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Guaranteed Transient Performance with $\mathcal{L}_1$ Adaptive Controller for Parametric Strict Feedback Systems

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Abstract— **This paper extends the $\mathcal{L}_1$ adaptive control architecture from [1], [2] to parametric strict feedback systems in the presence of unknown time-varying parameters and bounded disturbances, which are not required to have slow rate of variation. We prove that the $\mathcal{L}_1$ adaptive control architecture ensures guaranteed transient response for system's both signals, input and output, simultaneously. Simulations of a benchmark example conclude the paper.**

I. INTRODUCTION

Recent papers [1], [2] introduced a new paradigm for design of adaptive controllers that leads to guaranteed transient performance for system's both signals, input and output, simultaneously. The novel $\mathcal{L}_1$ adaptive control architectures adapt fast without generating high-frequencies in the control signal. This paper extends the results from [1], [2] to a class of strict parametric feedback systems. For simplicity of presentation of the main idea, the results in this paper are developed for second order systems. Extension to higher-order systems is straightforward and is not pursued in this paper.

The paper is organized as follows. Section II presents the mathematical preliminaries. Section III states the problem formulation. Section IV introduces the novel $\mathcal{L}_1$ adaptive control architecture, and Section V analyzes its properties. Simulations are presented in Section VI. Section VII concludes the paper.

II. PRELIMINARIES

In this Section, we recall some basic definitions and facts from linear systems theory, [3]–[5].

Definition 1: For a signal $\xi(t),\ t \geq 0,\ \xi \in \mathbb{R}^n$, its truncated $\mathcal{L}_\infty$ norm and $\mathcal{L}_\infty$ norm are defined as

$$\begin{aligned} \|\xi_t\|_{\mathcal{L}_\infty} &= \max_{i=1,..,n}\left(\sup_{0\leq\tau\leq t}|\xi_i(\tau)|\right), \\ \|\xi\|_{\mathcal{L}_\infty} &= \max_{i=1,..,n}\left(\sup_{\tau\geq 0}|\xi_i(\tau)|\right), \end{aligned}$$

where ξ_i is the i^{th} component of ξ.

Definition 2: The $\mathcal{L}_1$ gain of a stable proper single–input single–output system $H(s)$ is defined to be $||H(s)||_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$, where $h(t)$ is the impulse response of $H(s)$, computed via the inverse Laplace transform $h(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} H(s)e^{st}ds, t \geq 0$, in which the integration is done along the vertical line $x = \alpha > 0$ in the complex plane.

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Proposition: A continuous time LTI system (proper) with impulse response $h(t)$ is stable if and only if $\int_0^\infty |h(\tau)|d\tau < \infty$. A proof can be found in [3] (page 81, Theorem 3.3.2).

Definition 3: For a stable proper m input n output system $H(s)$ its $\mathcal{L}_1$ gain is defined as

$$\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,\cdots,n}\left(\sum_{j=1}^{m}\|H_{ij}(s)\|_{\mathcal{L}_1}\right), \qquad (1)$$

where $H_{ij}(s)$ is the i^{th} row j^{th} column element of $H(s)$.

The next lemma extends the results of Example 5.2 ([4], page 199) to general multiple input multiple output systems.

Lemma 1: For a stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have

$$\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}, \quad \forall\ t > 0.$$

Corollary 1: For a stable proper MIMO system $H(s)$, if the input $r(t) \in \mathbb{R}^m$ is bounded, then the output $x(t) \in \mathbb{R}^n$ is also bounded as $\|x\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}$.

Lemma 2: For a cascaded system $H(s) = H_2(s)H_1(s)$, where $H_1(s)$ is a stable proper system with m inputs and l outputs and $H_2(s)$ is a stable proper system with l inputs and n outputs, we have $\|H(s)\|_{\mathcal{L}_1} \leq \|H_2(s)\|_{\mathcal{L}_1}\|H_1(s)\|_{\mathcal{L}_1}$.

Theorem 1: ([4], Theorem 5.6)($\mathcal{L}_1$ **Small Gain Theorem**) The interconnected system $w_2(s) = \Delta(s)(w_1(s) - M(s)w_2(s))$ with input $w_1(t)$ and output $w_2(t)$ is stable if $\|M(s)\|_{\mathcal{L}_1}\|\Delta(s)\|_{\mathcal{L}_1} < 1$.

III. PROBLEM FORMULATION

Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= \theta_1(t)x_1(t) + \sigma_1(t) + x_2(t), \\ \dot{x}_2(t) &= \theta_2^\top(t)x(t) + \sigma_2(t) + u(t), \end{aligned} \qquad (2)$$

where $x(t) = [x_1(t),\, x_2(t)]^\top$ is the measurable state vector, $x(0) = x_0$, $u(t)$ is the control signal, $\theta_1(t)$, $\theta_2(t)$, $\sigma_1(t)$ and $\sigma_2(t)$ are bounded time-varying unknown parameters and disturbances. Without loss of generality, we assume that

$$\theta_i(t) \in \Theta_i, \quad \sigma_i(t) \in \Sigma_i, \quad t \geq 0,\ i = 1, 2, \qquad (3)$$

where Θ_1, Θ_2, Σ_1 and Σ_2 are known sets. We further assume that $\theta_i(t)$ and $\sigma_i(t)$, $i = 1, 2$ are continuously differentiable, and their derivatives are uniformly bounded:

$$\begin{aligned} \|\dot{\theta}_i(t)\| &\leq b_{\theta_i} < \infty, \quad \forall\ t \geq 0, \\ \|\dot{\sigma}_i(t)\| &\leq b_{\sigma_i} < \infty, \quad \forall\ t \geq 0, \end{aligned} \qquad (4)$$

where the numbers $b_{\theta_i}, b_{\sigma_i}$, $i = 1, 2$, can be arbitrarily large. The control objective is to design an adaptive controller

$u(t)$ to ensure that $x_1(t)$ tracks a given bounded continuous reference input $r(t)$ with finite derivative.

IV. $\mathcal{L}_1$ ADAPTIVE CONTROLLER

The elements of $\mathcal{L}_1$ adaptive controller are introduced next:

State Predictor: We consider the following state predictor:

$$\begin{aligned}\dot{\hat{x}}_1(t) &= -a_1\tilde{x}_1(t) + \hat{\theta}_1(t)x_1(t) + \hat{\sigma}_1(t) + x_2(t)\,,\\ \dot{\hat{x}}_2(t) &= -a_2\tilde{x}_2(t) + \hat{\theta}_2(t)^\top x(t) + \hat{\sigma}_2(t) + u(t)\,,\\ \hat{x}(0) &= x_0\,,\end{aligned} \quad (5)$$

where $a_1 > 0$ and $a_2 > 0$ are positive gains,

$$\tilde{x}(t) = \left[\begin{array}{c}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{array}\right] = \left[\begin{array}{c}\hat{x}_1(t) - x_1(t)\\ \hat{x}_2(t) - x_2(t)\end{array}\right] \quad (6)$$

is the vector of the prediction errors, while $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$, $\hat{\sigma}_1(t)$ and $\hat{\sigma}_2(t)$ are the adaptive estimates.

Adaptive Laws: Adaptive estimates are governed by the following adaptive laws:

$$\begin{aligned}\dot{\hat{\theta}}_1(t) &= \Gamma\mathrm{Proj}(\hat{\theta}_1(t), -x_1(t)\tilde{x}^\top(t)P[1\ \ 0]^\top),\ \hat{\theta}_1(0) = \hat{\theta}_{1_0}\\ \dot{\hat{\theta}}_2(t) &= \Gamma\mathrm{Proj}(\hat{\theta}_2(t), -x(t)\tilde{x}^\top(t)P[0\ \ 1]^\top),\ \hat{\theta}_2(0) = \hat{\theta}_{2_0}\\ \dot{\hat{\sigma}}_1(t) &= \Gamma\mathrm{Proj}(\hat{\sigma}_1(t), -\tilde{x}^\top(t)P[1\ \ 0]^\top), \hat{\sigma}_1(0) = \hat{\sigma}_{1_0}\\ \dot{\hat{\sigma}}_2(t) &= \Gamma\mathrm{Proj}(\hat{\sigma}_2(t), -\tilde{x}^\top(t)P[0\ \ 1]^\top), \hat{\sigma}_2(0) = \hat{\sigma}_{2_0}\end{aligned} \quad (7)$$

where $\Gamma \in \mathbb{R}^+$ is the adaptation gain, and $P = P^\top > 0$ is the solution of the algebraic equation $A_m^\top P + PA_m = -Q$ for Hurwitz $A_m = \left[\begin{array}{cc}-a_1 & 0\\ 0 & -a_2\end{array}\right]$ and $Q > 0$.

Control Law: Let

$$\alpha_1(s) = -a_1(x_1(s) - r(s)) - C_1(s)r_1(s) + sr(s)\,, \quad (8)$$

where $r(s)$ and $r_1(s)$ are the Laplace transformations of $r(t)$ and $r_1(t) = \hat{\theta}_1(t)x_1(t) + \hat{\sigma}_1(t)$, and let the control signal be defined via:

$$u(s) = -a_2(x_2(s) - \alpha_1(s)) + s\alpha_1(s) - C_2(s)r_2(s)\,, \quad (9)$$

in which $r_2(s)$ is the Laplace transformation of

$$r_2(t) \;=\; \hat{\theta}_2^\top(t)x(t) + \hat{\sigma}_2(t) \quad (10)$$

with $C_1(s)$ and $C_2(s)$ being stable and strictly proper systems subject to $C_1(0) = C_2(0) = 1$. The relative degree of $C_1(s)$ is chosen to be ≥ 2 to ensure that $u(t)$ is a low-pass signal.

Further, let

$$\begin{aligned}L_1 &= \max_{\theta_1(t)\in\Theta_1} |\theta_1(t)|\,,\\ L_{2_i} &= \max_{\theta_{2_i}(t)\in\Theta_{2_i}} |\theta_{2_i}(t)|\,, i = 1, 2\,,\end{aligned} \quad (11)$$

where $\theta_{2_i}(t)$ is the i^{th} element of $\theta_2(t)$, Θ_1 and Θ_2 are the compact sets defined in (3). Define

$$L = \max\{L_1, L_{2_1} + L_{2_2}(1 + a_1 + L_1\|C_1(s)\|_{\mathcal{L}_1})\}\ , \quad (12)$$

$$A_g = \left[\begin{array}{cc}-a_1 & 1\\ 0 & -a_2\end{array}\right], \quad (13)$$

$$H_g(s) = (sI - A_g)^{-1}\,, \quad (14)$$

$$G(s) = H_g(s)\left[\begin{array}{cc}1 - C_1(s) & 0\\ 0 & 1 - C_2(s)\end{array}\right]. \quad (15)$$

We now state the $\mathcal{L}_1$-gain performance requirement that ensures stability of the entire system and desired transient performance.

$\mathcal{L}_1$-gain stability requirement: Design $C_1(s)$ and $C_2(s)$ to ensure that

$$\|G(s)\|_{\mathcal{L}_1} L < 1\,, \quad (16)$$

where $G(s)$ is defined in (15).

V. ANALYSIS OF $\mathcal{L}_1$ ADAPTIVE CONTROLLER

A. Closed-loop reference system

Consider the following closed-loop reference system:

$$\begin{aligned}\dot{x}_{r_1}(t) &= \theta_1(t)x_{r_1}(t) + \sigma_1(t) + x_{r_2}(t)\,, && (17)\\ \dot{x}_{r_2}(t) &= \theta_2^\top(t)x_r(t) + \sigma_2(t) + u_r(t)\,, && (18)\\ u_r(s) &= -a_2(x_{r_2}(s) - \alpha_{r_1}(s))\\ &\quad + s\alpha_{r_1}(s) - C_2(s)r_{r_2}(s)\,, && (19)\\ \alpha_{r_1}(s) &= -a_1(x_{r_1}(s) - r(s))\\ &\quad - C_1(s)r_{r_1}(s) + sr(s), && (20)\end{aligned}$$

where $r_{r_1}(s)$ and $r_{r_2}(s)$ are the Laplace transformation of signals

$$\begin{aligned}r_{r_1}(t) &= \theta_1(t)x_{r_1}(t) + \sigma_1(t)\,,\\ r_{r_2}(t) &= \theta_2^\top(t)x_r(t) + \sigma_2(t)\,.\end{aligned} \quad (21)$$

Lemma 3: If (16) holds, then the closed-loop reference system in (17)-(20) is stable.

Proof: Let

$$\begin{aligned}z_{r_1}(t) &= x_{r_1}(t) - r(t)\,,\\ z_{r_2}(t) &= x_{r_2}(t) - \alpha_{r_1}(t).\end{aligned} \quad (22)$$

It follows from (17)-(18) that

$$\begin{aligned}\dot{z}_{r_1}(t) &= -a_1 z_{r_1}(t) + z_{r_2}(t) + r_3(t)\,,\\ \dot{z}_{r_2}(t) &= -a_2 z_{r_2}(t) + r_4(t)\,,\end{aligned} \quad (23)$$

where $r_3(t)$ and $r_4(t)$ are signals with their Laplace transformations being:

$$r_3(s) = (1 - C_1(s))r_{r_1}(s)\,,\ r_4(s) = (1 - C_2(s))r_{r_2}(s)\,. \quad (24)$$

Let

$$z_r(s) = [z_{r_1}(s)\ \ z_{r_2}(s)]^\top\,.$$

It follows from (15), (21) and (23) that

$$z_r(s) = G(s)\left[\begin{array}{c}r_{r_1}(s)\\ r_{r_2}(s)\end{array}\right]. \quad (25)$$

From (21) and (22) we have

$$\begin{aligned}r_{r_1}(t) &= \theta_1(t)(z_{r_1}(t) + r(t)) + \sigma_1(t)\,,\\ r_{r_2}(t) &= \theta_{2_1}(t)(z_{r_1}(t) + r(t)) + \sigma_2(t)\\ &\quad + \theta_{2_2}(t)(z_{r_2}(t) + \alpha_{r_1}(t))\,.\end{aligned} \quad (26)$$

It follows from (20) that

$$\alpha_{r_1}(s) = -a_1 z_{r_1}(s) - C_1(s) r_{r_1}(s) + s r(s)\,. \tag{27}$$

Thus

$$\begin{bmatrix} r_{r_1}(t) \\ r_{r_2}(t) \end{bmatrix} = \begin{bmatrix} \theta_1(t) & 0 \\ \theta_{2_1}(t) - a_1\theta_{2_2}(t) & \theta_{2_2}(t) \end{bmatrix} \begin{bmatrix} z_{r_1}(t) \\ z_{r_2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -\theta_{2_2}(t) r_5(t) \end{bmatrix} + \begin{bmatrix} r_6(t) \\ r_7(t) \end{bmatrix}, \tag{28}$$

where

$$\begin{aligned} r_5(s) &= C_1(s) r_{10}(s)\,, r_{10}(s) = \theta_1(t) z_{r_1}(t)\,, \\ r_6(t) &= \theta_1(t) r(t) + \sigma_1(t)\,, \\ r_7(t) &= \theta_{2_1}(t) r(t) + \sigma_2(t) + \theta_{2_2}(t)\dot{r}(t) - \theta_{2_2}(t) r_8(t)\,, \\ r_8(s) &= C_1(s) r_9(s)\,, r_9(t) = \theta_1(t) r(t) + \sigma_1(t)\,. \end{aligned} \tag{29}$$

Since $r(t)$, $\dot{r}(t)$, $\sigma(t)$ and $\theta(t)$ are bounded, it can be verified straightforwardly that $r_6(t)$ and $r_7(t)$ are bounded. It follows from Lemma 1 and (12) that

$$\left\| \begin{array}{c} r_{r_1} \\ r_{r_2} \end{array} \right\|_{\mathcal{L}_\infty} \leq L \|z_r\|_{\mathcal{L}_\infty} + \left\| \begin{array}{c} r_6 \\ r_7 \end{array} \right\|_{\mathcal{L}_\infty}. \tag{30}$$

Theorem 1 ensures that the cascaded system in (25) is bounded, which completes the proof. □

B. Bounded Error Signal

Lemma 4: For the system in (2) and $\mathcal{L}_1$ adaptive controller in (5), (7) and (9), the tracking error between the system state and the state predictor is bounded as follows:

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \gamma_0\,, \tag{31}$$

where

$$\gamma_0 = \sqrt{\frac{\theta_m}{\lambda_{\min}(P)\Gamma}}\,, \tag{32}$$

$$\theta_m \triangleq 4(\max_{\theta\in\Theta_1}\theta^2 + \max_{\theta\in\Theta_2}\theta^\top\theta + \max_{\sigma\in\Sigma_1}\sigma^2 + \max_{\sigma\in\Sigma_2}\sigma^2) + 4\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\left(\sum_{i=1}^{2}\left(\max_{\theta\in\Theta_i}\|\theta\| b_{\theta_i} + \max_{\sigma\in\Sigma_i}\|\sigma\| b_{\sigma_i}\right)\right).$$

Proof. It follows from (2) and (5) that the error dynamics between the system and the predictor are

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + \begin{bmatrix} \tilde{\theta}_1(t)x_1(t) + \tilde{\sigma}_1(t) \\ \tilde{\theta}_2^\top(t)x(t) + \tilde{\sigma}_2(t) \end{bmatrix}. \tag{33}$$

Consider the following candidate Lyapunov function

$$V(t) = \tilde{x}^\top(t)P\tilde{x}(t) + \frac{1}{\Gamma}(\tilde{\theta}_1^2(t) + \tilde{\theta}_2^\top(t)\tilde{\theta}_2(t) + \tilde{\sigma}_1^2(t) + \tilde{\sigma}_2^2(t))\,, \tag{34}$$

where

$$\tilde{\theta}_i = \hat{\theta}_i(t) - \theta_i(t)\,,\quad \tilde{\sigma}_i(t) = \hat{\sigma}_i(t) - \sigma_i(t)\,,\ i = 1, 2. \tag{35}$$

The adaptive law in (7) ensures the following inequality:

$$\begin{aligned} \dot{V}(t) \leq\ & -\tilde{x}^\top(t)Q\tilde{x}(t) - 2\Gamma^{-1}\Big(\tilde{\theta}_1(t)\dot{\theta}_1(t) \\ & +\tilde{\theta}_2^\top(t)\dot{\theta}_2(t) + \tilde{\sigma}_1(t)\dot{\sigma}_1(t) + \tilde{\sigma}_2(t)\dot{\sigma}_2(t)\Big) \end{aligned} \tag{36}$$

The projection algorithm ensures that $\hat{\theta}_i(t) \in \Theta_i$, $\hat{\sigma}_i(t) \in \Sigma_i$, $i = 1, 2$ for all $t \geq 0$, and therefore

$$\max_{t\geq 0}\Gamma^{-1}(\tilde{\theta}_1^2(t) + \tilde{\theta}_2^\top(t)\tilde{\theta}_2(t) + \tilde{\sigma}_1^2(t) + \tilde{\sigma}_2^2(t)) \leq 4\left(\max_{\theta\in\Theta_1}\theta^2 + \max_{\theta\in\Theta_2}\theta^\top\theta + \max_{\sigma\in\Sigma_1}\sigma^2 + \max_{\sigma\in\Sigma_2}\sigma^2\right)/\Gamma \tag{37}$$

for any $t \geq 0$. If at any t

$$V(t) > \frac{\theta_m}{\Gamma}\,, \tag{38}$$

where θ_m is defined in (32), then it follows from (37) that

$$\tilde{x}^\top(t)P\tilde{x}(t) > 2\frac{\lambda_{\max}(P)}{\Gamma\lambda_{\min}(Q)}\left(\sum_{i=1}^{2}\left(\max_{\theta\in\Theta_i}\|\theta\| b_{\theta_i} + \max_{\sigma\in\Sigma_i}\|\sigma\| b_{\sigma_i}\right)\right), \tag{39}$$

and hence

$$\tilde{x}^\top(t)Q\tilde{x}(t) \geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\tilde{x}^\top(t)P\tilde{x}(t) > 4\frac{\sum_{i=1}^{2}\left(\max_{\theta\in\Theta_i}\|\theta\| b_{\theta_i} + \max_{\sigma\in\Sigma_i}\|\sigma\| b_{\sigma_i}\right)}{\Gamma}.$$

The upper bounds in (4) along with the projection based adaptive laws lead to the following upper bound:

$$-2\frac{\tilde{\theta}_1(t)\dot{\theta}_1(t) + \tilde{\theta}_2^\top(t)\dot{\theta}_2(t) + \tilde{\sigma}_1(t)\dot{\sigma}_1(t) + \tilde{\sigma}_2(t)\dot{\sigma}_2(t)}{\Gamma} \leq 4\frac{\sum_{i=1}^{2}\left(\max_{\theta\in\Theta_i}\|\theta\| b_{\theta_i} + \max_{\sigma\in\Sigma_i}\|\sigma\| b_{\sigma_i}\right)}{\Gamma}. \tag{40}$$

Hence, if $V(t) > \frac{\theta_m}{\Gamma}$, then from (36) we have

$$\dot{V}(t) < 0\,. \tag{41}$$

Since we have set $\hat{x}(0) = x(0)$, we can verify that

$$V(0) \leq \left(\max_{\theta\in\Theta_1}\theta^2 + \max_{\theta\in\Theta_2}\theta^2 + \max_{\sigma\in\Sigma_1}\sigma^2 + \max_{\sigma\in\Sigma_2}\sigma^2\right)/\Gamma < \frac{\theta_m}{\Gamma}.$$

It follows from (41) that $V(t) \leq \frac{\theta_m}{\Gamma}$ for any $t \geq 0$. Since $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \leq \tilde{x}^\top(t)P\tilde{x}(t) \leq V(t)$, then

$$||\tilde{x}(t)||^2 \leq \frac{\theta_m}{\lambda_{\min}(P)\Gamma}\,,$$

which concludes the proof. □

C. Transient and Steady State Performance

Let

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - r(t) \\ x_2(t) - \alpha_1(t) \end{bmatrix}, \tag{42}$$

and

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = z(t) - z_r(t)\,. \tag{43}$$

Theorem 2: Given the system in (2) and $\mathcal{L}_1$ adaptive controller defined via (5), (7) and (9) subject to (16), we have:

$$\begin{aligned} \|e\|_{\mathcal{L}_\infty} &\leq \gamma_1 \,, & (44)\\ \|x - x_r\|_{\mathcal{L}_\infty} &\leq \gamma_2 \,, & (45)\\ \|u - u_r\|_{\mathcal{L}_\infty} &\leq \gamma_3 \,, & (46) \end{aligned}$$

where

$$\begin{aligned} &\gamma_1 = \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L}\gamma_0 \,, & (47)\\ &\gamma_2 = \gamma_1 + a_1\gamma_1 + \|C_1(s)\|_{\mathcal{L}_1} L_1 \gamma_1 \\ &\quad + \|(s+a_1)C_1(s)\|_{\mathcal{L}_1}\gamma_0 \,, & (48)\\ &\gamma_3 = a_2\gamma_1 + \|(s+a_2)C_2(s)\|_{\mathcal{L}_1}\gamma_0 \\ &\quad + \|C_2(s)\|_{\mathcal{L}_1}(L_{2_1} + L_{2_2})\gamma_2 + a_1\gamma_4 \\ &\quad + \|s(s+a_1)C_1(s)\|_{\mathcal{L}_1}\gamma_0 + \|sC_1(s)\|_{\mathcal{L}_1} L_1\gamma_1 \,, & (49) \end{aligned}$$

while γ_0 is defined in (32), $\gamma_4 = (a_1 + 1 + L_1\|1 - C_1(s)\|_{\mathcal{L}_1})\gamma_1 + \|(s+a_1)C_1(s)\|_{\mathcal{L}_1}\gamma_0$, and

$$C(s) = \begin{bmatrix} C_1(s) & 0 \\ 0 & C_2(s) \end{bmatrix}. \quad (50)$$

Proof. Let

$$\begin{aligned} \bar{r}_1(t) &= \theta_1(t)x_1(t) + \sigma_1(t)\,,\\ \bar{r}_2(t) &= \theta_2^\top(t)x(t) + \sigma_2(t)\,,\\ \tilde{r}_1(t) &= \tilde{\theta}_1(t)x_1(t) + \tilde{\sigma}_1(t)\,,\\ \tilde{r}_2(t) &= \tilde{\theta}_2^\top(t)x(t) + \tilde{\sigma}_2(t)\,. \end{aligned} \quad (51)$$

It follows from (2), (8) and (9) that

$$\dot{z}(t) = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -a_1 z_1(t) + z_2(t) + \bar{r}_3(t) - \tilde{r}_3(t) \\ -a_2 z_2(t) + \bar{r}_4(t) - \tilde{r}_4(t) \end{bmatrix}, \quad (52)$$

where $\bar{r}_3(t)$, $\bar{r}_4(t)$, $\tilde{r}_3(t)$, $\tilde{r}_4(t)$ are signals with their Laplace transformations:

$$\begin{aligned} \bar{r}_3(s) &= (1 - C_1(s))\bar{r}_1(s)\\ \bar{r}_4(s) &= (1 - C_2(s))\bar{r}_2(s)\\ \tilde{r}_3(s) &= C_1(s)\tilde{r}_1(s)\\ \tilde{r}_4(s) &= C_2(s)\tilde{r}_2(s)\,. \end{aligned}$$

Let

$$\begin{aligned} e_{r_1}(t) &= \theta_1(t)(x_1(t) - x_{r_1}(t))\\ e_{r_2}(t) &= \theta_2^\top(t)(x(t) - x_r(t))\,. \end{aligned} \quad (53)$$

It follows from (23) and (52) that

$$\dot{e}(t) = \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{bmatrix} = \begin{bmatrix} -a_1 e_1(t) + e_2(t) + e_{r_3}(t) - \tilde{r}_3(t) \\ -a_2 e_2(t) + e_{r_4}(t) - \tilde{r}_4(t) \end{bmatrix}, \quad (54)$$

where $e_{r_3}(t)$ and $e_{r_4}(t)$ are signals with their Laplace transformations:

$$\begin{aligned} e_{r_3}(s) &= (1 - C_1(s))e_{r_1}(s)\\ e_{r_4}(s) &= (1 - C_2(s))e_{r_2}(s)\,. \end{aligned} \quad (55)$$

From (15) and (54) we have

$$e(s) = G(s)\begin{bmatrix} e_{r_1}(s) \\ e_{r_2}(s) \end{bmatrix} - (sI - A_m)^{-1}C(s)\begin{bmatrix} \tilde{r}_1(s) \\ \tilde{r}_2(s) \end{bmatrix}. \quad (56)$$

It follows from (22), (42), (43) and (53) that

$$\begin{aligned} e_{r_1}(t) &= \theta_1(t)e_1(t)\,,\\ e_{r_2}(t) &= \theta_{2_1}(t)e_1(t) + \theta_{2_2}(t)(e_2(t) + e_\alpha(t))\,, \end{aligned} \quad (57)$$

where

$$e_\alpha(s) = -a_1 e_1(s) - C_1(s)e_{r_1}(s)\,, \quad (58)$$

which eventually leads to

$$\begin{bmatrix} e_{r_1}(t) \\ e_{r_2}(t) \end{bmatrix} = \begin{bmatrix} \theta_1(t) & 0 \\ \theta_{2_1}(t) - a_1\theta_{2_2}(t) & \theta_{2_2}(t) \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -\theta_{2_2}(t)e_{r_5}(t) \end{bmatrix} \quad (59)$$

with

$$e_{r_5}(s) = C_1(s)e_{r_1}(s)\,. \quad (60)$$

Since $\|e_{r_1}\|_{\mathcal{L}_\infty} \leq L_1\|e_1\|_{\mathcal{L}_\infty}$, it follows from Lemma 1 that (30), we have

$$\left\| \begin{matrix} e_{r_1} \\ e_{r_2} \end{matrix} \right\|_{\mathcal{L}_\infty} \leq L\|e\|_{\mathcal{L}_\infty}\,. \quad (61)$$

It follows from (33) that

$$\tilde{x}(s) = (sI - A_m)^{-1}\begin{bmatrix} \tilde{r}_1(t) \\ \tilde{r}_2(t) \end{bmatrix}, \quad (62)$$

and hence

$$(sI - A_m)^{-1}C(s)\begin{bmatrix} \tilde{r}_1(t) \\ \tilde{r}_2(t) \end{bmatrix} = C(s)\tilde{x}(s)\,. \quad (63)$$

It follows from (56), (61) and (63) that

$$\|e\|_{\mathcal{L}_\infty} \leq \|G(s)\|_{\mathcal{L}_1} L\|e\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}\,,$$

which leads to

$$\|e\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L}\|\tilde{x}\|_{\mathcal{L}_\infty}\,. \quad (64)$$

The bound in (44) follows from Lemma 4 and (64) directly.

To prove the bound in (46), we notice that from (8) and (20) one can derive

$$\alpha_1(s) - \alpha_{r_1}(s) = -a_1 e_1(s) - C_1(s)e_{r_1}(s) - C_1(s)\tilde{r}_1(s)\,, \quad (65)$$

where $e_{r_1}(t)$ is defined in (53). Since $C_1(s)\tilde{r}_1(s) = (s + a_1)C_1(s)\tilde{x_1}(s)$ and $\|\tilde{x}_1\|_{\mathcal{L}_\infty} \leq \|e_1\|_{\mathcal{L}_\infty}$, we have

$$\begin{aligned} &\|\alpha_1 - \alpha_{r_1}\|_{\mathcal{L}_\infty} \leq a_1\gamma_1 + \|C_1(s)\|_{\mathcal{L}_1} L_1\gamma_1 + \\ &\|(s+a_1)C_1(s)\|_{\mathcal{L}_1}\gamma_0\,. \end{aligned} \quad (66)$$

Since $\|x - x_r\|_{\mathcal{L}_\infty} \leq \max\{\|e_1\|_{\mathcal{L}_\infty}, \|e_2\|_{\mathcal{L}_\infty} + \|\alpha_1 - \alpha_{r_1}\|_{\mathcal{L}_\infty}\}$ it follows from (48) and (66) that (45) is proved.

It follows from (9) and (19) that

$$\begin{aligned} u(s) - u_r(s) &= -a_2 e_2(s) + s(\alpha_1(s) - \alpha_{r_1}(s))\\ &\quad - C_2(s)\tilde{r}_2(s) - C_2(s)e_{r_2}(s)\,, \end{aligned}$$

which along with (65) leads to

$$\begin{aligned} &u(s) - u_r(s) = -a_2 e_2(s) - C_2(s)\tilde{r}_2(s) - C_2(s)e_{r_2}(s)\\ &+ s(-a_1 e_1(s) - C_1(s)e_{r_1}(s) - C_1(s)\tilde{r}_1(s))\,. \end{aligned} \quad (67)$$

Letting $e_d(t)=\dot{e}_1(t)$, we have

$$e_d(s)=se_1(s)\,, \tag{68}$$

and hence (67) can be rewritten as

$$\begin{aligned}u(s)-u_r(s)=&-a_2e_2(s)-C_2(s)\tilde{r}_2(s)-C_2(s)e_{r_2}(s)\\&-a_1e_d(s)-sC_1(s)(e_{r_1}(s)+\tilde{r}_1(s))\,.\end{aligned} \tag{69}$$

Since

$$e_d(t)=\dot{e}_1(t)=-a_1e_1(t)+e_2(t)+e_{r_3}(t)-\tilde{r}_3(t)\,, \tag{70}$$

and

$$\|e_i\|_{\mathcal{L}_\infty}\le\|e\|_{\mathcal{L}_\infty}\le\gamma_1\,, \tag{71}$$

we have

$$\|e_d\|_{\mathcal{L}_\infty}\le(a_1+1)\gamma_1+\|e_{r_3}\|_{\mathcal{L}_\infty}+\|\tilde{r}_3\|_{\mathcal{L}_\infty}\,. \tag{72}$$

We further notice that (44) and (45) imply that

$$\begin{aligned}\|e_{r_1}\|_{\mathcal{L}_\infty}&\le L_1\gamma_1\,,\\ \|e_{r_2}\|_{\mathcal{L}_\infty}&\le (L_{2_1}+L_{2_2})\gamma_2\,.\end{aligned} \tag{73}$$

Combining (44), (55) and (73), we have

$$\|e_{r_3}\|_{\mathcal{L}_\infty}\le L_1\|1-C_1(s)\|_{\mathcal{L}_1}\gamma_1\,. \tag{74}$$

Since

$$\tilde{r}_3(s)=C_1(s)\tilde{r}_1(s)=(s+a_1)C_1(s)\tilde{x}_1(s)\,,$$

we have

$$\|\tilde{r}_3\|_{\mathcal{L}_\infty}\le\|(s+a_1)C_1(s)\|_{\mathcal{L}_1}\gamma_0\,. \tag{75}$$

It follows from (72), (75) and (74) that

$$\begin{aligned}\|e_d\|_{\mathcal{L}_\infty}\le&\ (a_1+1+L_1\|1-C_1(s)\|_{\mathcal{L}_1})\gamma_1\\&+\|(s+a_1)C_1(s)\|_{\mathcal{L}_1}\gamma_0\,.\end{aligned} \tag{76}$$

Since

$$\begin{aligned}sC_1(s)\tilde{r}_1(s)&=s(s+a_1)C_1(s)\tilde{x}_1(s)\,,\\ C_2(s)\tilde{r}_2(s)&=(s+a_2)C_2(s)\tilde{x}_2(s)\,,\end{aligned} \tag{77}$$

it follows from (49), (69), and (73) that

$$\begin{aligned}&\|u-u_r\|_{\mathcal{L}_\infty}\le a_2\gamma_1+\|(s+a_2)C_2(s)\|_{\mathcal{L}_1}\gamma_0\\&+\|C_2(s)\|_{\mathcal{L}_1}(L_{2_1}+L_{2_2})\gamma_2+a_1\|e_d\|_{\mathcal{L}_\infty}\\&+\|s(s+a_1)C_1(s)\|_{\mathcal{L}_1}\gamma_0+\|sC_1(s)\|_{\mathcal{L}_1}L_1\gamma_1\,,\end{aligned}$$

which combining (76) proves (46). □

The following Corollary 2 follows from Theorem 2 directly.

Corollary 2: Given the system in (2) and the $\mathcal{L}_1$ adaptive controller defined via (5), (7) and (9) subject to (16), we have:

$$\begin{aligned}\lim_{\Gamma\to\infty}(x(t)-x_r(t))&=0\,,\quad\forall\ t\ge0, \qquad (78)\\ \lim_{\Gamma\to\infty}(u(t)-u_r(t))&=0\,,\quad\forall\ t\ge0\,. \qquad (79)\end{aligned}$$

Thus, the tracking error between $x(t)$ and $x_r(t)$, as well between $u(t)$ and $u_r(t)$, is uniformly bounded by a constant inverse proportional to Γ. This implies that during the transient one can achieve arbitrarily close tracking performance for both signals simultaneously by increasing Γ.

We note that the control law $u_r(t)$ in the closed-loop reference system, which is used in the analysis of $\mathcal{L}_\infty$ norm bounds, is not implementable since its definition involves the unknown parameters. Theorem 2 ensures that the $\mathcal{L}_1$ adaptive controller approximates $u_r(t)$ both in transient and steady state. So, it is important to understand how these bounds can be used for ensuring uniform transient response with *desired* specifications. If $C_1(s)=C_2(s)=1$, the control law in (19)-(20) becomes

$$\begin{aligned}u_{id}(s)&=-a_2(x_{id_2}(s)-\alpha_{id_1}(s))\\&\quad+s\alpha_{id_1}(s)-r_{id_2}(s)\,,\\ \alpha_{id_1}(s)&=-a_1(x_{id_1}(s)-r(s))\\&\quad-r_{id_1}(s)+sr(s),\end{aligned} \tag{80}$$

where $r_{id_1}(t)=\theta_1(t)x_{id_1}(t)+\sigma_1(t)$, $r_{r_2}(t)=\theta_2^\top(t)x_{id}(t)+\sigma_2(t)$, which is the *ideal* non-adaptive backstepping controller for the system in (17)-(18). Let

$$\begin{aligned}z_{id_1}(t)&=x_{id_1}(t)-r(t)\\ z_{id_2}(t)&=x_{id_2}(t)-\alpha_{id_1}(t).\end{aligned}$$

It follows from (80) that

$$\begin{aligned}\dot{z}_{id_1}(t)&=-a_1z_{id_1}(t)+z_{id_2}(t)\\ \dot{z}_{id_2}(t)&=-a_2z_{id_2}(t)\,,\end{aligned} \tag{81}$$

which can be rewritten as:

$$\dot{z}_{id}(t)=A_mz_{id}(t)\,,$$

where $z_{id}(t)=[z_{id_1}(t)\quad z_{id_2}(t)]^\top$. Since A_m is Hurwitz, $z_{id}(t)$ converges exponentially to the origin, and hence

$$\lim_{t\to\infty}(x_{id_1}(t)-r(t))=0\,. \tag{82}$$

In the closed-loop reference system (17)-(20), $u_{id}(t)$ is further low-pass filtered by $C_1(s)$ and $C_2(s)$ in (19)-(20) to have guaranteed low-frequency range. Thus, the reference system in (17)-(18) has a different response as compared to (81) with (80). In [2], for unknown constant parameters θ specific design guidelines are suggested for selection of $C(s)$ to achieve the desired response. In case of fast varying $\theta(t)$, it is obvious that the bandwidth of the controller needs to be matched correspondingly.

VI. Simulations

As an illustrative example, consider the system in (2), where the time-varying unknown parameters are:

$$\begin{aligned}\theta_1(t)&=0.5\sin(0.3t)\,,\\ \theta_2(t)&=[0.5\sin(0.3t)\ \ 0.2\sin(0.3t)+0.1\cos(0.2t)]^\top\,,\\ \sigma_1(t)&=\sin(0.3t)\,,\\ \sigma_2(t)&=\cos(0.3t)\,.\end{aligned}$$

For bounded $\theta_1(t)$ and $\theta_2(t)$, we assume the knowledge of the following conservative bounds:

$$L_1=1,\ L_{2_1}=1\,,\ L_{2_2}=0.5\,. \tag{83}$$

The control objective is to ensure that $x_1(t)$ tracks

$$r(t) = \cos(0.3t)$$

with guaranteed transient performance.

We implement $\mathcal{L}_1$ adaptive controller via (5), (7) and (9) with

$$C_1(s) = \frac{400}{s^2 + 28.28s + 400}\,, \quad C_2(s) = \frac{20}{s+20}\,.$$
$$a_1 = a_2 = 1\,,\ Q = I_{2\times 2}\,,\ \Gamma = 20000\,.$$

It can be verified numerically that

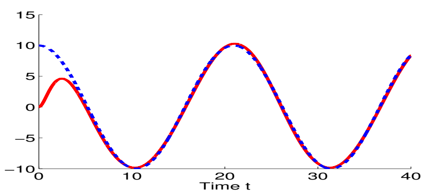
$$\|C_1(s)\|_{\mathcal{L}_1} = 1.087\,,\ \|G(s)\|_{\mathcal{L}_1} = 0.242\,,$$

and it follows from (83) that

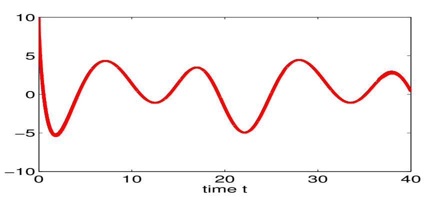
$$\|G(s)\|_{\mathcal{L}_1} L = 0.242 \times 3.54 = 0.857 < 1\,.$$

Hence, the $\mathcal{L}_1$ stability requirement in (16) holds.

The simulation results are shown in Figures 1(a)-1(b). Next, we consider the same controller for faster time-varying



(a) $y(t)$ (solid) and $r(t)$ (dashed)



(b) Time-history of $u(t)$

Fig. 1. Performance for $\sigma_1(t) = \sin(0.3t)$, $\sigma_2(t) = \cos(0.3t)$.

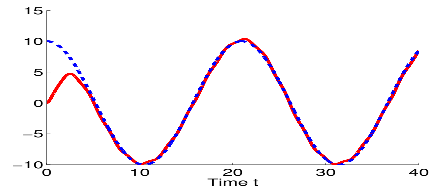
disturbances

$$\sigma_1(t) = \sin(3t), \quad \sigma_2(t) = \cos(3t) \tag{84}$$

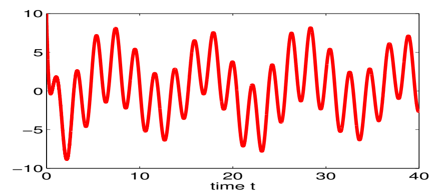
without any retuning. The system response and the control signal are plotted in Figs. 2(a)-2(b). Finally, we consider higher frequencies in the disturbance:

$$\sigma_1(t) = \sin(10t), \quad \sigma_2(t) = \cos(10t)\,. \tag{85}$$

The simulation results are shown in 3(a)-3(b). We note that $\mathcal{L}_1$ adaptive controller guarantees smooth and uniform transient performance in the presence of different unknown nonlinearities and time-varying disturbances. The controller frequencies are exactly matched with the frequencies of the disturbance that it is supposed to cancel out. We also notice that $x_1(t)$ and $\hat{x}_1(t)$ are almost the same in Figs. 1(a), 2(a) and 3(a).

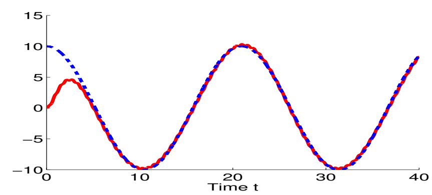


(a) $y(t)$ (solid) and $r(t)$ (dashed)

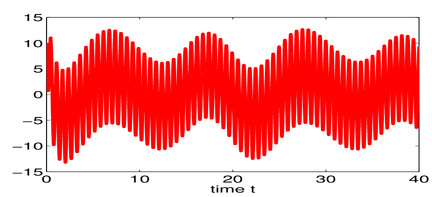


(b) Time-history of $u(t)$

Fig. 2. Performance for $\sigma_1(t) = \sin(3t)$, $\sigma_2(t) = \cos(3t)$.



(a) $y(t)$ (solid) and $r(t)$ (dashed)



(b) Time-history of $u(t)$

Fig. 3. Performance for $\sigma_1(t) = \sin(10t)$, $\sigma_2(t) = \cos(10t)$.

VII. Conclusion

A novel $\mathcal{L}_1$ adaptive control architecture is presented that has guaranteed transient response in addition to stable tracking for parametric strict feedback systems with time-varying unknown parameters and bounded disturbances. The control signal and the system response approximate the same signals of a closed-loop reference system, which can be designed to achieve desired specifications.

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