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Design and Analysis of a Novel $\mathcal{L}_1$ Adaptive Controller, Part I: Control Signal and Asymptotic Stability

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Abstract— **In this paper, we develop a novel adaptive control architecture that ensures that the input and output of an uncertain linear system track the input and output of a desired linear system during the transient phase, in addition to the asymptotic tracking. These features are established by first performing an equivalent reparametrization of MRAC, the main difference of which from MRAC is in definition of the error signal for adaptive laws. This new architecture, called companion model adaptive controller (CMAC), allows for incorporation of a low-pass filter into the feedback-loop that enables to enforce the desired transient performance by increasing the adaptation gain. For the proof of asymptotic stability, the $\mathcal{L}_1$ gain of a cascaded system, comprised of this filter and the closed-loop desired reference model, is required to be less than the inverse of the upper bound of the norm of unknown parameters used in projection based adaptation laws. Moreover, the new $\mathcal{L}_1$ adaptive controller is guaranteed to stay in the low-frequency range. Simulation results illustrate the theoretical findings.**

I. Introduction

Model Reference Adaptive Controller (MRAC) is developed conventionally to control linear systems with unknown coefficients [1], [2]. This architecture has been facilitated by the Lyapunov stability theory, which gives sufficient conditions for stable performance without characterizing the frequency properties of the resulting controller. Application of adaptive controllers was therefore largely restricted due to the fact that the system uncertainties during the transient have led to unpredictable/undesirebale situations, involving control signals of high-frequency or large amplitudes, large transient errors or slow convergence rate of tracking errors, to name a few.

Improvement of the transient performance of adaptive controllers has been addressed from various perspectives in numerous efforts [2]–[15], to name a few. An example presented in [12] demonstrated that the system output can have overly poor transient tracking behavior before ideal asymptotic convergence can take place. On the other hand, in [9] the author proved that it may not be possible to optimize $\mathcal{L}_2$ and $\mathcal{L}_\infty$ performance simultaneously by using a constant adaptation rate. Following these results, modifications of adaptive controllers were proposed in [5], [13] that render the tracking error arbitrarily small in terms of both mean-square and $\mathcal{L}_\infty$ bounds. Further, it was shown in [3] that the modifications proposed in [5], [13] could be derived as a linear feedback of the tracking error, and the improved performance was obtained only due to a nonadaptive high-gain feedback in that scheme. In [2], composite adaptive controller was proposed, which suggests a new adaptation law using both tracking error and prediction error that leads to less oscillatory behavior in the presence of high adaptation gains as compared to MRAC. In [14], it is shown that arbitrarily close transient bound can be achieved by enforcing parameter-dependent persistent excitation condition. In [10], computable $\mathcal{L}_2$ and $\mathcal{L}_\infty$ bounds for the output tracking error signals are obtained for a special class of adaptive controllers using backstepping. The underlying linear nonadaptive controller possesses a parametric robustness property, however, for a large parametric uncertainty it requires high gain. In [11], dynamic certainty equivalent controllers with unnormalized estimators were used for adaptation that permit to derive a uniform upper bound for the $\mathcal{L}_2$ norm of the tracking error in terms of initial parameter estimation error. In the presence of sufficiently small initial conditions, the author proved that the $\mathcal{L}_\infty$ norm of the tracking error is upper bounded by the $\mathcal{L}_\infty$ norm of the reference input. Finally, in [16] a new certainty equivalence based adaptive controller is presented using backstepping based control law with a normalized adaptive law to achieve asymptotic stability and guarantee performance bounds comparable with the tuning functions scheme, without the use of higher order nonlinearities.

As compared to the linear systems theory, several important aspects of the transient performance analysis seem to be missing in these papers. First, the bounds in these papers are computed for tracking errors only, and not for the control signals. Although the latter can be deduced from the former, it is straightforward to verify that the ability to adjust the former may not extend to the latter in case of nonlinear control laws. Second, since the purpose of adaptive control is to ensure stable performance in the presence of modeling uncertainties, one needs to ensure that the changes in reference input and unknown parameters due to possible faults or unexpected uncertainties do not lead to unacceptable transient deviations or oscillatory control signals, implying that a retuning of adaptive parameters is required. Finally, one needs to ensure that whatever modifications or solutions are suggested for performance improvement of adaptive controllers, they are not achieved via high-gain feedback.

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In Part I of this paper, we present a novel adaptive control architecture and its asymptotic stability analysis. This new architecture guarantees that the control signal is in low-frequency range. In Part II [17], complete analysis of the transient performance of both control signal and system response is presented, and design methods for achieving guaranteed transient response are given.

This paper is organized as follows. Section II states some preliminary definitions, and Section III gives the problem formulation. In Section IV, we introduce the Companion Model Adaptive Controller (CMAC), which is a reparameterization of the conventional MRAC. In Section V, the new $\mathcal{L}_1$ adaptive controller is presented. Stability and convergence results of the $\mathcal{L}_1$ adaptive controller are presented in Section VI. In section VII, simulation results are presented, and Section VIII concludes the paper. Proofs are in Appendix.

II. Preliminaries

In this Section, we recall some basic definitions from linear systems theory, [8], [18], [19].

Definition 1: For signal $\xi(t)$, $t \geq 0$, $\xi \in \mathbb{R}^n$, its truncated $\mathcal{L}_\infty$ norm and $\mathcal{L}_\infty$ norm are defined as: $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{0\leq\tau\leq t}|\xi_i(\tau)|)$, $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{\tau\geq 0}|\xi_i(\tau)|)$, where ξ_i is the i^{th} component of ξ.

Definition 2: The $\mathcal{L}_1$ gain of a stable proper single–input single–output system $H(s)$ is defined as: $\|H(s)\|_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$, where $h(t)$ is the impulse response of $H(s)$, computed via the inverse Laplace transform: $h(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} H(s)e^{st}ds, t \geq 0$, in which the integration is done along the vertical line $x = \alpha > 0$ in the complex plane.

Proposition: A continuous time LTI system (proper) with impulse response $h(t)$ is stable if and only if its $\mathcal{L}_1$ gain is finite: $\int_0^\infty |h(\tau)|d\tau < \infty$.

A proof can be found in [8] (page 81, Theorem 3.3.2).

Definition 3: For a stable proper m input n output system $H(s)$ its $\mathcal{L}_1$ gain is defined as $\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,..,n}(\sum_{j=1}^m \|H_{ij}(s)\|_{\mathcal{L}_1})$, where $H_{ij}(s)$ is the i^{th} row j^{th} column element of $H(s)$.

The next lemma extends the results of Ex. 5.2 ([18], page 199) to general multiple input multiple output systems.

Lemma 1: For a stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have $\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}, \forall\, t > 0$.

Corollary 1: For a stable proper MIMO system $H(s)$, if the input $r(t) \in \mathbb{R}^m$ is bounded, then the output $x(t) \in \mathbb{R}^n$ is also bounded: $\|x\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}$.

The proof follows from Lemma 1 directly.

Lemma 2: For a cascaded system $H(s) = H_2(s)H_1(s)$, where $H_1(s)$ is a stable proper system with m inputs and l outputs and $H_2(s)$ is a stable proper system with l inputs and n outputs, we have $\|H(s)\|_{\mathcal{L}_1} \leq \|H_2(s)\|_{\mathcal{L}_1}\|H_1(s)\|_{\mathcal{L}_1}$.

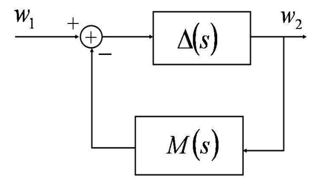


Fig. 1. Block diagram for $\mathcal{L}_1$ small gain theorem

We now state the $\mathcal{L}_1$ *Small Gain Theorem*. Consider an interconnected LTI system in Fig. 1, where $w_1 \in \mathbb{R}^{n_1}$, $w_2 \in \mathbb{R}^{n_2}$, $M(s)$ is a stable proper system with n_2 inputs and n_1 outputs, and $\Delta(s)$ is a stable proper system with n_1 inputs and n_2 outputs.

Theorem 1: ($\mathcal{L}_1$ **Small Gain Theorem**) The interconnected system in Fig. 1 is stable if $\|M(s)\|_{\mathcal{L}_1}\|\Delta(s)\|_{\mathcal{L}_1} < 1$.

The proof follows from small gain theorem in [18] (page 218, Theorem 5.6]), written for $\mathcal{L}_1$ gain.

Consider a linear time invariant system: $\dot{x}(t) = Ax(t) + bu(t)$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is Hurwitz, and assume that the transfer function $(s\mathbb{I}-A)^{-1}b$ is strictly proper and stable. Notice that it can be expressed as: $(s\mathbb{I}-A)^{-1}b = \frac{n(s)}{d(s)}$, where $d(s) = \det(s\mathbb{I}-A)$ is a n^{th} order stable polynomial, and $n(s)$ is a $n \times 1$ vector with its i^{th} element being a polynomial function: $n_i(s) = \sum_{j=1}^n n_{ij}s^{j-1}$.

Lemma 3: If $(A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n)$ is controllable, the matrix N with its i^{th} row j^{th} column entry n_{ij} is full rank.

Lemma 4: If (A, b) is controllable and $(s\mathbb{I}-A)^{-1}b$ is strictly proper and stable, there exists $c_o \in \mathbb{R}^n$ such that the transfer function $c_o^\top(s\mathbb{I}-A)^{-1}b$ is minimum phase with relative degree one.

III. Problem Formulation

Consider single-input single-output system dynamics:

$$\dot{x}(t) = Ax(t) + bu(t),\ y(t) = c^\top x(t),\ x(0) = x_0\,, \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal, $b, c \in \mathbb{R}^n$ are known constant vectors, A is an unknown $n \times n$ matrix, $y \in \mathbb{R}$ is the regulated output. The objective is to design a *low-frequency adaptive controller* to ensure that $y(t)$ tracks a given bounded continuous reference signal $r(t)$, while all other error signals remain bounded. Following the convention, we introduce the following matching assumption:

Assumption 1: There exist a Hurwitz matrix $A_m \in \mathbb{R}^{n\times n}$ and a vector of ideal parameters $\theta \in \mathbb{R}^n$ such that (A_m, b) is controllable and $A_m - A = b\,\theta^\top$. We further assume the unknown parameter θ belongs to a given compact convex set Θ, i.e. $\theta \in \Theta$.

IV. Companion Model Adaptive Controller

In this section, we present a control architecture, which is equivalent to conventional Model Reference Adaptive Control (MRAC) architecture. We further use this to develop a novel adaptive control architecture with guaranteed transient performance.

A. Companion Model Adaptive Controller

Theorem 2: [CMAC] Given a bounded reference input signal $r(t)$ of interest to track, the following direct adaptive feedback/feedforward controller

$$\begin{aligned} u(t) &= \hat{\theta}^\top(t)x(t) + k_g r(t), && (2)\\ \dot{\hat{\theta}}(t) &= \Gamma \mathrm{Proj}(\hat{\theta}(t), x(t)\tilde{x}^\top(t)Pb),\ \hat{\theta}(0) = \hat{\theta}_0\,, && (3)\end{aligned}$$

in which $\hat{\theta}(t) \in \mathbb{R}^n$ are the adaptive parameters, $\Gamma = \Gamma_c \mathbb{I}_{n\times n}$, $\Gamma_c > 0$ is the adaptation gain, $P = P^\top > 0$ is the solution of algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$ for arbitrary $Q > 0$, $\tilde{x}(t) = \hat{x}(t) - x(t)$ is the tracking error between system dynamics in (1) and the following companion system

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m\hat{x}(t) + b(u(t) - \hat{\theta}^\top(t)x(t))\,, \quad \hat{x}(0) = x_0\\ \hat{y}(t) &= c^\top \hat{x}(t)\,, && (4)\end{aligned}$$

ensures that $\lim_{t\to\infty} \tilde{x}(t) = 0$.
The proof is straightforward. Indeed, subject to Assumption 1, the system dynamics in (1) can be rewritten as:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b(u(t) - \theta^\top x(t))\,, x(0) = x_0 && (5)\\ y(t) &= c^\top x(t). \end{aligned}$$

Notice that the companion model in (4) shares the same structure with (5), while the control law in (2), (3) reduces the closed loop dynamics of the companion model to the desired reference model:

$$\dot{\hat{x}}(t) = A_m\hat{x}(t) + bk_g r(t)\,, \quad \hat{x}(0) = x_0. \qquad (6)$$

We also notice that the closed-loop tracking error dynamics are the same as in MRAC:

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) - b\tilde{\theta}^\top(t)x(t)\,, \quad \tilde{x}(0) = 0. \qquad (7)$$

Since the closed-loop companion model in (6) is bounded, from standard Lyapunov arguments and Barbalat's lemma it follows that $\lim_{t\to\infty} \tilde{x}(t) = 0$. Thus, the companion model adaptive control architecture is equivalent to MRAC in a sense that from the same initial condition they both lead to the same tracking error dynamics. The following remark is in order.

Remark 1: The matching assumption implies that the ideal tracking controller is given by the following linear relationship $u(t) = \theta^\top x(t) + k_g r(t)$, where

$$k_g = -1/(c^\top A_m^{-1} b)\,. \qquad (8)$$

The choice of k_g in (8) ensures that for constant r one has $\lim_{t\to\infty} y(t) = r$ in both MRAC and CMAC architectures.

B. Bounded Tracking Error Signal and Transient Performance

For both architectures MRAC and CMAC, one can prove that the tracking error can be rendered arbitrarily small by increasing the adaptive gain. The main result is given by the following lemma.

Lemma 5: Given the system in (5), one has $||\tilde{x}(t)|| \le \sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\min}(P)\Gamma_c}}$ for any $t \ge 0$, where

$$\bar{\theta}_{\max} = \max_{\theta\in\Theta} \sum_{i=1}^{n} 4\theta_i^2, \qquad (9)$$

and $\lambda_{\min}(P)$ is the minimum eigenvalue of P.

Theorem 2 states that the tracking error goes to zero asymptotically as $t \to \infty$, however it provides no guarantees for the transient performance.

V. $\mathcal{L}_1$ Adaptive Controller

In this section, we introduce a filtering technique for CMAC that enables to prove guaranteed transient performance for system's both signals, input and output, simultaneously, [17]. Letting

$$\bar{r}(t) = \hat{\theta}^\top(t)x(t), \qquad (10)$$

the companion model in (4) can be viewed as a low-pass system with $u(t)$ being the control signal, $\bar{r}(t)$ being a time-varying disturbance, which is not prevented from having high-frequency oscillations. Instead of (2) we will consider the following control design for (4)

$$u(s) = C(s)\big(\bar{r}(s) + k_g r(s)\big) - \hat{K}^\top(s)\hat{x}(s)\,, \qquad (11)$$

where $\bar{r}(s), r(s), u(s), \hat{x}(s)$ are the Laplace transformations of $\bar{r}(t), r(t), u(t), \hat{x}(t)$, k_g is a pre-specified design gain, $\hat{K}(s)$ is a feedback module, and $C(s)$ is a low-pass filter with low-pass gain 1. Below we introduce the design specifics of every component in the control law (11), i.e. $\hat{K}(s)$, $C(s)$, k_g in a systematic manner.

1) Design of the Feedback Module $\hat{K}(s)$: The design of $\hat{K}(s)$ is equivalent to design of a proper stable feedback controller for a Single-Input-Multiple-Output stable Linear Time Invariant (LTI) system

$$\dot{\bar{x}}(t) = A_m\bar{x}(t) + b\bar{u}(t)\,, \qquad (12)$$

where A_m is defined in Assumption 1, $\bar{x}(t) \in \mathbb{R}^n$ is the state vector (measurable), and $\bar{u}(t) \in \mathbb{R}$ is the control signal. Let the open loop transfer function of the system in (12) be $H_o(s) = (s\mathbb{I} - A_m)^{-1}b$. With the feedback module $\bar{u}(s) = \hat{r}(s) - \hat{K}^\top(s)\bar{x}(s)$, where $\hat{r}(s)$ is any reference input for the system in (12), the closed-loop transfer function between $\hat{r}(s)$ and $\bar{x}(s)$ takes the form:

$$H_c(s) = \Big(\mathbb{I} + H_o(s)\hat{K}^\top(s)\Big)^{-1} H_o(s). \qquad (13)$$

Since the open loop transfer function is already stable, the objective of the feedback module is to provide an opportunity for changing the bandwidth of the control signal if needed.

2) Design of the Low-pass Filter $C(s)$: Consider the closed-loop companion model with the control signal defined in (11). It can be viewed as an LTI system with two inputs $r(t)$ and $\bar{r}(t)$:

$$\hat{x}(s) = \bar{G}(s)\bar{r}(s) + G(s)r(s) \quad (14)$$

$$\bar{G}(s) = H_c(s)(C(s) - 1) \quad (15)$$

$$G(s) = k_g H_c(s)C(s)\,. \quad (16)$$

We note that $\bar{r}(t)$ is related to $\hat{x}(t)$, $u(t)$ and $r(t)$ via nonlinear relationships. Let

$$\theta_{\max} = \max_{\theta\in\Theta} \sum_{i=1}^{n} |\theta_i|\,, \quad (17)$$

where θ_i is the i^{th} element of θ, Θ is the compact set, where the unknown parameter lies. We now give the $\mathcal{L}_1$-gain requirement, which is needed for the design of the low-pass filter $C(s)$. This in turn ensures stability of the entire system as discussed later in Section VI.

$\mathcal{L}_1$ **gain requirement:** Design $C(s)$, $\hat{K}(s)$ to satisfy

$$\|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} < 1\,, \quad (18)$$

where $\|\bar{G}(s)\|_{\mathcal{L}_1}$ is the $\mathcal{L}_1$ gain of $\bar{G}(s)$ in (15), and $\theta_{\max}$ is defined in (17).

We can use the requirement in (18) to determine the bandwidth B of the low-pass filter $C(s)$.

Remark 2: We note that the $\mathcal{L}_1$ gain requirement can always be met by increasing the bandwidth B of the low-pass filter $C(s)$. Indeed, the system $\bar{G}(s)$ is a result of cascading a strictly proper, and hence low-pass, system $H_c(s)$ with a high-pass filter $C(s) - 1$. If the cut-off frequency of the high-pass filter is much larger than the bandwidth of the low-pass filter, then as a result one gets a no-pass filter, the $\mathcal{L}_1$ gain of which can easily satisfy (18). On the other hand, if one considers $h_c(t)$, the impulse response of $H_c(s)$, then $\|h_c\|_{\mathcal{L}_1}$, the $\mathcal{L}_1$ norm of $h_c(t)$, is always finite. Since $H_c(s)$ is a low-pass stable system, $h_c(t)$ is a low-frequency signal. If one passes $h_c(t)$ into a high pass filter $C(s)-1$, the $\mathcal{L}_1$ norm of the output can be arbitrarily small if the cutoff frequency (which is related to the bandwidth B of $C(s)$) of the high-pass filter is greater than the bandwidth of $h_c(t)$.

Remark 3: If the bandwidth of $H_c(s)$ is small and $H_c(s)$ decays fast, the bandwidth B of $C(s)$ needed to satisfy the $\mathcal{L}_1$ gain requirement is also small. This is one of the reasons for including a feedback module $\hat{K}(s)$ to have the opportunity to reduce the bandwidth of $H_o(s)$, if it is too large.

3) Setting the Design Gain k_g: To ensure zero steady state error for constant reference inputs, k_g is chosen as:

$$k_g = 1/(c^\top H_c(0)C(0))\,. \quad (19)$$

Since $C(s)$ is a low-pass filer with low-pass gain 1, then $\lim_{s\to 0} C(s) = 1$, and equation (19) can be simplified to $k_g = 1/(c^\top H_c(0))$.

The complete $\mathcal{L}_1$ adaptive controller consists of the companion model (4), the adaptive law (3) and the control law (11), which satisfies the $\mathcal{L}_1$ gain requirement in (18).

Simple Choice of the Feedback Module

One simple choice of $\hat{K}(s)$ is $\hat{K}(s) = 0$, which implies that $H_c(s) = H_o(s)$. For this "no feedback module design", the purpose of the low-pass filter $C(s)$ can be illustrated more clearly. Without $C(s)$, the entire signal $\bar{r}(t)$, including the high-frequency component is passed to the control channel exactly. With $C(s)$, upon Laplace transform the system in (5) takes the form:

$$x(s) = H_o(s)C(s)\bar{r}(s) - H_o(s)\theta^\top x(s) + H_o(s)C(s)k_g r(s)\,,$$

while the companion model in (4) takes the form:

$$\hat{x}(s) = H_o(s)(C(s) - 1)\bar{r}(s) + H_o(s)C(s)k_g r(s).$$

We note that $\bar{r}(t)$ is passed to both systems - the true one and the companion model. The true system gets only the "low-frequency filtered signal" $C(s)\bar{r}(s)$, while the "virtual" companion model, gets the "left-over high-frequency signal" $(C(s) - 1)\bar{r}(s)$.

VI. Stability and Performance of the $\mathcal{L}_1$ Adaptive Controller

Consider the Lyapunov function candidate:

$$V(\tilde{x}(t), \tilde{\theta}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t)\,. \quad (20)$$

It is straightforward to verify that for the systems in (5) and (4), we have

$$\dot{V}(t) \le -\tilde{x}^\top(t)Q\tilde{x}(t) \le 0\,. \quad (21)$$

Notice that the result in (21) is independent of the control design. However, one cannot deduce stability from it. One needs to prove in addition that with the $\mathcal{L}_1$ adaptive controller the state of the companion model will remain bounded. Boundedness of the system state will follow.

Theorem 3: Given the system in (5) and the $\mathcal{L}_1$ adaptive controller defined via (3), (4), (11) subject to (18), the tracking error $\tilde{x}(t)$ converges to zero asymptotically: $\lim_{t\to\infty} \tilde{x}(t) = 0$.

A. Tracking Performance

We now consider the steady state performance of the $\mathcal{L}_1$ adaptive controller for a constant reference input r. As $t \to \infty$, it follows from Theorem 3 and the adaptive law in (3) that

$$\lim_{t\to\infty} (\hat{x}(t) - x(t)) = 0 \quad (22)$$

$$\lim_{t\to\infty} \dot{\hat{\theta}}(t) = 0. \quad (23)$$

Equation (22) implies that we can replace $x(t)$ by $\hat{x}(t)$ for the steady state analysis. Since r is a constant, it follows from (23) that the derivatives of all signals approach zero as $t \to \infty$. Hence, the steady state can be analyzed using the transfer functions as $s \to 0$. Since $C(s)$ is a low-pass filter with low-pass gain 1, we have $\lim_{s\to 0} \bar{G}(s) = 0$, and therefore $\lim_{t\to\infty} \hat{y}(t) = \lim_{s\to 0} k_g c^\top H_c(s) r$. Using (19) this leads to $\lim_{t\to\infty} \hat{y}(t) = r$. Theorem 3 implies that $\lim_{t\to\infty} (\hat{y}(t) - y(t)) = 0$, and therefore $\lim_{t\to\infty} y(t) = r$.

The system response to a time-varying $r(t)$ and complete characterization of the transient performance of the $\mathcal{L}_1$ adaptive controller is presented in Part II [17].

B. Low-frequency Control Signal

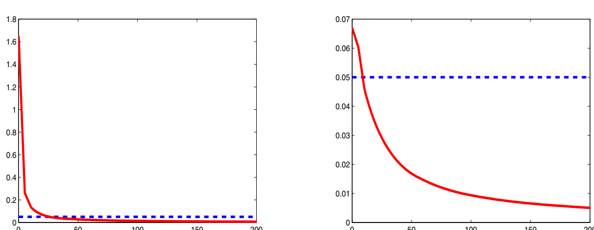
It follows from (11), (14), (15) and (16) that the $\mathcal{L}_1$ adaptive controller is $u(s) = (C(s) - \hat{K}^\top(s) H_c(s)(C(s) - 1))\bar{r}(s) + k_g(C(s) - \hat{K}^\top(s) H_c(s) C(s)) r(s)$. Since $C(s)$, $\hat{K}^\top(s) H_c(s)$ are low-pass filters, then $u(t)$ is a low-frequency signal, no matter if $\bar{r}(t)$ has high-frequency components. We compare it to the conventional adaptive controller $u(s) = \bar{r}(s) + k_g r(s)$, where the high-frequency signal $\bar{r}(t)$ goes into the control channel directly without passing through any low-pass filters.

VII. Simulations

For simulation purposes, the following system parameters have been selected: $A = \begin{bmatrix} 0 & 1 \\ -5 & 3.1 \end{bmatrix}$, $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where the unknown parameter $\theta = [4 \quad -4.5]^\top$ belongs to a given compact set: $\theta_i \in [-10, \ 10]$, $i = 1, 2$.

We give now the complete $\mathcal{L}_1$ adaptive controller for this system. We consider the companion model $\dot{\hat{x}}(t) = A_m \hat{x}(t) + b(u(t) - \hat{\theta}(t) x(t))$, and the control law $u(s) = C(s)(\bar{r}(s) + k_g r(s)) - \hat{K}^\top(s)\hat{x}(s)$. We implement two $\mathcal{L}_1$ adaptive control designs: one without any feedback module and another with a static feedback module. In both cases, we take the simplest low-pass filter in the form of a first order system:

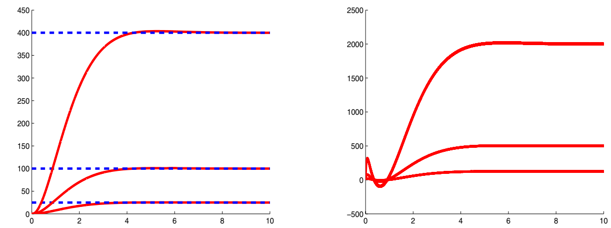
$$C(s) = \omega/(s + \omega). \tag{24}$$



(a) $\mathcal{L}_1$ design without $\hat{K}(s)$ (b) $\mathcal{L}_1$ design with $\hat{K}(s)$

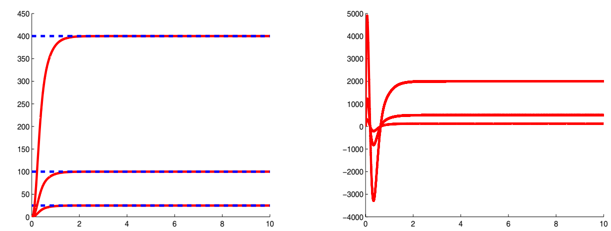
Fig. 2. $\|\bar{G}(s)\|_{\mathcal{L}_1}$ (solid) and $\frac{1}{\Theta_{\max}}$ (dotted) w.r.t. ω

At first, we design $\mathcal{L}_1$ adaptive controller with $\hat{K}(s) = 0$. It follows from (17) that $\frac{1}{\theta_{\max}} = 0.05$. Given a low-pass filter $C(s)$, we can compute $\|\bar{G}(s)\|_{\mathcal{L}_1}$ numerically. In Figure 2(a), we plot $\|\bar{G}(s)\|_{\mathcal{L}_1}$ with respect to ω, the latter being the bandwidth of the first order low-pass filter in (24), and we compare it with $\frac{1}{\theta_{\max}}$. We note that the $\mathcal{L}_1$ stability criteria can be met if one chooses $\omega > 27$. We choose $\omega = 30$, which makes $\|\bar{G}(s)\|_{\mathcal{L}_1} = 0.0426 < \frac{1}{\theta_{\max}} = 0.05$. Let the adaptive gain be $\Gamma_c = 10000$. Design of $C(s)$ in (24) with $\omega = 30$ leads to $k_g = 1$. The simulation results for $r = 25, 100, 400$ are plotted in Figures 3(a)-3(b).



(a) $y(t)$ (solid) and $r(t)$(*dashed*) (b) $u(t)$, $\Gamma_c = 10000$

Fig. 3. $\mathcal{L}_1$ adaptive controller without $\hat{K}(s)$ for $r = 25, 100, 400$



(a) $y(t)$ (solid) and $r(t)$(*dashed*) (b) $u(t)$, $\Gamma_c = 2000$

Fig. 4. $\mathcal{L}_1$ adaptive controller with $\hat{K}(s)$ for $r = 25, 100, 400$

Next we design the $\mathcal{L}_1$ controller with the use of $\hat{K}(s)$. We use a simple pole placement method to improve the $H_o(s)$, by setting the poles of desired closed-loop system $H_c(s)$ at $[-7 \quad -6]$. The choice of the static feedback module $\hat{K}(s) = [41 \quad 11.6]$ results in $H_c(s) = \begin{bmatrix} \frac{1}{s^2+13s+42} & \frac{s}{s^2+13s+42} \end{bmatrix}^\top$. In Figure 2(b), we plot $\|\bar{G}(s)\|_{\mathcal{L}_1}$ with respect to ω and compare it with $\frac{1}{\theta_{\max}}$. We notice that the $\mathcal{L}_1$ gain requirement can be satisfied if we choose $\omega > 8.9$. We set $\omega = 10$, which ensures that $\|\bar{G}(s)\|_{\mathcal{L}_1} = 0.047 < \frac{1}{\theta_{\max}} = 0.05$. Let the adaptive gain be $\Gamma_c = 2000$. The choice of $C(s)$ with $\omega = 10$ leads to $k_g = 42$ according to (19). The simulation results for $r = 25, 100, 400$ are plotted in Figures 4(a)-4(b).

As shown in the plots, $\mathcal{L}_1$ adaptive controller permits faster convergence by increasing the adaptation gain without generating high-frequency control signal. Also, notice that the $\mathcal{L}_1$ adaptive controller responds to a step-response similar to linear systems, i.e. scaled output response and control signal are obtained for a scaled reference input (compare Figures for $r = 25, 100, 400$). Figures 2(a) and 2(b) demonstrate the benefits of using $\hat{K}(s)$. We note that by using $\hat{K}(s)$, the $\mathcal{L}_1$ gain requirement enabled to accommodate the low-pass filter $C(s)$ with a smaller bandwidth.

VIII. Conclusion

In this paper, a novel $\mathcal{L}_1$ adaptive controller is presented that guarantees low-frequency control signal, as compared to the conventional adaptive controller, and ensures asymptotic convergence of the tracking error to zero. Moreover, the new control architecture tolerates high adaptation gains, leading to improved transient performance. In Part II [17], complete analysis of the transient performance is given. In [20], [21], the methodology is extended to systems with unknown time-varying parameters and bounded disturbances in the presence of unknown high-frequency gain, and stability margins are derived.

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Appendix

Proof. of Lemma 5: The candidate Lyapunov function, which can be used to prove asymptotic convergence of tracking error to zero in Theorem 2, is given by $V(\tilde{x}(t),\tilde{\theta}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t)$. The following upper bound is straightforward to derive:

$$\tilde{x}^\top(t)P\tilde{x}(t) \le V(t) \le V(0), \quad \forall t \ge 0. \tag{25}$$

Since the projection algorithm ensures that $\hat{\theta}(t) \in \Theta$ for ant $t \ge 0$, then

$$\max_{t\ge 0} \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t) \le (\bar{\theta}_{\max})/\Gamma_c, \quad \forall t \ge 0, \tag{26}$$

where $\bar{\theta}_{\max}$ is defined in (9). Since $\tilde{x}(0) = 0$, then $V(0) = \tilde{\theta}^\top(0)\Gamma^{-1}\tilde{\theta}(0)$, which along with (25), (26) leads to $\tilde{x}^\top(t)P\tilde{x}(t) \le (\bar{\theta}_{\max})/\Gamma_c, \quad t \ge 0$. Since $\lambda_{\min}(P)\|\tilde{x}\|^2 \le \tilde{x}^\top(t)P\tilde{x}(t), \quad t \ge 0$, then $||\tilde{x}(t)|| \le \sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\min}(P)\Gamma_c}}$ for all $t \ge 0$.

Proof of Theorem 3: Let $\lambda_{\min}(P)$ be the minimum eigenvalue of P. From (20) and (21) it follows that $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \le \tilde{x}^\top(t)P\tilde{x}(t) \le V(t) \le V(0)$, implying that

$$\|\tilde{x}(t)\|^2 \le V(0)/\lambda_{\min}(P),\ t \ge 0. \tag{27}$$

From Definition 1, $\|\tilde{x}\|_{\mathcal{L}_\infty} = \max_{i=1,..,n,t\ge 0} |\tilde{x}_i(t)|$. The relationship in (27) ensures that $\max_{i=1,..,n,t\ge 0} |\tilde{x}_i(t)| \le \sqrt{\dfrac{V(0)}{\lambda_{\min}(P)}}$, and therefore for any $t > 0$ we have $\|\tilde{x}_t\|_{\mathcal{L}_\infty} \le \sqrt{\dfrac{V(0)}{\lambda_{\min}(P)}}$. Hence, using the triangular relationship for norms, one can prove

$$|\ \|\hat{x}_t\|_{\mathcal{L}_\infty} - \|x_t\|_{\mathcal{L}_\infty}\ | \le \sqrt{V(0)/(\lambda_{\min}(P))}. \tag{28}$$

The projection technique in (3) ensures that $\hat{\theta}(t) \in \Theta, \forall t \ge 0$. From (10) and (17) it follows that $\|\bar{r}_t\|_{\mathcal{L}_\infty} \le \theta_{\max}\|x_t\|_{\mathcal{L}_\infty}$. Using (28), we have

$$\|\bar{r}_t\|_{\mathcal{L}_\infty} \le \theta_{\max}(\|\hat{x}_t\|_{\mathcal{L}_\infty} + \sqrt{V(0)/(\lambda_{\min}(P))}\ ). \tag{29}$$

It follows from Lemma 1 that $\|\hat{x}_t\|_{\mathcal{L}_\infty} \le \|\bar{G}(s)\|_{\mathcal{L}_1}\|\bar{r}_t\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}$, which along with (29) leads to

$$\begin{aligned}\|\hat{x}_t\|_{\mathcal{L}_\infty} &\le \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max}(\|\hat{x}_t\|_{\mathcal{L}_\infty} + \sqrt{V(0)/(\lambda_{\min}(P))}\ ) \\ &+\|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}.\end{aligned} \tag{30}$$

Let $\lambda = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max}$. From (18) it follows that $\lambda < 1$. The relationship in (30) can be written as $(1-\lambda)\|\hat{x}_t\|_{\mathcal{L}_\infty} \le \lambda\sqrt{\frac{V(0)}{\lambda_{\min}(P)}} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}$, and hence

$$\|\hat{x}_t\|_{\mathcal{L}_\infty} \le (\lambda\sqrt{\frac{V(0)}{\lambda_{\min}(P)}} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty})/(1-\lambda). \tag{31}$$

Since $V(0), \lambda_{\min}(P), \|G(s)\|_{\mathcal{L}_1}, \|r\|_{\mathcal{L}_\infty}, \lambda$ are all finite and $\lambda < 1$, the relationship in (31) implies that $\|\hat{x}_t\|_{\mathcal{L}_\infty}$ is finite for any $t > 0$, and hence $\hat{x}(t)$ is bounded. The relationship in (28) implies that $\|x_t\|_{\mathcal{L}_\infty}$ is also finite for all $t > 0$ and therefore $x(t)$ is bounded. The adaptive law in (3) ensures that the estimates $\hat{\theta}(t)$ are also bounded. Hence, it can be checked easily from (7) that $\dot{\tilde{x}}(t)$ is bounded, and it follows from Barbalat's lemma that $\lim_{t\to\infty} \tilde{x}(t) = 0$.