

REALIZABILITY AND THE AVRUNIN-SCOTT THEOREM FOR HIGHER-ORDER SUPPORT VARIETIES

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ABSTRACT. We introduce higher-order support varieties for pairs of modules over a commutative local complete intersection ring, and give a complete description of which varieties occur as such support varieties. In the context of a group algebra of a finite elementary abelian group, we also prove a higher-order Avrunin-Scott-type theorem, linking higher-order support varieties and higher-order rank varieties for pairs of modules.

1. INTRODUCTION

Support varieties for modules over commutative local complete intersections were introduced in [A] and [AB], inspired by the cohomological varieties of modules over group algebras of finite groups. These geometric invariants encode several homological properties of the modules. For example, the dimension of the variety of a module equals its complexity. In particular, a module has finite projective dimension if and only if its support variety is trivial.

In this paper, we define higher-order support varieties for pairs of modules over complete intersections. These varieties are defined in terms of Grassmann varieties of subspaces of the canonical vector space associated to the defining regular sequence of the complete intersection. Thus, for a fixed dimension d , the support varieties of order d are subsets of the Grassmann variety of d -dimensional subspaces of the canonical vector space, under a Plücker embedding into $\mathbb{P}^{\binom{c}{d}-1}$. For $d = 1$, we recover the classical support varieties: the varieties of order 1 are precisely the projectivizations of the support varieties defined in [AB].

We show that several properties which hold for classical support varieties also hold for the higher-order varieties. We also give a complete description of the closed subsets of the Grassmann variety that occur as higher-order support varieties. These results follow directly from perhaps the surprising fact that the higher order support varieties are completely determined by the classical (first-order) support varieties in a very explicit way (Proposition 2.3 below.) We also prove a higher-order Avrunin-Scott result for group algebras of finite elementary abelian groups. Namely, we extend the notion of r -rank varieties from [CFP] to higher-order rank varieties of pairs of modules and show that these varieties are isomorphic to the higher-order support varieties.

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In Section 2 we give our definition of higher-order support varieties, and prove some of their elementary properties. In particular, we show that they are well-defined, independent of the choice of corresponding intermediate complete intersection, and are in fact closed subsets of the Grassmann variety. In Section 3 we discuss the realizability question of which varieties can occur as higher-order support varieties of pairs of modules. In Section 4 we prove the higher-order Avrunin-Scott result.

2. HIGHER-ORDER SUPPORT VARIETIES

In this section and the next, we fix a regular local ring (Q, \mathfrak{n}, k) and an ideal I generated by a regular sequence of length c contained in \mathfrak{n}^2 . We denote by R the complete intersection ring

$$R = Q/I,$$

and by V the k -vector space

$$V = I/\mathfrak{n}I.$$

We assume throughout that k is an algebraically closed field. For an element $f \in I$, we let \bar{f} denote its image in V .

If the codimension of the complete intersection $R = Q/I$ is at least 2, then V has dimension at least 2, and it makes sense to consider subspaces W of V . Each such subspace has many corresponding complete intersections, in the following sense: if W is a subspace of V , then choosing preimages in I of a basis of W we obtain another regular sequence [BrH, Theorem 2.1.2(c,d)], and the ideal $J \subseteq I$ it generates. We thus get natural projections of complete intersections $Q \rightarrow Q/J \rightarrow R$. We call Q/J a *complete intersection intermediate to Q and R (corresponding to W)*, or when the context is clear, simply an *intermediate complete intersection*.

We now give our definition of higher-order support variety. We fix a basis of V , and let $G_d(V)$ denote the Grassmann variety of d -dimensional subspaces of V under the Plücker embedding into $\mathbb{P}^{\binom{c}{d}-1}$ with respect to the chosen basis of V . For any subset S of V closed under scalar multiplication, we let $\mathbb{P}(S)$ denote the projectivization of S with respect to the chosen basis of V . For $x \in S$ we denote by $[x]$ its equivalence class in $\mathbb{P}(S)$.

Definition 2.1. Let M and N be finitely generated R -modules. We set

$$V_R^d(M, N) = \{p_W \in G_d(V) \mid \text{Ext}_{Q/J}^i(M, N) \neq 0 \text{ for infinitely many } i\},$$

where W is a d -dimensional subspace of V , p_W is the corresponding point in the Grassmann variety $G_d(V)$, and Q/J is an intermediate complete intersection corresponding to W . We also define $V_R^d(M) = V_R^d(M, k)$.

Remark. From [AB, Theorem 2.5] we note that $V_R^1(M, N) = \mathbb{P}(V_R(M, N))$ is the projectivization of the original affine support variety $V_R(M, N)$ defined in [AB].

There are two aspects of the definition which warrant further discussion.

- (1) The definition is independent of the chosen intermediate complete intersection Q/J corresponding to W , and
- (2) $V_R^d(M, N)$ is a closed set in $G_d(V)$.

We next give proofs of these two statements.

Let Q/J and Q/J' be two complete intersections intermediate to Q and R . The condition that

$$(J + \mathfrak{n}I)/\mathfrak{n}I = (J' + \mathfrak{n}I)/\mathfrak{n}I$$

in V defines an equivalence relation on the set of such intermediate complete intersections (see [BeJ]). The following result addresses (1) above.

Proposition 2.2. *Suppose that Q/J and Q/J' are equivalent complete intersections intermediate to Q and R , that is, $(J + \mathfrak{n}I)/\mathfrak{n}I = (J' + \mathfrak{n}I)/\mathfrak{n}I$ in V . Then for all finitely generated R -modules M and N one has $\text{Ext}_{Q/J}^i(M, N) = 0$ for all $i \gg 0$ if and only if $\text{Ext}_{Q/J'}^i(M, N) = 0$ for all $i \gg 0$.*

Proof. Let $W = (J + \mathfrak{n}I)/\mathfrak{n}I$ and consider the natural map of k -vector spaces $\varphi_J : J/\mathfrak{n}J \rightarrow W \subseteq V$ defined by $f + \mathfrak{n}J \mapsto f + \mathfrak{n}I$. This is an isomorphism: it is onto by construction, and one-to-one since $J \cap \mathfrak{n}I = \mathfrak{n}J$. The condition that $(J + \mathfrak{n}I)/\mathfrak{n}I = (J' + \mathfrak{n}I)/\mathfrak{n}I$ is equivalent to $\varphi_J(J/\mathfrak{n}J) = \varphi_{J'}(J'/\mathfrak{n}J')$. By [BeJ, Proposition 3.2], one has the equality $\varphi_J(V_{Q/J}(M, N)) = \varphi_{J'}(V_{Q/J'}(M, N))$, where $V_{Q/J}(M, N)$ denotes the affine support variety of M and N over the complete intersection Q/J . By [AB, Proposition 2.4(1) and Theorem 2.5] one has that $\text{Ext}_{Q/J}^i(M, N) = 0$ for all $i \gg 0$ if and only if $V_{Q/J}(M, N) = \{0\}$. The same holds over Q/J' , and thus the result follows by the injectivity of φ_J . \square

Next, we address the second point in the remark. The following result is a vast generalization of [CFP, Proposition 1.4], and employs the Dade's Lemma-type result of Avramov and Buchweitz, [AB, Proposition 2.4(1)] together with [AB, Theorem 2.5].

Proposition 2.3. *For all finitely generated R -modules M and N one has that $V_R^d(M, N)$ is completely determined by $V_R^1(M, N)$. That is,*

$$p_W \in V_R^d(M, N) \text{ if and only if } \mathbb{P}(W) \cap V_R^1(M, N) \neq \emptyset$$

Proof. We have $x \in \mathbb{P}(W) \cap V_R^1(M, N)$ if and only if $\text{Ext}_{Q/(f)}^i(M, N) \neq 0$ for infinitely many i and for some $f \in I$ with $[\bar{f}] = x \in \mathbb{P}(W)$. By [AB, Proposition 2.4(1) and Theorem 2.5], this condition is the same as $\text{Ext}_{Q/J}^i(M, N) \neq 0$ for infinitely many i , where Q/J is an intermediate complete intersection corresponding to W . By definition, this happens if and only if $p_W \in V_R^d(M, N)$. \square

Corollary 2.4. *For all finitely generated R -modules M and N one has that $V_R^d(M, N)$ is a closed set in $G_d(V)$.*

Proof. This result follows from an incidence correspondence (see, for example, [Ha, Example 6.14]), as we now describe. Set

$$\Gamma = \{(p_W, x) \in G_d(V) \times G_1(V) \mid x \in \mathbb{P}(W) \cap V_R^1(M, N)\}.$$

Since Γ is an incidence correspondence, it is a closed subset of the product space $G_d(V) \times G_1(V)$. We have the two natural projections

$$\begin{array}{ccc} & G_d(V) \times G_1(V) & \\ \pi \swarrow & & \searrow \pi' \\ G_d(V) & & G_1(V) \end{array}$$

Now by classical results from elimination theory (see, for example, [E, Theorem 14.1]), the image of Γ under π is closed in $G_d(V)$. It suffices now to know that $\pi(\Gamma) = V_R^d(M, N)$, but this is just a restatement of Proposition 2.3. \square

Remarks. (1) Let $\mathcal{T} = \{(p_W, x) \in G_d(V) \times G_1(V) \mid x \in \mathbb{P}(W)\}$. Then the map $\tau : \mathcal{T} \rightarrow G_d(V)$ given by $\tau(p_W, x) = p_W$ is the tautological bundle over the Grassmann variety $G_d(V)$. For Γ as in the proof of Proposition 2.4, we have $\Gamma \subseteq \mathcal{T}$, and $\tau(\Gamma) = V_R^d(M, N)$. Thus $V_R^d(M, N)$ may be interpreted as the image under the tautological bundle of the fiber of $V_R^1(M, N)$ in \mathcal{T} .

(2) One could also have defined $G_d(V)$ as the set of d -dimensional subspaces of V , and $V_R^d(M, N)$ as those d -dimensional subspaces for which a corresponding intermediate complete intersection Q/J satisfies $\text{Ext}_{Q/J}^i(M, N) \neq 0$ for infinitely many i . One could then reformulate $V_R^d(M, N)$ as $p_d(p_1^{-1}(V_R^1(M, N)))$, where p_1 and p_d are the canonical forgetting morphisms from the flag variety $G_{1,d}(V)$ to $G_1(V)$ and $G_d(V)$, respectively.

(3) In the definition of $V_R^d(M, N)$, a specific basis of V was chosen. We remark that the definition is independent of the choice of basis, in the sense that if another basis of V is chosen, then the two higher-order support varieties are isomorphic. Indeed, this is true for the first order affine varieties $V_R(M, N)$ by [AB, Remark 2.3]. It then follows that the same is true for the projectivizations $V_R^1(M, N)$, namely, there is an automorphism $\xi : G_1(V) \rightarrow G_1(V)$ such that if $V_R^1(M, N)$ is the support variety with respect to the first basis, and $V_R^1(M, N)'$ is the support variety with respect to the second, then $\xi(V_R^1(M, N)) = V_R^1(M, N)'$. The general result for the higher-order support varieties follows from the incidence correspondence from the proof above.

We now give basic properties of higher-order support varieties, akin to those of the one-dimensional affine support varieties.

Theorem 2.5. *The following hold for finitely generated R -modules M and N .*

- (1) $V_R^d(k) = G_d(V)$.
- (2) $V_R^d(M, N) = V_R^d(N, M)$. For $d = 1$, we moreover have $V_R^1(M, N) = V_R^1(M) \cap V_R^1(N)$.
- (3) $V_R^d(M, M) = V_R^d(k, M) = V_R^d(M)$.
- (4) If M' is a syzygy of M and N' is a syzygy of N , then $V_R^d(M, N) = V_R^d(M', N')$.
- (5) If $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ and $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$ are short exact sequences of finitely generated R -modules, then for $\{h, i, j\} = \{1, 2, 3\}$ there are inclusions

$$V_R^d(M_h, N) \subseteq V_R^d(M_i, N) \cup V_R^d(M_j, N);$$

$$V_R^d(M, N_h) \subseteq V_R^d(M, N_i) \cup V_R^d(M, N_j).$$

- (6) If M is Cohen-Macaulay of codimension m , then

$$V_R^d(M) = V_R^d(\text{Ext}_R^m(M, R)).$$

In particular, if M is a maximal Cohen-Macaulay R -module, then $V_R^d(M) = V_R^d(\text{Hom}_R(M, R))$.

- (7) If x_1, \dots, x_d is an M -regular sequence, then

$$V_R^d(M) = V_R^d(M/(x_1, \dots, x_d)M).$$

Proof. The proof of properties (1)–(7) for the affine one-dimensional support varieties $V_R(M, N)$ are given in [AB] (see also [BeJ].) Since $V_R^1(M, N)$ is simply the projectivization of $V_R(M, N)$, the same properties also hold for these varieties. Finally, properties (1)–(7) for $d > 1$ follow from the $d = 1$ case, as we now indicate.

For a subset X of $G_1(V)$, we let

$$\Gamma(X) = \{(p_W, x) \in G_d(V) \times G_1(V) \mid x \in \mathbb{P}(W) \cap X\}.$$

The proofs make repeated use of the fact that $V_R^d(M, N) = \pi(\Gamma(V_R^1(M, N)))$, where π is as in the proof of Proposition 2.4. For example, for (1) we have $V_R^d(k) = \pi(\Gamma(V_R^1(k))) = \pi(G_1(V)) = G_d(V)$.

For (2), we use the fact that $V_R^1(M, N) = V_R^1(M) \cap V_R^1(N) = V_R^1(N, M)$. Therefore $V_R^d(M, N) = \pi(\Gamma(V_R^1(M, N))) = \pi(\Gamma(V_R^1(N, M))) = V_R^d(N, M)$.

To prove (3), we use the equalities $V_R^d(M, M) = \pi(\Gamma(V_R^1(M, M))) = \pi(\Gamma(V_R^1(k, M))) = V_R^d(k, M)$. The remaining equality and (4) are proved similarly.

To prove (5), we use the fact that for subsets X and Y of $G_1(V)$ one has $\Gamma(X \cup Y) = \Gamma(X) \cup \Gamma(Y)$. (We also use the fact that π preserves unions, and both π and Γ preserve containment.) Therefore

$$\begin{aligned} V_R^d(M_h, N) &= \pi(\Gamma(V_R^1(M_h, N))) \\ &\subseteq \pi(\Gamma(V_R^1(M_i, N) \cup V_R^1(M_j, N))) \\ &= \pi(\Gamma(V_R^1(M_i, N))) \cup \pi(\Gamma(V_R^1(M_j, N))) \\ &= V_R^d(M_i, N) \cup V_R^d(M_j, N). \end{aligned}$$

The proofs of (6) and (7) are analogous to the proofs of [AB, Theorem 5.6(10)] and [AI, 7.4] (see also [BeJ, Theorem 2.2(7) and (8)].) \square

We can extend Proposition 2.4(1) of [AB], to a sort of generalized Dade’s Lemma, in the projective context.

Proposition 2.6. *Fix $1 \leq d \leq c$. Then $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if $V_R^d(M, N) = \emptyset$.*

Proof. By [AB, Proposition 2.4(1) and Theorem 2.5], $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if $V_R^1(M, N) = \emptyset$. The latter holds if and only if $\Gamma = \Gamma(V_R^1(M, N)) = \emptyset$, which in turn holds if and only if $V_R^d(M, N) = \pi(\Gamma) = \emptyset$, where Γ and π are from the proof of Proposition 2.4. \square

3. REALIZABILITY

In this section we give a complete description of which closed subsets of $G_d(V)$ can possibly occur as the d th order support variety $V_R^d(M, N)$ of a pair of finitely generated R -modules (M, N) . The basis of the description is the following result in the first order case.

Theorem 3.1. *Every closed subset of $G_1(V)$ is the support variety of some finitely generated R -module. Specifically, if Z is a closed subset of $G_1(V)$, then there exists a finitely generated R -module M such that $Z = V_R^1(M, k)$.*

Proof. This is well-known in the affine case, see, for example, [Be]. Since every closed set in $G_1(V)$ is the projectivization of a cone in V , and $V_R^1(M, N)$ is the projectivization of $V_R(M, N)$, the result follows. \square

The framework of the proof of Proposition 2.4 allows us to complete the description of realizable higher-order varieties. Recall that π denotes the projection map $G_d(V) \times G_1(V) \rightarrow G_d(V)$.

Theorem 3.2. *For a subset Z of $G_1(V)$, set*

$$\Gamma(Z) = \{(p_W, x) \in G_d(V) \times G_1(V) \mid x \in \mathbb{P}(W) \cap Z\}.$$

Let Y be a closed subset of $G_d(V)$. Then $Y = V_R^d(M, N)$ for a pair of finitely generated R -modules (M, N) if and only if $Y = \pi(\Gamma(Z))$ for some closed subset Z of $G_1(V)$.

Proof. Suppose that $Y = V_R^d(M, N)$ for a pair of finitely generated R -modules (M, N) . Then the proof of Proposition 2.4 shows that $Y = \pi(\Gamma(V_R^1(M, N)))$.

Conversely, suppose that $Y = \pi(\Gamma(Z))$ for some closed subset Z of $G_1(V)$. Then Theorem 3.1 shows that $Z = V_R^1(M, N)$ for some pair of finitely generated R -modules (M, N) . Thus $Y = \pi(\Gamma(V_R^1(M, N))) = V_R^d(M, N)$, again from the proof of Proposition 2.4. \square

Theorem 3.2 shows that, in contrast to first order support varieties, the realizability of varieties in $G_d(V)$ for $d > 1$ as d th order support varieties of a pair of finitely generated R -modules is more restrictive. Indeed, consider a smallest non-trivial first order support variety $V_R^1(M, N)$, namely, one consisting of a single point x . Then $V_R^d(M, N)$ consists of all d -dimensional planes in V containing x . Changing the basis of V if necessary, we can assume that $x = (1, 0, \dots, 0) \in G_1(V)$. Then there is an obvious bijective correspondence between d -dimensional subspaces of V containing x , and $(d-1)$ -dimensional subspaces of a $(c-1)$ -dimensional k -vector space. Thus $\dim V_R^d(M, N) = \dim G_{d-1}(k^{c-1}) = (d-1)(c-d)$. In particular, we have $\dim V_R^{c-1}(M, N) = c-2$, which is of codimension one in $G_{c-1}(V)$, and this is when $V_R^1(M, N)$ is nontrivially as small as possible.

The following example illustrates the previous discussion.

Example. Let k be a field (of arbitrary characteristic), and $Q = k[[x_1, \dots, x_c]]$. Then Q is a regular local ring with maximal ideal $\mathfrak{n} = (x_1, \dots, x_c)$. For $I = (x_1^2, \dots, x_c^2)$, the quotient ring $R = Q/I$ is a codimension c complete intersection. Let $M = R/(x_1)$. Then it is not hard to show that relative to the basis $\overline{x_1^2}, \dots, \overline{x_c^2}$ of $V = I/\mathfrak{n}I$, the order 1 support variety of M is $V_R^1(M, k) = \{(1, 0, \dots, 0)\}$. Thus we have $\dim V_R^d(M, k) = (d-1)(c-d)$, for $1 \leq d \leq c-1$.

4. HIGHER-ORDER RANK VARIETIES AND A HIGHER-ORDER AVRUNIN-SCOTT THEOREM

In this final section we consider complete intersections of a special form, namely, those which arise as the group algebra kE of a finite elementary abelian p -group E , where k has characteristic p , and is algebraically closed. In this case one has

$$kE \cong k[x_1, \dots, x_c]/(x_1^p, \dots, x_c^p).$$

Therefore for the remainder of this section we assume that kE is the truncated polynomial ring $k[x_1, \dots, x_c]/(x_1^p, \dots, x_c^p)$. Note that by assigning $\deg x_i = 1$ for $1 \leq i \leq c$, the k -algebras Q and kE are standard-graded and commutative rings. We let Q_1 and kE_1 denote their degree one components, respectively. These are both k -vector spaces of dimension c . For any element u of kE_1 one has $u^p = 0$, and

thus the subalgebra $k[u]$ of kE generated by u is isomorphic to $k[x]/(x^p)$ (for x an indeterminate). Since $k[u]$ is a principal ideal ring, every finitely generated $k[u]$ -module is a direct sum of a free module and a torsion module. Recall from [C] that the *rank variety* $W_E(M)$ of a kE -module M is the set of those $u \in kE_1$ such that the torsion part of M as a $k[u]$ -module is nonzero. It was conjectured by Carlson [C] and subsequently proven by Avrunin and Scott [AvS] that the rank variety and the group cohomological support variety $V_{kE}(M)$ of a kE -module agree.

Recall that I denotes the ideal (x_1^p, \dots, x_c^p) , and V the k -vector space $I/\mathfrak{n}I$, where \mathfrak{n} is the maximal ideal (x_1, \dots, x_c) . We now want to show that the classical Avrunin-Scott theorem mentioned above is a special case of a more general result involving the higher-order varieties. We generalize the definition of d th order rank varieties from [CFP] (which they call d -rank varieties) to d th order rank varieties $W_E^d(M, N)$ of pairs of modules (M, N) . Fix a basis of kE_1 , and consider the Grassmann variety $G_d(kE_1)$ of d -dimensional subspaces of kE_1 under the Plücker embedding into $\mathbb{P}^{\binom{a}{d}-1}$ with respect to the chosen basis.

Definition 4.1. We set

$$W_E^d(M, N) = \{p_W \in G_d(kE_1) \mid \text{Ext}_{k[W]}^i(M, N) \neq 0 \text{ for infinitely many } i\}$$

where $G_d(kE_1)$ is the Grassmann variety of d -dimensional subspaces of kE_1 , p_W is the point in $G_d(kE_1)$ corresponding to the d -dimensional subspace W , and $k[W]$ is the subalgebra of kE generated by W .

Consider the Frobenius map $\Phi : k \rightarrow k$ given by $\Phi(a) = a^p$. Since k is algebraically closed, it contains p th roots, and so this map is an automorphism. We have a Φ -semilinear isomorphism of k -vector spaces

$$\alpha : kE_1 \rightarrow V$$

defined as follows. For $u \in kE_1$, we choose a preimage \tilde{u} in Q , and then we set $\alpha(u) = \tilde{u}^p + \mathfrak{n}I \in V$. It is clear that α is a Φ -semilinear homomorphism of k -vector spaces, which is defined independent of the choice of preimage. It is easy to see that α is onto. Since $\dim kE_1 = \dim V$, α is also one-to-one.

Taking as a basis for V the image under α of the chosen basis of kE_1 , we obtain an induced Φ -semilinear isomorphism of Grassmann varieties

$$\beta : G_d(kE_1) \rightarrow G_d(V)$$

with respect to these bases. Specifically, let p_W be a point in $G_d(kE_1)$, and W the associated d -dimensional subspace of kE_1 . Let \widetilde{W}^p denote the ideal of Q generated by the p th powers of preimages in Q_1 of a basis of W . Then $\beta(p_W)$ is the point in $G_d(V)$ (with respect to the chosen basis of V) corresponding to the subspace $\widetilde{W}^p + \mathfrak{n}I/\mathfrak{n}I$.

Theorem 4.2. *Given finitely generated kE -modules M and N , one has*

$$\beta(W_E^d(M, N)) = V_{kE}^d(M, N).$$

The proof relies on the following lemma, which is a statement extracted from the proof of [A, Theorem (7.5)]. For completeness we include the proof here.

For any non-zero $u \in kE_1$ we choose a preimage \tilde{u} in Q_1 , and define a homomorphism from $\mu : k[u] \rightarrow Q/(\tilde{u}^p)$ by sending u to $\tilde{u} + (\tilde{u}^p)$. Note that $Q/(\tilde{u}^p)$ is

free when regarded as module over $k[u]$ via μ . We have the commutative diagram of ring homomorphisms

$$\begin{array}{ccc} & & Q/(\tilde{u}^p) \\ & \nearrow \mu & \downarrow \\ k[u] & \hookrightarrow & kE \end{array}$$

where the vertical map is the natural projection. In particular, the action of $k[u]$ on a kE -module M factors through μ .

Lemma 4.3. *Let M be a finitely generated kE -module. Then M has finite projective dimension over $k[u]$ if and only if it has finite projective dimension over $Q/(\tilde{u}^p)$.*

Proof. The proof follows part of that of [A, Theorem (7.5)]. Suppose that M has finite projective dimension over $Q/(\tilde{u}^p)$. Since $Q/(\tilde{u}^p)$ is free over $k[u]$ any free resolution of M over $Q/(\tilde{u}^p)$ is also one of M over $k[u]$. Thus M has a finite free resolution over $k[u]$.

Conversely, suppose M is free as a $k[u]$ -module. Let F be a minimal free resolution of M over $Q/(\tilde{u}^p)$. Since F is also a free resolution of M over $k[u]$ and $\mathrm{Tor}_i^{k[u]}(M, k) = 0$ for all $i > 0$, we see that $F \otimes_{k[u]} k$ is a minimal free resolution of $M \otimes_{k[u]} k$ over $Q/(\tilde{u}^p) \otimes_{k[u]} k \cong Q/(\tilde{u})$. Since $Q/(\tilde{u})$ is regular and $F \otimes_{k[u]} k$ is a minimal, we must have that $F_c \otimes_{k[u]} k = 0$, and this implies $F_c = 0$. Thus F is a finite free resolution, and so M has finite projective dimension over $Q/(\tilde{u})$. \square

We now give a proof of Theorem 4.2.

Proof. Suppose that $p_W \in W_E^d(M, N)$. Then by definition there exist infinitely many nonzero $\mathrm{Ext}_{k[W]}^i(M, N)$. Therefore, by Dade's Lemma, there exist infinitely many nonzero $\mathrm{Ext}_{k[u]}^i(M, N)$ for some element $u \in W$. Thus both M and N have infinite projective dimension over $k[u]$. Therefore, by the lemma, both M and N have infinite projective dimension over $Q/(\tilde{u}^p)$, and so it follows from [AB, Proposition 5.12] that there exist infinitely many nonzero $\mathrm{Ext}_{Q/(\tilde{u}^p)}^i(M, N)$. This implies that there exist infinitely many nonzero $\mathrm{Ext}_{Q/(\tilde{W}^p)}^i(M, N)$, where \tilde{W}^p represents the ideal generated by the p th powers of preimages in Q_1 of a basis of W . This gives $\beta(p_W) \in V_{kE}^d(M, N)$.

For the reverse containment we just retrace our steps, noting that any $f \in I$ is equivalent mod $\mathfrak{n}I$ to an element of the form $a_1 x_1^p + \cdots + a_c x_c^p = (\sqrt[p]{a_1} x_1 + \cdots + \sqrt[p]{a_c} x_c)^p$, $a_i \in k$, and hence it is clear how to employ the previous lemma. \square

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