V. Conclusion

In this note, an explicit parametric solution to the generalized Sylvester matrix equation $AX + BY = EXF$ with the matrix F being an arbitrary square matrix has been provided in terms of the R-controllability matrix associated with the matrix triple (E, A, B) and an observability matrix associated with the matrix F and a free parameter matrix. The proposed solution offers all the degrees of freedom. The proposed results may bring new convenience in many applications related to the generalized matrix equation.

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$\mathcal{L}_1$ Adaptive Output Feedback Controller for Systems of Unknown Dimension

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***Abstract*—This note presents novel adaptive output feedback control methodology for systems of unknown dimension in the presence of unmodeled dynamics and time-varying uncertainties. The adaptive output feedback controller ensures uniformly bounded transient and asymptotic tracking for the system's both signals, input and output, simultaneously. The performance bounds can be systematically improved by increasing the adaptation rate. Simulations of an unstable nonminimum phase system verify the theoretical findings.**

***Index Terms*—Adaptive output feedback, guaranteed transient performance, nonminimum phase systems.**

I. Introduction

This note extends the results of [1]–[3] to an output feedback framework for a single-input signal-output (SISO) system of unknown dimension in the presence of time-varying disturbances. The methodology ensures uniformly bounded transient response for the system's both signals, input and output, simultaneously, in addition to asymptotic tracking. The $\mathcal{L}_\infty$ norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by increasing the adaptation gain.

Adaptive algorithms achieving arbitrarily improved transient performance in the case of constant unknown parameters are given in [4]–[14], and for unknown time-varying parameters have been given in [15]. While the results in [15] improved upon [16]–[18], by extending the class of systems beyond the slow time-variation of the unknown parameters and guaranteeing performance improvement to an arbitrary degree, they still did not provide means for regulating the frequency spectrum of the control signal during the transient. In [19] and [20], we developed a new architecture for control of uncertain systems, named $\mathcal{L}_1$ adaptive controller, which permits fast adaptation and yields the desired transient response for the system's both signals, input and output, simultaneously, in addition to asymptotic tracking. In this note, we extend the methodology to systems of unknown dimension in the presence of time-varying bounded disturbances without limiting the rate of their variation. By modifying the architecture correspondingly, we prove that the $\mathcal{L}_1$ adaptive controller ensures uniformly bounded

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transient response for the system's both signals, input and output, simultaneously, in addition to stable tracking. A closed-loop reference system is considered by introducing the filtered version of an ideal nominal controller. The $\mathcal{L}_\infty$ norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by increasing the adaptation rate.

The note is organized as follows. Section II states some preliminary definitions, and Section III gives the problem formulation. In Section IV, the closed-loop reference system is defined. In Section V, the novel $\mathcal{L}_1$ adaptive control architecture is presented. Stability and uniform transient tracking bounds of the $\mathcal{L}_1$ adaptive controller are presented in Section VI. Section VII provides a discussion on class of systems for which the proposed methodology can be implemented. In Section VIII, simulation results are presented, while Section IX concludes the note.

II. Preliminaries

In this Section, we recall basic definitions and facts from linear systems theory.

Definition 1: For a signal $\xi(t), t \geq 0, \xi \in \mathbb{R}^n$, its truncated $\mathcal{L}_\infty$ and $\mathcal{L}_\infty$ norms are $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{0\leq\tau\leq t}|\xi_i(\tau)|)$, $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,..,n}(\sup_{\tau\geq 0}|\xi_i(\tau)|)$, where ξ_i is the ith component of ξ.

Definition 2: The $\mathcal{L}_1$ gain of a bounded-input bounded-output (BIBO) stable proper SISO system is defined by $||H(s)||_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$, where $h(t)$ is the impulse response of $H(s)$.

We note that a transfer function is BIBO stable if and only if every pole has a negative real part.

Lemma 1: For a BIBO stable proper SISO system $H(s)$ with input $r(t)$ and output $x(t)$, we have $\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty} \quad \forall t \geq 0$.

III. Problem Formulation

Consider the following SISO system:

$$y(s) = A(s)(u(s) + d(s)), y(0) = 0, \tag{1}$$

where $u(t) \in \mathbb{R}$ is the system's input, $y(t) \in \mathbb{R}$ is the system's output, $A(s)$ is a strictly proper unknown transfer function, $d(s)$ is the Laplace transform of the time-varying uncertainties and disturbances $d(t) = f(t, y(t))$, while f is an unknown map, subject to the following assumptions.

Assumption 1: There exist constants $L < 0$ and $L_0 < 0$ such that the following inequalities $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|, |f(t,y)| \leq L|y| + L_0$ hold uniformly in $t \geq 0$.

Assumption 2: There exist constants $L_1 < 0$, $L_2 < 0$, and $L_3 < 0$ such that for all $t \geq 0$

$$|\dot{d}(t)| \leq L_1|\dot{y}(t)| + L_2|y(t)| + L_3. \tag{2}$$

We note that the numbers L, L_0, L_1, L_2, and L_3 can be arbitrarily large. Let $r(t)$ be a given bounded continuous reference input signal. The control objective is to design an adaptive output feedback controller $u(t)$ such that the system output $y(t)$ tracks the reference input following a desired reference model, i.e., $y(s) \approx M(s)r(s)$. In this note, we consider a first-order system, i.e.

$$M(s) = m/(s+m), \quad m > 0. \tag{3}$$

We note that the system in (1) can be rewritten as

$$y(s) = M(s)\left(u(s) + \sigma(s)\right) \tag{4}$$

$$\sigma(s) = ((A(s) - M(s))u(s) + A(s)d(s))/M(s). \tag{5}$$

IV. Closed-Loop Reference System

Consider the following closed-loop reference system:

$$y_{\mathrm{ref}}(s) = M(s)(u_{\mathrm{ref}}(s) + \sigma_{\mathrm{ref}}(s)) \tag{6}$$

$$\sigma_{\mathrm{ref}}(s) = \frac{(A(s) - M(s))u_{\mathrm{ref}}(s) + A(s)d_{\mathrm{ref}}(s)}{M(s)} \tag{7}$$

$$u_{\mathrm{ref}}(s) = C(s)(r(s) - \sigma_{\mathrm{ref}}(s)) \tag{8}$$

where $d_{\mathrm{ref}}(t) = f(t, y_{\mathrm{ref}}(t))$, and $C(s)$ is a strictly proper system with $C(0) = 1$. One simple choice would be

$$C(s) = \omega/(s+\omega). \tag{9}$$

We note that there is no algebraic loop involved in the definition of $\sigma(s)$, $u(s)$ and $\sigma_{\mathrm{ref}}(s)$, $u_{\mathrm{ref}}(s)$.

We will further restrict the choice of $C(s)$ and $M(s)$ to ensure that

$$H(s) = \frac{A(s)M(s)}{(C(s)A(s) + (1 - C(s))M(s))} \quad \text{is BIBO stable,} \tag{10}$$

and

$$\|G(s)\|_{\mathcal{L}_1} L < 1, \quad G(s) = H(s)(1 - C(s)). \tag{11}$$

The condition in (11) restricts the class of systems $A(s)$ in (1) that can be stabilized by the controller architecture in this note. However, as discussed in Section VII, the class of such systems is not empty. Letting

$$A(s) = \frac{A_n(s)}{A_d(s)}, \quad C(s) = \frac{C_n(s)}{C_d(s)}, \quad M(s) = \frac{M_n(s)}{M_d(s)} \tag{12}$$

it follows from (10) that

$$H(s) = \frac{C_d(s)M_n(s)A_n(s)}{M_d(s)C_n(s)A_n(s) + (C_d(s) - C_n(s))M_n(s)A_d(s)}. \tag{13}$$

We note that a strictly proper $C(s)$ implies that the order of $C_d(s) - C_n(s)$ and $C_d(s)$ is the same. Since the order of $A_d(s)$ is higher than that of $A_n(s)$, we note that the transfer function $H(s)$ is strictly proper. The next Lemma establishes the stability of the closed-loop system in (6)–(8).

Lemma 2: If $C(s)$ and $M(s)$ verify the conditions in (10) and (11), the closed-loop reference system in (6)–(8) is BIBO stable.

Proof: It follows from (7) and (8) that

$$u_{\mathrm{ref}}(s) = \frac{C(s)M(s)r(s) - C(s)A(s)d_{\mathrm{ref}}(s)}{C(s)A(s) + (1 - C(s))M(s)}. \tag{14}$$

It follows from (6) and (7) that

$$y_{\mathrm{ref}}(s) = A(s)(u_{\mathrm{ref}}(s) + d_{\mathrm{ref}}(s)). \tag{15}$$

Substituting (14) into (15), it follows from (10) that

$$y_{\mathrm{ref}}(s) = H(s)\left(C(s)r(s) + (1 - C(s))d_{\mathrm{ref}}(s)\right). \tag{16}$$

Since $H(s)$ is strictly proper and BIBO stable, $G(s)$ is also strictly proper and BIBO stable, and therefore

$$\|y_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \leq \|H(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}(L\|y_{\mathrm{ref}}\|_{\mathcal{L}_\infty} + L_0).$$

It follows from (11) and (17) that

$$\|y_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \leq \rho, \quad \rho = \frac{\|H(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}L_0}{1 - \|G(s)\|_{\mathcal{L}_1}L}. \tag{18}$$

Hence $\|y_{\mathrm{ref}}\|_{\mathcal{L}_\infty}$ is finite, which implies that the closed-loop reference system in (6)–(8) is BIBO stable.

Remark 1: We notice that the reference system in (6) is written in terms of the desired system behavior, defined by $M(s)$. The uncertainties due to $A(s)$ and $f(t, y_{\mathrm{ref}}(t))$ are lumped in the signal $\sigma_{\mathrm{ref}}(s)$. The control signal defined via (8) cancels the uncertainties within the bandwidth of $C(s)$, which eventually defines the tradeoff between performance and robustness.

V. $\mathcal{L}_1$ ADAPTIVE CONTROLLER

A. Notations

Choose arbitrary $P < 0$ and let $Q = 2mP$. Define

$$H_0(s) = \frac{A(s)}{C(s)A(s) + (1 - C(s))M(s)},$$
$$H_1(s) = \frac{(A(s) - M(s))C(s)}{C(s)A(s) + (1 - C(s))M(s)}. \tag{19}$$

Using (12) in (19), we have $H_0(s) = \frac{C_d(s)A_n(s)M_d(s)}{H_d(s)}$, and

$$H_1(s) = \frac{C_n(s)A_n(s)M_d(s) - C_n(s)A_d(s)M_n(s)}{H_d(s)} \tag{20}$$

where $H_d(s) = C_n(s)A_n(s)M_d(s) + M_n(s)A_d(s)(C_d(s) - C_n(s))$. Since the relative order between $C_d(s) - C_n(s)$ and $C_n(s)$ is greater than zero, the order of $M_n(s)A_d(s)(C_d(s) - C_n(s))$ is higher than $C_n(s)A_d(s)M_n(s)$. Similarly, since the relative order between $A_d(s)$ and $A_n(s)$ is greater than zero, while the relative order between $M_n(s)$ and $M_d(s)$ is -1, we note that the order of $M_n(s)A_d(s)(C_d(s) - C_n(s))$ is higher than that of $C_n(s)A_n(s)M_d(s)$. Therefore, $H_1(s)$ is strictly proper. We note from (13) and (20) that $H_1(s)$ has the same denominator as $H(s)$, and it follows from (10) that $H_1(s)$ is BIBO stable. Using similar arguments, it can be verified that $H_0(s)$ is proper and BIBO stable.

Let

$$\Delta = \|H_1(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H_0(s)\|_{\mathcal{L}_1}(L\rho + L_0) + \left(\left\|\frac{H_1(s)}{M(s)}\right\|_{\mathcal{L}_1} + L\|H_0(s)\|_{\mathcal{L}_1}\frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}\right)\bar{\gamma}$$

where $\bar{\gamma} < 0$ is an arbitrary constant. Since $H_1(s)$ is BIBO stable and strictly proper, $\|H_1(s)/M(s)\|_{\mathcal{L}_1}$ is finite, and hence, Δ is a finite number. Let

$$\beta_1 = 4\Delta\|H_0(s)\|_{\mathcal{L}_1}\left(L_1\beta_{01} + L_2\frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}\right)$$
$$\beta_2 = 4\Delta\|sH_1(s)\|_{\mathcal{L}_1}(\|r\|_{\mathcal{L}_\infty} + 2\Delta) + 4\Delta\|H_0(s)\|_{\mathcal{L}_1}\left(L_1\beta_{02} + L_3 + \rho L_2\right) \tag{21}$$

where ρ is defined in (18), and

$$\beta_{01} = \|sH(s)(1 - C(s))\|_{\mathcal{L}_1}\frac{L\,\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}$$
$$\beta_{02} = \|sH(s)C(s)\|_{\mathcal{L}_1}(\|r\|_{\mathcal{L}_\infty} + 2\Delta) + \|sH(s)(1 - C(s))\|_{\mathcal{L}_1}(L\rho + L_0). \tag{22}$$

Since $H(s)$ and $H_1(s)$ are strictly proper and BIBO stable, $\|sH_1(s)\|_{\mathcal{L}_1}$, $\|sH(s)C(s)\|_{\mathcal{L}_1}$ and $\|sH(s)(1 - C(s))\|_{\mathcal{L}_1}$ are finite. We further define

$$\beta_3 = P\beta_1/Q = \beta_1/(2m),$$
$$\beta_4 = 4\Delta^2 + P\beta_2/Q = 4\Delta^2 + \beta_2/(2m). \tag{23}$$

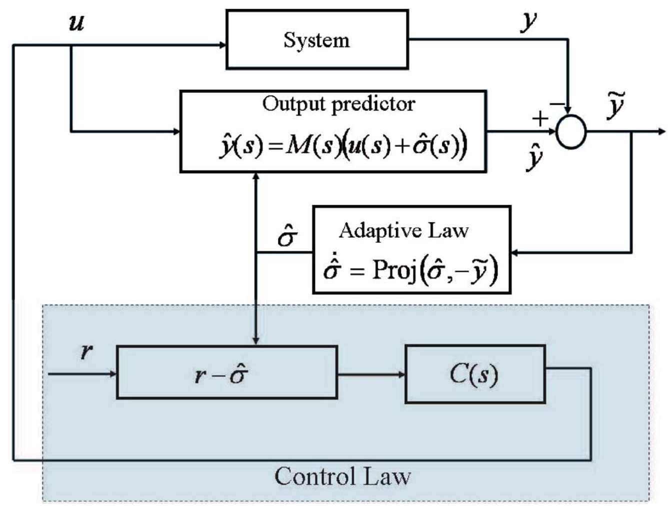


Fig. 1. Closed-loop system with $\mathcal{L}_1$ adaptive controller.

B. $\mathcal{L}_1$ Adaptive Controller

We consider the following output predictor:

$$\dot{\hat{y}}(t) = -m\hat{y}(t) + m\left(u(t) + \hat{\sigma}(t)\right), \hat{y}(0) = 0 \tag{24}$$

where the adaptive estimate $\hat{\sigma}(t)$ is governed by the following adaptation law:

$$\dot{\hat{\sigma}}(t) = \Gamma_c \mathrm{Proj}(\hat{\sigma}(t), -mP\tilde{y}(t)), \quad \tilde{y}(t) = \hat{y}(t) - y(t), \quad \hat{\sigma}(0) = 0 \tag{25}$$

with $\Gamma_c \in \mathbb{R}^+$ being the adaptation rate subject to the following lower bound:

$$\Gamma_c > \max\left\{\frac{\alpha\beta_3^2}{(\alpha - 1)^2\beta_4 P}, \frac{\alpha\beta_4}{P\bar{\gamma}^2}\right\} \tag{26}$$

in which $\alpha > 1$ is an arbitrary constant, while projection is confined to the following bound:

$$|\hat{\sigma}(t)| \le \Delta. \tag{27}$$

Letting

$$\gamma_0 = \sqrt{\alpha\beta_4/(\Gamma_c P)}, \tag{28}$$

it follows from (26) that $\bar{\gamma} \ge \gamma_0$, and hence

$$\Delta \ge \|H_1(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H_0(s)\|_{\mathcal{L}_1}(L\rho + L_0) + \left(\left\|\frac{H_1(s)}{M(s)}\right\|_{\mathcal{L}_1} + L\|H_0(s)\|_{\mathcal{L}_1}\frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1}L}\right)\gamma_0.$$

The control signal is generated by

$$u(s) = C(s)(r(s) - \hat{\sigma}(s)). \tag{29}$$

The complete $\mathcal{L}_1$ adaptive controller consists of (24), (25), and (29) subject to the $\mathcal{L}_1$-gain condition in (11). The closed-loop system is illustrated in Fig. 1.

VI. ANALYSIS OF $\mathcal{L}_1$ ADAPTIVE CONTROLLER

In this section, we analyze the stability and the performance of $\mathcal{L}_1$ adaptive controller. Let $H_2(s) = -M(s)C(s)/(C(s)A(s) + (1 - C(s))M(s))$. Using the definitions from (12), we have

$$H_2(s) = \frac{-C_n(s)A_d(s)M_n(s)}{C_n(s)A_n(s)M_d(s) + M_n(s)A_d(s)(C_d(s) - C_n(s))}. \tag{30}$$

Since the relative order between $C_d(s) - C_n(s)$ and $C_n(s)$ is greater than zero, it can be verified straightforwardly that $H_2(s)$ is strictly proper. We note from (13) and (30) that $H_2(s)$ has the same denominator as $H(s)$, and it follows from (10) that $H_2(s)$ is BIBO stable. Since $H_2(s)$ is strictly proper and BIBO stable, $H_2(s)/M(s)$ is BIBO stable and proper, and hence, its $\mathcal{L}_1$ gain is finite. It can be verified that $C(s)H(s)/M(s)$ is also strictly proper and BIBO stable, and hence, $\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}$ exists and is finite. Let $H_3(s) = H(s)C(s)/M(s)$.

Theorem 1: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller in (24), (25), and (29) subject to (11), we have

$$\|\tilde{y}\|_{\mathcal{L}_\infty} < \gamma_0 \tag{31}$$

$$\|y - y_{\rm ref}\|_{\mathcal{L}_\infty} \leq \gamma_1 \tag{32}$$

$$\|u - u_{\rm ref}\|_{\mathcal{L}_\infty} \leq \gamma_2 \tag{33}$$

where $\tilde{y}(t) = \hat{y}(t) - y(t)$, γ_0 is defined in (28), and

$$\begin{aligned}\gamma_1 &= \|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\,\gamma_0/(1 - \|G(s)\|_{\mathcal{L}_1} L),\\ \gamma_2 &= L\|H_3(s)\|_{\mathcal{L}_1}\gamma_1 + \|H_2(s)/M(s)\|_{\mathcal{L}_1}\,\gamma_0.\end{aligned} \tag{34}$$

Proof: Let $\tilde{\sigma}(t) = \hat{\sigma}(t) - \sigma(t)$, where $\sigma(t)$ is defined in (5). It follows from (29) that

$$u(s) = C(s)r(s) - C(s)(\sigma(s) + \tilde{\sigma}(s)) \tag{35}$$

and the system in (4) consequently takes the form

$$y(s) = M(s)\Big(C(s)r(s) + (1 - C(s))\sigma(s) - C(s)\tilde{\sigma}(s)\Big). \tag{36}$$

Substituting (35) into (5), it follows from the definition of $H(s), H_0(s)$, and $H_1(s)$ in (10) and (19) that

$$\sigma(s) = H_1(s)(r(s) - \tilde{\sigma}(s)) + H_0(s)d(s). \tag{37}$$

Substituting (37) into (36), we have

$$\begin{aligned}y(s) = M(s)&(C(s) + H_1(s)(1 - C(s)))(r(s) - \tilde{\sigma}(s))\\ &+ H_0(s)M(s)(1 - C(s))d(s).\end{aligned} \tag{38}$$

It can be verified from (10) and (19) that $M(s)(C(s) + H_1(s)(1 - C(s))) = H(s)C(s)$, $H(s) = H_0(s)M(s)$, and hence, (38) can be rewritten as

$$y(s) = H(s)(C(s)r(s) - C(s)\tilde{\sigma}(s)) + H(s)(1 - C(s))d(s). \tag{39}$$

Let $e(t) = y(t) - y_{\rm ref}(t)$. From (16) and (39), one has $e(s) = H(s)((1 - C(s))d_e(s) - C(s)\tilde{\sigma}(s))$, where $d_e(s)$ is introduced to denote the Laplace transform of $d_e(t) = f(t, y(t)) - f(t, y_{\rm ref}(t))$. Lemma 1 and Assumption 1 give the following upper bound:

$$\|e_t\|_{\mathcal{L}_\infty} \leq L\|H(s)(1 - C(s))\|_{\mathcal{L}_1}\|e_t\|_{\mathcal{L}_\infty} + \|r_{1_t}\|_{\mathcal{L}_\infty} \tag{40}$$

where $r_1(t)$ is the signal with its Laplace transformation $r_1(s) = C(s)H(s)\tilde{\sigma}(s)$. It follows from (4) and (24) that

$$\tilde{y}(s) = M(s)\tilde{\sigma}(s). \tag{41}$$

Therefore $r_1(s) = (C(s)H(s)/M(s))M(s)\tilde{\sigma}(s) = (C(s)H(s)/M(s))\tilde{y}(s)$, and $\|r_{1_t}\|_{\mathcal{L}_\infty} \leq \|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\|\tilde{y}_t\|_{\mathcal{L}_\infty}$. From (40) we have $\|e_t\|_{\mathcal{L}_\infty} \leq L\|H(s)(1 - C(s))\|_{\mathcal{L}_1}\|e_t\|_{\mathcal{L}_\infty} + \|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\|\tilde{y}_t\|_{\mathcal{L}_\infty}$, and hence

$$\|e_t\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L}\|\tilde{y}_t\|_{\mathcal{L}_\infty}.$$

First, we prove the bound in (31) by contradiction. Since $\tilde{y}(0) = 0$ and $\tilde{y}(t)$ is continuous, then assuming the opposite implies that there exists t' such that

$$\|\tilde{y}(t)\| < \gamma_0, \quad \forall\, 0 \leq t < t' \tag{43}$$

$$\|\tilde{y}(t')\| = \gamma_0 \tag{44}$$

which leads to

$$\|\tilde{y}_{t'}\|_{\mathcal{L}_\infty} = \gamma_0. \tag{45}$$

Since $y(t) = y_{\rm ref}(t) + e(t)$, it follows from (18) and (45) that

$$\begin{aligned}\|y_{t'}\|_{\mathcal{L}_\infty} &\leq \|y_{ref_{t'}}\|_{\mathcal{L}_\infty} + \|e_{t'}\|_{\mathcal{L}_\infty} \leq \rho\\ &\quad + \|C(s)H(s)/M(s)\|_{\mathcal{L}_1}\,\gamma_0/(1 - \|G(s)\|_{\mathcal{L}_1} L).\end{aligned} \tag{46}$$

It follows from (37) and (41) that $\sigma(s) = H_1(s)r(s) - H_1(s)\tilde{y}(s)/M(s) + H_0(s)d(s)$, and hence, (45) implies that $\|\sigma_{t'}\|_{\mathcal{L}_\infty} \leq \|H_1(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|H_1(s)/M(s)\|_{\mathcal{L}_1}\gamma_0 + \|H_0(s)\|_{\mathcal{L}_1}(L\|y_{t'}\|_{\mathcal{L}_\infty} + L_0)$, which along with (46) leads to

$$\|\sigma_{t'}\|_{\mathcal{L}_\infty} \leq \Delta. \tag{47}$$

Consider the following candidate Lyapunov function:

$$V(\tilde{y}(t), \tilde{\sigma}(t)) = P\tilde{y}^2(t) + \Gamma_c^{-1}\tilde{\sigma}^2(t). \tag{48}$$

The adaptive law in (25) ensures that for all $0 \leq t \leq t'$

$$\dot{V}(t) \leq -Q\tilde{y}^2(t) + 2\Gamma_c^{-1}|\tilde{\sigma}(t)\dot{\sigma}(t)|. \tag{49}$$

It follows from (37) that

$$\sigma_d(s) = sH_1(s)(r(s) - \tilde{\sigma}(s)) + H_0(s)d_d(s) \tag{50}$$

where $\sigma_d(s)$ and $d_d(s)$ are the Laplace transformations of $\dot{\sigma}(t)$ and $\dot{d}(t)$, respectively. From (27) and (47), we have

$$\|\tilde{\sigma}_{t'}\|_{\mathcal{L}_\infty} \leq 2\Delta. \tag{51}$$

It follows from (46) that

$$\|d_{t'}\|_{\mathcal{L}_\infty} \leq L\rho + \frac{L\,\|C(s)H(s)/M(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L}\gamma_0 + L_0. \tag{52}$$

From the definitions of β_{01} and β_{02} in (22), (39), and (52), we have $\|\dot{y}_{t'}\|_{\mathcal{L}_\infty} \leq \beta_{01}\gamma_0 + \beta_{02}$. It follows from Assumption 2 that

$$\|\dot{d}_{t'}\|_{\mathcal{L}_\infty} \leq L_2\|y_{t'}\|_{\mathcal{L}_\infty} + L_1(\beta_{01}\gamma_0 + \beta_{02}) + L_3. \tag{53}$$

From (46), (50), and (53) and the definitions of β_1 and β_2 in (21), it follows that

$$\|\dot{\sigma}_{t'}\|_{\mathcal{L}_\infty} \leq (\beta_1\gamma_0 + \beta_2)/(4\Delta). \tag{54}$$

Therefore, from (49), (51), and (54), we have

$$\dot{V}(t) \leq -Q\tilde{y}^2(t) + \Gamma_c^{-1}\Big(\beta_1\gamma_0 + \beta_2\Big), \qquad \forall\, 0 \leq t \leq t'. \tag{55}$$

The projection algorithm ensures that $|\hat{\sigma}(t)| \leq \Delta$ for all $t \geq 0$, and therefore

$$\max_{t' \geq t \geq 0} \Gamma_c^{-1}\tilde{\sigma}^2(t) \leq 4\Delta^2/\Gamma_c. \tag{56}$$

Let $\theta_{\max} \triangleq \beta_3\gamma_0 + \beta_4$, where β_3 and β_4 are defined in (23). If at any $t \in [0, t']$, $V(t) > \theta_{\max}/\Gamma_c$, then it follows from (48) and (56) that $P\tilde{y}^2(t) > P(\beta_1\gamma_0 + \beta_2)/(\Gamma_c Q)$, and hence

$$Q\tilde{y}^2 = (Q/P)P\tilde{y}^2 > (\beta_1\gamma_0 + \beta_2)/\Gamma_c. \tag{57}$$

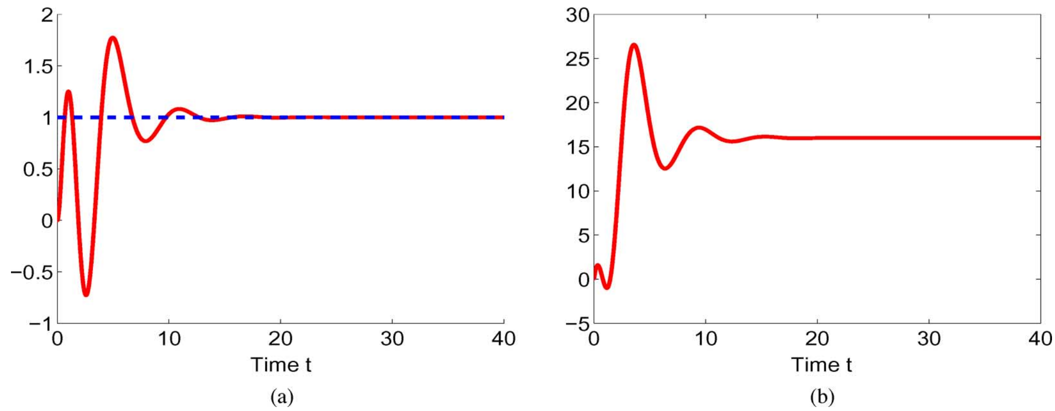


Fig. 2. Performance for $r(t) = 1$ and $d(t) = 0$. (a) $y(t)$ (solid) and $r(t)$ (dashed). (b) Time-history of $u(t)$.

From (55) and (57), it follows that if for some $t \in [0, t']$ $V(t) > \theta_{\max}/\Gamma_c$, then

$$\dot{V}(t) < 0. \tag{58}$$

Since $\tilde{y}(0) = 0$, we can verify that $V(0) \leq (\beta_3\gamma_0 + \beta_4)/\Gamma_c$. It follows from (58) that

$$V(t) \leq \theta_{\max}/\Gamma_c, \qquad 0 \leq t \leq t'. \tag{59}$$

Since $P|\tilde{y}(t)|^2 \leq V(t)$, then it follows from (59) that

$$|\tilde{y}(t)|^2 \leq (\beta_3\gamma_0 + \beta_4)/(\Gamma_c P), \qquad 0 \leq t \leq t'. \tag{60}$$

It follows from (45) and (60) that $\gamma_0^2 \leq (\beta_3\gamma_0 + \beta_4)/(\Gamma_c P)$, which along with (28) leads to $\alpha\beta_4 \leq \beta_3\gamma_0 + \beta_4$, and further implies

$$(\alpha - 1)^2\beta_4 \leq \alpha\beta_3^2/(\Gamma_c P). \tag{61}$$

Equation (61) limits the adaptive gain

$$\Gamma_c \leq \alpha\beta_3^2/((\alpha - 1)^2\beta_4 P) \tag{62}$$

which contradicts (26). Hence, (62) is not true which further implies that (44) does not hold. Therefore, (31) is true. It follows from (11), (31), and (42) that $\|e_t\|_{\mathcal{L}_\infty} \leq \|(C(s)H(s)/M(s)\|_{\mathcal{L}_1} / 1 - \|G(s)\|_{\mathcal{L}_1} L)\gamma_0$, which holds uniformly for all $t \geq 0$, and therefore, leads to (32).

It follows from (5) and (35) that

$$u(s) = \frac{M(s)(C(s)r(s) - C(s)\tilde{\sigma}(s)) - C(s)A(s)d(s)}{C(s)A(s) + (1 - C(s))M(s)}.$$

To prove the bound in (33), we notice that from (14) one can derive

$$\begin{aligned} u(s) - u_{\text{ref}}(s) &= -H_3(s)r_2(s) + H_2(s)\tilde{\sigma}(s) \\ &= -H_3(s)r_2(s) + (H_2(s)/M(s))M(s)\tilde{\sigma}(s) \end{aligned} \tag{63}$$

where $r_2(t) = f(t, y(t)) - f(t, y_{\text{ref}}(t))$. It follows from (41) and (63) that $\|u - u_{\text{ref}}\|_{\mathcal{L}_\infty} \leq L\|H_3(s)\|_{\mathcal{L}_1}\|y - y_{\text{ref}}\|_{\mathcal{L}_\infty} + \|H_2(s)/M(s)\|_{\mathcal{L}_1}\|\tilde{y}\|_{\mathcal{L}_\infty}$, which leads to (33). □

Thus, the tracking error between $y(t)$ and $y_{\text{ref}}(t)$, as well as between $u(t)$ and $u_{\text{ref}}(t)$, is uniformly bounded by a constant inverse proportional to Γ_c. This implies that during the transient, one can achieve arbitrarily close tracking performance for both signals simultaneously by increasing Γ_c.

We note that the control law $u_{\text{ref}}(t)$ in the closed-loop reference system, which is used in the analysis of $\mathcal{L}_\infty$ norm bounds, is not implementable since its definition involves the unknown parameters. Theorem 1 ensures that the $\mathcal{L}_1$ adaptive controller approximates $u_{\text{ref}}(t)$ both in transient and steady state. So, it is important to understand how these bounds can be used for ensuring uniform transient response with *desired* specifications. We notice that the following *ideal* control signal $u_{ideal}(t) = r(t) - \sigma(t)$ is the one that leads to desired system response

$$y_{ideal}(s) = M(s)r(s) \tag{64}$$

by cancelling the uncertainties exactly. Thus, the reference system in (6)–(8) has a different response as compared to (64). In [20], specific design guidelines are suggested for selection of $C(s)$ that lead to desired system response. Similar thinking can be applied in the case of this architecture as well.

VII. Discussion

In this section, we discuss the classes of systems that can satisfy (11) via the choice of $M(s)$ and $C(s)$. For simplicity, we consider the first-order $C(s)$ and $M(s)$ as pointed in (3) and (9). It follows from (3) and (9) that $H(s) = m(s + \omega)A_n(s)/(\omega(s + m)A_n(s) + msA_d(s))$. Stability of $H(s)$ is equivalent to stabilization of $A(s)$ by a PI controller, say, of the following structure $(\omega/m)((s + m)/s)$, where m and ω are the same as in (3) and (9). The open loop transfer function of the cascaded $A(s)$ with the PI controller will be $H_{PI}(s) = (\omega/m)((s + m)/s)A(s)$, leading to the following closed-loop system:

$$\omega(s + m)A_n(s)/(\omega(s + m)A_n(s) + msA_d(s)). \tag{65}$$

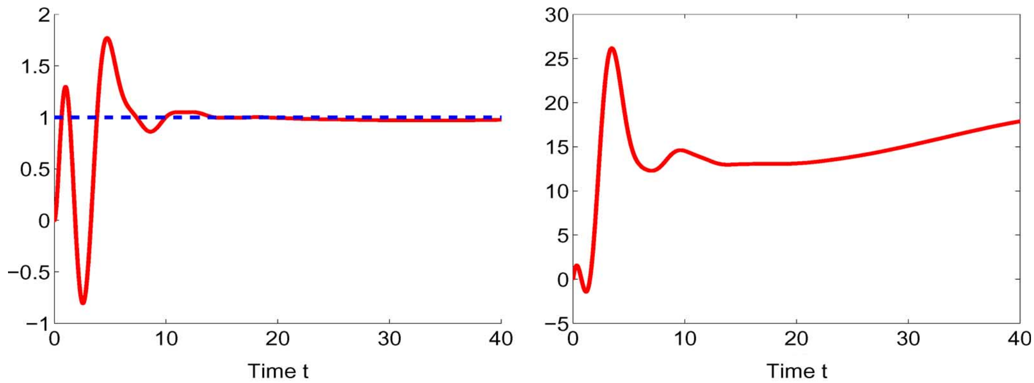
Hence, the stability of $H(s)$ is equivalent to that of (65), and the problem can be reduced to identifying the class of $A(s)$ that can be stabilized by a PI controller. It also permits the use of root locus methods for checking the stability of $H(s)$ via the open loop transfer function $H_{\text{PI}}(s)$. We note that the PI controller adds an open loop pole at the origin and an open loop zero at $-m$, while ω/m plays the role of the open-loop gain.

A. Minimum Phase Systems With Relative Degree 1 or 2

Consider a minimum phase system $H(s)$ with relative degree 1 or 2. Notice that the zeros of H_{PI} are located in the open left-half plane. By appropriate choice of the open-loop zero $-m$ and open-loop gain ω/m, it follows from the classical control theory that the closed-loop poles can be moved into the left-half plane. Hence, the transfer function in (65) is BIBO stable, and such is $H(s)$. We notice that the aforesaid discussions hold for any $A(s)$ with relative degree 1 or 2.

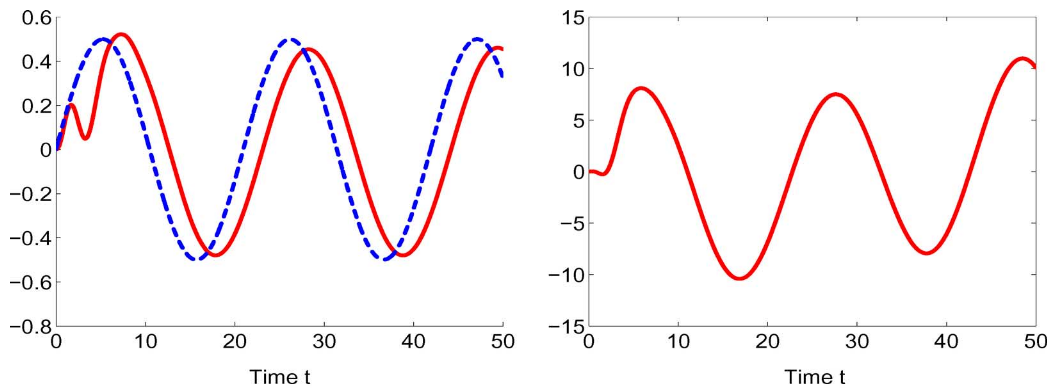
B. Other Systems

We note that nonminimum phase systems can also be stabilized by a PI controller. However, the choice of m and ω is not straightforward. In



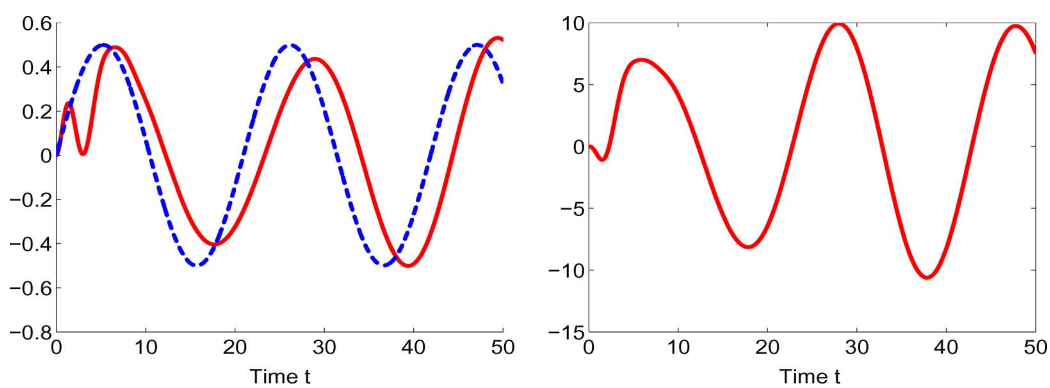
(a) (b)

Fig. 3. Performance for $r(t) = 1$ and $d(t) = \sin(0.1t)y(t) + 2\sin(0.1t)$. (a) $y(t)$ (solid) and $r(t)$ (dashed). (b) Time-history of $u(t)$.



(a) (b)

Fig. 4. Performance for $r(t) = 0.5\sin(0.3t)$ and $d(t) = \sin(0.1t)y(t) + 2\sin(0.1t)$. (a) $y(t)$ (solid) and $r(t)$ (dashed). (b) Time-history of $u(t)$.



(a) (b)

Fig. 5. Performance for $r(t) = 0.5\sin(0.3t)$ and $d(t) = \sin(0.1t)y(t) + 2\sin(0.4t)$. (a) $y(t)$ (solid) and $r(t)$ (dashed). (b) Time-history of $u(t)$.

the simulation example presented next, we demonstrate application of the $\mathcal{L}_1$ adaptive controller to an unknown nonminimum phase system in the presence of unknown nonlinear disturbances.

Remark 2: Finally, we notice that, in the light of the aforesaid discussion, a PI controller stabilizing $A(s)$, might also stabilize the system in the presence of the nonlinear disturbance $f(t, y(t))$. However, the transient performance cannot be quantified in the presence of unknown $A(s)$. The $\mathcal{L}_1$ adaptive controller will generate different low-pass control signals $u(t)$ for different unknown systems to ensure uniform transient performance for $y(t)$.

VIII. Simulation

As an illustrative example, consider the system in (1) with $A(s) = (s^2 - 0.5s + 0.5)/(s^3 - s^2 - 2s + 8)$. We note that $A(s)$ has both poles and zeros in the right-half plane, and hence, it is an unstable nonminimum phase system. We consider $\mathcal{L}_1$ adaptive controller defined via (24), (25), and (29), where $m = 3, \omega = 10, \Gamma_c = 500$. We set $\Delta = 100$. First, we consider the step response by assuming $d(t) = 0$. The simulation results of $\mathcal{L}_1$ adaptive controller are shown in Fig. 2(a) and (b). Next, we consider $d(t) = f(t, y(t)) =$

$\sin(0.1t)y(t) + 2\sin(0.1t)$, and apply the same controller without retuning. The control signal and the system response are plotted in Fig. 3(a) and (b). Further, we consider a time-varying reference input $r(t) = 0.5\sin(0.3t)$ and notice that, without any retuning of the controller, the system response and the control signal behave as expected [see Fig. 4(a) and (b)]. Fig. 5(a) and (b) plot the system response and the control signal for a different uncertainty $d(t) = f(t, y(t)) = \sin(0.1t)y(t) + 2\sin(0.4t)$ without any retuning of the controller.

We notice that in the case of minimum-phase systems, theoretically we can increase the bandwidth of $C(s)$ arbitrarily and cancel time-varying disturbances of arbitrary frequency. However, the bandwidth of $C(s)$ cannot be set arbitrarily large due to the bandwidth limitations in the control channels of the system. Also, a larger bandwidth of $C(s)$ can reduce the time-delay margin of the closed-loop system and imply that a higher adaptive gain is needed [19], [20].

IX. Conclusion

A novel $\mathcal{L}_1$ adaptive output feedback control architecture is presented in this note for systems of unknown dimension. It has guaranteed transient response for the system's both signals, input and output, simultaneously, in addition to stable tracking. The methodology has been used to augment a commercial autopilot for an unmanned aerial vehicle (UAV) to achieve an accurate path following for aggressive trajectories that the autopilot was not otherwise designed to follow [21]. It has been further used to support time-critical cooperation of UAVs under strict spatial constraints [22], [23].

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A Dissipation Inequality for the Minimum Phase Property

Christian Ebenbauer and Frank Allgöwer

***Abstract*—The minimum phase property is an important notion in systems and control theory. In this paper, a characterization of the minimum phase property of nonlinear control systems in terms of a dissipation inequality is derived. It is shown that this dissipation inequality is equivalent to the classical definition of the minimum phase property in the sense of Byrnes and Isidori, if the control system is affine in the input and the so-called input–output normal form exists.**

***Index Terms*—Dissipation inequality, minimum phase property, nonlinear systems.**

I. Introduction

Bode introduced the notion of minimum phase property in his seminal paper [3] more than 60 years ago. Today, the minimum phase property plays an important role in systems analysis and control design [9]–[11], [24]. For example, the notion of the minimum phase property can be used to describe fundamental performance limitations

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