptive controller

Let $\Delta = \|H_1(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H_0(s)\|_{\mathcal{L}_1}(L\rho + L_0) + \left(\|H_1(s)/M(s)\|_{\mathcal{L}_1} + L\|H_0(s)\|_{\mathcal{L}_1}\frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}\right)\bar{\gamma}$, where $\bar{\gamma} > 0$ is an arbitrary constant. Since $H_1(s)$ is stable and strictly proper, we note that $\|H_1(s)/M(s)\|_{\mathcal{L}_1}$ exists and, hence, Δ is a finite number. Let

$$\begin{aligned} \beta_1 &= 4\Delta\|H_0(s)\|_{\mathcal{L}_1}\left(L_1\beta_{01} + L_2\frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}\right) \\ \beta_2 &= 4\Delta\|sH_1(s)\|_{\mathcal{L}_1}\left(\|r\|_{\mathcal{L}_\infty} + 2\Delta\right) + \\ & \quad 4\Delta\|H_0(s)\|_{\mathcal{L}_1}\left(L_1\beta_{02} + L_3 + \rho L_2\right)\,, \quad (23) \end{aligned}$$

where ρ is defined in (19), and

$$\begin{aligned}\beta_{01} &= \|sH(s)(1-C(s))\|_{\mathcal{L}_1} \frac{L\,\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1} L} \\ \beta_{02} &= \|sH(s)C(s)\|_{\mathcal{L}_1} (\|r\|_{\mathcal{L}_\infty} + 2\Delta) \\ &\quad + \|sH(s)(1-C(s))\|_{\mathcal{L}_1} (L\rho + L_0)\,. \end{aligned} \tag{24}$$

Since $H(s)$ and $H_1(s)$ are strictly proper and stable, we note that $\|sH_1(s)\|_{\mathcal{L}_1}$, $\|sH(s)C(s)\|_{\mathcal{L}_1}$ and $\|sH(s)(1-C(s))\|_{\mathcal{L}_1}$ are finite. We further define

$$\begin{aligned}\beta_3 &= P/Q\beta_1 = \beta_1/(2m) \\ \beta_4 &= 4\Delta^2 + P\beta_2/Q = 4\Delta^2 + \beta_2/(2m)\,. \end{aligned} \tag{25}$$

2) $\mathcal{L}_1$ adaptive controller: The elements of $\mathcal{L}_1$ adaptive controller are introduced below.

Output Predictor: We consider the following output predictor:

$$\dot{\hat{y}}(t) = -m\hat{y}(t) + m\,(u(t) + \hat{\sigma}(t))\,, \quad \hat{y}(0) = 0\,, \tag{26}$$

where the adaptive estimate $\hat{\sigma}(t)$ is governed by the following adaptation law.

Adaptive Law: The adaptation of $\hat{\sigma}(t)$ is defined as:

$$\dot{\hat{\sigma}}(t) = \Gamma_c \mathrm{Proj}(\hat{\sigma}(t), -\tilde{y}(t)), \qquad \hat{\sigma}(0) = 0, \tag{27}$$

where $\tilde{y}(t) = \hat{y}(t) - y(t)$ is the error signal between the output of the system in (4) and the predictor in (26), $\Gamma_c \in \mathbb{R}^+$ is the adaptation rate subject to the following lower bound:

$$\Gamma_c > \max\left\{\alpha\beta_3^2/((\alpha-1)^2\beta_4 P), \alpha\beta_4/(P\bar{\gamma}^2)\right\} \tag{28}$$

with $\alpha > 1$ being an arbitrary constant, while projection is performed on the set

$$\hat{\sigma} \in \Delta. \tag{29}$$

Letting

$$\gamma_0 = \sqrt{\alpha\beta_4/(\Gamma_c P)}\,, \tag{30}$$

it follows from (28) that $\bar{\gamma} > \gamma_0$, and hence $\Delta \geq \|H_1(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H_0(s)\|_{\mathcal{L}_1}(L\rho + L_0) + \Big(\|H_1(s)/M(s)\|_{\mathcal{L}_1} + L\|H_0(s)\|_{\mathcal{L}_1} \frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1} L}\Big)\gamma_0$, and $|\hat{\sigma}(t)| \leq \Delta$ for any $t \geq 0$.

Control Law: The control signal is generated by:

$$u(s) = C(s)(r(s) - \hat{\sigma}(s))\,. \tag{31}$$

The complete $\mathcal{L}_1$ adaptive controller consists of (26), (27) and (31) subject to the $\mathcal{L}_1$-gain stability requirement in (11). The closed-loop system is illustrated in Fig. 1.

V. Analysis of $\mathcal{L}_1$ Adaptive Controller

In this section, we analyze stability and performance of $\mathcal{L}_1$ adaptive controller. Let $H_2(s) = -M(s)C(s)/(C(s)A(s) + (1-C(s))M(s))$. Using the definitions from (12), we have

$$H_2(s) = \frac{-C_n(s)A_d(s)M_n(s)}{C_n(s)A_n(s)M_d(s) + M_n(s)A_d(s)(C_d(s) - C_n(s))}\,. \tag{32}$$

Since the relative order between $C_d(s) - C_n(s)$ and $C_n(s)$ is greater than zero, it can be checked easily that $H_2(s)$ is strictly proper. We note from (13) and (32) that $H_2(s)$ has the same denominator as $H(s)$, and it follows from the $\mathcal{L}_1$-gain stability requirement that $H_2(s)$ is stable. Since $H_2(s)$ is strictly proper and stable, $H_2(s)/M(s)$ is stable and proper and, hence, its $\mathcal{L}_1$ gain is finite. It can be verified easily that $C(s)H(s)/M(s)$ is strictly proper and stable too and, hence, $\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}$ exists and is finite.

Theorem 1: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller in (26), (27), (31) subject to (11), we have:

$$\begin{aligned}\|\tilde{y}\|_{\mathcal{L}_\infty} &< \gamma_0 &(33)\\ \|y - y_{ref}\|_{\mathcal{L}_\infty} &\leq \gamma_1 &(34)\\ \|u - u_{ref}\|_{\mathcal{L}_\infty} &\leq \gamma_2, &(35)\end{aligned}$$

where $\tilde{y}(t) = \hat{y}(t) - y(t)$, γ_0 is defined in (30), and

$$\begin{aligned}\gamma_1 &= \|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\gamma_0/(1-\|G(s)\|_{\mathcal{L}_1}L) &(36)\\ \gamma_2 &= L\|H_3(s)\|_{\mathcal{L}_1}\gamma_1 + \|H_2(s)/M(s)\|_{\mathcal{L}_1}\gamma_0\,, &(37)\end{aligned}$$

where $H_3(s) = H(s)C(s)/M(s)$.

Proof. Let $\tilde{\sigma}(t) = \hat{\sigma}(t) - \sigma(t)$, where $\sigma(t)$ is defined in (5). It follows from (31) that

$$u(s) = C(s)r(s) - C(s)(\sigma(s) + \tilde{\sigma}(s))\,, \tag{38}$$

and the system in (4) consequently takes the form:

$$y(s) = M(s)\Big(C(s)r(s) + (1-C(s))\sigma(s) - C(s)\tilde{\sigma}(s)\Big)\,. \tag{39}$$

Substituting (38) into (5), it follows from the definition of $H(s)$, $H_0(s)$ and $H_1(s)$ in (10), (20), (21) that

$$\sigma(s) = H_1(s)(r(s) - \tilde{\sigma}(s)) + H_0(s)d(s)\,. \tag{40}$$

Substituting (40) into (39), we have

$$\begin{aligned}y(s) &= M(s)(C(s) + H_1(s)(1-C(s)))(r(s) - \tilde{\sigma}(s)) \\ &\quad + H_0(s)M(s)(1-C(s))d(s)\,. \end{aligned} \tag{41}$$

It can be verified from (10), (20) and (21) that $M(s)(C(s) + H_1(s)(1-C(s))) = H(s)C(s)$, $H(s) = H_0(s)M(s)$, and hence (41) can be rewritten as

$$y(s) = H(s)(C(s)r(s) - C(s)\tilde{\sigma}(s)) + H(s)(1-C(s))d(s)\,. \tag{42}$$

Let $e(t) = y(t) - y_{ref}(t)$. Then, using (16) and (42), one gets $e(s) = H(s)\left((1-C(s))d_e(s) - C(s)\tilde{\sigma}(s)\right)$, where $d_e(s)$ is introduced to denote the Laplace transform of $d_e(t) = f(t, y(t)) - f(t, y_{ref}(t))$. Lemma 1 and Assumption 1 give the following upper bound:

$$\|e_t\|_{\mathcal{L}_\infty} \leq L\|H(s)(1-C(s))\|_{\mathcal{L}_1}\|e_t\|_{\mathcal{L}_\infty} + \|r_{1_t}\|_{\mathcal{L}_\infty}\,, \tag{43}$$

where $r_1(t)$ is the signal with its Laplace transformation

$$r_1(s) = C(s)H(s)\tilde{\sigma}(s). \tag{44}$$

It follows from (4) and (26) that

$$\tilde{y}(s) = M(s)\tilde{\sigma}(s)\,. \tag{45}$$

It follows from (44) and (45) that $r_1(s) = \frac{C(s)H(s)}{M(s)}M(s)\tilde{\sigma}(s) = \frac{C(s)H(s)}{M(s)}\tilde{y}(s)$, and hence

$$\|r_{1_t}\|_{\mathcal{L}_\infty} \leq \|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\|\tilde{y}_t\|_{\mathcal{L}_\infty}\,. \tag{46}$$

From (43) and (46), we have $\|e_t\|_{\mathcal{L}_\infty} \leq L\|H(s)(1-C(s))\|_{\mathcal{L}_1}\|e_t\|_{\mathcal{L}_\infty}+\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\|\tilde{y}_t\|_{\mathcal{L}_\infty}$, and hence

$$\|e_t\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1}L}\|\tilde{y}_t\|_{\mathcal{L}_\infty}. \quad (47)$$

First we prove the bound in (33) by a contradiction argument. Since $\tilde{y}(0)=0$ and $\tilde{y}(t)$ is continuous, then assuming the opposite implies that there exists t' such that

$$\|\tilde{y}(t)\| < \gamma_0, \quad \forall\, 0 \leq t < t', \quad (48)$$
$$\|\tilde{y}(t')\| = \gamma_0, \quad (49)$$

which leads to

$$\|\tilde{y}_{t'}\|_{\mathcal{L}_\infty} = \gamma_0. \quad (50)$$

Since $y(t)=y_{ref}(t)+e(t)$, it follows from (18) and (50) that

$$\|y_{t'}\|_{\mathcal{L}_\infty} \leq \|y_{ref_{t'}}\|_{\mathcal{L}_\infty}+\|e_{t'}\|_{\mathcal{L}_\infty} \leq \rho + \|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\gamma_0/(1-\|G(s)\|_{\mathcal{L}_1}L). \quad (51)$$

It follows from (40) and (45) that $\sigma(s)=H_1(s)r(s)-H_1(s)\tilde{y}(s)/M(s)+H_0(s)d(s)$, and hence (50) implies that $\|\sigma_{t'}\|_{\mathcal{L}_\infty} \leq \|H_1(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}+\|H_1(s)/M(s)\|_{\mathcal{L}_1}\gamma_0+\|H_0(s)\|_{\mathcal{L}_1}(L\|y_{t'}\|_{\mathcal{L}_\infty}+L_0)$, which along with (51) leads to

$$\|\sigma_{t'}\|_{\mathcal{L}_\infty} \leq \Delta. \quad (52)$$

Consider the following candidate Lyapunov function:

$$V(\tilde{y}(t),\tilde{\sigma}(t)) = P\tilde{y}^2(t)+\Gamma_c^{-1}\tilde{\sigma}^2(t), \quad (53)$$

and the adaptive law in (27) ensures that for all $0 \leq t \leq t'$:

$$\dot{V}(t) \leq -Q\tilde{y}^2(t)-2\Gamma_c^{-1}\tilde{\sigma}(t)\dot{\sigma}(t). \quad (54)$$

It follows from (40) that

$$\sigma_d(s) = sH_1(s)(r(s)-\tilde{\sigma}(s))+H_0(s)d_d(s), \quad (55)$$

where $\sigma_d(s)$ and $d_d(s)$ are the Laplace transformations of $\dot{\sigma}(t)$ and $\dot{d}(t)$, respectively. From (29) and (52), we have

$$\|\tilde{\sigma}_{t'}\|_{\mathcal{L}_\infty} \leq 2\Delta. \quad (56)$$

It follows from (51) that

$$\|d_{t'}\|_{\mathcal{L}_\infty} \leq L\rho + \frac{L\,\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1}L}\gamma_0 + L_0. \quad (57)$$

From the definitions of β_{01} and β_{02} in (24), (42) and (57), we have

$$\|\dot{y}_{t'}\|_{\mathcal{L}_\infty} \leq \beta_{01}\gamma_0+\beta_{02}. \quad (58)$$

It follows from Assumption 2 and (58) that

$$\|\dot{d}_{t'}\|_{\mathcal{L}_\infty} \leq L_2\|y_{t'}\|_{\mathcal{L}_\infty}+L_1(\beta_{01}\gamma_0+\beta_{02})+L_3. \quad (59)$$

From (51), (55), (59) and the definitions of β_1 and β_2 in (23), it follows that

$$\|\dot{\sigma}_{t'}\|_{\mathcal{L}_\infty} \leq (\beta_1\gamma_0+\beta_2)/(4\Delta). \quad (60)$$

Therefore from (54), (56) and (60) we have

$$\dot{V}(t) \leq -Q\tilde{y}^2(t)+\Gamma_c^{-1}\Big(\beta_1\gamma_0+\beta_2\Big), \ \forall\, 0 \leq t \leq t'. \quad (61)$$

The projection algorithm ensures that $|\hat{\sigma}(t)| \leq \Delta$ for all $t \geq 0$, and therefore

$$\max_{t' \geq t \geq 0} \Gamma_c^{-1}\tilde{\sigma}^2(t) \leq 4\Delta^2/\Gamma_c. \quad (62)$$

Let $\theta_{\max} \triangleq \beta_3\gamma_0+\beta_4$, where β_3 and β_4 are defined in (25). If at any $t \in [0,\,t']$, $V(t) > \theta_{\max}/\Gamma_c$, then it follows from (53) and (62) that $P\tilde{y}^2(t) > P\left(\beta_1\gamma_0+\beta_2\right)/(\Gamma_c Q)$, and hence

$$Q\tilde{y}^2 = (Q/P)P\tilde{y}^2 > (\beta_1\gamma_0+\beta_2)/\Gamma_c. \quad (63)$$

From (61) and (63) it follows that if for some $t \in [0,t']$ $V(t) > \theta_{\max}/\Gamma_c$, then

$$\dot{V}(t) < 0. \quad (64)$$

Since $\tilde{y}(0)=0$, we can verify that

$$V(0) \leq \Big(\beta_3\gamma_0+\beta_4\Big)/\Gamma_c. \quad (65)$$

It follows from (64) that

$$V(t) \leq \theta_{\max}/\Gamma_c, \quad 0 \leq t \leq t'. \quad (66)$$

Since $P|\tilde{y}(t)|^2 \leq V(t)$, then it follows from (66) that

$$|\tilde{y}(t)|^2 \leq (\beta_3\gamma_0+\beta_4)/(\Gamma_c P), \quad 0 \leq t \leq t'. \quad (67)$$

It follows from (50) and (67) that $\gamma_0^2 \leq (\beta_3\gamma_0+\beta_4)/(\Gamma_c P)$, which along with (30) leads to $\alpha\beta_4 \leq \beta_3\gamma_0+\beta_4$, and further implies

$$(\alpha-1)^2\beta_4 \leq \alpha\beta_3^2/(\Gamma_c P). \quad (68)$$

Eq. (68) limits the adaptive gain

$$\Gamma_c \leq \alpha\beta_3^2/((\alpha-1)^2\beta_4 P), \quad (69)$$

which contradicts (28). Hence, (69) is not true which further implies that (49) does not hold. Therefore, (33) is true. It follows from the $\mathcal{L}_1$-gain requirement in (11), (33) and (47) that $\|e_t\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1-\|G(s)\|_{\mathcal{L}_1}L}\gamma_0$, which holds uniformly for all $t \geq 0$ and therefore leads to (34).

It follows from (5) and (38) that

$$u(s) = \frac{M(s)(C(s)r(s)-C(s)\tilde{\sigma}(s))-C(s)A(s)d(s)}{C(s)A(s)+(1-C(s))M(s)}.$$

To prove the bound in (35), we notice that from (14) one can derive

$$u(s)-u_{ref}(s) = -H_3(s)r_2(s)+H_2(s)\tilde{\sigma}(s), \quad (70)$$
$$= -H_3(s)r_2(s)+(H_2(s)/M(s))M(s)\tilde{\sigma}(s),$$

where $r_2(t)=f(t,y(t))-f(t,y_{ref}(t))$. It follows from (45) and (70) that $\|u-u_{ref}\|_{\mathcal{L}_\infty} \leq L\|H_3(s)\|_{\mathcal{L}_1}\|y-y_{ref}\|_{\mathcal{L}_\infty}+\|H_2(s)/M(s)\|_{\mathcal{L}_1}\|\tilde{y}\|_{\mathcal{L}_\infty}$, which leads to (35). □

The main result can be summarized as follows.

Theorem 2: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller in (26), (27), (31) subject to (11), we have:

$$\lim_{\Gamma_c\to\infty}(y(t)-y_{ref}(t)) = 0, \quad \forall\, t \geq 0, \quad (71)$$
$$\lim_{\Gamma_c\to\infty}(u(t)-u_{ref}(t)) = 0, \quad \forall\, t \geq 0. \quad (72)$$

The proof follows from Theorem 1 directly. Thus, the tracking error between $y(t)$ and $y_{ref}(t)$, as well between

$u(t)$ and $u_{ref}(t)$, is uniformly bounded by a constant inverse proportional to Γ_c. This implies that during the transient one can achieve arbitrarily close tracking performance for both signals simultaneously by increasing Γ_c.

We note that the control law $u_{ref}(t)$ in the closed-loop reference system, which is used in the analysis of $\mathcal{L}_\infty$ norm bounds, is not implementable since its definition involves the unknown parameters. Theorem 1 ensures that the $\mathcal{L}_1$ adaptive controller approximates $u_{ref}(t)$ both in transient and steady state. So, it is important to understand how these bounds can be used for ensuring uniform transient response with *desired* specifications. We notice that the following *ideal* control signal: $u_{ideal}(t) = r(t) - \sigma_{ref}(t)$, is the one that leads to desired system response:

$$y_{ideal}(s) \quad = \quad M(s)r(s) \tag{73}$$

by cancelling the uncertainties exactly. Thus, the reference system in (6)-(8) has a different response as compared to (73). In [4], specific design guidelines are suggested for selection of $C(s)$ that can lead to desired system response.

A. Discussion

In this section, we discuss the classes of systems that can satisfy (11) via the choice of $M(s)$ and $C(s)$. For simplicity, we consider first order $C(s)$ and $M(s)$ as pointed in (3) and (9). It follows from (3) and (9) that $H(s) = \frac{m(s+\omega)A_n(s)}{\omega(s+m)A_n(s)+msA_d(s)}$. Stability of $H(s)$ is equivalent to stabilization of $A(s)$ by a PI controller, say, of the following structure $(\omega/m)((s+m)/s)$, where m and ω are the same as in (3) and (9). The open loop transfer function of the cascaded $A(s)$ with the PI controller will be $H_{PI}(s) = (\omega/m)((s+m)/s)A(s)$, leading to the following closed-loop system:

$$\omega(s+m)A_n(s)/(\omega(s+m)A_n(s) + msA_d(s))\,. \tag{74}$$

Hence, the stability of $H(s)$ is equivalent to that of (74), and the problem can be reduced to identifying the class of $A(s)$ that can be stabilized by a PI controller. It also permits the use of root locus methods for checking the stability of $H(s)$ via the open loop transfer function $H_{PI}(s)$. We note that the PI controller adds an open loop pole at the origin and an open loop zero at $-m$, while ω/m plays the role of the open-loop gain.

1) Minimum phase system with relative degree 1 *or* 2*:* Consider a minimum phase system $H(s)$ with relative degree 1. Notice that the zeros of H_{PI} are located in the open left-half plane. As the open loop gain ω/m increases, it follows from the classical control theory that the closed-loop poles approach the open-loop zeros and tend to ∞. Since the system has relative degree 1, only one closed-loop pole can approach ∞ along the negative real axis, which implies that all the closed-loop poles are located in the open left-half plane. Hence, the transfer function in (74) is stable, such is $H(s)$. We notice that the above discussions hold for any $M(s)$ with relative degree 1.

For a minimum phase system with relative degree 2, with the increase of the open-loop gain ω/m, there are two closed-loop poles approaching ∞ along the direction of $-\pi/2$ and $\pi/2$ in the complex plane. Let the abscissa of the intersection of the asymptotes and the real axis be δ. We note that the two infinite poles approach $\delta \pm j\infty$. If δ is negative, the closed-loop system can be stabilized by increasing the open-loop gain. If the choice of $M(s)$ with relative degree 1 ensures negative δ, the closed-loop system can be stabilized by increasing the open-loop gain. We notice that any $M(s)$ with a denominator of order 2 and a nominator of order 1 places two open-loop zeros and one open-loop pole. By placing them appropriately, we can ensure negative δ. Therefore, by choosing appropriate $M(s)$, we can ensure stability of minimum phase systems with relative degree 1 or 2. We note here that the knowledge of the order of $A_d(s)$ and the relative degree of $A(s)$ are not required. As ω is large, which implies large open-loop gain, $C(s)$ approximates 1 and therefore $y(s) \approx M(s)r(s)$, which was the control objective.

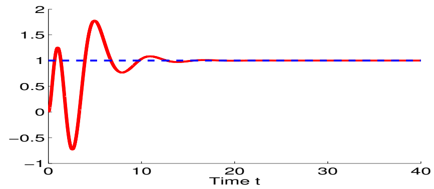
Remark 1: We notice that in the light of the above discussion, a PI controller, stabilizing $A(s)$, might also stabilize the system in the presence of the nonlinear disturbance $f(t, y(t))$. However, the transient performance cannot be quantified in the presence of unknown $A(s)$. The $\mathcal{L}_1$ adaptive controller will generate different low-pass control signals $u(t)$ for different unknown systems to ensure uniform transient performance for $y(t)$.

2) Other Systems: We note that non-minimum phase systems can also be stabilized by a PI controller. However, the choice of m and ω is not straightforward. In the simulation example below, we demonstrate the application of $\mathcal{L}_1$ adaptive controller for an unknown non-minimum phase system in the presence of unknown nonlinear disturbances.

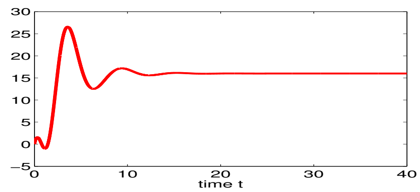
VI. Simulation

As an illustrative example, consider the system in (1) with $A(s) = (s^2 - 0.5s + 0.5)/(s^3 - s^2 - 2s + 8)$. We note that $A(s)$ has both poles and zeros in the right half plane and hence it is an unstable non-minimum phase system. We consider $\mathcal{L}_1$ adaptive controller defined via (26), (27) and (31), where $m = 3\,, \quad \omega = 10\,, \quad \Gamma_c = 500$. We set $\Delta = 100$. First, we consider the step response by assuming $d(t) = 0$. The simulation results of $\mathcal{L}_1$ adaptive controller are shown in Figures 2(a)-2(b). Next, we consider $d(t) = f(t, y(t)) = \sin(0.1t)y(t) + 2\sin(0.1t)$, and apply the same controller without retuning. The control signal and the system response are plotted in Figures 3(a)-3(b). Further, we consider a time-varying reference input $r(t) = 0.5\sin(0.3t)$ and notice that that without any retuning of the controller the system response and the control signal behave as expected, Figs. 4(a)-4(b). Figs. 5(a)-5(b) plot the system response and the control signal for a different uncertainty $d(t) = f(t, y(t)) = \sin(0.1t)y(t) + 2\sin(0.4t)$ without any retuning of the controller.

We notice that in the case of minimum-phase systems, theoretically we can increase the bandwidth of $C(s)$ arbitrarily and cancel time-varying disturbances of arbitrary frequency.

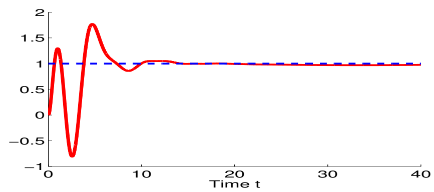


(a) $y(t)$ (solid) and $r(t)$ (dashed)

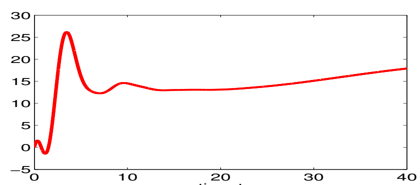


(b) Time-history of $u(t)$

Fig. 2. Performance for $r(t) = 1$ and $d(t) = 0$.

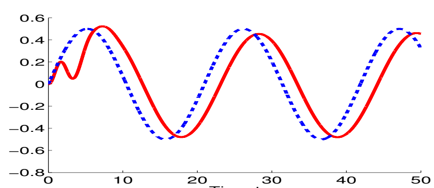


(a) $y(t)$ (solid) and $r(t)$ (dashed)

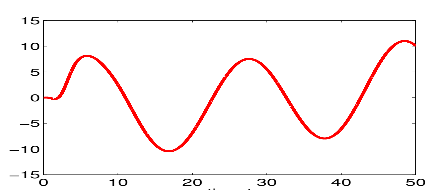


(b) Time-history of $u(t)$

Fig. 3. Performance for $r(t) = 1$ and $d(t) = \sin(0.1t)y(t) + 2\sin(0.1t)$.



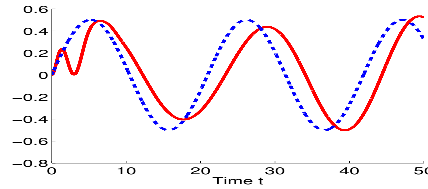
(a) $y(t)$ (solid) and $r(t)$ (dashed)



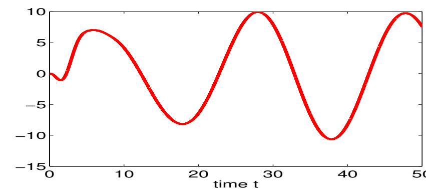
(b) Time-history of $u(t)$

Fig. 4. Performance for $r(t) = 0.5\sin(0.3t)$ and $d(t) = \sin(0.1t)y(t) + 2\sin(0.1t)$.

However, the bandwidth of $C(s)$ cannot be set arbitrarily large due to the bandwidth limitations in the control channels



(a) $y(t)$ (solid) and $r(t)$ (dashed)



(b) Time-history of $u(t)$

Fig. 5. Performance for $r(t) = 0.5\sin(0.3t)$ and $d(t) = \sin(0.1t)y(t) + 2\sin(0.4t)$.

of the system. Also, a larger bandwidth of $C(s)$ can reduce the time-delay margin of the closed-loop system and imply that a higher adaptive gain is needed [4], [5].

VII. Conclusion

A novel $\mathcal{L}_1$ adaptive output feedback control architecture is presented in this paper for systems of unknown dimension. It has guaranteed transient response for system's both signals, input and output, simultaneously, in addition to stable tracking. The methodology is verified for an unstable non-minimum phase system.

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