Robust Adaptive Control

UIUC MechSE 598

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Au lieu of Introduction

Many dynamic systems to be controlled have both parametric and dynamic uncertainties. For instance, robot manipulators may carry large objects with unknown inertial parameters. Power systems may be subjected to large variations in loading conditions. Fire-fighting aircraft may experience considerable mass changes as they load and unload large quantities of water. Adaptive control is an approach for controlling such systems. Today, techniques from adaptive control are being used to augment many of the existing controllers that have already a proven performance for a certain range of parameters, and adaptation is being used to improve the performance beyond those limits. The basic idea in adaptive control is to estimate the unknown parameters on-line based on measured system signals, and use the estimated parameters in the control input computation. An adaptive control system can be thus regarded as a control system with on-line parameter estimation. Since adaptive control systems are inherently nonlinear, their design and analysis is strongly connected with Lyapunov stability theory.

Research in adaptive control started in the early 1950's in connection with the design of autopilots for high-performance aircraft, which operate at a wide range of speeds and altitudes and thus experience large parameter variations. Adaptive control was proposed as a way of automatically adjusting the controller parameters in the face of changing aircraft dynamics. X-15 was the first aircraft tested with an adaptive autopilot in 1967.

X-15 was 50 feet long and had a wingspan of 22 feet. It weighed 33,000 pounds at launch and 15,000 pounds empty. Its flight control surfaces were hydraulically actuated and included all-moveable elevators, upper and lower rudders, speed brakes on the aft end of the fixed portion of the vertical fins, and landing flaps on the trailing edge of the wing. There were no ailerons; roll control was achieved by differential deflection of the horizontal tail.

All three X-15's were delivered with simple rate-feedback damping in all axes. The number three X-15, however, was extensively damaged during a ground run before it ever flew; when it was rebuilt it was fitted with a self-adaptive flight control system which included command augmentation, self-adaptive damper gains, several autopilot modes, and blended aerodynamic and ballistic controls.

The next flight resulting in the heat damage occurred in October 1967. In November of that year X-15 No.3 launched on what was planned to be a routine research flight to evaluate a boost guidance system and to conduct several other follow-on-experiments. During the boost, the airplane experienced an electrical problem that affected the flight control system and inertial displays. At peak altitude, the

X-15 began a yaw to the right, and it re-entered the atmosphere yawed crosswise to the flight path. It went into a spin and eventually broke up at 65,000 feet, killing the pilot Michael Adams. It was later found that the online adaptive control system was to be blamed for this incident.

Since the crash of X-15 more attention has been paid to stability analysis and robustness of adaptive controllers. While the main cause of that crash was parameter drift, as found out later, it was apparent that adaptive control theory was not ready for another flight test for the next 30 years.

It is only in the last decade that the interest in practical applications of adaptive control theory has revived due to the recent developments in *nonlinear control* theory. The theoretical advances along with the breakthroughs in computational technologies, availing cheap computation, have facilitated many applications of adaptive control, as in robotic manipulators, aircraft and rocket control, chemical processes, power systems, ship steering, bioengineering, etc. JDAM (Joint Direct Attack Munitions) flight tests and RESTORE program of highly unstable tailless aircraft X-36 are two examples from the aerospace community of successful application of adaptive control theory.

To get some practical insights into the control area that we are going to explore during this semester on a rigorous foundation, I would recommend you to get started from this simple example:

$$\dot{x}(t) = ax(t) + u(t), \quad x(0) = x_0\,.$$

Imagine that a is unknown, but you know certain bounds for it $a_{\min} \le a \le a_{\max}$, and you are interested in stabilization. Then a natural straightforward control choice would be

$$u_{lin}(t) = -kx(t), \quad k > a_{\max}\,,$$

leading to

$$\dot{x}(t) = -(k-a)x(t)\,.$$

You can simulate this system to convince yourself that it yields stabilization. However, there are two deficiencies to notice with this design:

- First, it is conservative. You need to know a conservative $a_{\max}$ to make sure that you select a control gain $k > a_{\max}$ to achieve a negative pole in the system $\dot{x}(t) = -(k-a)x(t)$. What if your estimation of $a_{\max}$ was not good enough for the entire flight envelope, and your choice of k fails to satisfy $k > a_{\max}$ all over sudden? Then obviously this control design will lead to instability.

- Second, to be on the safe side notice that you can select sufficiently conservative $a_{\max}$ to avoid the pitfall above. But this will require *maximum* control effort for the entire flight regime (leading to exhaustion of hardware), which might not be necessary if everything goes smooth (there are no winds, disturbances, etc.)

Adaptive control provides an alternative design for a case like this, and my recommendation would be for you to simulate an example as simple as this by selecting arbitrary numbers to convince yourself. Adaptive control offers the following solution

$$u_{ad}(t) = k(t)x(t),$$

where

$$\dot{k}(t) = -\gamma x^2(t)\,,$$

where $\gamma > 0$ is called adaptation rate and can be selected by you to be any positive number. This leads to the following closed-loop system instead:

$$\dot{x}(t) = a_m x(t) + \underbrace{(k(t) + a - a_m)}_{\tilde{k}(t)} x(t)\,, \quad a_m < 0\,,$$

where a_m is introduced (added and subtracted from the original system) to model the desired transient behavior (overshoot, settling time, etc.). Then you do not need even to know the bounds for a, and you can achieve stabilization independent of it. Thus, a time-varying $k(t)$ with a *correctly selected adaptation law* can achieve stabilization at a much lesser expense than a fixed-gain controller designed for conservative bounds for the unknown parameters. Try to code a simple system and plot both control signals to convince yourself in this. If you run into problems, I would be happy to share my codes with you.

The simple adaptive controller provided herein is an *inverse Lyapunov design*, i.e. it is derived from Lyapunov like analysis. To get familiar with it, we will start from the very beginning, foundations of systems of differential equations and analysis of the solutions of such systems.

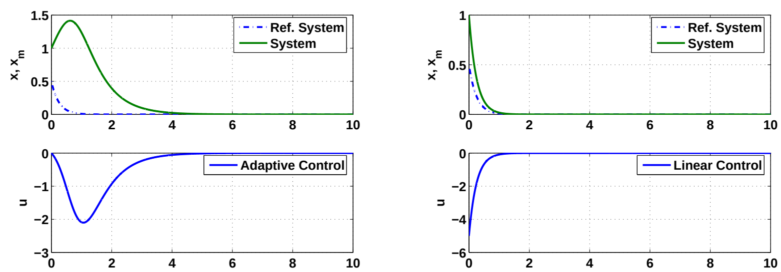


Fig. 1 Control history and tracking for $a = 1,\ a_m = -4,\ \gamma = 1,\ k = 5,\ x(0) = 1,\ x_m(0) = 0.5$**.**

1 Mathematical Preliminaries

Reading [11], pp. 87-95, and Appendices A, B & C.

1.1 Dynamical Systems, Existence and Uniqueness of Solutions, State Trajectory.

Dynamical systems are usually described via a system of differential equations

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is called state vector, $t \in [t_0, \infty)$ is the time-variable, and x_0 is the initial condition. The map $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is defined by the underlying physics of the problem, which, in most of the cases, is given via Newton's second law.

Definition 1.1. A continuous vector function $x : [t_0, t_1] \to \mathbb{R}^n$, satisfying $x(t_0) = x_0$, is called a solution of (1) over $t \in [t_0, t_1]$ if $\dot{x}(t)$ is defined $\forall\, t \in [t_0, t_1]$ and $\dot{x}(t) = f(t, x(t))\,, \forall\, t \in [t_0, t_1]$.

Example 1.1. The system

$$\dot{x}(t) = \sqrt{x(t)}\,, \quad x(0) = 0\,, \quad x \in \mathbb{R}^+\,, \quad t \geq 0 \tag{2}$$

has at least two solutions $x \equiv 0$ and $x(t) = \frac{1}{4}t^2$.

Question 1.1. How many solutions does the system in (1) have?

Lemma 1.1 (Theorem 3.1, p.88 [11]) If $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x, i.e.

$$\|f(t, x_1) - f(t, x_2)\| \leq L_r \|x_1 - x_2\|,\ \forall\, x_1, x_2 \in \mathcal{B}_r = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\},\ \forall\, t \in [t_0, t_1]\,,\ L_r > 0 \tag{3}$$

then there exists $\delta > 0$ such that the dynamical system in (1) has a <u>unique solution</u> for $t \in [t_0, t_0 + \delta]$.* (Read the proof in Appendix 1 of [11].)

*The Lipschitz constant L_r is indexed with r to indicate that it may change dependent on r. Assuming that beyond this point there should be no confusion with this, the Lipschitz constant will be indexed only when necessary.

The unique solution of system (1) defines the state trajectory. Since for every (t_0, x_0) there is a unique solution, it makes sense to denote this unique solution by $x(t, t_0, x_0)$. However, in most of the cases we will drop the initial conditions and simply write $x(t)$, unless there is a need to particularly specify the underlying initial conditions.

Several points about the phrasing of Lemma 1.1 are important to observe.

- The constant L_r in (3) is called *Lipschitz* constant, and the function $f(t,x)$ satisfying (3) is *Lipschitz* in x over the compact set $\mathcal{B}_r = \{x \in \mathbb{R}^n \mid \|x - x_0\| \le r\}$ *uniformly* in t over $t \in [t_0, t_1]$. In case if the condition in (3) holds for all $x \in \mathbb{R}^n$ with a universal constant L, then the function $f(t,x)$ is *globally Lipschitz* in x uniformly in t over $t \in [t_0, t_1]$. Notice that although the condition in (3) holds uniformly for all $t \in [t_0, t_1]$, the existence of a unique solution is guaranteed only for $t \in [t_0, t_0 + \delta]$, where $t_0 + \delta \le t_1$.

- It is also important to distinguish between open and closed sets, or otherwise saying between domains and compact sets. For example, the function $f(x) = \tan(x)$ is Lipschitz for every compact subset of $(-\pi/2, \pi/2)$, but it is not Lipschitz for $[-\pi/2, \pi/2]$. In such cases, one can say that $f(x) = \tan(x)$ is *locally Lipschitz* for $x \in (-\pi/2, \pi/2)$, because for every point of $x \in (-\pi/2, \pi/2)$ there is a constant L that would verify (3) in some neighborhood of that point, but there is no *universal* constant L that would serve the purpose for all the points of $x \in (-\pi/2, \pi/2)$. However, for arbitrary $0 < \epsilon \ll 1$, one can say that $f(x) = \tan(x)$ is Lipschitz on $x \in [-\pi/2 + \epsilon, \pi/2 - \epsilon]$, as a universal Lipschitz constant can be determined for that closed set $x \in [-\pi/2 + \epsilon, \pi/2 - \epsilon]$. (Try to determine that universal Lipschitz constant and prove that it depends upon ϵ!)

- If $f : \mathbb{R} \to \mathbb{R}$, then the Lipschitz condition can be written as:
$$\frac{|f(y) - f(x)|}{|y - x|} \le L\,.$$
This implies that L characterizes an upper bound for the slope of the function connecting the points x and y. If a function $f(x)$ has a bounded derivative $|f'(x)| \le k$ over $x \in [a, b]$, then it is Lipschitz on that interval with a Lipschitz constant k.

- Notice that as a notion Lipschitz continuity characterizes a smoothness property for the function over a domain. If a function is continuous on a compact set, then it is uniformly continuous![†] However, it may not be Lipschitz continuous. The function $x^{1/3}$ is uniformly continuous for $x \in [-1, 1]$, but is not locally Lipschitz (has unbounded derivative at the origin $x = 0$).

- In Example 1.1, the right hand side $\sqrt{x}$ is not differentiable at $x = 0$ (it has infinite slope!), hence it is not Lipschitz, and therefore the system in (1) is not guaranteed to have a unique solution with the given initial condition $x(0) = 0$. Notice that if one changes the initial condition to $x(0) = x_0$, where $x_0 \in \mathbb{R}^+$ is any positive number different from zero, then Lemma 1.1 states that there exists a neighborhood of x_0 and some $\delta > 0$ such that the system in (1) has a unique solution given by $x(t) = \frac{1}{4}(t + 2\sqrt{x_0})^2$ in that neighborhood of x_0 over the interval $t \in [0, \delta]$.

The next lemma relates the bound on the Jacobian of a function to its Lipschitz constant.

Lemma 1.2. Let $f(t, x) : [t_0, t_1] \times \mathcal{D} \to \mathbb{R}^n$ be continuous on the domain $\mathcal{D} \subset \mathbb{R}^n$. Suppose that $\frac{\partial f}{\partial x}(t, x)$ exists and is continuous on some compact subset $\Omega \subset \mathcal{D}$. Further assume that the Jacobian is uniformly bounded

$$\left\| \frac{\partial f}{\partial x}(t, x) \right\| \leq L$$

for all $[t_0, t_1] \times \Omega$. Then

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $t \in [t_0, t_1]$, $x, y \in \Omega$. (The proof can be found in [11], p.90.)

Lemma 1.3. If $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous on $[t_0, t_1] \times \mathbb{R}^n$, then f is globally Lipschitz in x *if and only if* $\frac{\partial f}{\partial x}(t, x)$ is uniformly bounded on $[t_0, t_1] \times \mathbb{R}^n$. (Try to prove it on your own!)

Remark 1.1. Notice that Lemma 1.3 states necessary and sufficient conditions for a class of systems that are continuous and have continuous derivative. In general, continuity of the derivative is

[†] A function $f(x) : \mathbb{R} \to \mathbb{R}$ is continuous at a point x_0, if for any $\epsilon > 0$ there exists $\delta(\epsilon, x_0)$ such that if $|x - x_0| \leq \delta(\epsilon, x_0)$, then $|f(x) - f(x_0)| \leq \epsilon$. If δ can be chosen independent of x_0 for all x from a given domain $\mathcal{D}$ so that for any $\epsilon > 0$ there exists $\delta(\epsilon)$ such that if $|x - y| \leq \delta(\epsilon)$ then $|f(x) - f(y)| \leq \epsilon$ for all $x, y \in \mathcal{D}$, then the function $f(x)$ is uniformly continuous on $\mathcal{D}$. Notice that uniform continuity is defined for a domain and/or set and cannot be defined for a point.

not required for the function to be Lipschitz. A Lipschitz function may have a discontinuous but bounded derivative. For example, $|x|$ is continuous, has discontinuous derivative at $x = 0$, but is globally Lipschitz with Lipschitz constant 1. The function $x^{1/3}$ is continuous at $x = 0$, but is not Lipschitz (has unbounded derivative). Roughly saying, the Lipschitz property of a function implies more than continuity, but less than continuous differentiability.

The next lemma states that a continuously differentiable function is locally Lipschitz.

Lemma 1.4. If $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous on $[t_0, t_1] \times \mathcal{D}$ for some open domain $\mathcal{D} \subset \mathbb{R}^n$, then f is locally Lipschitz in x on $[t_0, t_1] \times \mathcal{D}$. (Try to prove it on your own!)

Example 1.2. The scalar system

$$\dot{x}(t) = -x^2(t), \quad x(0) = -1 \tag{4}$$

has a locally Lipschitz right hand side for all $x \in \mathbb{R}$. The unique solution of it

$$x(t) = \frac{1}{t-1}$$

however is defined only for $t \in [0, 1)$. As $t \to 1$, $x(t) \to \infty$. This phenomenon is known as *finite escape time*, as x leaves any compact set <u>in finite time</u>.

So, this brings up the notion of the <u>maximum interval of existence of the solution</u>. Notice that in Lemma 1.1 the Lipschitz condition was required for all $x \in \mathcal{B}$ and for all $t \in [t_0, t_1]$, but the existence of a unique solution was guaranteed only for $t \in [t_0, t_0 + \delta]$ for some $\delta > 0$, $t_0 + \delta \leq t_1$.

Theorem 1.1. Suppose that $f(t, x)$ is piecewise continuous in t and is <u>globally Lipschitz</u> in x, i.e. $\exists\ L > 0$, such that

$$||f(t, x) - f(t, y)|| \leq L||x - y||$$

for all $x, y \in \mathbb{R}^n$, uniformly $\forall\ t \in [t_0, t_1]$. Then the system in (1) has a unique solution over $[t_0, t_1]$, where t_1 maybe arbitrarily large. (For the proof read Appendix C1 of [11].)

As stated in Lemma 1.3, a global Lipschitz condition for a function $f(x)$ is equivalent to having a globally bounded derivative. In the light of this, naturally the statement in Theorem 1.1 appears to be conservative. However, this should not scare, since it only states a sufficient condition. Indeed, the next example demonstrates that the function may not be globally Lipschitz, while a unique solution may exist for all $t \geq 0$.

Example 1.3. The system

$$\dot{x}(t) = -x^3(t)\,, \quad x(t_0) = x_0$$

does not have a globally Lipschitz right hand side, since $f(x) = -x^3$ has a derivative $f'(x) = x^2$, which is not bounded as $x \to \infty$. However, it has a unique solution well defined for all $t \geq 0$

$$x(t) = \operatorname{sgn}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}\,.$$

Remark 1.2. While it is reasonable to expect that most of the physical systems will have a *locally Lipschitz* right hand side, it is indeed too restrictive to expect *globally Lipschitz* right hand side. This assumption in most of the cases will not be verified.

Question 1.2. When can existence of a unique solution be guaranteed for all $t \geq 0$ in the presence of *locally Lipschitz* right hand side for the system in (1)?

Theorem 1.2. Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $\mathcal{D} \subset \mathbb{R}^n$. Let Ω be a compact subset of $\mathcal{D}$ such that $x_0 \in \Omega$, and suppose it is known that every solution of the system (1) lies entirely in Ω. Then, there is a unique solution that is defined for all $t \geq t_0$. (Read the proof in [11].)

Remark 1.3. It seems that there is no free lunch. One needs to know something about the behavior of the solution apriori, to be able to conclude its existence for all $t \geq 0$. As we will demonstrate next, this is not as scary as it may seem from the first glance.

Example 1.4. Getting back to the same example

$$\dot{x}(t) = -x^3(t)\,, \quad x(t_0) = x_0$$

notice that $f(x)$ is locally Lipschitz on $\mathbb{R}$. Moreover, if at any time instant $x(t)$ is positive, then its derivative will be negative, and vice versa. Therefore starting from any initial condition $x(0) = x_0$, the solution cannot leave the compact set $\{x \in \mathbb{R} \mid |x| \leq |x_0|\}$. Thus, without solving it explicitly, it is possible to apply Theorem 1.2 to draw a conclusion on the existence of a unique solution for all $t \geq 0$.

Remark 1.4. The example implies that one can verify the assumption in Theorem 1.2 without solving the system of differential equations. This type of philosophy lies in the basis of Lyapunov stability theorems that we will introduce in the subsequent sections and use intensively for analysis of adaptive control systems.

Lemma 1.5. If $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz, then $f_1 + f_2$, $f_1 f_2$ and $f_1 \circ f_2$ are locally Lipschitz. (Try to prove on your own as you will need it for your homework!)

The solution of the next exercise should help you to solve the homework problems.

Exercise 1.1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined as

$$f(x) = \begin{cases} \frac{1}{||Kx||} Kx, & \text{if} \quad g(x)||Kx|| \geq \mu > 0 \\ \frac{g(x)}{\mu} Kx, & \text{if} \quad g(x)||Kx|| < \mu \end{cases}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and nonnegative, and K is a constant matrix. Prove that $f(x)$ is Lipschitz on any compact subset of $\mathbb{R}^n$.

Solution 1.1. The function f can be written as $f(x) = g(x)\ Kx\ h(\psi(x))$, where

$$h(\psi) = \begin{cases} \frac{1}{\psi}, & \text{if} \quad \psi \geq \mu > 0 \\ \frac{1}{\mu}, & \text{if} \quad \psi < \mu \end{cases}$$

where $\psi(x) = g(x)||Kx||$. The norm function $||Kx||$ is Lipschitz since $\Big| ||Kx|| - ||Ky|| \Big| \leq ||Kx - Ky|| \leq ||K||\ ||x-y||$. From Lemma 1.5 it follows that $\psi(x)$ is Lipschitz on any compact set. Furthermore, $g(x)Kx$ is also Lipschitz. Thus, it remains only to show that $h(\psi)$ is Lipschitz in ψ over any compact set. Notice that $h(\psi)$ is a continuous, but non-smooth function. If $\psi_1 \geq \mu$ and $\psi_2 \geq \mu$, then we have

$$|h(\psi_2) - h(\psi_1)| = \Big|\frac{1}{\psi_2} - \frac{1}{\psi_1}\Big| = \Big|\frac{\psi_1 - \psi_2}{\psi_1 \psi_2}\Big| \leq \frac{1}{\mu^2}|\psi_2 - \psi_1|\,.$$

If $\psi_2 \geq \mu$ and $\psi_1 < \mu$, then

$$|h(\psi_2) - h(\psi_1)| = \Big|\frac{1}{\psi_2} - \frac{1}{\mu}\Big| = \frac{1}{\mu} - \frac{1}{\psi_2} = \frac{\psi_2 - \mu}{\mu\psi_2} \leq \frac{\psi_2 - \psi_1}{\mu\psi_2} \leq \frac{1}{\mu^2}|\psi_2 - \psi_1|\,.$$

Finally, if $\psi_1 \geq \mu$ and $\psi_2 < \mu$, then

$$|h(\psi_2) - h(\psi_1)| = \Big|\frac{1}{\mu} - \frac{1}{\mu}\Big| = 0 \leq \frac{1}{\mu^2}|\psi_2 - \psi_1|\,.$$

Thus, $h(\psi)$ is Lipschitz continuous with a Lipschitz constant $\frac{1}{\mu^2}$. Following Lemma 1.5, f is Lipschitz on any compact subset of $\mathbb{R}^n$.

Homework Problems 1.1. The following two exercises from [11] need to be solved completely.

- Exercise 3.2 from [11]. Solve any 4 problems out of given 6. Make sure that you include the 5^{th} problem on adaptive control in your selection, which is however in Section 1.2.6 and not 1.2.5, as indicated in the book.

- Exercise 3.5 from [11] (Hint: use the well known property of norms $c_1\|x\|_\alpha \leq \|x\|_\beta \leq c_2\|x\|_\alpha$).

1.2 Continuous Dependence on Initial Conditions, System Parameters and Comparison Principle.

Reading [11], pp. 95-110, and Appendices A, B & C.

When dealing with dynamical systems, it is quite natural to expect that the model equations that we derive have some kind of errors in parameters due to the approximations that we are making. Thus, a natural question arises if in this process of dealing with approximate equations we are drifting too far from the nominal motion of interest. Mathematically phrased, this question can be stated as follows.

Question 1.3. Under what type conditions the unique solution to the differential equation

$$\dot{x}(t) = f(t, x(t), \lambda)\,, \quad x(t_0) = x_0 \tag{5}$$

depends continuously on initial conditions t_0, x_0 and the parameters of the problem λ?

The equation in (5) can be otherwise presented as

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau), \lambda)d\tau \tag{6}$$

To prove that $x(t)$ continuously depends upon the initial time constant t_0, we need to recall Gronwall-Bellman lemma, which permits to derive an explicit bound for a function from an implicit inequality.

Lemma 1.6 (*Gronwall-Bellman lemma*) Let $\phi : [t_0, t_1] \to \mathbb{R}$ be continuous and $\psi : [t_0, t_1] \to \mathbb{R}$ be continuous and nonnegative. If a continuous function $y : [t_0, t_1] \to \mathbb{R}$ satisfies

$$y(t) \le \phi(t) + \int_{t_0}^{t} \psi(s)y(s)ds$$

for all $[t_0, t_1]$, then

$$y(t) \le \phi(t) + \int_{t_0}^{t} \phi(s)\psi(s)\exp\left[\int_{s}^{t} \psi(\tau)d\tau\right] ds$$

for all $[t_0, t_1]$. In particular, if $\phi(t) = \phi = const$, then

$$y(t) \le \phi \exp\left[\int_{s}^{t} \psi(\tau)d\tau\right] .$$

If in addition, $\psi(t) = \psi = const > 0$, then

$$y(t) \le \phi \exp\left[\psi(t - t_0)\right] .$$

(The proof can be found in [11], p.652.)

Before we give a complete answer to **Question 1.3**, let's prove continuous dependence of the unique solution upon t_0 first. Assume that (5) satisfies the assumptions for existence and uniqueness of a solution so that for every (t_0, x_0) there is a unique solution $x(t, t_0, x_0)$. Towards that end consider the following two solutions:

$$x(t,t_0,x_0) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau,t_0,x_0),\lambda)d\tau, \quad x(t,t_0',x_0) = x_0 + \int_{t_0'}^{t} f(\tau, x(\tau,t_0',x_0),\lambda)d\tau\,, \quad t_0' > t_0\,.$$

Then

$$x(t,t_0,x_0) - x(t,t_0',x_0) = \int_{t_0}^{t_0'} f(\tau, x(\tau,t_0,x_0),\lambda)d\tau + \int_{t_0'}^{t} \left[f(\tau, x(\tau,t_0,x_0),\lambda) - f(\tau, x(\tau,t_0',x_0),\lambda)\right] d\tau$$

Since the system in (5) satisfies the assumptions for existence and uniqueness of a solution, then for a given time interval $[t_0, t_1]$, we have that $||f(t, x(t), \lambda)|| \le M$ (from piece-wise continuity in t) and $||f(t, x, \lambda) - f(t, y, \lambda)|| \le L||x - y||$ (from Lipschitz continuity in x).[‡] Therefore

$$||x(t,t_0,x_0) - x(t,t_0',x_0)|| \le M(t_0' - t_0) + \int_{t_0'}^{t} L||x(\tau,t_0,x_0) - x(\tau,t_0',x_0)||d\tau\,.$$

By Gronwall-Bellman inequality

$$||x(t,t_0,x_0) - x(t,t_0',x_0)|| \le M(t_0' - t_0)\exp(L(t - t_0'))\,.$$

Hence, if on any compact interval $t \in [t_0, t_1]$ of time for arbitrary $\epsilon > 0$ one chooses $\delta(\epsilon) = \dfrac{\epsilon}{M\exp(L(t-t_0'))}$, then if $|t_0' - t_0| \le \delta$, one has $||x(t,t_0,x_0) - x(t,t_0',x_0)|| \le \epsilon$.

Theorem 1.3. [Bounded Perturbations of the State Equation.] Let $f(t, x)$ be piecewise continuous in t and Lipschitz in x on $[t_0, t_1] \times \mathcal{D}$ with Lipschitz constant L, where $\mathcal{D} \subset \mathbb{R}^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t, y), \quad y(t_0) = y_0$$

and

$$\dot{z} = f(t, z) + g(t, z), \quad z(t_0) = z_0\,,$$

[‡]Since we are interested in establishing the continuous dependence on t_0, then there is no loss of generality in fixing λ for the time being.

which are contained in $\mathcal{D}$ over the entire time interval $[t_0, t_1]$. Suppose that

$$\|g(t,x)\| \leq \mu, \quad \forall (t,x) \in [t_0, t_1] \times \mathcal{D}$$

for some $\mu > 0$. Then

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| \exp[L(t-t_0)] + \frac{\mu}{L}\left(\exp[L(t-t_0)] - 1\right), \quad \forall t \in [t_0, t_1].$$

Proof: The solutions $y(t)$ and $z(t)$ are given by

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$

$$z(t) = z_0 + \int_{t_0}^{t} [f(s, z(s)) + g(s, z(s))]ds$$

Subtracting and upper bounding

$$\begin{aligned} ||y(t) - z(t)|| &\leq ||y_0 - z_0|| + \textstyle\int_{t_0}^{t} ||f(s, y(s)) - f(s, z(s))||ds + \int_{t_0}^{t} ||g(s, y(s))||ds \\ &\leq \gamma + \mu(t - t_0) + \textstyle\int_{t_0}^{t} L||y(s) - z(s)||ds\,, \end{aligned}$$

where $\gamma = ||y_0 - z_0||$. Notice that we can apply Bellman-Gronwall lemma to derive an explicit bound for $||y(t) - z(t)||$:

$$||y(t) - z(t)|| \leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L\left[\gamma + \mu(s - t_0)\right] \exp[L(t - s)]ds$$

Integrating the right hand side by parts completes the proof. $\square$

The details can be found in [11]. The essence of the theorem is that a bounded perturbation of the state equation results in *at worst* an exponentially diverging difference in system trajectories. However, at least over a finite time interval the two solutions $y(t)$ and $z(t)$ are within a finite "distance" of one another.

Theorem 1.4. [Continuous Dependence on Initial Conditions and Parameters.] Let $f(x, \lambda, t)$ be continuous in its arguments and locally Lipschitz in x (uniformly in λ and t) over a domain $[t_0, t_1] \times \mathcal{D} \times \Lambda$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is an open connected set and $\Lambda = \{\lambda \mid \|\lambda - \lambda_0\| \leq c\}$ for some $c > 0$. Let $y(t, \lambda_0)$ be a solution of

$$\dot{x} = f(t, x, \lambda_0), \quad y(t_0, \lambda_0) = x_0 \in \mathcal{D}.$$

Further assume that $y(t, \lambda_0)$ is defined for all $[t_0, t_1]$ and belongs to $\mathcal{D}$ for all $[t_0, t_1]$. Then, for any $\epsilon > 0$ there exists $\delta > 0$ such that if

$$\|z_0 - y_0\| < \delta \quad \text{and} \quad \|\lambda - \lambda_0\| < \delta,$$

there is a unique solution $z(t, \lambda)$ of

$$\dot{x} = f(t, x, \lambda), \quad z(t_0, \lambda) = z_0,$$

defined on $[t_0, t_1]$ that satisfies

$$\|z(t, \lambda) - y(t, \lambda_0)\| < \epsilon$$

for all $t \in [t_0, t_1]$.

Proof: Since $y(t, \lambda_0)$ continuously depends upon t on the time interval $[t_0, t_1]$, then it is bounded for all $t \in [t_0, t_1]$. Define a tube around the solution $y(t, \lambda_0)$ by

$$\mathcal{U} = \{(t, x) \in [t_0, t_1] \times \mathbb{R}^n \mid ||x - y(t, \lambda_0)|| \leq \epsilon\}$$

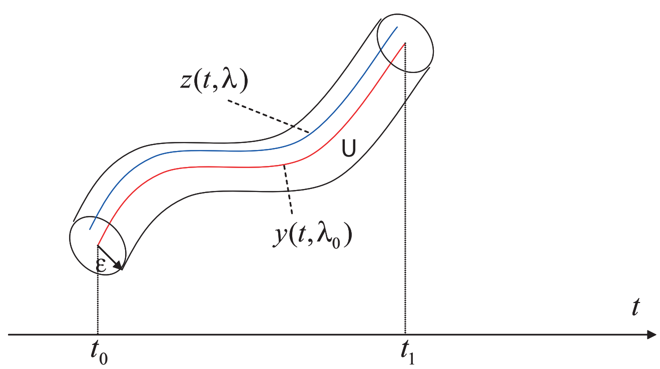


Fig. 2 Continuous dependence on initial states and parameters

Assume that ϵ is selected enough small so that $\mathcal{U} \subset [t_0, t_1] \times \mathcal{D}$. Since $\mathcal{U}$ is a compact subset of $\mathcal{D}$, then $f(t, x, \lambda)$ is Lipschitz in x on $\mathcal{U}$ with a Lipschitz constant L. Since f is continuous in λ, then for any α there is $\beta < c$ such that

$$||f(t, x, \lambda) - f(t, x, \lambda_0)|| < \alpha, \quad \forall\ (t, x) \in \mathcal{U}, \quad \forall\ ||\lambda - \lambda_0|| < \beta$$

Let's choose $\alpha \leq \epsilon$ and $||z_0 - y_0|| < \alpha$. Following the local existence and uniqueness result (Lemma 1.1), there exists a unique solution $z(t, \lambda)$ for some $[t_0, t_0 + \Delta]$. Since this solution starts in the tube, then Theorem 1.2 states that as long as the solution remains inside the tube, it can be extended for all $t \in [t_0, t_1]$. To prove this, assume that τ is the first time interval when $z(t, \lambda)$ leaves the tube, and let's prove that $\tau > t_1$. On the time interval $[t_0, \tau]$, the conditions of Theorem 1.3 are satisfied with $\mu = \alpha$. Hence,

$$||z(t,\lambda) - y(t,\lambda_0)|| < \alpha \exp[L(t-t_0)] + \frac{\alpha}{L}\left(\exp[L(t-t_0)] - 1\right) < \alpha\left(1 + \frac{1}{L}\right)\exp[L(t-t_0)]$$

Choosing $\alpha \leq \epsilon L \exp[-L(t_1 - t_0)]/(1 + L)$ ensures that the solution is not leaving the tube during the interval $[t_0, t_1]$. Therefore $z(t, \lambda)$ is defined for all $[t_0, t_1]$ and satisfies $||z(t, \lambda) - y(t, \lambda_0)|| < \epsilon$. Taking $\delta = \min\{\alpha, \beta\}$ completes the proof. □

The theorem states that under the given assumptions for any "acceptable range" around the nominal trajectory $y(t, \lambda_0)$ (i.e., a tube of arbitrarily small radius ϵ), there is an appropriately small range of initial states and parameters which give rise to trajectories that lie entirely inside the tube.

Talking of qualitative behavior of the solutions of the differential equations, a natural question to ask would be:

Question 1.4. Can one compute the bounds for the solution of $\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0$, without actually solving the equation?

Recall that Gronwall-Bellman lemma was one tool for addressing this issue. Now we will state the Comparison lemma as an alternative tool for it.

Lemma 1.7. [Comparison Lemma] Consider the scalar nonlinear system

$$\dot{x} = f(x,t), \qquad x(t_0) = x_0$$

where f is locally Lipschitz in x and continuous in t for all $t \geq 0$. Let $[t_0, T)$ be the maximal interval of existence of the solution $x(t)$ (where T could be infinity). Let $v(t)$ be a $\mathcal{C}^1$ function which satisfies the differential inequality

$$\dot{v} \leq f(v,t), \qquad v(t_0) \leq x_0.$$

Then $v(t) \leq x(t)$ for all $t \in [t_0, T)$.

In the statement of the lemma the requirement $\mathcal{C}^1$ on $v(t)$ can be relaxed, but to follow up on it you need familiarity with right and left derivatives from an advanced calculus course. For that general statement of the lemma, as well as a proof, see [11].

Example 1.5. Consider the first-order nonlinear equation

$$\dot{x} = -x - \tanh x, \qquad\qquad x(0) = x_0.$$

It is easy to see that this system has an equilibrium at the origin. We would like to know something about the other trajectories of the system, that is, those for which $x_0 \neq 0$. Consider a special function

$$V(x(t)) = \frac{1}{2}x^2(t)$$

which essentially measures the "squared distance" to x from the origin. Taking the derivative gives

$$\dot{V}(t) = x(t)\dot{x}(t) = x(t)\left(-x(t) - \tanh x(t)\right). \tag{7}$$

Notice that

$$\dot{V}(t) \leq -x^2(t) = -2V(x(t)). \tag{8}$$

This is a first order differential *inequality* with the initial condition $V(0) = V_0 = \frac{1}{2}x_0^2$. We may use the comparison lemma to bound the value of $V(t)$ by solving the differential *equation*

$$\dot{U}(t) = -2U(t), \quad U(0) = V_0.$$

Following the Comparison Lemma 1.7, we find that

$$V(t) \leq U(t) = V_0 e^{-2t}. \tag{9}$$

Now recall that $V(t) = \frac{1}{2}x^2(t)$. Then it is immediate from (9) that

$$|x(t)| \leq |x_0|e^{-t}.$$

That is, trajectories converge to the origin at least as fast as e^{-t}. We may therefore conclude that the equilibrium at the origin is *stable* (a notion which remains to be precisely defined). In fact, there is a general approach to stability analysis which is based on studying the behavior of a simple, scalar function such as V. The major strengths of the approach, known as Lyapunov stability analysis, are that it applies to systems of arbitrarily high order and it does not require explicit solution of the differential equations (something which is generally difficult or impossible for nonlinear systems).

The following exercises should help you with homework problems.

Exercise 1.2. Using Comparison Lemma, find an upper bound on the solution of the state equation

$$\dot{x}(t) = -x(t) + \frac{\sin t}{1 + x^2(t)}, \quad x(0) = 2.$$

Solution 1.2. Let $V(t) = \frac{1}{2}x^2(t)$. Then

$$\dot{V}(t) = x(t)\dot{x}(t) = -x^2(t) + \frac{x(t)\sin(t)}{1 + x^2(t)} \leq -2V(t) + 1\,.$$

Notice that $V(0) = 2$. To apply the comparison principle, let's look at this differential equation

$$\dot{u}(t) = -2u(t) + 1\,, \quad u(0) = 2\,.$$

Integrating by parts implies that

$$\int_0^u \frac{du}{1 - 2u} = \int_0^t dt + C\,,$$

where C is the integration constant and needs to be determined from the initial condition. Thus

$$-\frac{1}{2}\ln(1 - 2u) = t + C\,.$$

Since $u(0) = 2$, then

$$u(t) = \frac{1 + 3\mathrm{e}^{-2t}}{2}$$

Comparison lemma leads to the following upper bound

$$V(t) \leq u(t) = \frac{1 + 3\mathrm{e}^{-2t}}{2}$$

Thus

$$x(t) = \sqrt{2V(t)} \leq \sqrt{1 + 3\mathrm{e}^{-2t}}\,.$$

Exercise 1.3. Let $x_1 : \mathbb{R} \to \mathbb{R}^n$ and $x_2 : \mathbb{R} \to \mathbb{R}^n$ be differentiable functions such that

$$||x_1(a) - x_2(a)|| \leq \gamma, \quad ||\dot{x}_i(t) - f(t, x_i(t))|| \leq \mu_i, \quad i = 1, 2$$

for $a \leq t \leq b$. If f is Lipschitz continuous with Lipschitz constant L, then show that

$$||x_1(t) - x_2(t)|| \leq \gamma \mathrm{e}^{L(t-a)} + (\mu_1 + \mu_2)\left[\frac{\mathrm{e}^{L(t-a)} - 1}{L}\right], \quad a \leq t \leq b\,.$$

Solution 1.3. Let $y(t) = x_1(t) - x_2(t)$ and $\mu = \mu_1 + \mu_2$. Then

$$||\dot{y}(t)|| = ||\dot{x}_1(t) - \dot{x}_2(t) - f(t, x_1(t)) + f(t, x_1(t)) - f(t, x_2(t)) + f(t, x_2(t))||$$

Using the conditions of the problem, this can be upper bounded

$$||\dot{y}(t)|| \leq \mu_1 + \mu_2 + L||x_1(t) - x_2(t)|| = \mu + L||y(t)||\,,$$

which further leads to the following

$$||y(t)|| = \left\| y(a) + \int_a^t \dot{y}(s) ds \right\| \leq \gamma + \int_a^t ||\dot{y}(s)|| ds \leq \gamma + \mu(t-a) + \int_a^t L||y(s)|| ds$$

Gronwall-Bellman inequality yields

$$||y(t)|| \leq \gamma + \mu(t-a) + \int_a^t [\gamma + \mu(s-a)] L \mathrm{e}^{L(t-s)} ds\,.$$

Integrating the right-hand side by parts, and recalling that $y(t) = x_1(t) - x_2(t)$, $\mu = \mu_1 + \mu_2$, yields:

$$||x_1(t) - x_2(t)|| \leq \gamma \mathrm{e}^{L(t-a)} + (\mu_1 + \mu_2) \left[\frac{\mathrm{e}^{L(t-a)} - 1}{L} \right], \quad a \leq t \leq b\,.$$

Homework Problems 1.2. The following exercises from [11] need to be solved completely.

- Exercise 3.17.
- Exercise 3.24.
- Exercise 3.30.

2 Lyapunov Stability Theory for Autonomous Systems

Reading [11], pp. 111-133, Appendix C, and also [17], pp. 41-76.

2.1 System Equilibrium and the Notion of Stability for Time-Invariant Systems.

Consider the following general nonlinear system dynamics:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \tag{10}$$

$$y(t) = h(x(t)) \tag{11}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the system output, and f is locally Lipschitz continuous in x and u and piece-wise continuous in t. The control input is usually selected in a way so that a specific control objective is met. Commonly, this is a tracking objective for system states or output $y(t)$. Before introducing control design methods, let's focus on the unforced dynamics and introduce the notion of equilibrium and stability.

Towards that end, first we consider autonomous system dynamics in the following form:

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad x \in \mathbb{R}^n, \tag{12}$$

where f is locally Lipschitz in x. Recall that the system dynamics are called *autonomous*, if the right hand side *does not depend explicitly upon t*. Notice though that it implicitly depends upon t via $x(t)$. Since f does not depend explicitly upon t, it is quite fair to write the system dynamics for autonomous systems equivalently in the form:

$$\dot{x} = f(x), \quad x(0) = x_0, \quad x \in \mathbb{R}^n. \tag{13}$$

Definition 2.1. The system state $x = x_e$ is called equilibrium for (13), if once $x(t') = x_e$, it remains equal to x_e for all future time $t \geq t'$ on the maximum interval of existence of the solution.

Mathematically this means that the constant vector x_e satisfies

$$f(x_e) = 0.$$

The equilibrium points of the system can be found by solving the nonlinear algebraic equations $f(x_e) = 0$. The next question to ask would be if the equilibrium is stable or unstable. A good example of this is the pendulum that has stable equilibrium at the bottom position and an unstable one at the upright position. So, how to define and distinguish in between various equilibria of the system is the next question that we are going to explore now. Before then, however, let's state the following remark.

Remark 2.1. A nonlinear system may have finite number of isolated equilibria ($\dot{x} = (x-1)(x-2)$), infinite equilibria ($\dot{x} = \sin x$), or no equilibria at all ($\dot{x} = x^2 + 5$), while a linear system like $\dot{x} = Ax$ has either a single isolated equilibrium at the origin (in case of non-singular A) or an infinite number of those (in case of singular A, and in this case every point of the null-space of A is an equilibrium).

Example 2.1. A linear system

$$\ddot{x} + \dot{x} = 0$$

has infinite (continuum) equilibria. Indeed, a simple change of variables $x_1 = x,\ x_2 = \dot{x}$ leads to the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2\,, \end{aligned}$$

the equilibria of which should be determined from equating the right hand side (RHS) of both equations to zero $x_2 = 0$, which does not depend on x_1, and hence for this second order system it implies that every point on the x axis is an equilibrium.

Example 2.2. Consider the dynamics of pendulum with damping, Fig. 3:

$$MR^2\ddot{\theta}(t) + bR\dot{\theta}(t) + Mg\sin(\theta(t)) = 0\,.$$

It has infinite (countable) equilibria. To see this, consider the change of variables $x_1 = \theta,\ x_2 = \dot{\theta}$ to arrive at the state space representation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{MR}x_2 - \frac{g}{R^2}\sin(x_1)\,, \end{aligned}$$

the equilibria of which should be determined from equating the right hand side (RHS) of both equations to zero leading to $x_2 = 0$ and $\sin(x_1) = 0$, which implies that the points on the x axis with coordinates

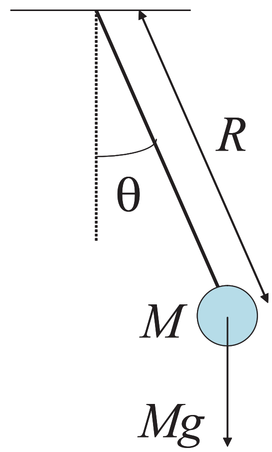


Fig. 3 Pendulum rotating in vertical plane

$[0,\ 0],\ [\pi,\ 0],\ [2\pi, 0],\ \cdots$ are the equilibria. These correspond to the top and bottom positions of the mass, one of those being stable equilibrium, while the other unstable.

In this and several subsequent lectures we will develop tools for stability analysis of the equilibria of the system. To streamline the subsequent material, without loss of generality, we will assume that the origin $x = 0$ is an equilibrium point for the system (13). If it is not, one can always consider change of variables to shift the arbitrary equilibrium to the origin. Indeed, assume that x^* is an equilibrium for (13), i.e. $f(x^*) = 0$, and consider change of variables $y(t) = x(t) - x^*$. Then $\dot{y}(t) = \dot{x}(t)$, since x^* is an equilibrium for the original system, and hence its derivative is zero. Thus, $\dot{y} = f(y + x^*)$, and the equilibrium is determined from $f(y + x^*) = 0$, implying that $y = 0$ is an equilibrium, since $f(x^*) = 0$.

Remark 2.2. Notice that if the original system is autonomous to begin with, like the one in (13), then from shifting the arbitrary equilibrium to the origin, the system still remains autonomous, i.e. no explicit dependence upon the time variable t appears from the change of variables in the right hand side of the modified dynamics. This is not true however if we are concerned about stability of a nominal *motion* for the same autonomous system.

To see this, let $x^*(t)$ be the nominal motion trajectory of (13) satisfying the initial condition $x^*(0) = x_0$. Let's perturb the initial condition $x(0) = x_0 + \delta x_0$ and denote the solution corresponding to this initial condition by $x(t)$. Let $e(t) = x(t) - x^*(t)$ be the error in between these two state trajectories. Since both $x(t)$ and $x^*(t)$ are solutions of the same equation (13), we can write

$$\dot{x}^* = f(x^*), \quad x^*(0) = x_0$$

$$\dot{x} = f(x), \quad x(0) = x_0 + \delta x_0$$

leading to the error dynamics

$$\dot{e} = \dot{x} - \dot{x}^* = f(x^*(t) + e) - f(x^*(t)) = g(e,t), \quad e(0) = \delta x_0\,, \tag{14}$$

where the explicit dependence upon t appeared due to the presence of $x^*(t)$, which corresponds to the nominal motion. Notice that $g(0,t) = 0$, which means that $e = 0$ is an equilibrium point for (14). Therefore instead of studying the deviation of $x(t)$ from the nominal motion $x^*(t)$, one can study the stability of the equilibrium at the origin for the error dynamics in (14). However, one needs to keep in mind that the error dynamics are *non-autonomous* despite the fact that the original system was *autonomous* to begin with, Fig. 4. This is a typical *nonlinear phenomenon*, and I recommend to convince yourself that for linear systems the error dynamics in (14) are still autonomous if the original linear system is autonomous to begin with!!!

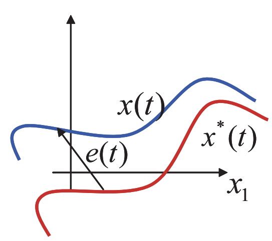


Fig. 4 The error changes with time

On the other hand, if one wants to keep everything in autonomous setting, then the dynamics in (14) needs to be combined with the $\dot{x}^*$ dynamics as:

$$\begin{aligned} \dot{e} &= f(x^*(t) + e) - f(x^*(t)) = g(e, x^*), \quad e(0) = \delta x_0\,, \\ \dot{x}^* &= f(x^*), \quad x^*(0) = x_0\,. \end{aligned}$$

The stability of the equilibrium of this system can be analyzed using tools for autonomous systems, however, it has higher order, and one needs to make sure that $\dot{x}^* = f(x^*)$ has equilibrium at the origin (one needs to shift the equilibrium of this system to the origin beforehand), so that the combined system has an equilibrium at the origin. This however will not eliminate the need for developing tools for non-autonomous systems, since these are more common in practice than autonomous systems. And for non-autonomous systems the nominal motion always satisfies a non-autonomous equation to begin with.

Another important feature of usually viewing equations and understanding their behavior, is to be precise about the functional dependencies. For example, $f(x) = x^3$ is an increasing function of x for all $x \in \mathbb{R}$, while if we view the same function on the system trajectories given by $x(t) = \frac{1}{t}$, then, as a function of t, $f(x(t))$ will be a decreasing function of t, since it equals $f(x(t)) = \frac{1}{t^3}$. Similarly, $f(t,x)$ may be increasing in both t and x, like $f(t,x) = t + x$, but along the trajectory $x(t) = -2t$, it appears to be decreasing function of t. Another example to consider would be $f_1(x_1) = x_1^2$, $x_1(t) = t^3$ and $f_2(x_2) = x_2^3$, $x_2(t) = t^2$, both leading to the same function of time $f_1(x_1(t)) = f_2(x_2(t)) = t^6$, but the underlying functions of x and t are completely different. In control problems it is very important to distinguish between functional dependencies carefully. In adaptive systems, this is even crucial, because as we will find out later through the course, when using neural networks for approximation of functions on compact sets, one ends up having some estimate of function approximation over time, but never in the space of x. It is important not to be confused in between two and always watch out for functional dependencies carefully.

We will henceforth introduce stability analysis methods for studying the stability of equilibria for both autonomous and non-autonomous systems. For the sake of systematic development, we will start with autonomous systems and consider the equilibrium at the origin.

Definition 2.2. The origin $x = 0$ of the system dynamics in (13) is a *locally stable equilibrium point* if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$||x(0)|| < \delta(\epsilon) \quad \Rightarrow \quad ||x(t)|| < \epsilon, \quad \forall\, t \geq 0\,.$$

Remark 2.3. Notice that f was required only to be locally Lipschitz in x, which implies that the existence of solution is guaranteed only for some Δt (see Lemma 1.1 and do not be confused to see here Δt instead of δ, since δ here has served different purpose), but the definition of stability is given for all $t \geq 0$. As you will shortly find out, Lyapunov's theorems for sufficient conditions on stability will be consistent with Theorem 1.2, requiring the solution to lie inside a domain. For definition of stability on the maximum interval of existence of solution refer to [15].

Definition 2.3. The origin $x = 0$ of the system dynamics in (13) is an *unstable equilibrium point* if it is not stable.

Let's learn how to negate Definition 2.2 for stability to get an elaborate definition for instability.

Definition 2.4. The origin $x = 0$ of the system dynamics in (13) is an *unstable equilibrium point* if there exists at least one value of $\epsilon > 0$ such that for every $\delta > 0$ there exists a finite $T(\delta) > 0$ so that

$$||x(0)|| < \delta \quad \Rightarrow \quad ||x(t)|| > \epsilon, \quad \forall\, t \geq T(\delta)\,.$$

It is important to distinguish between instability and the intuitive notion of "blowing up". In linear systems, instability is equivalent to blowing up, because unstable poles always lead to exponential growth of system states. However, for nonlinear systems, blowing up is only one way of instability. For example, the Van der Pol oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2)x_2 \end{aligned}$$

has unstable equilibrium at the origin, but the trajectories starting in the neighborhood of the origin tend to a limit cycle and do not go to infinity. This implies that if we choose ϵ in Definition 2.4 small enough for the circle of the radius ϵ to be inside this limit cycle, then system trajectories starting near the origin will eventually get out of this circle. Thus, even though the state of the system *does remain* around the origin in a certain sense (i.e. within the limit cycle), it cannot stay *arbitrarily close* to it, Fig. 5. This is the fundamental distinction between stability and instability.

Definition 2.5. The origin $x = 0$ of the system dynamics in (13) is a *locally asymptotic stable* (LAS) *equilibrium point* if it is stable and, in addition, there exists a $\delta > 0$ such that

$$||x(0)|| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0\,.$$

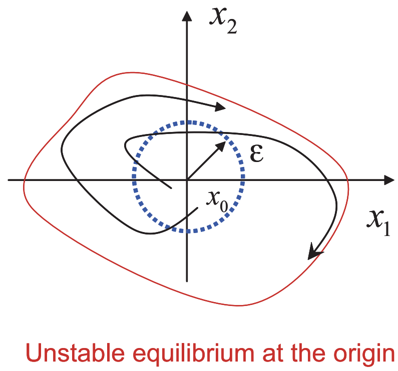


Fig. 5 Trajectories starting in the neighborhood of the origin tend to limit cycle

An example of a system which is stable, but not asymptotically, will be the pendulum without friction. Figure 6 illustrates these definitions in phase space.

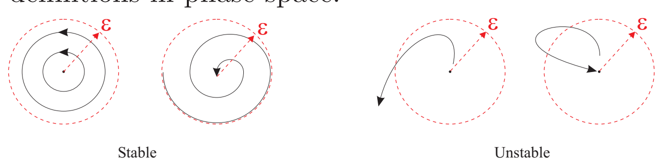


Fig. 6 Stability, asymptotic stability, "blow-up type" instability, instability

The last figure on the right illustrates a case, when the trajectories converge to the origin, but the system is *not* asymptotically stable. In fact, it is unstable.

Very often in realistic applications in case of asymptotic stability it is not sufficient to know that trajectories converge to the origin; one needs to know in addition the *rate* at which they converge. This brings up the notion of *exponential stability*.

Definition 2.6. The origin $x = 0$ of the system dynamics in (13) is a *locally exponentially stable* (LES) *equilibrium point* if there exist two strictly positive numbers α and λ such that for some $\delta > 0$

$$||x(0)|| \leq \delta \quad \Rightarrow \quad \forall\, t > 0 \quad ||x(t)|| \leq \alpha ||x(0)|| \mathrm{e}^{-\lambda t}\,.$$

This implies that the state of the system converges to the origin faster than an exponential function. The positive number $\lambda > 0$ is called rate of exponential convergence.

Example 2.3. The trajectories of the system $\dot{x}(t) = -(1 + \sin^2 x)x$ exponentially converge to the origin with a rate $\lambda = 1$. Indeed, its solution $x(t) = x(0)\exp(-\int_0^t (1+\sin^2(x(\tau)))d\tau)$, and therefore

$$|x(t)| \leq |x(0)|\mathrm{e}^{-t} .$$

Note that exponential stability implies asymptotic stability, but the opposite is not true.

Example 2.4. The unique solution of the system $\dot{x}(t) = -x^2,\ x(0) = 1$ is given by

$$x(t) = \frac{1}{1+t}$$

which is converging to the origin slower than any exponential function $\mathrm{e}^{-\lambda t}$ with $\lambda > 0$. So, the origin is asymptotically stable, but not exponentially. (Try to plot in Matlab these two curves to convince yourself in this!)

All the definitions up to this point were stated in local sense, i.e. in a neighborhood of the origin. Every theorem proving local stability result needs to provide at least a conservative estimate of the region of attraction so that to be *valuable* as a result at all!!! The theorems are usually stronger if they provide verifiable conditions for global stability.

Definition 2.7. The origin $x = 0$ of the system dynamics in (13) is a *globally asymptotically stable* (GAS) *equilibrium point* or *globally exponentially stable* (GES) *equilibrium point* if the definition of asymptotic stability (Definition 2.5) or the definition of exponential stability (Definition 2.6) hold for arbitrary $\delta \in [0, \infty)$ (i.e. the initial condition can be chosen arbitrarily large).

Figure 7 illustrates these definitions over time axis.

Linear time-invariant systems are either asymptotically stable, or marginally stable, or unstable. Linear asymptotic stability is always exponential and global. Linear instability always implies exponential blow up. The significance of the above stated definition is apparent for nonlinear systems, like demonstrated by the following example.

Example 2.5. Consider the following system

$$\dot{x} = -x + x^2, \quad x(0) = x_0 ,$$

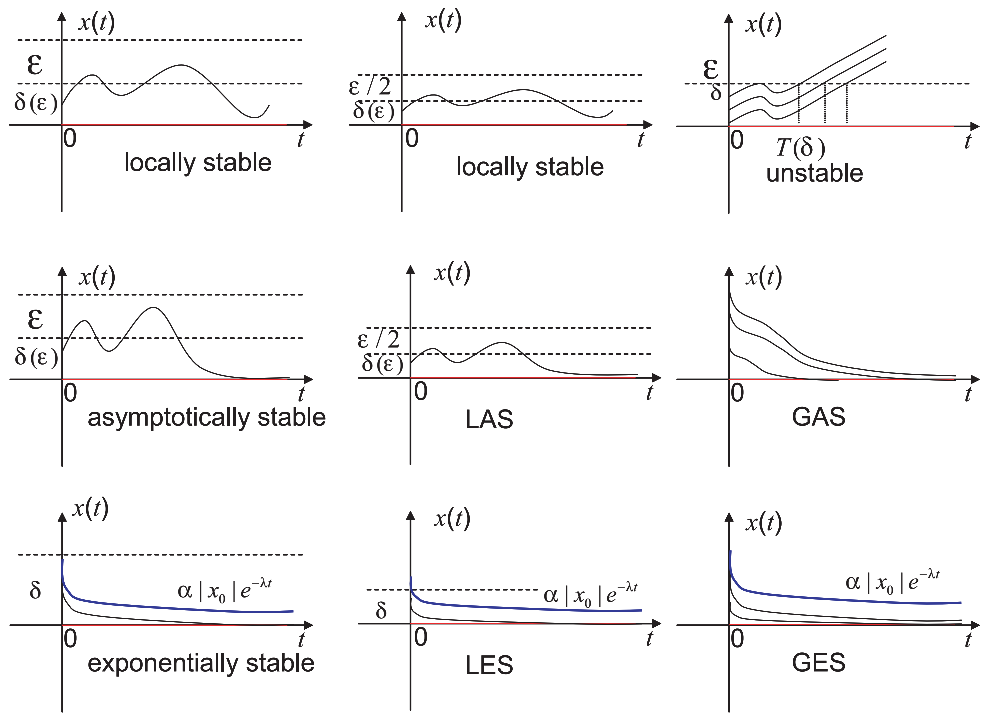


Fig. 7 Stability, asymptotic stability, exponential stability

that has an equilibrium at the origin. The linearization around origin results in the following linear system

$$\dot{x} = -x, \quad x(0) = x_0 \, .$$

Obviously the origin of the linearized system is globally exponentially stable (GES), since $x(t) = x_0 \mathrm{e}^{-t}$. However, the nonlinear system has two equilibria $x = 0$ (LAS) and $x = 1$ (unstable). (Try to solve explicitly the differential equation to convince yourself in this!).

Remark 2.4. It is important to observe that nonlinear systems, having isolated equilibria, will interlace the equilibria, i.e. there can never be two consecutive stable equilibria or two consecutive unstable equilibria, since these will contradict the definitions of stability and instability. Between two unstable equilibria, there is always one stable one, and vice versa. To get the physical intuition behind this, look at a regular sinusoid, for which the minimums are interlaced by maximums. A continuous function can never have two local minima or maxima one after another. There is always the opposite extremum in between. The peaks of the sinusoid are unstable equilibria, while the minimums are the stable ones. From the perspective of mechanical systems, there is energetic interpretation for equilibria and the relationship of it to stability. Stable equilibria are the minima of the energy.

There are several methods to investigate stability of equilibria of dynamical systems. The first and obvious one is to explicitly solve it and check the behavior of the solution by using tools from calculus (compute the limits, etc.). The second approach is to linearize the system around the origin and use the linearized system to investigate the stability of the nonlinear system. If the linearized system is asymptotically stable, then the nonlinear system is locally asymptotically stable around that equilibrium. Similarly if the linearized system is unstable around that origin, then so is the nonlinear system. If the linearized system is marginally stable, i.e. has eigenvalues on the imaginary axis, then nothing can be claimed about the nonlinear system in general. *Be careful with linearizations since the linearization of the nonlinear system around different equilibrium points will result in different linear systems*! A more powerful analysis tool that we will study in details will be Lyapunov's direct method that gives an opportunity to investigate stability of equilibria without explicitly solving the differential equations.

2.2 Lyapunov stability theorems.

Before introducing the main theorems for checking stability of equilibria without explicitly solving the differential equations, we need several definitions.

Definition 2.8. A continuous function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is called positive definite in a domain $\mathcal{D} \subset \mathbb{R}^n$ if

$$V(x) > 0 \quad \forall\, x \in \mathcal{D} - \{0\} \quad \text{and} \quad V(0) = 0\,.$$

It will be negative definite in case of opposite inequality.

Definition 2.9. A continuous function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is called positive semidefinite in a domain $\mathcal{D} \subset \mathbb{R}^n$ if

$$V(x) \geq 0 \quad \forall\, x \in \mathcal{D}\,.$$

It will be negative semidefinite in case of opposite inequality.

We can now formulate sufficient conditions for local stability, local asymptotic stability (LAS) and local exponential stability (LES).

Theorem 2.1. Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x), \quad x(0) = x_0\,, \quad x \in \mathbb{R}^n\,,$$

in which f is locally Lipschitz in x for $x \in \mathcal{D} \subset \mathbb{R}^n$, where $\mathcal{D}$ is a domain containing the origin. Suppose that there exists a continuously differentiable function $V(x) \in \mathcal{C}^1$, $V : \mathcal{D} \to \mathbb{R}$ such that $V(0) = 0$ and

$$V(x) > 0 \quad \forall\, x \in \mathcal{D} - \{0\}\,,$$

while

$$\dot{V}(x(t)) \leq 0 \quad \forall\, x \in \mathcal{D}\,.$$

Then the equilibrium $x = 0$ is stable. Moreover, if

$$\dot{V}(x(t)) < 0 \quad \forall\, x \in \mathcal{D} - \{0\}\,,$$

then $x = 0$ is asymptotically stable. Finally, if there exist positive scalars k_1, k_2, k_3 and $p \geq 1$ such that

$$k_1\|x\|^p \;\leq\; V(x) \;\leq\; k_2\|x\|^p\,, \quad x \in \mathcal{D} \tag{15}$$

$$\dot{V}(x(t)) \;\leq\; -k_3\, V(x(t))\,, \quad x \in \mathcal{D}\,, \tag{16}$$

then the equilibrium $x = 0$ is exponentially stable.

Proof: Before starting the proof, let's look into the intuition behind the required condition. Figure 8 plots cross sections of several level sets of the Lyapunov function. The condition $\dot{V}(t) = \frac{\partial V}{\partial x}\dot{x} < 0$ implies that the gradient of the Lyapunov function and the velocity vector have a negative inner product, which can happen only if the angle between the gradient of $V(x)$ and $\dot{x}$ is more than $90°$. Notice that in case of negative semidefinite $\dot{V}(t)$, this angle is allowed to be $90°$, which implies that the trajectory can stay on a level set.

Now, let's get to the formal proof. Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ small enough that

$$\mathcal{B}_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subset \mathcal{D}.$$

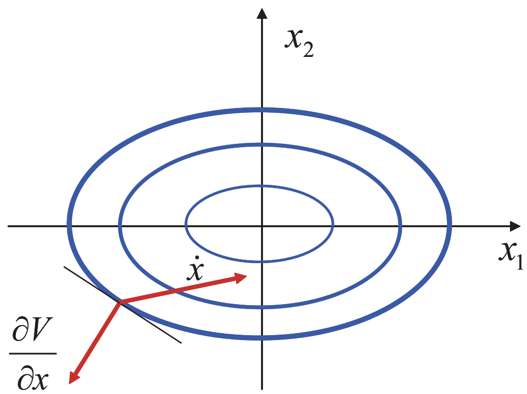


Fig. 8 Illustration of $\dot{V}(t) = \frac{\partial V}{\partial x}\dot{x} < 0$.

Let $\alpha = \min_{\|x\|=r} V(x)$ and note that $\alpha > 0$ because $V(x) > 0$, Fig. 9. Now let $\beta \in (0, \alpha)$ and define a set containing the origin which is bounded by a level surface of V and contained entirely within $\mathcal{B}_r$:

$$\Omega_\beta = \{x \in \mathcal{B}_r \mid V(x) \leq \beta\}.$$

We note that any trajectory starting in Ω_β at time $t = 0$ remains in Ω_β for all time $t \geq 0$ because

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta$$

for all $t \geq 0$. Furthermore, because Ω_β is a compact (*closed and bounded by definition*) set, any initial state in Ω_β gives rise to a unique solution that exists for all time (See Theorem 1.2). Continuity of V requires that there exists some $\delta > 0$ such that

$$\|x - 0\| < \delta \quad \Rightarrow \quad \|V(x) - V(0)\| < \beta, \quad \text{that is} \quad \|x\| < \delta \quad \Rightarrow \quad V(x) < \beta.$$

Using δ to define a ball $\mathcal{B}_\delta$ of radius δ, we see that

$$x(0) \in \mathcal{B}_\delta \quad \Rightarrow \quad x(0) \in \Omega_\beta \quad \Rightarrow \quad x(t) \in \Omega_\beta \quad \Rightarrow \quad x(t) \in \mathcal{B}_r \quad \Rightarrow \quad \|x(t)\| < r \leq \epsilon$$

for all $t \geq 0$. This proves that the existence of a positive definite function V with a negative semidefinite rate implies stability of the equilibrium at the origin.

Now suppose that $\dot{V}(x(t)) < 0$. We must show that

$$\lim_{t\to\infty} x(t) = 0,$$

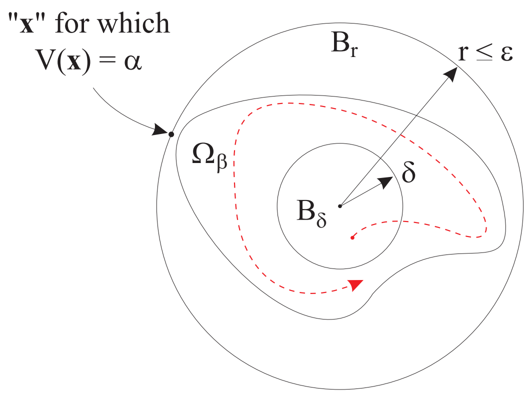


Fig. 9 Illustration for the proof of Lyapunov's stability theorem for autonomous systems

which is to say that for every $\tilde{r} > 0$, there exists a time $T > 0$ such that $\|x(t)\| < \tilde{r}$ for all time $t > T$. In fact, it is sufficient to show that

$$\lim_{t\to\infty} V(x(t)) = 0\,,$$

since $V = 0$ in $\mathcal{D}$ if and only if $x = 0$. Because V is bounded below and nonincreasing, we know that $V(x(t))$ approaches a constant value in the limit $t \to \infty$. (See "Convergence of Sequences" in Appendix A in [11].) Let

$$V(x(t)) \to c \geq 0 \qquad t \to \infty$$

Assume that $c > 0$, i.e. $c \neq 0$. Consider the ball $\mathcal{B}_d = \{x : ||x|| \leq d\}$ such that it lies inside $\Omega_c = \{x : V(x) \leq c\}$. Then $V(x(t)) \to c$ implies that the system trajectory $x(t)$ lies outside the ball $\mathcal{B}_d$ for all $t \geq 0$. Since $\dot{V}(x(t))$ is continuous, then it has a maximum over the compact set $\{x : d \leq ||x|| \leq r\}$, which we denote by $\gamma = \max\limits_{d\leq||x||\leq r} \dot{V}(x(t))$, where $\gamma < 0$, since $\dot{V}(x(t)) < 0$. By integration of the Lyapunov function

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0)) + \gamma t$$

Since $\gamma < 0$, then $V(x(0)) + \gamma t$ will decrease as $t \to \infty$, thus rendering the left hand side negative, which contradicts the fact that $V(x)$ is positive for all $x \neq 0$. Therefore $c = 0$, and this completes the LAS part of the proof.

Finally, to prove that the origin is LES, notice that the condition in (16) implies that

$$V(x(t)) \leq V(x(0))\mathrm{e}^{-k_3 t}\,.$$

Since (15) implies that $V(x(0)) \leq k_2||x(0)||^p$, and $k_1||x(t)||^p \leq V(x(t))$, then it follows that

$$k_1||x(t)||^p \leq k_2||x(0)||^p \mathrm{e}^{-k_3 t}\,,$$

which leads to the following

$$||x(t)|| \leq \left(\frac{\beta}{\alpha}\right)^{1/p} ||x(0)||\mathrm{e}^{(-k_3/p)t}\,,$$

proving exponential stability. $\square$

Comments on Lyapunov's Direct Method:

- One need not solve the differential equation in order to ascertain stability. One only needs to analyze the behavior of a single, scalar function under the given dynamics.
- Lyapunov's stability theorem is *sufficient* but *not necessary*. Just because one is unable to prove Lyapunov stability, does not mean that the equilibrium is unstable. On the other hand, it is sufficient to find one Lyapunov function satisfying the sufficient conditions of the theorem, and one can claim that the system is stable.
- There is no technique for constructing Lyapunov functions which is applicable to all nonlinear systems with stable equilibria. For mechanical systems, there are several powerful techniques, based on energy and momentum conservation laws. (Methods applicable to other classes of systems include the "variable gradient method" [11] and methods based on linearizing the dynamics.)

Example 2.6. Consider the nonlinear dynamical system representing a rigid spacecraft given by

$$\dot{x}_1(t) = I_{23}x_2(t)x_3(t),\ \ \dot{x}_2(t) = I_{31}x_3(t)x_1(t),\ \ \dot{x}_3(t) = I_{12}x_1(t)x_2(t),\ \ x_1(0) = x_{10},\ x_2(0) = x_{20},\ x_3(0) = x_{30},$$

where $I_{23} = (I_2 - I_3)/I_1,\ I_{31} = (I_3 - I_1)/I_2,\ I_{12} = (I_1 - I_2)/I_3$, and I_1, I_2, I_3 are the principal moments of inertia of the spacecraft such that $I_1 > I_2 > I_3 > 0$. To examine the stability of this system consider the Lyapunov function candidate $V(x_1, x_2, x_3) = \frac{1}{2}(\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2)$, where all $\alpha_i > 0$. The derivative of it is given by

$$\dot{V}(x_1, x_2, x_3) = x_1 x_2 x_3(\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12})\,.$$

Since $I_{31} = (I_3 - I_1)/I_2 < 0$ due to $I_1 > I_2 > I_3 > 0$, then one can easily select $\alpha_1, \alpha_2, \alpha_3$ to ensure that $\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12} = 0$, which will render $\dot{V}(t) = 0$ in the entire space. Hence the dynamical system is Lyapunov stable.

This example basically demonstrates how a Lyapunov function can be designed with some freedom to begin with, like having free parameters similar to α_i's here, and upon some preliminary analysis one can restrict these parameters to satisfy the sufficient conditions of the stability theorem.

Example 2.7. Consider the planar dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2 - a x_1^3 \\ \dot{x}_2 &= -x_1 - b x_2 \,, \end{aligned}$$

where $a > 0$ and $b > 0$. Note that this system has an equilibrium at the origin. To determine its stability, we choose a candidate Lyapunov function

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0, \quad V(0) = 0.$$

The rate of change of V is

$$\begin{aligned} \dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1\left(x_2 - a x_1^3\right) + x_2\left(-x_1 - b x_2\right) \\ &= -\left(a x_1^4 + b x_2^2\right) < 0. \end{aligned}$$

We conclude from Lyapunov's stability theorem that $V(x)$ is a Lyapunov function and the equilibrium at the origin is asymptotically stable.

Remark 2.5. Notice that we used the phrase "candidate Lyapunov function". Unless you do not prove that $V(x)$ satisfies the sufficient conditions of Lyapunov stability theorems, you cannot call it Lyapunov function.

Definition 2.10. Let $x = 0$ be an asymptotically stable equilibrium for the system in (13). Then the set given by

$$\mathcal{D}_0 = \{x_0 \in \mathcal{D} : \ \text{if} \ \ x(0) = x_0, \ \ \text{then} \ \ \lim_{t\to\infty} x(t) = 0\} \tag{17}$$

is called *domain of attraction of $x = 0$*.

Question 2.1. A question which naturally arises is: Under which conditions the above stated stability theorem (Theorem 2.1) can give sufficient conditions for global stability, or otherwise saying when can the entire space $\mathbb{R}^n$ be a domain of attraction?

A crucial point in the proof of Theorem 2.1 was that the level sets of Lyapunov function were bounded for $\beta \in [0, \alpha)$ for some $\alpha \in [0, \epsilon]$. While setting $\mathcal{D} = \mathbb{R}^n$ in Theorem 2.1 is necessary to extend the result for global stability, it is not sufficient. The level set Ω_β of the Lyapunov function needed to be contained inside $\mathcal{B}_r$! Can Ω_β be unbounded for some β?

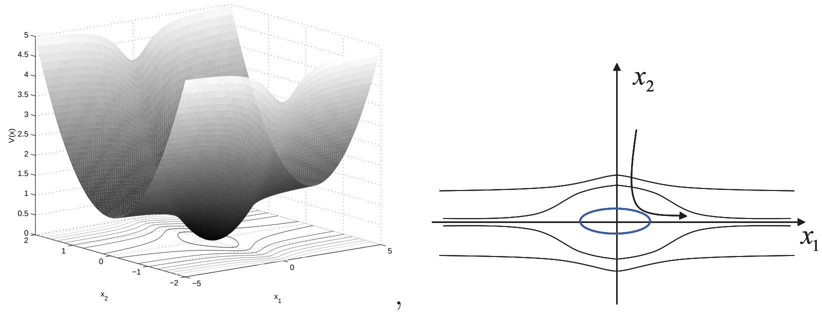


Fig. 10 Contour plot of $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$ **and its cross-sections.**

Let's look at the following positive definite function: $V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + x_2^2$. This function is positive definite and therefore can be used as Lyapunov function candidate. However, as can be seen from a contour plot, the set $\Omega_\beta = \{x \ : \ V(x) \leq \beta\}$ is unbounded for β large enough (in fact, for $\beta > 1$). Thus, it is possible for a trajectory starting in Ω_β to remain in Ω_β for some value of β (which it *must* following the proof of Theorem 2.1, if $V > 0$ and $\dot{V} < 0$) and yet still escape to infinity, thus leading to instability. We therefore impose one more condition (in addition to the requirement $\mathcal{D} = \mathbb{R}^n$). We require that V be *radially unbounded*: $V(x) \to \infty$ as $\|x\| \to \infty$. In the context of the proof of Theorem 2.1, this will imply that α *strictly grows* with r and tends to ∞ as $r \to \infty$. Any function $V(x)$ satisfying this condition will imply that $\Omega_\beta = \{x \ : \ V(x) \leq \beta\}$ is *a bounded set for every* $\beta = [0, \infty)$, and hence every trajectory starting in it will remain inside it for all $t \geq 0$. So, this property together with the requirement $\mathcal{D} = \mathbb{R}^n$) will lead to sufficient condition for global stability that we give without a proof.

Theorem 2.2. Let $x = 0$ be an equilibrium point for the system (13), where $f(x)$ is locally Lipschitz for all $x \in \mathbb{R}^n$, and suppose that there exists a continuously differentiable function $V(x)$, defined on $\mathbb{R}^n$. If

$$\begin{aligned} V(x) &> 0 \quad \forall\, x \neq 0, \quad V(0) = 0, \\ \dot{V}(x(t)) &< 0 \quad \forall\, x \neq 0, \text{ and} \\ V(x) \to \infty \quad &\text{as} \quad \|x\| \to \infty, \end{aligned}$$

then $x = 0$ is globally asymptotically stable (GAS). If alternatively the conditions in (15) and (16) hold globally for all $x \in \mathbb{R}^n$, then the origin is globally exponentially stable (GES).

Referring back to our previous example, we see that $\mathcal{D} = \mathbb{R}^2$ and $V = \frac{1}{2}(x_1^2 + x_2^2)$ is radially unbounded. We therefore conclude that $x_1 = x_2 = 0$ is a globally asymptotically stable equilibrium. Any trajectory converges to the origin as $t \to \infty$. Note that the strength of our conclusion depends on our choice of Lyapunov function. If we had used the Lyapunov function $V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + x_2^2$, we wouldn't have concluded global stability.

Remark 2.6. [Exercise 4.9 from [11].] In the requirement of radial unboundedness it is crucial to make sure that $V(x) \to \infty$ as $||x|| \to \infty$ <u>*holds uniformly in all directions*</u>. For example, the function

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

goes to ∞ along each principal axis

$$x_1 = 0 \quad \Rightarrow \quad V(x) = \frac{x_2^2}{1 + x_2^2} + x_2^2 \to \infty \ \text{ as } \ |x_2| \to \infty\,,$$

$$x_2 = 0 \quad \Rightarrow \quad V(x) = \frac{x_1^2}{1 + x_1^2} + x_1^2 \to \infty \ \text{ as } \ |x_1| \to \infty\,,$$

but it is not radially unbounded, since along the line $x_1 = x_2$ we have $V(x) = \frac{4x_1^2}{1+4x_1^2} \to 1$ as $|x_1| \to \infty$. So, it is not radially unbounded and therefore cannot be used for the proof of global asymptotic stability.

The following exercise analyzes an example without the use of Lyapunov functions.

Exercise 2.1. Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

Solution 2.1. Let $f(x) = ax^p + g(x)$. Near the origin, since $|g(x)| \leq k|x|^{p+1}$, then the term ax^p is dominant. Hence, $\mathrm{sgn}(f(x)) = \mathrm{sgn}(ax^p)$. Consider the case when $a < 0$ and p is odd. With $V(x) = \frac{1}{2}x^2$ as a Lyapunov function candidate, we have

$$\dot{V}(x(t)) = x(ax^p(t) + g(x(t))) \leq ax^{p+1}(t) + k|x^{p+2}(t)|$$

Near the origin, the term ax^{p+1} is dominant. Since p is odd, $p+1$ is even, and therefore $x^{p+1}(t)$ is always positive. Hence, $\dot{V}(x(t))$ is negative definite in the neighborhood of the origin, and therefore the origin is asymptotically stable.

Consider now the case when $a > 0$ and p is odd. In the neighborhood of the origin, since $a > 0$ is positive, p is odd and ax^p is dominant, then $\mathrm{sgn}(f(x)) = \mathrm{sgn}(x)$. Hence, a trajectory starting near $x = 0$ will be always moving away from $x = 0$. This shows that the origin is unstable. When p is even, a similar behavior will take place on one side of the origin; namely when $x > 0$ and $a > 0$, or when $x < 0$ and $a < 0$. Therefore the origin is unstable.

Homework Problems 2.1. The following problems from [11] need to be solved completely.

- Exercise 4.3 from [11].
- Exercise 4.4 from [11].

2.3 Invariant Set Theorems.

Reading [11], pp. 111-133, Appendix C, and also [17], pp. 41-76.

Theorem 2.1 stated that if there is a positive definite function $V(x)$ such that its derivative along the system trajectories is negative semidefinite, then the equilibrium is stable. It also stated that if the derivative is negative definite, then the origin is asymptotically stable. While very often the standard choice of the Lyapunov function leads to negative semidefinite derivative, one can still prove asymptotic stability of the origin from different considerations. These are so-called invariant set theorems. Let's start by a classical motivating example.

Example 2.8. Consider classical pendulum dynamics:

$$MR^2\ddot{\theta}(t) + k\dot{\theta}(t) + MgR\sin(\theta(t)) = 0\,, \tag{18}$$

where M is the mass, R is the length of the rod, k is the friction coefficient, θ is the angle of the rod from the vertical equilibrium. From your dynamics courses you should remember that if $k = 0$, then there is no friction, and the pendulum can continue swinging without ever stopping. But in the presence of friction it eventually gets back to its equilibrium and stops there. Let's see what can we get from a standard choice of a Lyapunov function, using its total energy for that purpose. The state space representation for the pendulum dynamics would be:

$$\dot{x}_1(t) = x_2(t) \tag{19}$$

$$\dot{x}_2(t) = -a\sin(x_1(t)) - bx_2(t)\,, \tag{20}$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$, $a = \frac{g}{R}$, and $b = \frac{k}{MR^2}$. We want to study the stability of the equilibrium at the origin $x_e = (x_1, x_2)^\top = (0, 0)^\top$. Consider the total energy of the system as a Lyapunov function:

$$V(x) = a(1 - \cos(x_1)) + \frac{x_2^2}{2}$$

It is obvious that $V(x)$ is <u>*locally positive definite*</u> around the origin. Its time derivative along the system trajectories is

$$\dot{V}(x(t)) = a\sin(x_1)\dot{x}_1(t) + x_2\dot{x}_2(t) = -bx_2^2(t) \le 0\,.$$

Notice that $\dot{V}(x(t))$ is negative semidefinite. It is <u>**not**</u> negative definite, because $\dot{V}(x(t)) = 0$ for all $x_2 = 0$ irrespective of the values of x_1. Therefore we can conclude that the origin is a stable equilibrium point, but not asymptotically stable. But we know that in the presence of friction the origin is asymptotically stable! So, the conclusion is that the total energy was not a good choice for a Lyapunov function. To circumvent the situation, there are two ways to go:

- change the Lyapunov function and consider a more general quadratic form:

$$V(x) = x^\top P x + a(1 - \cos(x_1))$$

  where $P = P^\top > 0$ is a positive definite matrix, and play with its terms to get a negative definite derivative in the entire space. (Read [11], page 119, Example 4.4 on this)

- Use invariant set theorems.

The physical intuition behind the invariant set theorems is the following. Looking at the derivative of the Lyapunov function, we see that $\dot{V}(x(t)) = 0$ for the entire line $x_2 = 0$. Now assume that $x_2 = 0$ is an equilibrium *independent of the value of x_1* and let's arrive at contradiction. Indeed, assuming that $x_2 = 0$ is an equilibrium, we have

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0\,.$$

From system dynamics it immediately follows that

$$\sin(x_1) \equiv 0 \Rightarrow x_1 \equiv 0\,.$$

Hence, on the interval $-\pi < x_1 < \pi$ of the line $x_2 \equiv 0$ the system can maintain the $\dot{V}(x(t)) = 0$ condition **only** at the origin $x = 0$. Therefore $V(x(t))$ must decrease towards 0, and consequently $x(t) \to 0$ as $t \to \infty$, which is consistent with the fact that due to friction energy cannot remain constant while the system is in motion.

This example showed that if the derivative of a candidate Lyapunov function is negative semidefinite, but in addition the dynamics imply that no trajectory can stay identically at the points where $\dot{V}(x(t)) = 0$, except for the origin, then the origin is asymptotically stable. This argument follows from *La-Salle's Invariance Principle, which holds only for autonomous systems*. Before introducing the principle itself, let's give the definitions for invariant and positive invariant sets. Towards that end, consider the following autonomous system dynamics:

$$\dot{x} = f(x), \quad x_0 = x(0)\,, \tag{21}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function of its argument.

Definition 2.11. Given a trajectory $x(t)$ of (21), a point $P \in \mathbb{R}^n$ is called a *positive limit point* (or an *accumulation point*) of $x(t)$, if there exists a sequence of times $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t) \to P$ as $n \to \infty$. The set of all positive limit points is called the *positive limit set* of system dynamics in (21).

Definition 2.12. A set M is called an *invariant set* with respect to the dynamics (21) if

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M \quad \forall\, t \in \mathbb{R}.$$

A set M is called *positively invariant* if

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M \quad \forall\, t \geq 0.$$

By definition, trajectories cannot leave an invariant set in forward or reverse time. Thus, trajectories can neither enter nor leave an invariant set. Trajectories may enter a positively invariant set, however; they just can not leave it (in forward time).

The notion of asymptotic stability is related to the notion of convergence of trajectories to an invariant set. To define convergence to a set, we must define distance to the set. Let the distance between the *point* P and the *set* M be

$$\mathrm{dist}(P, M) = \inf_{x \in M} \|x(P) - x\|\,,$$

where $x(P)$ denotes the vector of coordinates of the point P. Then a trajectory $x(t)$ *converges to the set* M as $t \to \infty$ if for every $\epsilon > 0$, there is a time $T > 0$ such that

$$\mathrm{dist}(P, M) < \epsilon \quad \forall\, t > T.$$

Before stating La-Salle's invariance principle, we need the following lemma.

Lemma 2.1. If the solution of (21) is bounded and belongs to $\mathcal{D}$ for all $t \geq 0$, then its positive limit set $\mathcal{L}^+$ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches $\mathcal{L}^+$ as $t \to \infty$.

(A proof can be found in [11], Appendix C3.)

Theorem 2.3. [La-Salle's invariance principle] Let $\Omega \subset \mathcal{D} \subset \mathbb{R}^n$ be a compact (i.e., closed and bounded)[§]set that is positively invariant with respect to the dynamics (21). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function on $\mathcal{D}$ such that $\dot{V}(x(t)) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x(t)) = 0$ and let M be the largest invariant set contained in E. Then every solution starting in Ω converges to M as $t \to \infty$. If $\mathcal{D} = \mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded and $\dot{V}(x(t)) \leq 0$ for all $x \in \mathbb{R}^n$, then every solution starting in $\mathbb{R}^n$ converges to M, i.e. the statement holds globally.

(See [11], page 128 for the proof and Fig.11 for illustration.)

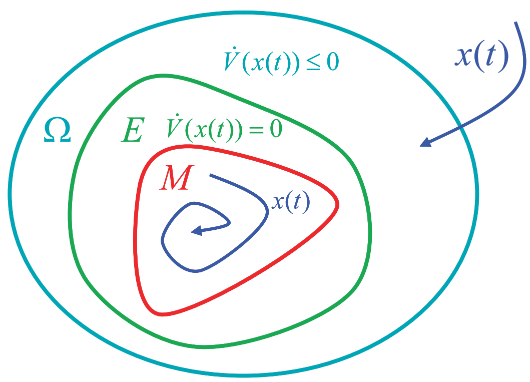


Fig. 11 Illustration of sets for La-Salle's Invariance Principle

Remark 2.7. Notice that La-Salle's invariance principle does not require $V(\cdot)$ be positive definite.

Remark 2.8. Since La-Salle's invariance principle strongly relies on phase plane analysis, by looking at the sets in the state space that are defined by the condition $\dot{V}(x(t)) = 0$ and/or are invariant sets of the system, it is important for the system to be <u>*autonomous*</u> so that $\dot{V}(x(t)) = \frac{\partial V}{\partial x}\dot{x}(t) = \frac{\partial V}{\partial x} f(x(t))$

[§]A set $\mathcal{S} \subset \mathbb{R}^n$ is *open* if every point in $\mathcal{S}$ is contained in an open ball of points which are also in $\mathcal{S}$. A set is *closed* if its complement is open. A set is *bounded* if the entire set can be contained in a closed ball of finite radius.

is only an *explicit function of x* and an *implicit function of t*.

Remark 2.9. The asymptotically stable equilibrium point is the positive limit set of every solution starting within its region of attraction. A stable limit cycle is a positive limit set of every solution starting from its region of attraction. Every solution approaches the limit cycle, as $t \to \infty$. However, it is important to notice that there is no specific limit point on the limit cycle. The limit cycle itself is a trajectory to which all other trajectories converge. The equilibrium point and the limit cycle are invariant sets, since every solution starting in these sets, remains in them for all $t \in \mathbb{R}$. In case of positive definite function $V(x)$ with $\dot{V}(x(t) \leq 0$ for all $x \in \Omega_c$, the set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is a positively invariant set, since as we saw in the proof of Theorem 2.1, every trajectory starting in it, remained in it for all $t \geq 0$. However, it is important to point out that in application of La-Salle's invariance principle the construction of the set Ω is not related to the function V. Any equilibrium point is an invariant set. The domain of attraction of an equilibrium point is also an invariant set. A trivial invariant set is the whole state space. For an autonomous system any of the trajectories in state space is an invariant set.

Remark 2.10. The sets in the La-Salle's principle are included as

$$M \subset E \subset \Omega \subset D \subset \mathbb{R}^n .$$

Most often our interest will be to show that the solution $x(t) \to 0$, as $t \to \infty$. For that we will need to show that the largest invariant set in E is the origin, i.e $M = \{0\}$. This is done by showing that no solution can stay identically in E other than the trivial solution $x = 0$.

The next two corollaries, known as Barbashin-Krasovskii's theorems, generalize La-Salle's invariance principle for positive definite functions $V(x)$ and can be viewed like extension of Lyapunov's theorems 2.1, 2.2.

Theorem 2.4. Let $x = 0$ be an equilibrium point for (21). Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable positive definite function on $\mathcal{D}$, where $\mathcal{D}$ contains the origin, such that $\dot{V}(x(t)) \leq 0$ for all $x \in \mathcal{D}$. Let $\mathcal{S} = \{x \in \mathcal{D} \mid \dot{V}(x(t)) = 0\}$ and suppose that no solution can stay identically in $\mathcal{S}$, other than the trivial solution $x(t) \equiv 0$. Then the origin is asymptotically stable.

Theorem 2.5. Let $x = 0$ be an equilibrium point for (21). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable positive definite radially unbounded function, such that $\dot{V}(x(t)) \leq 0$ for all $x \in \mathbb{R}^n$. Let $\mathcal{S} = \{x \in \mathbb{R}^n \mid \dot{V}(x(t)) = 0\}$ and suppose that no solution can stay identically in $\mathcal{S}$, other than the trivial solution $x(t) \equiv 0$. Then the origin is globally asymptotically stable.

Remark 2.11. When $\dot{V}(x(t))$ is negative definite, i.e. $\mathcal{S} = \{0\}$, then Theorems 2.4, 2.5 present particular cases of Theorems 2.1, 2.2.

Remark 2.12. La-Salle's theorems are applicable only to time-invariant systems.

Example 2.9. Consider the first order system that we had in Introduction

$$\dot{x}(t) = ax(t) + u(t)\,, \quad x(0) = x_0$$

and assume that we do not know a, but let's assume that we know some conservative bound for it so that $b > a$. We are interested in stabilization of this system. We construct the following feedback:

$$u(t) = -\hat{k}(t)x(t)\,,$$

where

$$\dot{\hat{k}}(t) \;=\; \gamma x^2(t)\,, \quad \hat{k}(0) = 0, \quad \gamma > 0\,.$$

Thus, we obtain the following closed-loop autonomous system:

$$\begin{aligned} \dot{x}(t) &= -(\hat{k}(t) - a)x(t)\,, & x(0) &= x_0 \\ \dot{\hat{k}}(t) &= \gamma x^2(t)\,, & \hat{k}(0) &= 0\,. \end{aligned}$$

The line $x = 0$ obviously represents the closed-loop system equilibrium set, irrespective of the values of $\hat{k}(t)$. We want to show that trajectories approach this equilibrium set, as $t \to \infty$. Take the following candidate Lyapunov function

$$V(x(t), \hat{k}(t)) = \frac{1}{2}x^2(t) + \frac{1}{2\gamma}(\hat{k}(t) - b)^2\,,$$

where $b > a$. The time derivative of $V((x(t), \hat{k}(t)))$ along the closed-loop system trajectories will be:

$$\dot{V}(x(t), \hat{k}(t)) = -x^2(t)(b - a) \leq 0\,.$$

Thus, $\dot{V}((x(t), \hat{k}(t)))$ is only negative semidefinite, hence only stability can be concluded at the offset. Let's apply La-Salle's principle to conclude asymptotic stability, instead of just stability. Notice that since $V(x, \hat{k})$ is positive definite, radially unbounded, the set $\Omega_c = \{(x, \hat{k}) \in \mathbb{R}^2 \mid V(x, \hat{k}) \leq c\}$ is compact and positive invariant for any value of $c > 0$. The set E in La-Salle's principle is given by $E = \{(x, \hat{k}) \in \Omega_c : \;\; x = 0\}$. Because every point on the line $x = 0$ is an equilibrium point, E is an invariant set. Therefore in this example $M = E$ in La-Salle's principle. From La-Salle's principle we conclude that every trajectory starting in Ω_c approaches E as $t \to \infty$, that is $x(t) \to 0$, as $t \to \infty$. Moreover, since $V(x, \hat{k})$ is radially unbounded, then this conclusion is global, i.e. it holds for arbitrary initial condition, otherwise saying c in the definition of Ω_c can be arbitrarily large.

Remark 2.13. Notice that due to the construction of E and M in this example, we could not conclude anything about $\hat{k}(t)$. We also could not apply Barbashin-Krasovskii's theorems. This was not the case in the pendulum example, where the right hand side of the system dynamics implied that M was comprised of one (equilibrium) point only, $M = \{0\}$, because of which the equilibrium $(x_1, x_2) = (0, 0)$ was asymptotically stable. In this closed-loop adaptive system M was comprised of a whole line (equilibrium set), and the conclusion we draw was only about $x(t)$, and not $\hat{k}(t)$. We will get to the issue of parameter convergence later in our course. Also, when getting to the tracking problems as opposed to this case of adaptive regulation, we will notice that the closed-loop system will not be autonomous, as this one was, and therefore La-Salle's principle will not be applicable. We will need Barbalat's lemma for that. Recall from (14) that the tracking error dynamics are non-autonomous.

Remark 2.14. Another important feature of this adaptive system was that we used a Lyapunov function for a system in which a was unknown. As long as an upper bound for a is known, b can be selected any number larger than that upperbound.

The following exercise should help you with the homework problem part c).

Exercise 2.2. Suppose that the set M in La-Salle's principle consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and is equal to one of these points.

Solution 2.2. According to La-Salle's principle, $x(t)$ approaches M as $t \to \infty$, which implies that given an $\epsilon > 0$ there exists a $T > 0$ such that $\inf_{y\in M} ||x(t) - y|| < \epsilon,\ \forall\ t > T$. Choose the ϵ so small that the neighborhood $N(P, 2\epsilon)$ of $P \in M$ contains no other points of M. We will prove that $||x(t)-y|| < \epsilon$, for all $t > T$ for some $P \in M$. Let's prove by contradiction. At $t = t_1$ let $P_1 \in M$ be a point for which $||x(t_1) - P_1|| < \epsilon$. Let $t_2 > t_1$ be the time instant at which $||x(t_2) - P_1|| = \epsilon$. Let P be any other point of M. Then

$$||x(t_2) - P|| = ||x(t_2) - P_1 + P_1 - P|| \geq ||P_1 - P|| - ||x(t_2) - P_1|| \geq 2\epsilon - \epsilon = \epsilon$$

This contradicts $\inf_{y\in M} ||x(t) - y|| < \epsilon,\ \forall\ t > T$. Therefore we arrive at contradiction. Given the fact that ϵ can be arbitrary small number, we proved that $x(t) \to P$ for some P.

Homework Problems 2.2. Exercise 4.21 from [11] needs to be solved completely.

3 System Analysis based on Lyapunov's Direct Method

Reading [11], pp. 135-144, and also [17], pp. 76-88.

With all the theorems and examples presented in the previous lectures we had the luxury of having an explicit Lyapunov function at hand. The question that we are trying to raise today and answer is how to find a Lyapunov function for a given system. While it would be naive to expect a complete and universal answer to this question, it is nevertheless possible to develop some "recipe" for certain classes of systems. Towards that end, we need to recall several definitions from linear algebra.

3.1 Preliminaries from linear algebra

Definition 3.1. A square matrix P is symmetric, if $P = P^\top$. It s skew-symmetric if $P = -P^\top$.

- Every square matrix can be presented as a sum of symmetric and skew-symmetric matrices:

$$P = \frac{P + P^\top}{2} + \frac{P - P^\top}{2}\,, \tag{22}$$

  where obviously the first term on the right is symmetric, while the second one is skew-symmetric.

- A quadratic function associated with a skew-symmetric matrix is always zero, since for a skew-symmetric matrix $P = -P^\top$ the following is true:

$$x^\top P x = -x^\top P^\top x\,,$$

  which implies that $x^\top P x = 0$.

- The above two facts lead to the following conclusion: arbitrary quadratic form $x^\top P x$ can be equivalently presented via a symmetric matrix. Indeed,

$$x^\top P x = x^\top \frac{P + P^\top}{2} x + x^\top \frac{P - P^\top}{2} x = x^\top \frac{P + P^\top}{2} x\,, \tag{23}$$

  since the second term is the skew-symmetric one and is zero.

Definition 3.2. A square matrix P is positive definite if for all $x \neq 0$ it implies $x^\top P x > 0$ (i.e. $x^\top P x$ is a positive definite function).

Definition 3.3. A square matrix P is positive semidefinite if for all x it implies $x^\top Px \geq 0$ (i.e. $x^\top Px$ is a positive semidefinite function).

Definition 3.4. A square time-varying matrix $M(t)$ is *uniformly* positive definite if for all $t \geq 0$ there exists $\alpha > 0$ such that $M(t) \geq \alpha\mathbb{I}$, i.e. $M(t) - \alpha\mathbb{I}$ is positive semidefinite for all $t \geq 0$.

Remark 3.1. If one substitutes the basis vectors in Definition 3.2, then it is straightforward to see that a *necessary condition* for a matrix to be positive definite will be positive elements on its diagonal. But this cannot serve as sufficient condition. If a real matrix P is *symmetric*, then Sylvester's criterion states a *necessary and sufficient* condition for positive definiteness, namely strict positivity of all its principal minors, or strict positivity of all its eigenvalues. As a consequence, a positive definite matrix is always invertible.¶

Recall from linear algebra that for any matrix A there exists a non-singular matrix B that transforms it into its Jordan form, given by $J = B^{-1}AB$, where the Jordan blocks of J are defined via the eigenvalues of A (for details on this refer to [1], [2]). For a positive definite matrix P, the following decomposition

$$P = T^\top \Lambda T,$$

where T is a matrix of eigenvectors satisfying $T^\top T = \mathbb{I}$, while Λ is a diagonal matrix of the eigenvalues of P, permits to write the following very much needed inequalities for our later analysis

$$\lambda_{\min}(P)||x||^2 \leq x^\top Px \leq \lambda_{\max}(P)||x||^2\,, \tag{24}$$

where $\lambda_{\min}(P), \lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of P. You should be able to prove (24) by noticing that

- $x^\top Px = x^\top T^\top \Lambda Tx = z^\top \Lambda z$, where $z = Tx\,$,
- $\lambda_{\min}(P)\mathbb{I} \leq \Lambda \leq \lambda_{\max}(P)\mathbb{I}\,$,
- $z^\top z = ||x||^2$.

¶Since arbitrary matrix can be decomposed into a sum of symmetric and skew-symmetric structure, then Sylvester's criterion is fairly general.

3.2 Lyapunov functions for Linear Time-Invariant (LTI) systems.

Consider the following linear time-invariant (LTI) system:

$$\dot{x}(t) = Ax(t)\,, \quad x(0) = 0, \quad x \in \mathbb{R}^n. \tag{25}$$

Obviously, this system has an equilibrium at the origin $x = 0$. If A is non-singular ($\det(A) \neq 0$), then this equilibrium is the unique equilibrium. If A is singular ($\det(A) = 0$), then every point in the entire non-trivial null-space of A is an equilibrium point, i.e. the system has an equilibrium subspace. Notice that a LTI system cannot have multiple isolated equilibria, since if x_1 and x_2 are equilibria, then, by the principle of superposition, every point on the line connecting x_1 and x_2 is an equilibrium point.

Theorem 3.1. The equilibrium point $x = 0$ of (25) is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}(\lambda_i) \leq 0$ and for every eigenvalue with $\mathrm{Re}(\lambda_i) = 0$ and algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i \mathbb{I}) = n - q_i$. The equilibrium point $x = 0$ of (25) is globally asymptotically stable (GAS) if and only if all eigenvalues of A satisfy $\mathrm{Re}(\lambda_i) < 0$. The equilibrium point $x = 0$ of (25) is unstable if at least one of the eigenvalues of A has $\mathrm{Re}(\lambda_i) > 0$.

The proof is straightforward and follows from the fact that the solution of (25) for any non-zero initial condition $x_0 \neq 0$ is given by $x(t) = \exp(tA)x_0$ (for details refer to [11]).

Remark 3.2. Notice that for GAS the theorem requires all the eigenvalues of A lie in open left-half plane. In that case the matrix A is called *Hurwitz*. If any of the eigenvalues is on the imaginary axis, then one cannot claim GAS, however, local stability can be guaranteed if the eigenvalue on the imaginary axis is a simple eigenvalue (i.e. with multiplicity 1) or the multiplicity q_i of that eigenvalue reduces the rank of $A - \lambda_i \mathbb{I}$ exactly to $n - q_i$. Also, if at least one of the eigenvalues lies in open right half-plane, then the origin is unstable.

Asymptotic stability of the origin of (25) can be also investigated via Lyapunov's direct method.

Theorem 3.2. The matrix A is Hurwitz if and only if for any given symmetric, positive definite matrix Q, there exists a symmetric, positive definite matrix $P = P^\top > 0$ satisfying

$$A^\top P + PA = -Q \tag{26}$$

Moreover, if A is Hurwitz, then P is the unique solution of the Lyapunov equation (26).

Proof: ("If") Assume that given symmetric, positive definite matrix Q, there exists a symmetric, positive definite matrix $P = P^\top > 0$ satisfying (26). Let's prove that A is Hurwitz. Notice from Theorem 3.1 that it is sufficient to prove that the origin of (25) is GAS. Define the Lyapunov function candidate

$$V(x(t)) = x^\top(t)Px(t)\,.$$

Obviously, $V(x) > 0$ for all $x \neq 0$, and $V(0) = 0$. Then

$$\dot{V}(x(t)) = x^\top(t)P\dot{x}(t) + \dot{x}^\top(t)Px(t) = x^\top(t)(PA + A^\top P)x(t) = -x^\top(t)Qx(t) < 0, \quad x \neq 0. \tag{27}$$

Thus, $x = 0$ is globally asymptotically stable equilibrium for the system in (25) (see Theorem 2.2). It follows from Theorem 3.1 that A is Hurwitz.

("Only if") Now, assume that A is Hurwitz and let Q be any symmetric, positive definite matrix. Let's prove that there exists a positive definite symmetric matrix $P = P^\top > 0$ solving (26). Define

$$P = \int_0^\infty \exp(A^\top t)\ Q\ \exp(At)dt\,. \tag{28}$$

Recall that matrix exponential is defined via the series expansion:

$$\exp(A) = \mathbb{I} + A + \frac{1}{2}A^2 + \cdots\,.$$

Since A is Hurwitz, then the series is convergent, and the integral is well-defined. It is positive definite, because

$$x^\top Px = \int_0^\infty x^\top \exp(A^\top t)\ Q\ \exp(At)xdt = \int_0^\infty (\exp(At)x)^\top\ Q\ (\exp(At)x)\,dt$$

is an integral of a positive definite quadratic function, which is zero if and only if $\exp(At)x = 0$ for all $t \geq 0$, and, for a nonsingular $\exp(At)$, this is satisfied only if $x = 0$.

Substituting (28) in equation (26) gives

$$\begin{aligned} A^\top P + PA &= \int_0^\infty A^\top \exp(A^\top t)\ Q\ \exp(At)dt + \int_0^\infty \exp(A^\top t)\ Q\ \exp(At)Adt \\ &= \int_0^\infty \frac{d}{dt}\Big(\exp(A^\top t)\ Q\ \exp(At)\Big)\,dt = \exp(A^\top t)\ Q\ \exp(At)|_0^\infty = -Q \quad (\exp(A^\infty t) = 0,\ \text{since } A \text{ is Hurwitz}). \end{aligned}$$

To prove uniqueness of P assume that for a given $Q > 0$ there exists another P_1 solving (26), i.e. $A^\top P_1 + P_1 A = -Q$, $A^\top P + PA = -Q$, and $P_1 \neq P$. Following the algebra of the last line of the above equation P_1 can be identically presented:

$$P_1 = -\int_0^\infty \frac{d}{dt}\Big(\exp(A^\top t)\ P_1\ \exp(At)\Big)\,dt$$

Expanding the derivatives gives:

$$P_1 = -\int_0^\infty \exp(A^\top t)\ (A^\top P_1 + P_1 A)\ \exp(At)dt = \int_0^\infty \exp(A^\top t)\ Q\ \exp(At)dt = P\,,$$

where the last equality follows from definition of P. □

Remark 3.3. The above theorem shows that any positive definite matrix Q can be used to determine the stability of a LTI system, i.e. stability of A in (25). A simple choice of Q is $\mathbb{I}$, which has also a surprising property of the best convergence rate. Even if it may seem now at the offset that there is no advantage to this result (since checking the eigenvalues of A can be viewed as an equivalent effort to solving (26) for P and determining the positive definiteness of P, which also amounts to checking for example the eigenvalues of P), you'll find out later that this theorem lies at the heart of the construction of the Lyapunov functions for most of the nonlinear systems, and especially for definition of the adaptive laws. The Lyapunov functions used for stability analysis of nonlinear adaptive systems will have $x^\top Px$ as the main state-dependent component, in addition to certain terms used for adaptive parameters.

Finally, we note that Theorem 3.1 lies in the heart of Lyapunov's indirect method, the key result of which we give without a proof.

Theorem 3.3. [Lyapunov's Indirect Method.] Let $x = 0$ be an equilibrium of the system

$$\dot{x} = f(x)\,, \quad x(0) = x_0\,, \tag{29}$$

where f is continuously differentiable in a neighborhood $\mathcal{D} \subset \mathbb{R}^n$ containing the origin. Define

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}.$$

Then, for the original nonlinear system in (29), the origin is

1. <u>*locally asymptotically stable*</u> if $\mathrm{Re}(\lambda) < 0$ for every eigenvalue λ of A.
2. unstable if $\mathrm{Re}(\lambda) > 0$ for at least one of the eigenvalues of A.

Key points to remember:

Remark 3.4. For the first time we require f be continuously differentiable, and not just Lipschitz. On the contrary, Lyapunov's direct method, presented in the past lecture, did not require linearization and could deal with the dynamics directly as long as sufficient conditions for the existence of a unique solution were satisfied (for which Lipschitz property was enough, which is a weaker requirement than continuous differentiability). From that perspective Lyapunov's direct method is a more powerful tool for stability analysis. The trade-off is that there is no straightforward recipe to determine a Lyapunov function for a given system. Talking of smoothness properties, Lyapunov's direct method requires the <u>*Lyapunov function to be continuously differentiable*</u>, i.e. $\dot{V}(x(t)) = \frac{\partial V}{\partial x}\dot{x}(t) = \frac{\partial V}{\partial x}f(x(t))$ be continuous, which is always guaranteed as long as $V(x)$ is selected to be a smooth function of x, and f is Lipschitz.

Remark 3.5. Note that the theorem says nothing about the case where $\mathrm{Re}(\lambda) \leq 0$ for every eigenvalue λ of A, i.e., the case where one or more eigenvalues lie on the imaginary axis. In this case, one cannot infer from the linearization whether the equilibrium of the nonlinear system is stable or unstable.

3.3 Krasovskii's Method

Let us now come back to the problem of finding Lyapunov functions for general nonlinear systems. Krasovskii's method ([17], p.84) suggests a simple form of Lyapunov function candidate, namely, $V = f^\top(x)f(x)$, for autonomous nonlinear systems of the form $\dot{x} = f(x)$. The basic idea of the method is simply to verify that this particular choice indeed leads to a Lyapunov function.

Theorem 3.4. (**Krasovskii**) Consider the autonomous system defined by

$$\dot{x} = f(x), \quad x(0) = x_0$$

with the equilibrium point of interest being the origin and $f(x)$ being continuously differentiable. Let

$A(x)$ denote the Jacobian of the system, i.e.,

$$A(x) = \frac{\partial f(x)}{\partial x}.$$

If the matrix $F(x) = A(x) + A^\top(x)$ is negative definite in a neighborhood Ω, then the equilibrium point at the origin is asymptotically stable. A Lyapunov function for this system is given by

$$V(x) = f^\top(x) f(x).$$

If Ω is the entire state space and, in addition, $V(x) \to \infty$ as $\|x\| \to \infty$, then the equilibrium point is globally asymptotically stable.

Proof: First, let us prove that the negative definiteness of $F(x)$ implies that $f(x) \neq 0$ for $x \neq 0$. Since the square matrix $F(x)$ is negative definite for non-zero x, one can show that the Jacobian matrix $A(x)$ is invertible, by contradiction. Indeed, assume that $A(x)$ is singular. Then one can find a non-zero vector u such that $A(x)u = 0$. Since

$$u^\top F(x) u = 2u^\top A(x) u$$

the singularity of $A(x)$ implies that $u^\top A(x) u = 0$, which contradicts the assumed negative definiteness of $F(x)$.

The invertibility and continuity of $A(x)$ guarantee that the inverse of $f(x)$ can be uniquely defined ([10], p.221). This implies that the system has only one equilibrium point in Ω (otherwise different equilibrium points would correspond to the same value of $f(x)$), i.e., that $f(x) \neq 0$ for $x \neq 0$.

We can now show asymptotic stability of the origin. Given the above result, the scalar function $V(x) = f^\top(x) f(x)$ is positive definite. Using the fact that $\dot{f}(x) = \frac{df(x)}{dt} = A(x)\dot{x} = A(x) f(x)$, the derivative of $V(x)$ can be written as

$$\dot{V}(x) = f^\top(x)\dot{f}(x) + \dot{f}^\top(x) f(x) = f^\top(x) A(x) f(x) + f^\top(x) A^\top(x) f(x) = f^\top(x) F(x) f(x).$$

The negative definiteness of $F(x)$ implies negative definiteness of $\dot{V}(x)$. Therefore, according to Lyapunov's direct method, the equilibrium state at the origin is asymptotically stable. The global asymptotic stability of the origin follows from radial unboundedness.

$\square$

Remark 3.6. While the use of Krasovskii's method is straightforward, its applicability is limited in practice, because the Jacobians of many systems do not satisfy the negative definiteness requirement. In addition, for systems of high order, it is difficult to check the negative definiteness of the matrix $F(x)$ for all x.

3.4 Performance Analysis

Lyapunov functions can be used not only for stability analysis, but also for analysis of the convergence rate for asymptotically stable systems. Before presenting the main lemma on convergence analysis, we give a small proposition on differential inequalities, similar to the Comparison lemma.

Proposition 3.1. If a real function $W(t)$ satisfies the inequality

$$\dot{W}(t) + \alpha W(t) \leq 0\,, \tag{30}$$

where α is a real number, then

$$W(t) \leq W(0)\exp(-\alpha t)\,.$$

Proof: Let us define a function $Z(t) \triangleq \dot{W}(t) + \alpha W(t)$. It follows from (30) that $Z(t) \leq 0$. Solving with respect to $W(t)$ implies

$$W(t) = W(0)\exp(-\alpha t) + \int_0^t \exp(-\alpha(t-\tau))Z(t)d\tau \leq W(0)\exp(-\alpha t)\,,$$

since the second (integral) term is non-positive. □

Remark 3.7. Proposition 3.1 implies that if $W(t)$ is a non-negative function satisfying (30) and $\alpha > 0$, then $W(t)$ converges to zero exponentially. Often in stability analysis, one can show that $\dot{V}(t) \leq -\gamma V(t)$, where $\alpha > 0$, which can be manipulated to show convergence of the system states to zero.

3.4.1 Convergence rate for linear systems.

Recall the choice of the Lyapunov function for linear systems, $V(x) = x^\top P x$, and let $\gamma = \dfrac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$, where $-Q = A^\top P + PA$, and A is Hurwitz, while the symbols $\lambda_{\min}(\cdot), \lambda_{\min}(\cdot)$ are used to denote minimum and maximum eigenvalues. Since $P \leq \lambda_{\max}(P)\mathbb{I}$, and $\lambda_{\min}(Q)\mathbb{I} \leq Q$, then

$$x^\top Q x \geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} x^\top \left(\lambda_{\max}(P)\mathbb{I}\right) x \geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} x^\top P x \geq \gamma V \,.$$

Since $\dot{V}(x) = -x^\top Q x$, then we have that

$$\dot{V}(t) \leq -\gamma V(t) \,,$$

which, following Proposition 3.1, consequently leads to

$$V(t) \leq V(0)\exp(-\gamma t) \,.$$

Recalling that $V(x) = x^\top P x$, and that $x^\top P x \geq \lambda_{\min}(P)||x||^2$ for all $t \geq 0$, implies that $x(t)$ converges to zero with *at least* the rate of $\gamma/2$.

It is interesting to note that the convergence rate estimate is the largest for $Q = \mathbb{I}$. Indeed, let P_0 be the solution of the Lyapunov equation corresponding to $Q = \mathbb{I}$, i.e. $A^\top P_0 + P_0 A = -\mathbb{I}$, and let P be the solution of the Lyapunov equation for some other Q_1, i.e. $A^\top P + PA = -Q_1$. Without loss of generality one can assume that $\lambda_{\min}(Q_1) = 1$. If this is not the case, then Q_1 can be always re-scaled to be such, which will correspondingly re-scale P, leaving the value of γ unchanged. Subtracting these two equations yields:

$$A^\top (P - P_0) + (P - P_0)A = -(Q_1 - \mathbb{I})$$

Since $\lambda_{\min}(Q_1) = 1 = \lambda_{\min}(\mathbb{I})$, the matrix $Q_1 - \mathbb{I}$ is positive semidefinite, and therefore $(P - P_0)$ is positive semidefinite, implying that $\lambda_{\max}(P) \geq \lambda_{\max}(P_0)$. Since $\lambda_{\min}(Q_1) = 1 = \lambda_{\min}(\mathbb{I})$, the convergence rate estimate $\gamma = \dfrac{\lambda_{\min}(\mathbb{I})}{\lambda_{\max}(P_0)}$ corresponding to $Q = \mathbb{I}$ is larger or equal to that $\gamma = \dfrac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)}$ corresponding to $Q = Q_1$.

The physical meaning of this optimal value of γ is easily interpreted for a symmetric matrix A. In that case, all eigenvalues of A are real, and there exists a change of coordinates such that A is diagonal in these new coordinates. In these new coordinates, $P = -\frac{1}{2}A^{-1}$ verifies the Lyapunov equation for $Q = \mathbb{I}$, and therefore the corresponding $\gamma/2$ is simply the absolute value of the dominant pole of the linear system. Moreover, γ is independent of the choice of the state coordinates.

3.4.2 Convergence rate for nonlinear systems.

For nonlinear systems there is no general recipe for estimating the convergence rate of state trajectories to zero. This should be investigated on a special case-by-case basis. The problem is that even if V is selected to be a quadratic function of system states, $\dot{V}(t)$ is not guaranteed to be such. However, for some special systems, it might be possible to manipulate $\dot{V}(t)$ to obtain some insights about the behavior of state trajectories.

Example 3.1. Consider the autonomous system, having equilibrium at the origin:

$$\begin{aligned} \dot{x}_1 &= x_1(x_1^2 + x_2^2 - 1) - 4x_1x_2^2 \\ \dot{x}_2 &= 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 1)\,. \end{aligned}$$

The candidate Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$ has a derivative $\dot{V}(t) = 2V(t)(V(t) - 1)$, which can be integrated:

$$V(t) = \frac{\alpha \exp(-2t)}{1 + \alpha \exp(-2t)}\,, \quad \alpha = \frac{V(0)}{1 - V(0)}\,.$$

Thus, if $||x(0)|| < 1$, which is equivalent to $V(0) \leq 1$, then $\alpha > 0$, and $V(t) < \alpha \exp(-2t)$, implying that the trajectories starting inside the unit circle converge to the origin exponentially with a rate of 1.

However, if the trajectory starts outside the unit circle, i.e. $\alpha < 0$, then this choice of Lyapunov function implies exponential blow-up.

The following exercise demonstrates the connection between optimal control and Lyapunov stability theory.

Exercise 3.1. Consider the closed-loop system under optimal stabilizing control:

$$\dot{x} = f(x) - kG(x)R^{-1}(x)G^\top(x)\left(\frac{\partial V}{\partial x}\right)^\top,$$

where $V(x)$ is a continuously differentiable positive definite function that satisfies the Hamilton-Jacobi equation

$$\frac{\partial V}{\partial x}f(x) + q(x) - \frac{1}{4}\frac{\partial V}{\partial x}G(x)R^{-1}(x)G^\top(x)\left(\frac{\partial V}{\partial x}\right)^\top = 0\,,$$

in which $q(x)$ is a positive semidefinite function, $R(x)$ is a non-singular matrix, and k is a positive constant. Using $V(x)$ as a Lyapunov function candidate, show that the origin is asymptotically stable when

1. $q(x)$ is positive definite and $k \geq \frac{1}{4}$.

2. $q(x)$ is positive semidefinite, $k \geq \frac{1}{4}$, and the only solution of $\dot{x} = f(x)$ that can stay identically in the set $\{q(x) = 0\}$ is the trivial solution $x(t) \equiv 0$.

When will the origin be globally asymptotically stable?

Solution 3.1. Using the system dynamics,

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) - k\frac{\partial V}{\partial x}G(x)R^{-1}(x)G^{\top}(x)\left(\frac{\partial V}{\partial x}\right)^{\top}.$$

Substituting for $\frac{\partial V}{\partial x}f(x)$ from the Hamilton-Jacobi equation, we get

$$\dot{V}(x) = -q(x) - \left(k - \frac{1}{4}\right)\frac{\partial V}{\partial x}G(x)R^{-1}(x)G^{\top}(x)\left(\frac{\partial V}{\partial x}\right)^{\top}.$$

1. If $q(x)$ is positive definite, and $k \geq \frac{1}{4}$, we conclude that $\dot{V}(x)$ is negative definite, and hence the origin is asymptotically stable.

2. If $q(x)$ is only positive semidefinite and $k \geq \frac{1}{4}$, we can only conclude that $\dot{V}(x)$ is negative semidefinite. Since $R(x)$ is a nonsingular matrix, then

$$\dot{V}(x) = 0 \;\Rightarrow\; \left\{q(x) = 0 \quad \text{and} \quad G^{\top}(x)\left(\frac{\partial V}{\partial x}\right)^{\top} = 0\right\} \;\Rightarrow\; \dot{x} = f(x).$$

   Since the only solution of $\dot{x} = f(x)$ that can stay identically in the set $\{q(x) = 0\}$ is the zero solution, we see that

$$q(x(t)) \equiv 0 \;\Rightarrow\; x(t) \equiv 0.$$

   By La-Salle's theorem we conclude that the origin is asymptotically stable.

The origin will be GAS, if all the assumptions hold globally and $V(x)$ is radially unbounded.

Homework Problems 3.1. The following two exercises from [11] need to be solved completely:

- Exercise 4.22
- Exercise 4.23.

4 Lyapunov Stability Theory for Nonautonomous Systems

Reading [11], pp. 144-156, and [17], pp. 100-126.

4.1 Stability of nonautonomous systems, uniformity, and comparison functions.

We will start by a motivating example to demonstrate how things can fundamentally change from the stability analysis perspective if the system is non-autonomous. Consider the following second-order mass-spring damper of unit mass system with time-varying damping:

$$\ddot{x}(t) + c(t)\dot{x}(t) + k_0 x(t) = 0, \quad x(0) = x_{10}\,, \quad \dot{x}(0) = x_{20}\,, \tag{31}$$

where $c(t) \geq 0$ is the time-varying damping coefficient and k_0 is the spring constant. Intuitively, one may suspect that as long as $c(t)$ is strictly larger than some positive constant, then the system should return to its equilibrium point $(x, \dot{x}) = (0, 0)$ asymptotically. This is the case with the *autonomous* second-order mass-spring damper system with *constant damping*:

$$\ddot{x}(t) + c_0\dot{x}(t) + k_0 x(t) = 0, \quad x(0) = x_{10}\,, \quad \dot{x}(0) = x_{20}\,, \quad c_0 \geq 0\,,$$

which can be easily demonstrated to have GAS at the origin similar to the example in (18) using invariant set arguments. However, if we select $c(t) = 2 + \mathrm{e}^t$ and $k_0 = 1$, then the solution of (31) from initial conditions $x(0) = 2\,, \dot{x}(0) = -1$ can be written as $x(t) = 1 + \mathrm{e}^{-t}$, which tends to $(x, \dot{x}) = (1, 0)$ instead of the equilibrium!!! The message is that the time-varying damping grows so fast that the system gets stuck at $x = 1$! We will get back to this example with a theorem at hand.

Another counter-intuitive example can be drawn from stability analysis of linear time-varying systems, by considering the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & \mathrm{e}^{2t} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_1(0) = x_{10}, \quad x_2(0) = x_{20}\,,$$

that has an $A(t)$ matrix with negative eigenvalues, equal to -1, for all $t \geq 0$, yet solving the second (decoupled) equation and substituting into the first one leads to

$$x_2(t) = x_2(0)\mathrm{e}^{-t}, \quad \dot{x}_1(t) + x_1(t) = x_2(0)\mathrm{e}^{t}\,,$$

which basically demonstrates that $x_1(t)$ can escape to infinity, since it is driven by an unbounded input.

Thus, one needs to be careful with stability analysis of non-autonomous systems. As we mentioned in the beginning, for trajectory tracking problems the error dynamics are non-autonomous, even if the nominal system is autonomous to begin with (see eq. (14) and the discussion around it). Therefore, non-autonomous systems present a more general class of systems, and therefore we need to generalize the concepts of equilibrium and definitions of stability for these systems as well. Towards that end, we consider the following general non-autonomous system dynamics:

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0\,, \quad x \in \mathbb{R}^n\,, \tag{32}$$

where $f : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0, \infty) \times \mathcal{D}$, where $\mathcal{D}$ is an open set containing the origin $x = 0$.

Definition 4.1. The equilibrium points x^* of (32) are defined via the solution of the following system of equation

$$f(t, x^*) \equiv 0, \quad t \geq t_0,$$

implying that if once the system state is at x^*, it remains there for all $t \geq t_0$.

- The linear time-varying system $\dot{x}(t) = A(t)x(t)$ has a unique equilibrium point at $x = 0$, unless $A(t)$ is singular for all $t \geq 0$.
- The nonlinear system $\dot{x}(t) = x^2(t) + b(t)$ with $b(t) \neq 0$ has no equilibria at all.
- The nonlinear system $\dot{x}(t) = (x(t) - 1)^2 b(t)$ with $b(t) \neq 0$ has a unique equilibrium at $x = 1$.
- The nonlinear system $\dot{x}(t) = (x(t) - 1)(x(t) - 2)b(t)$ with $b(t) \neq 0$ has two equilibrium points at $x = 1$ and $x = 2$.

The definitions of stability, AS, GAS, GES, etc. can be extended to time-varying systems with consideration of the initial time-instant. An important concept that needs to be kept in attention in analysis of time-varying systems is *uniformity*. Uniformity in time is important in order to ensure that the region of attraction does not vanish (tend to zero) with time. The following example demonstrates this intuition.

Example 4.1. The first order system

$$\dot{x}(t) = -\frac{x}{1+t}, \quad x(t_0) = x_0$$

has the general solution

$$x(t) = \frac{1+t_0}{1+t} x_0 \,,$$

which asymptotically converges to zero as $t \to \infty$, but not uniformly, because larger t_0 in the numerator requires longer time for convergence from the same initial condition x_0. To get the same time for convergence one needs to decrease $||x_0||$ uniformly with the increase of t_0, which implies shrinking of the domain of attraction with time.

Without loss of generality, we will assume that the origin $x = 0$ is an equilibrium point for (32), i.e. $f(t, 0) = 0$ for all $t \geq t_0$. Recall that arbitrary non-zero equilibrium can be translated to the origin via a change of coordinates (see eq. (14) and the discussion around it). We will therefore give definitions of stability for the equilibrium at the origin.

Definition 4.2. The equilibrium point $x = 0$ for the system (32) is

- *stable* if, for each $\epsilon > 0$, there exists $\delta(\epsilon, t_0) > 0$ such that

$$\|x(t_0)\| \leq \delta \quad \Rightarrow \quad \|x(t)\| \leq \epsilon \quad \forall\, t \geq t_0 \geq 0.$$

- *uniformly stable* if it is stable with δ independent of t_0.
- *unstable* if it is not stable.
- *asymptotically stable* if it is stable and there exists $c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty$ for all $\|x(t_0)\| < c(t_0)$.
- *uniformly asymptotically stable* if it is uniformly stable, asymptotically stable with c independent of t_0, and convergence of $x(t)$ to 0 is uniform in t_0. That is, for every $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that

$$\|x(t)\| \leq \epsilon \quad \forall\, t \geq t_0 + T(\epsilon) \;\text{ and }\; \|x(t_0)\| < c.$$

- *(uniformly) exponentially stable* if there exist two positive numbers α and λ such that for sufficiently small $x_0 = x(t_0)$

$$||x(t)|| \leq \alpha ||x_0|| \exp(-\lambda(t - t_0)), \quad \forall t \geq t_0.$$

- *globally uniformly asymptotically stable* if it is uniformly asymptotically stable and ϵ and c can be chosen arbitrarily large.

- *globally (uniformly) exponentially stable* if the definition of exponential stability holds for arbitrary x_0.

We immediately notice that the definition of exponential stability does not have the "non-uniform version". Recall that for stability analysis of autonomous systems we used positive definite functions, which we called Lyapunov functions. A natural extension of that concept to time-varying systems would be consideration of time-varying positive definite functions, given by the following definition.

Definition 4.3. A scalar time-varying function $V(t,x)$ is locally positive definite if $V(t,0) = 0$ and there exists a time-invariant positive definite function $W_1(x)$ such that

$$\forall\ t \geq t_0, \quad V(t,x) \geq W_1(x)\,.$$

The essence of the definition is that a time-varying function is locally positive definite if there exists a locally positive definite time-invariant function such that for all $t \geq t_0$ the time-varying function dominates the time-invariant one[‖]

Definition 4.4. A scalar time-varying function $V(t,x)$ is decrescent if $V(t,0) = 0$ and there exists a time-invariant positive definite function $W_2(x)$ such that $V(t,x) \leq W_2(x)$ for all $t \geq t_0$.

To generalize these concepts, we introduce the comparison functions.

Definition 4.5. A continuous function $\alpha : [0,a) \to \mathbb{R}^+$ belongs to *class $\mathcal{K}$* if it is strictly increasing and $\alpha(0) = 0$. It belongs to *class $\mathcal{K}_\infty$* if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

[‖]Similarly one can define global positive definiteness and semidefiniteness, etc.

Definition 4.6. A continuous function $\beta : [0, a) \times \mathbb{R}^+ \to \mathbb{R}^+$ belongs to *class $\mathcal{KL}$* if

- $\beta(r, s)$ is class $\mathcal{K}$ with respect to r for each fixed s, and
- $\beta(r, s)$ is decreasing in s for each fixed r and $\beta(r, s) \to 0$ as $s \to \infty$.

Examples:

- Class $\mathcal{K}$: $\alpha(r) = \tan(r)$ for $r \in [0, \frac{\pi}{2})$; $\alpha(r) = \tanh(r)$ for $r \in [0, \infty)$;
- Class $\mathcal{K}_\infty$: $\alpha(r) = kr$ for $k > 0$; $\alpha(r) = r^c$ for $c > 0$; $\alpha(r) = \sinh(r)$;
- Class $\mathcal{KL}$: $\beta(r, s) = kre^{-s}$ for $k > 0$.

Properties: Suppose α_1 and α_2 are class $\mathcal{K}$ on $[0, a)$, α_3 and α_4 are class $\mathcal{K}_\infty$, and β is class $\mathcal{KL}$.

- α_1^{-1} is class $\mathcal{K}$ on $[0, \alpha_1(a))$.
- α_3^{-1} is class $\mathcal{K}_\infty$.
- $\alpha_1 \circ \alpha_2$ is class $\mathcal{K}$ on $[0, a)$.
- $\alpha_3 \circ \alpha_4$ is class $\mathcal{K}_\infty$.
- $\alpha_1(\beta(\alpha_2(r), s))$ is class $\mathcal{KL}$.

Example 4.2. Given a locally Lipschitz, class $\mathcal{K}$ function α on $[0, a)$, one can construct a class $\mathcal{KL}$ function on $[0, a) \times [0, \infty)$ by solving the ODE

$$\dot{z} = -\alpha(z), \qquad z(t_0) = z_0$$

for $z_0 \in [0, a)$. For example, consider the class $\mathcal{K}_\infty$ function $\alpha(r) = kr$ where $k > 0$. Solving

$$\dot{z} = -kz, \qquad z(t_0) = z_0$$

gives

$$z(t) = z_0 e^{-k(t-t_0)}.$$

The function $\sigma(r, s) = re^{-ks}$ is class $\mathcal{KL}$.

The next lemma demonstrates the connection between positive definite functions and comparison functions.

Lemma 4.1. Let $V(x)$ be continuous and positive definite in a ball $\mathcal{B}_r \subset \mathbb{R}^n$. Then there exist locally Lipschitz, class $\mathcal{K}$ functions α_1 and α_2 on $[0, r)$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for all $x \in \mathcal{B}_r$. Moreover, if $V(x)$ is defined on all of $\mathbb{R}^n$ and is radially unbounded, then there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 such that the above inequality holds for all $x \in \mathbb{R}^n$.

An example that demonstrates these features can be drawn from the linear time-invariant system analysis that we presented in the previous lecture, by considering the Lyapunov function $V(x) = x^\top Px$ that verifies

$$\lambda_{\min}(P)||x||^2 \leq x^\top Px \leq \lambda_{\max}(P)||x||^2 \,.$$

Obviously the left and right hand side of this inequality are class $\mathcal{K}_\infty$ functions of $||x||$. Another important observation is that while Lyapunov functions are always defined on $\mathbb{R}^n$ or $[0, \infty) \times \mathbb{R}^n$, where n is the dimension of the state vector, so that $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, the comparison functions are defined to map at most $\mathbb{R}^2 \to \mathbb{R}$, i.e. are not related to the dimension of the state vector (they are not a function of vector argument).

These observations suggest that the stability definitions and theorems can be re-worded in terms of class $\mathcal{K}$ and $\mathcal{KL}$ functions. Let's look into the time-invariant case first. In the proof of Theorem 2.1, we needed to choose β and δ such that $\mathcal{B}_\delta \subset \Omega_\beta \subset \mathcal{B}_r$ so that a solution starting from $||x(0)|| \leq \delta$ verifies $||x(t)|| \leq \epsilon$ for all $t \geq 0$ to comply with the definition of stability at the origin. Now we know that a positive definite function $V(x)$ satisfies

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \,, \tag{33}$$

where $\alpha_i(\cdot)$ are class $\mathcal{K}$ functions. So, let's choose $\beta \leq \alpha_1(r)$, where $r < \epsilon$, and choose $\delta \leq \alpha_2^{-1}(\beta)$. Then from (33) we have

$$||x_0|| \leq \delta \quad \Rightarrow \quad V(x_0) \leq \alpha_2(\delta) \leq \beta$$

and $V(x) \leq \beta$ implies

$$\alpha_1(||x||) \leq V(x) \leq V(x_0) \leq \beta \leq \alpha_1(r) \quad \Rightarrow \quad \alpha_1(||x||) \leq \alpha_1(r) \quad \Leftrightarrow \quad ||x|| \leq r \,.$$

In the same proof we wanted to show that when $\dot{V}(x(t))$ is negative definite, then the solution tends to zero as $t \to \infty$. Since $\dot{V}(x(t))$ is negative definite, then there exists a class K function α_3 such that $\dot{V}(x) \leq -\alpha_3(||x||)$. Further, $V(x) \leq \alpha_2(||x||)$ implies that $\alpha_2^{-1}(V(x)) \leq ||x||$, and hence $\alpha_3(\alpha_2^{-1}(V(x))) \leq \alpha_3(||x||)$, which leads to $-\alpha_3(||x||) \leq -\alpha_3(\alpha_2^{-1}(V(x)))$. Hence $\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V))$. Comparison lemma then states that $V(t)$ is bounded by the solution of a scalar differential equation

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \quad y(0) = V(x(0))\,.$$

From the properties of class $\mathcal{K}$ functions we know that $\alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function, while Example 4.2 demonstrated that the solution of $\dot{y} = -\alpha_3(\alpha_2^{-1}(y))$ should be class $\mathcal{KL}$ function, i.e. $y(t) = \beta(y(0), t)$, and β is class $\mathcal{KL}$. Consequently, $V(x(t)) \leq \beta(V(x(0)), t)$, which shows that $V(x(t))$ goes to zero as $t \to \infty$. Moreover, taking this viewpoint of revising the results of Theorem 2.1, we can give the rates of convergence and get further insights into the notion of exponential stability. Indeed, $V(x(t)) \leq V(x(0))$ implies

$$\alpha_1(\|x(t)\|) \leq V(x(t)) \leq V(x(0)) \leq \alpha_2(\|x(0)\|)\,,$$

which leads to the following obvious upper bound

$$||x(t)|| \leq \alpha^{-1}(\alpha_2(||x(0)||))$$

where $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function. This is the stability. On the other hand $V(x(t)) \leq \beta(V(x(0)), t)$ implies that $\alpha_1(||x(t)||) \leq V(x(t)) \leq \beta(V(x(0)), t) \leq \beta(\alpha_2(||x(0)||, t)$, which leads to $||x(t)|| \leq \alpha_1^{-1}(\beta(\alpha_2(||x(0)||, t))$, where $\alpha_1^{-1}(\beta(\alpha_2(r), t))$ is a class $\mathcal{KL}$ function, implying asymptotic stability. It is straightforward to see that if $\alpha_1(||x||) = k_1||x||^p$ and $\alpha_2(||x||) = k_2||x||^p$ with obviously $k_2 > k_1$, then the derivations above imply exponential stability.

4.2 Lyapunov stability theorem for nonautonomous systems.

Before stating verifiable sufficient conditions for stability theorems for nonautonomous systems, we give a lemma establishing necessary and sufficient conditions for stability of equilibria of nonautonomous systems in terms of class $\mathcal{K}$ and $\mathcal{KL}$ functions.

Lemma 4.2. The equilibrium point $x = 0$ for the system (32) is

- *uniformly stable* if and only if there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a constant $c > 0$, independent of t_0, such that
$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall\, t \geq t_0 \geq 0 \;\text{ and }\; \|x(t_0)\| < c. \tag{34}$$

- *uniformly asymptotically stable* if and only if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a constant $c > 0$, independent of t_0, such that
$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \geq t_0 \geq 0 \;\text{ and }\; \|x(t_0)\| < c. \tag{35}$$

- *globally uniformly asymptotically stable* if it is uniformly asymptotically stable and (35) holds for any $x(t_0)$.

Remark 4.1. If in the definition of uniform asymptotic stability $\beta(\cdot,\cdot)$ takes the form $\beta(r, s) = kr\mathrm{e}^{-\lambda s}$, then one recovers the definition of exponential stability.

Theorem 4.1. [Lyapunov's Stability Theorem for Nonautonomous Systems.] Consider the system (32) with an equilibrium at the origin. Let $V : [0, \infty) \times \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{36}$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) \leq 0 \tag{37}$$

for all $t \geq 0$ and $x \in \mathcal{D}$, where W_1 and W_2 are continuous and positive definite. Then $x = 0$ is *uniformly stable*.

If the assumption in (37) can be verified as

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -W_3(x)\,, \tag{38}$$

where W_3 is a continuous and positive definite function in $\mathcal{D}$, then $x = 0$ is *uniformly asymptotically stable*. Moreover, letting $\mathcal{B}_r = \{x \mid ||x|| \leq r\} \subset \mathcal{D}$ and $c < \min_{||x||=r} W_1(x)$, every trajectory starting in $\{x \in \mathcal{B}_r \mid W_2(x) \leq c\}$ satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \quad \forall\ t \geq t_0 \geq 0$$

for some class $\mathcal{KL}$ function β. If $\mathcal{D} = \mathbb{R}^n$, and $W_1(x)$ is radially unbounded, then $x = 0$ is *globally uniformly asymptotically stable*.

Finally, if $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$ can be selected to verify

$$k_1\|x\|^p \leq V(t,x) \leq k_2\|x\|^p\,, \qquad t \in [0,\infty), \quad x \in \mathcal{D} \tag{39}$$

$$\dot{V}(t,x) \leq -k_3\|x\|^p\,, \qquad t \in [0,\infty), \quad x \in \mathcal{D} \tag{40}$$

for some positive constants k_1, k_2, k_3, p, *where the norm and the power are the same in (39), (40)*, then the origin is *locally (uniformly) exponentially stable*. If V is continuously differentiable for all $[0,\infty) \times \mathbb{R}^n$, and (39), (40) hold for all $[0,\infty) \times \mathbb{R}^n$, then the origin is *globally (uniformly) exponentially stable*.

Proof: *Uniform Stability:* Choose $r > 0$ small enough that $\mathcal{B}_r \subset \mathcal{D}$ and choose c such that

$$0 < c < \alpha = \min_{\|x\|=r} W_1(x).$$

Then

$$\mathcal{S}_{\text{outer}} = \{x \in \mathcal{B}_r \mid W_1(x) \leq c\}$$

is in the interior of $\mathcal{B}_r$. Let

$$\Omega_{t,c} = \{x \in \mathcal{B}_r \mid V(t,x) \leq c\}$$

and note that $\Omega_{t,c} \subseteq \mathcal{S}_{\text{outer}}$ because $V(t,x) \leq c$ implies that $W_1(x) \leq c$. On the other hand,

$$\mathcal{S}_{\text{inner}} = \{x \in \mathcal{B}_r \mid W_2(x) \leq c\} \subseteq \Omega_{t,c}$$

because $W_2(x) \leq c$ implies that $V(t,x) \leq c$. To recap,

$$\mathcal{S}_{\text{inner}} \subseteq \Omega_{t,c} \subseteq \mathcal{S}_{\text{outer}} \subset \mathcal{B}_r \subset \mathcal{D}$$

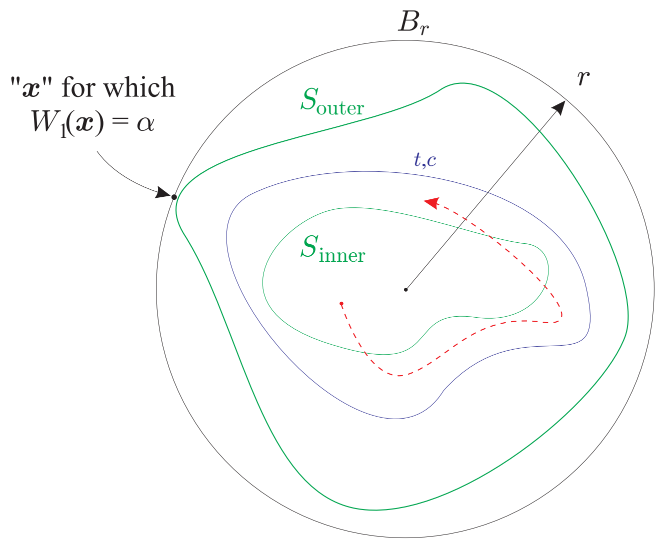


Fig. 12 The sets in the proof of stability theorem for non-autonomous systems

for all $t \geq 0$. See Figure 12, which is similar to Figure 9 with the only difference that $\Omega_{t,c}$ is time-varying.

Now, since $\dot{V}(t,x) \leq 0$ in $\mathcal{D}$, any trajectory $x(t)$ for which $x(t_0) = x_0 \in \Omega_{t_0,c}$ remains in $\Omega_{t,c}$ for all $t \geq t_0$. (See Theorem 3.3 in [11].) Thus, any trajectory starting in the (time-invariant) set $\mathcal{S}_{\text{inner}}$ remains in $\Omega_{t,c} \subset \mathcal{S}_{\text{outer}}$ for all $t \geq t_0$. The trajectory is therefore bounded and exists for all $t \geq t_0$.

We know that there exist locally Lipschitz class $\mathcal{K}$ functions α_1 and α_2 such that

$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|).$$

It follows that

$$\begin{aligned} \|x(t)\| &\leq \alpha_1^{-1}(V(t,x)) \\ &\leq \alpha_1^{-1}(V(t_0,x_0)) \quad (\text{since } \dot{V} \leq 0) \\ &\leq \alpha_1^{-1}(\alpha_2(\|x_0\|)). \end{aligned}$$

Because $\alpha_1^{-1} \circ \alpha_2$ is class $\mathcal{K}$, it follows from Lemma 4.2 that the origin is *uniformly stable.*

Uniform Asymptotic Stability: In the case where W_3 is positive definite in $\mathcal{D}$, there exists a locally Lipschitz class $\mathcal{K}$ function α_3 such that

$$W_3(x) \geq \alpha_3(\|x\|).$$

We therefore have

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \quad \text{and} \quad \dot{V} \leq -\alpha_3(\|x\|)$$

It follows that $\alpha_2^{-1}(V(t,x)) \leq \|x\|$, and hence $\alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(\|x\|)$, which upon sign reversal leads to

$$\dot{V} \leq -\alpha_3(\|x\|) \leq -\alpha_3(\alpha_2^{-1}(V)) =: -\alpha(V)$$

where $\alpha = \alpha_3 \circ \alpha_2^{-1}$ is class $\mathcal{K}$ and can be assumed to be locally Lipschitz without loss of generality. By the comparison lemma, one finds that for any $V(t_0, x_0) \in [0, c]$

$$V(t,x) \leq \sigma(V(x(t_0), t_0), t - t_0) \leq \sigma(\alpha_2(\|x(t_0)\|), t - t_0)$$

where $\sigma(r,s)$ is class $\mathcal{KL}$. It follows that any solution starting in $\{x \in \mathcal{B}_r \mid W_2(x) \leq c\}$ satisfies

$$\|x(t)\| \leq \alpha_1^{-1}\left(V(t, x(t))\right) \leq \alpha_1^{-1}\left(\sigma(\alpha_2(\|x(t_0)\|), t - t_0)\right) =: \beta(\|x(t_0)\|, t - t_0) \tag{41}$$

where β is class $\mathcal{KL}$. This implies that $x = 0$ is uniformly asymptotically stable.

Global Uniform Asymptotic Stability: Global uniform asymptotic stability follows from the assumption that $W_1(x)$ is radially unbounded. Indeed, in that case c can be chosen arbitrary large to include any initial state in $\{W_2(x) \leq c\}$. Then (41) can be proven for any initial state, assuming that $\alpha_1, \alpha_2, \alpha_3$ are also defined on $[0, \infty)$ and hence are independent of c.

Exponential Stability: The set $\{W_2(x) \leq c\}$ reduces to $\{k_2||x||^p \leq c\}$. The conditions in (39), (40) imply that

$$\dot{V}(t,x) \leq -\frac{k_3}{k_2} V(t,x).$$

Application of comparison lemma leads to the following upper bound

$$V(t,x) \leq V(x(t_0), t - t_0)\mathrm{e}^{-\frac{k_3}{k_2}(t - t_0)}.$$

And since $k_1\|x\|^p \leq V(t,x)$, we find that

$$\|x(t)\| \leq \left(\frac{1}{k_1} V(x(t_0), t - t_0)\mathrm{e}^{-\frac{k_3}{k_2}(t-t_0)}\right)^{\frac{1}{p}} \tag{42}$$

$$\leq \left(\frac{k_2}{k_1}\|x(t_0)\|^p \mathrm{e}^{-\frac{k_3}{k_2}(t-t_0)}\right)^{\frac{1}{p}} \tag{43}$$

$$= \|x(t_0)\| \left(\frac{k_2}{k_1}\mathrm{e}^{-\frac{k_3}{k_2}(t-t_0)}\right)^{\frac{1}{p}}, \tag{44}$$

proving exponential stability.

Global Exponential Stability: If the assumptions hold globally, then c can be chosen arbitrary large to imply global exponential stability. $\square$

Example 4.3. Let's prove that the origin of the scalar system

$$\dot{x} = -(1+\sin^2(t))x^3$$

is globally uniformly asymptotically stable. Consider the following candidate Lyapunov function $V(x) = \frac{1}{2}x^2$. Then

$$\dot{V} = -(1+\sin^2(t))x^4 \leq -x^4.$$

Thus, letting $W_1(x) = W_2(x) = \frac{1}{2}x^2$ and letting $W_3(x) = x^4$, we see that all conditions of the preceding theorem for global uniform asymptotic stability are satisfied.

Remark 4.2. The example basically demonstrates that there is nothing wrong with the selection of a time-invariant Lyapunov function for a time-varying system to prove stability. All one has to ensure that the conditions of Theorem 4.1 are satisfied.

Example 4.4. Now let's get back to the example with time-varying damping:

$$\ddot{x}(t) + c(t)\dot{x}(t) + k_0 x(t) = 0, \quad x(0) = x_{10}\,, \quad \dot{x}(0) = x_{20}\,, \tag{45}$$

and consider the following positive definite candidate Lyapunov function

$$V(t,x) = \frac{1}{2}(\dot{x}(t) + \alpha x(t))^2 + \frac{1}{2}b(t)x^2(t)\,,$$

where $\alpha < \sqrt{k_0}$ is any positive constant, and $b(t) = k_0 - \alpha^2 + \alpha c(t)$. Then

$$\dot{V}(t) = (\alpha - c(t))\dot{x}^2(t) + \frac{1}{2}\alpha(\dot{c}(t) - 2k_0)x^2(t)\,.$$

Thus, if one requires existence of positive numbers α and β such that $c(t) > \alpha > 0$ and $\dot{c}(t) \leq \beta < 2k_0$, then $\dot{V}(t)$ will be negative semidefinite leading to stability of the origin. If, in addition, one requires $c(t)$

to be upper bounded (guaranteeing decrescence of V), the above requirements will lead to asymptotic stability.

In fact, in [15] the assumption on the upper boundedness of $c(t)$ has been relaxed. The system $\ddot{x} + (2 + 8t)\dot{x} + 5x = 0$ is shown to have an asymptotically stable equilibrium point at the origin.

Finally, let's see what is required for the linear time-varying system to have an exponentially stable origin.

Theorem 4.2. A linear time-varying system

$$\dot{x}(t) = A(t)x(t)\,, \quad x(t_0) = x_0$$

has a globally exponentially stable equilibrium at the origin, if the eigenvalues of the symmetric matrix $A(t) + A^\top(t)$ remain strictly in the open left-half complex plane for all $t \geq 0$:

$$\exists\ \lambda > 0, \quad s.t.\ \ \forall\ i, \quad \forall\ t \geq 0, \quad \lambda_i(A(t) + A^\top(t)) \leq -\lambda\,.$$

Proof: Indeed, the following Lyapunov function

$$V(x(t)) = x^\top(t)x(t)$$

has negative definite derivative

$$\dot{V}(x(t)) = x^\top(t)\dot{x}(t) + \dot{x}^\top(t)x(t) = x^\top(t)(A(t) + A^\top(t))x(t) \leq -\lambda x^\top(t)x(t) = -\lambda V(x(t))\,,$$

so that for all $t \geq 0$ one has

$$0 \leq x^\top(t)x(t) = V(x(t)) \leq V(0)\mathrm{e}^{-\lambda t}\,.$$

Hence, the state converges to zero exponentially with a rate of at least $\lambda/2$. □

Remark 4.3. First notice that the proof also goes through when $A(t)$ depends explicitly upon the state as well, like $A(t, x(t))$. Also, it is a sufficient condition: some systems may fail to verify it, but you may still be able to prove stability of the origin. (See, for example, Example 4.8 on page 115 in [17].

4.3 Barbalat's lemma.

You should recall that for autonomous systems we introduced La-Salle's principle, which helped us to claim asymptotic stability for the cases when the derivative of the Lyapunov function was negative semidefinite only. For time-varying systems, La-Salle's principle cannot be applied, and instead one uses Barbalat's lemma.

In more details, consider the preceding theorem on stability of nonautonomous systems and suppose it is only known that $\dot{V}(t,x) \leq 0$. Unlike in the time-invariant case, we can not apply La-Salle's invariance principle. To do so would require defining a compact, positively invariant set Ω and a set $E = \{x \in \Omega \mid \dot{V} = 0\}$. Because the system is time-varying, however, it is not clear how to define the set Ω or the set E. (Recall from the proof of the previous stability theorem that the set $\Omega_{t,c}$ is *time-varying*.)

Barbalat's lemma provides a tool, which is similar to La-Salle's principle in a sense that it gives conditions under which $\dot{V} \to 0$ and V converges to a constant value, say zero. We might then conclude that trajectories converge to the set where $V = 0$. This would tell us something about the behavior of trajectories, although it would not tell us as much as La-Salle's principle which says, further, that trajectories converge to the *largest invariant set* contained in the set where $V = 0$.

To illustrate some of the difficulties, consider the following comments based on the discussion in [17]. Consider a differentiable function $f(t)$.

1. If $\dot{f} \to 0$ as $t \to \infty$, it does *not* follow that f converges to a limit. As an example,

$$f(t) = \sin(\log t) \quad \Rightarrow \quad \dot{f}(t) = \frac{\cos(\log t)}{t} \to 0 \quad \text{as} \quad t \to \infty.$$

2. If f converges to a limit as $t \to \infty$, it does *not* follow that $\dot{f} \to 0$. As an example,

$$f(t) = e^{-t}\sin^2(e^{2t}) \to 0 \quad \text{as} \quad t \to \infty,$$

   however

$$\dot{f}(t) = 2e^{t}\sin(2e^{2t}) - e^{-t}\sin^2(e^{2t})$$

   does not tend to zero.

3. If $\dot{f} \leq 0$ and f is lower-bounded , then f has a limit as $t \to \infty$.

Lemma 4.3. [Barbalat's Lemma.] If the differentiable function $f(t)$ converges to a finite limit as $t \to \infty$ and if $\dot{f}$ is uniformly continuous, then $\dot{f} \to 0$ as $t \to \infty$.

An alternative formulation of Barbalat's lemma is the following.

Lemma 4.4. [Barbalat's Lemma.] If $f(t)$ is uniformly continuous for all $t \in [0, \infty)$, and

$$\lim_{t \to \infty} \int_0^t f(\tau) d\tau$$

exists and is finite, then

$$f(t) \to 0 \quad \text{as} \quad t \to \infty \,.$$

Recall that a function $f(t)$ is *continuous* on $[0, \infty)$ if for every $t_1 \geq 0$ and every $\epsilon > 0$ there exists a $\delta(t_1, \epsilon) > 0$ such that

$$|t - t_1| < \delta \quad \Rightarrow \quad |f(t) - f(t_1)| < \epsilon. \tag{46}$$

The function is called *uniformly continuous* on $[0, \infty)$ if δ does not depend on t_1. That is, a function is uniformly continuous if, given $\epsilon > 0$, the same $\delta(\epsilon)$ satisfies (46) at any time $t_1 \geq 0$. A sufficient condition for a function to be uniformly continuous is that its derivative be bounded.

Proof: Indeed, assume that $f(t) \nrightarrow 0$ as $t \to \infty$. Then there should exist a positive constant $k_1 > 0$ such that for every $T > 0$ we can find $T_1 \geq T$ with $|f(T_1)| \geq k_1$. Since $f(t)$ is uniformly continuous, there is a positive constant k_2 such that $|f(t + \tau) - f(t)| < k_1/2$ for all $t \geq 0$ and all $0 \leq \tau \leq k_2$. Hence,

$$|f(t)| = |f(t) - f(T_1) + f(T_1)| \geq |f(T_1)| - |f(t) - f(T_1)| > k_1 - \frac{1}{2}k_1 = \frac{1}{2}k_1, \quad \forall\, t \in [T_1, T_1 + k_2]\,.$$

Therefore

$$\Big| \int_{T_1}^{T_1+k_2} f(t) dt \Big| = \int_{T_1}^{T_1+k_2} |f(t)| dt > \frac{1}{2} k_1 k_2 \,,$$

where the equality holds, since $f(t)$ retains the same sign for $T_1 \leq t \leq T_1 + k_2$. Thus, $\int_0^t f(\tau) d\tau$ cannot converge to a finite limit as $t \to \infty$, which is a contradiction. $\square$

Corollary 4.1. If a scalar function $V(t, x)$ satisfies the conditions

- $V(t, x)$ is lower bounded,

- $\dot{V}(t,x)$ is negative semidefinite, and
- $\dot{V}(t,x)$ is uniformly continuous in time,

then $\dot{V}(t,x) \to 0$ as $t \to \infty$.

The corollary says that trajectories converge to a set E within which $\dot{V} = 0$. This conclusion is much weaker than La-Salle's invariance principle; we can not conclude that trajectories converge to the largest invariant set contained in E.

Example 4.5. Consider the closed-loop adaptive control system for the tracking problem:

$$\begin{aligned} \dot{x}(t) &= -x(t) + \theta(t)r(t) \\ \dot{\theta}(t) &= -x(t)r(t)\,, \end{aligned}$$

where $x(t)$ represents the state, $\theta(t)$ the adaptive parameter, and $r(t)$ is a reference input. Consider the quadratic function

$$V(x(t),\theta(t)) = \frac{1}{2}x^2(t) + \frac{1}{2}\theta^2(t).$$

Differentiating,

$$\dot{V}(t) = x(t)(-x(t) + \theta(t)r(t)) + \theta(t)(-x(t)r(t)) = -x^2(t) \leq 0$$

Because of the reference signal $r(t)$, the system is time-varying and we may not use La-Salle's invariance principle. To verify uniform continuity of $\dot{V}(t)$, we check that $\ddot{V}(t) = -2x(t)(-x(t) + \theta(t)r(t))$ is bounded. First note that, because $V(t)$ is positive definite and nonincreasing, $x(t)$ and $\theta(t)$ are bounded:

$$\left\| \begin{pmatrix} x(t) \\ \theta(t) \end{pmatrix} \right\| \leq V(0).$$

Assuming that the reference signal $r(t)$ is also bounded, it follows that $\ddot{V}$ is bounded and therefore that $\dot{V}$ is uniformly continuous. It then follows from the previous corollary to Barbalat's lemma that $x \to 0$ as $t \to \infty$. Note that we can conclude nothing more about the behavior of $\theta(t)$, except that it is bounded.

4.4 Linear Time-Varying Systems and Linearization

Next we introduce a couple of theorems specifically for the stability analysis of linear time-varying systems.

Lemma 4.5. Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0 \tag{47}$$

that has an equilibrium at $x = 0$. Let $A(t)$ be continuous for all $t \geq 0$. Suppose there exists a continuously differentiable, symmetric, bounded, positive definite matrix $P(t)$ such that

$$0 < c_1\mathbb{I} \leq P(t) \leq c_2\mathbb{I}, \quad \forall\, t \geq 0\,, \quad c_1 > 0, \quad c_2 > 0,$$

which satisfies the matrix differential equation

$$-\dot{P}(t) = P(t)A(t) + A^\top(t)P(t) + Q(t)\,, \tag{48}$$

where $Q(t)$ is continuous, symmetric and positive definite:

$$Q(t) \geq c_3\mathbb{I} > 0, \quad \forall\, t \geq 0, \quad c_3 > 0.$$

Then $x = 0$ is an exponentially stable equilibrium point for the system.

Proof: Indeed, $V(t,x) = x^\top P(t)x$ verifies

$$c_1||x||^2 \leq V(t,x) \leq c_2||x||^2\,,$$

and its derivative along the system trajectories $\dot{x}(t) = A(t)x(t)$ verifies

$$\dot{V}(t,x) = \dot{x}^\top P(t)x + x^\top \dot{P}(t)x + x^\top P(t)\dot{x} = -x^\top Q(t)x \leq -c_3||x||^2\,.$$

Thus, $V(t,x)$ verifies (39), (40), which is required for exponential stability. □

The stability of the linear time-varying system in (47) can be completely characterized by its state-transition matrix. From linear systems theory it is known that the solution to (47) is given by

$$x(t) = \Phi(t,t_0)x(t_0)\,,$$

where $\Phi(t,t_0)$ is the state transition matrix.

Lemma 4.6. The equilibrium $x = 0$ of (47) is GUAS, *if and only if* the state transition matrix satisfies the inequality

$$||\Phi(t, t_0)|| \leq k\mathrm{e}^{-\lambda(t-t_0)}, \quad \forall\, t \geq t_0 \geq 0$$

for some positive constants k and λ.

You can read the proof in [11]. Meantime, the key point to notice, is that **for linear time-varying systems uniform asymptotic stability of the origin is equivalent to exponential stability.** Since the knowledge of the state transition matrix requires solving the ODEs, this theorem is not as useful as the eigenvalue condition for LTI systems (all eigenvalues of A need to lie in open left-half plane).

Theorem 4.3. Let $x = 0$ be the exponentially stable equilibrium point of (47). Assume that $A(t)$ is continuous and bounded. Then for any $Q(t)$, which is continuous, symmetric, bounded and uniformly positive definite, there exists a continuously differentiable, bounded, positive definite, symmetric matrix $P(t)$ that satisfies (48). Then $V(t, x) = x^\top(t)P(t)x(t)$ is a Lyapunov function for the system that satisfies the conditions (39), (40).

Indeed, verification of (39), (40) follows from considering

$$P(t) = \int_t^\infty \Phi^\top(\tau, t)Q(\tau)\Phi(\tau, t)d\tau\,,$$

where $\Phi(t, t_0)$ is the state transition matrix and using the result of Lemma 4.5.

Finally, we formulate a sufficient condition for local exponential stability of the origin for the nonlinear time-varying system:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0\,. \tag{49}$$

Theorem 4.4. Let $x = 0$ be an equilibrium for (49), where $f : [0, \infty) \times \mathcal{B}_r \to \mathbb{R}^n$ is continuously differentiable, $\mathcal{B}_r = \{x \in \mathbb{R}^n \mid ||x|| < r\}$, and the Jacobian $\partial f/\partial x$ is bounded and Lipschitz on $\mathcal{B}_r$, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}\Big|_{x=0}\,.$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system, if it is an exponentially stable equilibrium point for the linear system

$$\dot{x}(t) = A(t)x(t).$$

Homework Problems 4.1. Exercises 4.38, 4.39 from [11] need to be solved completely.

5 Boundedness and Ultimate Boundedness.

[Read [11, p.168]]

To motivate the further derivations, consider the linear scalar system subject to an exogenous input:

$$\dot{x}(t) = -x(t) + \alpha \sin t, \quad x(t_0) = a, \quad a > \alpha > 0\,. \tag{50}$$

The solution is given by:

$$x(t) = ae^{-(t-t_0)} + \alpha \int_{t_0}^{t} e^{(t-\tau)} \sin \tau d\tau\,. \tag{51}$$

The solution can be bounded:

$$|x(t)| \le ae^{-(t-t_0)} + \alpha \int_{t_0}^{t} e^{(t-\tau)} d\tau = ae^{-(t-t_0)} + \alpha \left[1 - e^{-(t-t_0)}\right] \le a, \quad \forall\; t \ge t_0\,, \tag{52}$$

which shows that the solution is *uniformly bounded* for all $t \ge t_0$, uniformly in t_0, i.e. with a bound independent of t_0. While this bound is valid for all $t \ge t_0$, it does not take into consideration the exponentially decaying term, therefore it is a conservative bound. On the other hand, for any number b, satisfying $\alpha < b < a$, the following is true:

$$|x(t)| \le b, \qquad \forall t \ge t_0 + \ln\left(\frac{a-\alpha}{b-\alpha}\right) \tag{53}$$

The bound b, which again is independent of t_0, gives a better estimate of the solution after a transient period has passed. In this case, the solution is said to be *uniformly ultimately bounded*, and b is called the ultimate bound. Of course, if the solution is bounded, then it is ultimately bounded, and vice versa.

Properties of system boundedness and ultimate boundedness can be established via Lyapunov analysis similar to stability, asymptotic/exponential stability. To this end, consider the following Lyapunov function candidate $V = x^2/2$, and compute its derivative:

$$\dot{V}(t) = x\dot{x} = -x^2 + x\alpha \sin t \le -x^2 + \alpha|x| \tag{54}$$

Then $\dot{V}(t) < 0$ for all x outside the compact set $\{|x| \le \alpha\}$. For any $c > \alpha^2/2$, solutions starting in the set $\{V(x) \le c\}$ will remain therein for all future time, since $\dot{V}(t) < 0$ on the boundary $V = c$. Hence the solutions are uniformly bounded. Moreover, for any number ε, satisfying $\alpha^2/2 < \varepsilon < c$, $\dot{V}(t)$ will be negative in the set $\{\varepsilon \le V \le c\}$, which shows that in this set V will decrease monotonically until the solution enters the set $\{V \le \varepsilon\}$. From that time on, the solution cannot leave the set $\{V \le \varepsilon\}$, because $\dot{V}(t) < 0$ on the boundary $V = \varepsilon$. Therefore, the solution is uniformly ultimately bounded with ultimate bound $|x| \le \sqrt{2\varepsilon}$, where ε is arbitrary number satisfying $\varepsilon > \alpha^2/2$.

Definition 5.1. The solutions of the nonlinear system

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \tag{55}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ is the state of the system, $\mathcal{D}$ is an open set containing the origin, and $f(t, x) : \mathbb{R} \times \mathbf{D} \to \mathbb{R}^n$ is locally Lipschitz continuous function of its arguments, are

- *uniformly bounded* if there exists a positive constant γ, independent of t_0, such that for every $\delta \in (0, \gamma)$, there is $\beta = \beta(\delta) > 0$, independent of t_0, such that $\|x_0\| \le \delta$ implies $\|x(t)\| \le \beta$, $t \ge t_0$.

- *globally uniformly bounded* if for every $\delta \in (0, \infty)$, there is $\beta = \beta(\delta) > 0$, independent of t_0, such that $\|x_0\| \le \delta$ implies $\|x(t)\| \le \beta$, $t \ge t_0$.

- *uniformly ultimately bounded with ultimate bound* $b > 0$ if there exists $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists $T = T(\delta, b) > 0$ such that $\|x_0\| \le \delta$ implies $\|x(t)\| \le b$, $t \ge T$.

- *globally uniformly ultimately bounded*, if for every $\delta \in (0, \infty)$, there exists $T = T(\delta, b) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t)\| < b$, $t \ge T$.

The main difference of this definition from the definition of stability, is that ε is not required to be arbitrarily small! In the definition of stability, we start by arbitrarily small ε and require existence of $\delta(\varepsilon)$. A system that is ultimately bounded is not necessarily Lyapunov stable.

In case of autonomous systems the word "uniformly" can be dropped, since the solution depends only upon $t - t_0$.

To apply Lyapunov analysis for the study of boundedness and ultimate boundedness, consider a continuously differentiable, positive definite Lyapunov function candidate $V(x)$ and the set Λ, Fig. 13:

$$\Lambda = \{\varepsilon \le V(x) \le c\}, \qquad \varepsilon \le c\,. \tag{56}$$

For simplicity of the analysis below, we consider here a more limited class of time-invariant Lyapunov functions $V(x)$, as opposed to a more general structure $V(t, x)$ associated with the non-autonomous *system in (55) in all our previous theorems. This is only done for the sake of simplifying the introduction of new concepts. Extension to more general class of Lyapunov functions $V(t,x)$ follows in straightforward manner as previously done for stability proofs.*

Suppose that along the trajectories of the system (55) we have

$$\dot{V}(x) \leq -W(x), \qquad \forall\, x \in \Lambda,\ \forall\, t \geq t_0\,, \tag{57}$$

where $W(x)$ is a continuous positive definite function. Inequality (57) implies that the sets $\Omega_c = \{V(x) \leq c\}$ and $\Omega_\varepsilon = \{V(x) \leq \varepsilon\}$ are positively invariant, since on the boundaries $\partial\Omega_c$ and $\partial\Omega_\varepsilon$ the derivative is negative, $\dot{V}(t) < 0$. While in Λ, the conditions of the theorem on uniform asymptotic stability ([11], Theorem 4.8, 4.9) are satisfied, therefore

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \tag{58}$$

for some class $\mathcal{KL}$ function β. The function $V(x(t))$ is decreasing in Λ until the trajectory enters Ω_ε in *finite time*. To show that the trajectory enters Ω_ε in *finite time* and stays therein for all future time, consider the minimum $k = \min\limits_{x \in \Lambda} W(x) > 0$. The minimum exists because $W(x)$ is continuous and Λ is compact. Hence,

$$W(x) \geq k, \quad x \in \Lambda\,. \tag{59}$$

Thus, it follows from (57) that

$$\dot{V}(x) \leq -k, \qquad \forall\, x \in \Lambda,\ \ \forall\, t \geq t_0\,. \tag{60}$$

Therefore

$$V(x(t)) \leq V(x(t_0)) - k(t - t_0) \leq c - k(t - t_0)\,, \tag{61}$$

which shows that $V(x(t))$ reduces to ε within the *finite* time interval $[t_0, t_0 + (c - \varepsilon)/k]$.

It is the case in many of the proofs that the inequality (57) is being presented rather through norm inequalities than through definition of the set Λ, as we will see in the forthcoming argument:

$$\dot{V}(t,x) \leq -W(x), \qquad \forall\, x: \quad \mu \leq \|x\| \leq r,\ \ \forall\, t \geq t_0\,. \tag{62}$$

Assume that r is sufficiently large in comparison to μ, so that one can choose c and ε such that the set Λ is non-empty and contained in $\{\mu \leq \|x\| \leq r\}$. Let α_1 and α_2 be class $\mathcal{K}$ class functions s.t.

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)\,. \tag{63}$$

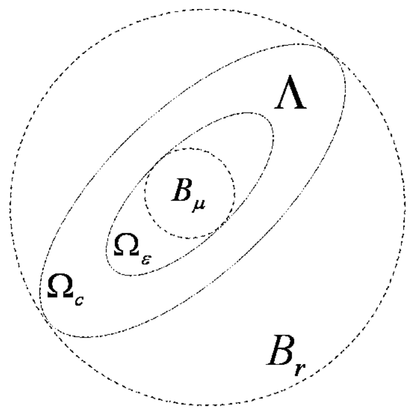


Fig. 13 Geometric Representation of the Sets in the Space of Variables.

From the left inequality of (63), we have

$$V(x) \le c \;\Rightarrow\; \alpha_1(\|x\|) \le c \quad\Leftrightarrow\quad \|x\| \le \alpha_1^{-1}(c)\,. \tag{64}$$

Therefore taking $c = \alpha_1(r)$ ensures that $\Omega_c \subset \mathcal{B}_r = \{x : \; \|x\| \le r\}$. On the other hand, from the right inequality of (63), we have

$$\|x\| \le \mu \;\Rightarrow\; V(x) \le \alpha_2(\mu)\,. \tag{65}$$

Consequently, taking $\varepsilon = \alpha_2(\mu)$ ensures that $\mathcal{B}_\mu \subset \Omega_\varepsilon, \;\; \mathcal{B}_\mu = \{x : \; \|x\| \le \mu\}$. To ensure $\varepsilon < c$, one needs to take $\mu < \alpha_2^{-1}(\alpha_1(r))$.

Thus, when the (57) is given via norm bounds on x, like in (62), instead of level sets of Lyapunov function, one needs to define the *maximum* level set of the Lyapunov function that lies inside the outer ball, and the *minimum* level set of the Lyapunov function lying outside the inner ball, and the set Λ is the one in between. Then it is easy to show that all the trajectories starting in Ω_c enter Ω_ε within a finite time T. Notice that if the trajectory starts in Ω_ε, then $T = 0$.

Indeed, from (65) and (63) it follows that

$$V(x) \le \varepsilon \;\Rightarrow\; \alpha_1(\|x\|) \le \varepsilon \quad\Leftrightarrow\quad \|x\| \le \alpha_1^{-1}(\varepsilon) \tag{66}$$

Since $\varepsilon = \alpha_2(\mu)$, then

$$x \in \Omega_\varepsilon \Leftrightarrow \|x\| \le \alpha_1^{-1}(\alpha_2(\mu)) \tag{67}$$

Therefore the ultimate bound can be taken as $b = \alpha_1^{-1}(\alpha_2(\mu))$.

The above considerations lead to the following statement of sufficient conditions for uniform ultimate boundedness and ultimate boundedness that we formulate for general Lyapunov function $V(t,x)$.

Theorem 5.1. [11] Consider the nonlinear system (55). Let $\mathcal{D}$ be a domain that contains the origin and $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function, $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be class $\mathcal{K}$ functions, and $W : \mathcal{D} \to \mathbb{R}$ be a positive-definite function such that:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad x \in \mathcal{D} \subset \mathbb{R}^n, \tag{68}$$

$$\dot{V}(t,x) \leq -W(x), \quad x \in \mathcal{D} \subset \mathbb{R}^n, \quad \|x\| > \mu, \tag{69}$$

where

$$\mu < \alpha_2^{-1}(\alpha_1(r)), \tag{70}$$

and r is the radius of the ball $\mathcal{B}_r = \{x : \|x\| \leq r\} \subset \mathcal{D}$. Then there exists a class $\mathcal{KL}$ function β such that for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$ there is $T \geq 0$ (dependent on $x(t_0)$ and μ) such that the solution of (55) satisfies

$$\|x(t)\| \quad \leq \quad \beta(\|x(t_0)\|, t - t_0), \qquad \forall t_0 \leq t \leq t_0 + T \tag{71}$$

$$\|x(t)) \quad \leq \quad \alpha_1^{-1}(\alpha_2(\mu)), \qquad \forall t \geq t_0 + T \tag{72}$$

Moreover, if $\mathcal{D} = \mathbb{R}^n$ and α_1 belongs to class K_∞, then (71) and (72) hold for any initial state $x(t_0)$ no matter how large is μ, i.e. the results are global.

The proof is similar to the proofs of stability, asymptotic stability and can be found in [11].

Theorem 5.1 requires existence of a positive definite Lyapunov function, having a negative definite derivative outside a compact set, to ensure boundedness of the system trajectories. Notice that it is important to have the compact set defined in the space of the entire argument of the Lyapunov function! If the system dynamics is given by

$$\dot{x}_1 \quad = \quad f_1(x_1, x_2) \tag{73}$$

$$\dot{x}_2 \quad = \quad f_2(x_1, x_2), \tag{74}$$

then the derivative of a possible Lyapunov function candidate $V(x_1, x_2)$ should satisfy

$$\dot{V}(t) < 0 \qquad \forall \ \|x\| > \mu, \tag{75}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$. If one obtains by the end of the proof

$$\dot{V}(t) < 0 \qquad \forall \; \|x_1\| > \mu \,, \tag{76}$$

which defines a stripe along the x_2 axis in the space of (x_1, x_2), then practically x_2 is allowed to escape to infinity along that axis and can destabilize the system. This requires to be especially careful when doing proofs on boundedness as opposed to stability. The proofs on stability in the previous sections ended with $\dot{V}(t) = -e^\top Qe < 0$, which was negative semidefinite in the space of the variables (e, θ) originally defining the Lyapunov function $V(e, \theta)$, therefore the solutions were Lyapunov stable, no matter that the last expression in $\dot{V}(t)$ was *independent* of the parameter errors. Loss of the negative square of one of the variables in $\dot{V}(t)$ expression was taking away the option of claiming asymptotic stability, which we were able to recover with the help of La Salle's principle or Barbalat's lemma. But at least we had the stability guaranteed!!!

The proofs on ultimate boundedness require to have the negative squares of *all the variables* by the end in the $\dot{V}(t)$ expression, to ensure **the existence of a compact set.** Loss of the negative square of one of the variables in $\dot{V}(t)$ expression can lead to a much worse situation (destabilization), like we observed during parameter drift. Lack of one of the negative squares takes away the opportunity to define the compact level sets and balls in the proof, and thus the validity of the claims leading to the main statement.

An alternative is the $\text{Proj}(\cdot, \cdot)$ operator that guarantees the boundedness of the parameter errors by definition of the adaptive law. Other modifications include e-modification and σ-modification of adaptive laws that we will introduce in systematic manner, one after another.

6 Converse Lyapunov Theorems

Until now, we have presented stability theorems, requiring existence of a Lyapunov function $V(t,x)$ that would verify certain (sufficient) conditions for the given system dynamics. In every case, we had to search for a candidate Lyapunov function with certain properties. Now we would like to pose and answer the inverse question: when does such a function exist and how to find it? The answer to this is given by *converse Lyapunov theorems*. Converse Lyapunov theorems prove existence of a Lyapunov function for stable systems and characterize some of its properties. Some of the converse theorems give even a constructive answer, but these constructive answers depend upon the actual solution of the system, and therefore in most of the cases these are not useful for practical analysis. First, we give the converse theorem for exponentially stable systems.

Theorem 6.1. Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{77}$$

where $f : [0, \infty) \times \mathcal{B}_r \to \mathbb{R}^n$ is continuously differentiable, $\mathcal{B}_r = \{x \in \mathbb{R}^n \mid ||x|| < r\}$, and the Jacobian matrix $\partial f/\partial x$ is bounded on $\mathcal{B}_r$, uniformly in t. Let k, λ and r_0 be positive constants with $r_0 < r/k$. Let $\mathcal{B}_0 = \{x \in \mathbb{R}^n \mid ||x|| < r_0\}$. Assume that the trajectories of the system satisfy

$$||x(t)|| \leq k||x(t_0)||\mathrm{e}^{-\lambda(t-t_0)}, \quad \forall\, x(t_0) \in \mathcal{B}_0, \quad \forall\, t \geq t_0 \geq 0\,. \tag{78}$$

Then, there exists a function $V : [0, \infty) \times \mathcal{B}_0 \to \mathbb{R}$ that satisfies

$$c_1||x||^2 \leq V(t,x) \leq c_2||x||^2 \tag{79a}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3||x||^2 \tag{79b}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4||x|| \tag{79c}$$

for some positive constants c_1, c_2, c_3, c_4. Moreover, if $r = \infty$, and the origin is GES (globally exponentially stable), then $V(t,x)$ is defined and satisfies the aforementioned inequalities on $\mathbb{R}^n$. Furthermore, if the system is autonomous, V can be chosen independent of t.

Proof: Let $\phi(t; t_0, x_0)$ be the unique solution corresponding to (t_0, x_0) initial condition, and let $\phi(\tau; t, x)$ denote unique solution starting at (t, x) so that $\phi(t; t, x) = x$. For all $x \in \mathcal{B}_0$ we have

$\phi(\tau; t, x) \in \mathcal{B}_r$ for all $\tau \geq t$. Let

$$V(t,x) = \int_t^{t+\delta} \phi^\top(\tau; t, x)\phi(\tau; t, x)d\tau\,,$$

where $\delta > 0$ is a design constant to be yet selected. From (78) it follows that the trajectories are exponentially decaying, which can be used to upper bound $\phi(\tau; t, x)$ in $V(t, x)$ as follows:

$$V(t,x) \leq \int_t^{t+\delta} k^2 \mathrm{e}^{-2\lambda(\tau-t)} d\tau \; ||x||^2 = \frac{k^2}{2\lambda}(1 - \mathrm{e}^{-2\lambda\delta})||x||^2\,.$$

On the other hand, since the Jacobian is bounded on $\mathcal{B}_r$, then

$$||f(t,x)|| \leq L||x||,$$

where L is the uniform bound for the Jacobian on $\mathcal{B}_r$. This leads to the following lower bound

$$||\phi(\tau; t, x)|| \geq ||x||^2 \mathrm{e}^{-2L(\tau-t)}\,,$$

which can be used to arrive at

$$V(t,x) \geq \frac{1}{2L}(1 - \mathrm{e}^{-2L\delta})||x||^2\,.$$

It remains only to notice that one can choose

$$c_1 = \frac{1}{2L}(1 - \mathrm{e}^{-2L\delta})\,, \quad c_2 = \frac{k^2}{2\lambda}(1 - \mathrm{e}^{-2\lambda\delta})\,,$$

to verify (79a).

The inequality in (79b) can be verified by explicitly computing $\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x)$, and choosing $\delta = \frac{\ln(2k^2)}{2\lambda}$ and $c_3 = \frac{1}{2}$ for upper bounding.

The inequality in (79c) can be verified by explicitly computing $\frac{\partial V}{\partial x}$ and choosing $c_4 = \frac{2k}{(\lambda - L)}(1 - \mathrm{e}^{-(\lambda-L)\delta})$ for upper bounding.

If all assumptions hold globally, then clearly r_0 can be chosen arbitrarily large. If the system is autonomous, then $\phi(\tau; t, x)$ depends only on $(\tau - t)$, i.e.

$$\phi(\tau; t, x) = \psi(\tau - t; x)\,.$$

Then

$$V(t,x) = \int_t^{t+\delta} \psi^\top(\tau - t; x)\phi(\tau - t; x)d\tau = \int_0^{\delta} \psi^\top(s; x)\phi(s; x)ds\,,$$

which is obviously independent of t. □

You are highly advised to read the detailed proof from [11].

Remark 6.1. We proved the theorem by explicit construction of the Lyapunov function, as an integral of the square of system's trajectory. We notice that the exponentially stable solution led to upper and lower bounds for $V(t,x)$ and $\dot{V}(t,x)$ in terms of powers of the system state.

Theorem 6.1 can be used to prove Lyapunov's indirect method in an elegant way.

Theorem 6.2. Let $x = 0$ be an equilibrium point for the nonlinear system (77), where $f : [0,\infty) \times \mathcal{B}_r \to \mathbb{R}^n$ is continuously differentiable, $\mathcal{B}_r = \{x \in \mathbb{R}^n \mid ||x|| < r\}$, and the Jacobian matrix $\partial f/\partial x$ is bounded and Lipschitz on $\mathcal{B}_r$, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}(t,x)\Big|_{x=0}.$$

Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if it is an exponentially stable equilibrium point for the linear time-varying system:

$$\dot{x}(t) = A(t)x(t)\,.$$

Proof: **IF:** The **IF** part follows from Theorem 4.4.

ONLY IF: To prove the **ONLY IF** part, notice that the linearized system can be equivalently represented as:

$$\dot{x}(t) = A(t)x(t) = f(t,x) - \underbrace{(f(t,x) - A(t,x))}_{g(t,x)},$$

where $g(t,x)$ lumps the higher order terms, truncated during the linearization:

$$f(t,x) = A(t)x + g(t,x)\,,$$

and can be upper bounded

$$||g(t,x)|| \leq L||x||^2, \quad \forall\; t \geq 0, \quad \forall\; x \in \mathcal{B}_r\,.$$

Read [11], pages 160-161, on linearization of functions of vector argument and the corresponding upper bounding.

Since the origin is an exponentially stable equilibrium of the nonlinear system, there exist positive constants such that

$$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)}, \quad \forall\, t \geq 0, \ \ \forall\, ||x(t_0)|| \leq c\,.$$

Choosing $r_0 < \min\{c, r/k\}$, we notice that all the conditions of Theorem 6.1 are satisfied. Then, there exists a Lyapunov function $V(t,x)$ that verifies the claims of Theorem 6.1. Let's compute

$$\dot{V}(t,x(t)) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}A(t)x(t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) - \frac{\partial V}{\partial x}g(t,x)\,.$$

Using conditions (79b) and (79c), and the upper bound from $||g(t,x)|| \leq L||x||^2$, we can further derive the following **local upper bound**:

$$\dot{V}(t,x(t)) \leq -c_3||x||^2 + c_4L||x||^3 < -(c_3 - c_4L\rho)||x||^2, \quad \forall\, ||x|| < \rho\,.$$

Choosing $\rho < \min\{r_0, c_3/(c_4L)\}$ will ensure that $\dot{V}(t,x)$ is negative definite in $||x|| < \rho$. Thus, all the conditions (39), (40) are satisfied locally, and therefore the origin is a locally exponentially stable equilibrium for the linear time-varying system. □

Corollary 6.1. Let $x = 0$ be an equilibrium point of the nonlinear system $\dot{x} = f(x)$, where $f(x)$ is continuously differentiable in some neighborhood of $x = 0$. Let

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}.$$

Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system **if an only if** A is Hurwitz.

Example 6.1. Recall Example 1.4, where we had

$$\dot{x}(t) = -x^3(t)\,.$$

Linearization of this system around the origin results in the linear system $\dot{x} = 0$, whose A matrix is not Hurwitz. From Theorem 6.2 we conclude that the origin is **not exponentially stable**, although it **is asymptotically stable** as it can be verified via the candidate Lyapunov function $V = x^4$ using Lyapunov's direct method. Indeed $\dot{V} = 4x^3\dot{x} = -4x^6 < 0, \ \forall\, x \neq 0$.

The converse theorems on local uniform asymptotic stability for time-varying systems and global asymptotic stability for time-invariant systems have more involved proofs, and therefore we will present them without the proofs, emphasizing the key features and differences from the one on exponential stability.

Theorem 6.3. Let $x = 0$ be an equilibrium point for the nonlinear system (77), where $f : [0, \infty) \times \mathcal{B}_r \to \mathbb{R}^n$ is continuously differentiable, $\mathcal{B}_r = \{x \in \mathbb{R}^n \mid ||x|| < r\}$, and the Jacobian matrix $\partial f/\partial x$ is bounded on $\mathcal{B}_r$, uniformly in t. Let β be a class $\mathcal{KL}$ function and r_0 be a positive constant such that $\beta(r_0, 0) < r$. Let $\mathcal{B}_0 = \{x \in \mathbb{R}^n \mid ||x|| < r_0\}$. Assume that the origin is a uniformly asymptotically stable equilibrium point, i.e. the trajectories of the system satisfy

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \quad \forall\, x(t_0) \in \mathcal{B}_0, \quad \forall\, t \geq t_0 \geq 0\,.$$

Then there is a continuously differentiable function $V : [0, \infty) \times \mathcal{B}_0 \to \mathbb{R}$ that satisfies the inequalities

$$\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||) \tag{80a}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(||x||) \tag{80b}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_4(||x||)\,, \tag{80c}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t.

Remark 6.2. In fact, the assumption on continuous differentiability of $f(t, x)$ can be relaxed to locally Lipschitz condition, and the interested reader is referred to [25] on this (Theorem 14 in that reference). A theorem on global uniform asymptotic stability is summarized in Theorem 23 in [25].

Theorem 6.4. Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(x),$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz, and $\mathcal{D}$ is a domain that contains the origin. Let $\Omega \in \mathcal{D}$ be the region of attraction of $x = 0$. Then there is a continuously differentiable positive definite function $V(x)$

and a continuous positive definite function $W(x)$, both defined for all $x \in \Omega$, such that

$$V(x) \to \infty, \quad \text{as} \quad x \to \partial\Omega \tag{81a}$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall\, x \in \Omega, \tag{81b}$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of Ω. When $\Omega = \mathbb{R}^n$, $V(x)$ is radially unbounded.

7 Model Reference Adaptive Control

A model-reference adaptive control (MRAC) system can be schematically represented by Fig. 14. It is composed of four parts: a system containing unknown parameters, a reference model for specifying

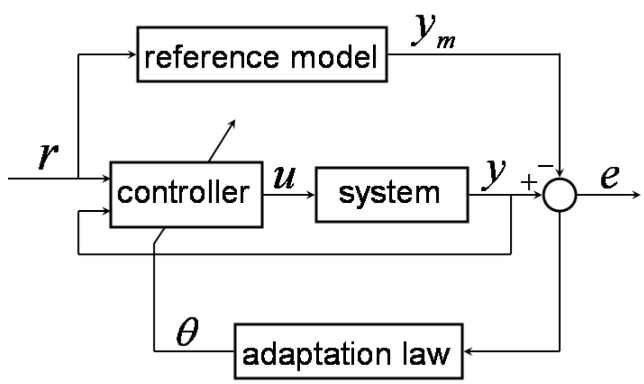


Fig. 14 MRAC

the desired output of the control system, a feedback control law containing adjustable parameters, and an adaptation mechanism for updating the adjustable parameters.

The *system* is assumed to have a known structure with unknown parameters.

The *reference model* is used to specify the ideal response of the adaptive control system to the external command. It defines the ideal system behavior that the adaptation mechanism should seek in adjusting the parameters. The choice of the reference model is a part of the adaptive control system design. The choice has to satisfy two requirements: i) it should reflect the performance specification in the control objective, like rise time, settling time, overshoot, etc. ii) it should be achievable for the adaptive control system with its structural characteristics, like its order, relative degree of the regulated outputs, etc.

The *controller* is usually parameterized by a number of adjustable parameters. The controller should have perfect tracking capacity. That is, when the system parameters are exactly known, the corresponding controller parameters should make the system output identical to that of the reference model. When the system parameters are not known, the adaptation mechanism will adjust the controller parameters so that perfect tracking is asymptotically achieved. If the control law is linear in terms of adjustable parameters, it is said to be linearly parameterized.

The *adaptation mechanism* is used to adjust the parameters in the control law. The objective of the adaptation is to make the tracking error converge to zero. Main difference from the conventional control lies in the structure of this mechanism. The main issue is to synthesize an adaptation mechanism which will guarantee that the control system remains stable and the tracking error converges to zero as the parameters are varied. Adaptation law is not necessarily uniquely defined.

7.1 Direct and Indirect MRAC

There are two ways of thinking about the adaptation law, known as direct MRAC and indirect MRAC. Direct MRAC directly adjusts the controller parameters, while indirect MRAC estimates the plant parameters and uses those for control design. To get better insight into each of these schemes, let's first look at conventional Model Reference Control (MRC) without adaptation, i.e. for known linear systems that can be represented via transfer functions, Fig. 15.

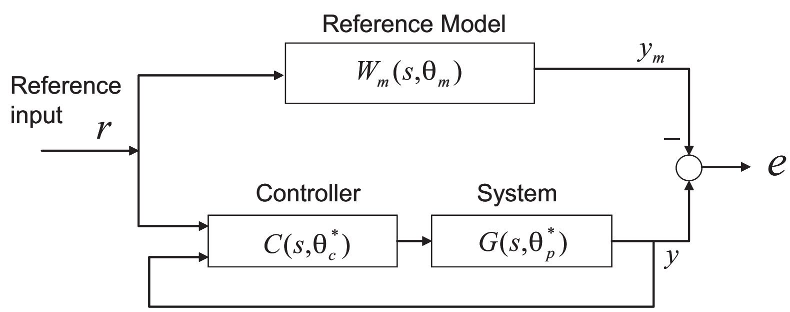


Fig. 15 Model Reference Control

Here $G(s, \theta_p^*)$ represents the transfer function of the linear system, where θ_p^* are the coefficients of it, while $C(s, \theta_c^*)$ is the linear controller that via the choice of θ_c^* needs to achieve asymptotic tracking of $e(t)$ to zero, where $e(t)$ represents the tracking error between the system output $y(t)$ and the output $y_m(t)$ of the desired reference model $W_m(s, \theta_m)$. The desired reference model $W_m(s, \theta_m)$ is selected in a way that achieves tracking of the reference input $r(t)$ with desired transient characteristics (rise time, settling time, overshoot, etc.). So, in general, one tries to choose a law

$$\theta_c^* = F(\theta_p^*, \theta_m)\,,$$

so that to achieve

$$\frac{y(s)}{r(s)} = \frac{y_m(s)}{r(s)}\,.$$

Of course, to achieve this, $G(s,\theta_p^*)$ and $W_m(s,\theta_m)$ have to satisfy certain type of matching assumptions: $G(s,\theta_p^*)$ must be controllable and $W_m(s,\theta_m)$ must be "achievable" for $G(s,\theta_p^*)$ via some control law.

When the plant parameters θ_p^* are unknown, then there are two ways to go. The first and straightforward one is called *indirect adaptive control*, which estimates θ_p^* at every time instant to have $\theta_p(t)$ and uses it in the same non-adaptive control law (see Fig.16) to arrive at

$$\theta_c(t) = F(\theta_p(t), \theta_m)\,.$$

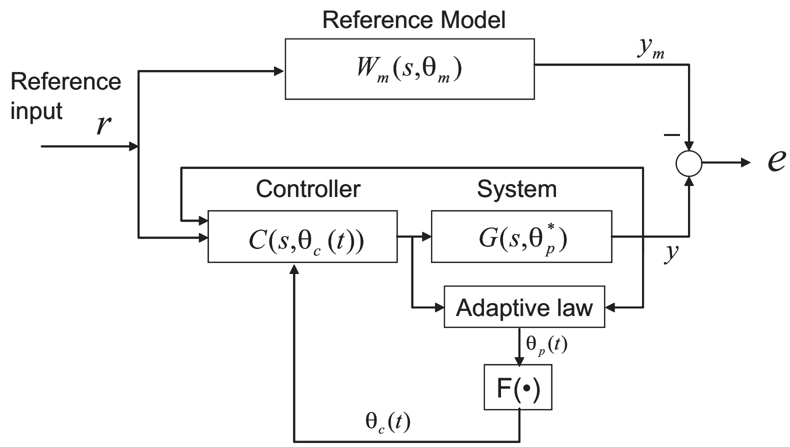


Fig. 16 Indirect Model Reference Adaptive Control

The second path is called *direct adaptive control*, which uses direct estimates of the controller parameters. For that the system is first being parameterized in terms of controller parameters to get

$$G(s,\theta_p^*) = G(s, F^{-1}(\theta_c^*, \theta_m)) = \mathcal{G}(s, \theta_m, \theta_c^*)\,.$$

Then, an adaptive law is developed to estimate $\theta_c(t)$, Fig. 17.

While for linear systems with unknown parameters, in the context of MRC, direct and indirect MRAC can be shown to be equivalent, for systems that cannot be written in parametric structure indirect adaptive control is often the only approach that can be used.

7.2 Proportional Integral Control as the Simplest Possible Adaptive Scheme

Adaptive Control is a generalization of the well-known integral control from linear systems theory. If you recall, integral control is being introduced to remove the steady-state error. Let's now look into

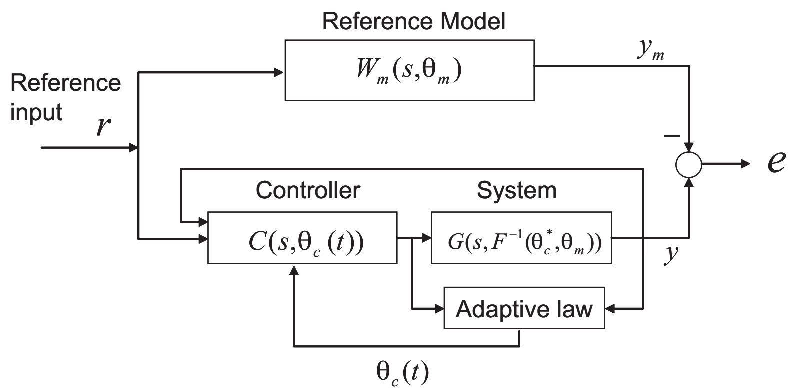


Fig. 17 Direct Model Reference Adaptive Control

this argument from the perspective of adaptive control. Consider the following scalar system

$$\dot{x}(t) = x(t) + \theta + u(t), \quad x(0) = x_0\,, \tag{82}$$

where θ is an unknown constant. The control objective is to determine a control law $u(x,t)$ such that $x(t)$ converges to zero as $t \to \infty$ for any given initial condition, or otherwise saying that achieves GAS of the origin.

Let's follow the philosophy of direct MRAC and obtain parametrization of the unknown system in terms of the controller parameter and estimate it online. If we knew θ, we would design

$$u^*(t) = -2x(t) - \theta\,,$$

to achieve stabilization. Since we do not know θ, we design the controller to be

$$u(t) = -2x(t) - \hat{\theta}(t)\,, \tag{83}$$

where $\hat{\theta}(t)$ is the estimate of θ, which leads to the following closed-loop system

$$\dot{x}(t) = -x(t) - \tilde{\theta}(t), \quad x(0) = x_0\,, \tag{84}$$

where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$. Let's look at the following Lyapunov function candidate

$$V(x(t), \tilde{\theta}(t)) = \frac{1}{2}x^2(t) + \frac{1}{2\gamma}\tilde{\theta}^2(t)\,,$$

where $\gamma > 0$ is any positive number defining the adaptation rate. Taking the time derivative leads to

$$\dot{V}(x(t), \tilde{\theta}(t)) = x(t)\dot{x}(t) + \frac{1}{\gamma}\tilde{\theta}(t)\dot{\tilde{\theta}}(t) = x(t)(-x(t) - \tilde{\theta}(t)) + \frac{1}{\gamma}\tilde{\theta}(t)\dot{\tilde{\theta}}(t)\,.$$

We observe that if we select the following adaptation law

$$\dot{\hat{\theta}}(t) = \dot{\tilde{\theta}}(t) = \gamma x(t)\,, \quad \theta(0) = \theta_0\,, \tag{85}$$

then using (84) and (85)

$$\dot{V}(x(t), \tilde{\theta}(t)) = -x^2(t) \leq 0\,.$$

Since $\dot{V}(x(t), \tilde{\theta}(t)) = 0$ for all $x = 0$, i.e. for an entire line in the $(x, \tilde{\theta})$ space and not just at the origin $(x, \tilde{\theta}) = (0, 0)$, then it is negative semidefinite. To prove GAS of the origin for the system dynamics recall application of La-Salle's invariance principle, Example 2.9, and notice that in this case you can get $\tilde{\theta}(t) \to 0$ as $t \to \infty$. We basically demonstrated how to select an adaptive law to achieve stabilization at the origin. If we substitute (85) into (83), we see that the latter is nothing else than the well-known PI control from linear systems theory:

$$u(t) = -2x(t) - \gamma \int_0^t x(\tau) d\tau\,, \tag{86}$$

that you are used to implementing for removing the steady state error caused by unknown constant θ in (82). Now you saw the Lyapunov perspective for it. It all collapsed to linear PI control, because the unknown constant θ in (82) was not multiplying the system state!

7.3 Adaptive Stabilization

Now let's look at the following scalar system

$$\dot{x}(t) = ax(t) + u(t), \quad x(0) = x_0\,,$$

where the unknown constant a is multiplying the system state. The control objective is to determine a control law $u(x, t)$ such that $x(t)$ converges to zero as $t \to \infty$ for any given initial condition, or otherwise saying that achieves GAS of the origin.

Let's follow the philosophy of direct MRAC and obtain parametrization of the unknown system in terms of the controller parameter and estimate it online. So, let's assume that the controller has the conventional linear structure

$$u(t) = -k^* x(t)\,,$$

that achieves the desired pole placement, i.e. $a - k^* = a_m$, and $a_m < 0$. Since we do not know a, we cannot compute k^*. But we can write the system dynamics equivalently

$$\dot{x}(t) = ax(t) - k^* x(t) + k^* x(t) + u(t)\,, \quad x(0) = x_0\,.$$

So, if we want $a - k^* = a_m$, then this is equivalent to

$$\dot{x}(t) = a_m x(t) + k^* x(t) + u(t)\,, \quad x(0) = x_0\,.$$

Now, we want to have a control law that estimates k^* online. So, let's consider

$$u(t) = -k(t)x(t)$$

and substitute it into the system dynamics:

$$\dot{x}(t) = a_m x(t) - \tilde{k}(t)x(t)\,, \quad x(0) = x_0\,, \tag{87}$$

where $\tilde{k}(t) = k(t) - k^*$. Now we have to come up with an adaptive law to determine $k(t)$ in a way so that to achieve $x(t) \to 0$ as $t \to \infty$. Before then, notice that since $k^* = const$, then

$$\tilde{k}(t) = k(t) - k^* \quad \Rightarrow \quad \dot{\tilde{k}}(t) = \dot{k}(t)$$

Let's look at the following Lyapunov function candidate

$$V(x(t), \tilde{k}(t)) = \frac{1}{2}x^2(t) + \frac{1}{2\gamma}\tilde{k}^2(t)\,,$$

where $\gamma > 0$ is any positive number. Taking the time derivative leads to

$$\dot{V}(x(t), \tilde{k}(t)) = x(t)\dot{x}(t) + \frac{1}{\gamma}\tilde{k}(t)\dot{\tilde{k}}(t) = x(t)(a_m x(t) - \tilde{k}(t)x(t)) + \frac{1}{\gamma}\tilde{k}(t)\dot{\tilde{k}}(t)\,.$$

We observe that if we select the following adaptation law

$$\dot{k}(t) = \dot{\tilde{k}}(t) = \gamma x^2(t)\,, \quad k(0) = k_0\,, \tag{88}$$

then

$$\dot{V}(x(t), \tilde{k}(t)) = a_m x^2(t) \leq 0\,,$$

since $a_m < 0$. In (88), γ is called adaptation rate. Since $\dot{V}(x(t), \tilde{k}(t)) = 0$ for all $x = 0$, i.e. for an entire line in the $(x, \tilde{k})$ space and not just at the origin $(x, \tilde{k}) = (0, 0)$, then it is negative semidefinite. To prove GAS of the origin for the system dynamics recall application of La-Salle's invariance principle, Example 2.9. We basically demonstrated how to select an adaptive law to achieve stabilization at the origin. Of course, this selection of the adaptive law is not the only one.

In this simple case of adaptive regulation, the entire closed-loop system consists of two equations (87), (88) that can be solved explicitly:

$$x(t) = \frac{2c\mathrm{e}^{-ct}}{c + k_0 - a + (c - k_0 + a)\mathrm{e}^{-2ct}} x_0$$

$$k(t) = a + \frac{c[(c + k_0 - a)\mathrm{e}^{2ct} - (c - k_0 + a)]}{(c + k_0 - a)\mathrm{e}^{2ct} + (c - k_0 + a)} ,$$

where $c^2 = \gamma x_0^2 + (k_0 - a)^2$. We can compute the limits

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} k(t) = a + c = a + \sqrt{\gamma x_0^2 + (k_0 - a)^2} .$$

We see that the limit of $k(t)$ depends upon γ, x_0, k_0, which can equal k^* only for a very special set of these parameters. Any change in these parameters will affect the transient performance, since the closed loop pole of (87) converges to $-c = -\sqrt{\gamma x_0^2 + (k_0 - a)^2}$ as $t \to \infty$, which depends upon initial conditions and adaptation gain.

We note that there are two free design parameters $k_0, \gamma > 0$ in this simple case of adaptive regulation. For any value of these parameters, the above derived Lyapunov proof is guaranteeing asymptotic convergence of tracking error to zero and boundedness of the parameter error. Larger γ implies faster adaptation, but can lead to oscillations in tracking performance. Also, increasing γ is limited by the hardware, since it requires smaller integration step for the adaptive law, which becomes "stiff" in the presence of large γ. Determining the "optimal" γ is a matter of success with tuning.

7.4 Direct MRAC of First Order Systems

Let the dynamics of first order system propagate according to the following differential equations:

$$\dot{x}(t) = ax(t) + bu(t) , \quad x(0) = x_0 , \tag{89}$$

where $x \in \mathbb{R}$ is the state of the system, a and b are unknown constants, while $\mathrm{sgn}(b)$ is known, $u \in \mathbb{R}$ is the control input. Consider the following reference model dynamics:

$$\dot{x}_m(t) = a_m x_m(t) + b_m r(t), \quad a_m < 0, \quad x_m(0) = x_{m0} \tag{90}$$

where $x_m \in \mathbb{R}$ is the state of the reference model, $r(t)$ is a uniformly continuous bounded input signal of interest to track, while a_m, b_m specify the desired performance metrics for tracking $r(t)$.

Objective. *For a given uniformly bounded input $\{r \in \mathbb{R} : |r(t)| \leq r_{\max}\}$ define an adaptive feedback signal $u(t)$ such that the state $x(t)$ of the system (89) tracks the state $x_m(t)$ of the reference model asymptotically, while all the signals remain bounded.*

The direct adaptive model reference feedback is defined as:

$$u(t) = k_x(t)x(t) + k_r(t)r(t) \tag{91}$$

where $k_x(t),\ k_r(t)$ are adaptive gains to be defined through stability proof.

Substituting (91) into (89), implies the following closed-loop system dynamics:

$$\dot{x}(t) = (a + bk_x(t))x(t) + bk_r(t)r(t) \tag{92}$$

Comparing (90) with system dynamics in (92), assumptions are formulated that guarantee existence of the adaptive feedback signal in (91) for model matching.

Assumption 7.1. (Matching assumption)

$$\begin{aligned} &\exists\ k_x^*, \quad bk_x^* = a_m - a \\ &\exists\ k_r^*, \quad bk_r^* = b_m \end{aligned} \tag{93}$$

Remark 7.1. The true knowledge of the ideal gains k_x^*, k_r^* is not required, only their existence is assumed. Also notice that in scalar case these assumptions are obviously satisfied. These are called matching assumptions, because in case of knowledge of these ideal gains, one could substitute them in (92) instead of $k_x(t), k_r(t)$ to obtain the reference system in (90) directly.

Let $e(t) = x(t) - x_m(t)$ be the tracking error signal. Then the tracking error dynamics can be written:

$$\dot{e}(t) = a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t)\,, \quad e(0) = e_0\,, \tag{94}$$

where $\Delta k_x(t) = k_x(t) - k_x^*$, $\Delta k_r(t) = k_r(t) - k_r^*$ are introduced for parameter errors. Consider the following adaptation laws:

$$\begin{aligned} \dot{k}_x(t) &= -\gamma_x x(t)e(t)\mathrm{sgn}(b)\,, \quad k_x(0) = k_{x0} \\ \dot{k}_r(t) &= -\gamma_r r(t)e(t)\mathrm{sgn}(b)\,, \quad k_r(0) = k_{r0}, \end{aligned} \tag{95}$$

where $\gamma_x > 0, \gamma_r > 0$ are adaptation gains, k_{x0}, k_{r0} can be the "best possible guess" of the ideal values of unknown parameters,[**] and the following Lyapunov function candidate:

$$V(e(t), \Delta k_x(t), \Delta k_r(t)) = e^2(t) + \left(\gamma_x^{-1}\Delta k_x^2(t) + \gamma_r^{-1}\Delta k_r^2(t)\right)|b| \tag{96}$$

Its derivative along the system trajectories (94), (95) will be:

$$\begin{aligned}
\dot{V}(t) &= 2e(t)\left(a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t)\right) \\
&\quad +2\gamma_x^{-1}|b|\Delta k_x(t)\Delta\dot{k}_x(t) + 2\gamma_r^{-1}|b|\Delta k_r(t)\Delta\dot{k}_r(t) \\
&= -2|a_m|e^2(t) \\
&\quad +2|b|\Delta k_x(t)\left(x(t)e(t)\mathrm{sgn}(b) + \gamma_x^{-1}\Delta\dot{k}_x(t)\right) \\
&\quad +2|b|\Delta k_r(t)\left(r(t)e(t)\mathrm{sgn}(b) + \gamma_r^{-1}\Delta\dot{k}_r(t)\right) \\
&= -2|a_m|e^2(t) \le 0
\end{aligned} \tag{97}$$

Hence the equilibrium of (94), (95) is Lyapunov stable, i.e. the signals $e(t)$, $\Delta k_x(t)$, $\Delta k_r(t)$ are bounded. Since $x(t) = e(t) + x_m(t)$, and $x_m(t)$ is the state of a stable reference model, then $x(t)$ is bounded. This consequently implies that $\dot{e}(t)$ in (94) is bounded. Compute the second derivative of $V(e(t), \Delta k_x(t), \Delta k_r(t))$:

$$\ddot{V} = -4|a_m|e(t)\dot{e}(t) \tag{98}$$

From the above considerations, it follows that $\ddot{V}$ is bounded, and hence $\dot{V}(t)$ is uniformly continuous. Application of Barbalat's lemma immediately yields $\dot{V}(t) \to 0$ as $t \to \infty$, which consequently proves convergence of the tracking error to zero asymptotically. Notice however, that due to the structure of the expression $\dot{V}(t) = -2|a_m|e^2(t)$, asymptotic convergence of parameter errors to zero is **NOT** guaranteed. The parameter errors are guaranteed only to stay bounded.

Remark 7.2. It is important to notice that the adaptive laws are selected in a way to render certain terms in the $\dot{V}(t)$ expression zero, so that to ensure $\dot{V}(t) \le 0$. Therefore adaptive control is very often called *inverse Lyapunov design*. It is the limitation of the Lyapunov stability theory that one cannot easily remove the assumption on the knowledge of $\mathrm{sgn}(b)$. Had we assumed that b with its sign is completely unknown in (89), it would have been impossible to complete the stability proof with

[**] We will later discuss in more details how adaptive laws can be initialized based on optimality considerations.

this Lyapunov function. Relaxation of this assumption can be achieved by invoking the concept of Nussbaum-gain, [18]. Briefly, a function $v(\cdot)$ is called Nussbaum-type if it has the following properties:

$$\lim_{k\to\infty}\sup\frac{1}{k}\int_0^k v(s)ds = \infty\,, \quad \lim_{k\to\infty}\inf\frac{1}{k}\int_0^k v(s)ds = -\infty\,.$$

One simple choice of Nussbaum-type function verifying this property would be $v(s) = s^2\cos s$, and an adaptive controller can be selected as $u(t) = v(k(t))x(t)$, $\dot{k}(t) = \gamma x^2(t)$. Throughout this course we will assume that $\mathrm{sgn}(b)$ is known. It is up to your curiosity to explore what exists beyond the classroom and your teaching notes.

7.4.1 Passive Identifier based Reparameterization of Direct MRAC

Let us obtain the same result via a slightly different order of the steps in the above algorithm. Let the dynamics be again given by:

$$\dot{x}(t) = \underbrace{(a_m - bk_x^*)}_{a}x(t) + bu(t)\,, \quad x(0) = x_0\,, \quad a_m < 0\,, \tag{99}$$

where $x \in \mathbb{R}$ is the state of the system, $a_m < 0$ is the desired pole of the *ideal* reference system dynamics, k_x^* is unknown, and for simplicity b is known, $u \in \mathbb{R}$ is the control input. Instead of the reference model dynamics in (90), let's consider a *passive identifier*:

$$\dot{\hat{x}}(t) = a_m\hat{x}(t) - b\hat{k}_x(t)x(t) + bu(t)\,, \quad \hat{x}(0) = x_0\,, \tag{100}$$

where $\hat{x} \in \mathbb{R}$ is the state of the passive identifier, and $a_m < 0$ is selected to achieve desired properties for the resulting error dynamics. The dynamics in (100) is quite often referred to as *state predictor*. It replicates the system's structure, with the unknown parameter k_x^* replaced by its estimate $\hat{k}_x(t)$. By subtracting (100) from (99), we obtain the *identification error dynamics* or the *prediction error dynamics*, independent of control choice:

$$\dot{\tilde{x}}(t) = a_m\tilde{x}(t) - b\Delta k_x(t)x(t), \quad \tilde{x}(0) = 0\,, \quad \tilde{x}(t) \triangleq \hat{x}(t) - x(t), \quad \Delta k_x(t) \triangleq \hat{k}_x(t) - k_x^*\,.$$

It is obvious that using the adaptive law

$$\dot{\hat{k}}_x(t) = \gamma b\tilde{x}(t)x(t)$$

with the following Lyapunov function candidate

$$V(\tilde{x}(t), \Delta k_x(t)) = \tilde{x}^2(t) + \frac{1}{\gamma}\Delta k_x^2(t)$$

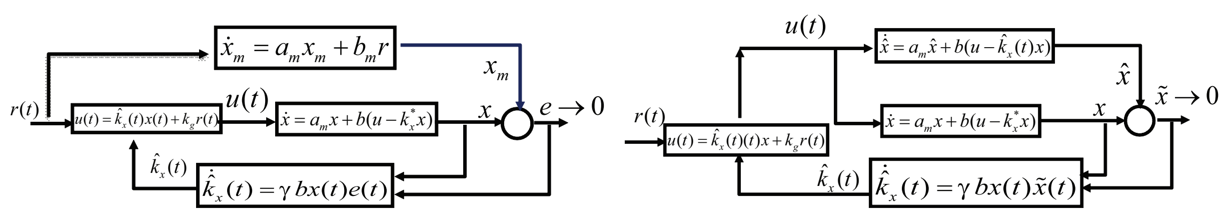
will give $\dot{V}(t) = 2a_m\tilde{x}^2(t) \leq 0$, implying that all errors $(\tilde{x}(t), \Delta k_x(t))$ are uniformly bounded. However, due to the open-loop nature of (100), one cannot conclude asymptotic stability for $\tilde{x}(t)$ (both states $x(t)$ and $\hat{x}(t)$ can drift to infinity with the same rate, keeping the error $\tilde{x}(t)$ bounded). Remember that application of Barbalat's lemma was well justified due to the *boundedness of the reference system*, which was used for derivation of the error dynamics. However, if we close the loop, using the same adaptive feedback structure

$$u(t) = \hat{k}_x(t)x(t) + k_g r(t)\,,$$

where k_g is a feedforward gain and can be selected to achieve asymptotic tracking of step reference inputs (as compared to (90) one can say that $bk_g = b_m$), then the closed-loop *passive identifier* or the *state predictor* will replicate the reference system of (90):

$$\dot{\hat{x}}(t) = a_m\hat{x}(t) + bk_g r(t)\,, \quad \hat{x}(0) = x(0)\,,$$

allowing for application of Barbalat's lemma for concluding asymptotic tracking of $\tilde{x}(t) \to 0$.



a) Direct MRAC **b)** Passive identifier based reparameterization

Fig. 18 Implementation architectures for MRAC and passive identifier based reparameterization

We see that from the same initial conditions the direct MRAC and the passive identifier-based reparameterization of it lead to the same error dynamics, implying that all error signals remain bounded, and the tracking error converges to zero asymptotically. The fundamental difference in between these schemes is illustrated in Fig. 18, where, the control signal is provided as input to both systems, plant and predictor, in case of passive-identifier based reparameterization, while in the case of direct MRAC, the control signal serves as input only to the plant. This feature will be later exploited in the context of $\mathcal{L}_1$ adaptive control, towards defining an architecture with *quantifiable performance bounds*.

7.5 Indirect MRAC of First Order Systems.

In an indirect MRAC scheme we estimate the plant parameters and not the controller parameters. Consider the dynamics of the same first order system as in (89):

$$\dot{x}(t) = ax(t) + bu(t)\,, \quad x(0) = x_0\,, \tag{101}$$

where $x \in \mathbb{R}$ is the state of the system, a and b are unknown constants, $\mathrm{sgn}(b)$ is known, $u \in \mathbb{R}$ is the control input. Consider the same reference model dynamics as in (90):

$$\dot{x}_m(t) = a_m x_m(t) + b_m r(t), \quad a_m < 0\,, \quad x_m(0) = x_{m0}\,, \tag{102}$$

where $x_m \in \mathbb{R}$ is the state of the reference model, $r(t)$ is a uniformly continuous bounded input signal of interest to track, a and b are unknown, but a conservative lower bound for $|b|$ is known in addition to $\mathrm{sgn}(b)$. Without loss of generality, let's assume that $b > \bar{b} > 0$, where $\bar{b}$ is known.

Objective. *For a given uniformly bounded input $\{r \in \mathbb{R} : |r(t)| \leq r_{\max}\}$ define an adaptive feedback signal $u(t)$ such that the state $x(t)$ of the system (101) tracks the state $x_m(t)$ of the reference model (102) asymptotically, while all the signals remain bounded.*

The indirect adaptive feedback is defined as:

$$u(t) = \frac{1}{\hat{b}(t)}\left(-\hat{a}(t)x(t) + a_m x(t) + b_m r(t)\right) \tag{103}$$

where $\hat{a}(t)$, $\hat{b}(t)$ are adaptive gains that need to be defined via stability proof subject to the constraint that $\hat{b}(t) \neq 0$ for all $t \geq 0$. Rewrite the system dynamics in (101) in the following way:

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) + \hat{a}(t)x(t) + \hat{b}(t)u(t) - \hat{a}(t)x(t) - \hat{b}(t)u(t) \\ &= \hat{a}(t)x(t) + \hat{b}(t)u(t) - \Delta a(t)x(t) - \Delta b(t)u(t) \end{aligned} \tag{104}$$

where $\Delta a(t) = \hat{a}(t) - a$, $\Delta b(t) = \hat{b}(t) - b$ are parametric errors. Substituting (103) into (104), implies the following closed-loop system dynamics:

$$\dot{x}(t) = a_m x(t) + b_m r(t) - \Delta a(t)x(t) - \Delta b(t)u(t) \tag{105}$$

Let $e(t) = x(t) - x_m(t)$ be the tracking error signal. Then the tracking error dynamics can be written:

$$\dot{e}(t) = a_m e(t) - \Delta a(t)x(t) - \Delta b(t)u(t) \tag{106}$$

Consider the following adaptation laws[††]

$$\begin{aligned}
\dot{\hat{a}}(t) &= \gamma_a x(t)e(t)\,, && \hat{a}(0) = \hat{a}_0 \\
\dot{\hat{b}}(t) &= \begin{cases} \gamma_b u(t)e(t), & \text{if} \quad \hat{b}(t) \geq \bar{b} \\ \gamma_b u(t)e(t) + \frac{\bar{b}-\hat{b}(t)}{\hat{b}(t)-\bar{b}+\epsilon}, & \text{if} \quad \hat{b}(t) < \bar{b} \end{cases}\,, && \hat{b}(0) = \hat{b}_0 > \bar{b}
\end{aligned} \tag{107}$$

where $\gamma_a > 0, \gamma_b > 0$ are adaptation gains, $\epsilon > 0$ is a sufficiently small number so that $\bar{b} - \epsilon > 0$,[‡‡] and the initialization of $\hat{b}(0) = \hat{b}_0 > \bar{b}$ is done with correct sign respecting the known conservative lower bound. This adaptive law has the following features:

- It starts from a positive value due to the initial condition $\hat{b}(0) = \hat{b}_0 > \bar{b}$;
- If $\gamma_b u(t)e(t) \geq 0$, then $\hat{b}(t)$ will not decrease;
- If $\gamma_b u(t)e(t) < 0$, then $\hat{b}(t)$ will decrease;

  - As $\hat{b}(t)$ takes values less than $\bar{b}$, but still remains larger than $\bar{b}-\epsilon$, the *positive term* $\frac{\bar{b}-\hat{b}(t)}{\hat{b}(t)-\bar{b}+\epsilon}$ corrects for the derivative of $\hat{b}(t)$ to slowly turn it back;

  - As $\hat{b}(t) \to \bar{b} - \epsilon$, then $\frac{\bar{b}-\hat{b}(t)}{\hat{b}(t)-\bar{b}+\epsilon} \to \infty$, forcing $\hat{b}(t)$ to grow.

Therefore $\hat{b}(t)$ never reaches zero. In fact, for all $t \geq 0$, one has $\hat{b}(t) > \bar{b} - \epsilon > 0$, and the control signal in (103) is well-defined. It also guarantees that $\dot{\hat{b}}(t)$ is Lipschitz. Since it is Lipschitz, then i) a unique solution exists for $\hat{b}(t)$, and ii) $\dot{\hat{b}}(t)$ is also continuous, which ensures that the derivative of the Lyapunov function candidate, specified below, is continuous. Therefore Theorem 2.1 can be applied. So, consider the following Lyapunov function candidate:

$$V(e(t), \Delta a(t), \Delta b(t)) = e^2(t) + \gamma_a^{-1}\Delta a^2(t) + \gamma_b^{-1}\Delta b^2(t) \tag{108}$$

[††]This is the scalar case of the projection operator that we will define in more details later in the course.

[‡‡]It is called tolerance of the projection and is specified by the designer.

Its derivative along the system trajectories (106), (107) will be:

$$
\begin{aligned}
\dot{V}(t) &= 2e(t)\left(a_m e(t) - \Delta a(t)x(t) - \Delta b(t)u(t)\right) + 2\gamma_a^{-1}\Delta a(t)\Delta\dot{a}(t) + 2\gamma_b^{-1}\Delta b(t)\Delta\dot{b}(t) \\
&= -2|a_m|e^2(t) + 2\Delta a(t)\left(-x(t)e(t) + \gamma_a^{-1}\Delta\dot{a}(t)\right) + 2\Delta b(t)\left(-u(t)e(t) + \gamma_b^{-1}\Delta\dot{b}(t)\right) \\
&= \begin{cases} -2|a_m|e^2(t), & \text{if} \quad \hat{b}(t) \geq \bar{b} \\ -2|a_m|e^2(t) + 2\underbrace{\Delta b(t)}_{<0}\underbrace{\dfrac{\bar{b} - \hat{b}(t)}{\hat{b}(t) - \bar{b} + \epsilon}}_{>0}, & \text{if} \quad \hat{b}(t) < \bar{b} \end{cases} \\
&\leq 0
\end{aligned}
$$

Hence the equilibrium of (106), (107) is Lyapunov stable. Following similar arguments like in direct MRAC case, one can apply Barbalat's lemma to prove asymptotic convergence of $e(t)$ to zero and boundedness of parameter errors.

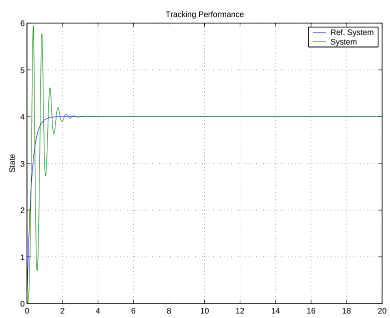
Example. Consider a first order system

$$\dot{x}(t) = 5x(t) + 3u(t)\,, \quad x(0) = x_0\,. \tag{109}$$

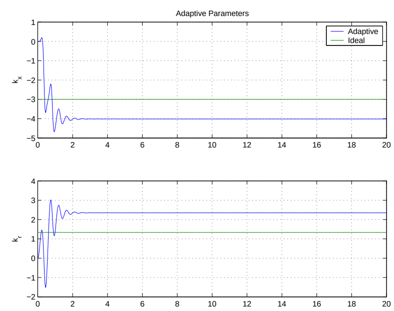
Let the reference model be

$$\dot{x}_m(t) = -4x_m(t) + 4r(t)\,, \quad x_m(0) = x_{m0}\,. \tag{110}$$

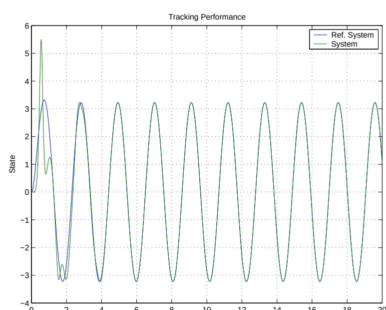
The plots 19(a)-19(d) show the tracking performance and parameter convergence with direct MRAC for two different reference inputs: $r(t) = 4$ and $r = 4\sin(3t)$. The plots clearly demonstrate the asymptotic convergence of the tracking error to zero for both commands, while the parameter convergence takes place only in the case of sinusoidal input and is not guaranteed for the step command. On an intuitive level the explanation is that simple commands like step can be tracked with less effort, without requiring the controller to find the ideal parameters, while a bit challenging command like the sinusoid can be tracked only if the controller parameters converge to the ideal ones.



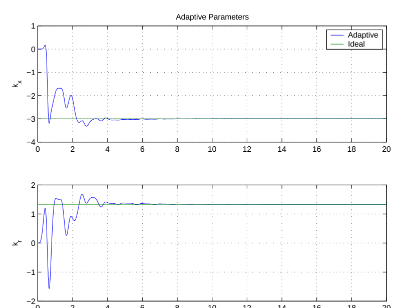
a) $r = 4$, tracking



b) $r = 4$, parameter convergence



c) $r = 4\sin(3t)$, tracking



d) $r = 4\sin(3t)$, parameter convergence

Fig. 19 Comparison of tracking performance for various reference inputs.

8 Parameter Convergence: Persistency of Excitation or Uniform Complete Observability

In order to gain insights about the convergence of the estimated parameters to their true values, let's examine the equation:

$$\dot{e}(t) = a_m e(t) + b(\Delta k_x(t)x(t) + \Delta k_r(t)r(t)) \tag{111}$$

We have shown that $e(t) \to 0$ as $t \to \infty$. We have shown that $\Delta k_x(t), \Delta k_r(t)$ are bounded. Moreover, from (95) it follows that $\dot{k}_x(t), \dot{k}_r(t)$ are bounded, implying that $k_x(t), k_r(t)$ are uniformly continuous, and therefore $\Delta k_x(t), \Delta k_r(t)$ are also uniformly continuous. From (92) it follows that $\dot{x}(t)$ is bounded, and therefore $x(t)$ is uniformly continuous. Assuming that $r(t)$ is also uniformly continuous, we have that the right hand side of (111) is uniformly continuous, i.e. $\dot{e}(t)$ is uniformly continuous. We have shown that $e(t) \to 0$ as $t \to \infty$. So, application of Barbalat's lemma implies that $\dot{e}(t) \to 0$ as $t \to \infty$.

Consequently, we have

$$\Delta k_x(t)x(t) + \Delta k_r(t)r(t) \to 0, \quad t \to \infty\,. \tag{112}$$

In vector form, this can be written:

$$v^\top(t)\Delta\theta(t) \to 0, \quad t \to \infty\,, \tag{113}$$

where $v(t) = [\; x(t) \quad r(t) \;]^\top$, $\Delta\theta(t) = [\; \Delta k_x(t) \quad \Delta k_r(t) \;]^\top$. The issue of parameter convergence is reduced to the question of what conditions the vector $v(t) = [x(t)\; r(t)]^\top$ should satisfy in order for the equation (113) to imply that $\Delta\theta(t) \to 0$.

To gain further insights into the problem, notice that if $r(t) = r_0 = const$, then, due to the stability of the reference model, for large values of t we have

$$x \cong x_m \cong b_m r_0\,. \tag{114}$$

Thus, for large values of t we have

$$v^\top = [\; x \quad r_0 \;] \cong r_0[\; b_m \quad 1 \;]\,. \tag{115}$$

The limiting relationship in (113) takes the form:

$$b_m \Delta k_x + \Delta k_r \cong 0\,. \tag{116}$$

This clearly implies that the parameter errors instead of converging to zero, converge to a straight line in the parameter space. In case if $b_m = 1$, then the steady state errors of two parameter errors should be of equal magnitude but opposite sign.

However, when $r(t)$ is such that the corresponding signal vector $v(t)$ satisfies the *persistency of excitation* (PE) condition, one can prove that the adaptive laws (95) will guarantee parameter convergence, i.e.

$$\lim_{t\to\infty} \Delta\theta(t) = 0. \tag{117}$$

Definition 8.1. The signal $v(t)$ is said to be persistently exciting if there exist $\alpha > 0$ and $T > 0$ such that for all $t \geq 0$

$$\int_t^{t+T} v(\tau)v^\top(\tau)d\tau > \alpha\mathbb{I} \tag{118}$$

where $\mathbb{I}$ is the identity matrix.

Indeed, since the tracking error converges to zero, then it follows from (95) that

$$\lim_{t\to\infty} (\Delta\theta(t+\tau') - \Delta\theta(t)) = 0 \tag{119}$$

for any $\tau' \in [0,\ T]$. Hence, we have

$$\begin{aligned}
&\lim_{t\to\infty} \int_t^{t+T} \Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(\tau)d\tau - \lim_{t\to\infty} \int_t^{t+T} \Delta\theta^\top(t)v(\tau)v^\top(\tau)\Delta\theta(t)d\tau\,, \\
&= \lim_{t\to\infty} \int_t^{t+T} \Big(\Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(\tau) - \Delta\theta^\top(t)v(\tau)v^\top(\tau)\Delta\theta(t)\Big)\, d\tau \\
&= \lim_{t\to\infty} \int_t^{t+T} \Big(\Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(\tau) - \Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(t) \\
&+\Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(t) - \Delta\theta^\top(t)v(\tau)v^\top(\tau)\Delta\theta(t)\Big)d\tau \\
&= \lim_{t\to\infty} \int_t^{t+T} \Big(\Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\left(\Delta\theta(\tau) - \Delta\theta(t)\right) + \Big(\Delta\theta^\top(\tau) - \Delta\theta^\top(t)\Big)\, v(\tau)v^\top(\tau)\Delta\theta(t)\Big)\, d\tau \\
&= \int_t^{t+T} \lim_{t\to\infty} \Big(\Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\left(\Delta\theta(\tau) - \Delta\theta(t)\right) + \Big(\Delta\theta^\top(\tau) - \Delta\theta^\top(t)\Big)\, v(\tau)v^\top(\tau)\Delta\theta(t)\Big)\, d\tau
\end{aligned}$$

It follows from (119) that $\lim_{t\to\infty} (\Delta\theta(\tau) - \Delta\theta(t)) = 0$, where $\tau \in [t, t+T]$, and therefore

$$\lim_{t\to\infty} \int_t^{t+T} \Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(\tau)d\tau - \lim_{t\to\infty} \int_t^{t+T} \Delta\theta^\top(t)v(\tau)v^\top(\tau)\Delta\theta(t)d\tau = 0\,.$$

If (117) is not true, there exists $\epsilon > 0$ such that for any t', there exists $t > t'$ which satisfies

$$\|\Delta\theta(t)\|^2 \geq \epsilon\,,$$

and hence,

$$\int_t^{t+T} \Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(\tau)d\tau \geq \alpha\epsilon\,. \tag{120}$$

Multiply both sides of (111) by $\Delta\theta^\top(t)v(t)$, we have

$$\dot{e}(t)\Delta\theta^\top(t)v(t) = -a_m e(t)\Delta\theta^\top(t)v(t) + \Delta\theta^\top(t)v(t)v^\top(t)\Delta\theta(t). \tag{121}$$

Integrating (121) from t to $t+T$, we have

$$\begin{aligned}
&e(t+T)\Delta\theta^\top(t+T)v(t+T) - e(t)\Delta\theta^\top(t)v(t) + \int_t^{t+T} e(t)\frac{d}{dt}\left[(\Delta\theta^\top(\tau)v(\tau))\right]d\tau \\
&= \int_t^{t+T} -a_m e(\tau)\Delta\theta^\top(\tau)v(\tau)d\tau + \int_t^{t+T} \Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(\tau)d\tau\,.
\end{aligned}$$

As $t \to \infty$, we have that $e(t) \to 0$ and therefore, taking into consideration the finite interval for integration, we have the following upper bounds:

$$\begin{aligned}
&\lim_{t\to\infty}\Big(e(t+T)\Delta\theta^\top(t+T)v(t+T) - e(t)\Delta\theta^\top(t)v(t)\Big) = 0 \\
&\left|\lim_{t\to\infty}\left(\int_t^{t+T} e(t)\frac{d}{dt}\left[\Delta\theta^\top(\tau)v(\tau)\right]d\tau\right)\right| \leq \lim_{t\to\infty}\left(\max_{\tau\in[t,t+T]}||e(\tau)||\,\left|\Delta\theta^\top(\tau)v(\tau)\Big|_t^{t+T}\right| T\right) = 0 \\
&\left|\lim_{t\to\infty}\int_t^{t+T} -a_m e(\tau)\Delta\theta^\top(\tau)v(\tau)d\tau\right| \leq |a_m|\lim_{t\to\infty}\max_{\tau\in[t,t+T]}\Big(||e(\tau)||\,\left|\Delta\theta^\top(\tau)v(\tau)\right|\Big)\;T = 0
\end{aligned}$$

This consequently implies that

$$\lim_{t\to\infty}\int_t^{t+T} \Delta\theta^\top(\tau)v(\tau)v^\top(\tau)\Delta\theta(\tau)d\tau = 0\,,$$

which contradicts (120).

Intuitively, persistency of excitation implies that the vectors $v(t)$ corresponding to different times t cannot always be linearly dependent. This relates to the notion of *uniform complete observability* for linear time-varying systems, [19].

Definition 8.2. The linear time-varying system

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t), \quad x(t_0) = x_0 \\
y(t) &= C(t)x(t)
\end{aligned}$$

is *uniformly completely observable*, if its observability Grammian is uniformly positive definite, i.e. it verifies the following inequality for some $\alpha > 0$ and $T > 0$:

$$W(t, t+\delta) = \int_t^{t+T} \Phi^\top(\tau, t) C^\top(\tau) C(\tau) \Phi(\tau, t) d\tau \geq \alpha \mathbb{I}, \quad \forall\, t \geq 0\,,$$

where $\Phi(\tau, t)$ is the state transition matrix, defining the unique solution for the given initial condition:

$$x(t) = \Phi(t, t_0) x_0\,.$$

To see the relationship of the PE condition to the notion of uniform complete observability, notice that the closed-loop error dynamics of the adaptive system (94), (95) can be rewritten as a linear time-varying system:

$$\begin{bmatrix} \dot{e}(t) \\ \Delta\dot{k}_x(t) \\ \Delta\dot{k}_r(t) \end{bmatrix} = \begin{bmatrix} a_m & bx(t) & br(t) \\ -\gamma_x x(t)\mathrm{sgn}(b) & 0 & 0 \\ -\gamma_r r(t)\mathrm{sgn}(b) & 0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \Delta k_x(t) \\ \Delta k_r(t) \end{bmatrix}$$

for which we have already proven boundedness of the $A(t)$, by proving boundedness of all its elements. One can prove that if $v(t) = [x(t)\ r(t)]^\top$ is persistently exciting, then this linear time-varying system is uniformly completely observable. Using Lyapunov arguments, it can be further proven that this leads to exponential convergence of tracking errors and parameter errors to zero.

The remaining question is the relation between $r(t)$ and the persistent excitation of $v(t)$. It can be shown that in case of first order systems with two unknown parameters, persistency of excitation is guaranteed as long as $r(t)$ contains at least one sinusoidal input. In case of linear systems, usually m sinusoidal inputs ensure convergence of $2m$ parameters [3]. For nonlinearly parameterized systems, there is no definite answer to this question in general case, and it depends upon the structure of the system, [4].

Homework Problems 8.1.

- Try to reproduce the plots of my example without asking for the code from me.
- Try to apply indirect MRAC to the same example and compare the performance.
- For both designs, direct and indirect MRAC:

- Change the adaptation gains and write your insights about the observations that you can make by changing the adaptation gains. What do you see? If you think that including a plot can help to better illustrate, then you can do so. Otherwise I do not want to see plots; I want to read the results of your observations.

- You need to get a feel of when you can get an "oscillations-free" transient.

- Analyze the trade-offs as you change the adaptation gains.

- If you find anything interesting that you can generalize, let me know.

The following proofs can be done towards extra credit:

- Prove that the PE condition in (118) indeed ensures uniform complete observability of the closed-loop error dynamics.
- Prove that m distinct sinusoidal inputs ensure convergence of $2m$ parameters.

9 Adaptive Control in the Presence of Input Constraints

All the adaptive control schemes presented until today did not account for actuator position limits. The resulting adaptive controller could have had arbitrary magnitude, since its derivation was based on the philosophy of *inverse Lyapunov design*. By which we mean that until now we have explored **structures**, for which a candidate Lyapunov function could have been determined for derivation of adaptive laws to ensure stability, and with the help of Barbalat's lemma we could establish asymptotic convergence of tracking error to zero. Let's see what happens if our adaptive controller violates the actuator position limits.

Let the dynamics of first order system propagate according to the following differential equations:

$$\dot{x}(t) = ax(t) + bu(t) \tag{122}$$

where $x \in \mathbb{R}$ is the state of the system, a and b are unknown constants, while $\mathrm{sgn}(b)$ is known, $u \in \mathbb{R}$ is the control input, subject to the following constraint:

$$u(t) = u_{\max}\,\mathrm{sat}\left(\frac{u_c(t)}{u_{\max}}\right) = \begin{cases} u_c(t), & |u_c(t)| \leq u_{\max} \\ u_{\max}\,\mathrm{sgn}\left(u_c(t)\right), & |u_c(t)| \geq u_{\max} \end{cases} \tag{123}$$

Here $u_c(t)$ is the commanded control input, while $u_{\max} > 0$ defines the amplitude saturation level. Rewrite the system dynamics in (122) in the following form:

$$\dot{x}(t) = ax(t) + bu_c(t) + b\Delta u(t) \tag{124}$$

where $\Delta u(t) = u(t) - u_c(t)$ is a measure of the control deficiency.

For a given uniformly bounded input $\{r \in \mathbb{R} : |r(t)| \leq r_{\max}\}$, define the reference model dynamics and an adaptive feedback signal $u_c(t)$, such that the state $x(t)$ of the corresponding closed-loop system tracks the state $x_m(t)$ of the reference model dynamics asymptotically, and all the signals remain bounded.

The direct adaptive model reference feedback, following the convention, is defined as:

$$u_c(t) = k_x(t)x(t) + k_r(t)r(t)\,, \tag{125}$$

where $k_x(t),\ k_r(t)$ are adaptive gains. Substituting (125) into (124), implies the following closed-loop system dynamics:

$$\dot{x}(t) = (a + bk_x(t))x(t) + bk_r(t)r(t) + b\Delta u(t) \tag{126}$$

We see that as compared to (92), we get an extra term in (126) due to the control deficiency $\Delta u(t)$. Without this term, we knew that using the matching assumptions from (93) and the corresponding adaptive laws from (95), we could achieve asymptotic tracking of the state of the reference system in (90), keeping all the parameter errors bounded. Now, if we try to derive the same error dynamics in (94), the term $bu(t)$ will pop-up in it. Since $\Delta u(t) = u(t) - u_c(t)$ depends upon the control signal, which in its turn depends upon the system states, we have no guarantee of its boundedness. So, we have to somehow remove its effects from the error dynamics in (94).

One way to do this, is to modify the reference dynamics in a way so that in the error dynamics we get one more adaptive law to go with $b\Delta u(t)$, which can be processed through a Lyapunov proof [22,23]. So, instead of (90), let's consider

$$\dot{x}_m(t) = a_m x_m(t) + b_m(r(t) + k_u(t)\Delta u(t)), \quad a_m < 0\,, \tag{127}$$

where $k_u(t)$ is another adaptive gain to be determined through stability proof. Comparing (127) with system dynamics in (126), we immediately see that its ideal value k_u^* is defined as:

$$b_m k_u^* = b\,. \tag{128}$$

Comparing this with the ideal value of k_r^* in (93) implies that

$$k_r^* k_u^* = 1\,. \tag{129}$$

If b is known, then k_u^* can be determined from (128). Consequently, there is no need for adaptation of k_u, and it can be immediately set to its ideal value $k_u^* = b/b_m$.

The rest follows as always. Let $e(t) = x(t) - x_m(t)$ be the tracking error signal. Then the tracking error dynamics can be written:

$$\dot{e}(t) = a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t) - b_m\Delta k_u(t)\Delta u(t)\,, \tag{130}$$

where $\Delta k_x(t) = k_x(t) - k_x^*$, $\Delta k_r(t) = k_r(t) - k_r^*$, $\Delta k_u(t) = k_u(t) - k_u^*$ are introduced for parameter errors. Consider the following adaptation laws:

$$\begin{aligned}
\dot{k}_x(t) &= -\gamma_x x(t)e(t)\mathrm{sgn}(b) \\
\dot{k}_r(t) &= -\gamma_r r(t)e(t)\mathrm{sgn}(b) \\
\dot{k}_u(t) &= \gamma_u \Delta u(t)e(t)b_m\,,
\end{aligned} \tag{131}$$

where $\gamma_x > 0, \gamma_r > 0, \gamma_u > 0$ are adaptation gains, and the following Lyapunov function candidate:

$$V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta k_u(t)) = e^2(t) + \left(\gamma_x^{-1}\Delta k_x^2(t) + \gamma_r^{-1}\Delta k_r^2(t)\right)|b| + \gamma_u^{-1}\Delta k_u^2(t)\,. \tag{132}$$

Its derivative along the system trajectories (130), (131) will be:

$$\begin{aligned}
\dot V &= 2e(t)\left(a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t) - b_m\Delta k_u(t)\Delta u(t)\right) \\
&\quad + 2\gamma_x^{-1}|b|\Delta k_x(t)\Delta\dot k_x(t) + 2\gamma_r^{-1}|b|\Delta k_r(t)\Delta\dot k_r(t) + \gamma_u^{-1}\Delta k_u(t)\Delta\dot k_u(t) \\
&= -2|a_m|e^2(t) \\
&\quad + 2|b|\Delta k_x(t)\left(x(t)e(t)\mathrm{sgn}(b) + \gamma_x^{-1}\Delta\dot k_x(t)\right) \\
&\quad + 2|b|\Delta k_r(t)\left(r(t)e(t)\mathrm{sgn}(b) + \gamma_r^{-1}\Delta\dot k_r(t)\right) \\
&\quad + 2\Delta k_u(t)\left(-\Delta u(t)e(t)b_m + \gamma_u^{-1}\Delta\dot k_u(t)\right) \\
&= -2|a_m|e^2(t) \le 0\,.
\end{aligned} \tag{133}$$

Hence the equilibrium of (130), (131) is Lyapunov stable, i.e. the signals $e(t),\ \Delta k_x(t),\ \Delta k_r(t),\ \Delta k_u(t)$ are bounded. Consequently, there exist $\Delta k_x^{\max}$, $\Delta k_r^{\max}$, such that $|\Delta k_x| < \Delta k_x^{\max}$, $\quad |\Delta k_r| < \Delta k_r^{\max} = \alpha\Delta k_x^{\max}$, $\ \forall t > t_0$, where $\alpha = \sqrt{\gamma_r/\gamma_x}$.

However, notice that we cannot conclude stability of the closed-loop system from this. Due to the adaptive modification of the reference model in (127), we do not have the *granted bonus* of the stable reference model, as it was in the absence of saturation. This implies that while all the errors $e(t),\ \Delta k_x(t),\ \Delta k_r(t),\ \Delta k_u(t)$ remain bounded, both the system state and the reference model state can be drifting to infinity with the same rate!!! So, our Lyapunov analysis does not give us anything regarding the system stability, despite the fact that the errors remain bounded! It is the right time to acknowledge the value of Barbalat's lemma that you were always able to apply due to the stability of the reference model, which was helping you to conclude boundedness of the system state upon your $\dot V(t) \le 0$ proof, and consequently asymptotic convergence of the tracking error to zero. Now, the question is, did we indeed lose all of this????

Luckily, not! It is now time to apply some intuition. Control saturation implies that we have limited control authority, so we cannot do everything, but we should be able to do something within our limits. So, most probably, all we can prove is a local result, and right now we need to define the local domain of attraction of the system state, so that starting from those initial conditions, with our control action, the system state remains bounded. Once we have boundedness of the system state, and the $\dot{V}(t) \leq 0$, we can conclude that the reference model state remains bounded, which ultimately can help us to apply Barbalat's lemma to prove local asymptotic stability. **Remember that when dealing with three signals, subject to a relationship $e(t) = x(t) - x_m(t)$, at least two of them need to be proven to be bounded, so that boundedness of the third one can be concluded! In the absence of saturation, $x_m(t)$ is bounded by definition, and $\dot{V}(t) \leq 0$ gives boundedness of $e(t)$, so that boundedness of $x(t)$ follows. Once we have adaptive modification of the reference model via a feedback from the main system, boundedness of $x_m(t)$ is not granted and needs to be proved. But instead we construct local domain of attraction of the system state, $x(t)$, so that starting form those initial conditions, the system state remains bounded with our feedback. Then boundedness of the state $x_m(t)$ of the reference model follows from boundedness of $e(t)$, which was concluded based on $\dot{V}(t) \leq 0$.**

Since now we are interested in proving boundedness of the system state, we consider the following Lyapunov function candidate for the system dynamics:

$$W(x) = \frac{1}{2}\, x^2(t)\,. \tag{134}$$

Staring your controller from zero initial conditions, your $\Delta u(0) = 0$, and as long as your $|u_c(t)| \leq u_{\max}$, then everything is the same as previously done. If $\Delta u(t) \neq 0$, then $|u_c(t)| > u_{\max}$, and $u(t) = u_{\max}\mathrm{sgn}(u_c(t))$ as it follows from (123). Then the system dynamics in (122) becomes:

$$\dot{x}(t) = ax(t) + bu_{\max}\mathrm{sgn}(u_c(t))\,. \tag{135}$$

Consequently

$$\begin{aligned}\dot{W}(x(t)) &= ax^2(t) + bx(t)u_{\max}\mathrm{sgn}(u_c(t)) \\ &= ax^2(t) + u_{\max}|bx(t)|\mathrm{sgn}(u_c(t))\mathrm{sgn}(bx(t))\,.\end{aligned} \tag{136}$$

For asymptotically stable systems, i.e. when $a < 0$, it immediately follows that $\dot{W} < 0$ if $|x| > u_{\max}|b|/|a|$. Therefore, the system state remains bounded, and Barbalat's lemma can be applied to ensure global asymptotic stability of the error dynamics in (130).

For unstable systems, i.e. when $a > 0$, consider two cases as in [23]:

1. $\mathrm{sgn}(u_c(t)) = -\mathrm{sgn}(bx(t))$.

2. $\mathrm{sgn}(u_c(t)) = \mathrm{sgn}(bx(t))$.

In the first case, it follows from (136) that $\dot{W} = ax^2(t) - u_{\max}\,|bx(t)|$. Therefore $\dot{W}(x(t)) < 0$, if

$$|x| < \frac{u_{\max}|b|}{|a|}\,. \tag{137}$$

In the second case, when $\mathrm{sgn}(u_c(t)) = \mathrm{sgn}(bx(t))$, it follows from (136) that $\dot{W} = ax^2(t) + u_{\max}\,|bx(t)| \geq 0$. Now, let's find a region, where this actually holds, i.e. we need to identify the region in the state space where $\mathrm{sgn}(u_c(t)) = \mathrm{sgn}(bx(t))$ indeed leads to $\dot{W} \geq 0$. Let's use the fact that $|u_c(t)| \geq u_{\max}$ and rewrite

$$\begin{aligned} 0 \leq \dot{W} &= ax^2(t) + u_{\max}|bx(t)| \leq ax^2(t) + |u_c(t)||bx(t)| = ax^2(t) + u_c(t)bx(t)\mathrm{sgn}(u_c(t))\mathrm{sgn}(bx(t)) \\ &= ax^2(t) + u_c(t)\ bx(t) = ax^2(t) + (k_x(t)x(t) + k_r(t)r(t))bx(t) \\ &= (a + bk_x(t))x^2(t) + bx(t)k_r(t)r(t)\,. \end{aligned}$$

Noting that $a = -|a_m| - bk_x^*$, we further get:

$$\begin{aligned} 0 &\leq (-|a_m| + b\Delta k_x(t))x^2(t) + bx(t)(k_r^* + \Delta k_r(t))r(t) \\ &\leq (-|a_m| + |b|\Delta k_x^{\max})x^2(t) + |b||x(t)|(|k_r^*| + \Delta k_r^{\max})r_{\max}\,. \end{aligned}$$

This implies

$$|a_m||x|\left(\left(1 - \frac{|b|\Delta k_x^{\max}}{|a_m|}\right)|x| - \frac{|b|(\Delta k_r^{\max} + |k_r^*|)}{|a_m|}r_{\max}\right) \leq 0\,. \tag{138}$$

Notice that since $V(e, \Delta k_x, \Delta k_r, \Delta k_u)$ is radially unbounded, and its derivative is negative $\dot{V}(t) \leq 0$, then the maximal values of all errors, including $\Delta k_x^{\max}, \Delta k_r^{\max}$, lie on the level set of Lyapunov function $V = V_0 = V(0)$. Therefore, if we enforce

$$\sqrt{V(0)} \leq \sqrt{\frac{|b|}{\gamma_x}\ \frac{|a_m| - |k_r^*||a|\frac{r_{\max}}{u_{\max}}}{\alpha|a|\frac{r_{\max}}{u_{\max}} + |b|}} \tag{139}$$

then we have

$$\Delta k_x^{\max} \leq \frac{|a_m| - |k_r^*|\frac{r_{\max}}{u_{\max}}|a|}{|b| + \alpha\frac{r_{\max}}{u_{\max}}|a|}\,. \tag{140}$$

This in turn ensures that $|a_m| - |b|\Delta k_x^{\max} \geq 0$. Therefore, it follows from (138) that if

$$|x| \leq \left(\frac{\Delta k_r^{\max} r_{\max} + |k_r^*| r_{\max}}{|a_m| - |b|\Delta k_x^{\max}} \right) |b| \,, \tag{141}$$

then $\dot{W}(x(t)) \geq 0$. On the other hand, (140) implies that

$$\frac{\Delta k_r^{\max} r_{\max} + |k_r^*| r_{\max}}{|a_m| - |b|\Delta k_x^{\max}} < \frac{u_{\max}}{|a|} \,. \tag{142}$$

Consequently, our analysis of the closed-loop system dynamics reveals that when $\Delta u(t) \neq 0$:

$$\dot{W}(x(t)) < 0, \quad \forall x \in \mathcal{A} \triangleq \left\{ \left(\frac{\Delta k_r^{\max} r_{\max} + |k_r^*| r_{\max}}{|a_m| - |b|\Delta k_x^{\max}} \right) |b| \leq |x| \leq \frac{u_{\max}|b|}{|a|} \right\} . \tag{143}$$

In other words, subject to (139), as long as the system initial conditions lie in the annulus region $\mathcal{A}$, then the system state remains bounded, and Barbalat's lemma can be applied to ensure asymptotic convergence of the tracking error to zero and boundedness of all the signals.

We basically proved the following theorem.

Theorem 9.1. Let a and b in (122), $u_{\max}$ in (123) and $r_{\max}$ be such that

$$\frac{|a_m|}{|b_m|} > \frac{|a|}{|b|} \frac{r_{\max}}{u_{\max}} \tag{144}$$

If the system initial condition and Lyapunov function in (132) satisfy:

$$|x(0)| < \frac{|b|}{|a|} u_{\max} \tag{145}$$

$$\sqrt{V(0)} < \sqrt{\frac{|b|}{\gamma_x}} \, \frac{|a_m| - |k_r^*||a| \frac{r_{\max}}{u_{\max}}}{\alpha |a| \frac{r_{\max}}{u_{\max}} + |b|} \tag{146}$$

then the adaptive system in (130), (131) has bounded solutions $\forall r,\ |r(t)| \leq r_{\max}$, and the tracking error $e(t)$ goes to zero asymptotically.

Remark 9.1. The condition in (144) ensures that the numerator in (146) is positive.

Remark 9.2. Theorem 9.1 implies that if the initial conditions of the state and parameter errors lie within certain bounds (145), (146), then the adaptive system will have bounded solutions, and the tracking error will go to zero asymptotically. The local nature of the result for *unstable* systems is due to the limitations on the control input. For stable systems the results are global. For neutrally stable systems, i.e. $a = 0$, the upper bound of the annulus region goes to infinity, therefore the convergence of the tracking error to zero is also global.

Remark 9.3. In the context of flight control applications, such modification guarantees that if the guidance system is issuing commands $r(t)$ that the control system cannot implement due to the control surface saturation limits, then one needs to scale the guidance command proportional to the control deficiency, like $r(t) + k_u(t)\Delta u(t)$.

Homework Problems 9.1. Take one of your scalar systems, enforce saturation, do adaptive modification of the reference model dynamics and implement the adaptive controller.

10 Direct MRAC for Nonlinear Systems with Matched Structured Nonlinearities

Development of adaptive control techniques for general class of nonlinear systems is one of the challenging problems in nonlinear control and is not solved up today. One class of systems, for which adaptive control techniques are derived and proofs for global stability exist, are the nonlinear systems with matched nonlinearities that can be linearly parameterized in unknown parameters, i.e. the nonlinearities in the system can be presented as $b\lambda W^\top \Phi(x)$, where $\Phi(x)$ is a vector of *known* continuous bounded functions, while W is a vector of *unknown* parameters, b is a *known* constant vector, λ is an *unknown* constant of known sign. Another class is the class of systems suitable for backstepping type of designs that we will cover later in the course [16]. In either case the system is assumed to be ideally parameterized in unknown parameters.

10.1 Direct MRAC for Nonlinear Systems with Matched Structured Nonlinearities.

Let the system dynamics propagate according to the following differential equation:

$$\dot{x}(t) = Ax(t) + b\lambda(u(t) + W^\top \Phi(x(t)))\,, \quad x(0) = x_0\,, \tag{147}$$

where $x \in \mathbb{R}^n$ is the state of the system, A is an unknown matrix, b is a known constant vector, $\lambda \neq 0$ is an unknown constant of known sign, $u \in \mathbb{R}$ is the control input, $\Phi(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a $(m \times 1)$-dimensional vector of *known* continuous functions, while W represents a $(m \times 1)$-dimensional vector of *unknown* constant parameters.

Let the reference model of interest for tracking be given:

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t)\,, \quad x_m(0) = x_{m0}\,, \tag{148}$$

where $x_m \in \mathbb{R}^n$ is the state of the reference model, A_m is a Hurwitz $(n \times n)$-dimensional matrix, $b_m \in \mathbb{R}^n$ is a constant vector, $r(t) \in \mathbb{R}$ is a uniformly bounded continuous input.

Direct adaptive model reference feedback is defined as:

$$u(t) = k_x^\top(t)x(t) + k_r(t)r(t) - \hat{W}^\top(t)\Phi(x(t)) \tag{149}$$

where $k_x(t) \in \mathbb{R}^n$, $k_r(t) \in \mathbb{R}$ are the adaptive gains defined through the stability proof, $\hat{W}(t) \in \mathbb{R}^m$ is the estimate of W. Substituting (149) into (147), yields the following closed-loop system dynamics:

$$\dot{x}(t) \quad = \quad (A + b\lambda k_x^\top(t))x(t) + b\lambda k_r(t)r(t) - b\lambda \Delta W^\top(t)\Phi(x) \tag{150}$$

where $\Delta W(t) = \hat{W}(t) - W$ is the parameter estimation error.

Comparing (148) with the system dynamics in (150), assumptions are formulated that guarantee existence of the adaptive feedback signal.

Assumption 10.1. (Reference model matching conditions)

$$\begin{aligned} &\exists\ k_x^*, \quad b\lambda (k_x^*)^\top = A_m - A \\ &\exists\ k_r^*, \quad b\lambda k_r^* = b_m \end{aligned} \tag{151}$$

Remark 10.1. The knowledge of the gains k_x^*, k_r^* is not required, only their existence is assumed.

Let $e(t) = x(t) - x_m(t)$ be the tracking error signal. Then the tracking error dynamics can be written:

$$\dot{e}(t) = A_m e(t) + b\lambda \left(\Delta k_x^\top(t) x(t) + \Delta k_r(t) r(t) - \Delta W^\top(t) \Phi(x(t)) \right), \quad e(0) = e_0\,, \tag{152}$$

where $\Delta k_x(t) = k_x(t) - k_x^*$, $\Delta k_r(t) = k_r(t) - k_r^*$ denote parameter errors. Consider the following adaptation laws:

$$\begin{aligned} \dot{k}_x(t) &= -\Gamma_x x(t) e^\top(t) P b \operatorname{sgn}(\lambda), \quad k_x(0) = k_{x0}\,, \\ \dot{k}_r(t) &= -\gamma_r r(t) e^\top(t) P b \operatorname{sgn}(\lambda), \quad k_r(0) = k_{r0}\,, \\ \dot{\hat{W}}(t) &= \Gamma_W \Phi(x(t)) e^\top(t) P b \operatorname{sgn}(\lambda), \quad \hat{W}(0) = \hat{W}_0\,, \end{aligned} \tag{153}$$

where $\Gamma_x = \Gamma_x^\top > 0$, $\Gamma_W = \Gamma_W^\top > 0$, $\gamma_r > 0$ are the adaptation gains. Define the following Lyapunov function candidate:

$$\begin{aligned} &V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta W) \qquad (154) \\ = \ &e^\top(t) P e(t) + |\lambda| \left(\Delta k_x^\top(t) \Gamma_x^{-1} \Delta k_x(t) + \gamma_r^{-1} (\Delta k_r(t))^2 + \Delta W^\top(t) \Gamma_W^{-1} \Delta W(t) \right) \end{aligned}$$

where $P = P^\top > 0$ solves the algebraic Lyapunov equation

$$A_m^\top P + P A_m = -Q \tag{155}$$

for arbitrary $Q > 0$. The time derivative of the Lyapunov function in (154) along the system trajectories

(152), (153) is:

$$\begin{aligned}
\dot{V}(t) &= -e^\top(t)Qe(t) + 2e^\top(t)Pb\lambda\left(\Delta k_x^\top(t)x(t) + \Delta k_r(t)r(t) - \Delta W^\top(t)\Phi(x(t))\right) \\
&\quad +2|\lambda|\Delta k_x^\top(t)\Gamma_x^{-1}\Delta\dot{k}_x(t) + 2|\lambda|\Delta k_r(t)\gamma_r^{-1}\Delta\dot{k}_r(t) + 2|\lambda|\Delta W^\top(t)\Gamma_W^{-1}\Delta\dot{W}(t) \\
&= -e^\top(t)Qe(t) \qquad (156)\\
&\quad +2|\lambda|\Delta k_x^\top(t)\left(e^\top(t)Pbx(t)\mathrm{sgn}(\lambda) + \Gamma_x^{-1}\dot{k}_x(t)\right) \\
&\quad +2|\lambda|\Delta k_r(t)\left(e^\top(t)Pbr(t)\mathrm{sgn}(\lambda) + \gamma_r^{-1}\dot{k}_r(t)\right) \\
&\quad +2|\lambda|\Delta W^\top(t)\left(-e^\top(t)Pb\Phi(x(t))\mathrm{sgn}(\lambda) + \Gamma_W^{-1}\dot{\hat{W}}(t)\right) \\
&= -e^\top(t)Qe(t) \le 0 \qquad (157)
\end{aligned}$$

Hence, the derivative of the Lyapunov function candidate is negative semidefinite, therefore all signals are bounded. Application of Barbalat's lemma implies asymptotic convergence of tracking error to zero, but the same cannot be claimed for parameter errors unless PE conditions are enforced.

Remark 10.2. Notice that the matching assumptions in (151) permit to write the system dynamics in (147) equivalently in the form:

$$\begin{aligned}
\dot{x}(t) &= (A_m - b\lambda(k_x^*)^\top)x(t) + b\lambda(u(t) + W^\top\Phi(x(t))) \\
&= A_m x(t) + b\lambda(u(t) - (k_x^*)^\top x(t) + W^\top\Phi(x(t)))\,. \qquad (158)
\end{aligned}$$

From this structure it is obvious why the conditions in (151) are called matching conditions, because the uncertainties are all in the span of b.

10.2 Choice of Optimal Reference Model and Augmentation of Nominal Design.

Reading [11], pp. 579-589.

Augmenting any existing control scheme with an additional controller for performance improvement is called *Lyapunov redesign*. In designing adaptive controllers for industry it is quite common to have a requirement that the nominal design cannot be touched. What serves as nominal design in the majority of applications is an LQR controller with good stability margins, which is being designed using linearized models. Thus, addition of adaptive controllers into the loop has to take the form of augmentation. It has to have conservative guidelines for it in a sense that when you add the adaptive controller, at first the adaptation gain has to be set equal to 0 to make sure that the nominal performance is not violated. Then by slowly increasing the adaptation gain you need to see improvement in the performance in the presence of uncertainties. To understand how such implementation is being done, we need first to define a reference model of interest to track.

The reference model is one of the "free design parameters" in the adaptive controller along with the adaptation gains $\Gamma_x, \gamma_r, \Gamma_W$ and choice of the matrix Q. They are all connected in the adaptation laws (153). While $\Gamma_x, \gamma_r, \Gamma_W$ are free for tuning, P is related to your choice of A_m via the Lyapunov equation (155), which in turn has the Q as a free design parameter. You may recall that in linear systems analysis, we proved that $Q = \mathbb{I}$ is the best choice giving the fastest convergence rate. Here unfortunately such a result cannot be proven, and very often you may need to choose Q with different eigenvalues.

A common choice for a reference system accepted by engineers in industry is the optimal control for the nominal linear system. In that case, some linearized model of the nonlinear system is used to design an LQR controller, which defines a closed-loop linear reference system with desired transient characteristics, like overshoot, settling time, rise time, etc. So, assume that your nonlinear system in (147) is given to you in the form:

$$\dot{x}(t) = A_0 x(t) + b\lambda(u(t) + (k_x^*)^\top x(t) + W^\top \Phi(x(t)))\,, \quad x(0) = x_0\,, \tag{159}$$

where A_0 is known, b is known, k_x^* represents the *matched* uncertainties in the state matrix, λ represents the uncertainties in the control effectiveness and can be used to model actuator failures, $\mathrm{sgn}(\lambda)$ is known, W is a vector of unknown constants, $\Phi(x)$ is a vector of known nonlinear functions. Notice that $(k_x^*)^\top x(t)$ can be grouped with $W^\top \Phi(x(t))$, by assuming that x is one of the components of the known vector of nonlinearities given by $\Phi(x)$, while k_x^* can be appended with W. Therefore there is no loss of generality in assuming that the A matrix is known to begin with, since its matched uncertainties can always be lumped with the uncertainties in the span of b. In the absence of actuator failures (i.e. $\lambda = 1$) and uncertainties (i.e. $k_x^* = 0, W = 0$), your nominal system is given by a linear system

$$\dot{x}(t) = A_0 x(t) + bu(t)\,, \quad x(0) = x_0\,. \tag{160}$$

For this model you can design an LQR controller to meet your control design specifications, like overshoot, settling time, rise time, etc. Let this LQR based linear tracking controller be given by

$$u_{lin}(t) = -k_{LQR}^\top x(t) + k_g r(t)\,, \tag{161}$$

where the feedforward gain k_g is usually selected to achieve asymptotic tracking for a step input dependent upon the particular regulated output of interest, while k_{LQR} is computed using the positive (semi-)definite symmetric solution of corresponding Riccati equation. This leads to the following closed-loop nominal linear system that has the desired specifications:

$$\dot{x}_m(t) = A_m x(t) + b_m r(t)\,, \quad x_m(0) = x_{m0}\,, \tag{162}$$

where $A_m = A_0 - bk_{LQR}^\top$, $b_m = bk_g$. We immediately notice that this way derived reference model satisfies the matching assumptions in (151). Indeed,

$$A_m - A = A_0 - bk_{LQR}^\top - (A_0 + b\lambda(k_x^*)^\top) = -bk_{LQR}^\top - b\lambda(k_x^*)^\top = b\lambda(-k_{LQR}^\top/\lambda - (k_x^*)^\top)\,,$$

where $-(k_{LQR}^\top/\lambda + (k_x^*)^\top)$ plays the role of the k_x^* in (151). Similarly, $b_m = bk_g$ can be written as:

$$b_m = bk_g = b\lambda k_r^*\,,$$

so that for any value of λ there exists a constant k_r^* such that $\lambda k_r^* = k_g$.

Now, following (149) let's design an adaptive controller as:

$$u(t) \;=\; (k_x^\top(t) - k_{LQR}^\top)x(t) + (k_r(t) + k_g)r(t) - \hat{W}^\top(t)\Phi(x(t))\,. \tag{163}$$

Notice that it is equivalent to augmenting a baseline LQR controller, since it can be equivalently written as:

$$u(t) \;=\; u_{lin}(t) + k_x^\top(t)x(t) + k_r(t)r(t) - \hat{W}^\top(t)\Phi(x(t))\,. \tag{164}$$

The structure in (163) is *no different* from the one in (149) in a sense, that the adaptive laws that are defined for $k_x(t)$ and $k_r(t)$ in (153) will be the same for $k_x(t) - k_{LQR}$ and $k_r(t) + k_g$, since k_{LQR} and k_g are constants anyway. However, to make sure that we are indeed augmenting an LQR baseline controller for its performance improvement in the presence of uncertainties, we need to initialize the adaptive laws for $k_x(t)$ and $k_r(t)$ in (153) at 0. In that case, in the absence of uncertainties in the system when you do not need any adaptation, the adaptation gains $\Gamma_x, \gamma_r, \Gamma_W$ should be set to zero to reduce the controller in (163) to the LQR one in (161) to recover the nominal closed-loop performance of (162).

Control system design steps can be summarized as follows:

- Select your desired reference model based on your nominal values of uncertain parameters. You can do this, for example, by using the LQR theory or simple pole placement arguments. It would be helpful to study the linear servomechanism theory for obtaining good baseline design.

- Verify that it indeed achieves the desired control objective with desired specifications. You need to get good tracking with good stability margins, since you're still within linear system's theory.

- Insert the **matched** uncertainties into your nominal system and convince yourself that the performance of LQR is violated in the presence of uncertainties.

- Augment your LQR controller with an adaptive controller and verify its performance in the presence of uncertainties. Start from zero values of adaptation gains and increase those incrementally. Convince yourself that you cannot increase the adaptation gains too much, because the system runs into oscillations. Tune the adaptive gains for the best performance.

Homework Problems 10.1. Please do this in groups so that you can help each other. I don't mind grading one homework and giving the same grade to all the names listed on that homework.

1. Substitute the controller from (163) into (159), write the error dynamics using (162), specify the matching assumptions explicitly, do a Lyapunov proof to show that the tracking error goes to zero asymptotically. Be careful with the initial conditions of the adaptation laws. Convince yourself that when $\lambda = 1$ and $k_x^* = 0$, setting the adaptation gains to zero, you recover the nominal performance. Tuning of the adaptive gains starts from zero and goes up incrementally.

2. Use any second order system of your choice (pendulum, spring-mass, Van der Pol, short-period, etc.) to illustrate your findings by simulation.

3. While doing simulations, follow the steps suggested above.

10.3 Augmentation of a Nominal PI Controller.

As you may know, for tracking step inputs it is always better to use PI (proportional integral) controller so that not to have a steady-state error. This is also a common approach in industry: the integral state is always built into the system. Now let's see how can we augment a nominal PI controller. Consider the following system dynamics:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\lambda(u(t) + W^\top \Phi(x(t)))\,, \quad x(0) = x_0 \\ y(t) &= c^\top x(t)\,, \end{aligned} \tag{165}$$

where W is unknown, $A, b, c, \Phi(x)$ are known, λ is an unknown constant of known sign, $y \in \mathbb{R}$ is the regulated output of interest, $x \in \mathbb{R}^n$ is the n-dimensional state vector, $u \in \mathbb{R}$ is the control input. Notice that we directly lumped the matched uncertainties of the state matrix into $W^\top \Phi(x(t))$. The control objective is to design a full state feedback adaptive controller so that $y(t)$ tracks a given smooth trajectory $y_c(t)$ asymptotically, while all other signals remain bounded.

So, let's introduce the integral error as an additional state into the system dynamics and write:

$$\tilde{y}(t) = y(t) - y_c(t) = c^\top x(t) - y_c(t)\,,$$

so that

$$y_I(t) = \int_0^t \tilde{y}(\tau) d\tau\,.$$

Augmented with this additional state, the system dynamics in (165) can be written as:

$$\begin{bmatrix} \dot{y}_I(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & c^\top \\ 0 & A \end{bmatrix} \begin{bmatrix} y_I(t) \\ x(t) \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ b \end{bmatrix} (u(t) + W^\top \Phi(x(t))) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} y_c(t) \tag{166}$$

In the absence of uncertainties, this reduces to the following system, called <u>*idealized linear system*</u> for (166):

$$\begin{bmatrix} \dot{y}_I(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & c^\top \\ 0 & A \end{bmatrix} \begin{bmatrix} y_I(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} y_c(t)$$

For this system you can use any of the linear control design methods (like LQR, pole placement, etc.) to achieve stabilization and asymptotic tracking of $y_c(t)$. Let that linear controller be given by:

$$u_{lin}(t) = -k^\top \begin{bmatrix} y_I(t) \\ x(t) \end{bmatrix}.$$

The resulting closed-loop system is your *desired reference system* for adaptive tracking:

$$\begin{bmatrix} \dot{y}_{Im}(t) \\ \dot{x}_m(t) \end{bmatrix} = \underbrace{\left(\begin{bmatrix} 0 & c^\top \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 \\ b \end{bmatrix} k^\top \right)}_{A_{ref}} \begin{bmatrix} y_{Im}(t) \\ x_m(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} y_c(t)\,, \tag{167}$$

where A_{ref} is Hurwitz. The total control for (166) is formed as

$$u(t) = u_{lin}(t) + u_{ad}(t)\,, \tag{168}$$

where $u_{lin}(t) = -k^\top \begin{bmatrix} y_I(t) \\ x(t) \end{bmatrix}$ will already operate over the states of the uncertain system and not the idealized linear system. **This is a key point to remember!** The idealized linear system is used only for computation of the controller gain k.

Comparing (167) with (166), where in the latter system the total control (168) is substituted, error dynamics can be formed:

$$\begin{aligned} \begin{bmatrix} \dot{e}_I(t) \\ \dot{e}(t) \end{bmatrix} &= \begin{bmatrix} \dot{y}_I(t) - \dot{y}_{Im}(t) \\ \dot{x}(t) - \dot{x}_m(t) \end{bmatrix} \\ &= A_{ref} \begin{bmatrix} e_I(t) \\ e(t) \end{bmatrix} + (1-\lambda) \begin{bmatrix} 0 \\ b \end{bmatrix} k^\top \begin{bmatrix} y_I(t) \\ x(t) \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ b \end{bmatrix} (u_{ad}(t) + W^\top \Phi(x(t))) \\ &= A_{ref} \begin{bmatrix} e_I(t) \\ e(t) \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ b \end{bmatrix} \left(u_{ad}(t) + \frac{1-\lambda}{\lambda} k^\top \begin{bmatrix} y_I(t) \\ x(t) \end{bmatrix} + W^\top \Phi(x(t)) \right) \end{aligned}$$

We notice that due to the integral action the reference trajectory $y_c(t)$ disappeared from error dynamics so that **we do not need adaptation on feedforward gain!!!** The adaptive controller thus can be designed in a much simpler way:

$$u_{ad}(t) = -\hat{k}_x^\top(t) \begin{bmatrix} y_I(t) \\ x(t) \end{bmatrix} - \hat{W}^\top(t)\Phi(x(t))\,,$$

leading to the following form of closed-loop dynamics:

$$\begin{bmatrix} \dot{e}_I(t) \\ \dot{e}(t) \end{bmatrix} = A_{ref} \begin{bmatrix} e_I(t) \\ e(t) \end{bmatrix} - \lambda \begin{bmatrix} 0 \\ b \end{bmatrix} \left(\Delta k_x^\top(t) \begin{bmatrix} y_I(t) \\ x(t) \end{bmatrix} + \Delta W^\top(t)\Phi(x(t)) \right)\,,$$

where

$$\Delta k_x(t) = \hat{k}_x(t) - \frac{1-\lambda}{\lambda} k, \quad \Delta W(t) = \hat{W}(t) - W$$

are the parametric errors.

Homework Problems 10.2. Write the adaptive laws and finish the stability proof. Take your second order example from the previous homework and do PI+LQR+adaptive design. Using your own simulations, compare the performance of the two different control designs. Try different inputs, like step and sinusoid, or combinations of steps and sinusoids. Convince yourself that the integral helps to get better performance.

11 Robustness of MRAC: Parameter Drift

As with every control scheme, an important question to ask will be robustness of adaptive controllers to disturbances and measurement noise. Let's get back to our scalar direct MRAC scheme to investigate this issue. So, the system dynamics is given by:

$$\dot{x}(t) = ax(t) + bu(t)\,, \quad x(0) = x_0\,, \tag{169}$$

where $x \in \mathbb{R}$ is the state of the system, a and b are unknown constants, while $\mathrm{sgn}(b)$ is known, $u \in \mathbb{R}$ is the control input. Consider the following reference model dynamics:

$$\dot{x}_m(t) = a_m x_m(t) + b_m r(t), \quad a_m < 0, \quad x_m(0) = x_{m0}\,, \tag{170}$$

where $x_m \in \mathbb{R}$ is the state of the reference model, $r(t)$ is a uniformly continuous bounded input signal of interest to track, while a_m, b_m specify the desired performance metrics for tracking $r(t)$.

The direct adaptive model reference feedback in the presence of imperfect measurement will take the form:

$$u(t) = k_x(t)x_d(t) + k_r(t)r(t)\,, \tag{171}$$

where $x_d(t) = x(t)+d(t)$, and $d(t)$ models the disturbance in measurement, which is bounded $|d(t)| \le d_0$, and its derivative is also bounded $|\dot{d}(t)| \le d^*$. To verify its impact on the entire design, let's repeat the same steps of the Lyapunov proof again.

Substitute (171) into (169) to get the following closed-loop system dynamics:

$$\dot{x}(t) = (a + bk_x(t))x(t) + bk_r(t)r(t) + bk_x(t)d(t) \tag{172}$$

Let $e(t) = x_d(t) - x_m(t)$ be the tracking error signal. Then the tracking error dynamics can be written:

$$\dot{e}(t) = a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t) + (bk_x(t) - a_m)d(t) + \dot{d}(t)\,, \quad e(0) = e_0\,, \tag{173}$$

where $\Delta k_x(t) = k_x(t) - k_x^*$, $\Delta k_r(t) = k_r(t) - k_r^*$ are introduced for parameter errors. If we consider the same old adaptation laws from (95):

$$\begin{aligned} \dot{k}_x(t) &= -\gamma_x(x(t) + d(t))e(t)\mathrm{sgn}(b)\,, \quad k_x(0) = k_{x0} \\ \dot{k}_r(t) &= -\gamma_r r(t)e(t)\mathrm{sgn}(b)\,, \quad k_r(0) = k_{r0}, \end{aligned} \tag{174}$$

where **<u>we have taken into consideration that the measurement of $x(t)$ is not perfect</u>**, and the same Lyapunov function candidate from (96):

$$V(e(t), \Delta k_x(t), \Delta k_r(t)) = e^2(t) + \left(\gamma_x^{-1}\Delta k_x^2(t) + \gamma_r^{-1}\Delta k_r^2(t)\right)|b|\,, \tag{175}$$

then the derivative along the system trajectories (173), (174) will be:

$$\begin{aligned}
\dot{V}(t) &= 2e(t)\left(a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t) + (bk_x(t) - a_m)d(t) + \dot{d}(t)\right) \\
&\quad +2\gamma_x^{-1}|b|\Delta k_x(t)\Delta\dot{k}_x(t) + 2\gamma_r^{-1}|b|\Delta k_r(t)\Delta\dot{k}_r(t) \\
&= -2|a_m|e^2(t) + 2e(t)\left((bk_x(t) - a_m)d(t) + \dot{d}(t)\right) \\
&\quad +2|b|\Delta k_x(t)\left(x(t)e(t)\mathrm{sgn}(b) + \gamma_x^{-1}\Delta\dot{k}_x(t)\right) \\
&\quad +2|b|\Delta k_r(t)\left(r(t)e(t)\mathrm{sgn}(b) + \gamma_r^{-1}\Delta\dot{k}_r(t)\right) \\
&= -2|a_m|e^2(t) + 2e(t)\left((bk_x(t) - a_m)d(t) + \dot{d}(t)\right) \\
&\quad +2|b|\Delta k_x(t)\left(x(t)e(t)\mathrm{sgn}(b) - (x(t) + d(t))e(t)\mathrm{sgn}(b)\right) \\
&\quad +2|b|\Delta k_r(t)\left(r(t)e(t)\mathrm{sgn}(b) - r(t)e(t)\mathrm{sgn}(b)\right) \\
&= -2|a_m|e^2(t) + 2e(t)\left(bk_x(t) - a_m - b\Delta k_x(t)\right)d(t) + 2e(t)\dot{d}(t) \\
&= -2|a_m|e^2(t) - 2ae(t)d(t) + 2e(t)\dot{d}(t) \leq -2|a_m||e(t)|^2 + 2|a||e(t)||d(t)| + 2|e(t)||\dot{d}(t)| \\
&\leq -2|a_m||e(t)|^2 + 2|a||e(t)|d_0 + 2|e(t)|d^*
\end{aligned}$$

We can conclude that $\dot{V}(t) \leq 0$, if

$$|e(t)| > \frac{1}{|a_m|}\left(|a|d_0 + d^*\right)\,.$$

So, for large tracking errors, it seems that everything is fine, and $e(t)$ will be decreasing. But when

$$|e(t)| < \frac{1}{|a_m|}\left(|a|d_0 + d^*\right)\,,$$

then the sign of the derivative of the Lyapunov function is positive, $\dot{V}(t) > 0$. What does this imply?

First of all you need to notice that for the first time you got a different effect as opposed to all your previous exercises. In a number of your previous exercises you were able to show that $\dot{V}(t) \leq 0$ in a neighborhood of the origin. This was helping you to conclude local stability for a specific domain of attraction.

Now you got exactly the opposite: $\dot{V}(t) > 0$ in a neighborhood of the origin, but is negative far away in one particular direction. So, what's going on? As you can see from Figure 20, the parameter errors can diverge easily while the tracking error remains small. This effect is called **parameter drift**. Since the ideal values of all parameters are constant, the growth in parameter errors implies that the parameter estimates (which directly enter into the control definition) grow unboundedly. This will consequently require larger and larger (growing to infinity) control effort (which is impossible to provide by the hardware), eventually leading the system into instability. This was exactly the reason of that historical crash of X-15 leading to the death of the pilot, Michael Adams.

This argument is assumed to emphasize one more time the significance of the level sets of the Lyapunov function in all kinds of Lyapunov analysis. If we had $\dot{V}(t) \leq 0$ outside a compact set and not a strip, then once the parametric errors had crossed those bounds during their divergence, they would have faced the $\dot{V}(t) \leq 0$ environment, which would have forced them to come back.

You already know two ways of preventing this. One is the PE condition. If we had persistently exciting reference input, then our parameters would have converged, and the parametric error would have gone to zero. Another way is the projection-type adaptation law that we introduced in indirect MRAC to ensure that our parameters stay within prespecified regions by us, so that when they hit these bounds, the adaptive law forces them to go back. These tricks are called robustification of adaptive laws. Other modifications include σ-modification and e-modification that directly give negative definite derivative for the candidate Lyapunov function outside a compact set, like $\dot{V}(t) \leq -2|a_m||e(t)|^2 - \gamma||\tilde{\theta}(t)||^2 + c$. We will get to these modifications of the adaptive laws later in the course.

Homework Problems 11.1.

- Take the first order scalar system of the MRAC and insert disturbance into it and convince yourself that it can lead to parameter divergence, which will require larger control effort and lead the system into instability.

- Do projection type modification of the adaptive law and convince yourself that it prevents instability. Notice that in this case your projection type modification of the adaptive law will need bounds from both sides, and not only from below as I had provided in indirect adaptive scheme. You need to bound the parameters from both sides, do the stability proof and implement in simulations to convince yourself in this.

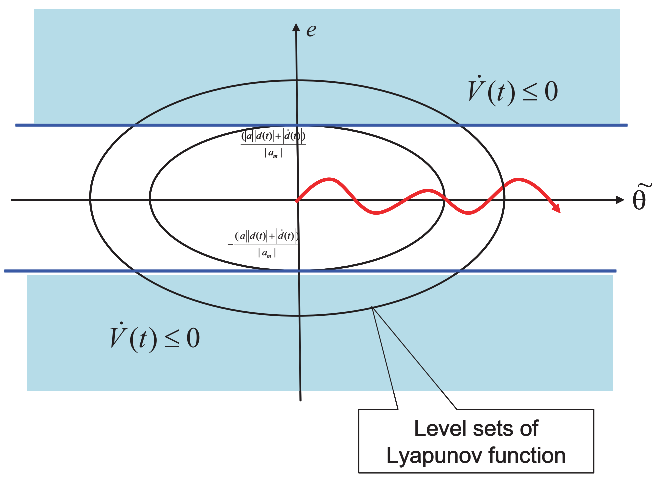


Fig. 20 Parameter divergence: nothing prevents the parameter error $\tilde{\theta}$ to grow

12 Projection Based Adaptation

Definition 12.1. $\Omega \subset \mathbb{R}^n$ is a convex set if $\forall\, x, y \in \Omega \subset \mathbb{R}^n$ the following holds:

$$\lambda\, x + (1-\lambda)\; y \in \Omega, \quad \forall\, 0 \le \lambda \le 1\,. \tag{176}$$

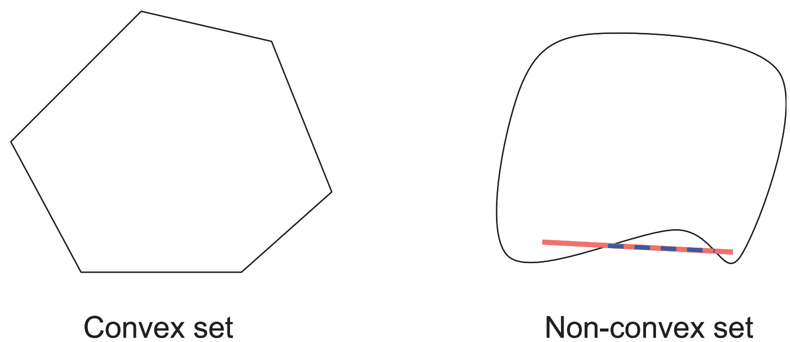


Fig. 21 Illustration of convex and non-convex sets

Definition 12.2. $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function if $\forall\, x, y \in \mathbb{R}^n$ the following holds:

$$f\left(\lambda\, x + (1-\lambda)\; y\right) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall\, 0 \le \lambda \le 1\,. \tag{177}$$

A sketch of a convex function is presented on Fig.22.

Lemma 12.1. Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then for any constant $\delta > 0$ the set $\Omega_\delta = \{\theta \in \mathbb{R}^n | f(\theta) \le \delta\}$ is convex. The set Ω_δ is called the sublevel set.

The proof is elementary. Given $\theta_1, \theta_2 \in \Omega_\delta$, i.e. $f(\theta_i) \le \delta$, it follows that for arbitrary $0 \le \lambda \le 1$

$$f\Bigg(\underbrace{\lambda\, \theta_1 + (1-\lambda)\; \theta_2}_{\theta}\Bigg) \le \lambda \underbrace{f(\theta_1)}_{\le\delta} + (1-\lambda) \underbrace{f(\theta_2)}_{\le\delta} \le \lambda\delta + (1-\lambda)\delta = \delta\,. \tag{178}$$

Since $f(\lambda\, \theta_1 + (1-\lambda)\; \theta_2) \le \delta$, then $\lambda\, \theta_1 + (1-\lambda)\; \theta_2 \in \Omega_\delta$. Since $\theta_1, \theta_2 \in \Omega_\delta$, then Ω_δ is convex.

Lemma 12.2. Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function. Choose a constant δ and consider the convex set $\Omega_\delta = \{\theta \in \mathbb{R}^n | f(\theta) \le \delta\} \subset \mathbb{R}^n$. Let $\theta, \theta^* \in \Omega_\delta$ and $f(\theta^*) < \delta$

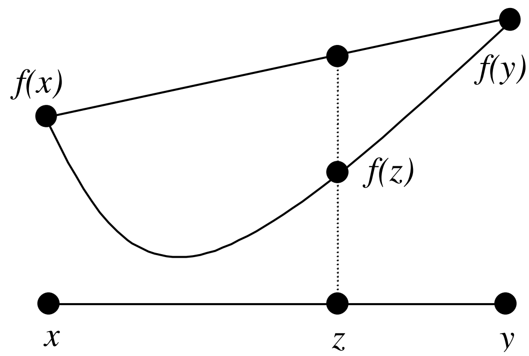


Fig. 22 Illustration of a convex function

and $f(\theta) = \delta$ (i.e. θ^* is **not** on the boundary of Ω_δ, while θ **is** on the boundary of Ω_δ). Then the following inequality takes place:

$$(\theta^* - \theta)^\top \nabla f(\theta) \leq 0\,, \tag{179}$$

where $\nabla f(\theta) = \begin{pmatrix} \frac{\partial f(\theta)}{\partial \theta_1} & \dots & \frac{\partial f(\theta)}{\partial \theta_n} \end{pmatrix}^\top \in \mathbb{R}^n$ is the gradient vector of $f(\theta)$ evaluated at θ.

Relation (179) is illustrated on Fig.23. It shows that the gradient vector evaluated at the boundary of a convex set always points away from the set.

Proof. Since $f(x)$ is convex function, then

$$f\left(\lambda\ \theta^* + (1-\lambda)\ \theta\right) \leq \lambda f(\theta^*) + (1-\lambda) f(\theta), \quad \forall\ 0 \leq \lambda \leq 1\,, \tag{180}$$

which can equivalently be rewritten:

$$f\left(\theta + \lambda\ (\theta^* - \theta)\ \right) \leq f(\theta) + \lambda\left(f(\theta^*) - f(\theta)\right)\,. \tag{181}$$

Then for any nonzero $0 < \lambda \leq 1$ we have

$$\frac{f(\theta + \lambda(\theta^* - \theta)) - f(\theta)}{\lambda} \leq f(\theta^*) - f(\theta) < \delta - \delta = 0\,. \tag{182}$$

Notice that the expression in the numerator on the left side $f(\theta + \lambda(\theta^* - \theta))$, being a scalar function of vector argument θ, can be simultaneously viewed as a scalar function of scalar argument $F(\lambda)$. In

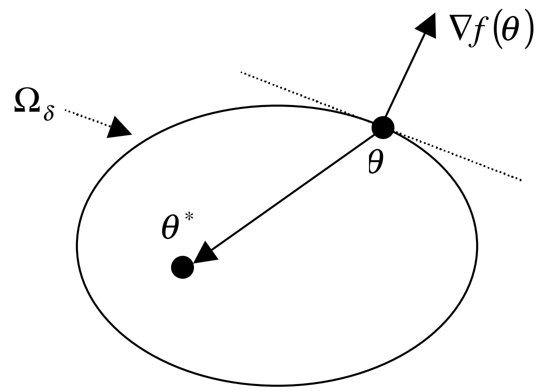


Fig. 23 Gradient and convex set

that case, the following is true:

$$F(\lambda) = F(0) + F'(0)\lambda + O(\lambda^2)\,. \tag{183}$$

Notice that $F(0) = f(\theta)$. Further, $F'(\lambda) = \left[\nabla f(\theta + \lambda(\theta^* - \theta))\right]^\top (\theta^* - \theta)$, where ∇f denotes the differentiation of f with respect to its whole vector argument, thus giving the gradient, while $(\theta^* - \theta)$ comes out of differentiation of the argument with respect to λ. Hence, $F'(0) = (\theta^* - \theta)^\top \nabla f(\theta)$. Therefore

$$f(\theta + \lambda(\theta^* - \theta)) = f(\theta) + (\theta^* - \theta)^\top \nabla f(\theta)\lambda + O(\lambda^2)\,. \tag{184}$$

Substituting into (182), taking the limit as $\lambda \to 0$ implies $(\theta^* - \theta)^\top \nabla f(\theta) \leq 0$ and completes the proof.

Remark 12.1. It is possible that for some convex function $f(x)$ and some δ, there may not exist $\theta \in \Omega_\delta$ such that $f(\theta) = \delta$, or $\theta^* \in \Omega_\delta$ such that $f(\theta^*) < \delta$. (e.g. $f(x) \equiv 1 \quad \forall x \in \mathbb{R}^n$: when $\delta > 1$, $f(\theta) < \delta,\ \forall\, \theta \in \mathbb{R}^n$; when $\delta = 1,\ f(\theta) = \delta\ \ \forall\, \theta \in \mathbb{R}^n$.) In this case we have a more general version of Lemma 12.2, which can be stated like this: Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function. Then for any $\theta,\ \theta^* \in \mathbb{R}^n$, the following inequality holds:

$$(\theta^* - \theta)^\top \nabla f(\theta) \leq f(\theta^*) - f(\theta).$$

Moreover, if $f(x)$ is radially unbounded, i.e., $f(x) \to \infty$ as $||x|| \to \infty$, then the sublevel set Ω_δ is bounded. The proof of the last fact can be done by contradiction: indeed, if Ω_δ is not bounded, then

$\exists\ \theta \in \Omega_\delta$ with $||\theta||$ arbitrarily large, for which $f(\theta)$ is arbitrarily large (due to radial unboundedness), which contradicts the fact $f(\theta) \leq \delta$.

Definition 12.3. [12] Consider a convex compact set with a smooth boundary given by:

$$\Omega_c \triangleq \{\theta \in \mathbb{R}^n \mid f(\theta) \leq c\}, \quad 0 \leq c \leq 1,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the following smooth convex function:

$$f(\theta) = \frac{\theta^\top\theta - \theta_{\max}^2}{\epsilon_\theta \theta_{\max}^2}, \tag{185}$$

where $\theta_{\max}$ is the norm bound imposed on the parameter vector θ, and ϵ_θ denotes the convergence tolerance of our choice. Let the true value of the parameter θ, denoted by θ^*, belong to Ω_0, i.e. $\theta^* \in \Omega_0$. The projection operator is defined as:

$$\mathrm{Proj}(\theta, y) \triangleq \begin{cases} y & \text{if } f(\theta) < 0, \\ y & \text{if } f(\theta) \geq 0 \text{ and } \nabla f^\top y \leq 0, \\ y - \underbrace{\dfrac{\nabla f}{\|\nabla f\|}}_{unit\ vector} \underbrace{\left\langle \dfrac{\nabla f^\top}{\|\nabla f\|}, y \right\rangle}_{projection} \underbrace{f(\theta)}_{scaling} & \text{if } f(\theta) \geq 0 \text{ and } \nabla f^\top y > 0 \end{cases} \tag{186}$$

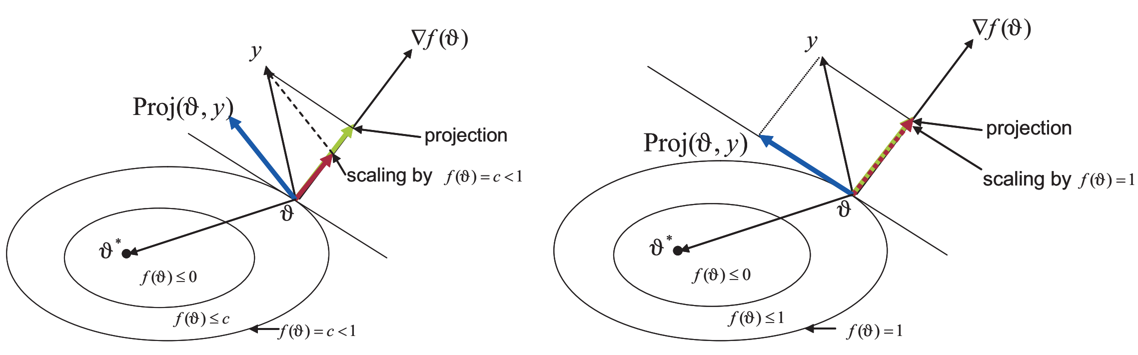


Fig. 24 Illustration of the projection operator

Property 12.1. [12] The projection operator $\mathrm{Proj}(\theta, y)$ as defined in (186) does not alter y if θ belongs to the set $\Omega_0 = \{\theta \in \mathbb{R}^n \mid f(\theta) \leq 0\}$. In the set $\{0 \leq f(\theta) \leq 1\}$, if $\nabla f^\top y > 0$, the $\mathrm{Proj}(\theta, y)$

operator subtracts a vector normal to the boundary of $\Omega_c = \{\theta \in \mathbb{R}^n \mid f(\theta) = c\}$ so that we get a smooth transformation from the original vector field y to an inward or tangent vector field for $c = 1$. Thus, if θ is the adaptive parameter, and $\dot{\theta}(t) = \mathrm{Proj}(\theta(t), y(t))$, then θ never leaves Ω_1.

Property 12.2. Given the vectors $y = [\begin{array}{ccc} y_1 & \cdots & y_n \end{array}]^\top \in \mathbb{R}^n$ and $\theta = [\begin{array}{ccc} \theta_1 & \cdots & \theta_n \end{array}]^\top \in \mathbb{R}^n$, we have:

$$(\theta - \theta^*)^\top (\mathrm{Proj}(\theta, y) - y) = \sum_{i=1}^{n} (\theta_i - \theta_i^*)^\top (\mathrm{Proj}(\theta_i, y_i) - y_i) \leq 0, \tag{187}$$

where θ^* is the true value of the parameter θ.

Indeed,

$$(\theta^* - \theta)^\top (y - \mathrm{Proj}(\theta, y)) = \begin{cases} 0 & \text{if } \; f(\theta) < 0, \\ 0 & \text{if } \; f(\theta) \geq 0 \; \text{ and } \; \nabla f^T y \leq 0, \\ \dfrac{\underbrace{(\theta^* - \theta)^\top \nabla f}_{\leq 0} \; \underbrace{\nabla f^T y}_{\geq 0} \; \underbrace{f(\theta)}_{\geq 0}}{\|\nabla f\|^2} & \text{if } \; f(\theta) \geq 0 \; \text{ and } \; \nabla f^T y > 0 \end{cases}$$

Changing the signs on the left side, one gets (187).

Remark 12.2. The special structure of the function f in (185) needs to be interpreted in the following way: if you solve $f(\theta) \leq 1$, which defines the boundaries of your outer set, then you get that $\theta^\top \theta \leq (1 + \epsilon_\theta)\theta_{\max}^2$. It is obvious that ϵ_θ specifies the maximum tolerance that you would allow your adaptive parameter to exceed as compared to its maximum conservative value selected by you.

Recommendation. Implement the projection operator in Matlab for 2D and 3D cases. In one case you will have two circles corresponding to $c = 0$ and $c = 1$, and in the other you'll have two spheres. Do the 2D and 3D plots to convince yourself in the change of the direction of the vector due to projection. When you hit the outer set, you must see a tangent vector. The easiest would be to implement the adaptive laws, by setting the true value of the unknown parameter outside the outer convex set and having excitation in the reference input. Then your adaptive parameter will tend to find the true value and will be eager to cross the convex sets to converge to it, but projection will not let it reach its limit, by changing the direction of adaptation on the outer set to its tangent. If you ever face a situation like this, it means that your choice of sets did not have the true value of the unknown parameter in it, i.e. your conservative knowledge of the value of the true unknown parameter was not "well conservative". When doing projection, it is always important to choose the sets sufficiently large so that a situation like this would not appear in real implementation. It will ensure parameter boundedness, but can disturb good behavior of the tracking error convergence.

13 Adaptive Control in the Presence of Uniformly Bounded Residual Nonlinearity

Let the system dynamics propagate according to the following differential equation:

$$\dot{x}(t) = Ax(t) + b\lambda(u(t) + W^\top(t)\Phi(x(t)) + d(t, x(t))), \quad x(0) = x_0, \tag{188}$$

where $x \in \mathbb{R}^n$ is the state of the system, A is an unknown matrix, b is a known constant vector, λ is an unknown constant of known sign, $d(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous **globally and uniformly bounded** function of state, $d(t,x) \in \mathcal{C}[\mathbb{R} \times \mathbb{R}^n, \mathbb{R}]$, $|d(t,x)| \le d^*$. We will call it residual nonlinearity.

For a given uniformly bounded continuous input $r(t)$, consider the following reference model:

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t), \quad x(0) = x_{m0}, \tag{189}$$

where $x_m \in \mathbb{R}^n$ is the state of the reference model, A_m is Hurwitz and is chosen to meet the performance specifications. The control objective is to design an adaptive controller to achieve tracking $x(t) \to x_m(t)$ as $t \to \infty$.

We consider the following direct model reference adaptive feedback:

$$u(t) = k_x^\top(t)x(t) + k_r(t)r(t) - \hat{W}^\top(t)\Phi(x), \tag{190}$$

where $k_x(t) \in \mathbb{R}^n$, $k_r(t) \in \mathbb{R}$, $\hat{W}(t)$ are the adaptive gains to be defined through the stability proof. Substituting (190) into (188), yields the following closed-loop system dynamics:

$$\dot{x}(t) = (A + b\lambda k_x^\top(t))x(t) + b\lambda k_r(t)r(t)) - b\lambda(\Delta W^\top(t)\Phi(x(t)) - d(t, x(t))), \tag{191}$$

where $\Delta W(t) = \hat{W}(t) - W$ is the parameter estimation error. Comparing (189) with the system dynamics in (191), assumptions are formulated that guarantee existence of the adaptive feedback.

Assumption 13.1. (Reference model matching conditions)

$$\begin{aligned} &\exists\ k_x^*, \quad b\lambda(k_x^*)^\top = A_m - A \\ &\exists\ k_r^*, \quad b\lambda k_r^* = b_m\,. \end{aligned} \tag{192}$$

Remark 13.1. The knowledge of the gains k_x^*, k_r^* is not required, only their existence is assumed.

Let $e(t) = x(t) - x_m(t)$ be the tracking error signal. Then the tracking error dynamics can be written:

$$\dot{e}(t) = A_m e(t) + b\lambda \left(\Delta k_x^\top(t) x(t) + \Delta k_r(t) r(t) - \Delta W^\top(t) \Phi(x(t)) + d(t, x(t)) \right) , \tag{193}$$

where $\Delta k_x(t) = k_x(t) - k_x^*$, $\Delta k_r(t) = k_r(t) - k_r^*$ denote parameter errors. Consider the following adaptation laws:

$$\begin{aligned} \dot{k}_x(t) &= \Gamma_x \text{Proj}\left(k_x(t), -x(t) e^\top(t) P b \text{sgn}(\lambda) \right) \\ \dot{k}_r(t) &= \gamma_r \text{Proj}\left(k_r(t), -r(t) e^\top(t) P b \text{sgn}(\lambda) \right) \\ \dot{\hat{W}}(t) &= \Gamma_W \text{Proj}\left(\hat{W}(t), \Phi(x(t)) e^\top(t) P b \text{sgn}(\lambda) \right) , \end{aligned} \tag{194}$$

where $\Gamma_x = \Gamma_x^\top > 0$, $\Gamma_W = \Gamma_W^\top > 0$, $\gamma_r > 0$ are the adaptation gains, and $\text{Proj}(\cdot, \cdot)$ defines the projection operator [12]. The latter ensures boundedness of all parameters, by definition. Define the following Lyapunov function candidate:

$$\begin{aligned} & V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta W) \\ = \; & e^\top(t) P e(t) + |\lambda| \left(\Delta k_x^\top(t) \Gamma_x^{-1} \Delta k_x(t) + \gamma_r^{-1} (\Delta k_r(t))^2 + \Delta W^\top(t) \Gamma_W^{-1} \Delta W(t) \right) , \end{aligned} \tag{195}$$

where $P = P^\top > 0$ solves the algebraic Lyapunov equation

$$A_m^\top P + P A_m = -Q \tag{196}$$

for arbitrary $Q > 0$. The time derivative of the Lyapunov function in (154) along the system trajectories (152), (153) is computed and evaluated using Property 12.2:

$$\begin{aligned} \dot{V}(t) &= -e^\top(t) Q e(t) + 2 e^\top(t) P b \lambda \left(\Delta k_x^\top(t) x(t) + \Delta k_r(t) r(t) - \Delta W^\top(t) \Phi(x(t)) + d(t, x(t)) \right) \\ &\quad + 2|\lambda| \Delta k_x^\top(t) \Gamma_x^{-1} \Delta \dot{k}_x(t) + 2|\lambda| \Delta k_r(t) \gamma_r^{-1} \Delta \dot{k}_r(t) + 2|\lambda| \Delta W^\top(t) \Gamma_W^{-1} \Delta \dot{W}(t) \\ &= -e^\top(t) Q e(t) + 2 e^\top(t) P b \lambda + d(t, x(t)) \\ &\quad + 2|\lambda| \underbrace{\Delta k_x^\top(t)}_{(k_x(t) - k_x^*)} \left(\underbrace{e^\top(t) P b x(t) \text{sgn}(\lambda)}_{-y} + \underbrace{\Gamma_x^{-1} \dot{k}_x(t)}_{\text{Proj}(k_x, y)} \right) \\ &\quad + 2|\lambda| \Delta k_r(t) \left(e^\top(t) P b r(t) \text{sgn}(\lambda) + \gamma_r^{-1} \dot{k}_r(t) \right) \\ &\quad + 2|\lambda| \Delta W^\top(t) \left(-e^\top(t) P b \Phi(x(t)) \text{sgn}(\lambda) + \Gamma_W^{-1} \dot{\hat{W}}(t) \right) \\ &= -e^\top(t) Q e(t) + 2 e^\top(t) P b \lambda + d(t, x(t)) \le -\lambda_{\min}(Q) \|e\|^2 + 2 \|e\| \|P b\| \; |\lambda| \; d^* \\ &\le -\|e\| \left[\lambda_{\min}(Q) \|e\| - 2 \|P b\| |\lambda| d^* \right] . \end{aligned} \tag{197}$$

Hence

$$\dot{V}(t) \le 0 \qquad \text{if} \qquad \|e\| \ge \frac{2|\lambda|\|Pb\| \ d^*}{\lambda_{\min}(Q)} \triangleq e_{\max} \,. \tag{198}$$

Thus, with account of bounds ensured by Projection operator, we have $\dot{V}(t) \le 0$, if

$$\|e\| \ge e_{\max} \cap \|\theta\| \ge \theta_{\max} \,,$$

where θ represents all the parametric errors, Fig. 25. Consider the smallest Lyapunov set Ω_ϵ around

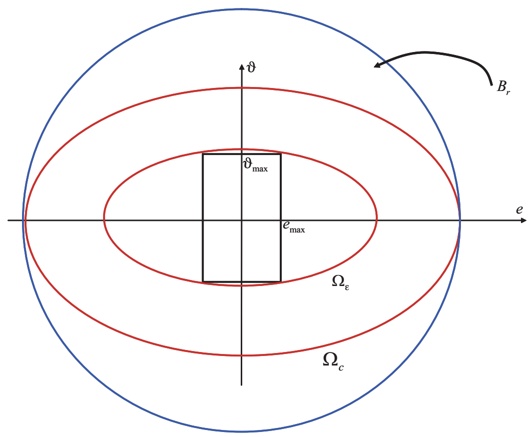


Fig. 25 Uniform Ultimate boundedness

this compact set and refer to Fig. 13 for the proof on ultimate boundedness. The only difference between Figures 13 and 25 is the rectangle in Fig. 25 instead of the ball $\mathcal{B}_\mu$ in Fig. 13. However, notice that in the consequent analysis it is the level set of the Lyapunov function Ω_ε that plays the key role. Thus, the globally uniformly bounded residual nonlinearity $d(t, x)$ led to global ultimate boundedness instead of asymptotic stability.

Remark 13.2. Notice that if $||\theta(0)|| > ||\theta_{\max}||$, then this corresponds to initializing the adaptation laws outside the outer convex set Ω_1 used in the definition of the Projection operator. In this case, since outside Ω_1 one has $f(\theta) > 1$, the projection operator will result in a "larger vector" in the result of the scaling, and thus upon subtraction in the same definition the direction of adaptation will change towards Ω_1.

Homework Problems 13.1. Alternative adaptive laws that ensure boundedness of the parameter errors are so called sigma-modification and e-modification [14]. The sigma-modification based adaptive laws are defined with an additional damping term as:

$$\begin{aligned}
\dot{k}_x(t) &= -\Gamma_x \left(x(t)e^\top(t)Pb\mathrm{sgn}(\lambda) + \sigma_x k_x(t) \right) \\
\dot{k}_r(t) &= -\gamma_r \left(r(t)e^\top(t)Pb\mathrm{sgn}(\lambda) + \sigma_r k_r(t) \right) \\
\dot{\hat{W}}(t) &= \Gamma_W \left(\Phi(x(t))e^\top(t)Pb\mathrm{sgn}(\lambda) + \sigma_W \hat{W}(t) \right) .
\end{aligned} \tag{199}$$

The e-modification based adaptive laws are defined as:

$$\begin{aligned}
\dot{k}_x(t) &= -\Gamma_x \left(x(t)e^\top(t)Pb\mathrm{sgn}(\lambda) + \sigma_x \|e^\top(t)Pb\| k_x(t) \right) \\
\dot{k}_r(t) &= -\gamma_r \left(r(t)e^\top(t)Pb\mathrm{sgn}(\lambda) + \sigma_r \|e^\top(t)Pb\| k_r(t) \right) \\
\dot{\hat{W}}(t) &= \Gamma_W \left(\Phi(x(t))e^\top(t)Pb\mathrm{sgn}(\lambda) + \sigma_W \|e^\top(t)Pb\| \hat{W}(t) \right) .
\end{aligned} \tag{200}$$

Please prove local boundedness using these adaptive laws and simulate your same system from the previous homework with all three adaptive laws (projection, e-modification, σ-modification).

When substituting these adaptive laws into the derivative of Lyapunov function, you'll need to do completion of squares in $\dot{V}(t)$ expression to get $\dot{V}(t) \le 0$ outside a compact set directly. You need to group terms and to do algebraic modifications, like $2ab \le a^2 + b^2$, and different variations of this with different scaling. Your final expression for $\dot{V}(t)$ will look like $\dot{V}(t) \le -(e-a)^2 - (\theta - b)^2 + c$, defining an ellipse in the error space, which does not need to be necessarily a level set of the original Lyapunov function.

14 Disturbance Rejection

Consider nonlinear system of the following type

$$\dot{x}(t) = Ax(t) + b\lambda(u(t) + W^\top \Phi(x(t)) + d(t)) , \quad x(0) = x_0 , \tag{201}$$

where the new term for you is the time-varying disturbance $d(t)$. The disturbances in the system that won't violate stability are usually classified into three types:

1. constant all the time like $d(t) = d_0 = const$, for which you already know how to use integral control;
2. vanishing in time so that $d(t) \to 0$ as $t \to \infty$;
3. uniformly bounded like $|d(t)| \leq d_0$ for all $t \geq 0$.

Any unbounded disturbance will obviously lead to instability, since there is no opportunity to reproduce infinite control effort over time to overcome it. Let's first convince ourselves that vanishing disturbances do not violate stability. Then we will introduce extra disturbance rejection type of controller to reject uniformly bounded disturbances.

First we need to recall some definitions from real analysis that can be found as well in [11], pp.195-196. For simplicity, here we tailor those to a scalar function.

Definition 14.1. The function $d(t)$ is said to belong to the space $\mathcal{L}_\infty$, if it is uniformly bounded

$$\sup_{t\geq 0} |d(t)| \leq d_0 < \infty , \quad d_0 > 0$$

Definition 14.2. The function $d(t)$ is said to belong to the space $\mathcal{L}_p$, $p \geq 1$, if the integral is bounded

$$\left(\int_0^\infty |d(t)|^p dt \right)^{1/p} < \infty .$$

For example, the function $\sin t \in \mathcal{L}_\infty$ on the interval $[0, \infty)$, but $\sin t \bar{\in} \mathcal{L}_2$ on the same interval, since $\int_0^\infty \sin^2 t \; dt$ does not exist, while the function $\dfrac{1}{t+1}$ is both in $\mathcal{L}_\infty$ and $\mathcal{L}_2$ on the interval $[0, \infty)$ and can be denoted like $\dfrac{1}{t+1} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

14.1 Vanishing disturbances

To convince ourselves that vanishing disturbances do not violate stability, we need to recall a corollary of Barbalat's lemma from [3](p.19). Let's rewrite Barbalat's lemma (Lemma 4.4) so that the corollary will be straightforward.

Lemma 14.1. If for a uniformly continuous function $f(t)$ the following limit

$$\lim_{t\to\infty}\int_0^t f(\tau)d\tau$$

exists and is finite, then $f(t) \to 0$ as $t \to \infty$.

Corollary 14.1. If $g \in \mathcal{L}_p$ for some $p \geq 1$, and $g, \dot{g} \in \mathcal{L}_\infty$, then $g(t) \to 0$ as $t \to \infty$.

The proof is straightforward and follows from the lemma, if one considers $f(t) = |g(t)|^p$. Boundedness of $g(t)$ and $\dot{g}(t)$ imply that $f(t)$ is uniformly continuous, while $g \in \mathcal{L}_p$ implies that the integral $\lim\limits_{t\to\infty}\int_0^t f(\tau)d\tau$ exists and is finite.

So, we first consider the system in (201), when $d \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{d} \in \mathcal{L}_\infty$. Hence

$$d(t) \to 0, \quad t \to \infty\,. \tag{202}$$

Let the reference model of interest for tracking be given:

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t)\,, \quad x_m(0) = x_{m0}\,, \tag{203}$$

where $x_m \in \mathbb{R}^n$ is the state of the reference model, A_m is a Hurwitz $(n \times n)$-dimensional matrix, $b_m \in \mathbb{R}^n$ is a constant vector, $r(t) \in \mathbb{R}$ is a uniformly bounded continuous input. Direct adaptive model reference feedback as earlier can be defined:

$$u(t) = k_x^\top(t)x(t) + k_r(t)r(t) - \hat{W}^\top(t)\Phi(x(t)) \tag{204}$$

where $k_x(t) \in \mathbb{R}^n,\ k_r(t) \in \mathbb{R}$ are the adaptive gains defined through the stability proof, $\hat{W}(t) \in \mathbb{R}^m$ is the estimate of W. Substituting (204) into (201), yields the following closed-loop system dynamics:

$$\dot{x}(t) \;=\; (A + b\lambda k_x^\top(t))x(t) + b\lambda k_r(t)r(t) - b\lambda\Delta W^\top(t)\Phi(x(t)) + b\lambda d(t)\,, \tag{205}$$

where $\Delta W(t) = \hat{W}(t) - W$ is the parameter estimation error. Subject to the same matching assumptions in Assumption 10.1, the following error dynamics can be written for the signal $e(t) = x(t) - x_m(t)$:

$$\dot{e}(t) = A_m e(t) + b\lambda \left(\Delta k_x^\top(t)x(t) + \Delta k_r(t)r(t) - \Delta W^\top(t)\Phi(x(t)) + d(t)\right), \quad e(0) = e_0\,, \tag{206}$$

where $\Delta k_x(t) = k_x(t) - k_x^*$, $\Delta k_r(t) = k_r(t) - k_r^*$ denote parameter errors as before. From Parameter Drift section we know that $d(t)$ can cause parameter drift, therefore we write the adaptation laws using Projection operator to ensure their boundedness by definition:

$$\begin{aligned}
\dot{k}_x(t) &= \Gamma_x \text{Proj}\left(k_x(t), -x(t)e^\top(t)Pb\,\text{sgn}(\lambda)\right)\\
\dot{k}_r(t) &= \gamma_r \text{Proj}\left(k_r(t), -r(t)e^\top(t)Pb\,\text{sgn}(\lambda)\right)\\
\dot{\hat{W}}(t) &= \Gamma_W \text{Proj}\left(\hat{W}(t), \Phi(x(t))e^\top(t)Pb\,\text{sgn}(\lambda)\right),
\end{aligned} \tag{207}$$

where $\Gamma_x = \Gamma_x^\top > 0$, $\Gamma_W = \Gamma_W^\top > 0$, $\gamma_r > 0$ are the adaptation gains. Define the following Lyapunov function candidate:

$$\begin{aligned}
&V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta W) \\
= \; & e^\top(t)Pe(t) + |\lambda|\left(\Delta k_x^\top(t)\Gamma_x^{-1}\Delta k_x(t) + \gamma_r^{-1}(\Delta k_r(t))^2 + \Delta W^\top(t)\Gamma_W^{-1}\Delta W(t)\right),
\end{aligned} \tag{208}$$

where $P = P^\top > 0$ solves the algebraic Lyapunov equation

$$A_m^\top P + PA_m = -Q \tag{209}$$

for arbitrary $Q > 0$. The time derivative of the Lyapunov function in (208) along the system trajectories (206), (207) is computed and evaluated using Property 12.2:

$$\begin{aligned}
\dot{V}(t) &= -e^\top(t)Qe(t) + 2e^\top(t)Pb\lambda\left(\Delta k_x^\top(t)x(t) + \Delta k_r(t)r(t) - \Delta W^\top(t)\Phi(x(t)) + d(t)\right)\\
&\quad +2|\lambda|\Delta k_x^\top(t)\Gamma_x^{-1}\Delta\dot{k}_x(t) + 2|\lambda|\Delta k_r(t)\gamma_r^{-1}\Delta\dot{k}_r(t) + 2|\lambda|\Delta W^\top(t)\Gamma_W^{-1}\Delta\dot{W}(t)\\
&= -e^\top(t)Qe(t) + 2e^\top(t)Pb\lambda d(t)\\
&\quad +2|\lambda| \underbrace{\Delta k_x^\top(t)}_{(k_x(t)-k_x^*)}\left(\underbrace{e^\top(t)Pbx(t)\text{sgn}(\lambda)}_{-y} + \underbrace{\Gamma_x^{-1}\dot{k}_x(t)}_{\text{Proj}(k_x,y)}\right)\\
&\quad +2|\lambda|\Delta k_r(t)\left(e^\top(t)Pbr(t)\text{sgn}(\lambda) + \gamma_r^{-1}\dot{k}_r(t)\right)\\
&\quad +2|\lambda|\Delta W^\top(t)\left(-e^\top(t)Pb\Phi(x(t))\text{sgn}(\lambda) + \Gamma_W^{-1}\dot{\hat{W}}(t)\right)\\
&= -e^\top(t)Qe(t) + 2e^\top(t)Pb\lambda d(t) \le -\lambda_{\min}(Q)\|e\|^2 + 2\|e\|\|Pb\|\;|\lambda|\;|d(t)|\\
&\le -\|e\|\left[\lambda_{\min}(Q)\|e(t)\| - 2\|Pb\||\lambda|d_0\right]
\end{aligned} \tag{210}$$

Hence

$$\dot{V}(t) \leq 0 \qquad \text{if} \qquad \|e\| \geq \frac{2|\lambda|\|Pb\|\ d_0}{\lambda_{\min}(Q)}\,. \tag{211}$$

Boundedness of adaptive parameters is ensured by the $\mathrm{Proj}(\cdot,\cdot)$ operator. Therefore $\dot{V}(t) \leq 0$ outside a compact set in the entire error space of tracking error and parameter errors. Therefore, the tracking error and parameter errors are *globally ultimately bounded*.

Now let us ensure that in this case we can achieve more than the ultimate boundedness of the tracking error. We have seen above that the derivative of the Lyapunov function satisfies the inequality

$$\dot{V}(t) \leq -\lambda_{\min}(Q)\|e(t)\|^2 + 2\|Pb\lambda\|\|e(t)\||d(t)| \tag{212}$$

Completing the squares ($2ab \leq a^2 + b^2$) in (212) yields

$$\dot{V}(t) = -\lambda_{\min}(Q)\|e(t)\|^2 + 2\frac{\|e(t)\|}{\sqrt{c_1}}\sqrt{c_1}\|Pb\lambda\||d(t)| \leq -(\lambda_{\min}(Q) - c_1)\|e(t)\|^2 + c_2|d(t)|^2\,, \tag{213}$$

where c_1 is any positive number and $c_2 = \frac{\|Pb\lambda\|^2}{c_1}$. The choice of the parameter c_1 and matrix Q are such that $\lambda_{\min}(Q) - c_1 > 0$. Rearranging the inequality in (213) and integrating on the interval $[0,\infty)$ results in

$$(\lambda_{\min}(Q) - c_1)\int_0^\infty \|e(t)\|^2 dt \leq V(0) - V(t) + c_2\int_0^\infty |d(t)|^2 dt\,, \tag{214}$$

Since $V(t)$ is positive definite we can ignore the negative term on the right and write

$$(\lambda_{\min}(Q) - c_1)\int_0^\infty \|e(t)\|^2 dt \leq V(0) + c_2\int_0^\infty |d(t)|^2 dt\,, \tag{215}$$

Since $d(t) \in \mathcal{L}_2$ it follows that $\int_0^\infty |d(t)|^2 dt < \infty$, therefore

$$\int_0^\infty \|e(t)\|^2 dt < \infty\,, \tag{216}$$

implying $e(t) \in \mathcal{L}_2$. Since all the signals in the system (206), (207) are bounded, it follows that $\dot{e}(t)$ is also bounded, that is $e(t), \dot{e}(t) \in \mathcal{L}_\infty$. Application of Corollary 14.1 implies that $e(t) \to 0, \quad t \to \infty$, and thus, one more time, Barbalat's lemma implies asymptotic stability.

Remark 14.1. Notice that despite the vanishing nature of disturbances we wouldn't have been able to conclude asymptotic convergence of tracking error to zero without the use of Projection operator.

If we had just used the simple structure of adaptive laws from (153), we would have had no guarantee of boundedness of parameter errors, which would have in its turn prevented the conclusion on uniform continuity of $e(t)$, which is crucial in application of Barbalat's lemma. The uniform continuity of $e(t)$ always follows from boundedness of parameter errors in (206), which we guaranteed by projection. Similarly, we could have used other adaptation laws, like e-modification, σ-modification to ensure boundedness of parameter errors.

Remark 14.2. Notice that in the context of our proof all that we needed from $d(t)$ was $d \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ property, while $\dot{d} \in \mathcal{L}_\infty$ was not utilized, i.e. uniform continuity of $d(t)$ was not required to prove stability, since all we needed for application of Barbalat's lemma's was uniform continuity of $e(t)$, which we were able to conclude from $d \in L_\infty$, because $\dot{e}(t)$ dynamics involved only $d(t)$, and never $\dot{d}(t)$. Thus, we have defined a disturbance rejection scheme for all disturbances from the class of $d \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. The $\dot{d} \in L_\infty$ feature helped us to conclude that $d(t) \to 0$ as $t \to \infty$, i.e. that the disturbances vanish. Without imposing $\dot{d} \in L_\infty$, it is not clear what physical interpretation should be given to disturbances of the class $d \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. This requires a little bit more than just boundedness, namely the integral of the square over infinite time horizon be also bounded.

Homework Problems 14.1. Take your same system from your the earlier homework, and insert vanishing disturbances into it. Simulate the same adaptive controller to convince yourself that it performs as expected. Check different adaptation laws, like e-modification, σ-modification, etc.

14.2 Non-vanishing disturbances and adaptive bounding

Now assume that the disturbance in the system (201) is bounded, i.e. $d \in \mathcal{L}_\infty$. An example of this will be $d(t) = \sin t$. For disturbances like this, the conventional theory of adaptive control has no ready recipes to offer and borrows tools from robust control, using bang-bang type signals. First notice that in the absence of any modifications to the control signal in (204) one can still prove global ultimate boundedness of error signals, by using the bound d_0, which will show up itself during upper bounding of the derivative of the Lyapunov function candidate (recall the proof on ultimate boundedness with the use of RBFs, where ε^* appeared in the ultimate bound!). This will be the most conservative approach.

A less conservative approach would be to adapt to the unknown bound d_0, which enables to obtain an *adjustable ultimate bound* that can be reduced to arbitrarily small magnitude via proper selection of design parameters. This technique is known as *adaptive bounding*.

Just to give you an idea, recently we have developed a novel adaptive control architecture that copes with bounded disturbances via a truly adaptive scheme without involving any robustifying signals, and you're welcome to request copies of [5, 6] for deeper understanding of the power of adaptive control. Within the framework of conventional theory, the adaptive controller from (204) is augmented via the *adaptive bounding* term as follows:

$$u(t) = k_x^\top(t)x(t) + k_r(t)r(t) - \hat{W}^\top(t)\Phi(x(t)) - \psi(t)\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right), \tag{217}$$

where $\psi(t)$ is another adaptive gain, δ is a design parameter, the role of which will be shortly clarified. Substituting this back into the system dynamics in (201), and comparing to (203), the error dynamics take the form:

$$\dot{e}(t) = A_m e(t) + b\lambda\left(\Delta k_x^\top(t)x(t) + \Delta k_r(t)r(t) - \Delta W^\top(t)\Phi(x) + d(t) - \psi(t)\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)\right) \tag{218}$$

We already know how to derive adaptive laws for $k_x(t), k_r(t), W(t)$ to clean the first portion of the error dynamics. We can use the same projection type adaptive laws as in (207). Now, we need to derive an adaptive law for $\psi(t)$ so that $d(t) - \psi(t)\tanh\left(\frac{e^\top(t)Pb\mathrm{sgn}(\lambda)}{\delta}\right)$ will be nicely upper bounded in the Lyapunov function derivative.

To this end consider the following Lyapunov function candidate:

$$\begin{aligned} & V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta W) \\ = \quad & e^\top(t)Pe(t) + |\lambda|\left(\Delta k_x^\top(t)\Gamma_x^{-1}\Delta k_x(t) + \gamma_r^{-1}(\Delta k_r(t))^2 + \Delta W^\top(t)\Gamma_W^{-1}\Delta W(t) + \gamma_\psi^{-1}(\Delta\psi(t))^2\right), \end{aligned} \tag{219}$$

where $\Delta\psi(t) = \psi(t) - d_0$ and $P = P^\top > 0$ solves the algebraic Lyapunov equation

$$A_m^\top P + PA_m = -Q \tag{220}$$

for arbitrary $Q > 0$. The time derivative of the Lyapunov function in (219) along the system trajectories

(218), (207) is computed as follows:

$$
\begin{aligned}
\dot{V}(t) &= -e^\top(t)Qe(t) \\
&\quad +2e^\top(t)Pb\lambda\left(\Delta k_x^\top(t)x(t)+\Delta k_r(t)r(t)-\Delta W^\top(t)\Phi(x(t))+d(t)-\psi(t)\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)\right) \\
&\quad +2|\lambda|\left(\Delta k_x^\top(t)\Gamma_x^{-1}\Delta\dot{k}_x(t)+\Delta k_r(t)\gamma_r^{-1}\Delta\dot{k}_r(t)+\Delta W^\top(t)\Gamma_W^{-1}\Delta\dot{W}(t)+\gamma_\psi^{-1}\Delta\psi(t)\Delta\dot{\psi}(t)\right) \\
&= -e^\top(t)Qe(t)+2e^\top(t)Pb\lambda\left(d(t)-\psi(t)\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)\right)+2|\lambda|\gamma_\psi^{-1}\Delta\psi(t)\Delta\dot{\psi}(t) \\
&\quad +2|\lambda|\underbrace{\Delta k_x^\top(t)}_{(k_x(t)-k_x^*)}\left(\underbrace{e^\top(t)Pbx(t)\mathrm{sgn}(\lambda)}_{-y}+\underbrace{\Gamma_x^{-1}\dot{k}_x(t)}_{\mathrm{Proj}(k_x,y)}\right) \\
&\quad +2|\lambda|\Delta k_r(t)\left(e^\top(t)Pbr(t)\mathrm{sgn}(\lambda)+\gamma_r^{-1}\dot{k}_r(t)\right) \\
&\quad +2|\lambda|\Delta W^\top(t)\left(-e^\top(t)Pb\Phi(x(t))\mathrm{sgn}(\lambda)+\Gamma_W^{-1}\dot{\hat{W}}(t)\right)
\end{aligned}
$$

Substituting the adaptive laws from (207) and using Property 12.2 we obtain:

$$
\begin{aligned}
\dot{V}(t) &\le -e^\top(t)Qe(t)+2e^\top(t)Pb\lambda\left(d(t)-\psi(t)\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)\right)+2|\lambda|\gamma_\psi^{-1}\Delta\psi(t)\Delta\dot{\psi}(t) \\
&\le -e^\top(t)Qe(t)+2|\lambda|\left(|e^\top(t)Pb|d_0-\mathrm{sgn}(\lambda)e^\top(t)Pb\;\psi(t)\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)+\gamma_\psi^{-1}\Delta\psi(t)\Delta\dot{\psi}(t)\right) \\
&= -e^\top(t)Qe(t)+2|\lambda|\Big[d_0|e^\top(t)Pb|-d_0\mathrm{sgn}(\lambda)e^\top(t)Pb\;\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right) \\
&+ d_0\mathrm{sgn}(\lambda)e^\top(t)Pb\;\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)-\mathrm{sgn}(\lambda)\psi(t)e^\top(t)Pb\;\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right) \\
&+ \gamma_\psi^{-1}\Delta\psi(t)\Delta\dot{\psi}(t)\Big]=-e^\top(t)Qe(t)+2|\lambda|\Big[d_0\left(|\mathrm{sgn}(\lambda)e^\top(t)Pb|-\mathrm{sgn}(\lambda)e^\top(t)Pb\;\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)\right) \\
&- \Delta\psi(t)\mathrm{sgn}(\lambda)e^\top(t)Pb\;\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)+\gamma_\psi^{-1}\Delta\psi(t)\Delta\dot{\psi}(t)\Big]
\end{aligned}
$$

One can easily verify that for any $\delta > 0$ the following inequality holds

$$0 \le |\eta| - \eta\tanh\left(\frac{\eta}{\delta}\right) \le \kappa\delta \tag{221}$$

where $\kappa = 0.2785$ is the solution of the equation $\kappa = \mathrm{e}^{-\kappa-1}$ (Just expand the $\tanh(x)$ into its equivalent exponential representation to convince yourself!). Then, defining the adaptive law for the estimate $\psi(t)$ via projection operator to eliminate the last terms in the upper bound of the derivative of the candidate

Lyapunov function

$$\dot{\psi}(t) = \gamma_\psi \mathrm{Proj}\left(\psi(t), e^\top(t)Pb\ \tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)\mathrm{sgn}(\lambda)\right) \tag{222}$$

and using the inequality in (221) for the second term in the derivative of the candidate Lyapunov function, the following upper bound can be derived:

$$\dot{V}(t) \quad \leq \quad -e^\top(t)Qe(t) + 2|\lambda|\kappa\delta d_0 \leq -\lambda_{\min}(Q)\|e\|^2 + 2|\lambda|\kappa\delta d_0 \tag{223}$$

Hence

$$\dot{V}(t) \leq 0 \qquad \text{if} \qquad \|e\| \geq \sqrt{\frac{2|\lambda|\kappa\delta\ d_0}{\lambda_{\min}(Q)}}\,. \tag{224}$$

Boundedness of adaptive parameters is ensured by the $\mathrm{Proj}(\cdot,\cdot)$ operator. Therefore $\dot{V}(t) \leq 0$ outside a compact set in the entire error space of tracking error and parameter errors. Therefore, the tracking error and parameter errors are *globally ultimately bounded*.

Remark 14.3. We notice that without the use of adaptive bounding we would have obtained

$$\dot{V}(t) \leq 0 \qquad \text{if} \qquad \|e\| \geq \sqrt{\frac{2|\lambda|d_0}{\lambda_{\min}(Q)}}\,, \tag{225}$$

which would have been comparable to d_0. The presence of δ in (224) gives an opportunity to regulate this lower bound to smaller values. Although, one should keep in mind that the ultimate bound for the tracking performance is defined via the value of the Lyapunov function on the minimum level set, embracing the rectangle formed by this bound and the bound of parameter errors in the result of the projection operator. The shape of the ellipses of the Lyapunov function depend on the choice of the adaptation gain and the choice of the matrix Q used in algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$.

Remark 14.4. As $\delta \to 0$, the adaptive bounding term approximates a $\mathrm{sgn}(\cdot)$ function:

$$\lim_{\delta\to 0}\tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right) = \mathrm{sgn}\left(e^\top(t)Pb\mathrm{sgn}(\lambda)\right) = \mathrm{sgn}\left(e^\top(t)Pb\right)\mathrm{sgn}(\lambda)\,, \tag{226}$$

and hence

$$\begin{aligned}
&\lim_{\delta\to 0}\left[|\mathrm{sgn}(\lambda)e^\top(t)Pb| - e^\top(t)Pb\ \tanh\left(\frac{e^\top(t)Pb}{\delta}\mathrm{sgn}(\lambda)\right)\mathrm{sgn}(\lambda)\right] \\
&= |e^\top(t)Pb| - e^\top(t)Pb\ \mathrm{sgn}\left(e^\top(t)Pb\right) = 0\,.
\end{aligned} \tag{227}$$

Therefore the derivative of the Lyapunov function candidate can be upper bounded as

$$\dot{V}(t) \quad \leq \quad -e^\top(t)Qe(t) \leq 0\,. \tag{228}$$

Recall that our sufficient conditions on stability and asymptotic stability always assumed existence of a continuously differentiable Lyapunov function (see e.g. Theorems 2.1, 4.1)). With (226), the right hand side of our error dynamics becomes discontinuous, and hence the candidate Lyapunov function is non-smooth. In this case the solutions of corresponding differential equations need to be understood in Filippov's sense, and stability needs to be investigated using generalized gradients. I recommend you to look into the work of Shevitz and Paden [20] to have an idea as stability analysis needs to be done for discontinuous systems. For adaptive systems, existence and uniqueness of solutions for systems with discontinuous right hand sides is proven in [21]. So, in the limit, with (226), one can apply Barbalat's lemma and prove asymptotic convergence of tracking error to zero instead of just boundedness. Notice, that in this case the adaptive law in (222) reduces to

$$\dot{\psi}(t) = \gamma_\psi \mathrm{Proj}\left(\psi(t), |e^\top(t)Pb|\right)\,. \tag{229}$$

Finally, with (226), the adaptive bounding term is just multiplication of the solution $\psi(t)$ of (229) by bang-bang type signal, well-known from optimal control.

Homework Problems 14.2. Take your same system from your earlier homework, and insert non-vanishing bounded disturbances into it. Simulate adaptive bounding type controller to convince yourself that it performs as expected. It would be helpful if you check different adaptation laws, like e-modification, σ-modification, etc.

15 Input-to-State Stability

Consider the system

$$\dot{x}(t) = f(t, x(t), u(t))\,, \tag{230}$$

where f is piece-wise continuous in t and locally Lipschitz in x and u, and let $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. The input $u(t)$ can be any piece-wise continuous bounded function of t for all $t \geq 0$. Suppose the unforced system

$$\dot{x} = f(t, x, 0) \tag{231}$$

has a globally uniformly asymptotically stable equilibrium point at the origin $x = 0$. The question is what can be say about the behavior of the system in (230) in the presence of a <u>*bounded*</u> input $u(t)$.

To get the feeling of how hard is the answer to this question for nonlinear systems, let's look at linear time-invariant systems first. Let A be a Hurwitz matrix and consider the system

$$\dot{x} = Ax + Bu\,.$$

The solution to this system is given by:

$$x(t) = \mathrm{e}^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau\,,$$

which can be upper bounded as follows:

$$||x(t)|| \leq k\mathrm{e}^{-\lambda(t-t_0)}||x(t_0)|| + \int_{t_0}^{t} k\mathrm{e}^{-\lambda(t-\tau)}||B||||u(\tau)||d\tau \leq k\mathrm{e}^{-\lambda(t-t_0)}||x(t_0)|| + \frac{k||B||}{\lambda}\sup_{t_0\leq\tau\leq t}||u(\tau)||\,.$$

This shows that if the zero input response decays exponentially fast (which is due to A being Hurwitz), then the zero state response is bounded for every bounded input, and the bound is proportional to the bound on the input.

The question that we are trying to answer is the following: **How much of this behavior should we expect for the nonlinear system in (230) if its unforced system (231) has a GUAS equilibrium at the origin?**

The answer is unfortunately negative as the following counter example demonstrates: the system

$$\dot{x} = -3x + (1 + 2x^2)u$$

has a GAS equilibrium at the origin when $u = 0$, but with $u = 1$ from the initial condition $x(0) = 2$ it has a solution $x(t) = (3 - \mathrm{e}^t)/(2 - 2\mathrm{e}^t)$ demonstrating finite escape time.

Thus, we constructed a nonlinear system, such that in the absence of control input its origin is GES, but in the presence of a bounded input like $u = 1$ there is an initial condition $x(0) = 2$ such that the unique trajectory of the system starting from that point escapes to infinity in finite time. The first question that would be natural to ask would be: and what about "smaller" initial conditions? Intuitively, from the experience with linear systems theory one would assume that under certain set of assumptions the negative definite derivative of the Lyapunov function for the system (231) should remain negative definite outside a compact set around the origin if the system in (230) is driven by a bounded input $u(t)$. This leads to the definition of **input-to-state stability**, known as ISS, introduced by Eduardo Sontag, one of the greatest mathematicians of our century, in [27]. I highly recommend you to download the paper [27] and read it. Please visit the website

http://www.math.rutgers.edu/∼sontag/

and download the presentation

http://www.math.rutgers.edu/∼sontag/bode-with-narrative.pdf

which was delivered by E. Sontag for his Bode lecture award (one of the most prestigious awards of IEEE CSS). You will learn lot from this lecture!

The notion of ISS merges **Lyapunov** stability theory, developed for dynamical systems $\dot{x} = f(x)$, with input-output stability theory, developed by **G. Zames** and others starting mid 1960s, for input/output operators $y(\cdot) = F(u(\cdot))$. The main result of G. Zames can be found in [28].

Definition 15.1. The system in (230) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $u(t)$ the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) + \gamma\Big(\sup_{t_0 \leq \tau \leq t} ||u(\tau)||\Big)\,. \tag{232}$$

Instead of getting scared right now, let's look into each of these terms separately and understand what does this definition imply. If $u \equiv 0$, then the relationship in (232) reduces to

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \tag{233}$$

which basically states that the equilibrium of the unforced system in (231) is globally uniformly asymptotically stable (Recall Definition 4.2 to see this). It further guarantees that for any bounded input $u(t)$

the state of the system will always be bounded by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} ||u(t)||$. From the properties of class $\mathcal{KL}$ functions it follows that as $t \to \infty$ one has $\beta \to 0$, so that the bound on the system state depends **only** on the bound of the control input. The presence of $||x(t_0)||$ in $\beta(||x(t_0)||, t - t_0)$ allows for the transient overshoot, Fig.26.

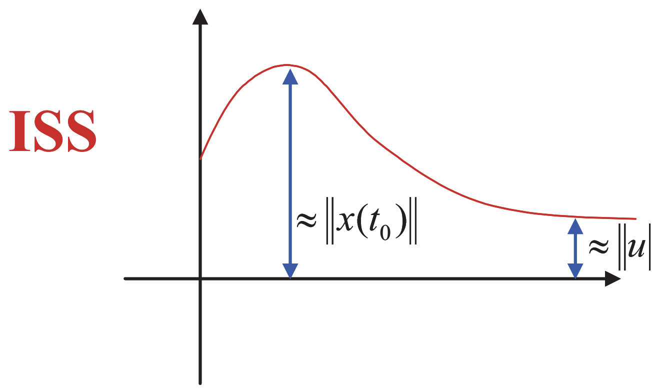


Fig. 26 State trajectory of a ISS system

Let's look back into our linear system to convince ourselves that this definition is simply a nonlinear generalization of what we derived in the linear case:

$$||x(t)|| \leq \underbrace{k \mathrm{e}^{-\lambda(t-t_0)} ||x(t_0)||}_{\beta(||x(t_0)||, t-t_0)} + \underbrace{\frac{k||B||}{\lambda} \sup_{t_0 \leq \tau \leq t} ||u(\tau)||}_{\gamma\left(\sup_{t_0 \leq \tau \leq t} ||u(\tau)||\right)} .$$

A sufficient condition for verification of the ISS property of the system is given via the following theorem:

Theorem 15.1. Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ be a continuously differentiable, positive definite function such that

$$\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||) \tag{234}$$

$$\dot{V}(t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W(x), \quad \forall \; ||x|| \geq \rho(||u||) > 0 \tag{235}$$

for all $(t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$, where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, $W(x)$ is a continuous positive definite function on $\mathbb{R}^n$. Then the system (230) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Proof: Application of Theorem 5.1 implies that $x(t)$ exists and satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) + \gamma\Big(\sup_{\tau \geq t_0} ||u(\tau)||\Big), \quad \forall\ t \geq t_0$$

without any restriction on $x(t_0)$. Since $x(t)$ depends only on $u(\tau)$ for $t_0 \leq \tau \leq t$, the supremum on the right-hand side can be taken over $[t_0, t]$, which immediately yields (232). □

The converse Lyapunov theorem on exponential stability leads to a more conservative result.

Lemma 15.1. If $f(t, x, u)$ is continuously differentiable and globally Lipschitz in (x, u), uniformly in t, and the unforced system in (231) has GES at the origin $x = 0$, then the system in (230) is ISS.

Proof: From converse Lyapunov Theorem 6.1 it follows that there exists a Lyapunov function $V(t, x)$ for the unforced system in (231) that satisfies (39), (40) globally. Due to the uniform global Lipschitz property of f, we have

$$||f(t, x, u) - f(t, x, 0)|| \leq L||u||, \quad \forall\ t \geq t_0, \ \ \forall\ (x, u)\,.$$

Computing the derivative of $V(t, x)$, we can upperbound:

$$\dot{V}(t, x(t)) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, 0) + \frac{\partial V}{\partial x}(f(t, x, u) - f(t, x, 0)) \leq -c_3||x||^2 + c_4||x||L||u||\,.$$

Further, we can do algebraic manipulations to obtain

$$\dot{V}(t, x(t)) \leq -c_3(1 - \theta)||x||^2 - c_3\theta||x||^2 + c_4||x||L||u||\,,$$

where $0 < \theta < 1$. Then,

$$\dot{V}(t, x(t)) \leq -c_3(1 - \theta)||x||^2, \quad \forall\ ||x|| \geq \frac{c_4 L||u||}{c_3\theta}\,,$$

for all (t, x, u). Hence, the conditions of Theorem 15.1 are satisfied with $\alpha_1(r) = c_1 r^2$, $\alpha_2(r) = c_2 r^2$, and $\rho(r) = \frac{c_4 L}{c_3\theta} r$, and therefore the system is ISS with $\gamma(r) = \sqrt{\frac{c_2}{c_1}}\, \frac{c_4 L}{c_3\theta} r$. □

Lemma 15.1 requires two strict conditions to ensure global ISS property for the system: globally Lipschitz $f(t, x, u)$ and GES of the origin of the unforced system. Next, we demonstrate the significance of these two conditions, by constructing two simple examples of non-ISS systems, when one of these two conditions does not hold.

Example 15.1.

1. First let's look back into our original system

$$\dot{x} = -3x + (1 + x^2)u\,.$$

   It is definitely not globally Lipschitz. We already know that it has finite escape time, so it cannot be ISS.

2. The system

$$\dot{x} = -\frac{x}{1+x^2} + u \triangleq f(x,u)$$

   on the opposite has globally Lipschitz f both in x and u, since both partials are globally bounded. The origin of the unforced system

$$\dot{x} = -\frac{x}{1+x^2} = f(x,0)$$

   is GAS, and this can be proved via the Lyapunov function $V(x) = x^2/2$, the derivative of which $\dot{V} = -x^2/(1+x^2)$ is negative definite for all $x \in \mathbb{R}^n$ (i.e. globally). Further, it is locally exponentially stable (LES), since the linearization of it is $\dot{x} = -x$. However, it is not GES. And, indeed, if $u(t) \equiv 1$, then $f(x,u) \geq 1/2$. Hence, $x(t) \geq x(t_0) + t/2$ for all $t \geq 0$, which shows that the solution is unbounded, and therefore cannot be ISS.

However, in the absence of GES or globally Lipschitz f, one may still be able to assert ISS by using Theorem 15.1. Next, we construct several examples to demonstrate this.

Example 15.2.

1. The system $\dot{x} = -x^3 + u$ has a GAS at the origin, when $u = 0$. (Recall that $\dot{x} = -x^3$ is not GES!!!) Taking $V = x^2/2$, we have

$$\dot{V} = -x^4 + xu = -(1-\theta)x^4 - \theta x^4 + xu \leq -(1 - theta)x^4, \quad |x| \geq \left(\frac{|u|}{\theta}\right)^{1/3}, \quad 0 < \theta < 1.$$

   Therefore the system is ISS with $\gamma(r) = (r/\theta)^{1/2}$.

2. The system $\dot{x} = f(x,u) = -x - 2x^3 + (1+x^2)u^2$ has a GES at the origin if $u = 0$, but f is not globally Lipschitz. Taking $V = x^2/2$, we obtain

$$\dot{V} = -x^2 - 2x^4 + x(1+x^2)u^2 \le -x^4, \quad \forall\, |x| > u^2\,,$$

and therefore the system is ISS with $\gamma(r) = r^2$.

An interesting application of the concept of ISS arises in stability analysis of cascaded systems:

$$\dot{x}_1 = f_1(t, x_1, x_2), \quad x_1(t_0) = x_{10} \tag{236a}$$

$$\dot{x}_2 = f_2(t, x_2), \quad x_2(t_0) = x_{20}\,, \tag{236b}$$

where $f_1 : [0,\infty) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ and $f_2 : [0,\infty) \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ are piecewise continuous in t and locally Lipschitz in $x = [x_1\ x_2]^\top$. Assume that the disconnected systems (i.e. when the first one is unforced)

$$\dot{x}_1 = f_1(t, x_1, 0) \tag{237a}$$

$$\dot{x}_2 = f_2(t, x_2) \tag{237b}$$

have GUAS equilibria at their respective origins. The next lemma shows that if (236a) with x_2 viewed as input is ISS, then the combined cascaded system in (236) will have GUAS at the origin.

Lemma 15.2. If the system (236a) with x_2 viewed as input is ISS, and the origin of (236b) is GUAS, then the origin of the combined cascaded system in (236) is GUAS.

Proof: Since the system (236a) with x_2 viewed as input is ISS, and the origin of (236b) is GUAS, then

$$||x_1(t)|| \le \beta_1(||x_1(s)||, t-s) + \gamma_1\Big(\sup_{s\le\tau\le t} ||x_2(\tau)||\Big)$$

and

$$||x_2(t)|| \le \beta_2(||x_2(s)||, t-s)\,,$$

where $t_0 \le s$ *let*, β_1, β_2 are class $\mathcal{KL}$ class functions, and γ_1 is a class $\mathcal{K}$ function. Letting $s = (t+t_0)/2$, we get

$$||x_1(t)|| \le \beta_1\left(\left\|x_1\left(\frac{t+t_0}{2}\right)\right\|, \frac{t-t_0}{2}\right) + \gamma_1\Big(\sup_{\frac{t+t_0}{2}\le\tau\le t} ||x_2(\tau)||\Big)\,, \tag{238}$$

while letting $s = t_0$ and $t = (t+t_0)/2$, we obtain

$$\left\| x_1\left(\frac{t+t_0}{2}\right)\right\| \leq \beta_1\Big(\|x_1(t_0)\|, \frac{t-t_0}{2}\Big) + \gamma_1\Big(\sup_{t_0 \leq \tau \leq \frac{t+t_0}{2}} ||x_2(\tau)||\Big)\,.$$

Similarly, we can obtain

$$\sup_{t_0 \leq \tau \leq \frac{t+t_0}{2}} ||x_2(\tau)|| \leq \beta_2(||x_2(t_0)||, 0)$$

and

$$\sup_{\frac{t+t_0}{2} \leq \tau \leq t} ||x_2(\tau)|| \leq \beta_2\Big(||x_2(t_0)||, \frac{t-t_0}{2}\Big)\,.$$

Using the last three inequalities in (238), and taking into consideration that $||x(t)|| \leq ||x_1(t)|| + ||x_2(t)||$, we get

$$||x(t)|| \leq \beta(||x(t_0)||, t-t_0)\,,$$

where

$$\beta(r,s) = \beta_1(\beta_1(r,s/2) + \gamma_1(\beta_2(r,0)), s/2) + \gamma_1(\beta_2(r,s/2)) + \beta_2(r,s)$$

is class $\mathcal{KL}$ function for all $r \geq 0$. Hence, the origin of (236) is GUAS. $\square$

16 Adaptive Backstepping

[**Read [11], pp.589-603, and [16], pp.1-121.**]

The entire set of tools, presented until today, assumed matched uncertainties. We had matched parametric uncertainties, matched uncertain nonlinearities, matched disturbances, etc. Of course, in reality, many of the systems do not satisfy this assumption. Backstepping is a technique that allows to control special class of systems with unmatched uncertainties. Backstepping is a recursive procedure that interlaces the choice of a Lyapunov function with the design of feedback control. It breaks a design problem for the full system into a sequence of design problems for lower order subsystems. Adaptive backstepping is a method for controlling systems with unmatched parametric uncertainties. A simple system that does not verify the matching assumption would be

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \theta\varphi(x_1), \\
\dot{x}_2 &= u,
\end{aligned}
\tag{239}
$$

where θ is an unknown parameter, while $\varphi(x_1)$ is a known nonlinear function. It is obvious that this system cannot be put into the well-known for you form

$$
\dot{x}(t) = Ax(t) + b(u(t) + W^\top \Phi(x(t)))\,.
$$

The nonlinearity cannot be factored by the system's b matrix! A typical example from aerospace applications is the problem of controlling the angle of attack:

$$
\dot{\alpha} = -L_\alpha(\alpha)\alpha + q \tag{240}
$$

$$
\dot{q} = M_0(\alpha, q) + u\,, \tag{241}
$$

where α is the angle of attack, q is the pitch rate, L_α models the unknown lift coefficient, while $M_0(\alpha, q)$ is the unknown pitching moment. Thus, the uncertainties in $L_\alpha(\alpha)$ are not matched by the control input.

Systems like this can be adaptively controlled by backstepping. But before jumping directly to adaptive backstepping, we need to review non-adaptive backstepping to understand what is the philosophy behind backstepping, and how recursive Lyapunov analysis is being developed.

16.1 Review of Backstepping

We start with the simplest structure known as integrator backstepping. Consider the system

$$\dot{x} = f(x) + g(x)\xi, \tag{242a}$$

$$\dot{\xi} = u, \tag{242b}$$

where $[x\ \xi]^\top \in \mathbb{R}^2$ is the system state, $u \in \mathbb{R}$ is the control input, $f : \mathcal{D} \to \mathbb{R}$ and $g : \mathcal{D} \to \mathbb{R}$ are known and smooth functions in the domain $\mathcal{D} \subset \mathbb{R}$ that contains the origin $x = 0$, and $f(0) = 0$. We want to design a state feedback control law to achieve asymptotic stability at the origin $(x = 0, \xi = 0)$.

For the time being, let's forget about the second equation (242b) and look at the first equation in (242a) and treat ξ as control input for it. Suppose the component (242a) can be stabilized by a smooth state feedback control law $\xi = \xi_c(x)$ with $\xi_c(0) = 0$, i.e. the origin of

$$\dot{x} = f(x) + g(x)\xi_c(x) \tag{243}$$

is asymptotically stable. For the case when $g(x) \neq 0$, the simplest feedback would be given by a dynamic inversion scheme

$$\xi_c(x) = \frac{1}{g(x)}(-k_1 x - f(x)), \quad k_1 > 0$$

leading to

$$\dot{x} = -k_1 x\,.$$

Notice that if at some points in the state space $g(x) = 0$, then the system loses controllability at those points. Indeed, if $g(x) = 0$, then the first and second equations are completely decoupled, and the first one cannot be controlled by the input from the second equation. Thus, the control input in the second equation needs to be designed to ensure that ξ tracks ξ_c. One might naively think that the feedback $u = \dot{\xi}_c - k_2(\xi - \xi_c)$, where $k_2 > 0$, will do the job, since it achieves asymptotically stable error dynamics $\Delta\dot{\xi} = -k_2\Delta\xi$, where $\Delta\xi = \xi - \xi_c$. And having $\dot{x} = -k_1 x$, both equations are stabilized in a nice decoupled way, and therefore ξ tracks ξ_c, while $x \to 0$ as $t \to \infty$. But this is wrong!!! And unacceptable!!! Why???

Can you make a guess what did go wrong here?

We treated ξ as a **true** control input for the first equation and substituted for it our solution $\xi_c(x) = \frac{1}{g(x)}(-k_1 x - f(x))$ directly. We forgot that ξ was a state, while the **only** true input is u.

This is the difference between the **true control** that you know and the concept of **virtual or pseudo-control** that comes up in backstepping design.

By treating ξ as virtual input for the first equation, one needs to account for the error between ξ and ξ_c, which is simply a stabilizing function for the first equation. Towards that end, we rewrite the system (242a) **equivalently** as:

$$\begin{aligned}\dot{x} &= (f(x) + g(x)\xi_c(x)) + g(x)(\xi - \xi_c(x)),\\ \dot{\xi} &= u.\end{aligned}$$

Now, there is nothing wrong to substitute for ξ_c in the first part of the equation, since the error $\xi - \xi_c(x)$ explicitly shows up in the second half of the equation. So, letting

$$\Delta\xi = \xi - \xi_c\,, \tag{244}$$

the system can be rewritten like

$$\begin{aligned}\dot{x} &= [f(x) + g(x)\xi_c(x)] + g(x)\Delta\xi,\\ \Delta\dot{\xi} &= v,\end{aligned} \tag{245}$$

in which the states are x and $\Delta\xi$, while $v = u - \dot{\xi}_c$, where $\dot{\xi}_c$ should be computed as

$$\dot{\xi}_c = \frac{\partial \xi_c}{\partial x}\dot{x} = \frac{\partial \xi_c}{\partial x}[f(x) + g(x)\xi]. \tag{246}$$

The system in (245) has the same structure as the system in (242) that we started from, except that now the first component has an asymptotically stable origin when the input $\Delta\xi$ is zero. Indeed, $\Delta\xi = 0$ corresponds to $\xi = \xi_c$ (look at (244), and recall that $\xi_c(x)$ was achieving asymptotic stability of the origin of $\dot{x} = f(x) + g(x)\xi_c(x)$). This feature will be exploited in the design of v to stabilize the overall system, while the actual control input will be computed as

$$u = v + \dot{\xi}_c\,.$$

So, let's use our choice of ξ_c

$$\xi_c(x) = \frac{1}{g(x)}(-k_1 x - f(x)), \quad k_1 > 0$$

in (245) and see what we get

$$\begin{aligned}\dot{x} &= -k_1 x + g(x)\Delta\xi,\\ \Delta\dot{\xi} &= v.\end{aligned} \tag{247}$$

So, what did we get??? We got an extra error $g(x)\Delta\xi$ as compared to $\dot{x} = -k_1 x$ that we would have had if instead of ξ we had actual control input in the first equation and had done the straightforward substitution using ξ_c. This is the price of backstepping. Now, our design of v in the second equation needs to make sure that the combined two-dimensional system has asymptotically stable error dynamics. So, design of v will be:

$$v = -k_2\Delta\xi - g(x)x\,,$$

to give a combined error dynamics with its state matrix being a sum of a **skew-symmetric matrix** ($A_1(x)$) and a **Hurwitz matrix** (A_2):

$$\begin{bmatrix} \dot{x} \\ \Delta\dot{\xi} \end{bmatrix} = \underbrace{\begin{bmatrix} -k_1 & g(x) \\ -g(x) & -k_2 \end{bmatrix}}_{A_1+A_2} \begin{bmatrix} x \\ \Delta\xi \end{bmatrix}, \quad A_1(x) = \begin{bmatrix} 0 & g(x) \\ -g(x) & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix}$$

Due to the skew-symmetric structure of the $A_1(x)$ matrix and Hurwitz property of A_2, stability of such system can be immediately concluded from Krasovskii's method given in theorem 3.4, Our system $\dot{x} = f(x) = A(x)x$ has equilibrium at the origin, where $A(x) = A_1(x) + A_2$ is the Jacobian of $f(x)$. Since $A_1(x)$ is skew-symmetric, and A_2 is Hurwitz, it immediately verifies the sufficient condition on $A(x)+A^\top(x) = 2A_2$ being negative definite. Then by Krasovskii's method, the equilibrium at the origin is asymptotically stable, and a Lyapunov function is given by $V(x) = f^\top(x)f(x) = x^\top A^\top(x)A(x)x$. Thus, $x \to 0$ as $t \to \infty$, and $\Delta\xi \to 0$ as $t \to \infty$, which implies that $\xi \to \xi_c$. So, our actual control input is

$$u = -k_2(\xi - \xi_c) - g(x)x + \frac{\partial\xi_c}{\partial x}[f(x) + g(x)\xi]\,, \quad k_2 > 0\,,$$

where $\xi_c(x) = \frac{1}{g(x)}(-k_1 x - f(x))$ and $k_1 > 0$. Now, it is important to pay attention to the fact that design of v in

$$v = -k_2\Delta\xi - g(x)x$$

was motivated by the fact to generate a skew-symmetric matrix in the combined error dynamics to ensure stability of its origin. So, a choice like $v = -k_2\Delta\xi$ would have stabilized only a decoupled $\Delta\dot{\xi} = \dot{\xi} - \dot{\xi}_c$, which does not exist on its own, since ξ_c involves coupling of the states from the first equation. Thus, understanding of stability theory, being aware of Krasosvskii's method, helped to complete the derivation of actual control input u. It was important to get the skew-symmetric matrix with negative off-diagonal elements. You have here the price and the beauty of backstepping!!!

Once we showed the link to Lyapunov theory and stability tools, it is time to formulate backstepping as a general design philosophy. Given the fact that $\xi_c(x)$ is supposed to stabilize the system $\dot{x} = f(x) + g(x)\xi$, it is fair to assume that we know a smooth, positive definite Lyapunov function $V(x)$ that satisfies

$$\frac{\partial V}{\partial x}[f(x) + g(x)\xi_c(x)] \le -W(x), \quad \forall\, x \in \mathcal{D}, \tag{248}$$

where $W(x)$ is positive definite. Existence of such a Lyapunov function for (243) can be deduced from converse Lyapunov theorems (see [11], p.162).

Consider the following positive definite candidate Lyapunov function for the system in (245):

$$\mathcal{V}(x, \Delta\xi) = V(x) + \frac{1}{2}(\Delta\xi)^2.$$

This is a Lyapunov function candidate for (245) because $\mathcal{V}(0,0) = 0$ (notice that $V(0) = 0$, since $V(x)$ was a Lyapunov function for proving asymptotic stability of the origin of $\dot{x} = f(x) + g(x)\xi_c(x)$) and $\mathcal{V} > 0$ otherwise. We can compute

$$\begin{aligned} \dot{\mathcal{V}} &= \frac{\partial V}{\partial x}[f(x) + g(x)\xi_c(x)] + \frac{\partial V}{\partial x} g(x)\Delta\xi + \Delta\xi v \\ &\le -W(x) + \frac{\partial V}{\partial x} g(x)\Delta\xi + \Delta\xi v. \end{aligned}$$

Choosing

$$v = -\frac{\partial V}{\partial x} g(x) - k\Delta\xi, \quad k > 0 \tag{249}$$

yields

$$\dot{\mathcal{V}} \le -W(x) - k(\Delta\xi)^2,$$

which shows that the origin $(x = 0, \Delta\xi = 0)$ is asymptotically stable. Since $\Delta\xi = 0$ corresponds to $\xi = \xi_c(x)$ and $\xi_c(0) = 0$, we conclude that the origin $x = 0, \xi = 0$ is asymptotically stable. Substituting for $v, \Delta\xi, \dot{\xi}_c$, we obtain the state feedback control law (from equations (244), (246), and (249))

$$u = \dot{\xi}_c + v = \underbrace{\frac{\partial \xi_c}{\partial x}[f(x) + g(x)\xi]}_{\dot{\xi}_c} - \frac{\partial V}{\partial x} g(x) - k\underbrace{[\xi - \xi_c(x)]}_{\Delta\xi}. \tag{250}$$

If all the assumptions hold globally and $V(x)$ is radially unbounded, we can conclude that the origin is globally asymptotically stable (GAS). The conclusion can be summarized in the next lemma.

Lemma 16.1. Consider the system in (242). Let $\xi_c(x)$ be a stabilizing state feedback control law for (242a) with $\xi_c(0) = 0$, and $V(x)$ be a Lyapunov function that satisfies (248) with some positive definite function $W(x)$. Then, the state feedback control law (250) stabilizes the origin of (242), with $V(x) + \frac{1}{2}(\xi - \xi_c(x))^2$ as a Lyapunov function. Moreover, if all the assumptions hold globally and $V(x)$ is radially unbounded, the origin will be globally asymptotically stable.

Let us move from systems (242) to the more general system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, && (251a) \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u, && (251b) \end{aligned}$$

where f_2 and g_2 are smooth functions of their arguments, $f_1(0) = 0$. If $g_2(x_1, x_2) \neq 0$ over the domain of interest, then the simple dynamic inversion type feedback

$$u = \frac{1}{g_2(x_1, x_2)}[v - f_2(x_1, x_2)] \tag{252}$$

will linearize (251b) to the following structure $\dot{x}_2 = v$. Therefore, if a stabilizing state feedback control law $x_{2_c}(x_1)$ and a Lyapunov function $V(x_1)$ exist such that the conditions of Lemma 16.1 are satisfied for (251a), then Lemma 16.1 and (252) yield

$$u = \frac{1}{g_2(x_1, x_2)} \left\{ \frac{\partial x_{2_c}}{\partial x_1}[f_1(x_1) + g_1(x_1)x_2] - \frac{\partial V}{\partial x_1} g_1(x_1) - k(x_2 - x_{2_c}) - f_2(x_1, x_2) \right\}, \quad k > 0 \tag{253}$$

as a stabilizing state feedback law and

$$\mathcal{V}(x_1, x_2) = V(x_1) + \frac{1}{2}(x_2 - x_{2_c})^2$$

as a Lyapunov function, respectively, for the overall system (251).

By recursively applying backstepping, we can stabilize *strict-feedback* systems of the form

$$\begin{aligned} \dot{z}_1 &= f_1(z_1) + g_1(z_1)z_2 \\ \dot{z}_2 &= f_2(z_1, z_2) + g_2(z_1, z_2)z_3 \\ &\vdots \\ \dot{z}_{k-1} &= f_{k-1}(z_1, z_2, \ldots, z_{k-1}) + g_{k-1}(z_1, z_2, \ldots, z_{k-1})z_k \\ \dot{z}_k &= f_k(z_1, z_2, \ldots, z_{k-1}, z_k) + g_k(z_1, z_2, \ldots, z_{k-1}, z_k)u \end{aligned} \tag{254}$$

under the assumption that $g_i(z_1, \ldots, z_i) \neq 0$ for $1 \leq i \leq k$. It is important to notice that backstepping philosophy can be applied only to specific structures. In this case the first equation depends linearly upon the second state with a coefficient bounded away from zero, the second equation linearly depends upon the third state with a coefficient bounded away from zero, etc. Another important feature of this structure is that the nonlinearities have *cascaded* structure, i.e. over each level only one more variable is allowed to enter into the nonlinearity. In the first equation, the nonlinearities can depend only upon the first state, while the second state enters linearly. In the second equation the nonlinearities can depend upon the first and second state, while the third state enters linearly, etc. This cascaded structure is crucial for deriving the skew-symmetric matrix for the final error dynamics. If you look carefully, you'll see that so that to treat x_2 as pseudo-control in the first equation, the rest of the entries in that equation should only depend on x_1, otherwise it won't be a well-defined input/output relationship.

In general, for nonlinear systems of the most general form there is no universal control method or theory. Every nonlinear control method has a special structure of the nonlinear system to go with it.

If you got to this point and it is still not clear what is backstepping doing, you better read it again before going to the adaptive section. I would recommend to implement it for better feeling of it.

16.2 Adaptive Regulation

In the above section, we reviewed the philosophy of backstepping, in which the key idea was to interlace the choice of a Lyapunov function with the design of the feedback control law. Again, before going to the adaptive backstepping, let us consider the following first-order and second-order systems and see how that philosophy can be implemented:

$$\dot{x} = u + \theta\varphi(x), \tag{255}$$

where θ is an unknown constant parameter, $\varphi(x)$ is locally Lipschitz known nonlinear function, and

$$\begin{aligned} \dot{x}_1 &= \varphi_1(x_1) + x_2, \\ \dot{x}_2 &= \theta\varphi_2(x) + u, \end{aligned} \tag{256}$$

where θ is an unknown constant parameter, $\varphi_1(x_1), \varphi_2(x)$ are locally Lipschitz known nonlinear functions, $x = [x_1,\ x_2]^\top$, $\varphi_1(0) = 0$. Notice that both systems verify the <u>*matched uncertainty*</u> assumption, since the unknown parameter θ is in the span of the input $u(t)$. However, in the first equation of the second system there is known nonlinearity $\varphi_1(x_1)$ which is not in the span of control. The presence of $\varphi_1(x_1)$ outside the span of control $u(t)$, even though known, makes a difference from the previous structures that we have explored by adaptive control methods.

The control objective is to stabilize $x(t)$ to the origin in both systems.

- First, consider the system in (255). It is straightforward to design a control input $u(t)$ as

$$u(t) = -k_1 x(t) - \hat{\theta}(t)\varphi(x(t)). \tag{257}$$

The derivative of $V(x, \tilde{\theta}) = \frac{1}{2}x^2 + \frac{1}{2\gamma}\tilde{\theta}^2$ with $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$ and $\gamma > 0$ gives

$$\begin{aligned} \dot{V}(t) &= x(t)\dot{x}(t) + \frac{1}{\gamma}\tilde{\theta}(t)\dot{\tilde{\theta}}(t) = x(t)(u(t) + \theta\varphi(x(t))) + \frac{1}{\gamma}\tilde{\theta}(t)\dot{\tilde{\theta}}(t) \\ &= x(t)(-k_1 x(t) - \hat{\theta}(t)\varphi(x(t)) + \theta\varphi(x(t))) + \frac{1}{\gamma}\tilde{\theta}(t)\dot{\tilde{\theta}}(t) \\ &= -k_1 x^2(t) + \tilde{\theta}(t)\left(\frac{1}{\gamma}\dot{\tilde{\theta}}(t) + x(t)\varphi(x(t))\right). \end{aligned}$$

Choosing the adaptive law as

$$\dot{\hat{\theta}}(t) = \gamma x(t)\varphi(x(t)), \quad \hat{\theta}(0) = \hat{\theta}_0 \tag{258}$$

yields

$$\dot{\tilde{\theta}}(t) = -\gamma x(t)\varphi(x(t)).$$

Consequently $\dot{V}(t) = -k_1 x^2(t) \le 0$ is negative semidefinite, and $x(t), \tilde{\theta}(t)$ are bounded. The closed-loop system is

$$\begin{aligned} \dot{x} &= -k_1 x + \tilde{\theta}\varphi(x), \\ \dot{\tilde{\theta}} &= -\gamma x\varphi(x). \end{aligned} \tag{259}$$

Since $\dot{V}(t) \le 0$, and $V(0,0) = 0$, the equilibrium $(x = 0, \tilde{\theta} = 0)$ of (259) is globally stable. Since the closed-loop system in (259) is autonomous, application of LaSalle's principle implies that if $\varphi(0) \ne 0$, then $x \to 0$ and $\tilde{\theta} \to 0$ as $t \to \infty$. If $\varphi(0) = 0$, then $x \to 0$, but $\tilde{\theta} \to 0$ is not guaranteed.

Remark 16.1. In fact, $x \to 0$ as $t \to \infty$ can be concluded without application of La-Salle's principle. A more general result, known as La-Salle–Yoshizawa lemma, helps to prove $x(t) \to 0$ directly from the fact that $\dot{V}(t) = -k_1 x^2(t) \le 0$. It can be used also in analysis of non-autonomous systems, instead of applying Barbalat's lemma. La-Salle's principle and Barbalat's lemma were convenient tools to explain you the difference between autonomous systems and non-autonomous systems. Also, in this particular system with our choice of the Lyapunov function candidate, $\tilde{\theta} \to 0$, in the case of $\varphi(0) \neq 0$, can be concluded only from application of La-Salle's principle. La-Salle–Yoshizawa lemma won't help in that. However, since now as you managed to grow through the course, we state La-Salle–Yoshizawa lemma for giving you the broader perspective.

Lemma 16.2. [La-Salle–Yoshizawa lemma] Let $x = 0$ be an equilibrium point of the system

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = 0\,,$$

where $x \in \mathbb{R}^n$ is the system state, and f is piece-wise continuous in t, locally Lipschitz in x uniformly in t. Let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a continuously differentiable, positive definite and radially unbounded function such that

$$\dot{V}(t) = \frac{\partial V}{\partial x} f(t, x) \leq -W(x(t)) \leq 0, \quad \forall\, t \geq 0, \quad \forall\, x \in \mathbb{R}^n\,,$$

where $W(x)$ is a continuous function. Then all solutions of the system $\dot{x}(t) = f(t, x(t))$ are globally uniformly bounded and satisfy

$$\lim_{t \to \infty} W(x(t)) = 0\,.$$

In addition, if $W(x)$ is positive definite, then the equilibrium $x = 0$ of the system $\dot{x}(t) = f(t, x(t))$ is globally uniformly asymptotically stable (GUAS).

- Now consider the system in (256). Before getting into the details, let us examine this system. Notice that the system in (256) has a structure similar to (251), but with an unknown parameter θ. However, the unknown parameter is matched, i.e. it is in the span of control. So, if we are interested in stabilizing $x_1(t)$ via the input $u(t)$, we need to couple our knowledge on adaptive control of systems with matched uncertainties with our recently reviewed backstepping philosophy. Backstepping will be illustrated first using this example. Upon that, we will extend it to unmatched uncertainties, like stated in the beginning of this section.

Another feature that I would like to point out for you is that in (251) we had to assume that $f(0) = 0$ (you need to look back to see how crucial was this for the control design), while in (255) we did not have to assume that $\varphi(0) = 0$. This is because in the latter system, $\varphi(x)$ is multiplying the unknown parameter, for which we are writing an adaptive law. In fact, it is even beneficial not to have $\varphi(0) = 0$, since then we lose parameter convergence. Thus, in (256) we assume that $\varphi_1(0) = 0$, which resembles $f(0) = 0$ in (251), while we impose no assumption on $\varphi_2(0)$. By the end of the proof, further clarification will be given on this.

Now, getting to actual control design, notice that we have no control of the first equation in (256), so we need to control it from the second equation. Viewing x_2 as a virtual input, a stabilizing feedback function for x_1 can be designed as

$$x_{2_c}(x_1) = -k_1 x_1 - \varphi_1(x_1). \tag{260}$$

Thus, repeating our well-known steps

$$\begin{aligned} \dot{x}_1 &= x_{2_c} + \varphi_1(x_1) + \underbrace{x_2 - x_{2_c}}_{\Delta x_2} = -k_1 x_1 + \Delta x_2, \\ \Delta\dot{x}_2 &= \dot{x}_2 - \dot{x}_{2_c} = \dot{x}_2 - \frac{\partial x_{2_c}}{\partial x_1}\dot{x}_1 = u + \theta\varphi_2(x) + \left(k_1 + \frac{\partial \varphi_1(x_1)}{\partial x_1}\right)(\Delta x_2 - k_1 x_1). \end{aligned} \tag{261}$$

If θ were known, the derivative of the following candidate Lyapunov function

$$V(x_1, \Delta x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(\Delta x_2)^2$$

given by

$$\dot{V}(x_1, \Delta x_2) \quad = \quad x_1(\Delta x_2 - k_1 x_1) + \Delta x_2 \left(u + \theta\varphi_2(x) + \left(k_1 + \frac{\partial \varphi_1(x_1)}{\partial x_1}\right)(\Delta x_2 - k_1 x_1)\right)$$

would be rendered negative definite

$$\dot{V} = -k_1 x_1^2 - k_2(\Delta x_2)^2 < 0$$

by the control law

$$u = -k_2 \Delta x_2 - x_1 - \left(k_1 + \frac{\partial \varphi_1(x_1)}{\partial x_1}\right)(\Delta x_2 - k_1 x_1) - \theta\varphi_2(x(t))\,. \tag{262}$$

Substitution in (261) gives a (skew-symmetric+Hurwitz) state-space matrix:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 + \Delta x_2, \\ \Delta \dot{x}_2 &= -x_1 - k_2 \Delta x_2 \,. \end{aligned} \tag{263}$$

If θ is unknown, it is natural to replace it by its estimate in the above control law:

$$u = -k_2 \Delta x_2 - x_1 - \left(k_1 + \frac{\partial \varphi_1(x_1)}{\partial x_1} \right) (\Delta x_2 - k_1 x_1) - \hat{\theta}(t) \varphi_2(x(t)) \,. \tag{264}$$

Now, we have to derive an adaptive law for $\hat{\theta}(t)$. It is straightforward to see that the closed-loop system with adaptive feedback retains the skew-symmetric structure of the state matrix and takes the form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -k_1 & 1 \\ -1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \varphi_2(x(t)) \end{bmatrix} \tilde{\theta}(t), \tag{265a}$$

where $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$. The rest is straightforward: we augment the original candidate Lyapunov function by an additional quadratic term of parametric error:

$$\mathcal{V}(x_1, \Delta x_2, \tilde{\theta}) = V(x_1, \Delta x_2) + \frac{1}{2\gamma} \tilde{\theta}^2 = \frac{1}{2} x_1^2 + \frac{1}{2} (\Delta x_2)^2 + \frac{1}{2\gamma} \tilde{\theta}^2 \,.$$

Its derivative is:

$$\dot{\mathcal{V}}(x_1, \Delta x_2, \tilde{\theta}) \;=\; -k_1 x_1^2(t) - k_2 (\Delta x_2(t))^2 + \tilde{\theta}(t) \left(\frac{1}{\gamma} \dot{\tilde{\theta}}(t) + \Delta x_2(t) \varphi_2(x(t)) \right) .$$

So, choosing

$$\dot{\hat{\theta}}(t) = \gamma \Delta x_2(t) \varphi_2(x(t)), \tag{266}$$

yields

$$\dot{\tilde{\theta}}(t) = -\gamma \Delta x_2(t) \varphi_2(x(t)), \tag{267}$$

and consequently $\dot{V}(t) = -k_1 x_1^2(t) - k_2 (\Delta x_2(t))^2 \le 0$. Thus the closed-loop adaptive system has a globally stable equilibrium at the origin $(x_1 = 0, \Delta x_2 = 0)$, and application of La-Salle–Yoshizawa lemma implies that $x_1, \Delta x_2 \to 0$ as $t \to \infty$, i.e. $x_1(t) \to 0,\; x_2(t) \to x_{2_c}(0) = 0$ as $t \to \infty$. Now, you see why $\varphi_1(0) = 0$ was important, otherwise we wouldn't have had $x_{2_c}(0) = 0$, and hence convergence of x_2 to the origin would have been in question. Application of La-Salle's principle implies that if $\varphi_2(0) \neq 0$, then $\tilde{\theta} \to 0$ as $t \to \infty$, i.e. we get in addition parameter convergence.

16.3 Adaptive Backstepping

In the previous section, we used the backstepping philosophy to control systems with matched uncertainty. Backstepping was only performed for the **unmatched known nonlinearity**, while for the adaptation purposes we used our old known philosophy, since the uncertain parameter was matched. Now, we will develop the adaptive backstepping, by which we mean that we will back-step adaptively to compensate for the unmatched unknown parameters.

Now we consider systems of the following type:

$$\dot{x}_1 = x_2 + \theta\varphi(x_1), \tag{268a}$$

$$\dot{x}_2 = u\,, \tag{268b}$$

where θ is the unknown parameter, u is the control input. Notice that this system is similar to (251). However, now the uncertainty is unmatched, since the unknown parameter is not in the span of the input u. The system (268) falls into the class of systems with *extended matching* with level of uncertainty being one. This system

$$\dot{x}_1 = x_2 + \theta\varphi(x_1), \tag{269a}$$

$$\dot{x}_2 = x_3 \tag{269b}$$

$$\dot{x}_3 = u \tag{269c}$$

has level of uncertainty two, because the uncertain parameter is separated from the control input by two integrators.

We will just develop the adaptive backstepping for (268), hoping that it should be straightforward to generalize. Viewing x_2 as a virtual input, we can design the following stabilizing function

$$x_{2_c}(x_1, \hat{\theta}) = -k_1 x_1 - \hat{\theta}(t)\varphi(x_1). \tag{270}$$

Notice that due to the presence of the unknown parameter θ in the first equation, the stabilizing function for the first equation now depends upon the adaptive parameter $\hat{\theta}(t)$.

Thus, the derivative of x_1 gets an extra term as compared to (261):

$$\dot{x}_1 = x_{2_c} + \theta\varphi_1(x_1) + \underbrace{x_2 - x_{2_c}}_{\Delta x_2} = -k_1 x_1 + \Delta x_2 + \tilde{\theta}(t)\varphi(x_1), \quad \tilde{\theta}(t) = \theta - \hat{\theta}_1(t), \tag{271}$$

while computation of $\Delta\dot{x}_2$ needs to account for its dependence upon $\hat{\theta}$:

$$\Delta\dot{x}_2 = \dot{x}_2 - \dot{x}_{2_c} = \dot{x}_2 - \frac{\partial x_{2_c}}{\partial x_1}\dot{x}_1 - \frac{\partial x_{2_c}}{\partial\hat{\theta}}\dot{\hat{\theta}} = u + \underbrace{\left(k_1 + \hat{\theta}(t)\frac{\partial\varphi_1(x_1)}{\partial x_1}\right)}_{-\frac{\partial x_{2_c}}{\partial x_1}}(x_2 + \theta\varphi(x_1)) + \varphi(x_1)\dot{\hat{\theta}}. \quad (272)$$

Consider the following candidate Lyapunov function

$$V(x_1, \Delta x_2, \tilde{\theta}) = \frac{1}{2}x_1^2 + \frac{1}{2}(\Delta x_2)^2 + \frac{1}{2\gamma}(\tilde{\theta})^2.$$

Its derivative will be

$$\begin{aligned}
\dot{V}(x_1, \Delta x_2, \tilde{\theta}) &= -k_1x_1^2 + x_1\Delta x_2 + \tilde{\theta}\left(\varphi(x_1)x_1 + \frac{1}{\gamma}\dot{\tilde{\theta}}\right) \\
&+ \Delta x_2\left(u - \frac{\partial x_{2_c}}{\partial x_1}(x_2 + \hat{\theta}\varphi(x_1) + \tilde{\theta}\varphi(x_1)) + \varphi(x_1)\dot{\hat{\theta}}\right) \\
&= -k_1x_1^2 + \tilde{\theta}\left(\varphi(x_1)x_1 - \frac{1}{\gamma}\dot{\hat{\theta}} - \frac{\partial x_{2_c}}{\partial x_1}\ \Delta x_2\ \varphi(x_1)\right) \\
&+ \Delta x_2\left(x_1 + u - \hat{\theta}\frac{\partial x_{2_c}}{\partial x_1}\varphi(x_1) - \frac{\partial x_{2_c}}{\partial x_1}x_2 + \varphi_1(x_1)\dot{\hat{\theta}}\right).
\end{aligned}$$

Choosing

$$\dot{\hat{\theta}} = \gamma\left(\varphi(x_1)x_1 - \frac{\partial x_{2_c}}{\partial x_1}\varphi(x_1)\Delta x_2\right)$$

and

$$u = -x_1 - k_2\Delta x_2 + \hat{\theta}\frac{\partial x_{2_c}}{\partial x_1}\varphi(x_1) + \frac{\partial x_{2_c}}{\partial x_1}x_2 - \varphi_1(x_1)\dot{\hat{\theta}},$$

the derivative of the candidate Lyapunov function can be rendered negative semidefinite

$$\dot{V}(x_1, \Delta x_2, \tilde{\theta}) = -k_1x_1^2 - k_2(\Delta x_2)^2 \leq 0.$$

From La-Salle–Yoshizawa's lemma we have that $x_1, \Delta x_2 \to 0$ as $t \to \infty$, which implies that $x_2 \to x_{2_c}$. Looking at the closed-system error dynamics

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1 \\ -1 & -k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \varphi(x_1) \\ \left(k_1 + (\theta - \tilde{\theta})\frac{\partial\varphi_1(x_1)}{\partial x_1}\right)\varphi(x_1) \end{bmatrix}\tilde{\theta}, \quad (273a)$$

$$\dot{\tilde{\theta}} = -\gamma\left[\varphi(x_1) \quad \left(k_1 + (\theta - \tilde{\theta})\frac{\partial\varphi_1(x_1)}{\partial x_1}\right)\varphi(x_1)\right]\begin{bmatrix} x_1 \\ \Delta x_2 \end{bmatrix}, \quad (273b)$$

we see that it is autonomous (due to the stabilization problem as opposed to tracking). So, La-Salle's invariance principle can be applied to conclude that $\tilde{\theta} \to 0$ as $t \to \infty$. Indeed, $\dot{V}(x_1, \Delta x_2, \tilde{\theta}) = -k_1 x_1^2 - k_2(\Delta x_2)^2 = 0$, implies that $x_1 = \Delta x_2 = 0$, which can **identically** take place only if $\tilde{\theta} = 0$, provided that $\varphi(0) \neq 0$. If $\varphi(0) = 0$, of course, no parameter convergence can be claimed. If $\varphi(0) \neq 0$, then $x_2 \to \theta\varphi(0)$ as $t \to \infty$.

Thus, backstepping led to a new type of control signal that involves the derivative of the adaptive parameter in addition to the adaptive parameter, which enters both linearly and with its square. Let's rewrite the adaptive controller in its full glory:

$$u = -x_1 - k_2(x_2 - k_1 x_1 - \hat{\theta}\varphi(x_1)) - \left(k_1 + \hat{\theta}\frac{\partial \varphi_1(x_1)}{\partial x_1}\right)(x_2 + \hat{\theta}\varphi(x_1)) - \varphi_1(x_1)\dot{\hat{\theta}}\,. \tag{274}$$

We notice correspondingly that both the adaptive law and the error dynamics have new structure. The parametric error in the state error equation is multiplied by the same vector that is used for definition of the adaptive laws. The closed-loop architecture is given in Figure 27.

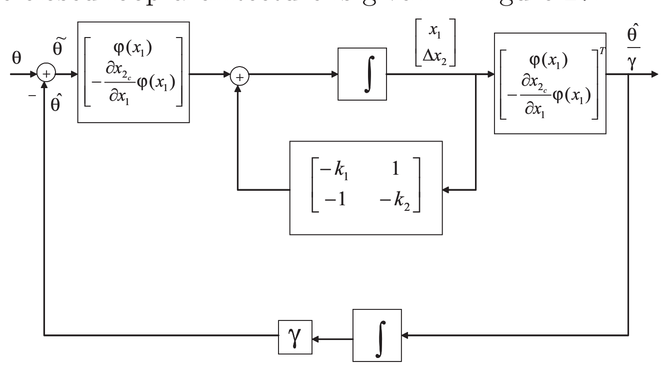


Fig. 27 Closed-loop adaptive system with backstepping

It is important to compare and see how much changed in the design when moving from the system of the type (256) to a system of the type (268). It should be obvious that if we had retained $\varphi_2(x)$ in the second equation of (268) without any unknown coefficient, then we would have needed to imply subtract it from our ultimate control definition in (274). It was dropped for simplicity of derivations. If we had retained $\varphi_2(x)$ in the second equation in (268) with another unknown coefficient, then it should have been straightforward to derive another update law for it, since it would be matched uncertainty and should reduce to the previous case.

Homework Problems 16.1.

- Consider implementing backstepping for a 2-D system. It would be good to do it incrementally, i.e. start from a system of the type (256) and move to a system of the type (268).

- **Challenge problem:** Write the backstepping scheme for the following system:

$$\begin{aligned} \dot{x}_1 &= \theta_1 \varphi_1(x_1) + x_2, \\ \dot{x}_2 &= \theta_2 \varphi_2(x) + u, \end{aligned} \tag{275}$$

  in which both θ_1 and θ_2 are unknown parameters. There is no technical challenge in this, but you need patience to go through.

17 Adaptive Output Feedback Control

The adaptive control methods presented in the previous sections were relying on the knowledge of full state feedback. To extend tools to output feedback, we need i) to be able to write a control signal definition using only available measured outputs and reference inputs, ii) to write the adaptive laws using only measured outputs and reference inputs. So that to formulate adaptive control using only output feedback, we need to recall the concept of strictly positive real transfer functions. First we consider linear systems and later present generalization to class of nonlinear systems, for which adaptive output feedback can be determined with asymptotic convergence properties.

17.1 Positive Real Systems

Reading [17], pp. 126-131.

Definition 17.1. A transfer function $H(s)$ is called positive real, if $\mathrm{Re}[H(s)] \geq 0$ for all $\mathrm{Re}[s] \geq 0$. It is strictly positive real (SPR), if $H(s-\epsilon)$ is positive real for some $\epsilon > 0$.

Example 17.1. The transfer function

$$H(s) = \frac{1}{s+\lambda},$$

where $\lambda > 0$, is obviously strictly positive real since for any complex number $s = \sigma + j\omega$ with $\sigma > 0$ we have

$$H(\sigma + j\omega) = \frac{1}{\sigma + j\omega + \lambda} = \frac{\sigma + \lambda - j\omega}{(\sigma+\lambda)^2 + \omega^2},$$

which has positive real part if one selects $\epsilon = \lambda/2$.

Theorem 17.1. A transfer function $H(s)$ is SPR, if and only if

- $H(s)$ is strictly stable transfer function;
- The real part of $H(s)$ is strictly positive along $j\omega$ axis, i.e.

$$\forall\, \omega \geq 0 \quad \mathrm{Re}[H(j\omega)] > 0\,.$$

Remark 17.1. The above theorem gives several <u>*necessary conditions*</u> for checking the SPR property of a transfer function:

- $H(s)$ is strictly stable;
- The Nyquist plot of $H(j\omega)$ lies entirely in the right half complex plane. Or equivalently saying, the phase shift of the system in response to sinusoidal input is always less than $90°$ degrees.
- $H(s)$ has relative degree 0 or 1, i.e. the difference of its poles and zeros is never more than 1 at most;
- $H(s)$ is strictly minimum phase, i.e. all its zeros lie in the open left-half plane.

The first and second necessary conditions are immediate from the theorem, while the third and fourth one are a consequence of the second condition.

Example 17.2.

- $H_1(s) = \frac{s-1}{s^2+as+b}$ cannot be SPR, because it is non-minimum phase.
- $H_2(s) = \frac{s+1}{s^2-s+1}$ cannot be SPR because it is unstable;
- $H_3(s) = \frac{1}{s^2+as+b}$ cannot be SPR because it has relative degree 2;
- $H_4(s) = \frac{s+1}{s^2+s+1}$ is strictly stable, minimum phase and has relative degree 1. It is SPR, because $H_4(j\omega) = \frac{j\omega+1}{-\omega^2+j\omega+1} = \frac{(j\omega+1)(-\omega^2-j\omega+1)}{(1-\omega^2)^2+\omega^2}$ has <u>*positive*</u> real part equal to $\mathrm{Re}[H_4(j\omega)] = \frac{1}{(1-\omega^2)^2+\omega^2}$.
- $H(s) = \frac{1}{s}$ is PR (positive real), but not SPR.

The main difference between PR and SPR functions is that PR functions can tolerate poles on the imaginary axis, while SPR ones cannot.

Lemma 17.1. [Kalman-Yakubovich lemma.] Consider a controllable linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= c^\top x(t) \,. \end{aligned}$$

The transfer function $H(s) = c^\top[s\mathbb{I} - A]^{-1}b$ is SPR if and only if there exist positive definite matrices P and Q such that

$$A^\top P + PA = -Q\,, \quad Pb = c\,.$$

For different extensions of this lemma see [17], pp.132-133.

17.2 Adaptive Laws for SPR Systems

Adaptive output feedback control with a global asymptotic stability proof can be done only if the transfer function of the error dynamics is SPR. If it is not SPR, then additional filters need to be introduced to render it SPR [7]. The key result, enabling adaptive output feedback control, is given by the following lemma. This lemma gives you a structure to be used for adaptive output feedback control design. So, whatever control or adaptive law we define later, we have to make sure that it fits into this structure.

Lemma 17.2. Consider the following system

$$\begin{aligned} \dot{e}(t) &= Ae(t) + b\lambda\tilde{\theta}^\top(t)v(t) \\ \tilde{y}(t) &= c^\top e(t)\,, \end{aligned} \tag{276}$$

where $\tilde{y} \in \mathbb{R}$ is the only scalar measurable output signal ($e \in \mathbb{R}^n$ is not fully measurable), $H(s) = c^\top(s\mathbb{I} - A)^{-1}b$ is a strictly positive real transfer function, λ is an unknown constant with known sign, $\tilde{\theta}(t)$ is a $m \times 1$ vector function of time (usually modeling the parametric errors), and $v(t)$ is measurable $m \times 1$ vector. If the vector $\tilde{\theta}(t)$ varies according to

$$\dot{\tilde{\theta}}(t) = -\mathrm{sgn}(\lambda)\gamma\tilde{y}(t)v(t) \tag{277}$$

with $\gamma > 0$ being a positive constant for adaptation rate, then $\tilde{y}(t)$ and $\tilde{\theta}(t)$ are globally bounded. Furthermore, if $v(t)$ is bounded, then

$$\tilde{y}(t) \to 0 \qquad \text{as} \qquad t \to \infty\,. \tag{278}$$

Proof: Since $H(s)$ is SPR, it follows from KYP lemma that there exist symmetric positive definite matrices Q and P such that

$$\begin{aligned} A^\top P + PA &= -Q && (279) \\ Pb &= c\,. && (280) \end{aligned}$$

Let $V(e,\tilde{\theta})$ be a positive definite function of the form:

$$V(e,\tilde{\theta}) = e^\top Pe + \frac{|\lambda|}{\gamma}\tilde{\theta}^\top\tilde{\theta}\,. \tag{281}$$

Its time derivative along the trajectories of the system, **<u>due to the SPR condition</u>**, will be:

$$\dot{V}(e(t),\tilde{\theta}(t)) = e^\top(t)(PA + A^\top P)e(t) + 2\,\underbrace{e^\top \underbrace{Pb}_{c}}_{\tilde{y}}(\lambda\tilde{\theta}^\top(t)v(t)) - 2\tilde{\theta}^\top(t)(\lambda\tilde{y}(t)v(t)) = -e^\top(t)Qe(t) \le 0\,.$$

Therefore the equilibrium at the origin ($e = 0,\ \tilde{\theta} = 0$) is globally stable.

Further, if the signal $v(t)$ is bounded, $\dot{e}(t)$ is also bounded (see (276)). From

$$\ddot{V}(t) = -2e^\top(t)Q\dot{e}(t) \tag{282}$$

it follows that $\ddot{V}(t)$ is also bounded. This implies uniform continuity of $\dot{V}(t)$. Since $V(e(t),\tilde{\theta}(t))$ is bounded, and $\dot{V}(t)$ is uniformly continuous, application of Barbalat's lemma indicates that $\dot{V}(t) \to 0$ as $t \to \infty$, implying that $e(t) \to 0$ as $t \to \infty$. Consequently this leads to asymptotic convergence of $\tilde{y}(t)$ to zero. $\square$

Remark 17.2. The **key difference** of the adaptive law in (277) from the conventional ones is that it does not have the P matrix in it, which was coming from the solution of the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$ for the reference matrix A_m. Thus, all that is required for this new adaptive law is the output error, and not the $e^\top Pb$. This fact is the **<u>fundamental</u>** point in the development of the adaptive output feedback scheme, since we are going to augment the main system dynamics with additional poles and zeros to achieve the SPR property for it. We will prove that this is feasible, and this augmented system matrix with this "unknown" coefficients will constitute a reference system with SPR property. Thus, the error dynamics will have the SPR property, while the augmented reference matrix, used for error dynamics, will be unknown.

Remark 17.3. In the above presented adaptive output feedback scheme the key point was to write the adaptive law via only measurable signals, i.e. system output. It was possible due to the SPR condition that helped to replace the conventional $e^\top(t)Pb$ in the adaptive law by its equivalent $y(t)$, because in the presence of SPR $Pb = c$ (Kalman-Yakubovich lemma), and hence $e^\top(t)Pb = e^\top(t)c = y(t)$. Without the SPR condition such a replacement would have been impossible.

Also, in our conventional error dynamics the signal $v(t)$ is not measurable. It usually depends upon the state vector x. So, even if we employ additional filters to render it SPR, we may need to do state estimation to have an implementable control law.

17.3 The Simplest System with Adaptive Output Feedback

Let's first see what would be the simplest system that would fit in the above described scheme. Firts of all let's acknowledge the fact that once we are talking about output feedback, the simplest system cannot be scalar!!! So, the minimum dimension is $n = 2$, for which we would like to demonstrate an adaptive output feedback control solution. So, let's see what do we need: we need an SPR triple (A, b, c) in the error dynamics in (276) and a measurable signal $v(t)$. An SPR triple implies having an SPR reference system:

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + b_m r(t) \\ y_m(t) &= c^\top x_m(t)\,, \end{aligned} \tag{283}$$

such that (A_m, b_m, c) is SPR, and $x_m \in \mathbb{R}^n$, $y_m \in \mathbb{R}$. So, let's construct our system to see what can we afford in terms of uncertainties in our system to benefit from Lemma 17.2. Recalling that we always needed some kind of matching with A_m, let's consider

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\lambda(u(t) - k_y^* y(t)) \\ y(t) &= c^\top x(t)\,, \end{aligned} \tag{284}$$

where k_y^* is the only unknown (scalar) parameter in addition to unknown λ, and $x \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$. The sign of λ needs to be known. Define the adaptive output feedback controller as:

$$u(t) = k_y(t)y(t) + k_r(t)r(t)\,.$$

Upon substitution in (284), we get

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\lambda(k_y(t)y(t) + k_r(t)r(t) - k_y^* y(t)) \\ y(t) &= c^\top x(t)\,. \end{aligned} \tag{285}$$

Let $\Delta k_y(t) = k_y(t) - k_y^*$. Then we have the following closed-loop system

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\lambda(\Delta k_y(t)y(t) + k_r(t)r(t)) \\ y(t) &= c^\top x(t)\,. \end{aligned} \tag{286}$$

If we compare this to the reference system in (283), we need one more matching condition:

$$\exists\ k_r^* : \quad b\lambda k_r^* = b_m\,.$$

Subtracting (286) from (283), and denoting $e(t) = x(t) - x_m(t)$, $\tilde{y}(t) = y(t) - y_m(t)$, $\Delta k_r(t) = k_r(t) - k_r^*$, we obtain:

$$\begin{aligned} \dot{e}(t) &= A_m e(t) + b\lambda\Delta k_y(t)y(t) + b\lambda\Delta k_r(t)r(t) \\ \tilde{y}(t) &= c^\top e(t)\,, \end{aligned} \tag{287}$$

which we can "pack" similar to (276), if we denote $\tilde{\theta}(t) = [\Delta k_y(t)\ \Delta k_r(t)]^\top$, $v(t) = [y(t)\ r(t)]^\top$:

$$\begin{aligned} \dot{e}(t) &= A_m e(t) + b\lambda\tilde{\theta}^\top(t)v(t) \\ \tilde{y}(t) &= c^\top e(t)\,. \end{aligned} \tag{288}$$

Thus, if we compare the system in (284) to our conventional one in (159), for which we developed state feedback adaptive control design methods, we see that Lemma 17.2 allowed us to have only SPR reference system and only output dependent uncertainties. The matching assumption consequently implied that the uncertain system must also have SPR transfer function from its input to its output. For the system in (288), obviously the adaptive law can be written as in Lemma 17.2.

Remark 17.4. It is straightforward to notice that the above design could have been easily extended to systems with output dependent nonlinearities:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\lambda(u(t) - (k_y^*)^\top \Phi(y(t))) \\ y(t) &= c^\top x(t)\,, \end{aligned}$$

where k_y^* is a vector of unknown constants, while $\Phi(y)$ is a vector of known nonlinear functions, by modifying the adaptive controller to be $u(t) = k_y^\top(t)\Phi(y(t)) + k_r(t)r(t)$.

17.4 Adaptive Output Feedback Control for Systems with Relative Degree One

Now let's consider a general linear system, in which the uncertainties can depend also upon the state of the system. However, we require the transfer function from the input to output be minimum-phase and have relative degree 1, i.e. all its zeros be in the open left-half plane and the difference between its poles and zeros be 1:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \quad x(0) = x_0 \\ y(t) &= c^\top x(t) \,. \end{aligned}$$

Let's denote:

$$G(s) = c^\top(s\mathbb{I} - A)^{-1}b = k_p \frac{Z_p(s)}{R_p(s)}\,,$$

where k_p is called high-frequency gain (the sign of which we always assumed to be known), while $Z_p(s)$ and $R_p(s)$ are polynomials, representing the *stable* zeros and (whatever) poles of the system respectively, and the order of $Z_p(s)$ is just one less than the order of $R_p(s)$, so that the system has relative degree 1. Let the desired reference model be given by

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + b_m r(t), \quad x_m(0) = x_{m_0} \\ y_m(t) &= c_m^\top x_m(t) \,. \end{aligned}$$

Denote

$$\frac{y_m(s)}{r(s)} = W_m(s) = k_m \frac{Z_m(s)}{R_m(s)}\,,$$

where k_m is the high -frequency gain of the reference system, while $Z_m(s)$ and $R_m(s)$ are polynomials, associated with the zeros and poles of the reference system. When the reference system is SPR, one can derive globally convergent stable adaptive controllers. However, let's first recall the Model Reference Control structure (MRC) and see how this can be achieved via output feedback in case if all parameters are known. Then we will introduce adaptive laws to adapt for parametric uncertainties. For a general MRC with known parameters, SPR and relative degree 1 are not required neither for the system, nor for the reference system.

17.4.1 Model Reference Control for known parameters

We notice that if the system had also stable poles, in addition to stable zeros, then a straightforward controller to achieve perfect tracking would be

$$u(s) = \frac{k_m}{k_p}\frac{Z_m(s)}{R_m(s)}\frac{R_p(s)}{Z_p(s)}r(s)\,, \tag{289}$$

which will cancel both zeros and poles of the system and replace those by the ones from the reference model. However, the poles of the system **do not have to be stable**, and feedback can be used to shift those. Since feedback cannot be used to change the zeros, it is important to assume that the system is minimum phase. This will allow the controller poles to cancel the system zeros.

So, first let's formulate the general assumptions for MRC.

Assumptions for the system to be controlled:

- $Z_p(s)$ is monic Hurwitz polynomial of degree m_p;
- An upper bound n of the degree n_p of $R_p(s)$ is known;
- The relative degree $n^* = n_p - m_p$ of $G(s)$ is known;
- $\mathrm{sgn}(k_p)$ is known.

Assumptions for the reference system:

- $Z_m(s), R_m(s)$ are monic Hurwitz polynomials of degree q_m, p_m, respectively, $p_m \leq n$;
- The relative degree $n_m^* = p_m - q_m$ of $W_m(s)$ is the same as n^*, i.e. $n_m^* = n^*$.

Remark 17.5. Notice that we allow the system be uncontrollable and unobservable, as long as it is detectable and stabilizable. Since all the zeros are assumed to be in open-left half plane, any pole-zero cancellation can take place only in open left-half plane.

So, in case of **known** parameters, instead of (289), let's consider the following ideal controller

$$u^*(s) = (\theta_1^*)^\top \frac{\alpha(s)}{\Lambda(s)}u^*(s) + (\theta_2^*)^\top \frac{\alpha(s)}{\Lambda(s)}y(s) + \theta_0^* y(s) + k^* r(s)\,, \tag{290}$$

where $\alpha(s) \triangleq \alpha_{n-2}(s) = [s^{n-2},\ s^{n-3},\ ..., s, 1]^\top$ if $n \geq 2$, and $\alpha(s) = 0$, if $n = 1$ (this corresponds to the simplest adaptive system considered in the previous subsection), $k^*, \theta_0^* \in \mathbb{R}$, and $\Lambda(s)$ is an arbitrary monic Hurwitz polynomial of degree $n-1$ that contains $Z_m(s)$ as a factor, i.e.

$$\Lambda(s) = \Lambda_0(s) Z_m(s)\,,$$

which implies that $\Lambda_0(s)$ is monic, Hurwitz and of degree $n_0 = n - 1 - q - m$. The control objective is to choose the parameter vector

$$\theta^* = [(\theta_1^*)^\top\ (\theta_2^*)^\top\ \theta_0^*\ k^*]^\top \in \mathbb{R}^{2n}$$

in a way so that

$$\frac{y(s)}{r(s)} = W_m(s)\,.$$

Obviously

$$k^* = \frac{k_m}{k_p}\,.$$

Existence of such $\theta_1^*, \theta_2^*, \theta_0^*, k^*$ is ensured via the following lemma ([26], p.336):

Lemma 17.3. Let the degrees of $R_p(s), Z_p(s), \Lambda(s), \lambda_0(s)$ and $R_m(s)$ be specified as above. Then, there always exists θ^* such that $u^*(s)$ achieves $\frac{y(s)}{r(s)} = W_m(s)$. In addition, if $R_p(s), Z_p(s)$ are coprime and $n = n_p$, then the solution θ^* is unique.

Let's now consider the state space representation of the ideal controller to analyze the effect of the initial conditions. All we stated above is equivalence of transfer functions which will lead to identical response in case of identical initial conditions. However, if our system and reference system have different initial conditions, then having the same transfer functions, we still need to characterize the initial transient. Towards that end consider the following state-space realization of that ideal controller

$$u^*(t) = (\theta^*)^\top \omega(t)$$

where $\theta^* = [(\theta_1^*)^\top\ (\theta_2^*)^\top\ \theta_0^*\ k^*]^\top$, $\omega(t) = [\omega_1^\top(t)\ \omega_2^\top(t)\ y(t)\ r(t)]^\top$, and

$$\begin{aligned} \dot{\omega}_1(t) &= F\omega_1(t) + gu^*(t) \\ \dot{\omega}_2(t) &= F\omega_2(t) + gy(t)\,, \end{aligned} \tag{291}$$

where $\omega_1, \omega_2 \in \mathbb{R}^{n-1}$ and

$$F = \begin{bmatrix} -\lambda_{n-2} & -\lambda_{n-3} & -\lambda_{n-4} & \cdots & -\lambda_0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and λ_i are coefficients of

$$\Lambda(s) = s^{n-1} + \lambda_{n-2}s^{n-2} + \cdots + \lambda_1 s + \lambda_0 = \det(s\mathbb{I} - F)$$

where (F, g) is the state space realization of $\frac{\alpha(s)}{\Lambda(s)}$, i.e.

$$(s\mathbb{I} - F)^{-1}g = \frac{\alpha(s)}{\Lambda(s)}\,.$$

The closed-loop system with this dynamic controller has the following state-space realization:

$$\begin{aligned} \dot{X}(t) &= A_c X(t) + b_c k^* r(t), \quad X(0) = X_0 \\ y(t) &= c_c^\top X(t)\,, \end{aligned}$$

where

$$X = \begin{bmatrix} x \\ \omega_1 \\ \omega_2 \end{bmatrix} \in \mathbb{R}^{n_p+2n-2}, \quad A_c = \begin{bmatrix} A + b\theta_0^* c^\top & b(\theta_1^*)^\top & b(\theta_2^*)^\top \\ g\theta_0^* c^\top & F + g(\theta_1^*)^\top & g(\theta_2^*)^\top \\ gc^\top & 0 & F \end{bmatrix}, \quad b_c = \begin{bmatrix} b \\ g \\ 0 \end{bmatrix}, \quad c_c = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$$

Since the transfer function from $r(t)$ to $y(t)$ is $W_m(s)$, then

$$c_c^\top (s\mathbb{I} - A_c)^{-1} b_c k^* = W_m(s)\,.$$

A non-minimal state-space realization of the reference model is given by

$$\begin{aligned} \dot{X}_m(t) &= A_c X_m(t) + b_c k^* r(t), \quad X_m(0) = X_{m_0} \\ y_m(t) &= c_c^\top X_m(t)\,, \end{aligned}$$

where $X_m \in \mathbb{R}^{n_p+2n-2}$. Letting $e(t) = X(t) - X_m(t)$, and $\tilde{y}(t) = y(t) - y_m(t)$, we have

$$\begin{aligned} \dot{e}(t) &= A_c e(t), \quad e(0) = e_0 \\ \tilde{y}(t) &= c_c^\top e(t)\,, \end{aligned}$$

which we can integrate to find the dependence of the tracking error on initial condition

$$\tilde{y}(t) \quad = \quad c_c^\top \exp(A_c t)\, e_0\,.$$

This proves that we have exponential convergence of the tracking error to zero.

17.4.2 Direct MRAC via output feedback for unknown parameters

Now we limit the reference system to be SPR, and the system to be minimum phase and have relative degree $n^* = n_p - m_p = 1$. The system does not have to be SPR. We can achieve this property for the system by shifting the poles. We notice that now θ^* cannot be calculated exactly, but Lemma 17.3 guarantees its existence. This was always the case in MRAC schemes. Existence of ideal parameters is required, while their knowledge is not needed to derive adaptive laws. So, our ideal controller $u^*(t) = (\theta^*)^\top \omega(t)$ is not implementable. Let's consider now the adaptive version of it

$$u(t) = \theta^\top(t)\omega(t)$$

where $\omega_1(t)$ and $\omega_2(t)$ are defined via the same filters:

$$\begin{aligned} \dot{\omega}_1(t) &= F\omega_1(t) + gu(t) \\ \dot{\omega}_2(t) &= F\omega_2(t) + gy(t)\,. \end{aligned}$$

The augmented open-loop state-space representation will take the form:

$$\begin{aligned} \dot{X}(t) &= A_0 X(t) + b_c u(t), \quad X(0) = X_0 \\ y(t) &= c_c^\top X(t)\,, \\ u(t) &= \theta^\top(t)\omega(t) \end{aligned}$$

where

$$X = \begin{bmatrix} x \\ \omega_1 \\ \omega_2 \end{bmatrix} \in \mathbb{R}^{n_p+2n-2}, \quad A_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & F & 0 \\ gc^\top & 0 & F \end{bmatrix}, \quad b_c = \begin{bmatrix} b \\ g \\ 0 \end{bmatrix}, \quad c_c = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}.$$

As we did for state-feedback case, we can add and subtract the desired input to rewrite the above system as

$$\dot{X}(t) \quad = \quad A_0 X(t) + b_c(\theta^*)^\top \omega(t) + b_c(u(t) - (\theta^*)^\top \omega(t)), \quad X(0) = X_0\,,$$

which can be ultimately written into the form:

$$\begin{aligned}\dot{X}(t) &= A_c X(t) + b_c k^* r(t) + b_c(u(t) - (\theta^*)^\top \omega(t)), \quad X(0) = X_0\,, \\ y(t) &= c_c^\top X(t)\,,\end{aligned}$$

where A_c has been defined earlier. Letting $e(t) = X(t) - X_m(t)$, and $\tilde{y}(t) = y(t) - y_m(t)$, we obtain the tracking error dynamics:

$$\begin{aligned}\dot{e}(t) &= A_c e(t) + b_c(u(t) - (\theta^*)^\top \omega(t)), \quad X(0) = X_0\,, \\ \tilde{y}(t) &= c_c^\top e(t)\,.\end{aligned}$$

Since

$$c_c^\top (s\mathbb{I} - A_c)^{-1} b_c k^* = W_m(s)\,,$$

which was assumed to be SPR, we have

$$\tilde{y}(s) = W_m(s)\frac{1}{k^*}\mathcal{L}\{(u(t) - (\theta^*)^\top \omega(t))\}\,,$$

where $\mathcal{L}\{\cdot\}$ is used to denote the Laplace transform of the symbol. Having defined

$$u(t) = \theta^\top(t)\omega(t)\,,$$

the error dynamics in frequency domain takes the form:

$$\tilde{y}(s) = W_m(s)\frac{1}{k^*}\mathcal{L}\{\tilde{\theta}^\top(t)\omega(t))\}\,.$$

We can now use Lemma 17.2 to write the adaptive laws. The state-space representation of the error dynamics will be:

$$\begin{aligned}\dot{e}(t) &= A_c e(t) + \frac{\bar{b}_c}{k^*}\tilde{\theta}^\top(t)\omega(t), \quad \bar{b}_c \triangleq b_c k^*, \quad X(0) = X_0\,, \\ \tilde{y}(t) &= c_c^\top e(t)\,.\end{aligned}$$

Following Lemma 17.2, the adaptive law is

$$\dot{\tilde{\theta}}(t) = \dot{\theta}(t) = -\gamma \tilde{y}(t)\omega(t)\mathrm{sgn}(k^*)\,,$$

which leads to negative semi-definite derivative of the candidate Lyapunov function

$$V(e, \tilde{\theta}) = \frac{1}{2}e^\top P_c e + \frac{1}{2\gamma|k^*|}\tilde{\theta}^2\,,$$

where P_c solves

$$A_c^\top P_c + P_c A_c = -Q, \quad P_c b_c = c_c\,,$$

since the reference system was SPR. We notice that (A_c, b_c, c_c) **was not the minimal realization** of the reference system, since it has pole-zero cancellations. However, since $c_c^\top(s\mathbb{I} - A_c)^{-1} b_c k^* = W_m(s)$, where $W_m(s)$ represents the original SPR reference system, Kalman-Yakubovich lemma holds also for (A_c, b_c, c_c), despite the fact that computation of $c_c^\top(s\mathbb{I} - A_c)^{-1} b_c$ involves pole-zero cancellations.

Remark 17.6. In this simplest adaptive output feedback scheme we observed the following: to achieve matching we needed to augment the system state by a dynamic controller, which in case of known parameters, can be computed following Lemma 17.3. In case of known parameters, this ideal dynamic controller closes the system loop to define an augmented ideal reference system, which is a non-minimal realization of the original SPR reference system. In case if system parameters are unknown, the ideal dynamic controller is turned into an adaptive one, by replacing the constant parameters by their adaptive estimates. This leads to an error dynamics, in which uncertainties are matched, and depend only upon the states of the controller, the system output and reference input. Since the controller dynamics are introduced by us, the states of this dynamic controller can be used to write adaptive laws. The SPR property of the reference system leads to SPR error dynamics, for which adaptive laws are written using Lemma 17.2. The augmented reference system is unknown.

Remark 17.7. Adaptive output feedback control for non-minimum phase systems with higher relative degree requires more complex control laws that due to time-restrictions we could not cover within one semester. You're advised to read on this in [26].

18 The Theory of Fast and Robust Adaptation

18.1 Mathematical Preliminaries on $\mathcal{L}$ stability

18.1.1 Norms for vectors and functions: $\mathcal{L}_p$ spaces

Let's first recall the definition of **norms for vectors**. A positive number is called a norm for a vector $u \in \mathbb{R}^n$ and is denoted by $||u||$, if it has the following four properties:

1. $||u|| \geq 0$;

2. $||u|| = 0$ if and only if $u \equiv 0$;

3. $||au|| = |a| \; ||u||$, for all $a \in \mathbb{R}$;

4. $||u + v|| \leq ||u|| + ||v||$.

It is straightforward to verify that the following definitions satisfy the four properties mentioned above.

1. ∞**-norm** for a vector $u = [u_1, \cdots, u_m] \in \mathbb{R}^m$ is defined as

$$||u||_\infty = \max_i |u_i|.$$

2. **2-norm** for a vector $u = [u_1, \cdots, u_m] \in \mathbb{R}^m$ is defined as

$$||u||_2 = \sqrt{u^\top u}.$$

3. **1-norm** for a vector $u = [u_1, \cdots, u_m] \in \mathbb{R}^m$ is defined as

$$||u||_1 = \sum_{i=1}^{i=m} |u_i|.$$

4. p**-norm** for a vector $u = [u_1, \cdots, u_m] \in \mathbb{R}^m$ is defined as

$$||u||_p = \Big(\sum_{i=1}^{i=m} |u_i|^p \Big)^{1/p}.$$

Thus, the norm is not defined uniquely: it can be defined in numerous different ways.

All the norms are equivalent, in a sense that if $||u||_p$ and $||u||_q$ are two different norms, then there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1||u||_p \leq ||u||_q \leq c_2||u||_p \,.$$

Next, using the definition of the norms of vectors, we consider the space of piecewise continuous bounded functions $u(t) : [0, +\infty) \to \mathbb{R}^m$, and define **norms for functions**.

1. **$\mathcal{L}_\infty$-norm and $\mathcal{L}_\infty$ space:** For the space of piecewise continuous bounded functions mapping $[0, \infty)$ into $\mathbb{R}^m$, which we denote by $\mathcal{L}_\infty^m$ space, we introduce the **$\mathcal{L}_\infty$-norm** of $u(t)$ as the least upper bound of its vector norm:

$$||u||_{\mathcal{L}_\infty} = \sup_{t\geq 0} ||u(t)|| < \infty.$$

   The vector norm on the right hand side can be any of the vector norms introduced above. For our analysis, when referring to $\mathcal{L}_\infty$-norm, we will use ∞-norm for the vector on the right, which gives

$$||u||_{\mathcal{L}_\infty} = \max_{i=1,\ldots,m} (\sup_{t\geq 0} |u_i(t)|), \tag{292}$$

   where $u_i(t)$ is the i^{th} component of $u(t)$.

2. **$\mathcal{L}_2$-norm and $\mathcal{L}_2$ space:** For the space of piecewise continuous square integrable functions mapping $[0, \infty)$ into $\mathbb{R}^m$, which we denote by $\mathcal{L}_2^m$ space, we introduce the **$\mathcal{L}_2$-norm** of $u(t)$ as:

$$||u||_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^\top(t)u(t)dt} < \infty,$$

   where on the right hand side we have used the 2-norm of the vector. In general, like mentioned above, there is nothing wrong to use any of the vector norms on the right hand side, however, for the $\mathcal{L}_2$-norm it is quite common to use the 2-norm of the vector.

3. **$\mathcal{L}_p$-norm and $\mathcal{L}_p$ space:** In general, the space $\mathcal{L}_p^m$ for $1 \leq p < \infty$ is defined as the set of piecewise continuous functions mapping $[0, \infty)$ into $\mathbb{R}^m$, for which the **$\mathcal{L}_p$-norm** is finite:

$$||u||_{\mathcal{L}_p} = \left(\int_0^\infty ||u(t)||^p dt\right)^{1/p} < \infty.$$

where $||u(t)||$ can be any norm of $u(t)$. It is also important to note that if we had functions mapping $(-\infty, \infty)$ into $\mathbb{R}^m$, then the integration has to be taken over $(-\infty, \infty)$. When p and m are clear from the context, we may drop one or both of them and refer to the space simply as $\mathcal{L}_p$, $\mathcal{L}^m$, or $\mathcal{L}$.

Example 18.1. Consider the piecewise continuous function

$$u(t) = \begin{cases} 1/\sqrt{t}, & 0 < t \leq 1 \\ 0, & t > 1\,. \end{cases}$$

It has finite $\mathcal{L}_1$-norm:

$$||u||_{\mathcal{L}_1} = \int_0^1 \frac{1}{\sqrt{t}} dt = 2\,.$$

Its $\mathcal{L}_\infty$-norm does not exist, since $||u||_{\mathcal{L}_\infty} = \sup_{t \geq 0} |u(t)| = \infty$, and its $\mathcal{L}_2$-norm is unbounded because the integral of $1/t$ is divergent. Thus, $u \in \mathcal{L}_1$, but $u \notin \mathcal{L}_2 \cup \mathcal{L}_\infty$.

Example 18.2. Next, consider the continuous function

$$u(t) = \frac{1}{1+t}$$

It has finite $\mathcal{L}_\infty$-norm and finite $\mathcal{L}_2$-norm:

$$||u||_{\mathcal{L}_\infty} = \sup_{t \geq 0} \left| \frac{1}{1+t} \right| = 1\,, \quad ||u||_{\mathcal{L}_2} = \left(\int_0^\infty \frac{1}{(1+t)^2} dt \right)^{1/2} = 1.$$

Its $\mathcal{L}_1$-norm does not exist, since

$$||u||_{\mathcal{L}_1} = \int_0^\infty \frac{1}{1+t} dt = \lim_{t \to \infty} \ln(1+t) \to \infty.$$

Thus $u \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, but $u \notin \mathcal{L}_1$.

Notice that the requirement for the $\mathcal{L}_p$-norm be finite restricts the class of functions that can belong to the $\mathcal{L}_p$ space. We see that for function norms the principle of equivalence doesn't hold (some of the norms can be finite, while some can be unbounded, i.e., not defined). So as not to be restricted by this, we consider the **extended space** $\mathcal{L}_e^m$, defined as the space of functions

$$\mathcal{L}_e^m = \{u \mid u_\tau(t) \in \mathcal{L}^m, \;\; \forall\, \tau \in [0, \infty)\}\,,$$

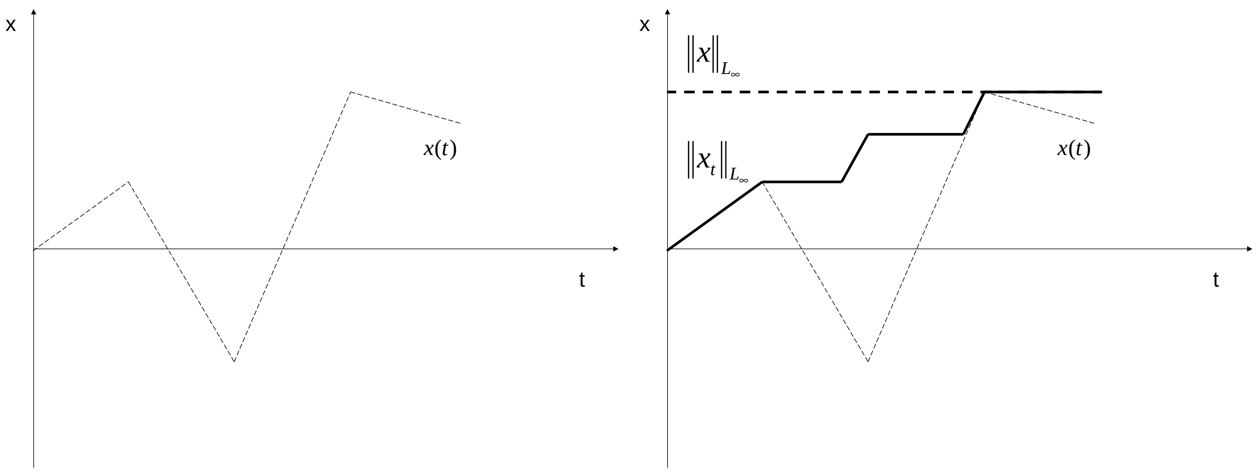


Fig. 28 Comparison between $\mathcal{L}_\infty$ and $\mathcal{L}_{\infty e}$ norms

where $u_\tau(t)$ is the truncation of function $u(t)$ defined by

$$u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}.$$

The **truncated $\mathcal{L}$-norm** of $u(t)$ is defined by the corresponding $\mathcal{L}$ norm of $u_\tau(t)$, i.e., $||u_\tau||_{\mathcal{L}}$.

Thus, every function that does not have finite escape time belongs to the extended space $\mathcal{L}_e$. The extended space $\mathcal{L}_e$ is a larger space that contains the unextended space $\mathcal{L}_p$ as its subset. These notions can be defined and generalized for any $\mathcal{L}_p$ norm. Therefore the index p has be dropped.

Getting back to examples 18.1 and 18.2, we can say that $\frac{1}{\sqrt{t}} \in \mathcal{L}_{1e}$, but $\frac{1}{\sqrt{t}} \notin \mathcal{L}_{2e} \cup \mathcal{L}_{\infty e}$; $\frac{1}{1+t} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and also $\frac{1}{1+t} \in \mathcal{L}_{1e}$.

Fig. 28 plots a signal $x(t)$ and its $\mathcal{L}_\infty$ and $\mathcal{L}_{\infty e}$ norms.

Property 18.1. If $||u||_{\mathcal{L}_1} < \infty$ and $||u||_{\mathcal{L}_\infty} < \infty$, then $||u||^2_{\mathcal{L}_2} \le ||u||_{\mathcal{L}_\infty} ||u||_{\mathcal{L}_1} < \infty$.

The proof is straightforward.

$$||u||^2_{\mathcal{L}_2} = \int_0^\infty u^\top(t)u(t)dt \le \int_0^\infty ||u(t)||\ ||u(t)||dt \le \int_0^\infty \left(\sup_{t\ge 0} ||u(t)||\right) ||u(t)||dt$$
$$= ||u||_{\mathcal{L}_\infty} \int_0^\infty ||u(t)||dt = ||u||_{\mathcal{L}_\infty} ||u||_{\mathcal{L}_1} < \infty\,.$$

Property 18.2. (**Hölder's Inequality**) If $f \in \mathcal{L}_{pe}$ and $g \in \mathcal{L}_{qe}$, where $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\tau |f(t)g(t)|dt \le \left(\int_0^\tau |f(t)|^p dt\right)^{1/p} \left(\int_0^\tau |g(t)|^q dt\right)^{1/q} \tag{293}$$

for every $\tau \in [0, \infty)$.

When $p = q = 2$, the inequality becomes the *Schwartz inequality*, i.e.,

$$\int_0^\tau |f(t)g(t)|dt \le \left(\int_0^\tau |f(t)|^2 dt\right)^{1/2} \left(\int_0^\tau |g(t)|^2 dt\right)^{1/2} \tag{294}$$

We see that Schwartz inequality can be expressed as $\|f(t)g(t)\|_{\mathcal{L}_1} \le \|f(t)\|_{\mathcal{L}_2}\|g(t)\|_{\mathcal{L}_2}$. The proof of Hölder's inequality can be found in any standard book on real analysis such as [8].

Proof: ([9]) Let A and B be the two factors on the right of (293), i.e., $A = \left(\int_0^\tau |f(t)|^p dt\right)^{1/p}$ and $B = \left(\int_0^\tau |g(t)|^q dt\right)^{1/q}$. The cases that A or B is 0 or ∞ are trivial. So we just need to consider the case $0 < A < \infty$ and $0 < B < \infty$. Let

$$F(t) = \frac{|f(t)|}{A}, \quad G(t) = \frac{|g(t)|}{B}.$$

This gives

$$\int_0^\tau F^p(t)dt = \int_0^\tau G^q(t)dt = 1.$$

Now we need to take advantage of the convexity of e.g. exponential function as an intermediate step. Notice that for every $t \in (0, \tau)$ we have $0 < F(t) < \infty$ and $0 < G(t) < \infty$, and therefore there exist real numbers r and s such that $F(t) = \mathrm{e}^{r/p}$, $G(t) = \mathrm{e}^{s/q}$ (of course, r and s are different for every t). Since $1/p + 1/q = 1$, the convexity of the exponential function implies that

$$\mathrm{e}^{r/p+s/q} \le \frac{\mathrm{e}^r}{p} + \frac{\mathrm{e}^s}{q}.$$

It follows that for every $t \in (0, \tau)$

$$F(t)G(t) \le \frac{F^p(t)}{p} + \frac{G^q(t)}{q}.$$

Integration of the above equality yields

$$\int_0^\tau F(t)G(t)dt = \frac{1}{AB}\int_0^\tau |f(t)||g(t)|dt \le \frac{1}{p} + \frac{1}{q} = 1.$$

Thus we have

$$\int_0^\tau |f(t)g(t)|dt \le \int_0^\tau |f(t)||g(t)|dt \le \left(\int_0^\tau |f(t)|^p dt\right)^{1/p} \left(\int_0^\tau |g(t)|^q dt\right)^{1/q}$$

□

18.1.2 Norms for matrices

For a matrix A the norm can be defined in two different ways.

- The matrix $A_{m\times n}$ can be viewed as an $m\times n$ dimensional vector, for which a p-norm can be defined similar to vectors as
$$||A||_p = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\Big)^{1/p}$$
This is usually called vector-norm. For $p=2$, this is called Frobenius norm and for a matrix A of real entries it is given by
$$||A||_2^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2 = \mathrm{tr}(A^\top A)$$
To show the second equality, let $A=[A_1\ A_2\ \ldots A_n]$ where $A_i\in\mathbb{R}^m$, $i=1,2,\ldots,n$, are column vectors. Then
$$A^\top A = \begin{bmatrix} A_1^\top \\ \vdots \\ A_n^\top \end{bmatrix}\begin{bmatrix} A_1 & \ldots & A_n \end{bmatrix} = \begin{pmatrix} A_1^\top A_1 & \ldots & A_1^\top A_n \\ \vdots & \ddots & \vdots \\ A_n^\top A_1 & \ldots & A_n^\top A_n \end{pmatrix}.$$
By definition we have
$$\mathrm{tr}(A^\top A) = \sum_{j=1}^{n} A_j^\top A_j = \sum_{j=1}^{n}\sum_{i=1}^{m}|a_{ij}|^2$$

- The matrix $A_{m\times n}$ can be viewed as an operator that maps $\mathbb{R}^n$, the space of n-dimensional vectors, into $\mathbb{R}^m$, the space of m-dimensional vectors. The operator norm or the induced p-norm of a matrix is defined as
$$||A||_p = \sup_{x\neq 0}\frac{||Ax||_p}{||x||_p} = \sup_{||x||_p=1}||Ax||_p\,.$$
The proof of the last equality is straightforward. Indeed, if $x\neq 0$, then
$$\frac{||Ax||_p}{||x||_p} = \frac{\frac{||Ax||_p}{||x||_p}}{\frac{||x||_p}{||x||_p}} = \frac{||A\frac{x}{||x||_p}||_p}{1}\,.$$
Taking the sup of both sides proves the last equality above. The three most popular induced norms are
$$||A||_1 = \max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}|\ (\text{column sum}),\quad ||A||_2 = \sqrt{\lambda_{\max}A^\top A},\quad ||A||_\infty = \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|\ (\text{row sum}),$$

Induced norms are important as they define properties of a map or a system from input space to output space. We can view $y = Ax$ as a relationship between input vector x and output vector y. Similarly, we view $y(s) = G(s)x(s)$ as an input-output relationship defined by the transfer function $G(s)$.

18.1.3 Norm of transfer function or induced norm of a system

Consider a linear system

$$Y(s) = G(s)U(s)\,, \tag{295}$$

where $G(s)$ has impulse response $g(t)$ so that the linear system can be equivalently presented via a convolution integral:

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau\,. \tag{296}$$

To show that (295) and (296) are equivalent, we introduce unit step function $1(t-\tau)$ where it is zero for $t < \tau$. Then we have

$$\begin{aligned}
\mathfrak{L}\left[\int_0^t g(\tau)u(t-\tau)d\tau\right] &= \mathfrak{L}\left[\int_0^\infty g(\tau)u(t-\tau)1(t-\tau)d\tau\right] \\
&= \int_0^\infty e^{-st}\left[\int_0^\infty g(\tau)u(t-\tau)1(t-\tau)d\tau\right]dt \\
&= \int_0^\infty g(\tau)d\tau\int_0^\infty u(t-\tau)1(t-\tau)e^{-st}dt\,,
\end{aligned}$$

where $\mathfrak{L}$ is used for Laplace transform. Changing the order of integration is valid here since $g(t)$ and $u(t)$ are both Laplace transformable, giving convergent integrals. If we substitute $\lambda = t - \tau$ into this last equation, the result is

$$\begin{aligned}
\mathfrak{L}\left[\int_0^t g(\tau)u(t-\tau)d\tau\right] &= \int_0^\infty g(\tau)e^{-s\tau}d\tau\int_0^\infty u(\lambda)e^{-s\lambda}d\lambda \\
&= G(s)U(s)
\end{aligned}$$

Definition 18.1. We say that the linear system in (295) or (296) is $\mathcal{L}_p$ stable if $u \in \mathcal{L}_p$ implies that $y \in \mathcal{L}_p$ and there exists some constant $c \geq 0$ such that $||y||_{\mathcal{L}_p} \leq c||u||_{\mathcal{L}_p}$ for any $u \in \mathcal{L}_p$. When $p = \infty$, then $\mathcal{L}_\infty$ stability is referred to as BIBO (bounded-input bounded-output) stability.

Theorem 18.1. (Parseval's Theorem [30]). For a causal signal $y \in \mathcal{L}_2$,

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2 d\omega\right)^{\frac{1}{2}} = \left(\int_0^{\infty}|g(t)|^2 dt\right)^{\frac{1}{2}}. \tag{297}$$

where $G(j\omega)$ is the Fourier transform of $g(t)$ ($g(t)$ is the impulse response of $G(s)$).

We now define three $\mathcal{L}_p$-norms of transfer functions and explain their meanings in "induced norm" sense.

1. **$\mathcal{L}_1$-norm of a transfer function** (or $\mathcal{L}_\infty/\mathcal{L}_\infty$-induced norm of the system) is given by:

$$||G(s)||_{\mathcal{L}_1} \triangleq ||g||_{\mathcal{L}_1} = \int_0^{\infty}|g(t)|dt\,,$$

   where $g(t)$ is the impulse response of $G(s)$.

2. **$\mathcal{L}_\infty$-norm of a transfer function** (or $\mathcal{L}_2/\mathcal{L}_2$-induced norm of the system) is given by:

$$||G(s)||_{\mathcal{L}_\infty} = \sup_{\omega}|G(j\omega)|$$

3. **$\mathcal{L}_2$-norm of the transfer function** (or $\mathcal{L}_2/\mathcal{L}_\infty$-induced norm of the system) is given by:

$$||G(s)||_{\mathcal{L}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2 d\omega\right)^{1/2}$$

Note that Parseval's Theorem (297) immediately implies that $\|G(s)\|_{\mathcal{L}_2} = \|g\|_{\mathcal{L}_2}$.

We now reveal the meaning of the $\mathcal{L}_1$-norm in "induced norm" sense. Suppose that the inputs u are signals of $\mathcal{L}_\infty$-norm ≤ 1. We will find the **least upper bound** on the $\mathcal{L}_\infty$ norm of the output y, that is, $\sup_{t\geq 0}\{\|y(t)\|_{\mathcal{L}_\infty} : \|u(t)\|_{\mathcal{L}_\infty} \leq 1\}$. We call this least upper bound the $\mathcal{L}_\infty/\mathcal{L}_\infty$ induced system norm, which will be exactly the $\mathcal{L}_1$-norm of the transfer function.

First, we show that $\|G(s)\|_{\mathcal{L}_1}$ is an upper bound on the $\mathcal{L}_\infty/\mathcal{L}_\infty$ system norm:

$$\begin{aligned}
|y(t)| &= \left|\int_0^\infty g(\tau)u(t-\tau)d\tau\right| \\
&\leq \int_0^\infty |g(\tau)u(t-\tau)|d\tau \\
&\leq \int_0^\infty |g(\tau)||u(t-\tau)|d\tau \\
&\leq \int_0^\infty |g(\tau)|\|u(t)\|_{\mathcal{L}_\infty} d\tau \\
&= \|u(t)\|_{\mathcal{L}_\infty} \int_0^\infty |g(\tau)|d\tau \\
&= \|G(s)\|_{\mathcal{L}_1}\|u(t)\|_{\mathcal{L}_\infty} \\
&\leq \|G(s)\|_{\mathcal{L}_1}, \quad \text{when} \quad \|u(t)\|_{\mathcal{L}_\infty} \leq 1. \qquad (298)
\end{aligned}$$

Then we show that $\|G(s)\|_{\mathcal{L}_1}$ is the least upper bound. Fix t and set $u(t-\tau) = \text{sgn}(g(\tau)),\ \forall\ \tau$. Then $\|u(t)\|_{\mathcal{L}_\infty} = 1$. We have

$$\begin{aligned}
y(t) &= \int_0^\infty g(\tau)u(t-\tau)d\tau \\
&= \int_0^\infty |g(\tau)|d\tau = \|G(s)\|_{\mathcal{L}_1} \qquad (299)
\end{aligned}$$

Thus $\sup\{\|y(t)\|_{\mathcal{L}_\infty} : \|u(t)\|_{\mathcal{L}_\infty} \leq 1\} \geq \|G(s)\|_{\mathcal{L}_1}$. Notice that from (298) it can be seen that $\sup\{\|y(t)\|_{\mathcal{L}_\infty} : \|u(t)\|_{\mathcal{L}_\infty} \leq 1\} \leq \|G(s)\|_{\mathcal{L}_1}$. So $\|G(s)\|_{\mathcal{L}_1}$ is the least upper bound on the $\mathcal{L}_\infty$ norm of the output.

For different input/output spaces, there are other ways of defining induced norms similar to the $\mathcal{L}_1$-norm we just defined. Suppose inputs u are signals of $\mathcal{L}_2$-norm ≤ 1. We will find the least upper bound on the $\mathcal{L}_2$-norm of the output y, that is, $\sup_{t\geq 0}\{\|y(t)\|_{\mathcal{L}_2} : \|u(t)\|_{\mathcal{L}_2} \leq 1\}$. We call this least upper bound the $\mathcal{L}_2/\mathcal{L}_2$ induced system norm.

Indeed, let's compute the $\mathcal{L}_2/\mathcal{L}_2$-induced norm. First we show that $\|G(s)\|_{\mathcal{L}_\infty}$ is an upper bound on the $\mathcal{L}_2$ norm of the output. Using Parseval's theorem we can upper bound

$$
\begin{aligned}
\|y(t)\|_{\mathcal{L}_2}^2 &= \|Y(s)\|_{\mathcal{L}_2}^2 = \|G(s)U(s)\|_{\mathcal{L}_2}^2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 |U(j\omega)|^2 d\omega \\
&\leq \|G(s)\|_{\mathcal{L}_\infty}^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(j\omega)|^2 d\omega \\
&= \|G(s)\|_{\mathcal{L}_\infty}^2 \|U(s)\|_{\mathcal{L}_2}^2 \\
&= \|G(s)\|_{\mathcal{L}_\infty}^2 \|u(t)\|_{\mathcal{L}_2}^2 \\
&\leq \|G(s)\|_{\mathcal{L}_\infty}^2, \quad \text{when} \quad \|u(t)\|_{\mathcal{L}_2} \leq 1
\end{aligned} \tag{300}
$$

Then we show it is the least upper bound. Choose a frequency ω_0 where $|G(j\omega)|$ is maximum, that is, $|G(j\omega_0)| = \|G(s)\|_{\mathcal{L}_\infty}$. Next choose the input u such that

$$
|U(j\omega)| = \begin{cases} c, & \text{if } |\omega - \omega_0| < \epsilon \text{ and } |\omega + \omega_0| < \epsilon \\ 0, & \text{otherwise} \end{cases}
$$

where ϵ is a small enough positive number and c is chosen so that u has unit $\mathcal{L}_2$ norm. Taking $c = \sqrt{\frac{\pi}{2\epsilon}}$ it can be seen that

$$
\begin{aligned}
\|u\|_{\mathcal{L}_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(j\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi}{2\epsilon} d\omega \\
&= \frac{1}{2\pi}\left(\frac{\pi}{2\epsilon}\int_{-\omega_0-\epsilon}^{-\omega_0+\epsilon} d\omega + \frac{\pi}{2\epsilon}\int_{\omega_0-\epsilon}^{\omega_0+\epsilon} d\omega\right) \\
&= \frac{1}{2\pi}\left(2 \cdot \frac{\pi}{2\epsilon} \cdot (2\epsilon) d\omega\right) \\
&= 1
\end{aligned}
$$

Then

$$
\begin{aligned}
\|Y(s)\|_{\mathcal{L}_2}^2 &= \frac{1}{2\pi}\left(\int_{-\omega_0-\epsilon}^{-\omega_0+\epsilon} |G(j\omega)\sqrt{\pi/2\epsilon}|^2 d\omega + \int_{\omega_0-\epsilon}^{\omega_0+\epsilon} |G(j\omega)\sqrt{\pi/2\epsilon}|^2 d\omega\right) \\
&\approx \frac{1}{2\pi}\left[|G(-j\omega_0)|^2\pi + |G(j\omega_0)|^2\pi\right] \\
&= |G(j\omega_0)|^2 \\
&= \|G(s)\|_{\mathcal{L}_\infty}^2.
\end{aligned}
$$

Then $\|G(s)\|_{\mathcal{L}_\infty}$ is the least upper bound on the $\mathcal{L}_2$-norm of the output.

Following the same steps, we show that the $\mathcal{L}_2$-norm of the transfer function is the induced norm of the map from $\mathcal{L}_2$ to $\mathcal{L}_\infty$. This is an application of the Schwarz inequality (294):

$$\begin{aligned} |y(t)| &= \left| \int_0^\infty g(t-\tau)u(\tau)d\tau \right| \\ &\leq \int_0^\infty |g(t-\tau)u(\tau)|\, d\tau \\ &\leq \left(\int_0^\infty |g(t-\tau)|^2 d\tau \right)^{1/2} \left(\int_0^\infty |u(\tau)|^2 d\tau \right)^{1/2} \\ &= \|G(s)\|_{\mathcal{L}_2} \|u(t)\|_{\mathcal{L}_2} \\ &\leq \|G(s)\|_{\mathcal{L}_2}, \quad \text{when} \quad \|u(t)\|_{\mathcal{L}_2} \leq 1 \end{aligned} \tag{301}$$

To show that $\|G(s)\|_{\mathcal{L}_2}$ is the least upper bound, apply the input $u(t) = g(-t)/\|G(s)\|_{\mathcal{L}_2}$, then

$$\|u(t)\|^2_{\mathcal{L}_2} = \int_0^\infty \frac{|g(-t)|^2}{\|G(s)\|^2_{\mathcal{L}_2}} dt = \frac{\|G(s)\|^2_{\mathcal{L}_2}}{\|G(s)\|^2_{\mathcal{L}_2}} = 1$$

$$|y(0)| = \left| \int_0^\infty \frac{g(-\tau)g(-\tau)}{\|G(s)\|_{\mathcal{L}_2}} d\tau \right| = \left| \frac{\|G(s)\|^2_{\mathcal{L}_2}}{\|G(s)\|_{\mathcal{L}_2}} \right| = \|G(s)\|_{\mathcal{L}_2}$$

Thus, $\|y(t)\|_{\mathcal{L}_\infty} \geq \|G(s)\|_{\mathcal{L}_2}$. Also from (301), $\|y(t)\|_{\mathcal{L}_\infty} \leq \|G(s)\|_{\mathcal{L}_2}$. We can see that $\|G(s)\|_{\mathcal{L}_2}$ is the $\mathcal{L}_2/\mathcal{L}_\infty$ induced system norm.

Next we show the bound on the $\mathcal{L}_2$-norm of the output when the input signal is in $\mathcal{L}_\infty$ space. Consider a sinusoidal input $u = \sin(\omega t)$ of unit amplitude and frequency ω such that $j\omega$ is not a zero of $G(s)$. Then $\|u(t)\|_{\mathcal{L}_\infty} = 1$. With this input the output is $Y(j\omega) = |G(j\omega)| \sin[\omega t + \angle G(j\omega)]$. Then

$$\|y(t)\|^2_{\mathcal{L}_2} = \frac{1}{2\pi} \int_{-\infty}^\infty |G(j\omega)|^2 |\sin[\omega t + \angle G(j\omega)]|^2 d\omega = \infty.$$

Hence, generally speaking, if the input is any signal in $\mathcal{L}_\infty$ space, we cannot find a finite bound on the $\mathcal{L}_2$-norm of the output.

18.2 Connection between Lyapunov stability and $\mathcal{L}$-stability

Next, it is of interest to us build a connection between $\mathcal{L}$-stability and Lyapunov stability theory.

Definition 18.2. A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable, if there exist a class $\mathcal{K}$ function α, defined on $[0, \infty)$, and a nonnegative constant β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \alpha(\|u_\tau\|_{\mathcal{L}}) + \beta \tag{302}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$. It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \gamma\|u_\tau\|_{\mathcal{L}} + \beta \tag{303}$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$. Here $\|\cdot\|_{\mathcal{L}}$ is any kind of the $\mathcal{L}$-norm we have defined.

The constant β in (302) or (303) is called bias. When inequality (303) is satisfied, we are usually interested in the smallest possible γ for which there is β such that (303) is satisfied. As we have shown in section 18.1.3, for linear systems the smallest γ for $\mathcal{L}_\infty$ stability is the $\mathcal{L}_1$-norm of the transfer function $H(s)$ (or the $\mathcal{L}_\infty/\mathcal{L}_\infty$ induced norm of the system), and $\beta = 0$ in that case.

Remark 18.1. In [11], the smallest γ for $\mathcal{L}_p$ stability is called $\mathcal{L}_p$-gain with the same p (p can be ∞). For example, when a linear system satisfies $\|(Hu)_\tau\|_{\mathcal{L}_\infty} \leq \gamma\|u_\tau\|_{\mathcal{L}_\infty} + \beta$, the smallest γ is called $\mathcal{L}_\infty$-gain, which is identical to the above mentioned induced $\mathcal{L}_\infty/\mathcal{L}_\infty$ norm of the system or $\mathcal{L}_1$-norm of the transfer function.

Theorem 18.2. Consider the time-invariant nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + G(x)u, \qquad x(0) = x_0 \\ y &= h(x) \end{aligned} \tag{304}$$

where $f(x)$ is locally Lipschitz, and $G(x)$, $h(x)$ are continuous over $\mathbb{R}^n$. The matrix $G(x)$ is $n \times m$ and $h : \mathbb{R}^n \to \mathbb{R}^q$. The function f and h vanish at the origin; that is, $f(0) = 0$ and $h(0) = 0$. Let γ be a positive number and suppose there exists a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the inequality

$$\mathcal{H}(V, f, G, h, \gamma) \triangleq \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} G(x)G^\top(x)\left(\frac{\partial V}{\partial x}\right)^\top + \frac{1}{2}h^\top(x)h(x) \leq 0 \tag{305}$$

for all $x \in \mathbb{R}^n$. Then, for each $x_0 \in \mathbb{R}^n$, the system (304) is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$-gain is less than or equal to γ. $\diamond$

Proof: For any $\gamma \neq 0$, we have the algebraic relationship

$$\begin{aligned}\frac{\partial V}{\partial x}f(x) + \frac{\partial V}{\partial x}G(x)u &= -\frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2}G^\top(x)\left(\frac{\partial V}{\partial x}\right)^\top \right\|_2^2 + \frac{\partial V}{\partial x}f(x) \\ &\quad + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}G(x)G^\top(x)\left(\frac{\partial V}{\partial x}\right)^\top + \frac{1}{2}\gamma^2\|u\|_2^2 \end{aligned} \tag{306}$$

Substituting (305) yields

$$\frac{\partial V}{\partial x}f(x) + \frac{\partial V}{\partial x}G(x)u \;\leq\; -\frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2}G^\top(x)\left(\frac{\partial V}{\partial x}\right)^\top \right\|_2^2 + \frac{1}{2}\gamma^2\|u\|_2^2 - \frac{1}{2}\|y\|_2^2 \tag{307}$$

Hence,

$$\frac{\partial V}{\partial x}f(x) + \frac{\partial V}{\partial x}G(x)u \leq \frac{1}{2}\gamma^2\|u\|_2^2 - \frac{1}{2}\|y\|_2^2$$

Notice that left hand side of above inequality is the derivative of V along the trajectories of the system (304). Integrating yields

$$V(x(\tau)) - V(x_0) \leq \frac{1}{2}\gamma^2 \int_0^\tau \|u(t)\|_2^2 dt - \frac{1}{2}\int_0^\tau \|y(t)\|_2^2 dt$$

where $x(t)$ is the solution of (304) for a given $u \in \mathcal{L}_{2e}$. Using $V(x) \geq 0$ we obtain

$$\int_0^\tau \|y(t)\|_2^2 dt \leq \gamma^2 \int_0^\tau \|u(t)\|_2^2 dt + 2V(x_0)$$

Taking the square roots and using the inequality $\sqrt{a^2+b^2} \leq a + b$ for nonnegative numbers a and b, we obtain

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma\|u_\tau\|_{\mathcal{L}_2} + \sqrt{2V(x_0)}$$

which completes the proof. □

Inequality (305) is known as the *Hamilton-Jacobi inequality.* Let's look at an example now ([11], page 212).

Example 18.3. Consider the SISO system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1^3 - kx_2 + u \\ y &= x_2\end{aligned} \tag{308}$$

where a and k are positive constants. We can choose $V(x) = \alpha(ax_1^4/4 + x_2^2/2)$ with $\alpha > 0$ as a candidate for the solution of the *Hamilton-Jacobi inequality*. It can be shown that

$$\mathcal{H}(V, f, G, h, \gamma) = \left(-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2}\right) x_2^2$$

To satisfy (305) we need to choose $\alpha > 0$ and $\gamma > 0$ such that $-\alpha k + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2} \leq 0$. We rewrite this equality as

$$\gamma^2 \geq \frac{\alpha^2}{2\alpha k - 1}$$

Since we are interested in the smallest possible γ, we choose α to minimize the right-hand side of the preceding inequality. The minimum value $1/k^2$ is achieved at $\alpha = 1/k$. Thus, choosing $\gamma = 1/k$ we conclude that the system is finite-gain $\mathcal{L}_2$ stable and the $\mathcal{L}_2$-gain is less than or equal to $1/k$.

From this example we can see how to obtain $\mathcal{L}_2$-stability results by looking for a Lyapunov function, which can satisfy the inequality (305).

18.3 Small-gain Theorem

Let us consider in particular the $\mathcal{L}_\infty$ stability of interconnected feedback system. For that we need to recall the definitions of $\mathcal{L}_\infty$ norm of signals from (292). For a piecewise continuous signal $\xi(t), t \geq 0, \xi(t) \in \mathbb{R}^n$, its truncated $\mathcal{L}_\infty$ norm and $\mathcal{L}_\infty$ norm are

$$\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,..,n} \big(\sup_{0 \leq \tau \leq t} |\xi_i(\tau)| \big) \tag{309}$$

$$\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,..,n} \big(\sup_{\tau \geq 0} |\xi_i(\tau)| \big), \tag{310}$$

where ξ_i is the i^{th} component of ξ. We notice that here the $||\cdot||_\infty$ norm is used for the vectors on the right, as was mentioned in (292).

Proposition 18.1. A continuous time LTI (proper) system $H(s)$ with impulse response $h(t)$ is BIBO stable if and only if its $\mathcal{L}_1$-norm is bounded, i.e. $||H(s)||_{\mathcal{L}_1} = ||h||_{\mathcal{L}_1} = \int_0^\infty |h(\tau)| d\tau < \infty$, or otherwise saying $h \in \mathcal{L}_1$.

In fact, recalling Definition 18.1 (a system is BIBO stable if $u \in \mathcal{L}_\infty$ implies that $y \in \mathcal{L}_\infty$), and the definition of $\mathcal{L}_1$-norm of a transfer function in terms of the $\mathcal{L}_\infty/\mathcal{L}_\infty$ induced norm of the system, the proof is straightforward.

Proof: *Sufficiency.* Assume that $\int_0^\infty |h(\tau)|d\tau \le k < \infty$,

$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau.$$

Let the input be bounded, i.e. let $|u(t-\tau)| \le M < \infty$, then $y(t)$ is bounded:

$$|y(t)| \le M \int_0^t |h(\tau)|d\tau \le Mk < \infty.$$

Necessity. To show necessity, we will prove that if the $\mathcal{L}_1$-norm of $h(t)$ is not bounded, i.e. if $\lim\limits_{t\to\infty} \int_0^t |h(\tau)|d\tau = \infty$, then there exists at least one bounded input that will force the input $y(t)$ to diverge. Fix t and set

$$u(t) = \begin{cases} +1, & \text{if } \ h(\tau) \ge 0 \\ -1, & \text{if } \ h(\tau) < 0. \end{cases}$$

Then $h(\tau)u(t-\tau) = |h(\tau)|$, which implies that

$$y(t) = \int_0^t |h(\tau)|d\tau.$$

Thus, while $u(t-\tau)$ is bounded, $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} \int_0^t |h(\tau)|d\tau = \infty$ by assumption. This implies that $y(t)$ is not bounded for all bounded inputs, and therefore the system is not BIBO stable. $\square$

Remark 18.2. Next, let's assume that $h \in \mathcal{L}_{1e}$, i.e. it is bounded for every $t < \infty$, but is not bounded uniformly:

$$||h_\tau||_{\mathcal{L}_1} = \int_0^\infty |h_\tau(\sigma)d\sigma = \int_0^\tau |h(\sigma)d\sigma < \infty\,.$$

For any $\tau \ge t$, if $u \in \mathcal{L}_{\infty e}$, a straightforward upper bounding gives the following result:

$$|y(t)| \le \int_0^t |h(t-\sigma)||u(\sigma)|d\sigma \le \sup_{0\le\sigma\le\tau} |u(\sigma)| \int_0^t |h(t-\sigma)d\sigma = \sup_{0\le\sigma\le\tau} |u(\sigma)| \int_0^t |h(s)|ds < \infty\,.$$

Consequently

$$||y_\tau||_{\mathcal{L}_\infty} \le ||h_\tau||_{\mathcal{L}_1} ||u_\tau||_{\mathcal{L}_\infty}, \quad \forall\, \tau \in [0,\infty)\,.$$

We notice that if the system $H(s)$ is stable, i.e. its $\mathcal{L}_1$-norm is uniformly bounded, then we get

$$||y_\tau||_{\mathcal{L}_\infty} \le ||H(s)||_{\mathcal{L}_1} ||u_\tau||_{\mathcal{L}_\infty}\,,$$

which can be viewed as an extension of Proposition 18.1 for a broader class of signals $u \in \mathcal{L}_{\infty e}$.

Definition 18.3. For a stable proper m input n output system $H(s)$ its $\mathcal{L}_1$-norm is defined as

$$\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,..,n} \left(\sum_{j=1}^{m} \|H_{ij}(s)\|_{\mathcal{L}_1} \right), \tag{311}$$

where $H_{ij}(s)$ is the i^{th} row j^{th} column element of $H(s)$.

An extension to multi-input multi-output systems is given next.

Lemma 18.1. For a stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have

$$\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty}, \quad \forall\, t \geq 0.$$

Proof. Let $x_i(t)$ be the i^{th} element of $x(t)$, $r_j(t)$ be the j^{th} element of $r(t)$, $H_{ij}(s)$ be the i^{th} row j^{th} element of $H(s)$, and $h_{ij}(t)$ be the impulse response of $H_{ij}(s)$. Then for any $t' \in [0, t]$, we have

$$x_i(t') = \int_0^{t'} \left(\sum_{j=1}^{m} h_{ij}(t' - \tau) r_j(\tau) \right) d\tau. \tag{312}$$

From (312) it follows that

$$\begin{aligned} |x_i(t')| &\leq \int_0^{t'} \left(\sum_{j=1}^{m} |h_{ij}(t'-\tau)||r_j(\tau)| \right) d\tau \leq \int_0^{t'} \left(\sum_{j=1}^{m} |h_{ij}(t'-\tau)| \right) d\tau \left(\max_{j=1,..,m} \sup_{0 \leq \tau \leq t'} |r_j(\tau)| \right) \\ &\leq \sum_{j=1}^{m} \left(\int_0^{t'} |h_{ij}(\tau)| d\tau \right) \left(\max_{j=1,..,m} \sup_{0 \leq \tau \leq t'} |r_j(\tau)| \right), \end{aligned}$$

and hence $\|x_{i_t}\|_{\mathcal{L}_\infty} \leq \left(\sum_{j=1}^{m} \|H_{ij}(s)\|_{\mathcal{L}_1} \right) \|r_t\|_{\mathcal{L}_\infty}$. It follows from (311) that

$$\|x_t\|_{\mathcal{L}_\infty} = \max_{i=1,..,n} \|x_{i_t}\|_{\mathcal{L}_\infty} \leq \max_{i=1,..,n} \left(\sum_{j=1}^{m} \|H_{ij}(s)\|_{\mathcal{L}_1} \right) \|r_t\|_{\mathcal{L}_\infty} = \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty}$$

for any $t \geq 0$. The proof is complete. $\square$

Similar to Proposition 18.1, we have the following result for MIMO systems.

Proposition 18.2. For a stable proper MIMO system $H(s)$, if the input $r(t) \in \mathbb{R}^m$ is bounded, then the output $x(t) \in \mathbb{R}^n$ is also bounded, and $\|x\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}$.

Lemma 18.2. For a cascaded system $H(s) = G_2(s)G_1(s)$, where $G_1(s)$ is a stable proper system with m inputs and l outputs and $G_2(s)$ is a stable proper system with l inputs and n outputs, we have

$$\|H(s)\|_{\mathcal{L}_1} \leq \|G_2(s)\|_{\mathcal{L}_1} \|G_1(s)\|_{\mathcal{L}_1} .$$

Proof. Let $y(t) \in \mathbb{R}^n$ be the output of $H(s) = G_1(s)G_2(s)$ in response to input $r(t) \in \mathbb{R}^m$, i.e. $y(s) = H(s)r(s) = G_2(s)G_1(s)r(s)$. Thus, on one hand from Proposition 18.2 it follows that

$$||y||_{\mathcal{L}_\infty} \leq ||H(s)||_{\mathcal{L}_1} ||r||_{\mathcal{L}_\infty}. \tag{313}$$

On the other hand, letting $\mathtt{r}(s) = G_2(s)r(s)$, and $y(s) = G_1(s)\mathtt{r}(s)$, from application of Proposition 18.2 twice, it follows that for any bounded $r(t)$ we have $||\mathtt{r}||_{\mathcal{L}_\infty} \leq ||G_2(s)||_{\mathcal{L}_1} ||r||_{\mathcal{L}_\infty}$, and

$$||y||_{\mathcal{L}_\infty} \leq \|G_1(s)\|_{\mathcal{L}_1} ||\mathtt{r}||_{\mathcal{L}_\infty} \leq \|G_1(s)\|_{\mathcal{L}_1} \|G_2(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} . \tag{314}$$

We need to prove that the upper bound in (313) is less than the upper bound in (314). Let $H_i(s), i = 1, .., n$ be the i^{th} row of the system $H(s)$. It follows from (311) that there exists i such that

$$\|H(s)\|_{\mathcal{L}_1} = \|H_i(s)\|_{\mathcal{L}_1}. \tag{315}$$

Then

$$y_i(s) = H_i(s)r(s) ,$$

and $||y||_{\mathcal{L}_\infty} = ||y_i||_{\mathcal{L}_\infty}$. Let $h_{ij}(t)$ be the j^{th} element of the impulse response of the system $H_i(s)$. For any T, let

$$r_j(t) = \mathrm{sgn} h_{ij}(T - t), \qquad t \in [0, T] , \ \forall j = 1, .., m. \tag{316}$$

It follows from (310) that $\|r\|_{\mathcal{L}_\infty} = 1$, and hence from (314) we have

$$|y_i(t)| \leq \|y_i\|_{\mathcal{L}_\infty} = \|y\|_{\mathcal{L}_\infty} \leq \|G_1(s)\|_{\mathcal{L}_1} \|G_2(s)\|_{\mathcal{L}_1}, \quad \forall\ t \geq 0. \tag{317}$$

For $r(t)$ satisfying (316), we have

$$y_i(T) = \int_{t=0}^{T} \sum_{j=1}^{m} h_{ij}(T - t) r_j(t) dt = \int_{t=0}^{T} \sum_{j=1}^{m} |h_{ij}(T - t)| dt = \sum_{j=1}^{m} \left(\int_{\sigma=0}^{T} |h_{ij}(\sigma)| d\sigma \right) .$$

Since (317) holds for all $t \geq 0$, we obtain the following bound, true for any T:

$$y_i(T) = \sum_{j=1}^{m} \left(\int_{\sigma=0}^{T} |h_{ij}(\sigma)| d\sigma \right) \leq \|G_1(s)\|_{\mathcal{L}_1} \|G_2(s)\|_{\mathcal{L}_1} .$$

Since the right-hand side of this bound is uniform, as $T \to \infty$, it follows from (315) that

$$\|H(s)\|_{\mathcal{L}_1} = \|H_i(s)\|_{\mathcal{L}_1} = \lim_{T\to\infty} \sum_{j=1}^{m} \left(\int_{t=0}^{T} |h_{ij}(t)| dt \right) \leq \|G_2(s)\|_{\mathcal{L}_1} \|G_1(s)\|_{\mathcal{L}_1} ,$$

and this completes the proof. □

Consider the interconnected LTI system in Fig. 29, where $w_1 \in \mathbb{R}^{n_1}$, $w_2 \in \mathbb{R}^{n_2}$, $M(s)$ is a stable proper system with n_2 inputs and n_1 outputs, and $\Delta(s)$ is a stable proper system with n_1 inputs and n_2 outputs.

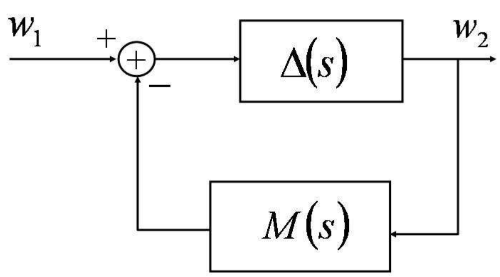


Fig. 29 Interconnected systems

Theorem 18.3. [Theorem 5.6 ([11], page 218)] ($\mathcal{L}_1$ **Small Gain Theorem**) The interconnected system in Fig. 29, which can also be expressed as

$$w_2(s) = \Delta(s) \underbrace{(w_1(s) - M(s)w_2(s))}_{\xi(s)}, \tag{318}$$

is stable if

$$\|M(s)\|_{\mathcal{L}_1} \|\Delta(s)\|_{\mathcal{L}_1} < 1. \tag{319}$$

Proof. Indeed, let $\xi(s) = w_1(s) - M(s)w_2(s)$ and $\xi(t) = \mathfrak{L}^{-1}\{\xi(s)\}$. Then it follows from Lemma 18.1 that

$$\begin{aligned} \|w_{2_t}\|_{\mathcal{L}_\infty} &\leq \|\Delta(s)\|_{\mathcal{L}_1}\|\xi_t\|_{\mathcal{L}_\infty} \\ &\leq \|\Delta(s)\|_{\mathcal{L}_1}\left(\|w_{1_t}\|_{\mathcal{L}_\infty} + \|M(s)\|_{\mathcal{L}_1}\|w_{2_t}\|_{\mathcal{L}_\infty}\right), \quad \forall\, t \geq 0, \end{aligned} \tag{320}$$

which along with (319) leads to

$$\left(1 - \|M(s)\|_{\mathcal{L}_1}\|\Delta(s)\|_{\mathcal{L}_1}\right)\|w_{2_t}\|_{\mathcal{L}_\infty} < \|\Delta(s)\|_{\mathcal{L}_1}\|w_{1_t}\|_{\mathcal{L}_\infty}, \qquad \forall\, t \geq 0. \tag{321}$$

Therefore,

$$\|w_{2_t}\|_{\mathcal{L}_\infty} < \frac{\|\Delta(s)\|_{\mathcal{L}_1}}{1 - \|M(s)\|_{\mathcal{L}_1}\|\Delta(s)\|_{\mathcal{L}_1}}\|w_{1_t}\|_{\mathcal{L}_\infty}, \quad \forall\, t \geq 0\,, \tag{322}$$

and it follows from the condition in (319) that $w_2(t)$ is bounded for all $t \geq 0$, if $w_1(t)$ is bounded. $\square$

Remark 18.3. Notice that (318) can be solved for $w_2(s)$, leading to

$$w_2(s) = \left(\mathbb{I} + \Delta(s)M(s)\right)^{-1}\Delta(s)w_1(s),$$

which implies that $\left(\mathbb{I} + \Delta(s)M(s)\right)^{-1}$ is stable.

Consider a linear time-invariant system:

$$\dot{x}(t) = Ax(t) + bu(t)\,, \qquad x(t_0) = x_0\,, \tag{323}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is Hurwitz, and assume that the transfer function $(s\mathbb{I} - A)^{-1}b$ is strictly proper and stable. Notice that it can be expressed as:

$$(s\mathbb{I} - A)^{-1}b = \frac{n(s)}{d(s)}\,, \tag{324}$$

where $d(s) = \det(s\mathbb{I} - A)$ is a n^{th} order stable polynomial, and $n(s)$ is a $n \times 1$ vector with its i^{th} element being a polynomial function:

$$n_i(s) = \sum_{j=1}^{n} n_{ij}s^{j-1} \tag{325}$$

Lemma 18.3. If $(A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n)$ is controllable, the matrix N with its i^{th} row j^{th} column entry n_{ij} is full rank.

Proof. Controllability of (A, b) for the LTI system in (323) implies that given an initial condition $x(t_0) = 0$ and arbitrary $x_{t_1} \in \mathbb{R}^n$ and arbitrary t_1, there exists $u(\tau), \tau \in [t_0,\ t_1]$, such that $x(t_1) = x_{t_1}$. If N is not full rank, then there exists a non-zero vector $\mu \in \mathbb{R}^n$, such that $\mu^\top n(s) = 0$. Since $x(s) = (s\mathbb{I}-A)^{-1}bu(s)+(s\mathbb{I}-A)^{-1}x(t_0) = \frac{n(s)}{d(s)}u(s)+(s\mathbb{I}-A)^{-1}x(t_0)$, for $x(t_0) = 0$ one has $\mu^\top x(\tau) = 0,\ \forall \tau > t_0$. This contradicts $x(t_1) = x_{t_1}$, in which $x_{t_1} \in \mathbb{R}^n$ is assumed to be an arbitrary point. Therefore, N must be full rank, and the proof is complete. □

Lemma 18.4. If (A, b) is controllable and $(s\mathbb{I} - A)^{-1}b$ is strictly proper and stable, there exists $c \in \mathbb{R}^n$ such that the transfer function $c^\top(s\mathbb{I}-A)^{-1}b$ is minimum phase with relative degree one, i.e. all its zeros are located in the left half plane, and its denominator is one order larger than its numerator.

Proof. It follows from (324) that for arbitrary vector $c \in \mathbb{R}^n$

$$c^\top(s\mathbb{I} - A)^{-1}b = \frac{c^\top N[s^{n-1}\ \cdots\ 1]^\top}{d(s)}, \tag{326}$$

where $N \in \mathbb{R}^{n\times n}$ is the matrix with its i^{th} row j^{th} column entry n_{ij} introduced in (325). Since (A, b) is controllable, it follows from Lemma 18.3 that N is full rank. Consider an arbitrary vector $\bar{c} \in \mathbb{R}^n$ such that $\bar{c}^\top[s^{n-1}\ \cdots\ 1]^\top$ is a stable $n-1$ order polynomial, and let $c = (N^{-1})^\top\bar{c}$. Then $c^\top(s\mathbb{I} - A)^{-1}b = \frac{\bar{c}^\top[s^{n-1}\ \cdots\ 1]^\top}{d(s)}$ has relative degree 1 with all its zeros in the left half plane. □

18.4 Proportional Integral Controller and Its Modification

Until now we have studied adaptive control architectures, for which we only proved asymptotic stability in *"absolute sense"*, i.e. convergence of tracking error to zero as $t \to \infty$, without analyzing the robustness of these schemes or the so-called *"relative stability"*. From linear system's theory we know that robustness is related to stability margins, which we can compute using Bode plots or Nyquist diagrams. Since for nonlinear systems these tools are not readily available, in most of the cases the robustness analysis is being resolved in some type of Monte-Carlo runs. From your homework problems you should have noticed that for determining the *best adaptive gain* there is no universal recipe and for every system it can be different. Intuitively one may suspect that the faster you adapt, the better you should be able to do, but if we examine our direct MRAC architecture for that, we can see that "the better" is not that true. Let's see what's going on. Let's go back to our first adaptive scheme of the proportional integral controller. The system was given by

$$\dot{x}(t) = x(t) + \theta + u(t), \quad x(0) = 0\,, \tag{327}$$

where θ was the unknown constant, and the controller was given by

$$u(t) = -2x(t) - \hat{\theta}(t)\,,$$

with $\hat{\theta}(t)$ being the estimate of θ, governed by

$$\dot{\hat{\theta}}(t) = \dot{\tilde{\theta}}(t) = \gamma x(t)\,, \quad \theta(0) = \theta_0\,, \quad \gamma > 0\,.$$

The resulting closed-loop system was given by

$$\dot{x}(t) = -x(t) - \tilde{\theta}(t), \quad x(0) = 0\,,$$

where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$. Since for the candidate Lyapunov function

$$V(x(t), \tilde{\theta}(t)) = \frac{1}{2}x^2(t) + \frac{1}{2\gamma}\tilde{\theta}^2(t)$$

we proved that $\dot{V}(t) \leq 0$, we can immediately write the following upper bound:

$$\max_{t\in[0,\infty)} ||x(t)|| \leq \sqrt{2V(t)} \leq \sqrt{2V(0)} = \sqrt{x^2(0) + \frac{1}{\gamma}\tilde{\theta}^2(0)} = \sqrt{\frac{1}{\gamma}\tilde{\theta}^2(0)}\,,$$

where we have assumed that $x(0) = 0$ (in the context of tracking problems this is equivalent to initializing the system state and the reference system state at the same position, which is feasible in case of full state feedback control laws). Thus, since our original objective was to drive $x(t)$ to zero, i.e. stabilization, then it is obvious that if we increase γ, then $\max_{t\in[0,\infty)} ||x(t)||$ will be reduced for all $t \geq 0$ *uniformly*, thus ensuring that $x(t)$ remains close to zero during the transient as well.

Since in this case the closed-loop system is linear, we can write the corresponding open-loop transfer function for the phase margin analysis. If we consider the system as a negative feedback closed loop system where θ is a disturbance signal, Fig. 30, the plant transfer function is $P(s) = \frac{1}{s-1}$, and the compensator transfer function is $C(s) = \frac{2s+\gamma}{s}$. We break the loop at the point where the input signal is going to enter the plant, and the open loop transfer function from u to u is just $P(s)C(s)$. Notice that when we consider the input-to-input open loop transfer function, we just take θ as zero. Then

$$H_o(s) = P(s)C(s) = \frac{2s+\gamma}{s(s-1)} \tag{328}$$

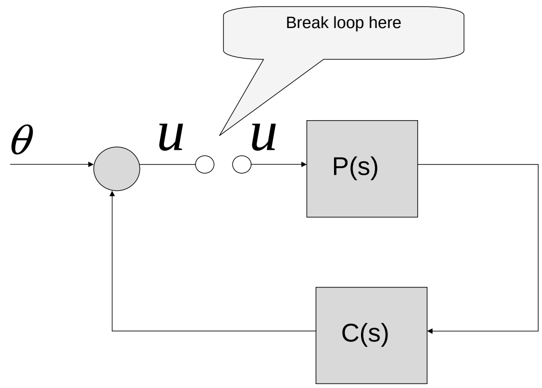


Fig. 30 The open loop system for phase margin analysis

and we can convince ourselves that increasing γ corresponds to increasing the cross-over frequency, which reduces the phase margin. So, if increasing the speed of adaptation, defined by γ, improves the tracking for all $t \geq 0$, including the transient phase, then it is obviously hurting the robustness, or the relative stability. For this simple linear system, it is straightforward to understand this with the help of classical control tools as how to trade off between tracking and robustness, but for nonlinear closed-loop system as it is the case for adaptive stabilization

$$\dot{x}(t) = a_m x(t) - \underbrace{\tilde{k}(t)x(t)}_{nonlinearity}\;, \quad \dot{k}(t) = \dot{\tilde{k}}(t) = \gamma \underbrace{x^2(t)}_{nonlinearity}$$

it is not obvious how to proceed. For nonlinear systems, a related notion will be the time-delay margin, which is defined as the maximum delay τ^* at the plant input, for which the system is not losing its stability. The related open-loop transfer function for the time-delay margin analysis of our above mentioned PI scheme will be

$$H_o(s) = \frac{2s+\gamma}{s(s-1)}\mathrm{e}^{-s\tau}\,, \tag{329}$$

and the time-delay margin will be defined from the solution of the characteristic equation:

$$\frac{2j\omega+\gamma}{j\omega(j\omega-1)}\mathrm{e}^{-j\omega\tau^*} = -1$$

resulting in

$$\tau^*(\gamma) = \frac{\angle H_o(j\omega)}{\omega}\,,$$

where ω is the cross-over frequency, while $\angle H_o(j\omega)$ stands for the phase margin of $H_o(s)$. It is obvious that the time-delay margin and phase margin, both are reduced, as one increases γ.

Thus, one thing is obvious that the basic structure has some deficiency that hurts robustness in the presence of fast adaptation. We know that tracking and robustness cannot be achieved simultaneously, so there is nothing surprising with this, but on the other hand one would like to explore the benefits of fast adaptation, if possible, by modifying the architecture so that the trade-off between tracking and robustness will be resolved differently, as opposed to being affected by γ.

Let's look at a different structure for the same problem of stabilization in the presence of constant unknown parameter, i.e. let's consider a modification of the PI controller in a way so that its robustness won't suffer from fast adaptation. Let's write a predictor (or *passive identifier*) for the system dynamics to derive an error signal for the adaptive law. Given the system

$$\dot{x}(t) = x(t) + \theta + u(t), \quad x(0) = 0\,, \tag{330}$$

where θ was the unknown constant, we will consider the following general structure for the controller

$$u(t) = -2x(t) + u_{ad}(t)\,,$$

where the adaptive signal $u_{ad}(t)$ is yet to be determined. This leads to the following partially closed-loop dynamics:

$$\dot{x}(t) = -x(t) + \theta + u_{ad}(t), \quad x(0) = 0\,. \tag{331}$$

We consider the passive identifier (or state predictor):

$$\dot{\hat{x}}(t) = -\hat{x}(t) + \hat{\theta}(t) + u_{ad}(t), \quad x(0) = 0\,, \tag{332}$$

which mimics the structure of the partially closed-loop system, except only for the unknown parameter being replaced by its estimate, and leads to the following error dynamics for $\tilde{x}(t) = \hat{x}(t) - x(t)$:

$$\dot{\tilde{x}}(t) = -\tilde{x}(t) + \tilde{\theta}(t), \quad \tilde{x}(0) = 0\,, \quad \tilde{\theta}(t) = \hat{\theta}(t) - \theta\,. \tag{333}$$

We notice that if we choose

$$\dot{\hat{\theta}}(t) = \dot{\tilde{\theta}}(t) = -\gamma\tilde{x}(t)\,, \quad \theta(0) = \theta_0\,, \quad \gamma > 0\,,$$

then for the candidate Lyapunov function

$$V(\tilde{x}(t), \tilde{\theta}(t)) = \frac{1}{2}\tilde{x}^2(t) + \frac{1}{2\gamma}\tilde{\theta}^2(t)$$

we can again prove that $\dot{V}(t) \leq 0$, and immediately write the same upper bound:

$$\max_{t\in[0,\infty)} ||\tilde{x}(t)|| \leq \sqrt{2V(t)} \leq \sqrt{2V(0)} = \sqrt{\tilde{x}^2(0) + \frac{1}{\gamma}\tilde{\theta}^2(0)} = \sqrt{\frac{1}{\gamma}\tilde{\theta}^2(0)}\,.$$

However, all of this is done without defining the adaptive control signal signal so far. It has been canceled out while forming the error dynamics (333)!!! So far we have only proved stability, i.e. boundedness of error signals, but we are not sure if our system state or the state of the predictor will remain bounded. Thus, we cannot apply Barbalat's lemma to conclude asymptotic stability (both states of the system and the predictor can drift to infinity with the same rate keeping the error between them bounded). We still need to define the adaptive signal and prove that either the state predictor or the system state remains bounded with it. Stability of the other will follow from boundedness of the tracking error signal. We had similar arguments back in Section 7.4.1.

Let's choose the adaptive signal to be the solution of the following ODE

$$\dot{u}_{ad}(t) = -u_{ad}(t) - \hat{\theta}(t)\,, \quad u_{ad}(0) = 0\,,$$

which can otherwise be written, using frequency domain tools, as an output of a strictly proper stable low-pass filter:

$$u_{ad}(s) = -C(s)\hat{\theta}(s)\,,$$

where

$$C(s) = \frac{1}{1+s}\,.$$

Since $C(s)$ is a strictly proper stable low-pass filter, and $\hat{\theta}(t)$ is bounded, it follows that $u_{ad}(t)$ will be bounded, which implies that both $\hat{x}(t)$ and $x(t)$ will be bounded, and therefore $\tilde{x}(t) \to 0$ as $t \to \infty$.

Thus, the entire closed-loop system is defined by

$$\begin{aligned}
\dot{x}(t) &= -x(t) + \theta + u_{ad}(t), & x(0) &= 0\,. \\
\dot{\hat{x}}(t) &= -\hat{x}(t) + \hat{\theta}(t) + u_{ad}(t), & x(0) &= 0\,, \\
\dot{\hat{\theta}}(t) &= -\gamma\tilde{x}(t)\,, & \gamma &> 0\,, \\
u_{ad}(s) &= -C(s)\hat{\theta}(s)\,, & C(s) &= \tfrac{1}{1+s}\,, \quad u_{ad}(0) = 0\,.
\end{aligned}$$

The open-loop transfer functions for the phase margin analysis and the time-delay margin analysis are:

$$H_{opm}(s) = \frac{2(s^2+s+\gamma)+\gamma(s-1)C(s)}{[s^2+s+(1-C(s))\gamma](s-1)}, \tag{334}$$

$$H_o(s) = \frac{2(s^2+s+\gamma)+\gamma(s-1)C(s)}{[s^2+s+(1-C(s))\gamma](s-1)}e^{-s\tau}. \tag{335}$$

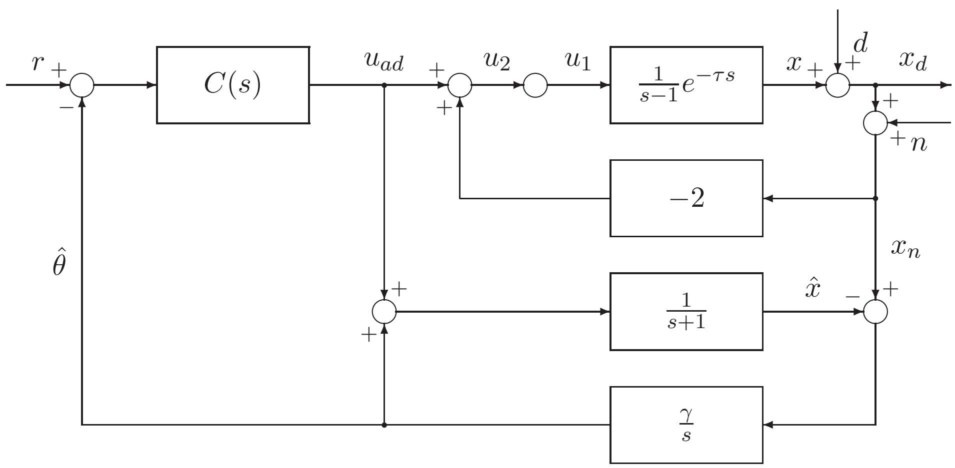


Fig. 31 Time-delay analysis for $\mathcal{L}_1$ controller

Details: Following are some major steps for arriving at the above equation. The diagram of the complete closed-loop system with time-delay is shown in Figure 31, in which we break the loop at the plant input and analyze the transfer function from $u_1(s)$ to $u_2(s)$ with $r = n = d = 0$. First we have:

$$x(s) = \frac{1}{s-1}\left(u_1(s)e^{-s\tau}+\theta\right) \tag{336}$$

$$\hat{x}(s) = \frac{1}{s+1}\Big(2x(s)+u_2(s)+\hat{\theta}(s)\Big) \tag{337}$$

$$\hat{\theta}(s) = -\frac{\gamma}{s}(\hat{x}(s)-x(s)) \tag{338}$$

$$u_2(s) = -2x(s)+C(s)(-\hat{\theta}(s)) \tag{339}$$

Then express $\hat{\theta}(s)$ in terms of $u_1(s)$, $u_2(s)$ and θ:

$$\hat{\theta}(s) = \frac{\gamma}{s^2+s+\gamma}\left(e^{-s\tau}u_1(s)-u_2(s)+\theta\right) \tag{340}$$

Plug (340) and (336) into (339) and let $\theta = 0$, then we have

$$u_2(s) = \frac{\gamma C(s)}{s^2+s+\gamma}\left(e^{-s\tau}u_1(s)-u_2(s)\right)+\frac{-2}{s-1}u_1(s)e^{-s\tau}$$

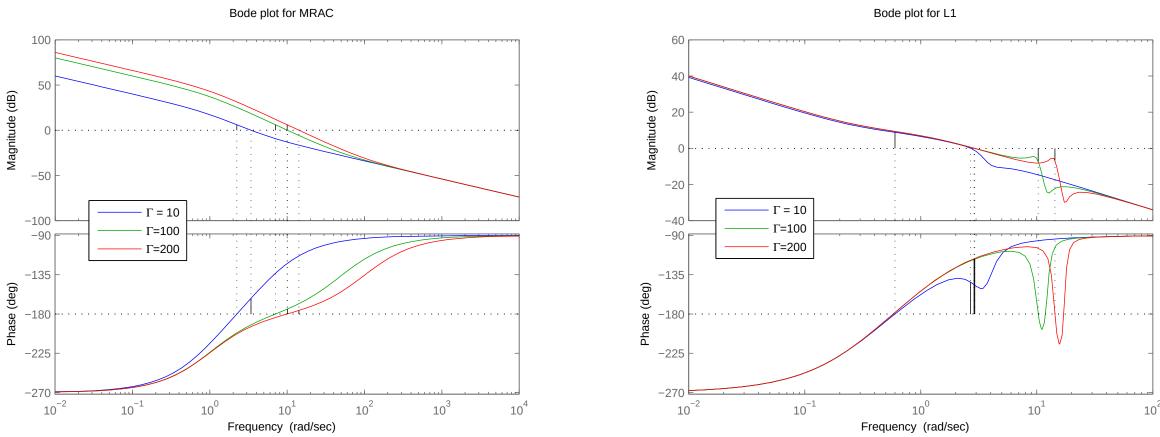


Fig. 32 Bode plot for the PI and the filtered PI control schemes (the filtered PI operates on the prediction error and not on the state)

Thus

$$u_2(s) = \underbrace{\frac{-2(s^2+s+\gamma)-\gamma(s-a)C(s)}{[s^2+s+(1-C(s))\gamma](s-a)}e^{-s\tau}}_{-H_o(s)} u_1(s) \tag{341}$$

Then we have (335). When there is no time-delay, i.e. $\tau = 0$, we get (334).

Notice that when there is no low-pass filter in the controller (339), i.e. $C(s) = 1$, this control scheme becomes a PI control scheme, and (334) and (335) are reduced to (328) and (329), respectively.

Although (335) looks more complex than (329) for PI controller, the Bode plots show that this filtered version of PI has better robustness than the conventional PI in that the phase margin and time delay margin are not affected by high gain. In Figure 32 we can see that the cross-over frequency for the modified PI controller does not change with high gain. Figure 33 shows that while the time-delay margin for PI is killed by high gain, the modified PI has guaranteed time-delay margin.

To show this quantitatively, write (334) as:

$$\begin{aligned} H_o(\omega) &= \frac{2j\omega(j\omega+1)^2+2(j\omega+1)\gamma+(j\omega-1)\gamma}{(j\omega^2-j\omega)(j\omega+1)^2+j\omega^2\gamma-j\omega\gamma} \\ &\to \frac{2(j\omega+1)\gamma+(j\omega-1)\gamma}{(j\omega)^2\gamma-j\omega\gamma} \quad \text{as } \gamma\to\infty \\ &= \frac{1+3j\omega}{-\omega^2-j\omega} \\ &= \frac{\sqrt{1+9\omega^2}}{\sqrt{\omega^4+\omega^2}}\exp\left(j(\arctan(3\omega)-\arctan\frac{1}{\omega}-\pi)\right) \end{aligned}$$

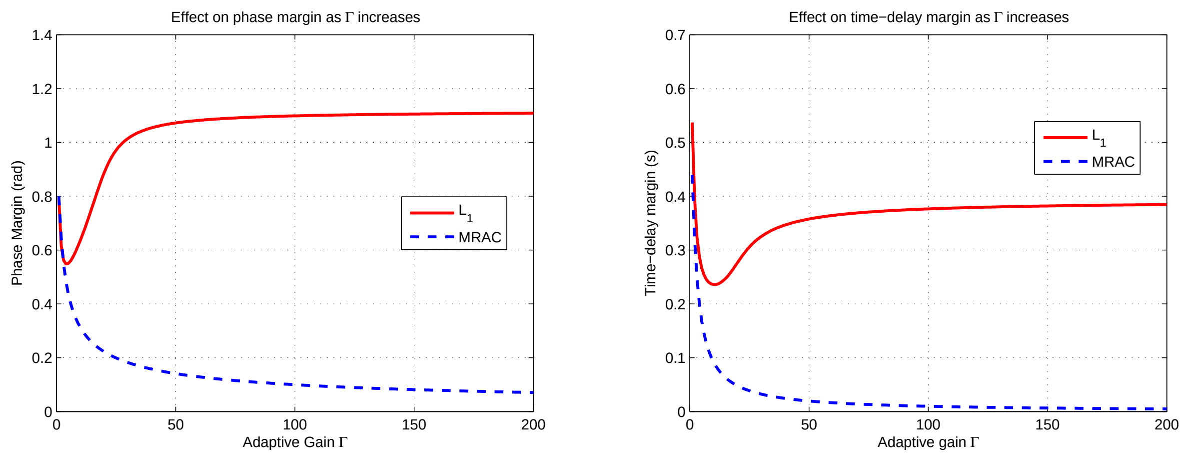


Fig. 33 Effect of high gain on the phase margin and time-delay margin for the PI and the filtered PI control schemes (the filtered PI operates on the prediction error and not on the state)

So, when $\gamma \to \infty$, the cross-over frequency is given by

$$\omega_c \approx \sqrt{8} = 2.83,$$

and the phase margin is given by

$$Pm \approx \arctan(3\omega_c) - \arctan\frac{1}{\omega} = 1.11.$$

Thus, the time-delay margin is given by:

$$\lim_{\gamma\to\infty} \tau^* = \frac{Pm}{\omega_c} \approx 0.4$$

The question now to ask would be: if we could get better tracking and robustness by increasing γ with this filtering type modification, then what do we lose and where do we lose? We know that we cannot have both of them simultaneously. We will lose robustness if we increase the bandwidth of $C(s)$. But let's observe all of this via appropriate nonlinear analysis.

18.5 $\mathcal{L}_1$ Adaptive Control Architecture for Systems with Known High-Frequency Gain

18.5.1 Paradigm Shift: Achievable Control Objective

Let's recall the development of the conventional MRAC architecture. Consider the following single-input single-output system dynamics:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + bu(t), \quad x(0) = x_0 \\
y(t) &= c^\top x(t)\,,
\end{aligned}
\tag{342}
$$

where $x \in \mathbb{R}^n$ is the system state vector (measured), $u \in \mathbb{R}$ is the control signal, $b, c \in \mathbb{R}^n$ are known constant vectors, A is an unknown $n \times n$ matrix, $y \in \mathbb{R}$ is the regulated output.

Consider the matching assumption:

Assumption 18.1. There exist a Hurwitz matrix $A_m \in \mathbb{R}^{n\times n}$ and a vector of ideal parameters $\theta \in \mathbb{R}^n$ such that (A_m, b) is controllable and $A_m - A = \; b\,\theta^\top$. We further assume the unknown parameter θ belongs to a given compact convex set Θ, i.e.

$$
\theta \in \Theta. \tag{343}
$$

Subject to this assumption, the system dynamics in (342) can be rewritten as:

$$
\begin{aligned}
\dot{x}(t) &= A_m x(t) + b(u(t) - \theta^\top x(t)), \quad x(0) = x_0 \\
y(t) &= c^\top x(t).
\end{aligned}
\tag{344}
$$

The MRAC paradigm proceeds by considering the so-called **ideal controller**, defined via the ideal value of θ

$$
u_{nom}(t) = \theta^\top x(t) + k_g r(t)\,, \tag{345}
$$

leading to the **desired reference system** behavior:

$$
\begin{aligned}
\dot{x}_m(t) &= A_m x_m(t) + bk_g r(t), \quad x_m(0) = x_0 \\
y_m(t) &= c^\top x_m(t),
\end{aligned}
\tag{346}
$$

In (345), $k_g = \lim\limits_{s\to 0} \dfrac{1}{c^\top (s\mathbb{I} - A_m)^{-1} b} = \dfrac{1}{-c^\top A_m^{-1} b}$. It ensures zero steady state error for constant inputs. Indeed, if $r(t)$ is a constant, the relationship $\dfrac{y_m(s)}{r(s)} = k_g c^\top (s\mathbb{I} - A_m)^{-1} b_m$ along with the final value theorem yields $\lim\limits_{t\to\infty} y_m(t) = r$.

It later proceeds by considering the adaptive version of (345):

$$u(t) = \hat{\theta}^\top(t)x(t) + k_g r(t), \tag{347}$$

$$\dot{\hat{\theta}}(t) = \Gamma \mathrm{Proj}(-x(t)\tilde{x}^\top(t)Pb, \hat{\theta}(t)), \quad \hat{\theta}(0) = \hat{\theta}_0 \tag{348}$$

where $\hat{\theta}(t) \in \mathbb{R}^n$ are the adaptive parameters, $\tilde{x}(t) = x(t) - x_m(t)$ is the tracking error, $\Gamma \in \mathbb{R}^{n\times n}$ is a positive definite matrix of adaptation gains, and $P = P^\top > 0$ is the solution of the algebraic equation $A_m^\top P + PA_m = -Q$ for arbitrary $Q > 0$. This leads to the tracking error dynamics:

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + b\tilde{\theta}^\top(t)x(t), \quad \tilde{x}(0) = 0, \quad \tilde{\theta}(t) = \hat{\theta}(t) - \theta. \tag{349}$$

Using standard Lyapunov arguments and Barbalat's lemma, one can prove that $\lim_{t\to\infty} \tilde{x}(t) = 0$.

As we see, Barbalat's lemma in conjunction with a Lyapunov proof straightforwardly leads to asymptotic stability, but the issue of the transient analysis remains open. It is time to question how the transient can be addressed with the above structure of adaptive control, and where do the complications lie. Let's consider the *so-called* reference system of MRAC in (346), which is defined in conjunction with the *ideal* nominal controller (344). This reference system is "*algebraically*" achievable, by substitution of the *ideal* nominal controller (344) into the system (342). This however does not imply that the controller in (344) is "implementable" for all possible values of θ, which naturally questions the feasibility of the reference system (346). Moreover, if one considers time-varying $\theta(t)$ with extremely high frequencies, it is obvious that during the implementation of the nominal controller in case of even completely known $\theta(t)$, one can only "*pass as much to the system*" as the control channel bandwidth permits. Thus, the reference system (346) may appear to be an overly ambitious goal even in the layout of the nominal design paradigm and on the level of control objective. If we further assume that we do NOT know $\theta(t)$, and we are trying to estimate, then the initial value of the adaptive controller $u_{ad}(0) = \hat{\theta}^\top(0)x(0) + k_g r(t)$ depends upon the initial guess of $\hat{\theta}(0) = \hat{\theta}_0$, which may be very far from the nominal value of θ. This in turn implies that at $t = 0$ the closed loop system, defined via (342), (347), (348), is very far from the reference system (346). Of course, with the help of Barbalat's lemma in case of constant θ one proves asymptotic convergence of $\tilde{x}(t) \to 0$ as $t \to \infty$, however asymptotic convergence in its nature has no guarantee for the rate of convergence, as we know already. It is only *exponential convergence* that can give some guarantee for the transient, but we do NOT have exponential convergence without the PE condition. A paper by Zang & Bitmead shows that arbitrary bad transients can happen in closed-loop adaptive systems before asymptotic convergence takes place, [32].

The paradigm shift here is to give up on the nice clean reference system in (346) and to *reduce the control objective*, by rendering it *feasible* so that from $t = 0$ one has *an implementable nominal controller* and *achievable reference system*. Given our control channel specification, or *the bandwidth*, one can consider the following nominal controller

$$u_{ref}(s) = C(s)\left(k_g r(s) + \theta^\top x_{ref}(s)\right), \tag{350}$$

where $C(s)$ is a strictly proper stable low-pass filter with DC gain 1, $C(0) = 1$, and its realization assumes zero initialization. This controller is implementable *independent of* θ and leads to an *achievable, albeit not clean, reference system*:

$$\begin{aligned} x_{ref}(s) &= H_o(s)\left(k_g C(s) r(s) + (C(s)-1)\theta^\top x_{ref}(s)\right) + (s\mathbb{I} - A_m)^{-1} x_0, \qquad x_{ref}(0) = x_0 \\ y_{ref}(s) &= c^\top x_{ref}(s), \end{aligned} \tag{351}$$

where $H_o(s) = (s\mathbb{I} - A_m)^{-1} b$. Since due to the presence of $C(s)$ the uncertainties do not cancel out completely, we need to state a sufficient condition, under which this reference system will be stable.

Towards that end, notice that (351) can be explicitly solved for

$$x_{ref}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1} G(s) r(s) + x_{in}(s),\ x_{in}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(s\mathbb{I} - A_m)^{-1} x_0, \tag{352}$$

where $\bar{G}(s) = H_o(s)(C(s) - 1)$, and $G(s) = H_o(s) k_g C(s)$. Let the choice of $C(s)$ verify

$$\|\bar{G}(s)\|_{\mathcal{L}_1} \theta_{\max} < 1, \tag{353}$$

where

$$\theta_{\max} = \max_{\theta \in \Omega} \sum_{i=1}^{n} |\theta_i|. \tag{354}$$

Lemma 18.5. If the condition in (353) holds, i.e. $\|\bar{G}(s)\|_{\mathcal{L}_1} \theta_{\max} < 1$, then $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$ and consequently $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1} G(s)$ are stable.

Proof. It follows from (311) that

$$\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} = \max_{i=1,..,n} \left(\|\bar{G}_i(s)\|_{\mathcal{L}_1} \left(\sum_{j=1}^{n} |\theta_j| \right) \right),$$

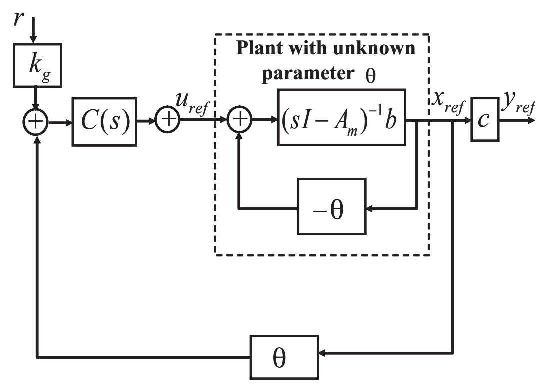


Fig. 34 Closed-loop reference LTI system

where $\bar{G}_i(s)$ is the i^{th} element of $\bar{G}(s)$, and θ_j is the j^{th} element of θ. From (354) we have $\sum_{j=1}^{n} |\theta_j| \leq \theta_{\max}$, and hence

$$\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} \leq \max_{i=1,..,n} \Big(\|\bar{G}_i(s)\|_{\mathcal{L}_1}\Big)\theta_{\max} = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max}, \quad \forall\, \theta \in \Omega. \tag{355}$$

The relationship in (353) implies that $\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} < 1$. Letting $\Delta(s) = -1$ and $M(s) = \bar{G}(s)\theta^\top$, we have

$$(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1} = (\mathbb{I} + \Delta(s)M(s))^{-1}.$$

The Small Gain Theorem 18.3 and Remark 18.3 ensure that $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$ is stable. Since $G(s)$ is stable, Lemma 18.2 implies that $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)$ is stable. □

Thus, we obtained a sufficient condition on the choice of $C(s)$ via (353) such that the reference system (351) of *the appropriately reduced control objective* remains stable. In the absence of $C(s)$, it simply reduces to the nominal reference system of MRAC, given by (346). The closed-loop system (342) with the controller (350) is given in Fig. 34.

Remark 18.4. Notice that since this reference system depends upon the unknown parameters θ, it cannot serve the purpose of introducing the specs (overshoot, rise time, settling time, etc.), although it is an LTI system. It merely defines an *achievable control objective*. So, we still need to determine an LTI system for the specs. Thus, you may well question why do we call it a *reference system* then? It is because as we will prove shortly, we can compute performance bounds with respect to this system and show that we can reduce these bounds by increasing the adaptation gain Γ_c.

18.5.2 Adaptive Structure

Next, let's explore the corresponding adaptive structure for achieving our *achievable reference system* from $t = 0$. From the structure of (351), it is obvious that we need some type of filtering also for the corresponding adaptive controller, so that in the absence of the filter everything collapses back to MRAC. Recall that in Section 7.4.1 we had an equivalent reparameterization of MRAC via *passive identifier*. If we look into the structure of direct MRAC, then low-pass filtering of the control signal will destroy the structure of the error dynamics and the corresponding adaptive laws, as at the output of the system in the result of filtering one will get a state vector of higher dimension, which will not be subtractable form the state of the reference system, Fig. 18 (left). On the other hand, in the passive identifier based reparameterization the structure of the error dynamics is independent of the control signal, and therefore allows for low-pass filtering without destroying the Lyapunov proof or the adaptive laws Fig. 18 (right). Of course, the low-pass filtering requires additional boundedness proof for the predictor before Barbalat's lemma can be applied for asymptotic stability, but the fact that it preserves the Lyapunov proof for the boundedness of the error dynamics and the parametric errors, is critical to the overall analysis. Thus, for the system dynamics is (344), we consider the following passive identifier:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m\hat{x}(t) + b(u(t) - \hat{\theta}^\top(t)x(t))\,, \quad \hat{x}(0) = x_0 \\ \hat{y}(t) &= c^\top\hat{x}(t)\,, \end{aligned} \tag{356}$$

where $\hat{\theta}(t)$ are parameter estimates governed by the same adaptive law in (348) with $\tilde{x}(t)$ being $\tilde{x}(t) = \hat{x}(t) - x(t)$ (detailed shortly). The only difference of the passive identifier (or state predictor) in (356) from the main system dynamics in (344) is that the unknown parameters are replaced by their adaptive estimates.

Letting $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$, the dynamics of the tracking error are the same as in (349):

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) - b\tilde{\theta}^\top(t)x(t), \quad \tilde{x}(0) = x_0. \tag{357}$$

Using the same projection type adaptive laws with $\tilde{x}(t) = \hat{x}(t) - x(t)$, we have

$$\dot{\hat{\theta}}(t) = \Gamma\mathrm{Proj}(x(t)\tilde{x}^\top(t)Pb, \hat{\theta}(t)), \tag{358}$$

and using standard Lyapunov arguments one can conclude that $\tilde{x}(t)$ and $\tilde{\theta}(t)$ are bounded.

(We notice that due to the sign difference for the parametric error in the error dynamics in (349) and (357), we have different signs in the projection operator in the adaptive laws.)

The corresponding adaptive control signal will be defined as an output of the low-pass filter

$$u(s) = C(s)(\bar{r}(s) + k_g r(s)), \quad k_g = 1/(c^\top H_o(0)), \quad H_o(s) = (s\mathbb{I} - A_m)^{-1} b, \tag{359}$$

where $\bar{r}(s)$ stands for Laplace transformation of $\bar{r}(t) = \hat{\theta}^\top(t)x(t)$, and the adaptive law for $\hat{\theta}(t)$ is written via the prediction error as in (358).

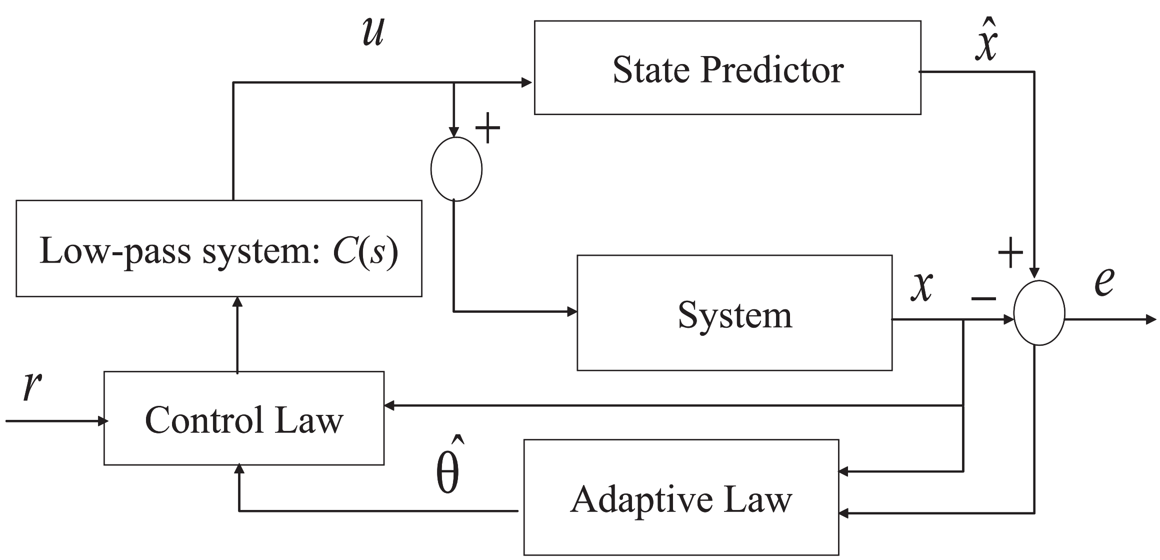


Fig. 35 Closed-loop system with $\mathcal{L}_1$ adaptive controller

Remark 18.5. To simplify for yourself the paradigm shift towards the design of $\mathcal{L}_1$ adaptive controller, I am requesting at this point that you take a different look at the blocks involved in the control design. Let's stop thinking of the reference model (or the closed-loop predictor at this stage) as a system that provides the desired specs, and look at it as a system that simply generates an error signal for the adaptive laws. If we look at it like a tool for giving us just an opportunity to generate an error signal for the adaptive law, then the reference system in MRAC looks like a pretty clean system free of any frequency abnormalities, while our system, which is driven by the **nonlinear adaptive controller**, faces all the unknown and possible high frequencies in the result of uncertainties. The philosophical question is: can we switch the roles? Can we make sure that in the entire closed-loop architecture the unknown and possible high frequencies go into a system, which is being used for the computation of the error signal in the adaptive law, while our system gets only nice clean low frequency signals with **predictable frequencies**? The phrase predictable frequency implies that if we define the control signal as an output of a low-pass filter, then we can appropriately band its frequencies, by selecting the bandwidth of the filter in a desirable way. And of course, once we are filtering the control signal, we had to use the state predictor based reparameterization, which helps to retain the structure of the adaptive law independent of the control signal definition. This is a key point, since its origin goes back to the gradient search for minimizing the quadratic form associated with the candidate Lyapunov function. So, a low-pass filtered control signal in (344) will retain a nice structure for the system with more-or-less predictable performance for its control signal, while passing the left-over high frequencies to the state predictor in (356), which is just a system in the computer for computing the error signal for the adaptive law. Of course, if we are filtering the control signal, we will need to ensure boundedness of the closed-loop predictor in (356), which will give us a condition on the bandwidth of the filter. Boundedness of the system state will follow from the boundedness of the tracking error signal, which is guaranteed from the Lyapunov proof and is independent of the control signal definition in the state predictor based architecture.

18.5.3 Stability of $\mathcal{L}_1$ adaptive control architecture: separation between adaptation and robustness

To prove the stability of the $\mathcal{L}_1$ adaptive control architecture, we will use the small-gain theorem. Rewrite the system dynamics

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b(u(t) - \theta^\top x(t)), \quad x(0) = x_0 \\ y(t) &= c^\top x(t), \end{aligned}$$

and the state predictor

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) + b(u(t) - \hat{\theta}^\top(t) x(t)), \quad \hat{x}(0) = x_0, \\ \hat{y}(t) &= c^\top \hat{x}(t), \end{aligned}$$

along with the control signal

$$u(s) = C(s)(\bar{r}(s) + k_g r(s))\,,$$

where $k_g = 1/(c^\top H_o(0))$, $H_o(s) = (s\mathbb{I} - A_m)^{-1}b$, $\bar{r}(t) = \hat{\theta}^\top(t)x(t)$, $\tilde{x}(t) = \hat{x}(t) - x$, and $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$.

Consider the following Lyapunov function candidate:

$$V(\tilde{x}(t), \tilde{\theta}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t)\,, \tag{360}$$

where $P = P^\top > 0$ solves $A_m^\top P + PA_m = -Q$ for some $Q > 0$, and Γ is the adaptive gain. Using the error dynamics

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) - b\tilde{\theta}^\top(t)x(t)\,, \quad \tilde{x}(0) = 0\,, \tag{361}$$

it is straightforward to verify that the projection based adaptive laws will lead to that

$$\dot{V}(t) \leq -\tilde{x}^\top(t)Q\tilde{x}(t) \leq 0\,. \tag{362}$$

Notice that the result in (362) is independent of $u(t)$, whether it is filtered or not, since until now it has been treated as a time-varying signal in the system dynamics and the state predictor and has been cancelled out while forming the error dynamics. Therefore from (362) one cannot conclude stability. Both states (of system and predictor) can drift to infinity with the same rate, keeping the error bounded. Thus, we need to prove in addition that with the $\mathcal{L}_1$ adaptive controller the state of the predictor will remain bounded. Upon that we will be able to apply Barbalat's lemma to conclude boundedness of the system state.

The next theorem formulates a sufficient condition for boundedness of the predictor state and helps to conclude that the tracking error $\tilde{x}(t) \to 0$ as $t \to \infty$. From the structure of predictor model it is straightforward to see that

$$\hat{x}(s) = \bar{G}(s)\bar{r}(s) + G(s)r(s) + (s\mathbb{I} - A_m)^{-1}x_0\,,$$

where $\bar{G}(s) = H_o(s)(C(s) - 1)$ and $G(s) = k_g H_o(s)C(s)$.

Theorem 18.4. Given the system in (342) and the $\mathcal{L}_1$ adaptive controller defined via (359), (356), (358) with $\tilde{x}(t) = \hat{x}(t) - x(t)$, subject to (353), the tracking error $\tilde{x}(t)$ converges to zero asymptotically:

$$\lim_{t\to\infty} \tilde{x}(t) = 0. \tag{363}$$

Proof. Let $\lambda_{\min}(P)$ be the minimum eigenvalue of P. From (360) and (362) it follows that $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \le \tilde{x}^\top(t)P\tilde{x}(t) \le V(t) \le V(0)$, implying that

$$\|\tilde{x}(t)\|^2 \le V(0)/\lambda_{\min}(P),\ t \ge 0. \tag{364}$$

From (309), $\|\tilde{x}\|_{\mathcal{L}_\infty} = \max\limits_{i=1,..,n,t\ge 0} |\tilde{x}_i(t)|$. The relationship in (364) ensures $\max\limits_{i=1,..,n,t\ge 0}|\tilde{x}_i(t)| \le \sqrt{V(0)/\lambda_{\min}(P)}$, and therefore for all $t \ge 0$ one has $\|\tilde{x}_t\|_{\mathcal{L}_\infty} \le \sqrt{V(0)/\lambda_{\min}(P)}$. Using the triangular relationship for norms implies that

$$|\ \|\hat{x}_t\|_{\mathcal{L}_\infty} - \|x_t\|_{\mathcal{L}_\infty}\ | \le \sqrt{V(0)/\lambda_{\min}(P)}. \tag{365}$$

The projection algorithm in (358) ensures that $\hat{\theta}(t) \in \Omega, \forall t \ge 0$. Recalling that $\bar{r}(t) = \hat{\theta}(t)x(t)$ we have $\|\bar{r}_t\|_{\mathcal{L}_\infty} \le \theta_{\max}\|x_t\|_{\mathcal{L}_\infty}$. Substituting for $\|x_t\|_{\mathcal{L}_\infty}$ from (365) leads to the following

$$\|\bar{r}_t\|_{\mathcal{L}_\infty} \le \theta_{\max}\left(\|\hat{x}_t\|_{\mathcal{L}_\infty} + \sqrt{V(0)/\lambda_{\min}(P)}\right). \tag{366}$$

It follows from Lemma 18.1 that

$$\|\hat{x}_t\|_{\mathcal{L}_\infty} \le \|\bar{G}(s)\|_{\mathcal{L}_1}\|\bar{r}_t\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty} + \|(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}||x_0||_{\mathcal{L}_1},$$

which along with (366) gives the following upper bound

$$\|\hat{x}_t\|_{\mathcal{L}_\infty} \le \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max}\left(\|\hat{x}_t\|_{\mathcal{L}_\infty} + \sqrt{V(0)/\lambda_{\min}(P)}\right) + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty} + \|(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}||x_0||_{\mathcal{L}_1}$$

Let

$$\lambda = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max}. \tag{367}$$

From (353) it follows that $\lambda < 1$, which allows for grouping the terms: $(1-\lambda)\|\hat{x}_t\|_{\mathcal{L}_\infty} \le \lambda\sqrt{V(0)/\lambda_{\min}(P)} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty} + \|(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}||x_0||_{\mathcal{L}_1}$, and hence

$$\|\hat{x}_t\|_{\mathcal{L}_\infty} \le (\lambda\sqrt{V(0)/\lambda_{\min}(P)} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty} + \|(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}||x_0||_{\mathcal{L}_1})/(1-\lambda). \tag{368}$$

Since $V(0), \lambda_{\min}(P), \|G(s)\|_{\mathcal{L}_1}, \|r_t\|_{\mathcal{L}_\infty}, \lambda, \|(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}, ||x_0||_{\mathcal{L}_1}$ are all finite and $\lambda < 1$, the relationship in (368) implies that $\|\hat{x}_t\|_{\mathcal{L}_\infty}$ is finite for all $t \ge 0$, and hence $\hat{x}(t)$ is *uniformly* bounded. The relationship in (365) implies that $\|x_t\|_{\mathcal{L}_\infty}$ is also finite for all $t \ge 0$, and therefore $x(t)$ is also *uniformly* bounded. The adaptive law in (358) ensures that the estimates $\hat{\theta}(t)$ are also bounded. From (361) it follows that $\dot{\tilde{x}}(t)$ is bounded, and it follows from Barbalat's lemma that $\lim\limits_{t\to\infty} \tilde{x}(t) = 0$. $\square$

Remark 18.6. The above stability proof *clearly decoupled adaptation from robustness*: adaptive laws were derived from Lyapunov-like analysis using gradient-minimization philosophy, while the stability conclusion was drawn from the small-gain theorem. This treatment of the problem introduced clear *separation between adaptation and robustness*, which we will shortly observe in the structure of the **guaranteed, decoupled and uniform** performance bounds.

18.6 Transient and Steady-State Performance

18.6.1 Asymptotic Performance

Now it is the time to compute the performance bounds to justify why do we call this a reference system. Letting

$$r_1(t) = \tilde{\theta}^\top(t)x(t), \tag{369}$$

and recalling that $\bar{r}(t) = \hat{\theta}^\top(t)x(t)$ and $\tilde{x}(t) = \hat{x}(t) - x(t)$, we can rewrite it as

$$\bar{r}(t) = \theta^\top(\hat{x}(t) - \tilde{x}(t)) + r_1(t)\,.$$

Hence, the closed-loop state predictor can be rewritten as

$$\hat{x}(s) = \bar{G}(s)\left(\theta^\top\hat{x}(s) - \theta^\top\tilde{x}(s) + r_1(s)\right) + G(s)r(s) + (s\mathbb{I} - A_m)^{-1}x_0,$$

where $r_1(s)$ is the Laplace transformation of $r_1(t)$ defined in (369), and further put into the form:

$$\hat{x}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}\Big(- \bar{G}(s)\theta^\top\tilde{x}(s) + \bar{G}(s)r_1(s) + G(s)r(s)\Big) + x_{in}(s). \tag{370}$$

It follows from (361) and (369) that $\dot{\tilde{x}}(t) = A_m\tilde{x}(t) - br_1(t)$, and hence

$$\tilde{x}(s) = -H_o(s)r_1(s). \tag{371}$$

Recalling the definition of $\bar{G}(s)$, the state of the predictor can be presented as

$$\hat{x}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s) + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}\left(-\bar{G}(s)\theta^\top\tilde{x}(s) - (C(s)-1)\tilde{x}(s)\right) + x_{in}(s).$$

Using $x_{ref}(s)$ from (352) and recalling the definition of $\tilde{x}(s) = \hat{x}(s) - x(s)$, one arrives at

$$x(s) = x_{ref}(s) - \Big(\mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}\big(\bar{G}(s)\theta^\top + (C(s)-1)\mathbb{I}\big)\Big)\tilde{x}(s). \tag{372}$$

The expressions in (359) and (350) lead to the following expression of the control signal

$$u(s) = u_{ref}(s) + C(s)r_1(s) + C(s)\theta^\top(x(s) - x_{ref}(s))\,. \tag{373}$$

Theorem 18.5. Given the system in (342) and the $\mathcal{L}_1$ adaptive controller defined via (359), (356), (358), subject to (353), we have:

$$\lim_{t\to\infty} \|x(t) - x_{ref}(t)\| = 0\,, \tag{374}$$

$$\lim_{t\to\infty} |u(t) - u_{ref}(t)| = 0\,. \tag{375}$$

Proof. Let

$$r_2(t) = x_{ref}(t) - x(t)\,. \tag{376}$$

It follows from (372) that

$$r_2(s) = \left(\mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}\left(\bar{G}(s)\theta^\top + (C(s)-1)\mathbb{I}\right)\right)\tilde{x}(s). \tag{377}$$

The signal $r_2(t)$ can be viewed as the response of the LTI system

$$H_2(s) = \mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}\left(\bar{G}(s)\theta^\top + (C(s)-1)\mathbb{I}\right) \tag{378}$$

to the bounded error signal $\tilde{x}(t)$. It follows from Lemma 18.5 that $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$, $\bar{G}(s)$, $C(s)$ are stable and, therefore, $H_2(s)$ is stable. Hence, from (363) we have $\lim_{t\to\infty} r_2(t) = 0$. Let

$$r_3(s) = C(s)r_1(s) + C(s)\theta^\top(x(s) - x_{ref}(s)).$$

It follows from (373) that $r_3(t) = u(t) - u_{ref}(t)$. Since $\tilde{\theta}(t)$ is bounded, it follows from (361) and (363) that $\lim_{t\to\infty} r_1(t) = 0$. Since $C(s)$ is a stable proper system, it follows from (374) that $\lim_{t\to\infty} r_3(t) = 0$. □

Lemma 18.6. Given the system in (342) and the $\mathcal{L}_1$ adaptive controller defined via (356), (358), (359) subject to (353), if $r(t)$ is constant, then $\lim_{t\to\infty} y(t) = r$.

Proof. Since

$$y_{ref}(t) = c^\top x_{ref}(t),$$

it follows from (374) that

$$\lim_{t\to\infty} (y(t) - y_{ref}(t)) = 0.$$

From (352) it follows that

$$y_{ref}(s) = c^\top(\mathbb{I}-\bar{G}(s)\theta^\top)^{-1}G(s)r(s)+c^\top x_{in}(s) = c^\top(\mathbb{I}-\bar{G}(s)\theta^\top)^{-1}G(s)r(s)+c^\top(\mathbb{I}-\bar{G}(s)\theta^\top)^{-1}(s\mathbb{I}-A_m)^{-1}x_0. \tag{379}$$

Since the second term is the response to the initial condition for a Hurwitz matrix A_m, which is exponentially decaying, application of the end value theorem along with the definition of k_g ensures

$$\lim_{t\to\infty} y_{ref}(t) = \lim_{s\to 0} c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r = c^\top H_o(0)C(0)k_g r = r.$$

18.6.2 Transient Performance

First we notice that in the proof of Theorem 18.4 we obtained the following upper bound:

$$||\tilde{x}(t)|| \leq \sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\min}(P)\Gamma_c}}, \quad \bar{\theta}_{\max} \triangleq \max_{\theta\in\Omega}\sum_{i=1}^{n} 4\theta_i^2, \qquad \forall\ t \geq 0, \tag{380}$$

where $\lambda_{\min}(P)$ is the minimum eigenvalue of P. Indeed, this follows from the upper bound in (364), if we notice that with $\tilde{x}(0) = 0$, we have

$$V(0) = \tilde{\theta}^\top(0)\Gamma^{-1}\tilde{\theta}(0),$$

and further use the fact that the projection algorithm ensures that $\hat{\theta}(t) \in \Omega,\ \forall t \geq 0$, and therefore

$$\max_{t\geq 0} \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t) \leq \frac{\bar{\theta}_{\max}}{\Gamma_c}, \quad \forall t \geq 0, \tag{381}$$

where $\bar{\theta}_{\max}$ is defined in (380). Since $\lambda_{\min}(P)\|\tilde{x}\|^2 \leq \tilde{x}^\top(t)P\tilde{x}(t)$, then $||\tilde{x}(t)|| \leq \sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\min}(P)\Gamma_c}}$.

Further, it follows from Lemma 18.4 that there exists $c_o \in \mathbb{R}^n$ and stable polynomials $N_d(s)$ and $N_n(s)$ such that

$$c_o^\top H_o(s) = N_n(s)/N_d(s)\,, \tag{382}$$

where the order of $N_d(s)$ is one more than the order of $N_n(s)$. The next theorem gives the transient and steady state performance bounds with respect to the LTI reference system.

Theorem 18.6. Given the system in (342) and the $\mathcal{L}_1$ adaptive controller defined via (359), (356), (358), subject to (353), we have:

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_1/\sqrt{\Gamma_c}\,, \tag{383}$$

$$\|y - y_{ref}\|_{\mathcal{L}_\infty} \leq \|c^\top\|_{\mathcal{L}_1}\gamma_1/\sqrt{\Gamma_c}\,, \tag{384}$$

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_2/\sqrt{\Gamma_c}\,, \tag{385}$$

where $\|c^\top\|_{\mathcal{L}_1}$ is the $\mathcal{L}_1$-norm of $c^\top$ and

$$\gamma_1 = \|H_2(s)\|_{\mathcal{L}_1}\sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\max}(P)}}, \tag{386}$$

$$\gamma_2 = \left\|C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\max}(P)}} + \|C(s)\theta^\top\|_{\mathcal{L}_1}\gamma_1\,. \tag{387}$$

Proof. It follows from (377), (378) and Lemma 18.2 that $\|r_2\|_{\mathcal{L}_\infty} \le \|H_2(s)\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}$, while (380) implies that

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \le \sqrt{\bar{\theta}_{\max}/(\lambda_{\max}(P)\Gamma_c)}\,. \tag{388}$$

Therefore,

$$\|r_2\|_{\mathcal{L}_\infty} \le \|H_2(s)\|_{\mathcal{L}_1}\sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\max}(P)\Gamma_c}}, \tag{389}$$

which leads to (383). The upper bound in (384) follows from (383) and Lemma 18.2 directly. From (371) we have

$$\begin{aligned} r_3(s) &= C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top H_o(s) r_1(s) + C(s)\theta^\top(x(s) - x_{ref}(s)) \\ &= -C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top \tilde{x}(s) + C(s)\theta^\top(x(s) - x_{ref}(s))\,, \end{aligned}$$

where c_o is introduced in (382). It follows from (382) that $C(s)\frac{1}{c_o^\top H_o(s)} = C(s)\frac{N_d(s)}{N_n(s)}$, where $N_d(s)$, $N_n(s)$ are stable polynomials and the order of $N_n(s)$ is one less than the order of $N_d(s)$. Since $C(s)$ is stable and strictly proper, the complete system $C(s)\frac{1}{c_o^\top H_o(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$-norm exists and is finite. Hence, we have $\|r_3\|_{\mathcal{L}_\infty} \le \left\|C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top\right\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty} + \|C(s)\theta^\top\|_{\mathcal{L}_1}\|x - x_{ref}\|_{\mathcal{L}_\infty}$. Using (388), the upper bound in (385) is straightforward to derive. □

Theorem 18.6 states that $x(t)$, $y(t)$ and $u(t)$ follow $x_{ref}(t)$, $y_{ref}(t)$ and $u_{ref}(t)$ not only asymptotically but also during the transient, provided that the adaptive gain is selected sufficiently large. Thus, the control objective is reduced to designing $C(s)$ to ensure that the reference LTI system has the desired response in terms of control specifications.

Remark 18.7. Notice that if we set $C(s) = 1$, then the $\mathcal{L}_1$ adaptive controller degenerates into a MRAC type. In that case $\left\|C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top\right\|_{\mathcal{L}_1}$ cannot be finite, since $H_o(s)$ is strictly proper. Therefore, from (387) it follows that $\gamma_2 \to \infty$, and hence for the control signal in MRAC one can not reduce the bound in (385) by increasing the adaptive gain.

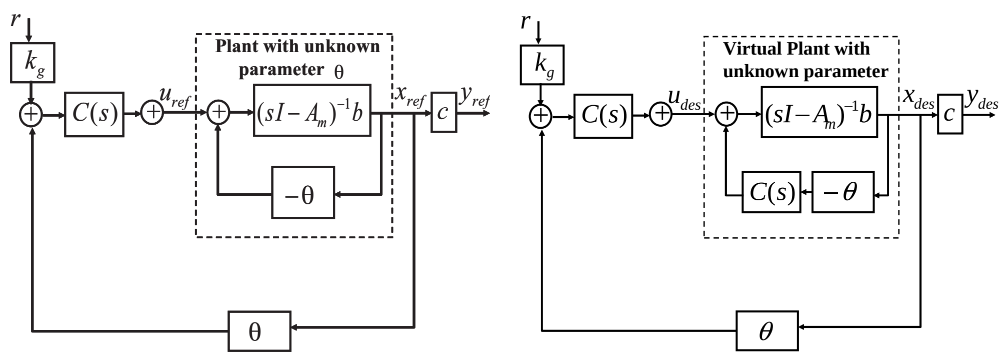
18.7 Design Guidelines for Achieving Desired Specifications

18.7.1 Design System

We proved that the error between the state and the control signal of the closed-loop system with $\mathcal{L}_1$ adaptive controller in (342), (356), (358), (359) (Fig. 35) and the state and the control signal of the closed-loop reference system in (350), (352) (Fig. 34) can be rendered arbitrarily small by choosing large adaptive gain. Therefore, the control objective is reduced to determining $C(s)$ to ensure that the reference system in (350), (352) (Fig. 34) has the desired response from $r(t)$ to $y_{ref}(t)$. Since this reference system depends upon the unknown parameter θ, it cannot serve the purpose of introducing the specs. Let's look at another system, called *design system*:

$$x_{des} = H_o(s)C(s)\left(k_g r(s) + \theta^\top x_{des}(s)\right) + H_o(s)C(s)(-\theta^\top x_{des}(s)) + x_{in}(s),$$

the output of which is free from unknown parameters, Fig. 36(b). Notice that in the design system



a) Reference system **b)** Design system

Fig. 36 Comparison of reference system and design system

a low pass filter $C(s)$ is added into the inner feedback loop to render its output independent of the unknown parameter. Therefore it can serve the purpose of introducing the specs. This leads to the following signals:

$$y_{des}(s) = c^\top G(s)r(s) + c^\top x_{in}(s) = C(s)k_g c^\top H_o(s)r(s) + c^\top x_{in}(s)\,, \tag{390}$$

$$u_{des}(s) = k_g C(s)\left(1 + C(s)\theta^\top H_o(s)\right)r(s). \tag{391}$$

We note that $u_{des}(t)$ depends on the unknown parameter θ, while $y_{des}(t)$ does not. This is intuitively correct: if one wants a UNIFORM output response, the control signals MUST depend upon the parameters of the system.

Lemma 18.7. Subject to (353), the following upper bounds hold:

$$\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} \leq \frac{\lambda}{1-\lambda}\|c^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}, \tag{392}$$

$$\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} \leq \frac{1}{1-\lambda}\|c^\top\|_{\mathcal{L}_1}\|h_3\|_{\mathcal{L}_\infty}, \tag{393}$$

$$\|u_{ref} - u_{des}\|_{\mathcal{L}_\infty} \leq \frac{\lambda}{1-\lambda}\|C(s)\theta^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}, \tag{394}$$

$$\|u_{ref} - u_{des}\|_{\mathcal{L}_\infty} \leq \frac{1}{1-\lambda}\|C(s)\theta^\top\|_{\mathcal{L}_1}\|h_3\|_{\mathcal{L}_\infty}, \tag{395}$$

where λ is defined in (367), and $h_3(t)$ is the inverse Laplace transformation of

$$H_3(s) = (C(s)-1)C(s)r(s)k_g H_o(s)\theta^\top H_o(s). \tag{396}$$

Proof. It follows from (351) and (352) that $y_{ref}(s) = c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s)$. Following Lemma 18.5, the condition in (353) ensures the stability of the reference LTI system. Since $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$ is stable, then one can expand it into convergent series:

$$y_{ref}(s) = c^\top\left(\mathbb{I} + \sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^i\right)G(s)r(s) + c^\top x_{in}(s) = y_{des}(s) + c^\top\left(\sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^i\right)G(s)r(s) + c^\top x_{in}(s) \tag{397}$$

Let

$$r_4(s) = c^\top\left(\sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^i\right)G(s)r(s).$$

Then $r_4(t) = y_{ref}(t) - y_{des}(t)$. The relationship in (355) implies that $\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} \leq \lambda$, and it follows from Lemma 18.2 that

$$\|r_4\|_{\mathcal{L}_\infty} \leq \left(\sum_{i=1}^{\infty}\lambda^i\right)\|c^\top\|_{\mathcal{L}_1}\|G\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} = \frac{\lambda}{1-\lambda}\|c^\top\|_{\mathcal{L}_1}\|G\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}. \tag{398}$$

From (397) we have

$$y_{ref}(s) = y_{des}(s) + c^\top\Big(\sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^{i-1}\Big)\bar{G}(s)\theta^\top G(s)r(s),$$

which along with (396) and recalling the definition of $\bar{G}(s)$ leads to

$$y_{ref}(s) = y_{des}(s) + c^\top\left(\sum_{i=1}^{\infty}(\bar{G}(s)\theta^\top)^{i-1}\right)H_3(s).$$

Lemma 18.2 immediately implies that

$$\|r_4\|_{\mathcal{L}_\infty} \leq \Big(\sum_{i=1}^{\infty} \lambda^{i-1}\Big) \|c^\top\|_{\mathcal{L}_1} \|h_3\|_{\mathcal{L}_\infty}.$$

Comparing $u_{des}(s)$ in (391) to $u_{ref}(s)$ in (350) it follows that $u_{des}(s)$ can be written as

$$u_{des}(s) = k_g C(s) r(s) + C(s)\theta^\top x_{des}(s)\,,$$

where $x_{des}(s) = C(s) k_g H_o(s) r(s)$. Therefore

$$u_{ref}(s) - u_{des}(s) = (C(s)\theta^\top)(x_{ref}(s) - x_{des}(s)).$$

Hence, it follows from Lemma 18.1 that

$$\|u_{ref} - u_{des}\|_{\mathcal{L}_\infty} \leq \|C(s)\theta^\top\|_{\mathcal{L}_1} \|x_{ref} - x_{des}\|_{\mathcal{L}_\infty}.$$

Using the same steps as for $\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty}$, we have

$$\|x_{ref} - x_{des}\|_{\mathcal{L}_\infty} \leq \frac{\lambda}{1-\lambda} \|G(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty},$$

$$\|x_{ref} - x_{des}\|_{\mathcal{L}_\infty} \leq \frac{1}{1-\lambda} \|h_3\|_{\mathcal{L}_\infty}\,,$$

which leads to (394) and (395). □

Thus, the problem is reduced to finding a strictly proper stable $C(s)$ to ensure that $\lambda < 1$ or $\|h_3\|_{\mathcal{L}_\infty}$ are sufficiently small, and $y_{des}(s) \approx D_d(s)r(s)$, where $D_d(s)$ has the desired specifications (rise time, settling time, overshoot, etc.). Then, Theorem 18.6 and Lemma 18.7 will imply that the output $y(t)$ of the system in (342) and the $\mathcal{L}_1$ adaptive control signal $u(t)$ will follow $y_{des}(t)$ and $u_{des}(t)$ both in transient and steady state with quantifiable bounds, given in (384), (385) and (392)-(395).

Notice that $\lambda < 1$ is required for stability. From (390)-(395), it follows that for achieving $y_{des}(s) \approx D_d(s)r(s)$ it is desirable to ensure that λ or $\|h_3\|_{\mathcal{L}_\infty}$ are sufficiently small and, in addition, $C(s)c^\top H_o(s) \approx D_d(s)$. We notice that these requirements are not in conflict with each other. So, using Lemma 18.2, one can consider the following conservative upper bound

$$\lambda = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} = \|H_o(s)(C(s)-1)\|_{\mathcal{L}_1}\theta_{\max} \leq \|H_o(s)\|_{\mathcal{L}_1}\|C(s)-1\|_{\mathcal{L}_1}\theta_{\max}. \tag{399}$$

Thus, minimization of λ can be achieved from two different perspectives: i) fix $C(s)$ and minimize $\|H_o(s)\|_{\mathcal{L}_1}$, ii) fix $H_o(s)$ and minimize the $\mathcal{L}_1$-norm of one of the cascaded systems $\|H_o(s)(C(s)-1)\|_{\mathcal{L}_1}$, $\|(C(s)-1)r(s)\|_{\mathcal{L}_1}$ or $\|C(s)(C(s)-1)\|_{\mathcal{L}_1}$ via the choice of $C(s)$.

As in MRAC, assume

$$k_g c^\top H_o(s) \approx D_d(s). \tag{400}$$

Lemma 18.8. Let

$$C(s) = \frac{\omega}{s+\omega}.$$

For any single input n-output strictly proper stable system $H_o(s)$ the following is true:

$$\lim_{\omega\to\infty} \|(C(s)-1)H_o(s)\|_{\mathcal{L}_1} = 0.$$

Proof. It follows that

$$(C(s)-1)H_o(s) = \frac{-s}{s+\omega}H_o(s) = \frac{-1}{s+\omega}sH_o(s).$$

Since $H_o(s)$ is strictly proper and stable, $sH_o(s)$ is stable and has relative degree ≥ 0, and hence $\|sH_o(s)\|_{\mathcal{L}_1}$ is finite. Since

$$\left\|\frac{-1}{s+\omega}\right\|_{\mathcal{L}_1} = \frac{1}{\omega},$$

it follows from (18.2) that

$$\|(C(s)-1)H_o(s)\|_{\mathcal{L}_1} \leq \frac{1}{\omega}\|sH_o(s)\|_{\mathcal{L}_1},$$

and the proof is complete. □

Lemma 18.8 states that if one chooses $k_g c^\top H_o(s) \approx D_d(s)$, then by increasing the bandwidth of the low-pass system $C(s)$, it is possible to render $\|\bar{G}(s)\|_{\mathcal{L}_1}$ arbitrarily small. With large ω, the pole $-\omega$ due to $C(s)$ is omitted, and $H_o(s)$ is the dominant reference system leading to $y_{ref}(s) \approx k_g c^\top H_o(s) r(s) \approx D_d(s)r(s)$. We note that $k_g c^\top H_o(s)$ is exactly the reference model of the MRAC design. Therefore this approach is equivalent to mimicking MRAC.

However, increasing the bandwidth of $C(s)$ is not the only choice for minimizing $\|\bar{G}(s)\|_{\mathcal{L}_1}$. Since $C(s)$ is a low-pass filter, its complementary $1 - C(s)$ is a high-pass filter with its cutoff frequency approximating the bandwidth of $C(s)$. Since both $H_o(s)$ and $C(s)$ are strictly proper systems, $\bar{G}(s) = H_o(s)(C(s)-1)$ is equivalent to cascading a low-pass system $H_o(s)$ with a high-pass system $C(s)-1$. If one chooses the cut-off frequency of $C(s)-1$ larger than the bandwidth of $H_o(s)$, it ensures that $\bar{G}(s)$ is a "no-pass" system, and hence its $\mathcal{L}_1$-norm can be rendered arbitrarily small. This can be achieved via higher order filter design methods. The illustration is given in Fig. 37.

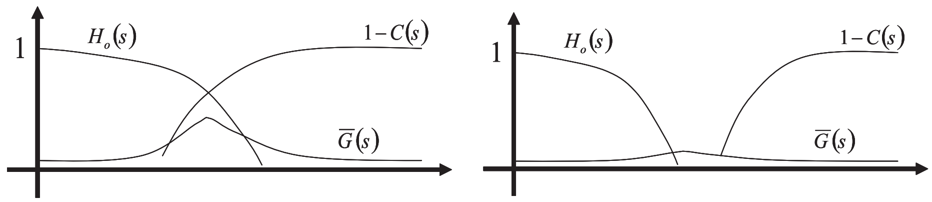


Fig. 37 Cascaded systems

To minimize $\|h_3\|_{\mathcal{L}_\infty}$, we note that $\|h_3\|_{\mathcal{L}_\infty}$ can be upperbounded in two ways: $\|h_3\|_{\mathcal{L}_\infty} \leq \|(C(s)-1)r(s)\|_{\mathcal{L}_1}\|h_4\|_{\mathcal{L}_\infty}$, where $h_4(t)$ is the inverse Laplace transformation of $H_4(s) = C(s)k_g H_o(s)\theta^\top H_o(s)$, and $\|h_3\|_{\mathcal{L}_\infty} \leq \|(C(s)-1)C(s)\|_{\mathcal{L}_1}\|h_5\|_{\mathcal{L}_\infty}$, where $h_5(t)$ is the inverse Laplace transformation of $H_5(s) = r(s)k_g H_o(s)\theta^\top H_o(s)$. We note that since $r(t)$ is a bounded signal and $C(s), H_o(s)$ are stable proper systems, $\|h_4\|_{\mathcal{L}_\infty}$ and $\|h_5\|_{\mathcal{L}_\infty}$ are finite. Therefore, $\|h_3\|_{\mathcal{L}_\infty}$ can be minimized by minimizing $\|(C(s)-1)r(s)\|_{\mathcal{L}_1}$ or $\|(C(s)-1)C(s)\|_{\mathcal{L}_1}$. Following the same arguments as above and assuming that $r(t)$ is in low-frequency range, one can choose the cut-off frequency of $C(s)-1$ to be larger than the bandwidth of the reference signal $r(t)$ to minimize $\|(C(s)-1)r(s)\|_{\mathcal{L}_1}$. For minimization of $\|C(s)(C(s)-1)\|_{\mathcal{L}_1}$ notice that if $C(s)$ is an ideal low-pass filter, then $C(s)(C(s)-1) = 0$ and hence $\|h_3\|_{\mathcal{L}_\infty} = 0$. Since an ideal low-pass filter is not physically implementable, one can minimize $\|C(s)(C(s)-1)\|_{\mathcal{L}_1}$ via appropriate choice of $C(s)$.

The above presented approaches ensure that $C(s) \approx 1$ in the bandwidth of $r(s)$ and $H_o(s)$. Therefore it follows from (390) that $y_{des}(s) = C(s)k_g c^\top H_o(s)r(s) \approx k_g c^\top H_o(s)r(s)$, which along with (400) yields $y_{des}(s) \approx D_d(s)r(s)$.

Remark 18.8. From Theorem 18.6 and Lemma 18.7 it follows that the $\mathcal{L}_1$ adaptive controller can generate a system response to track (390) and (391) both in transient and steady state if we set the adaptive gain large and minimize λ or $\|h_3\|_{\mathcal{L}_\infty}$. Notice that $u_{des}(t)$ in (391) depends upon the unknown parameter θ, while $y_{des}(t)$ in (390) does not. This implies that for different values of θ, the $\mathcal{L}_1$ adaptive controller will generate different control signals (dependent on θ) to ensure uniform system response (independent of θ). This is natural, since different unknown parameters imply different systems, and to have similar response for different systems the control signals have to be different. Here is the obvious advantage of the $\mathcal{L}_1$ adaptive controller in a sense that it controls a partially known system as an LTI feedback controller would have done if the unknown parameters were known.

Remark 18.9. It follows from Theorem 18.6 that in the presence of large adaptive gain the $\mathcal{L}_1$ adaptive controller and the closed-loop system state with it approximate $u_{ref}(t), y_{ref}(t)$. Therefore, we conclude from (352) that $y(t)$ approximates the response of the LTI system $c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)$ to the input $r(t)$, hence its transient performance specifications, such as overshoot and settling time, can be derived for every value of θ. If we further minimize λ or $\|h_3\|_{\mathcal{L}_\infty}$, it follows from Lemma 18.7 that $y(t)$ approximates the response of the LTI system $C(s)c^\top H_o(s)$. In this case, the $\mathcal{L}_1$ adaptive controller leads to uniform transient performance of $y(t)$ independent of the value of the unknown parameter θ. For the resulting $\mathcal{L}_1$ adaptive control signal one can characterize the transient specifications such as its amplitude and rate change for every $\theta \in \Omega$, using $u_{des}(t)$.

18.7.2 Decoupled (guaranteed) performance bounds

If we put the results of Theorem 18.6 and Lemma 18.7 together, we will observe the following **uniform, guaranteed** and **decoupled** performance bounds with respect to the signals $y_{des}(t)$ and $u_{des}(t)$:

$$\|y - y_{des}\|_{\mathcal{L}_\infty} \leq \frac{\lambda}{1-\lambda}\|c^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \gamma_1/\sqrt{\Gamma_c}, \tag{401}$$

$$\|u - u_{des}\|_{\mathcal{L}_\infty} \leq \frac{\lambda}{1-\lambda}\|C(s)\theta^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \gamma_2/\sqrt{\Gamma_c}\,. \tag{402}$$

The first bound in this structure on the right is in charge of robustness, while the second bound is in charge of adaptation. The first one can be minimized via the choice of λ, while the second one can be minimized by increasing the adaptation rate. Thus, recalling the discussion in Section 18.4, where we observed the adverse effects of large adaptive gain on robustness, we see that with this type of filtering we *decoupled adaptation from robustness.* Adaptation and robustness can be tuned separately and independently without affecting each other.

18.7.3 Comparison with high-gain controller

We use a scalar system to compare the performance of the $\mathcal{L}_1$ adaptive controller and a linear high-gain controller. Towards that end, let $\dot{x}(t) = -\theta x(t) + u(t)$, where $x \in \mathbb{R}$ is the measurable system state, $u \in \mathbb{R}$ is the control signal and $\theta \in \mathbb{R}$ is unknown, which belongs to a given compact set $[\theta_{\min},\ \theta_{\max}]$. Let $u(t) = -kx(t) + kr(t)$, leading to the following closed-loop system $\dot{x}(t) = (-\theta - k)x(t) + kr(t)$. We need to choose $k > -\theta_{\min}$ to guarantee stability. We note that both the steady state error and the transient performance depend on the unknown parameter value θ. By further introducing a proportional-integral controller, one can achieve zero steady state error. If one chooses $k \gg \max\{|\theta_{\max}|, |\theta_{\min}|\}$, it leads to high-gain system $x(s) = \dfrac{k}{s-(-\theta-k)}r(s) \approx \dfrac{k}{s+k}r(s)$.

To apply the $\mathcal{L}_1$ adaptive controller, let the desired system response be $x(s) \approx \frac{2}{s+2}r(s)$, implying $H_o(s) = \frac{1}{s+2}$. This leads to the following control design $u(t) = -2x(t) + u_{ad}(t)$, where $u_{ad}(s) = C(s)(\bar{r}(s) + k_g r(s)) = C(s)(\mathfrak{L}\{\hat{\theta}^\top(t)x(t)\})$, $k_g = 2$. Choosing $C(s) = \frac{\omega_n}{s+\omega_n}$ with large ω_n, and setting the adaptive gain Γ_c large, it follows from Theorem 18.6 that

$$x(s) \approx x_{ref}(s) = C(s)k_g H_o(s)r(s) \approx \frac{\omega_n}{s+\omega_n}\frac{2}{s+2}r(s) \approx \frac{2}{s+2}r(s) \tag{403}$$

$$u(s) \approx u_{ref}(s) = (-2+\theta)x_{ref}(s) + 2r(s). \tag{404}$$

The relationship in (403) implies that the control objective is met, while the relationship in (404) states that the $\mathcal{L}_1$ adaptive controller approximates $u_{ref}(t)$, which cancels the unknown θ.

18.8 Simulation Example

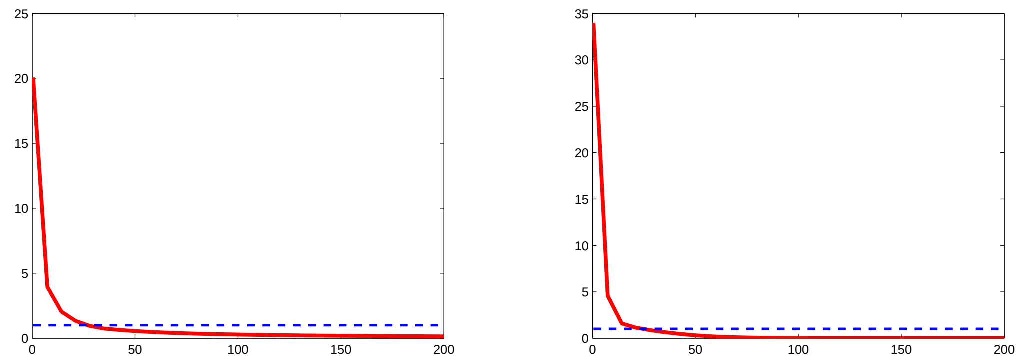
Consider the system in (344) with the following parameters:

$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}, \quad b = [0 \quad 1]^\top, \quad c = [1 \quad 0]^\top, \quad \theta = [4 \quad -4.5]^\top.$$

We further assume that the unknown parameter θ belongs to a known compact set $\Theta = \{\theta \in \mathbb{R}^2 \mid \theta_1 \in [-10, 10], \theta_2 \in [-10, 10]\}$. Letting $\Gamma_c = 10000$, we implement the $\mathcal{L}_1$ adaptive controller following (356), (358) and (359). It follows from (354) that $\theta_{\max} = 20$, while $\|\bar{G}(s)\|_{\mathcal{L}_1}$ can be calculated numerically. In Fig. 38(a), we plot

$$\lambda = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} \tag{405}$$

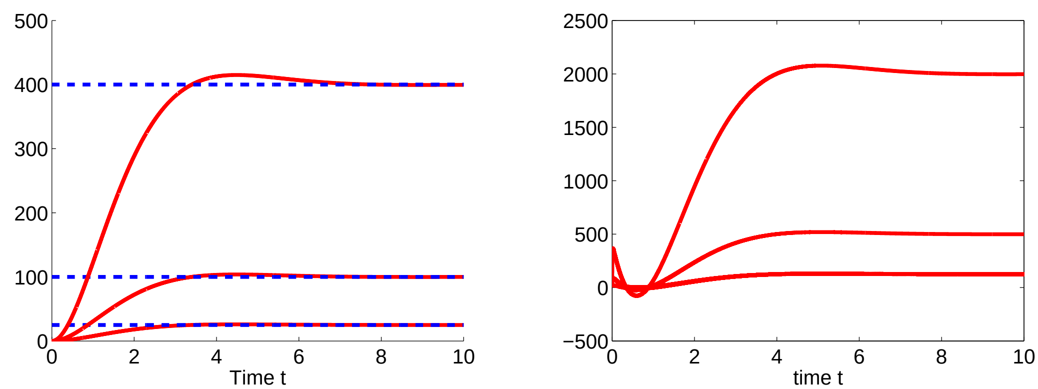
with respect to ω and compare it to 1. We notice that for $\omega > 30$, we have $\lambda < 1$. Choosing $C(s) = \dfrac{160}{s+160}$ gives $\lambda = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} = 0.1725 < 1$, which leads to improved performance bounds in (392)-(395). The simulation results of the $\mathcal{L}_1$ adaptive controller are shown in Figs. 39(a)-39(b) for reference inputs $r = 25,\ 100,\ 400$, respectively. We note that it leads to scaled control inputs and scaled system outputs for scaled reference inputs. Figs. 40(a)-40(b) show the system response and the control signal for reference input $r(t) = 100\cos(0.2t)$, without any retuning of the controller. Figs. 41(a)-41(b) show the system response and the control signal for reference input $r(t) = 100\cos(0.2t)$ and time varying $\theta(t) = [2 + 2\cos(0.5t) \quad 2 + 0.3\cos(0.5t) + 0.2\cos(t/\pi)]^\top$, without any retuning of the



a) λ (solid) defined in (405) **b)** λ (solid) defined in (406)

Fig. 38 **λ (solid) with respect to ω and constant 1 (dashed)**

controller. We note that the $\mathcal{L}_1$ adaptive controller leads to almost identical tracking performance for both constant or time-varying unknown parameters. The control signals are different since they are adapting to different uncertainties to ensure uniform transient response.



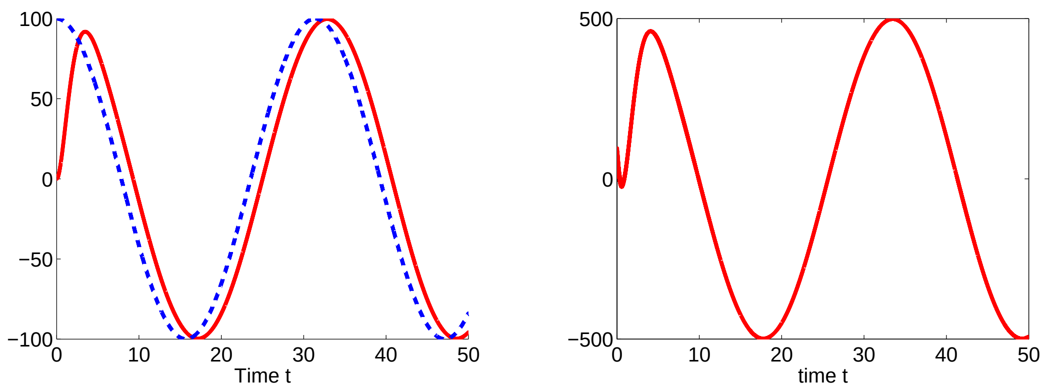
a) $y(t)$ (solid) and $r(t)$ (dashed) **b)** Time-history of $u(t)$

Fig. 39 **Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = \frac{160}{s+160}$ for $r = 25,\ 100,\ 400$**

Next, we consider a higher order filter with low adaptive gain $\Gamma_c = 400\,,\ C(s) = \frac{3\omega^2 s+\omega^3}{(s+\omega)^3}$. In Fig. 38(b), we plot

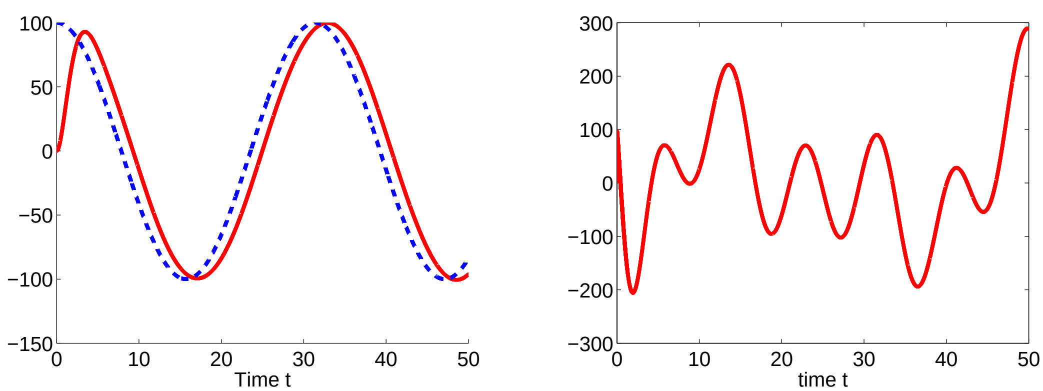
$$\lambda = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} \tag{406}$$

with respect to ω and compare it to 1. We notice that when $\omega > 25$, we have $\lambda < 1$ and the $\mathcal{L}_1$-norm upper bound in (353) is satisfied. Letting $\omega = 50$ leads to $\lambda = 0.3984$. The simulation results of the $\mathcal{L}_1$



a) $y(t)$ (solid) and $r(t)$ (dashed) **b)** Time-history of $u(t)$

Fig. 40 Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = \frac{160}{s+160}$ for $r = 100\cos(0.2t)$



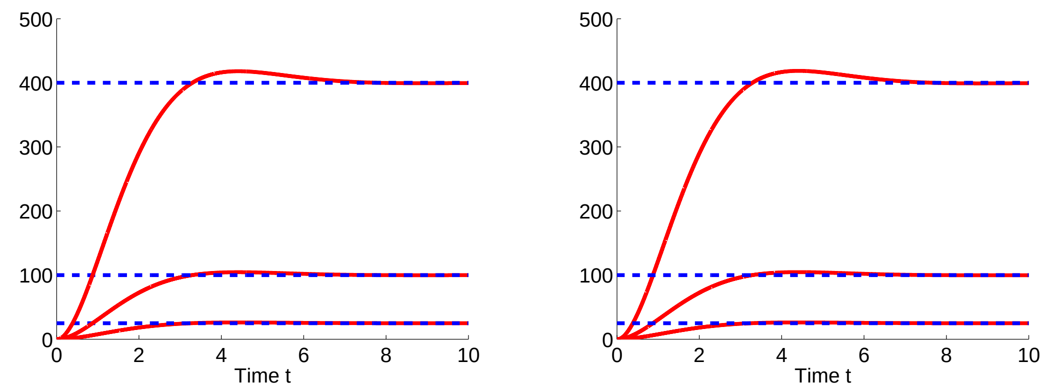
a) $y(t)$ (solid) and $r(t)$ (dashed) **b)** Time-history of $u(t)$

Fig. 41 Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = \frac{160}{s+160}$ for $r = 100\cos(0.2t)$ with time-varying $\theta(t) = [2 + 2\cos(0.5t) \quad 2 + 0.3\cos(0.5t) + 0.2\cos(t/\pi)]^\top$

adaptive controller are shown in Figs. 42(a)-42(b) for reference inputs $r = 25,\ 100,\ 400$, respectively. We note that it again leads to scaled control inputs and scaled system outputs for scaled reference inputs.

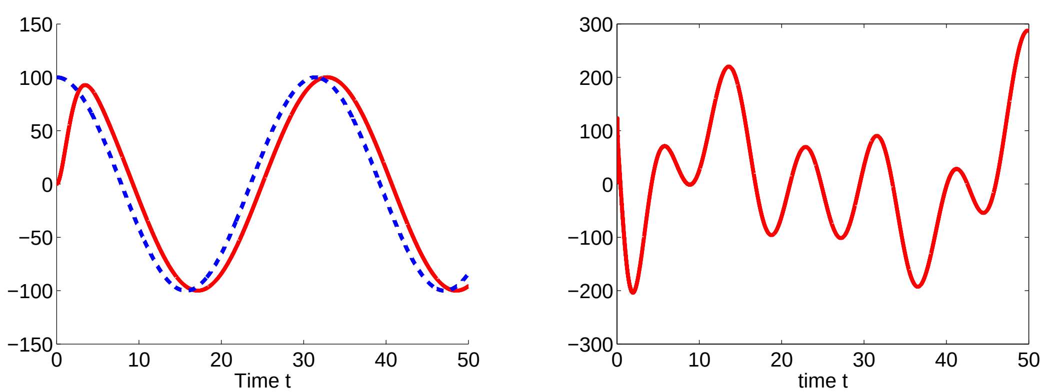
We further show that with time-varying unknown parameters the $\mathcal{L}_1$ adaptive controller following (356), (358) and (359) can still achieve good performance. More rigorous treatment to this case is elaborated in next section. Basically, we prove that, in this case, by increasing the adaptation rate one can ensure uniform transient response for systems both signals, input and output, simultaneously.

Figs. 43(a)-43(b) show the system response and control signal for reference input $r(t) = 100\cos(0.2t)$ and time-varying $\theta(t) = [2 + 2\cos(0.5t) \quad 2 + 0.3\cos(0.5t) + 0.2\cos(t/\pi)]^\top$, without any retuning of the controller. In addition, we notice that this performance is achieved by a much smaller adaptive gain as compared to the design with the first order $C(s)$.



a) $y(t)$ (solid) and $r(t)$ (dashed) **b)** Time-history of $u(t)$

Fig. 42 Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = \frac{7500s+50^3}{(s+50)^3}$ for $r = 25,\ 100,\ 400$



a) $y(t)$ (solid) and $r(t)$ (dashed) **b)** Time-history of $u(t)$

Fig. 43 Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = \frac{7500s+50^3}{(s+50)^3}$ for $r = 100\cos(0.2t)$ with time-varying $\theta(t) = [2 + 2\cos(0.5t) \quad 2 + 0.3\cos(0.5t) + 0.2\cos(t/\pi)]^\top$

Remark 18.10. The simulations pointed out that with higher order filter $C(s)$ one could use relatively small adaptive gain. While a rigorous relationship between the choice of the adaptive gain and the order of the filter cannot be derived, an insight into this can be gained from the following analysis. It follows from (344) and (359) that $x(s) = G(s)r(s) + H_o(s)\theta^\top x(s) + H_o(s)C(s)\bar{r}(s)$, while the state predictor in (356) can be rewritten as $\hat{x}(s) = G(s)r(s) + H_o(s)\left(C(s) - 1\right)\bar{r}(s)$. We note that $\bar{r}(t)$ is divided into two parts. Its low-frequency component $C(s)\bar{r}(s)$ is what the system gets, while the complementary high-frequency component $\left(C(s) - 1\right)\bar{r}(s)$ goes into the state predictor. If the bandwidth of $C(s)$ is large, then it can suppress only the high frequencies in $\bar{r}(t)$, which appear only in the presence of large adaptive gain. A properly designed higher order $C(s)$ can be more effective to serve the purpose of filtering with reduced tailing effects, and, hence can generate similar λ with smaller bandwidth. This further implies that similar performance can be achieved with smaller adaptive gain.

18.9 Extension to Systems with Unknown High-frequency Gain

Once we enabled fast adaptation, we can think of considering systems in the presence of rapidly varying uncertainties. In this section, we consider uncertain systems in the presence of unknown high-frequency gain, time-varying unknown parameters and time-varying bounded disturbances.

18.9.1 Problem Formulation

Consider the following system dynamics:

$$\dot{x}(t) = A_m x(t) + b\left(\omega u(t) + \theta^\top(t)x(t) + \sigma(t)\right), \quad y(t) = c^\top x(t), \quad x(0) = x_0\,, \tag{407}$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the regulated output, $b, c \in \mathbb{R}^n$ are known constant vectors, A_m is a known Hurwitz $n \times n$ matrix, $\omega \in \mathbb{R}$ is an unknown constant with known sign, $\theta(t) \in \mathbb{R}^n$ is a vector of time-varying unknown parameters, while $\sigma(t) \in \mathbb{R}$ is a time-varying disturbance. Without loss of generality, we assume that

$$\omega \in \Omega_0 = [\omega_{l_0},\ \omega_{u_0}]\,, \quad \theta(t) \in \Theta, \quad |\sigma(t)| \le \Delta_0\,, \quad t \ge 0\,, \tag{408}$$

where $\omega_{u_0} > \omega_{l_0} > 0$ are known (conservative) upper and lower bounds, Θ is a known compact set and $\Delta_0 \in \mathbb{R}^+$ is a known (conservative) bound of $\sigma(t)$. We further assume that $\theta(t)$ and $\sigma(t)$ are continuously differentiable and their derivatives are uniformly bounded:

$$\|\dot{\theta}(t)\| \le d_\theta < \infty, \quad |\dot{\sigma}(t)| \le d_\sigma < \infty, \quad \forall\ t \ge 0\,, \tag{409}$$

where $\|\cdot\|$ denotes the 2-norm, while the numbers d_θ, d_σ can be arbitrarily large.

The control objective is to design a full-state feedback adaptive controller to ensure that $y(t)$ tracks a given bounded reference signal $r(t)$ *both in transient and steady state*, while all other error signals remain bounded.

18.9.2 $\mathcal{L}_1$ Adaptive Controller

Let's look at a novel adaptive control architecture for the system in (407) that permits complete transient characterization for both $u(t)$ and $x(t)$. The elements of the $\mathcal{L}_1$ adaptive controller are introduced next:

State Predictor: We consider the following state predictor:

$$\dot{\hat{x}}(t) = A_m\hat{x}(t) + b\left(\hat{\omega}(t)u(t) + \hat{\theta}^\top(t)x(t) + \hat{\sigma}(t)\right), \quad \hat{y}(t) = c^\top\hat{x}(t), \quad \hat{x}(0) = x_0, \tag{410}$$

which has the same structure as the system in (407). The only difference is that the unknown parameters $\omega, \theta(t), \sigma(t)$ are replaced by their adaptive estimates $\hat{\omega}(t), \hat{\theta}(t), \hat{\sigma}(t)$. Adaptive parameters are governed by the following adaptation laws.

Adaptive Laws: Adaptive estimates are given by:

$$\begin{aligned}
\dot{\hat{\theta}}(t) &= \Gamma_\theta \text{Proj}(-x(t)\tilde{x}^\top(t)Pb, \hat{\theta}(t)), \quad \hat{\theta}(0) = \hat{\theta}_0 && (411)\\
\dot{\hat{\sigma}}(t) &= \Gamma_\sigma \text{Proj}(-\tilde{x}^\top(t)Pb, \hat{\sigma}(t)), \qquad \hat{\sigma}(0) = \hat{\sigma}_0 && (412)\\
\dot{\hat{\omega}}(t) &= \Gamma_\omega \text{Proj}(-\tilde{x}^\top(t)Pbu(t), \hat{\omega}(t)), \quad \hat{\omega}(0) = \hat{\omega}_0, && (413)
\end{aligned}$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$, $\Gamma_\theta = \Gamma_c \mathbb{I}_{n\times n} \in \mathbb{R}^{n\times n}$, $\Gamma_\sigma = \Gamma_\omega = \Gamma_c > 0$ are the adaptation rates, and $P = P^\top > 0$ is the solution of the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$, $Q > 0$. In the implementation of the projection operator we use the compact set Θ as given in (408), while we replace Δ_0, Ω_0 by larger sets Δ and $\Omega = [\omega_l,\ \omega_u]$ such that

$$\Delta_0 < \Delta, \qquad 0 < \omega_l < \omega_{l_0} < \omega_{u_0} < \omega_u, \tag{414}$$

the purpose of which will be clarified in the analysis of the time-delay and gain margins.

Next we introduce a different controller structure:

Control Law: The control signal is generated through gain feedback of the following system:

$$\chi(s) = D(s)r_u(s), \quad u(s) = -k\chi(s), \tag{415}$$

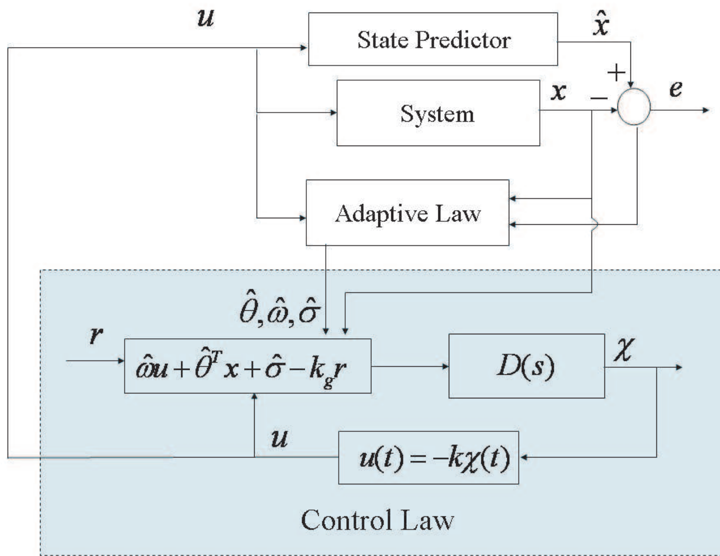


Fig. 44 Closed-loop system with $\mathcal{L}_1$ adaptive controller

where $k > 0$ is a feedback gain, $r_u(s)$ is the Laplace transformation of $r_u(t) = \hat{\omega}(t)u(t) + \bar{r}(t)$, $\bar{r}(t) = \hat{\theta}^\top(t)x(t) + \hat{\sigma}(t) - k_g r(t)$, $k_g = -1/(c^\top A_m^{-1} b)$, while $D(s)$ is any transfer function that leads to strictly proper stable

$$C(s) = \omega k D(s)/(1 + \omega k D(s)) \tag{416}$$

with low-pass gain $C(0) = 1$. One simple choice is $D(s) = 1/s$, which yields a first order strictly proper $C(s)$ in the following form: $C(s) = \omega k/(s + \omega k)$. Further, let

$$L = \max_{\theta(t)\in\Theta} \sum_{i=1}^{n} |\theta_i(t)| \,, \tag{417}$$

where $\theta_i(t)$ is the i^{th} element of $\theta(t)$, Θ is the compact set defined in (408).

The $\mathcal{L}_1$ adaptive controller consists of (410), (411)-(413), (415) subject to the following $\mathcal{L}_1$-norm upper bound:

$$\|G(s)\|_{\mathcal{L}_1} L < 1\,, \quad G(s) = (s\mathbb{I} - A_m)^{-1} b(1 - C(s))\,. \tag{418}$$

The closed loop system is illustrated in Fig. 44.

In case of constant $\theta(t)$ and $\sigma(t)$, the stability requirement of the $\mathcal{L}_1$ adaptive controller can be simplified. For the specific choice of $D(s) = 1/s$, the stability requirement of $\mathcal{L}_1$ adaptive controller is reduced to

$$A_g = \begin{bmatrix} A_m + b\theta^\top & b\omega \\ -k\theta^\top & -k\omega \end{bmatrix} \tag{419}$$

being Hurwitz for all $\theta \in \Theta$, $\omega \in \Omega$.

18.9.3 Closed-loop Reference System

We now consider the following closed-loop (not LTI system in general) reference system with its control signal and system response being defined as follows:

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + b\left(\omega u_{ref}(t) + \theta^\top(t) x_{ref}(t) + \sigma(t)\right), \; x_{ref}(0) = x_0 \tag{420}$$

$$u_{ref}(s) = C(s)\frac{\bar{r}_{ref}(s)}{\omega}, \quad y_{ref}(t) = c^\top x_{ref}(t)\,, \tag{421}$$

where $\bar{r}_{ref}(s)$ is the Laplace transformation of the signal $\bar{r}_{ref}(t) = -\theta^\top(t)x_{ref}(t) - \sigma(t) + k_g r(t)\,.$

Lemma 18.9. If $D(s)$ verifies the condition in (418), the reference system in (420)-(421) is stable.

Proof. Let $H(s) = (s\mathbb{I} - A_m)^{-1}b\,.$ It follows from (420)-(421) that $x_{ref}(s) = G(s)r_1(s) + H(s)C(s)k_g r(s)\,,$ where $r_1(s)$ is the Laplace transformation of $r_1(t) = \theta^\top(t)x_{ref}(t) + \sigma(t)$ subject to the following bound: $\|r_1\|_{\mathcal{L}_\infty} \leq L\|x_{ref}\|_{\mathcal{L}_\infty} + \|\sigma\|_{\mathcal{L}_\infty}\,.$ Since $D(s)$ verifies the condition in (418), then Theorem 18.3 ensures that the closed-loop system in (420)-(421) is stable, if we consider $\Delta(s) = G(s)$ and $M(s) = \theta^\top(s)$. $\square$

Lemma 18.10. If $\theta(t)$ is constant, and $D(s) = 1/s$, then the closed-loop reference system in (420)-(421) is stable *iff* the matrix A_g in (419) is Hurwitz.

Proof. In case of constant $\theta(t)$, the state space form of the closed-loop system in (420)-(421) is given by:

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + b\left(\omega u_{ref}(t) + \theta^\top x_{ref}(t) + \sigma(t)\right),$$

$$\dot{u}_{ref}(t) = -\omega k u_{ref}(t) + k\left(-\theta^\top x_{ref}(t) - \sigma(t) + k_g r(t)\right).$$

Letting $\zeta(t) = [x_{ref}(t)\; u_{ref}(t)]^\top$, it can be rewritten as $\dot{\zeta}(t) = A_g\zeta(t) + [b\sigma(t) \quad kk_g r(t) - k\sigma(t)]^\top$, which is stable iff A_g is Hurwitz. $\square$

18.9.4 Transient and Steady State Performance

To prove uniform transient and steady state tracking between the closed-loop adaptive system with $\mathcal{L}_1$ adaptive controller (407), (410), (411)-(413), (415) and the reference system in (420)-(421), we first need to quantify the prediction error performance that is used in the adaptive law.

Lemma 18.11. For the system in (407) and the $\mathcal{L}_1$ adaptive controller in (410), (411)-(413) and (415), the prediction error between the system state and the predictor is bounded

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_m}{\lambda_{\min}(P)\Gamma_c}},$$

where

$$\theta_m \triangleq \max_{\theta\in\Theta} \sum_{i=1}^{n} 4\theta_i^2 + 4\Delta^2 + 4\left(\omega_u - \omega_l\right)^2 + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\left(\max_{\theta\in\Theta} \|\theta\| d_\theta + d_\sigma \Delta\right).$$

Proof: Consider the candidate Lyapunov function:

$$V(\tilde{x}(t), \tilde{\theta}(t), \tilde{\omega}(t), \tilde{\sigma}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \Gamma_c^{-1}\tilde{\theta}^\top(t)\tilde{\theta}(t) + \Gamma_c^{-1}\tilde{\omega}^2(t) + \Gamma_c^{-1}\tilde{\sigma}^2(t)\,,$$

where $\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta(t),\ \tilde{\sigma}(t) \triangleq \hat{\sigma}(t) - \sigma(t),\ \tilde{\omega}(t) \triangleq \hat{\omega}(t) - \omega$. It follows from (407) and (410) that

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + b(\tilde{\omega}(t)u(t) + \tilde{\theta}^\top(t)x(t) + \tilde{\sigma}(t)), \quad \tilde{x}(0) = 0. \tag{422}$$

Using the projection based adaptation laws from (411)-(413), one has the following upper bound:

$$\dot{V}(t) \leq -\tilde{x}^\top(t)Q\tilde{x}(t) + \Gamma_c^{-1}\tilde{\theta}^\top(t)\dot{\theta}(t) + \Gamma_c^{-1}\tilde{\sigma}(t)\dot{\sigma}(t)\,. \tag{423}$$

The projection algorithm ensures that $\hat{\theta}(t) \in \Theta$, $\hat{\omega}(t) \in \Omega$, $\hat{\sigma}(t) \in \Delta$ for all $t \geq 0$, and therefore

$$\max_{t\geq 0}\left(\Gamma_c^{-1}\tilde{\theta}^\top(t)\tilde{\theta}(t) + \Gamma_c^{-1}\tilde{\omega}^2(t) + \Gamma_c^{-1}\tilde{\sigma}^2(t)\right) \leq \left(\max_{\theta\in\Theta}\sum_{i=1}^{n} 4\theta_i^2 + 4\Delta^2 + 4\left(\omega_u - \omega_l\right)^2\right)/\Gamma_c \tag{424}$$

for any $t \geq 0$. If $V(t) > \theta_m/\Gamma_c$ at some t, then it follows from (424) that

$$\tilde{x}^\top(t)P\tilde{x}(t) > 2\frac{\lambda_{\max}(P)}{\Gamma_c\lambda_{\min}(Q)}\left(\max_{\theta\in\Theta}\|\theta\|d_\theta + d_\sigma\Delta\right),$$

and hence $\tilde{x}^\top(t)Q\tilde{x}(t) > \lambda_{\min}(Q)\tilde{x}^\top(t)P\tilde{x}(t)/\lambda_{\max}(P) > 2(\max_{\theta\in\Theta}\|\theta\|d_\theta + d_\sigma\Delta)/\Gamma_c$. The upper bounds in (409) along with the projection based adaptive laws lead to the following upper bound:

$$\tilde{\theta}^\top(t)\dot{\theta}(t) + \tilde{\sigma}(t)\dot{\sigma}(t) \leq 2\max_{\theta\in\Theta}\|\theta\|d_\theta + d_\sigma\Delta\,.$$

Hence, if $V(t) > \theta_m/\Gamma_c$, then from (423) we have

$$\dot{V}(t) < 0\,. \tag{425}$$

Since we have set $\hat{x}(0) = x(0)$, we can verify that

$$V(0) \leq \Big(\max_{\theta\in\Theta} \sum_{i=1}^{n} 4\theta_i^2 + 4\Delta^2 + 4\left(\omega_u - \omega_l\right)^2 \Big)/\Gamma_c < \theta_m/\Gamma_c.$$

It follows from (425) that $V(t) \leq \dfrac{\theta_m}{\Gamma_c}$ for any $t \geq 0$. Since $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \leq \tilde{x}^\top(t)P\tilde{x}(t) \leq V(t)$, then $||\tilde{x}(t)||^2 \leq \dfrac{\theta_m}{\lambda_{\min}(P)\Gamma_c}$, which concludes the proof. □

Notice that here we only claim boundedness of the tracking error between the system state and the predictor state. This is due to the rate of change of time-varying signals $\theta(t)$ and $\sigma(t)$ appearing in $\dot{V}(\tilde{x}, \tilde{\theta}, \tilde{\omega}, \tilde{\sigma})$. For constant θ and σ asymptotic convergence can be claimed as it is shown in the next section. We further notice that this bound is proportional to the rate of variation of uncertainties and is inverse proportional to the adaptation gain.

Recalling that $H(s) = (s\mathbb{I} - A_m)^{-1}b$, it follows from Lemma 18.4 that there exists $c_o \in \mathbb{R}^n$ s.t.

$$c_o^\top H(s) = N_n(s)/N_d(s)\,, \tag{426}$$

where $\deg(N_d(s)) - \deg(N_n(s)) = 1$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials. The next two theorems are in charge for both transient and steady-state performance of the $\mathcal{L}_1$ adaptive controller.

Theorem 18.7. Given the system in (407) and the $\mathcal{L}_1$ adaptive controller defined via (410), (411)-(413) and (415) subject to (418), we have:

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_1\,, \quad \|u - u_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_2\,, \tag{427}$$

where

$$\gamma_1 = \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|H(s)(1 - C(s))\|_{\mathcal{L}_1} L} \sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}, \quad \gamma_2 = \left\| \frac{C(s)}{\omega} \right\|_{\mathcal{L}_1} L\gamma_1 + \left\| \frac{C(s)}{\omega} \frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}.$$

Proof: Let $\tilde{r}(t) = \tilde{\omega}(t)u(t) + \tilde{\theta}^\top(t)x(t) + \tilde{\sigma}(t)\,,\ r_2(t) = \theta^\top(t)x(t) + \sigma(t)\,.$ It follows from (415) that $\chi(s) = D(s)(\omega u(s) + r_2(s) - k_g r(s) + \tilde{r}(s))\,,$ where $\tilde{r}(s)$ and $r_2(s)$ are the Laplace transformations of signals $\tilde{r}(t)$ and $r_2(t)$. Consequently

$$\chi(s) = \frac{D(s)}{1 + k\omega D(s)}(r_2(s) - k_g r(s) + \tilde{r}(s))\,, \quad u(s) = -\frac{kD(s)}{1 + k\omega D(s)}(r_2(s) - k_g r(s) + \tilde{r}(s))\,.$$

Using the definition of $C(s)$ from (416), we can write

$$\omega u(s) = -C(s)(r_2(s) - k_g r(s) + \tilde{r}(s))\,, \tag{428}$$

and the system in (407) consequently takes the form:

$$x(s) = H(s)\left((1 - C(s))r_2(s) + C(s)k_g r(s) - C(s)\tilde{r}(s)\right). \tag{429}$$

It follows from (420)-(421) that $x_{ref}(s) = H(s)\left((1 - C(s))r_1(s) + C(s)k_g r(s)\right)$, where $r_1(s)$ is the Laplace transformation of the signal $r_1(t)$. Let $e(t) = x(t) - x_{ref}(t)$. Then, using (429), one gets

$$e(s) = H(s)\left((1 - C(s))r_3(s) - C(s)\tilde{r}(s)\right), \quad e(0) = 0\,, \tag{430}$$

where $r_3(s)$ is the Laplace transformation of the signal

$$r_3(t) = \theta^\top(t)e(t)\,. \tag{431}$$

Lemma 18.11 gives the following upper bound:

$$\|e_t\|_{\mathcal{L}_\infty} \leq \|H(s)(1 - C(s))\|_{\mathcal{L}_1}\|r_{3_t}\|_{\mathcal{L}_\infty} + \|r_{4_t}\|_{\mathcal{L}_\infty}\,, \tag{432}$$

where $r_4(t)$ is the signal with its Laplace transformation being $r_4(s) = C(s)H(s)\tilde{r}(s)$. From the relationship in (422) we have $\tilde{x}(s) = H(s)\tilde{r}(s)$, which leads to $r_4(s) = C(s)\tilde{x}(s)$, and hence $\|r_{4_t}\|_{\mathcal{L}_\infty} \leq \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. Using the definition of L in (417), one can verify easily that $\|(\theta^\top e)_t\|_{\mathcal{L}_\infty} \leq L\|e_t\|_{\mathcal{L}_\infty}$, and from (431) we have that $\|r_{3_t}\|_{\mathcal{L}_\infty} \leq L\|e_t\|_{\mathcal{L}_\infty}$. From (432) we have

$$\|e_t\|_{\mathcal{L}_\infty} \leq \|H(s)(1 - C(s))\|_{\mathcal{L}_1} L\|e_t\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}\,.$$

The upper bound from Lemma 18.11 and the $\mathcal{L}_1$-norm upper bound from (418) lead to the following upper bound

$$\|e_t\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|H(s)(1 - C(s))\|_{\mathcal{L}_1} L}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}},$$

which holds uniformly for all $t \geq 0$ and therefore leads to the first bound in (427).

To prove the second bound in (427), we notice that from (421) and (428) one can derive

$$u(s) - u_{ref}(s) = -\frac{C(s)}{\omega}\theta^\top(t)(x(s) - x_{ref}(s)) - r_5(s)\,,$$

where

$$r_5(s) = \frac{C(s)}{\omega}\tilde{r}(s).$$

Therefore, it follows from Lemma 18.11 that

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \le (L/\omega)\|C(s)\|_{\mathcal{L}_1}\|x - x_{ref}\|_{\mathcal{L}_\infty} + \|r_5\|_{\mathcal{L}_\infty}\,. \tag{433}$$

We have

$$r_5(s) = \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top H(s)\tilde{r}(s) = \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top \tilde{x}(s)\,,$$

where c_o is introduced in (426). Using the polynomials from (426), we can write that

$$\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} = \frac{C(s)}{\omega}\frac{N_d(s)}{N_n(s)}\,.$$

Since $C(s)$ is stable and strictly proper, the complete system $C(s)\frac{1}{c_o^\top H(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$-norm exists and is finite. Hence, we have

$$\|r_5\|_{\mathcal{L}_\infty} \le \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}\,.$$

Lemma 18.11 consequently leads to the upper bound:

$$\|r_5\|_{\mathcal{L}_\infty} \le \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}\,,$$

which, when substituted into (433), leads to the second bound in (427). $\square$

Theorem 18.8. For the closed-loop system in (407) with $\mathcal{L}_1$ adaptive controller defined via (410), (411)-(413) and (415), subject to (419), if $\theta(t)$ is (unknown) constant and $D(s) = \dfrac{1}{s}$, we have:

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \le \gamma_3\,, \quad \|u - u_{ref}\|_{\mathcal{L}_\infty} \le \gamma_4\,, \tag{434}$$

where

$$\gamma_3 = \left\|H_g(s)C(s)\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}},\ H_g(s) = (s\mathbb{I} - A_g)\begin{bmatrix} b \\ 0 \end{bmatrix},$$

$$\gamma_4 = \left\|\frac{C(s)}{\omega}\theta^\top\right\|_{\mathcal{L}_1}\gamma_3 + \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}.$$

Proof: Let

$$\zeta(s) = -\frac{C(s)}{\omega}\theta^\top e(s).$$

With this notation, (430) can be written as

$$e(s) = H(s)(\theta^\top e(s) + \omega\zeta(s) - C(s)\tilde{r}(s))$$

and further put into state space form as:

$$\begin{bmatrix} \dot{e}(t) \\ \dot{\zeta}(t) \end{bmatrix} = A_g \begin{bmatrix} e(t) \\ \zeta(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} r_6(t)\,, \tag{435}$$

where $r_6(t)$ is the signal with its Laplace transformation $r_6(s) = -C(s)\tilde{r}(s)$. Let

$$x_\zeta(t) = [e^\top(t)\ \zeta(t)]^\top.$$

Since A_g is Hurwitz, then $H_g(s)$ is stable and strictly proper. It follows from (435) that $x_\zeta(s) = -H_g(s)C(s)\tilde{r}(s)$. Therefore, we have

$$x_\zeta(s) = -H_g(s)C(s)\frac{1}{c_o^\top H(s)}c_o^\top H(s)\tilde{r}(s) = -H_g(s)C(s)\frac{1}{c_o^\top H(s)}c_o^\top \tilde{x}(s),$$

where c_o is introduced in (426). It follows from (426) that

$$H_g(s)C(s)\frac{1}{c_o^\top H(s)} = H_g(s)C(s)\frac{N_d(s)}{N_n(s)}\,. \tag{436}$$

Since both $H_g(s)$ and $C(s)$ are stable and strictly proper, the complete system $H_g(s)C(s)\frac{1}{c_o^\top H(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$-norm exists and is finite. Hence, we have

$$\|x_\zeta\|_{\mathcal{L}_\infty} \leq \left\|H_g(s)C(s)\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}\,. \tag{437}$$

The proof of (434) is similar to the proof of (427). □

Corollary 18.1. Given the system in (407) and the $\mathcal{L}_1$ adaptive controller defined via (410), (411)-(413) and (415) subject to (418), we have:

$$\lim_{\Gamma_c\to\infty}(x(t) - x_{ref}(t)) = 0\,,\ \lim_{\Gamma_c\to\infty}(u(t) - u_{ref}(t)) = 0\,,\ \forall\ t \geq 0\,.$$

Thus, the tracking error between $x(t)$ and $x_{ref}(t)$, as well between $u(t)$ and $u_{ref}(t)$, is uniformly bounded by a constant inverse proportional to Γ_c. This implies that during the transient phase one can achieve arbitrarily close tracking performance for both signals simultaneously by increasing the adaptation rate Γ_c.

Remark 18.11. We notice that the above analysis assumes zero trajectory initialization error, i.e. $\hat{x}_0 = x_0$, which is in the spirit of the methods for transient performance improvement in [16]. In [5], we have proved that non-zero trajectory initialization error leads only to an exponentially decaying term in both system state and control signal, without affecting the performance throughout.

18.9.5 Asymptotic Convergence

Since the bounds in (427) are uniform for all $t \geq 0$, they are in charge for both transient and steady state performance. In case of constant θ and σ one can prove in addition the following asymptotic result.

Lemma 18.12. Given the system in (407) with constant θ, σ and $\mathcal{L}_1$ adaptive controller defined via (410), (411)-(413) and (415) subject to (418), we have: $\lim\limits_{t\to\infty} \tilde{x}(t) = 0$.

Proof: It follows from Lemmas 18.9 and 18.11, and Theorem 18.7 that both $x(t)$ and $\hat{x}(t)$ in $\mathcal{L}_1$ adaptive controller are bounded for bounded reference inputs. The adaptive laws in (411)-(413) ensure that the estimates $\hat{\theta}(t)$, $\hat{\omega}(t)$, $\hat{\sigma}(t)$ are also bounded. Hence, it can be checked easily from error dynamics that $\dot{\tilde{x}}(t)$ is bounded, and it follows from Barbalat's lemma that $\lim\limits_{t\to\infty} \tilde{x}(t) = 0$. □

18.9.6 Design Guidelines

We note that the control law $u_{ref}(t)$ in the closed-loop reference system, which is used in the analysis of $\mathcal{L}_\infty$ norm bounds, is not implementable since its definition involves the unknown parameters. Theorem 18.7 ensures that the $\mathcal{L}_1$ adaptive controller approximates $u_{ref}(t)$ both in transient and steady state. So, it is important to understand how these bounds can be used for ensuring uniform transient response with *desired* specifications. We notice that the following *ideal* control signal

$$u_{ideal}(t) = \frac{k_g r(t) - \theta^\top(t) x_{ref}(t) - \sigma(t)}{\omega} \tag{438}$$

is the one that leads to desired system response:

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + b k_g r(t), \quad y_{ref}(t) = c^\top x_{ref}(t) \tag{439}$$

by cancelling the uncertainties exactly. In the closed-loop reference system (420)-(421), $u_{ideal}(t)$ is further low-pass filtered by $C(s)$ to have guaranteed low-frequency range. Thus, the reference system in (420)-(421) has a different response as compared to (439) achieved with (438). Similar to the case of constant unknown parameters, one can think of design guidelines for selection of $C(s)$ that ensure that the response of $x_{ref}(t)$ and $u_{ref}(t)$ can be made as close as possible to (439).

18.10 Analysis of Stability Margins

18.10.1 Time-delay Margin Analysis

In section 18.4 you have already seen time-delay analysis for one specific case of MRAC (PI controller) and $\mathcal{L}_1$ adaptive control. However, in a general sense, it is extremely difficult to analyze the time-delay margin of a conventional closed loop adaptive control system due to its nonlinear characteristics. Notice here that we talk about characterization of the time-delay margin of an adaptive control system, not adaptive control of time-delayed systems, which has been studied intensively.

For the $\mathcal{L}_1$ adaptive control scheme, we can derive the time-delay margin of it. We give a rigorous proof for a conservative lower bound for the time-delay that the closed-loop adaptive system with $\mathcal{L}_1$ adaptive controller can tolerate. We know that for linear systems the time-delay margin can be obtained from their phase margin. So, given the $\mathcal{L}_1$ adaptive controller, can we find an equivalent LTI system whose time-delay margin can be related to that of the closed loop adaptive system? The answer is YES.

We develop the time-delay margin analysis for the case of constant unknown parameters, i.e. when $\theta = const$. We rewrite the open-loop system in (407) as

$$x(s) = \bar{H}(s)(\omega u(s) + \sigma(s)), \tag{440}$$

where $\bar{H}(s) = (sI - A_m - b\theta^\top)^{-1} b$. Without loss of generality, we set:

$$x(0) = 0. \tag{441}$$

We further consider the following three systems.

System 1. Let $x_d(t)$ be the delayed signal of the open-loop state $x(t)$ of (440) by a constant time interval τ, i.e

$$x_d(t) = \begin{cases} x(t-\tau) & t \geq \tau\,, \\ 0 & t < \tau\,. \end{cases} \tag{442}$$

We close the loop of (440) with $\mathcal{L}_1$ adaptive controller (410), (411)-(413), (415), using $x_d(t)$ from (442) instead of $x(t)$ everywhere in the definition of (410), (411)-(413), (415). We denote the resulting control and the state trajectory of this closed-loop system by $u(t)$ and $x_d(t)$. We further notice that this closed-loop adaptive system has a UNIQUE solution. It is the stability of this closed-loop system that we are trying to determine dependent upon τ. It is important to point out that while applying the $\mathcal{L}_1$ adaptive controller (410), (411)-(413), (415) to the system in (440) using $x_d(t)$ from (442), one cannot derive the dynamics of the error signal between the system state and the predictor state, the boundedness of which is stated in Lemma 18.11. Neither Theorems 18.7, 18.8 are valid.

System 2. Next, we consider the following closed-loop system with the same zero initial conditions:

$$\dot{x}_q(t) = A_m x_q(t) + b\left(\omega u_q(t) + \theta^\top x_q(t) + \sigma(t) + \eta(t)\right)\,, \tag{443}$$

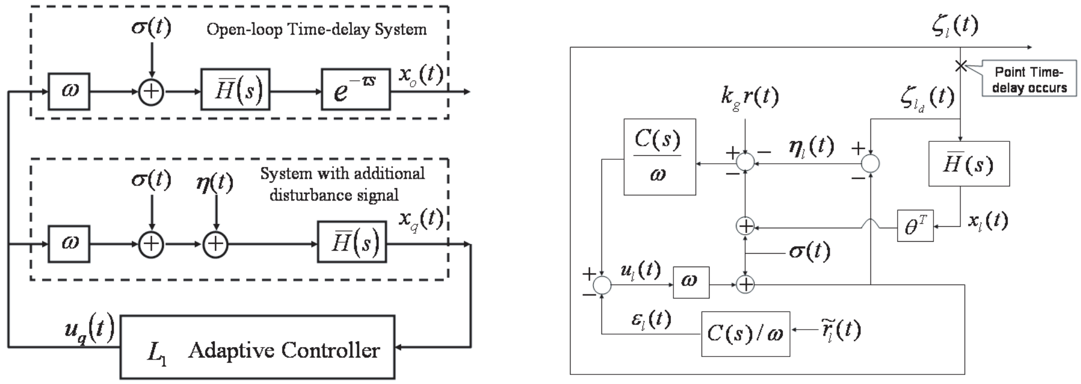
where $x_q(0) = x(0)$, $u_q(t)$ is defined via (410), (411)-(413) and (415) with $x(t)$ being replaced by $x_q(t)$, while $\eta(t)$ is a continuously differentiable bounded signal with uniformly bounded derivative. As compared to (407) or (440), the system in (443) has one more additional disturbance signal $\eta(t)$. If

$$|\sigma(t) + \eta(t)| \leq \Delta\,, \tag{444}$$

where Δ has been introduced in (414) (but not explicitly defined yet! Further construction of Δ is needed.), then application of $\mathcal{L}_1$ adaptive controller to the system in (443) is well defined, and hence the results of Theorem 18.7 are valid. We denote by $u_q(t)$ the time trajectory of the $\mathcal{L}_1$ adaptive controller, resulting from its application to (443).

System 3. Finally, we consider the open-loop system in (440)-(442) and apply $u_q(t)$ to it and look at its delayed output $x_o(t)$, where the subindex o is added to indicate the open-loop nature of this signal. It is important to notice that at this point we view $u_q(t)$ as a time-varying input signal for (440), and not as a feedback signal, so that (440) remains an open-loop system in this context.

Illustration of these last two systems is given in Fig. 45(a).



a) Systems 2 and 3 **b)** LTI system.

Fig. 45

Lemma 18.13. If the time-delayed output of the open-loop System 3 has the same time history as the closed-loop output of System 2, i.e.

$$x_o(t) = x_q(t), \ \forall \ t \geq 0 \,, \tag{445}$$

then $u(t) = u_q(t), \ x_d(t) = x_q(t), \forall \ t \geq 0$, where $u(t)$ and $x_d(t)$ denote the control and state trajectories of the closed-loop System 1 in (440)-(442) with $\mathcal{L}_1$ adaptive controller.

Proof. Eq. (445) implies that the open-loop time-delayed System 3 in (440)-(442) generates $x_q(t)$ in response to the input $u_q(t)$. When applied to (443), $u_q(t)$ leads to $x_q(t)$. Hence, $u_q(t)$ and $x_q(t)$ are also solutions of the closed-loop adaptive System 1 in (440)-(442) with (410), (411)-(413), (415). □

This Lemma consequently implies that to ensure stability of the System 1 in the presence of a given time-delay τ, it is sufficient to prove existence of $\eta(t)$ in System 2, satisfying (444) and verifying (445). Satisfying (444) can guarantee that the $\mathcal{L}_1$ adaptive controller is well defined for the System 2. Verifying (445) proves the equivalence between the time histories of the signals of System 1 and System 2. With that we are one step closer to our final goal of analyzing the time-delay margin of the closed-loop adaptive system with $\mathcal{L}_1$ adaptive controller.

We notice, however, that the closed-loop System 2 is a nonlinear system due to the nonlinear adaptive laws, so that the proof on existence of such $\eta(t)$ for this system and explicit construction of the set Δ is not straightforward. Moreover, we note that the condition in (445) relates the time-delay τ of System 1 (or System 3) to the signal $\eta(t)$ implicitly. We then introduce another equivalent LTI system that helps to prove existence of such $\eta(t)$ and leads to explicit construction of Δ. Since the equivalent system is a LTI, we can calculate its time-delay margin. Then we can study the relationship between this LTI system and System 1 in the presence of the same time-delay τ, where τ is within the time-delay margin of the LTI system. The time-delay margin of the equivalent LTI system serves as a conservative, but guaranteed, lower bound on time-delay margin of System 1. Definition of this LTI system is the key step in the overall time-delay margin analysis. It has an exogenous input that lumps the time trajectories of the nonlinear elements of the closed-loop System 2. We show only the main results in the notes, and the readers are encouraged to find more details in [31].

Consider the following closed-loop LTI system:

$$
\begin{aligned}
&x_l(s) = \bar{H}(s)\zeta_l(s), \;\; \epsilon_l(s) = (C(s)/\omega)\tilde{r}_l(s) \\
&u_l(s) = (1/\omega)C(s)(k_g r(s) - \theta^\top x_l(s) - \sigma(s) - \eta_l(s)) - \epsilon_l(s)
\end{aligned}
$$

where $\zeta_l(s) = \omega u_l(s) + \sigma(s)$, $\eta_l(s) = \zeta_l(s) - \omega u_l(s) - \sigma(s)$, $r(s)$ and $\sigma(s)$ are the Laplace transformations of the bounded signals $r(t)$ and $\sigma(t)$, respectively, $x_l(t)$, $u_l(t)$ and $\epsilon_l(t)$ are the states, $\zeta_l(t)$ is its output signal, and $\tilde{r}_l(s)$ is the Laplace transformation of an exogenous signal $\tilde{r}_l(t)$. We note that the system trajectories are uniquely defined once $\tilde{r}_l(t)$ is given.

Assume the system output $\zeta_l(t)$ experiences time-delay τ, so that in the presence of the time-delay we have:

$$x_l(s) = \bar{H}(s)\zeta_{l_d}(s) \tag{446}$$

$$u_l(s) = (C(s)/\omega)\Big(k_g r(s) - \theta^\top x_l(s) - \sigma(s) - \eta_l(s)\Big) - \epsilon_l(s) \tag{447}$$

$$\epsilon_l(s) = (C(s)/\omega)\tilde{r}_l(s) \tag{448}$$

$$\zeta_l(s) = \omega u_l(s) + \sigma(s)\,, \tag{449}$$

where $\zeta_{l_d}(t)$ is the time-delayed signal of $\zeta_l(t)$, i.e

$$\zeta_{l_d}(t) = \begin{cases} 0 & t < \tau\,, \\ \zeta_l(t-\tau) & t \geq \tau\,, \end{cases} \tag{450}$$

consequently leading to redefined $\eta_l(s)$:

$$\eta_l(s) = \zeta_{l_d}(s) - \omega u_l(s) - \sigma(s). \tag{451}$$

Let

$$x_l(0) = 0, \quad u_l(0) = 0\,, \quad \epsilon_l(0) = 0\,. \tag{452}$$

We notice that the system in (446)-(449) is highly coupled. Its diagram is plotted in Figure 45(b).

The phase margin of this LTI system can be determined by its open-loop transfer function from $\zeta_{l_d}(t)$ to $\zeta_l(t)$. It can be equivalently written as:

$$\begin{aligned} \zeta_l(s) &= \frac{1}{1-C(s)}\left(r_b(s) - r_f(s)\right)\,, \quad r_f(s) = C(s)(1+\theta^\top \bar{H}(s))\zeta_{l_d}(s)\,, \\ r_b(s) &= C(s)k_g r(s) + (1-C(s))\sigma(s) - \omega\epsilon_l(s)\,. \end{aligned} \tag{453}$$

Assume that $\tilde{r}_l(t)$ is such that $\epsilon_l(t)$ is bounded. Since $\sigma(t)$ and $r(t)$ are bounded, $C(s)$ is strictly proper and stable, then $r_b(t)$ is also bounded. The open-loop transfer function of the system in (453) is:

$$H_o(s) = \frac{C(s)}{1-C(s)}(1+\theta^\top \bar{H}(s))\,, \tag{454}$$

the phase margin $\mathcal{P}(H_o(s))$ of which can be derived from its Bode plot easily. Its time-delay margin is given by:

$$\mathcal{T}(H_o(s)) = \mathcal{P}(H_o(s))/\omega_c\,, \tag{455}$$

where $\mathcal{P}(H_o(s))$ is the phase margin of the open-loop system $H_o(s)$, and ω_c is the cross-over frequency of $H_o(s)$. The next lemma states a sufficient condition for boundedness of all the states in the system (446)-(449), including the internal states.

Lemma 18.14. Let

$$\tau < \mathcal{T}(H_o(s))\,, \tag{456}$$

and ϵ_b be any positive number such that $\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b$. Then the signals $\zeta_l(t)$, $x_l(t)$, $u_l(t)$, $\eta_l(t)$ are bounded.

We then construct the set Δ. For any $\tau < \mathcal{T}(H_o(s))$ and any $\epsilon_b > 0$, Lemma 18.14 guarantees that the map $\Delta_n : \mathbb{R}^+ \times [0, \mathcal{T}(H_o(s))) \to \mathbb{R}^+$

$$\Delta_n(\epsilon_b, \tau) \;=\; \max_{\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b} \|\sigma + \eta_l\|_{\mathcal{L}_\infty} \tag{457}$$

is well defined.

Lemma 18.15. Let τ comply with (456), and ϵ_b be any positive number. If $\tilde{r}_l(t)$ is such that the resulting $\epsilon_l(t)$ is bounded

$$\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b \,, \tag{458}$$

and

$$2\omega\|u_l\|_{\mathcal{L}_\infty} + 2L\|x_l\|_{\mathcal{L}_\infty} + 2\Delta \geq \|\tilde{r}_l\|_{\mathcal{L}_\infty} \,, \tag{459}$$

where

$$\Delta = \Delta_n(\epsilon_b, \tau) + \delta_1 \,, \tag{460}$$

with $\delta_1 > 0$ being arbitrary constant, then $\eta_l(t)$ has a uniformly bounded derivative.

For any $\tau < \mathcal{T}(H_o(s))$ and any $\epsilon_b > 0$, Lemma 18.15 guarantees that the following map $\Delta_d : \mathbb{R}^+ \times [0, \mathcal{T}(H_o(s))) \to \mathbb{R}^+$

$$\Delta_d(\epsilon_b, \tau) = \max_{\tilde{r}_l(t)} \|\dot{\sigma} + \dot{\eta}_l\|_{\mathcal{L}_\infty} \tag{461}$$

is well defined, where $\tilde{r}_l(t)$ complies with (458) and (459). Further, let

$$\theta_m(\epsilon_b, \tau) \triangleq \max_{\theta \in \Theta} \sum_{i=1}^{n} 4\theta_i^2 + 4\Delta^2 + 4\left(\omega_u - \omega_l\right)^2 + 2\lambda_{\max}(P)\Delta_d(\epsilon_b, \tau)\Delta/\lambda_{\min}(Q) \,, \tag{462}$$

$$\epsilon_c(\epsilon_b, \tau) = \left\| C(s)(c_o^\top H(s))^{-1} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\theta_m(\epsilon_b, \tau)/(\lambda_{\max}(P)\epsilon_b^2)} \,. \tag{463}$$

We notice that for any finite $\epsilon_b \in \mathbb{R}^+$ and any τ verifying (456), we have finite $\Delta_n(\epsilon_b, \tau)$ and $\Delta_d(\epsilon_b, \tau)$, and hence finite $\epsilon_c(\epsilon_b, \tau)$, if $\tilde{r}_l(t)$ complies with (458) and (459).

The main result is given by the following theorem.

Theorem 18.9. Consider the closed-loop adaptive system, comprised of System 1 in (440)-(442) with (410), (411)-(413), (415) and the LTI system in (446)-(449) in the presence of the same time delay τ. For any $\epsilon_b \in \mathbb{R}^+$ choose the set Δ as in (460) and let

$$\Gamma_c \geq \sqrt{\epsilon_c(\epsilon_b, \tau)} + \delta_2 \,, \tag{464}$$

where δ_2 is arbitrary positive constant. Then for every τ satisfying $\tau < \mathcal{T}(H_o(s))$, there exists an exogenous signal $\tilde{r}_l(t)$ ensuring that $\|\epsilon_l\|_{\mathcal{L}_\infty} < \epsilon_b$, and $x_l(t) = x_d(t)\,, u_l(t) = u(t)\,, \forall\, t \geq 0\,.$

Theorem 18.9 establishes the equivalence of state and control trajectories of the closed-loop adaptive system and the LTI system in (446)-(449) in the presence of the same time-delay. Therefore the time-delay margin of the system in (446)-(449) can be used as a conservative lower bound for the time-delay margin of the closed-loop adaptive system.

18.10.2 Gain Margin Analysis

We now analyze the gain margin of the system in (407) with $\mathcal{L}_1$ adaptive controller. By inserting a gain module g into the control loop, the system in (407) can be formulated as:

$$\dot{x}(t) = A_m x(t) + b\left(\omega_g u(t) + \theta^\top(t)x(t) + \sigma(t)\right)\,, \tag{465}$$

where $\omega_g = g\omega$. We note that this transformation implies that the set Ω in the application of the Projection operator for adaptive laws needs to increase accordingly. However, increased Ω will not violate the condition in (418). Thus, it follows from (414) that the gain margin of the $\mathcal{L}_1$ adaptive controller is determined by:

$$\mathcal{G}_m = [\omega_l/\omega_{l_0},\ \ \omega_u/\omega_{u_0}]. \tag{466}$$

If $g \in \mathcal{G}_m$, then the closed-loop system in (465) satisfies the $\mathcal{L}_1$ stability criterion in (418), implying that the entire closed-loop system is stable. We note that the lower-bound of $\mathcal{G}_m$ is greater than zero. Eq. (466) implies that arbitrary gain margin can be obtained through appropriate choice of Ω.

18.11 Final Words on Design Issues

The most important feature of $\mathcal{L}_1$ adaptive controller is its ability of fast adaptation. We have seen that fast adaptation with $\mathcal{L}_1$ adaptive controller ensures guaranteed transient response with respect to $x_{ref}(t)$ and $u_{ref}(t)$, and bounded away from zero time-delay margin for the closed-loop system with it. The remaining question is: under which conditions the signals $x_{ref}(t)$ and $u_{ref}(t)$ perform as desired, from the perspective of control design?

Take an extreme case as an example. If $C(s) = 1$, which means that the low pass filter has infinitely large bandwidth, then $x_{ref}(t)$ and $u_{ref}(t)$ are reduced to the ideal reference signals of MRAC defining the desired performance. However, as we notice from (454), increasing the bandwidth of $C(s)$ hurts the time-delay margin of the closed-loop adaptive system *in the presence of fast adaptation*. We have observed this with MRAC that increasing the adaptive gain leads to reduced time-delay margin. Thus, the choice of the $C(s)$ (or original $D(s)$) defines the trade-off between the desired (ideal) tracking performance and the robustness - the golden rule of the feedback. More details on selection of $C(s)$ as how to achieve the desired tracking performance and the corresponding performance bounds one can find in [31].

18.12 Simulation Example

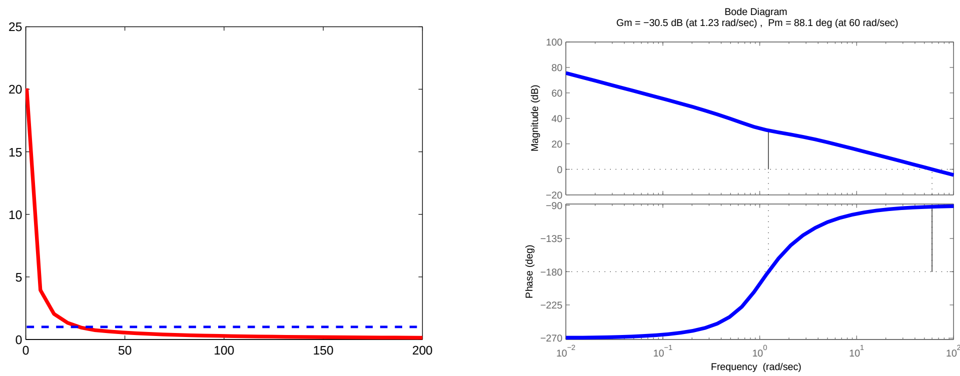
Consider the dynamics of a single-link robot arm rotating on a vertical plane:

$$I\ddot{q}(t) + \frac{Mgl\cos q(t)}{2} + F(t)\dot{q}(t) + F_1(t)q(t) + \bar{\sigma}(t) = u(t)\,, \tag{467}$$

where $q(t)$ and $\dot{q}(t)$ are measured angular position and velocity, respectively, $u(t)$ is the input torque, I is the unknown moment of inertia, M is the unknown mass, l is the unknown length, $F(t)$ is an unknown time-varying friction coefficient, $F_1(t)$ is position dependent external torque, and $\bar{\sigma}(t)$ is unknown bounded disturbance. The control objective is to design $u(t)$ to achieve tracking of bounded reference input $r(t)$ by $q(t)$. Let $x = [q\ \dot{q}]^\top$. The system in (467) can be presented in the state-space form as:

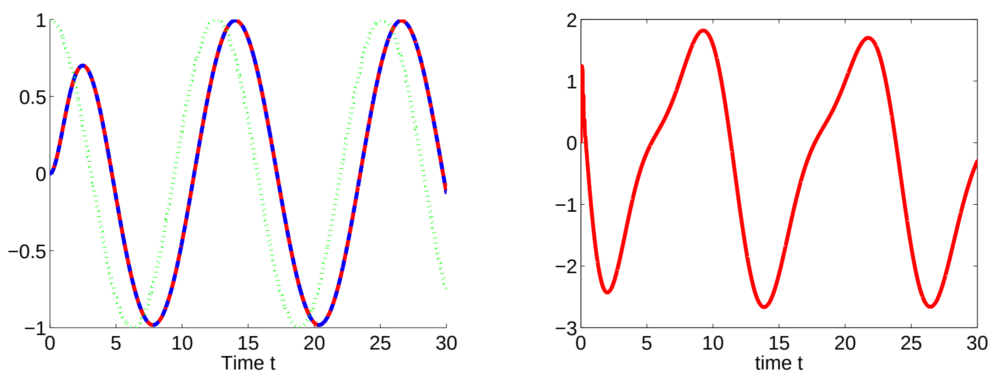
$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\Big(\frac{u(t)}{I} + \frac{Mgl\cos(x_1(t))}{2I} + \frac{\bar{\sigma}(t)}{I} + \frac{F_1(t)}{I}x_1(t) + \frac{F(t)}{I}x_2(t)\Big), \quad x(0) = x_0\,, \\ y(t) &= c^\top x(t)\,, \end{aligned} \tag{468}$$

where x_0 is the initial condition, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The system can be further put into the form: $\dot{x}(t) = A_m x(t) + b(\omega u(t) + \theta^\top(t)x(t) + \sigma(t))\,,\ y(t) = c^\top x(t)$, where $\omega = 1/I$ is the unknown control effectiveness, $\theta(t) = [1 + \frac{F_1(t)}{I}\ \ 1.4 + \frac{F(t)}{I}]^\top\,,\ \sigma(t) = \dfrac{Mgl\cos(x_1(t))}{2I} + \dfrac{\bar{\sigma}(t)}{I}$, and $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$. Let the unknown control effectiveness, time-varying parameters and disturbance be given by: $\omega = 1\,,\ \theta(t) = [2+\cos(\pi t)\quad 2+0.3\sin(\pi t)+0.2\cos(2t)]^\top\,,\ \sigma(t) = \sin(\pi t)$, so that the compact sets can be conservatively chosen as $\Omega = [0.5,\ 2],\ \Theta = [-10,\ 10],\ \Delta = [-10,\ 10]$. For implementation of the $\mathcal{L}_1$ adaptive controller (410), (411)-(413) and (415), we need to verify the $\mathcal{L}_1$ stability requirement in (418). Letting $D(s) = 1/s$, we have $G(s) = \frac{\omega k}{s+\omega k}H(s),\ H(s) = [\frac{1}{s^2+1.4s+1}\ \frac{s}{s^2+1.4s+1}]^\top$. We can check that $L = 20$ in (417). In Fig. 46(a), we plot $\|G(s)\|_{\mathcal{L}_1}L$ as a function of ωk and compare it to 1. We notice that for $\omega k > 30$, we have $\|G(s)\|_{\mathcal{L}_1}L < 1$. Since $\omega > 0.5$, we set $k = 60$. We set the adaptive gain $\Gamma_c = 10000$.



a) $\|G(s)\|_{\mathcal{L}_1} L$ with respect to ωk. **b)** Bode plot of $H_o(s)$ for $\theta = (2 \quad 2)^\top, \omega = 1$.

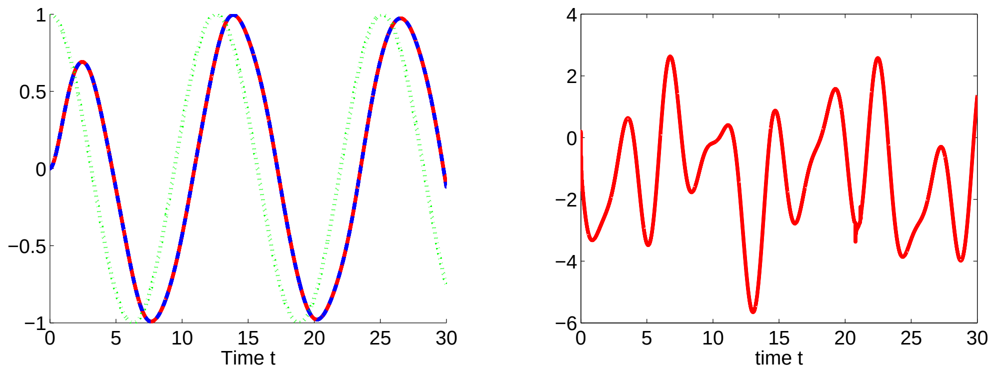
Fig. 46



a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted) **b)** Time-history of $u(t)$

Fig. 47 Performance of $\mathcal{L}_1$ adaptive controller for $\sigma(t) = \sin(\pi t)$

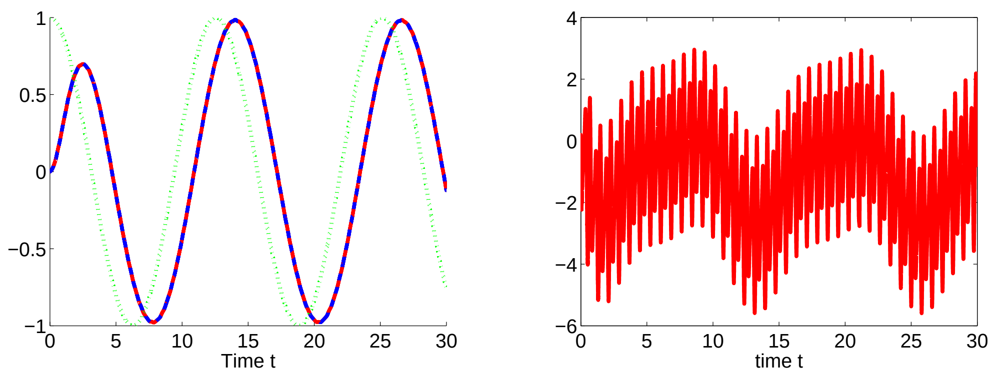
The simulation results of the $\mathcal{L}_1$ adaptive controller, without any retuning, are shown in Figures 47(a)-47(b) for reference input $r = \cos(\pi t)$. Next, we consider different disturbance signal: $\sigma(t) = \cos(x_1(t)) + 2\sin(10t) + \cos(15t)$. The simulation results are shown in 48(a)-48(b). Finally, we consider much higher frequencies in the disturbance: $\sigma(t) = \cos(x_1(t)) + 2\sin(100t) + \cos(150t)$. The simulation results are shown in 49(a)-49(b). We note that the $\mathcal{L}_1$ adaptive controller guarantees smooth and uniform transient performance in the presence of different unknown time-varying disturbances. The controller frequencies are exactly matched with the frequencies of the disturbance that it is supposed to cancel out. We also notice that $x_1(t)$ and $\hat{x}_1(t)$ are almost the same in Figs. 47(a), 48(a) and 49(a).



a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

b) Time-history of $u(t)$

Fig. 48 Performance of $\mathcal{L}_1$ adaptive controller for $\sigma(t) = \cos(x_1(t)) + 2\sin(10t) + \cos(15t)$



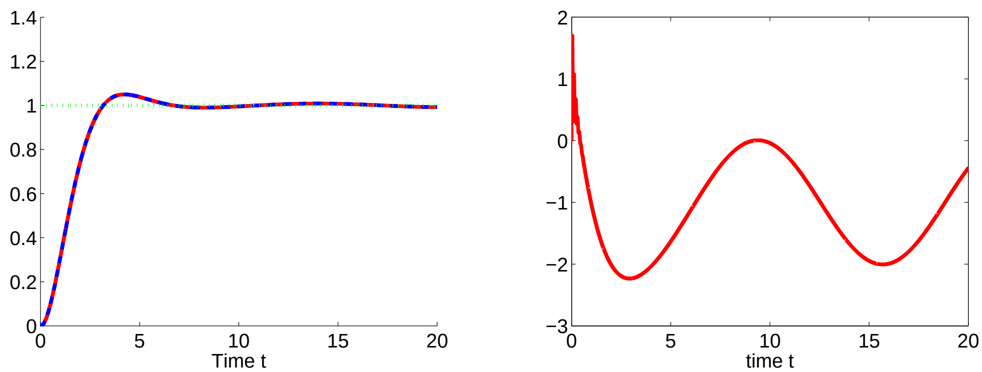
a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

b) Time-history of $u(t)$

Fig. 49 Performance of $\mathcal{L}_1$ adaptive controller for $\sigma(t) = \cos(x_1(t)) + 2\sin(100t) + \cos(150t)$

Next we verify the time-delay margin. Assuming constant $\theta(t)$, it can be cast into the form in (407). Let $\theta = [2 \;\; 2]^\top$, $\omega = 1$, $\sigma(t) = \sin(\pi t)$, so that the compact sets can be conservatively chosen as $\Omega_0 = [0.5, \;\; 2]$, $\Theta = [-10, \, 10]$, $\Delta_0 = [-10, \, 10]$, respectively. Next, we analyze the stability margins of the $\mathcal{L}_1$ adaptive controller for this system numerically.

For $\theta = [2 \;\; 2]^\top$, $\omega = 1$ we can derive $H_o(s)$ in (454) and look at its Bode plot in Fig. 46(b). It has phase margin $88.1^\circ(1.54\text{rad})$ at the cross frequency $9.55\text{Hz}(60\text{rad/s})$. Hence, the time-delay margin can be derived from (455) as: $\mathcal{T}(H_o(s)) = \frac{1.54\text{rad}}{60\text{rad/s}} = 0.0256$. We set $\Delta = [-1000 \quad 1000]^\top$, $\Gamma_c = 500000$,



a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

b) Time-history of $u(t)$

Fig. 50 Performance of $\mathcal{L}_1$ adaptive controller with time-delay $0.02s$

and run the $\mathcal{L}_1$ adaptive controller with time-delay $\tau = 0.02$. The simulations in Figs. 50(a)-50(b) verify our theoretical finding. As we stated, the time-delay margin of the LTI system in (454) provides only a conservative lower bound for the time-delay margin of the closed-loop adaptive system. So, we simulate the $\mathcal{L}_1$ adaptive controller in the presence of larger time-delay, like $\tau = 0.1$ sec., and observe that the system is not losing its stability. Since θ and ω are unknown to the controller, we derive the $\mathcal{T}(H_o(s))$ for all possible $\theta \in \Theta = [-10,\ 10]$ and $\omega \in \Omega_0 = [0.5,\ 2]$ and use the most conservative value. It gives $\mathcal{T}(H_o(s)) = 0.005s$. The gain margin can be arbitrarily large.

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