# $L^{1}$ SEMIGROUP GENERATION FOR FOKKER-PLANCK OPERATORS ASSOCIATED WITH GENERAL LÉVY DRIVEN SDES 

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#### Abstract

We prove a new generation result in $L^{1}$ for a large class of nonlocal operators with non-degenerate local terms. This class contains the operators appearing in Fokker-Planck or Kolmogorov forward equations associated with Lévy driven SDEs, i.e. the adjoint operators of the infinitesimal generators of these SDEs. As a byproduct, we also obtain a new elliptic regularity result of independent interest. The main novelty in this paper is that we can consider very general Lévy operators, including state-space depending coefficients with linear growth and general Lévy measures which can be singular and have fat tails.


## 1. Introduction

In this paper we prove an $L^{1}$ generation result for Fokker-Planck (FP) or Kolmogorov forward operators associated to autonomous Lévy driven SDEs. In their most general form such SDEs can be written as (cf. [27, 3, 21, 36])

$$
\begin{align*}
d Y_{t}= & b\left(Y_{t-}\right) d t+\sigma\left(Y_{t-}\right) d B_{t}  \tag{1.1}\\
& +\int_{|z|<1} p\left(Y_{t-}, z\right) \tilde{N}(d z, d t)+\int_{|z| \geq 1} p\left(Y_{t-}, z\right) N(d z, d t),
\end{align*}
$$

where $b: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \sigma: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times n}, p: \mathbb{R}^{d} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times m}, B_{t}$ is a $n$-dimensional Brownian motion, and $N$ and $\tilde{N}$ are $m$-dimensional Poisson and compensated Poisson random measures, respectively. Under suitable assumptions (cf. [39]), the solution $Y_{t}$ of (1.1) is a Markov process with infinitesimal generator $L^{*}$,

$$
\begin{align*}
L^{*} f(y)= & \sum_{i=1}^{d} b_{i}(y) \partial_{i} f(y)+\frac{1}{2} \sum_{i, j=1}^{d} a_{i j}(y) \partial_{i} \partial_{j} f(y)  \tag{1.2}\\
& +\sum_{k=1}^{m} \int_{|z|<1}\left[f\left(y+p_{k}(y, z)\right)-f(y)-D f(y) p_{k}(y, z)\right] \nu_{k}(d z) \\
& +\sum_{k=1}^{m} \int_{|z| \geq 1}\left[f\left(y+p_{k}(y, z)\right)-f(y)\right] \nu_{k}(d z)
\end{align*}
$$

[^0]where $a:=\sigma \sigma^{T}, \nu(d z) d t:=\mathbb{E} N(d z, d t)$. For convenience we assign $\nu(\{0\})=0$, $p=\left(p_{1}, \ldots, p_{m}\right)$, and $\nu=\left(\nu_{1}, \ldots, \nu_{m}\right)$.
In many cases, the process $Y_{t}$ admits a probability density function (PDF) $u(t, x)$, a function $u \geq 0$ such that $\mathbb{E} \phi\left(Y_{t}\right)=\int_{\mathbb{R}^{d}} \phi(x) u(t, x) d x$ for all $\phi \in C_{b}\left(\mathbb{R}^{d}\right)$. Formally the PDF $u$ solves the Fokker-Planck or forward Kolmogorov equation
\[

$$
\begin{equation*}
\partial_{t} u(t, x)=L u(t, x), \tag{1.3}
\end{equation*}
$$

\]

where $L$ is the adjoint of $L^{*}$. It is this operator $L$ that we call the Focker-Planck operator. To be precise, we define $L$ on the domain $D(L):=C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ as the adjoint operator of $L^{*}$ in (1.2), i.e.

$$
\begin{aligned}
L u(x)= & \frac{1}{2} \sum_{i, j=1}^{d} \partial_{i j}\left(a_{i j} u(x)\right)-\operatorname{div}(b(x) u(x))+\operatorname{div}\left(\int_{\{r \leq|z|<1\}} p(x, z) \nu(d z) u(x)\right) \\
& +\int_{|z|<r}[u(x-q(x, z))-u(x)+D u(x) q(x, z)] m(x, z) \nu(d z) \\
& +D u(x)^{T} \int_{|z|<r}[p(x, z)-q(x, z) m(x, z)] \nu(d z) \\
& +u(x) \int_{|z|<r}\left[m(x, z)+\operatorname{div}_{x} p(x, z)-1\right] \nu(d z)+J_{r} u(x)
\end{aligned}
$$

for $r>0$ small enough, $y(x, z)=x-p(y(x, z), z)=: x-q(x, z)$ and $m(x, z):=$ $\operatorname{det}\left(D_{x} y(x, z)\right)$, and $J_{r}$ is the adjoint of $J_{r}^{*} f(y):=\sum_{k=1}^{m} \int_{|z| \geq r}\left[f\left(y+p_{k}(y, z)\right)-\right.$ $f(y)] \nu_{k}(d z)$.
To obtain $L^{p}$ or Sobolev space theories for such complicated $x$-depending non-local operators, the literature resorts to the global invertibility assumption $[26,5]$,

$$
\begin{equation*}
0<C^{-1} \leq \operatorname{det}\left(1_{d}+D_{y} p(y, z)\right) \leq C \quad \text { for all } \quad y, z \in \mathbb{R}^{d} \tag{1.5}
\end{equation*}
$$

Such an assumption is crucial and e.g. allows one to show (under some further assumptions) that $L u$ belongs to $L^{p}$ for any $u \in C_{c}^{\infty}$ and $p \in[1, \infty]$, that $L$ is indeed the adjoint of $L^{*}$, and that $J_{r}$ then takes the explicit form (cf. Section 2.4 in [26])

$$
\begin{equation*}
J_{r} u(x)=\int_{|z| \geq r}[u(x-q(x, z)) m(x, z)-u(x)] \nu(d z) \tag{1.6}
\end{equation*}
$$

But assumption (1.5) is very restrictive and excludes many applications, including most $x$-depending cases of interest! One of the main contributions of this paper is to show how it can be dropped completely, even in the borderline $L^{1}$ setting. We will see that we can still work with $L$ even though e.g. $J_{r}$ now will be defined through duality only, without an explicit representation.

The main result of this paper is that under quite general assumptions, $L$ generates a strongly continuous contraction semigroup on $L^{1}\left(\mathbb{R}^{d}\right)$.
A standard consequence is then that there exists a unique mild solution in $L^{1}$ of the Cauchy problem for (1.3) [23], and under further assumptions, one can prove that this solution is the PDF of the process $Y_{t}[10,18,17]$. Here it is crucial that we work in the space $L^{1}$ since PDFs by definition belong to this space but in general not to $L^{p}$ for any $p>1$. An other application is the convergence of approximations and numerical methods. Many such results follow from Kato-Lie-Trotter or Chernoff
formulas where the generation result is a prerequisite [18, 14]. In [18] generation is the most difficult step of the proof, and in many cases, our new generation result provides the generation result needed in [14] (Assumption 6).
The assumptions of our generation result include a uniformly elliptic local part, unbounded coefficients with finite differentiability, and general Lévy measures (can be singular and have fat tails etc.). In particular the conditions on the non-local part are very general, covering most jump models in applications [3, 6, 21, 41]. The restrictive assumption is mainly the uniform ellipticity, which means the local part can not degenerate/vanish in any direction. In the literature, such ellipticity or weaker hypo-ellipticity are typically used to guarantee the existence of (smooth) PDFs.

The main tools of the proofs are taken from semigroup theory. We essentially use the Lumer-Phillips theorem to prove the semigroup generation of dissipative operators in $L^{1}$. This is not an easy task. The difficulty arises not only from the space $L^{1}$ being non-reflexive, as we have already encountered in the case without jumps [18], but also because of the complicated non-local terms in the FP operator. Since we treat very general Lévy models and unbounded coefficients, we can not use the standard global invertibility assumption (1.5) and show semigroup generation directly. In stead, our strategy is to write the operator as the sum of three parts that we analyze separately: the local part, the small jumps part, and the large jumps part. Through a non-trivial extension of the analysis of [18] (see below), we show that (the sum of) the two first parts generates a strongly continuous contraction semigroup on $L^{1}\left(\mathbb{R}^{d}\right)$. The presence of the third part is new in this setting and crucial for the analysis. We show that it is a bounded operator on $L^{1}\left(\mathbb{R}^{d}\right)$ and then treat it as a perturbation to the semigroup generated by the sum of the other two parts.

Note that in this new approach, no invertibility assumption is needed. This is true even though we need invertibility to handle the small jumps term. But since we have split of the large jumps, we only need local invertibility now. By localizing as much as we need (taking $r$ in (1.4) small enough), we observe that invertibility follows from a standard Lipschitz assumption on $p$ (cf. Proposition 3.2 (b) and proof). For this argument to work, we also have to handle the remaining large jumps term using only duality arguments.
A key next step in the generation argument is then to show that the first and the second parts of the FP operator are dissipative in $L^{1}$ and that their corresponding adjoints are dissipative in $L^{\infty}$. Both results rely on the negativity of the corresponding operators. In the $L^{1}$ setting it translates into the inequality $\int_{\{u \neq 0\}} L|u| d x \leq 0$ where $L$ denotes the FP operator. The proof is technical and involve separation and approximation of the domains $\{u>0\}$ and $\{u<0\}$ where $|u|$ is smooth. The non-local case is more difficult and requires additional arguments because the domains can no longer be separated as in the local case. On the other hand, to show dissipativity of the adjoint in $L^{\infty}$, we first prove that the maximal domain of the adjoint is contained in certain Sobolev spaces. To this end, we obtain new elliptic regularity results for non-local operators, extending recent local results in [45]. In the local case, dissipativity then follows from an argument using the Bony maximum principle for Sobolev functions [18]. Here this argument is extended to our non-local operators using additional ideas from [29].

Our elliptic regularity result is of independent interest: It applies to very general Lévy operators, operators with degenerate non-local parts, unbounded and variable coefficients, and general Lévy measures.
Let us now briefly discuss the background setting of our problem. Over the past decades, there has been a large number of publications in the field of stochastic dynamics and its various application areas - including physics, engineering, and finance. In these fields, the response of dynamical systems to stochastic excitation is studied, and the typical model is (a system of) stochastic differential equations (SDEs). Traditionally, the driving noise has been Gaussian, but there is a large and increasing number of applications that need more general Lévy driving noise like e.g. anomalous diffusions in physics and biology and advanced market models in finance and insurance $[3,6,21,41,33,35]$. A common feature and difficulty of such models are that the corresponding processes may have sudden jumps and hence discontinuous realizations or sample paths.

Then we take a look the literature related to the semigroup generation result. For local forward equations and SDEs driven by Brownian motion, many classical generation results are given e.g. in [23]. More recent results for $L^{1}$ and unbounded coefficients can be found in [25]. Also for many non-local operators like fractional Laplacian or generators of Lévy processes such generation results are classical, see e.g. Theorem 3.4.2 in [3]. That book also gives generation results in $C_{0}$ for more complicated generators of Lévy driven SDEs in Theorem 6.7.4. When it comes to generation in $L^{1}$, we have only been able to find one paper on non-local operators with variable coefficients. Theorem 1.1 in [42] gives such a result for the operator $L=-(-\Delta)^{\alpha / 2}+b(x) \cdot \nabla$. Note well that these results do not apply to the FP operator directly, but to its adjoint. In the local case, the regularity of the coefficients allows us to rewrite the FP operator as an adjoint operator plus a (possibly unbounded) zero-order term. Hence generation may follow from results for this augmented "adjoint" operator as discussed in [18]. However, in the nonlocal case, this trick is not available unless we assume also the very restrictive global invertibility assumption (1.5).

Generation results can also be obtained in a completely different way as a consequence of so-called heat kernal analysis. There the aim is to obtain sharp bounds on the heat kernals or transition probabilities $p(t, x ; s, y)$ of the Markov process defined by (1.1). The semigroup $P_{t}$ generated by $L$, can then be explicitly defined as $P_{t} f(x)=\int_{\mathbb{R}^{d}} \rho(t, x ; 0, y) f(y) d y$ for suitable functions $f$. This research area dates back to [4], and more recently also includes jump processes and non-local operators (e.g. [7, 11, 20]). We will focus on [19] which seems to have the most general results that apply to Lévy driven SDEs with variable coefficients. The assumptions include uniform local ellipticity, "bounded" coefficients, and a non-local part that satisfies some moment condition and is comparable (from one side) to the fractional Laplacian. In this case $P_{t}$ is a strongly continuous contraction semigroup on $L^{1}$ (and $L^{p}$ for any $\rho \in[1, \infty]$ ) by Theorem 1.1 (1), (2), (5), and (6) of [19] and the application of standard arguments.
Compared with existing results, our generation result applies to FP operators with much more general jump/non-local parts and unbounded coefficients. Moreover, we do not use heat kernel analysis, but rather a direct semigroup approach.

Outline. In Section 2 we state the assumptions and the main result. Then we prove our main results in Section 3. In Section 5 we prove that the generator of the SDE and its adjoint are dissipative. Many required properties of the nonlocal operators are obtained in Section 4, including that the long jump part of the operator is bounded on $L^{1}$. Finally, Section 6 is devoted to the proof of the elliptic regularity result.

Notation. The following notation will be used throughout the paper: $\partial_{t}:=\frac{\partial}{\partial t}$, $D=D_{x}:=\left(\frac{\partial}{\partial x_{1}}, \cdots, \frac{\partial}{\partial x_{d}}\right)^{T}=:\left(\partial_{1}, \cdots, \partial_{d}\right)^{T},\|\cdot\|_{1}:=\|\cdot\|_{L^{1}\left(\mathbb{R}^{d}\right)},\|\cdot\|_{\infty}:=\|\cdot\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}$, essinf is the essential infimum, $\mathbb{E}$ denotes the mathematical expectation; $1_{d}$ the identity matrix in $\mathbb{R}^{d \times d} ; C_{b}^{k}\left(\mathbb{R}^{d}\right)$ and $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ the spaces of functions with bounded continuous derivatives up to $k$-th order and smooth compactly supported functions, respectively; $\mathcal{D}^{\prime}\left(\mathbb{R}^{d}\right)$ the dual space of $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.

The following abbreviations are used: PDF - probability density function, SDE stochastic differential equation, FP - Fokker-Planck.

## 2. SEmigroup generation

In this section, we state the assumptions, our main result on semigroup generation, a related elliptic regularity result, and remarks. Elliptic regularity is needed for our proof of generation. The properties of the operator $L$ and the proof of the generation result will be given the next section.

We will use the following assumptions:
(H1) $b \in C^{1}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ and $\sigma \in C^{2}\left(\mathbb{R}^{d}, \mathbb{R}^{d \times n}\right)$, and there exists a constant $K>0$ such that for all $x \in \mathbb{R}^{d}, j=1, \ldots, n$, and $j, k=1, \ldots, d$,

$$
\left|\partial_{k} \sigma_{i j}(x)\right|+\left|\partial_{k} b_{i}(x)\right| \leq K
$$

For all $k=1, \cdots, m$,
(H2) $p_{k}: \mathbb{R}^{d} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is Borel measurable, $C^{1}$ in $y$, and for $\nu$-a.e. $|z|<1$, $p_{k}(\cdot, z)$ is $C^{2}$ in $y$ and

$$
\begin{array}{ll}
\left|p_{k}(y, z)\right| \leq K(1+|y|)|z| & \text { for all } y \in \mathbb{R}^{d} \\
\left|D_{y} p_{k}(y, z)\right| \leq K|z| & \text { for all } y \in \mathbb{R}^{d} \\
\left|D_{y}^{2} p_{k}(y, z)\right| \leq C_{R}|z| & \text { for all }|y| \leq R .
\end{array}
$$

(H3) $\nu_{k}$ is a non-negative Radon measure satisfying

$$
\int_{\mathbb{R}^{d}}\left(1 \wedge|z|^{2}\right) \nu_{k}(d z)<\infty
$$

We will also use the following more abstract assumption:
(E) (Elliptic regularity) Let $J_{r}^{*} f(y):=\sum_{k=1}^{m} \int_{|z| \geq r}\left[f\left(y+p_{k}(y, z)\right)-f(y)\right] \nu_{k}(d z)$, for some $r>0$ small enough. If

$$
\begin{equation*}
f, g \in L^{\infty}\left(\mathbb{R}^{d}\right) \quad \text { and } \quad\left(L^{*}-J_{r}^{*}\right) f=g \quad \text { in } \quad \mathcal{D}^{\prime}\left(\mathbb{R}^{d}\right) \tag{2.1}
\end{equation*}
$$

then $f \in W_{\mathrm{loc}}^{2, p}\left(\mathbb{R}^{d}\right)$ for some $p>d$.

Remark 2.1. Any Lévy measure $\nu_{k}$ and most $p$ 's from applications satisfy assumptions (H1) - (H3). E.g. the $\alpha$-stable processes with $p(y, z)=z$ and $\nu(d z)=\frac{c_{\alpha} d z}{|z|^{d+\alpha}}$, $\alpha \in(0,2)$. Unbounded $p$ 's appear in finance and insurance [21, 8, 35, 15], e.g. $p(y, z)=y z$ and $p(y, z)=y\left(e^{z}-1\right)$. The jump term $p$ is allowed to vanish on arbitrary large sets, and then the non-local part of the FP operator degenerates.

Assumptions (H1) - (H3) (except the $C^{2}$ regularity) are standard assumptions for the existence and uniqueness of strong solutions of Lévy driven SDEs (1.1) [3, 36]. They imply that the coefficients may be unbounded in $y$ (with linear growth), and the assumptions on the non-local operator are very general indeed: SDEs with arbitrary Lévy jump terms, even strongly degenerate ones, are included. In particular, we do not require any invertibility of $y+p_{k}(y, z)$ to define $L$ as the adjoint of the generator $L^{*}$ like in $[26,5]$ where the global assumption (1.5) is used. Note that this global condition is always satisfied when $p$ does not depend on $y$, and that this paper is probably the first work on semigroup generation not to explicitly or implicitly assume such a condition.
When it comes to assumption (E), it is most likely already satisfied under assumptions (H1) - (H3) if we assume also uniform ellipticity. See e.g. [18] for local operators. The general case seems not be covered in the literature, so we will prove that (E) holds under ellipticity and mild additional assumptions below.

Now we can state the main result of this paper:
Theorem 2.2 (Semigroup generation). Assume (H1) - (H3) and (E). Then the closure of $L$ generates a strongly continuous contraction semigroup on $L^{1}\left(\mathbb{R}^{d}\right)$.

We now give results verifying assumption (E) under uniform ellipticity and mild additional assumptions on the jump-terms:
(HE1) There exists $\alpha>0$ such that for all $x, y \in \mathbb{R}^{d}$,

$$
y^{T} a(x) y \geq \alpha|y|^{2}
$$

(HE2) There exists some $s \in[1,2)$ such that $\int_{|z|<1}|z|^{s} \nu_{k}(d z)<\infty$.
For all $k=1, \cdots, m$,
(H2') (H2) holds, and $p_{k}(\cdot, z)$ is $C^{3}$ in $y$ for $\nu_{k}$-a.e. $z$, and there exists $\tilde{p}_{k}(z) \geq 0$ such that for all $R>0,|y| \leq R$, and $\nu_{k}$-a.e. $z$,

$$
\left|p_{k}(y, z)\right|+\left|D_{y} p_{k}(y, z)\right|+\left|D_{y}^{2} p_{k}(y, z)\right|+\left|D_{y}^{3} p_{k}(y, z)\right| \leq C_{R}\left(|z| \wedge \tilde{p}_{k}(z)\right)
$$

(H3') (H3) holds, and $\tilde{C}:=\max _{k=1, \ldots, m} \int_{|z| \geq 1} \tilde{p}_{k}(z) \nu_{k}(d z)<\infty$.
Under (HE2), $s<2$ is the maximal (pseudo) differential order of the non-local part of the FP operator. Since the bound is only from above, the Lévy measures $\nu_{k}$ may be degenerate.

When the Lévy measure is not too singular $(s=1)$ we only need (HE1) and (HE2). In the general case all assumptions are needed.

Theorem 2.3 (Elliptic regularity). Assumption (E) holds if either one of the two sets of assumptions below hold:
(a) (H1), (H2), (H3), (HE1), and (HE2) with $s=1$.
(b) (H1), (H2'), (H3'), (HE1), and (HE2) with $s \in(1,2)$.

This result will be proved in Section 6.
Remark 2.4. (a) If $p \equiv 0$ and the operator is local, then Theorem 2.2 has been proven in [18] with assumption (HE1) replacing assumption (E).
(b) To do our generation proof (to prove the dissipativity of $L^{*}$ ) we need enough regularity for the equation $L^{*} u=f$ to hold a.e. for any $f \in L^{\infty}$ and some version of the Bony maximum principle to apply. This is encoded in (E), and such a condition can only be true under some sort of non-degeneracy conditions on the second order local terms (e.g. (HE1)).
(c) Assumption (E) can be relaxed when there are no second-order terms in the operator. Then the principal non-local term must be non-degenerate. To extend our proofs in this direction, new Bony type maximum principles are needed for fractional Sobolev spaces. We will not pursue this idea in this paper.
(d) Non-degeneracy conditions like (HE1) or weaker Hörmander conditions, along with smoothness assumptions on the coefficients, are standard assumptions in the literature to ensure the existence of (smooth) PDFs for (1.1), see e.g. [30, 31, 32, $9,16,46]$ and references therein.
(f) Elliptic regularity results are well-known for local operators (and PDEs), and results that cover the local part of our operators ( $L^{*}$ with $p \equiv 0$ ) can be found in the recent paper [45]. Theorem 2.3 (a) is essentially a corollary of results in [45] where the non-local term is treated as a lower-order perturbation. Part (b) is much more complicated and requires additional regularity on $p(x, z)$.

There are also very general results for pseudo-differential operators. These results require that the symbols are smooth and satisfy certain decay assumptions which are not in general satisfied by the operators we consider here, see e.g. Section 7.3.1 in [1].

In the rest of the paper we set $m=1$ and $p_{k}(x, z)=p(x, z)$ to simplify the notation. The general case is similar and will be omitted.

## 3. Properties of $L$ and proof of generation

In this section, we show that $L$ is well-defined and dissipative in $L^{1}$, that $L^{*}$ is dissipative in $L^{\infty}$, and use a version of the Lumer-Phillips theorem along with a perturbation result to show semigroup generation for $L$ in $L^{1}$.
To work with $L$, we decompose it along with $L^{*}$ into three parts. For any $r \in(0,1)$,

$$
L^{*}=A_{r}^{*}+I_{r}^{*}+J_{r}^{*},
$$

where

$$
\begin{aligned}
A_{r}^{*} f(y) & =b^{T}(y) D f(y)+\frac{1}{2} \sum_{i, j=1}^{d} a_{i j} \partial_{i} \partial_{j} f(y)+[D f(y)]^{T} \int_{\{r \leq|z|<1\}} p(y, z) \nu(d z), \\
I_{r}^{*} f(y) & =\int_{\{|z|<r\}}\left[f(y+p(y, z))-f(y)-[D f(y)]^{T} p(y, z)\right] \nu(d z), \\
J_{r}^{*} f(y) & =\int_{\{|z| \geq r\}}[f(y+p(y, z))-f(y)] \nu(d z) .
\end{aligned}
$$

By integration by parts and the change of variables $x=y+p(y, z)$ (assuming it is invertible), it follows that the adjoint

$$
L=A_{r}+I_{r}+J_{r}
$$

where

$$
\begin{align*}
A_{r} u(x)= & \frac{1}{2} \sum_{i, j=1}^{d} \partial_{i} \partial_{j}\left(a_{i j} u(x)\right)-\operatorname{div}\left[\left(b(x)+\int_{\{r \leq|z|<1\}} p(x, z) \nu(d z)\right) u(x)\right] \\
I_{r} u(x)= & \int_{|z|<r}[u(x-q(x, z))-u(x)+D u(x) q(x, z)] m(x, z) \nu(d z)  \tag{3.1}\\
& +(D u(x))^{T} \int_{|z|<r}[p(x, z)-q(x, z) m(x, z)] \nu(d z) \\
& +u(x) \int_{|z|<r}\left[m(x, z)+\operatorname{div}_{x} p(x, z)-1\right] \nu(d z)
\end{align*}
$$

for $y(x, z)=x-p(y(x, z), z)=: x-q(x, z)$ and $m(x, z):=\operatorname{det}\left(D_{x} y(x, z)\right)$. The derivation can be found in Section 2.4 in [26]. If we assume global invertibility of $y \mapsto y+p(y, z)$, assumption (1.5), then $J_{r}$ has the explicit form (1.6). One contribution of this paper is to relax this condition, and not work with a $J_{r}$ given by an explicit formula, but rather defined only by the duality $J_{r}=\left(J_{r}^{*}\right)^{*}$. Moreover, without global invertibility, the derivation of $I_{r}$ from $I_{r}^{*}$ only holds for $r$ small enough. In this case, we still get the (local) invertibility needed to do the abovementioned change of variables (see Proposition 3.2 and Section 4).
Note that $A_{r}, I_{r}, A_{r}^{*}, I_{r}^{*}$ are unbounded operators while $J_{r}$ and $J_{r}^{*}$ are bounded.
Remark 3.1. $J_{r}$ and $J_{r}^{*}$ can be defined on $L^{\infty}$ and $L^{1}$ respectively (see below). To make the integrands well-defined (Borel or $\nu$-measurable) for functions in $L^{1}$ and $L^{\infty}$, we always work with Borel representatives (cf. Remark 2.1 in [2]).

Now we show that our operators are well-defined on $L^{1}$.

## Proposition 3.2.

(a) Assume (H1) and $r>0$. Then $A_{r}: D(L) \rightarrow L^{1}\left(\mathbb{R}^{d}\right)$ is well-defined.
(b) Assume (H2) and (H3). Then there is $r_{0}<\frac{1}{4 d K}$ such that $I_{r}: D(L) \rightarrow L^{1}\left(\mathbb{R}^{d}\right)$ is well-defined for all $0<r<r_{0}$.
(c) Assume (H2) and (H3) and $r>0$. Then $J_{r}: L^{1}\left(\mathbb{R}^{d}\right) \rightarrow L^{1}\left(\mathbb{R}^{d}\right)$ is well-defined and bounded,

$$
\left\|J_{r}\right\| \leq 2 \nu(\{|z| \geq r\})
$$

It follows that $L: D(L) \rightarrow L^{1}\left(\mathbb{R}^{d}\right)$ is well-defined if $(\mathrm{H} 1)-(\mathrm{H} 3)$ holds. The proof will be given in Section 4.
Next, we show that the operators (and their adjoints) are dissipative in the sense of the following definition (see e.g. Section II. 3 of [23]):

Definition 3.3. A linear operator $(B, D(B))$ on a Banach space $(\mathbb{X},\|\cdot\|)$ is dissipative if $\|(\lambda-B) u\| \geq \lambda\|u\|$ for all $\lambda>0$ and all $u \in D(B)$.

## Theorem 3.4.

(a) Assume (H1) - (H3) and $r<r_{0}$, where $r_{0}$ is defined in Proposition 3.2. Then $A_{r}+I_{r}$ is dissipative on $D(L) \subset L^{1}\left(\mathbb{R}^{d}\right)$.
(b) Assume (H1) - (H3), and (E). Then $A_{r}^{*}+I_{r}^{*}$ is dissipative on $D\left(A_{r}^{*}+I_{r}^{*}\right) \subset$ $L^{\infty}\left(\mathbb{R}^{d}\right)$.
(c) Assume (H2) and (H3). Then $J_{r}$ is dissipative on $L^{1}\left(\mathbb{R}^{d}\right)$ for any $r>0$.
$D\left(A_{r}^{*}+I_{r}^{*}\right)$ is the maximal domain of $A_{r}^{*}+I_{r}^{*}$ on $L^{\infty}\left(\mathbb{R}^{d}\right)$. It will be characterized in Section 5.3. The proposition is proved in Sections $5.2-5.4$. These proofs and the proofs of related auxiliary results constitute the main technical innovation of this paper. They are highly non-trivial, and the PDE-inspired way of doing the proofs seems to be unconventional.
Remark 3.5. One can easily check that also $J_{r}^{*}$ is dissipative on $L^{\infty}\left(\mathbb{R}^{d}\right)$. Hence both $L$ and $L^{*}$ are dissipative by Section III. 2 in [23].

We are in a position to use the Lumer-Phillips theorem to prove the following preliminary generation result.

Proposition 3.6. Assume (H1) - (H3), (E), and $r<r_{0}$, where $r_{0}$ is defined in Proposition 3.2. Then $A_{r}+I_{r}$ generates a strongly continuous contraction semigroup on $L^{1}\left(\mathbb{R}^{d}\right)$.

Proof. Since $A_{r}+I_{r}$ and $A_{r}^{*}+I_{r}^{*}$ are dissipative by Proposition 3.4, and $A_{r}+I_{r}$ is densely defined $\left(D(L)\right.$ is dense in $\left.L^{1}\left(\mathbb{R}^{d}\right)\right), A_{r}+I_{r}$ generates a strongly continuous contraction semigroup $P_{t}$ on $L^{1}\left(\mathbb{R}^{d}\right)$ by a version of the Lumer-Phillips theorem see Corollary II.3.17 in [23].

To get a generation results for the full operator $L$, we view it as a bounded perturbation of $A_{r}+I_{r}$ and use the following result:

Theorem 3.7 (Theorem 3.3.4 in [37]). Let $B_{1}$ generate a contraction semigroup on a Banach space $(\mathbb{X},\|\cdot\|)$ and $B_{2}$ be dissipative. Assume $D\left(B_{1}\right) \subset D\left(B_{2}\right) \subset \mathbb{X}$ and there is a $\lambda>0$ such that

$$
\left\|B_{2} x\right\| \leq\left\|B_{1} x\right\|+\lambda\|x\| \quad \text { for all } x \in D\left(B_{1}\right)
$$

If $B_{2}^{*}$, the adjoint of $B_{2}$, is densely defined, then the closure of $B_{1}+B_{2}$ generates a strongly continuous contraction semigroups.

Proof of Theorem 2.2. Take $r<r_{0}$ where $r_{0}$ is defined in Proposition 3.2. Then note that $L-J_{r}=A_{r}+I_{r}$ generates a strongly continuous contraction semigroup on $L^{1}\left(\mathbb{R}^{d}\right)$ by Proposition 3.6 , that $J_{r}$ is bounded and dissipative on $L^{1}\left(\mathbb{R}^{d}\right)$ by

Propositions 3.2 (c) and 3.4 (c), and $J_{r}^{*}$ is bounded and hence an everywhere defined operator on $L^{\infty}=\left(L^{1}\right)^{*}$ by definition. Hence the result follows from Theorem 3.7 with $B_{1}=L-J_{r}, B_{2}=J_{r}$, and $\lambda \geq\left\|J_{r}\right\|$.

## 4. Operators in $L^{1}$ - proof of Proposition 3.2

In this section prove Proposition 3.2, i.e. we show that the operators $A_{r}$ and $I_{r}$ are well-defined from $D(L)$ into $L^{1}$ and $J_{r}$ well-defined and bounded on $L^{1}$. For $A_{r}$ this is immediate from the definition of this operator, so we will focus on the other two operators. If we assume the global invertibility (1.5), then the results follow from arguments similar to those given in Section 2.4 of [26]. However, the general case is more complicated and will be dealt with now.

We recall from Section 1 that

$$
y(x, z)=x-p(y(x, z), z)=: x-q(x, z) \quad \text { and } \quad m(x, z):=\operatorname{det}\left(D_{x} y(x, z)\right)
$$

and note that by the implicit function theorem

$$
\begin{equation*}
m(x, z)=\operatorname{det}\left(1_{d}-D_{x} q(x, z)\right)=\frac{1}{\operatorname{det}\left(1_{d}+\left(D_{y} p\right)(y(x, z), z)\right)} \tag{4.1}
\end{equation*}
$$

4.1. Proposition 3.2 (b) - the operator $I_{r}$. Throughout this section, we assume (H2) - (H3) with $r<1 /(4 d K)$, and we define the set

$$
U_{r}:=\mathbb{R}^{d} \times\left\{z \in \mathbb{R}^{d}:|z|<r\right\}
$$

Before we prove the result, we give a long list of technical results.
Lemma 4.1. $|y(x, z)| \leq 2|x|+1$ for $(x, z) \in U_{r}$.
Proof. Observe that for $(x, z) \in U_{r}$ with $r<1 /(4 d K)$,

$$
\begin{aligned}
|y(x, z)| & \leq|y(x, z)+q(x, z)|+|q(x, z)| \\
& =|y(x, z)+q(x, z)|+|p(y(x, z), z)| \\
& \leq|x|+K(1+|y(x, z)|)|z| \\
& \leq|x|+\frac{1}{4 d}(1+|y(x, z)|) \\
& \leq|x|+\frac{1}{2}(1+|y(x, z)|)
\end{aligned}
$$

and the result follows.
Lemma 4.2. For some $C>0$ and all $(x, z) \in U_{r}$,

$$
|q(x, z)| \leq C(1+|x|)|z| .
$$

Proof. Just note that

$$
|q(x, z)|=|p(y(x, z), z)| \leq K(1+|y(x, z)|)|z| \leq C(1+|x|)|z|
$$

by (H2) and Lemma 4.1.
Next we show that invertibility (1.5) holds if we restrict to the set $U_{r}$ (compare with (2.2.7) in [26]).

Lemma 4.3. There is $C>1$ such that for all $(x, z) \in U_{r}$, (1.5) holds and hence

$$
0<C^{-1} \leq m(x, z) \leq C \quad \text { for all } \quad(x, z) \in U_{r}
$$

Proof. Straightforward by definition, assumptions, and (4.1).
Lemma 4.4. Define $f(\cdot, z):=\left(\operatorname{div}_{y} p\right)(\cdot, z)=\sum_{k=1}^{d} f_{k}(\cdot, z)$. Then

$$
|f(x, z)-f(y(x, z), z)| \leq C_{R}|z|^{2}, \text { for all }|x| \leq R \text { and }|z|<r .
$$

where $C(x)>0$ locally bounded with respect to $x$.

Proof. Observe

$$
f(x, z)-f(y(x, z), z)=\sum_{k=1}^{d}\left[f_{k}(x, z)-f_{k}(y(x, z), z)\right]
$$

For each $k$,

$$
\begin{aligned}
f_{k}(x, z)-f_{k}(y(x, z), z) & =f_{k}(x, z)-f_{k}(x-q(x, z), z) \\
& =q^{T}(x, z)\left(D f_{k}\right)(x-\theta q(x, z), z)
\end{aligned}
$$

By Lemma 4.2 and (H2),

$$
\left|q^{T}(x, z)\left(D f_{k}\right)(x-\theta q(x, z), z)\right| \leq C(1+|x|)|z| K|z|=: C(x)|z|^{2}
$$

The result follows since $\left|f_{k}(x, z)-f_{k}(y(x, z), z)\right| \leq C(x)|z|^{2}$.
Lemma 4.5. Let $M=M(x, z):=\left(D_{y} p\right)(y(x, z), z)$, then

$$
\operatorname{det}\left(1_{d}+M\right)=1+\operatorname{tr}(M)+P(x, z)
$$

where $|P(x, z)| \leq C|z|^{2}$ for all $x \in \mathbb{R}^{d}$.
Proof. This is easily seen by the definition of determinant, the definition of $M$ and assumption (H2). One can also refer to Section 2 of [13]. The constant $C$ is uniform in $x$ by (H2).

Lemma 4.6. There exists $C>0$ such that

$$
|m(x, z)-1| \leq C|z| \quad \text { for all } \quad(x, z) \in U_{r}
$$

Proof. Denote

$$
D_{y}(y+p(y, z))=1_{d}+D_{y} p(y, z)=: 1_{d}+M
$$

Then by definition

$$
|m(x, z)-1|=\left|\frac{1}{\operatorname{det}\left(1_{d}+M\right)}-1\right|=\left|\frac{1-\operatorname{det}\left(1_{d}+M\right)}{\operatorname{det}\left(1_{d}+M\right)}\right|
$$

By Lemma 4.5 and $(\mathrm{H} 2), \operatorname{det}\left(1_{d}+M\right)=1+\operatorname{tr}(M)+P(x, z)$, and then for $|z|<r$,

$$
\left|\operatorname{det}\left(1_{d}+M\right)-1\right| \leq|\operatorname{tr}(M)+P(x, z)| \leq C|z|
$$

The proof is then complete if we can get a lower bound on $\left|\operatorname{det}\left(1_{d}+M\right)\right|$.
We claim that $\operatorname{det}\left(1_{d}+M\right) \geq 2^{-d}$. For each entry in $M$ and $|z|<r=\frac{1}{4 d K}$,

$$
\left|\left(\partial_{y_{i}} p_{j}\right)(y, z)\right| \leq K|z| \leq \frac{1}{4 d}
$$

and the matrix $1_{d}+M$ is diagonally dominant. By Theorem 1 in [38],

$$
\operatorname{det}\left(1_{d}+M\right) \geq \prod_{i=1}^{d}\left(\left|\alpha_{i i}\right|-\beta_{i}\right)
$$

where $\alpha_{i i}=1+\left(\partial_{y_{i}} p_{i}\right)(y, z)$ and $\beta_{i}=\sum_{j=i+1}^{d}\left|\left(\partial_{y_{i}} p_{j}\right)(y, z)\right|$. Hence $\left|\alpha_{i i}\right| \geq 1-\frac{1}{4 d}$ and $\beta_{i} \leq \frac{1}{4}$, and thus $\left|\alpha_{i i}\right|-\beta_{i} \geq 1-\frac{1}{4 d}-\frac{1}{4} \geq \frac{1}{2}$ and $\operatorname{det}\left(1_{d}+M\right) \geq 2^{-d}$. The proof is complete.

Proof of Proposition 3.2 (b). First note that since $r<\frac{1}{4 d K}$ and the supports of $u$ and $I_{r}$ are compact, we only need to consider $x, z$ on compact sets depending on $u$ (and $p, q$ - see below) but not on $r$. With this in mind we bound the different terms $I_{r}$, see (3.1). For the first integral,

$$
\begin{aligned}
& \left|\int_{|z|<r}[u(x-q(x, z))-u(x)+q(x, z) D u(x)] m(x, z) \nu(d z)\right| \\
& =\left|\int_{|z|<r} \int_{0}^{1}(1-\theta) q^{T}(x, z)\left[D^{2} u(x)\right] q(x, z) d \theta m(x, z) \nu(d z)\right| \\
& \leq C(u) \int_{|z|<r}|z|^{2} \nu(d z)<\infty .
\end{aligned}
$$

Here we also used that $q$ is bounded on compact sets. For the second integral, we keep in mind that $x=y+p(y, z)$. The integrand is then

$$
\begin{aligned}
& p(x, z)-q(x, z) m(x, z) \\
& =p(x, z)-q(x, z)+q(x, z)(1-m(x, z)) \\
& =p(x, z)-p(x-q(x, z), z)+q(x, z)(1-m(x, z)) \\
& =\left(D_{y} p\right)(x-\theta q(x, z), z) q(x, z)+q(x, z)(1-m(x, z)) \\
& =q(x, z)\left[\left(D_{y} p\right)(x-\theta q(x, z), z)+(1-m(x, z))\right] .
\end{aligned}
$$

Hence by (H2) and Lemmas 4.2 and 4.6 , for $x, z$ in the compact,

$$
|p(x, z)-q(x, z) m(x, z)| \leq C(u)|z|(|z|+|z|)
$$

and hence

$$
\left|\int_{|z|<r}(D u(x))^{T}[p(x, z)-q(x, z) m(x, z)] \nu(d z)\right| \leq C(u) \int_{|z|<r}|z|^{2} \nu(d z) .
$$

For the third integral, we take $f(\cdot, z):=\left(\operatorname{div}_{y} p\right)(\cdot, z)$, and note that integrand

$$
\begin{aligned}
& m(x, z)+\operatorname{div}_{x} p(x, z)-1=m(x, z)+f(x, z)-1 \\
& =[m(x, z)+f(y(x, z), z)-1]+[f(x, z)-f(y(x, z), z)]
\end{aligned}
$$

The last term can be estimated by Lemma 4.4,

$$
|f(x, z)-f(y(x, z), z)| \leq C(u)|z|^{2}
$$

For the first term, recall that $M=M(x, z)=\left(D_{y} p\right)(y(x, z), z)$ and note that $\operatorname{tr}(M)=f(y(x, z), z)$. Then by Lemma 4.5,

$$
\begin{aligned}
m(x, z)+f(y(x, z), z)-1 & =\operatorname{det}\left(1_{d}+M\right)^{-1}+\operatorname{tr}(M)-1 \\
& =\frac{1}{1+\operatorname{tr}(M)+P(x, z)}+\operatorname{tr}(M)-1 \\
& =\frac{-P(x, z)+\operatorname{tr}(M) P(x, z)+(\operatorname{tr}(M))^{2}}{1+\operatorname{tr}(M)+P(x, z)} .
\end{aligned}
$$

By Lemma 4.5 again,

$$
\left|-P(x, z)+\operatorname{tr}(M) P(x, z)+(\operatorname{tr}(M))^{2}\right| \leq C|z|^{2} \quad \text { and } \quad|M|+|P(x, z)| \leq C|z|
$$

where $C$ does not depend on $x$ and hence the support of $u$. We may therefore take a sufficiently small $r_{0}<\frac{1}{4 d K}$ (independently of $u$ ) such that for $|z|<r<r_{0}$,

$$
1+\operatorname{tr}(M)+P(x, z) \geq \frac{1}{2}
$$

Hence $|m(x, z)+f(y(x, z), z)-1| \leq C|z|^{2}$, and it follows that the third integral in (3.1) is well defined.

From the above estimtates and the compactness of the support, it then follows that there is $r_{0}>0$ such that $\left\|I_{r} u\right\|_{1}=\int_{\mathbb{R}^{d}}\left|I_{r} u(x)\right| d x<\infty$ for any $0<r<r_{0}$ and any $u \in D(L)$. The proof is complete.

Remark 4.7. In the proof of Lemma 2.4.3 in in [26], the authors claim that if $M$ is a symmetric matrix such that $\operatorname{det}\left(1_{d}+M\right) \neq 0$, then

$$
\left|\frac{1}{\operatorname{det}\left(1_{d}+M\right)}-1-\operatorname{tr}(M)\right| \leq C\|M\|^{2}
$$

If we could take $M=D_{y} p(y, z)$ in this inequality, it would simplify our proofs. However, in our setting $D_{y} p(y, z)$ is not symmertric in general.
4.2. Proposition 3.2 (c) - the operator $J_{r}$. We start by two auxilliary results.

Lemma 4.8. Assume (H3) and $u \in L^{1}\left(\mathbb{R}^{d}\right)$. Then $J_{r} u$ can be represented by a bounded, absolutely continuous, and finitely additive signed measure $\lambda_{u}$ such that

$$
\begin{equation*}
\left\langle J_{r} u, f\right\rangle=\int_{\mathbb{R}^{d}} f(x) \lambda_{u}(d x) \quad \text { for all } \quad f \in L^{\infty}\left(\mathbb{R}^{d}\right) \tag{4.2}
\end{equation*}
$$

Moreover, the total variation norm $\left|\lambda_{u}\right|\left(\mathbb{R}^{d}\right) \leq 2\|u\|_{L^{1}} \nu(\{|z|>r\})$.
Proof. This is quite standard. By the definition and $(\mathrm{H} 3), J_{r}^{*}$ is a bounded linear operator on $L^{\infty}\left(\mathbb{R}^{d}\right)$, and $\left\|J_{r}^{*}\right\| \leq 2 \nu(\{|z| \geq r\})$ since $\left|J_{r}^{*} f(y)\right| \leq 2\|f\|_{\infty} \nu(\{|z| \geq r\})$ for all $y \in \mathbb{R}^{d}$. Hence its adjoint operator $J_{r}$ is a bounded linear operator on the dual space of $L^{\infty}\left(\mathbb{R}^{d}\right)$ with $\left\|J_{r}\right\|=\left\|J_{r}^{*}\right\|$, cf. Theorem 3.3 in [40]. Hence also $\left\|J_{r}\right\| \leq 2 \nu(\{|z| \geq r\})$, and since $\|u\|_{\left(L^{\infty}\right)^{\prime}}=\|u\|_{L^{1}}$ for $u \in L^{1}$, we have

$$
\left\|J_{r} u\right\|_{\left(L^{\infty}\right)^{\prime}} \leq\left\|J_{r}\right\|\|u\|_{\left(L^{\infty}\right)^{\prime}} \leq 2\|u\|_{L^{1}} \nu(\{|z| \geq r\})
$$

Then by Theorem IV.8.16 in [22], there is an isometric isomorphism between the dual of $L^{\infty}\left(\mathbb{R}^{d}\right)$ and the bounded, absolutely continuous, finitely additive signed (ACFAS) measures. That is, $J_{r} u \in\left(L^{\infty}\left(\mathbb{R}^{d}\right)\right)^{\prime}$ corresponds uniquely to a ACFAS measure $\lambda_{u}$ such that (4.2) holds. The integral is here defined in the standard way
by first defining it for finitely(!) valued simple functions and then take the limit of total variation. The isometry part of the result means that the norm of $J_{r} u$ equals the total variation of $\lambda_{u},\left\|J_{r} u\right\|_{\left(L^{\infty}\right)^{\prime}}=\left|\lambda_{u}\right|\left(\mathbb{R}^{n}\right)$. The proof is complete.

We will need the following version of the Radon-Nikodym Theorem.
Theorem 4.9 (Theorem 10.39 in [43]). Let $\mu$ be a (finite and countably) additive set function. If $\mu$ is absolutely continuous with respect to the Lebesgue measure, then there exists an integrable function $w \in L^{1}\left(\mathbb{R}^{d}, d x\right)$, such that

$$
\mu(E)=\int_{E} w(x) d x \quad \text { for all } E \in \Sigma
$$

where $\Sigma$ is the $\sigma$-algebra of all Lebesgue measurable sets.
Proof of Proposition 3.2 (c).

1. By Lemma 4.8, $J_{r} u$ can be represented by a bounded, absolutely continuous, and finitely additive signed measure $\lambda_{u}$ such that for any measurable set $E \subset \mathbb{R}^{d}$,

$$
\begin{equation*}
\left|\lambda_{u}\right|(E)<\infty \quad \text { and } \quad \lambda_{u}(E)=\int_{\mathbb{R}^{d}} \chi_{E}(x) \lambda_{u}(d x)=\left\langle J_{r} u, \chi_{E}\right\rangle \tag{4.3}
\end{equation*}
$$

2. We check that $\lambda_{u}$ is in fact also countably additive. Suppose $\left\{A_{k} \subset \mathbb{R}^{d}: k=1,2,3, \ldots\right\}$ is a sequence of pair-wise disjoint Lebesgue measurable sets. Then $\chi_{\cup_{k} A_{k}}(y)=$ $\sum_{k} \chi_{A_{k}}(y)$, and by (4.3),

$$
\begin{aligned}
\lambda_{u}\left(\cup_{k} A_{k}\right) & =\left\langle J_{r} u, \chi_{\cup_{k} A_{k}}\right\rangle=\left\langle u, J_{r}^{*} \chi_{\cup_{k} A_{k}}\right\rangle \\
& =\int_{\mathbb{R}^{d}} u(x) \int_{|z| \geq r}\left[\chi_{\cup_{k} A_{k}}(x+p(x, z))-\chi_{\cup_{k} A_{k}}(x)\right] \nu(d z) d x \\
& =\int_{\mathbb{R}^{d}} u(x) \int_{|z| \geq r} \sum_{k}\left[\chi_{A_{k}}(x+p(x, z))-\chi_{A_{k}}(x)\right] \nu(d z) d x \\
& =\sum_{k} \int_{\mathbb{R}^{d}} u(x) \int_{|z| \geq r}\left[\chi_{A_{k}}(x+p(x, z))-\chi_{A_{k}}(x)\right] \nu(d z) d x \\
& =\sum_{k} \lambda_{u}\left(A_{k}\right) .
\end{aligned}
$$

In view of Remark 3.1, $\chi_{A_{k}}(x+p(x, z))$ is $\nu$-measurable for almost every $x$, and integration and summation commute by the dominated convergence theorem since $u \in L^{1}$ and (H3) holds.
3. By the Radon-Nikodym theorem, there is a unique $w_{u} \in L^{1}\left(\mathbb{R}^{d}\right)$ such that $\lambda_{u}(A)=\int_{A} w_{u}(x) d x$ for measurable $A \subset \mathbb{R}^{d}$. In other words, for any $u \in L^{1}\left(\mathbb{R}^{d}\right)$, we may identify $J_{r} u$ with $w_{u}$. Moreover, by the definition of total variation of $\lambda_{u}$ and Lemma 4.8 again, $\left\|w_{u}\right\|_{L^{1}}=\left|\lambda_{u}\right|\left(\mathbb{R}^{d}\right) \leq 2\|u\|_{L^{1}} \nu(\{|z|>r\})$. The proof is complete.

## 5. Dissipative operators - proof of Theorem 3.4

This whole section is devoted to the proof that the operators $A_{r}, I_{r}, J_{r}$ and their adjoints are dissipative, i.e. to prove Theorem 3.4.
5.1. Analysis on $I_{r}$. Consider $u \in D(L)$, and let

$$
V:=\left\{x \in \mathbb{R}^{d}: u(x)=0\right\}
$$

Denote $w:=|u|$, and decompose

$$
V^{c}=\{u(x)>0\} \cup\{u(x)<0\}=: V^{+} \cup V^{-}
$$

Denote $I_{r}:=S+T$, where $S$ is the principal non-Local term as

$$
\begin{equation*}
S u(x):=\int_{|z|<r}[u(x-q(x, z))-u(x)+q(x, z) D u(x)] m(x, z) \nu(d z) . \tag{5.1}
\end{equation*}
$$

Lemma 5.1. Assume (H2) and (H3). Then $A_{r} w, T w$, and $S w$ are well-defined on $V^{c}$. In addition, there exists a non-negative function $R(x)$ such that

$$
S w(x)= \begin{cases}R(x)+S u(x), & x \in V^{+}  \tag{5.2}\\ R(x)-S u(x), & x \in V^{-}\end{cases}
$$

Proof. Obvious $A_{r} w$ and $T w$ are well-defined on $V^{c}$, since they are local operators and $V^{c}$ is an open set where $w= \pm u$.

For $x \in V^{c}$. Recall $x=y+p(y, z)$, and $m(x, z) \geq 0$. Hence for any $|z|<r$, denote

$$
\begin{aligned}
F_{x}^{+} & :=\{z<|r|: u(x-q(x, z))>0\}, \\
F_{x}^{-} & :=\{z<|r|: u(x-q(x, z))<0\}, \\
F_{x}^{0} & :=\{z<|r|: u(x-q(x, z))=0\} .
\end{aligned}
$$

If $x \in V^{+}$, that is, $u(x)>0$, we observe that a neighborhood of 0 is contained in $F_{x}^{+}$. Then there holds

$$
\begin{aligned}
S w(x)= & \int_{|z|<r}[w(x-q(x, z))-w(x)+q(x, z) D w(x)] m(x, z) \nu(d z) \\
= & \int_{F_{x}^{+}}[u(x-q(x, z))-u(x)+q(x, z) D u(x)] m(x, z) \nu(d z) \\
& +\int_{F_{x}^{-}}[-u(x-q(x, z))-u(x)+q(x, z) D u(x)] m(x, z) \nu(d z) \\
& +\int_{F_{x}^{0}}[0-u(x)+q(x, z) D u(x)] m(x, z) \nu(d z) \\
= & \int_{F_{x}^{+}}[u(x-q(x, z))-u(x)+q(x, z) D u(x)] m(x, z) \nu(d z) \\
& +\int_{F_{x}^{-}}[u(x-q(x, z))-u(x)+q(x, z) D u(x) \\
& +\int_{F_{x}^{0}}[0-u(x)+q(x, z) D u(x)] m(x, z) \nu(d z) \\
= & S u(x)-2 \int_{F_{x}^{--}} u(x-q(x, z)) m(x, z) \nu(d z) .
\end{aligned}
$$

The last term is then non-negative and point-wisely finite.

Similarly, if $x \in V^{-}$,

$$
\begin{aligned}
S w(x)= & \int_{|z|<r}[w(x-q(x, z))-w(x)+q(x, z) D w(x)] m(x, z) \nu(d z) \\
= & \int_{|z|<r}[w(x-q(x, z))+u(x)-q(x, z) D u(x)] m(x, z) \nu(d z) \\
= & \int_{F_{x}^{+}}[u(x-q(x, z))+u(x)-q(x, z) D u(x)] m(x, z) \nu(d z) \\
& +\int_{F_{x}^{-}}[-u(x-q(x, z))+u(x)-q(x, z) D u(x)] m(x, z) \nu(d z) \\
& +\int_{F_{x}^{0}}[0+u(x)-q(x, z) D u(x)] m(x, z) \nu(d z) \\
= & -\int_{|z|<r}[u(x-q(x, z))-u(x)+q(x, z) D u(x)] m(x, z) \nu(d z) \\
& +2 \int_{F_{x}^{+}} u(x-q(x, z)) m(x, z) \nu(d z) \\
= & -S u(x)+2 \int_{F_{x}^{+}} u(x-q(x, z)) m(x, z) \nu(d z) .
\end{aligned}
$$

Therefore we obtained the following relationship

$$
\begin{aligned}
S w(x) & = \begin{cases}S u(x)-2 \int_{F_{x}^{-}} u(x-q(x, z)) m(x, z) \nu(d z), & x \in V^{+} \\
-S u(x)+2 \int_{F_{x}^{+}} u(x-q(x, z)) m(x, z) \nu(d z), & x \in V^{-}\end{cases} \\
& =: \begin{cases}S u(x)+R(x), & x \in V^{+}, \\
-S u(x)+R(x), & x \in V^{-}\end{cases}
\end{aligned}
$$

The proof is complete.
Lemma 5.2. Assume (H2) and (H3). Then $\int_{V^{c}} I_{r} w(x) d x \leq 0$.

Proof. By definition, we can write for $x \in V^{c}$

$$
\begin{aligned}
I_{r} w(x)= & \int_{|z|<r}([w(x-q(x, z))-w(x)+q(x, z) D w(x)] m(x, z) \\
& +(D w(x))^{T}[p(x, z)-q(x, z) m(x, z)] \\
& \left.+w(x)\left[m(x, z)+\operatorname{div}_{x} p(x, z)-1\right]\right) \nu(d z) \\
= & \int_{|z|<r} h(x, z) \nu(d z) .
\end{aligned}
$$

Next we note that there exist constants $C>0$ and $R>0$ such that

$$
\begin{equation*}
h(x, z) \geq-C|z|^{2} \chi_{\{|x| \leq R\}}(x)=:-g(x, z) \tag{5.3}
\end{equation*}
$$

This is true in view of $u \in D(L)$, Lemma 5.1, and the discussion in Section 4.1. Evidently by the assumptions, $0 \leq \int_{\mathbb{R}^{d}} \int_{|z|<r} g(x, z) \nu(d z) d x<\infty$. So $g$ is an integrable lower bound.

Then we truncate the integrand by defining

$$
h_{n}(x, z):=h(x, z) \chi_{\{r / n \leq|z|<r\}}(z), \quad n=1,2,3, \ldots
$$

Obviously $h_{n}(x, z) \geq \min \{h(x, z), 0\} \geq-g(x, z)$, and $\lim _{n} h_{n}(x, z)=h(x, z)$, for all $(x, z) \in V^{c} \times\{|z|<r\}$. Then we claim that for all $n=1,2,3, \ldots$

$$
\begin{equation*}
\int_{V^{c}} \int_{|z|<r} h_{n}(x, z) \nu(d z) d x \leq 0 . \tag{5.4}
\end{equation*}
$$

With (5.3), (5.4), and the integrable lower bound $g(x, z)$, we can apply Fatou's Lemma and prove Lemma 5.2 by

$$
\begin{aligned}
\int_{V^{c}} I_{r} w(x) d x & =\int_{V^{c}} \int_{|z|<r} h(x, z) \nu(d z) d x=\int_{V^{c}} \int_{|z|<r} \lim _{n} h_{n}(x, z) \nu(d z) d x \\
& \leq \liminf _{n} \int_{V^{c}} \int_{|z|<r} h_{n}(x, z) \nu(d z) d x \leq \liminf _{n} 0=0 .
\end{aligned}
$$

The rest of the proof will be used to prove Claim (5.4). Observe that by definition

$$
\int_{V^{c}} \int_{|z|<r} h_{n}(x, z) \nu(d z) d x=\int_{V^{c}} \int_{r / n \leq|z|<r} h(x, z) \nu(d z) d x
$$

and the Lévy measure $\nu$ is no longer singular on the set $\{r / n \leq|z|<r\}$. Hence

$$
\begin{align*}
& \int_{V^{c}} \int_{r / n \leq|z|<r} h(x, z) \nu(d z) d x \\
& 5)  \tag{5.5}\\
& =\int_{V^{c}} \int_{r / n \leq|z|<r}\left[w(x-q(x, z)) m(x, z)-w(x)+\operatorname{div}_{x}(w(x) p(x, z))\right] \nu(d z) d x
\end{align*}
$$

Then we consider the first two terms in (5.5).

$$
\begin{aligned}
& \int_{V^{c}} \int_{r / n \leq|z|<r}[w(x-q(x, z)) m(x, z)-w(x)] \nu(d z) d x \\
& =\int_{r / n \leq|z|<r} \int_{V^{c}}[w(x-q(x, z)) m(x, z)-w(x)] d x \nu(d z) \\
& =\int_{r / n \leq|z|<r}\left(\int_{\mathbb{R}^{d}}[w(x-q(x, z)) m(x, z)-w(x)] d x\right. \\
& \left.\quad-\int_{V}[w(x-q(x, z)) m(x, z)-w(x)] d x\right) \nu(d z) \\
& \leq \int_{r / n \leq|z|<r}\left(\int_{\mathbb{R}^{d}}[w(x-q(x, z)) m(x, z)-w(x)] d x\right) \nu(d z)
\end{aligned}
$$

since $w=|u|=0$ on $V$ and $w \geq 0$. For the last term, we observe

$$
\begin{aligned}
& \int_{r / n \leq|z|<r}\left(\int_{\mathbb{R}^{d}}[w(x-q(x, z)) m(x, z)-w(x)] d x\right) \nu(d z) \\
& =\int_{r / n \leq|z|<r}\left(\|u\|_{1}-\|u\|_{1}\right) \nu(d z)=0
\end{aligned}
$$

Now it remains to consider the third term in (5.5). By Fubini's theorem, we have

$$
\begin{aligned}
& \int_{V^{c}} \int_{r / n \leq|z|<r} \operatorname{div}_{x}(w(x) p(x, z)) \nu(d z) d x \\
& =\int_{r / n \leq|z|<r} \int_{V^{c}} \operatorname{div}_{x}(w(x) p(x, z)) d x \nu(d z)
\end{aligned}
$$

Since $V^{c}=\{u \neq 0\}$, we now claim that

$$
\begin{equation*}
\int_{V^{c}} \operatorname{div}_{x}(w(x) p(x, z)) d x=0 \tag{5.6}
\end{equation*}
$$

Then $\int_{V^{c}} \int_{r / n \leq|z|<r} \operatorname{div}_{x}(w(x) p(x, z)) \nu(d z) d x=\int_{r / n \leq|z|<r} 0 \nu(d z)=0$.
Finally, we are to show (5.6). Without loss of generality, we assume the set $\{u \neq 0\}$ has piece-wise $C^{1}$ boundary, otherwise we can approximate $V^{c}$ by sets $\left\{|u|>\varepsilon_{n}\right\}$, $0<\varepsilon_{n} \rightarrow 0$, with $C^{1}$ boundaries. This can be done since by Sard's theorem and the implicit function theorem, $\{|u|>\varepsilon\}$ has $C^{1}$ boundary for a.e. $0<\varepsilon<\max |u|$. More details can be found in the proof of Proposition 3.4 in [18].
Because $u=0$ on both $\partial\{u>0\}$ and $\partial\{u<0\}$, we have

$$
\begin{aligned}
& \int_{\{u \neq 0\}} \operatorname{div}_{x}(w(x) p(x, z)) d x \\
& =\int_{\{u>0\}} \operatorname{div}_{x}(u(x) p(x, z)) d x-\int_{\{u<0\}} \operatorname{div}_{x}(u(x) p(x, z)) d x \\
& =\int_{\partial\{u>0\}} u(x) p(x, z) \vec{n} d S-\int_{\partial\{u<0\}} u(x) p(x, z) \vec{n} d S \\
& =0-0=0,
\end{aligned}
$$

where $\vec{n}$ denotes the outer unit normal vector.
Now the proof of Claim (5.4) is complete.

### 5.2. Dissipativity of $A_{r}+I_{r}$.

Proof of Proposition 3.4 (a). The sum of dissipative operators are in general not necessarily dissipative operators. However in our case, we are able to show that for any $\lambda>0$, there holds the dissipativity inequality

$$
\left\|\left(\lambda-\left(A_{r}+I_{r}\right)\right) u\right\|_{1} \geq \lambda\|u\|_{1} .
$$

Recall the decomposition $I_{r}=S+T$ in Lemma 5.1. We rewrite

$$
\begin{aligned}
& \left\|\left(\lambda-\left(A_{r}+I_{r}\right)\right) u\right\|_{1}:=\int_{\mathbb{R}^{d}}\left|\lambda u-A_{r} u-I_{r} u\right| \\
& =\int_{\mathbb{R}^{d}}\left|\lambda u-A_{r} u-T u-S u\right| \\
& \geq \int_{V^{c}}\left|\lambda u-A_{r} u-T u-S u\right| \\
& \left.=\int_{V^{+}} \mid \lambda u-A_{r} u-T u-S u\right)\left|+\int_{V^{-}}\right| \lambda u-A_{r} u-T u-S u \mid
\end{aligned}
$$

Then we use the relationship (5.2) and yield

$$
\begin{aligned}
& \int_{V^{+}}\left|\lambda u(x)-A_{r} u(x)-T u(x)-S u(x)\right| d x \\
& \quad+\int_{V^{-}}\left|\lambda u(x)-A_{r} u(x)-T u(x)-S u(x)\right| d x \\
& =\int_{V^{+}}\left|\lambda w(x)-A_{r} w(x)-T w(x)-(S w(x)-R(x))\right| d x \\
& \quad+\int_{V^{-}}\left|\lambda(-w)(x)-A_{r}(-w)(x)-T(-w)(x)-(R(x)-S w(x))\right| d x \\
& =\int_{V^{+}}\left|\lambda w(x)-A_{r} w(x)-T w(x)-S w(x)+R(x)\right| d x \\
& \quad+\int_{V^{-}}\left|\lambda w(x)-A_{r} w(x)-T w(x)-S w(x)+R(x)\right| d x
\end{aligned}
$$

So we got the same integrand on $V^{+}$and $V^{-}$, that is

$$
\begin{aligned}
& \left.\int_{V^{+}} \mid \lambda w(x)-A_{r} w(x)-T w(x)-S w(x)+R(x)\right) \mid d x \\
& \left.\quad+\int_{V^{-}} \mid \lambda w(x)-A_{r} w(x)-T w(x)-S w(x)+R(x)\right) \mid d x \\
& =\int_{V^{c}}\left|\lambda w(x)-A_{r} w(x)-T w(x)-S w(x)+R(x)\right| d x \\
& =\int_{V^{c}}\left|\lambda w(x)-A_{r} w(x)-I_{r} w(x)+R(x)\right| d x
\end{aligned}
$$

Keep in mind that $R(x) \geq 0$ for all $x \in V^{c}$, and we can estimate that

$$
\begin{aligned}
& \int_{V^{c}}\left|\lambda w(x)-A_{r} w(x)-I_{r} w(x)+R(x)\right| d x \\
& \geq \int_{V^{c}}\left(\lambda w(x)-A_{r} w(x)-I_{r} w(x)+R(x)\right) d x \\
& \geq \int_{V^{c}}\left(\lambda w(x)-A_{r} w(x)-I_{r} w(x)\right) d x \\
& =\int_{\mathbb{R}^{d}} \lambda w(x)-\int_{V^{c}}\left(A_{r} w(x)+I_{r} w(x)\right) d x \\
& \geq \lambda\|u\|_{1}
\end{aligned}
$$

since $\int_{V^{c}} A_{r} w \leq 0$ from the proof of Proposition 3.4 in [18], and $\int_{V^{c}} I_{r} w \leq 0$ by Lemma 5.2.
5.3. Dissipativity of $A_{r}^{*}+I_{r}^{*}$. We specify the domain of the adjoint operator,

$$
\begin{aligned}
& D\left(A_{r}^{*}+I_{r}^{*}\right):= \\
& \left\{f \in L^{\infty}\left(\mathbb{R}^{d}\right): \exists g \in L^{\infty}\left(\mathbb{R}^{d}\right) \text { such that } \forall u \in D(L), \int_{\mathbb{R}^{d}} g u=\int_{\mathbb{R}^{d}} f\left(A_{r}+I_{r}\right) u\right\} .
\end{aligned}
$$

Then $\left(A_{r}^{*}+I_{r}^{*}\right) f=g$ in the distributional sense.

Proof of Proposition 3.4 (b). Consider an arbitrary function $f \in D\left(A_{r}^{*}+I_{r}^{*}\right)$, it then follows from Condition (E) that $f \in W_{l o c}^{2, p}\left(\mathbb{R}^{d}\right)$ for some $p>d$. Hence we construct $f_{n}(y):=f(y)-\frac{|y|^{2}}{n}=: f(y)-g_{n}(y)$. Then by Lemma 5.9 in [18], there exists a sequence $y_{n}^{\prime} \in \mathbb{R}^{d}$ such that $y_{n}^{\prime}$ is the global maximal point of $f_{n}$ with

$$
\begin{equation*}
\lim _{n}\left|f\left(y_{n}^{\prime}\right)\right|=\|f\|_{\infty}, \lim _{n}\left(1+\left|y_{n}^{\prime}\right|\right)\left|D f\left(y_{n}^{\prime}\right)\right|=0, \text { and } \lim _{n} \frac{\left|y_{n}^{\prime}\right|^{2}}{n}=0 \tag{5.7}
\end{equation*}
$$

Without loss of generality, we require $f\left(y_{n}^{\prime}\right) \geq 0$. Then in view of assumptions (H1) - (H3) and Lemma 4.3, for each $n$ we are able to apply a version of the Bony maximum principle - Proposition 3.1.14 in [26] - in any bounded neighborhood of $y_{n}^{\prime}$ and obtain

$$
\begin{equation*}
\lim _{\rho \rightarrow 0} \operatorname{ess}_{\inf }^{B\left(\rho, y_{n}^{\prime}\right)}\left(-A_{r}^{*}-I_{r}^{*}\right) f_{n}\left(y_{n}^{\prime}\right) \geq 0 \tag{5.8}
\end{equation*}
$$

(Note that Proposition 3.1.14 holds without uniform ellipticity as can easily be seen from its proof given in [29]). We also avoid the points where $D^{2} f$ is not defined and pick, for each $n$, another point $y_{n}$ such that

$$
\begin{equation*}
\left|y_{n}-y_{n}^{\prime}\right|+\left|f\left(y_{n}\right)-f\left(y_{n}^{\prime}\right)\right|+\left|D f\left(y_{n}\right)\right| \leq \frac{1}{n}, \quad\left(-A_{r}^{*}-I_{r}^{*}\right) f_{n}\left(y_{n}\right) \geq-\frac{1}{n} \tag{5.9}
\end{equation*}
$$

And we can always take

$$
\left\{y_{n}\right\} \subset\left\{y \in \mathbb{R}^{d}:\left|\lambda f(y)-A_{r}^{*} f(y)-I_{r}^{*} f(y)\right| \leq\left\|\lambda f-\left(A_{r}^{*}+I_{r}^{*}\right) f\right\|_{\infty}\right\}
$$

because the complement of the latter set has zero Lebesgue measure in $\mathbb{R}^{d}$. Hence

$$
\begin{aligned}
& \left(-A_{r}^{*}-I_{r}^{*}\right) f\left(y_{n}\right) \\
& =\left(-A_{r}^{*}-I_{r}^{*}\right) f_{n}\left(y_{n}\right)+\left(-A_{r}^{*}-I_{r}^{*}\right) g_{n}\left(y_{n}\right) \\
& \geq-\frac{1}{n}-\left|\left(A_{r}^{*}+I_{r}^{*}\right) g_{n}\left(y_{n}\right)\right| \\
& =-\frac{1}{n}-\frac{1}{n}\left|2\left\langle y_{n}, b\left(y_{n}\right)\right\rangle+\sigma^{T}\left(y_{n}\right) \sigma\left(y_{n}\right)+\int_{|z|<r} p^{T}\left(y_{n}, z\right) p\left(y_{n}, z\right) \nu(d z)\right| .
\end{aligned}
$$

By (5.7) and (5.9), (H1) - (H3), the right hand side of above tends to zero. Therefore

$$
\liminf _{n}\left(-A_{r}^{*}-I_{r}^{*}\right) f\left(y_{n}\right) \geq 0
$$

Finally for all $\lambda>0$, there holds

$$
\begin{aligned}
\lambda\|f\|_{\infty} & =\lambda \lim _{n} f\left(y_{n}\right)=\liminf _{n} \lambda f\left(y_{n}\right) \\
& \leq \liminf _{n} \lambda f\left(y_{n}\right)+\liminf _{n}\left(-A_{r}^{*}-I_{r}^{*}\right) f\left(y_{n}\right) \\
& \leq \liminf _{n}\left(\lambda f\left(y_{n}\right)-\left(A_{r}^{*}+I_{r}^{*}\right) f\left(y_{n}\right)\right) \\
& \leq\left\|\left(\lambda-A_{r}^{*}-I_{r}^{*}\right) f\right\|_{\infty}
\end{aligned}
$$

The proof is complete.
Remark 5.3. (a) Maximum principles like (5.8) first appeared in [12] for local operators with $p>d$ and the critical case $p=d$ was treated in [34], and the first treatment of non-local operators is found in [29] with $p(y, z)=z$, and the proof can be easily extended to functions $p(y, z)$ locally bounded in $y$, see Section 3.1 in [26].
(b) The infimum in (5.8) cannot be replaced by supremum. Indeed, let $f(x):=$ $x^{2}\left(\sin \left(\ln \left(x^{2}\right)\right)-2\right) \in W_{\text {loc }}^{2, \infty}(\mathbb{R})$, with global maxima $f(0)=0$. Then $f^{\prime \prime}(x)=$ $6 \cos \left(\ln \left(x^{2}\right)\right)-2 \sin \left(\ln \left(x^{2}\right)\right)-4$ and $\lim _{r \rightarrow 0} \operatorname{ess}_{\inf }^{B(r, 0)} f^{\prime \prime}=-10$ but on the other hand $\lim _{r \rightarrow 0} \operatorname{ess} \sup _{B(r, 0)} f^{\prime \prime}=2$.
5.4. Dissipativity of $J_{r}$. Proposition 3.2 (b) claims that $J_{r}$ is well defined on $L^{1}\left(\mathbb{R}^{d}\right)$, but in general it does not possess an explicit expression since (1.5) may no longer hold.
In view of Section 1.1.4 in [37], for each $u \in L^{1}\left(\mathbb{R}^{d}\right)$, we define its duality set

$$
\mathcal{J}(u):=\left\{f \in L^{\infty}\left(\mathbb{R}^{d}\right):\langle u, f\rangle=\|u\|_{1}^{2}=\|f\|_{\infty}^{2}\right\}
$$

An equivalent definition for dissipativity is that for all $u \in L^{1}\left(\mathbb{R}^{d}\right)$ there exists $f \in$ $\mathcal{J}(u)$ such that $\left\langle J_{r} u, f\right\rangle \leq 0$, where $\langle\cdot, \cdot\rangle$ stands for duality pairing, cf. Definition 1.4.1 and Theorem 1.4.2 in [37].

Proof of Theorem 3.4 (c). For any $u \in L^{1}\left(\mathbb{R}^{d}\right)$, take $f_{u}(x):=\|u\|_{1} \operatorname{sign} u(x)$.
Obviously $f_{u} \in \mathcal{J}(u)$. Moreover

$$
\begin{aligned}
\left\langle J_{r} u, f_{u}\right\rangle & =\left\langle u, J_{r}^{*} f_{u}\right\rangle=\int_{\mathbb{R}^{d}} u J_{r}^{*} f_{u} \\
& =\int_{\mathbb{R}^{d}} u(x)\|u\|_{1} \int_{|z| \geq r}[\operatorname{sign} u(x+p(x, z))-\operatorname{sign} u(x)] \nu(d z) d x \\
& =\|u\|_{1} \int_{\mathbb{R}^{d}} \int_{|z| \geq r}[u(x) \operatorname{sign} u(x+p(x, z))-|u|(x)] \nu(d z) d x \\
& \leq 0
\end{aligned}
$$

since the integrand is always non-positive.

## 6. Elliptic Regularity - proof of Proposition 2.3

The proof of part (a) is similar to the proof of part (b), but easier since the non-local operator can be treated as a lower order perturbation. In this case the proof follows from arguing as in the proof of Theorem 1.5 in [45] and applying Proposition 1.1 in [45] and Lemma 2.3.6 in [26].
By condition (HE) with $s=1$, we know $\int_{|z|<1}|z| \nu(d z)<\infty$. Hence for $C^{1}$ functions the following operators are well-defined.

$$
\begin{aligned}
I_{r}^{*} \phi(x)= & \tilde{I}_{r}^{*} \phi(x)+D \phi(x) \int_{|z|<r} p(x, z) \nu(d z) \\
\tilde{I}_{r}^{*} \phi(x): & =\int_{|z|<r}(\phi(x+p(x, z))-u(x)) \nu(d z) \\
\tilde{I}_{r} \phi(x): & =\left(\tilde{I}_{r}^{*}\right)^{*} \phi(x)=\int_{|z|<r}(\phi(x-q(x, z))-\phi(x)) m(x, z) \nu(d z) \\
& \quad+\phi(x) \int_{|z|<r}(m(x, z)-1) \nu(d z)
\end{aligned}
$$

where $m$ and $q$ are defined in the beginning of Section 4 .

Proof of Proposition 2.3 (a). Fix $R>0$ and let $B_{R}:=\left\{x \in \mathbb{R}^{d}:|x| \leq R\right\}$. By assumptions and (HE) with $s=1$, equation (2.1) is equivalent to

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} \sum_{i, j=1}^{d} f \partial_{i}\left(\frac{1}{2} a_{i j} \partial_{j} u\right)=\int_{\mathbb{R}^{d}}\left[g u+f H-f \tilde{I}_{r} u\right] \tag{6.1}
\end{equation*}
$$

where

$$
\begin{aligned}
H(x):= & \sum_{i=1}^{d} \partial_{i}\left(b_{i}(x) u(x)\right)-\frac{1}{2} \sum_{i, j=1}^{d} \partial_{i}\left(\partial_{j} a_{i j}(x) u(x)\right) \\
& +\int_{|z|<1} \operatorname{div}(u(x) p(x, z)) \nu(d z)
\end{aligned}
$$

With $r<1 /(4 d K)$ as in Section 4.1, one can apply Lemmas 4.2 and 4.6 and verify all assumptions of Lemma 2.3.6 in [26]. Hence for all $1 \leq q \leq \infty$,

$$
\begin{equation*}
\left\|\tilde{I}_{r} u\right\|_{L^{q}\left(B_{R}\right)} \leq C(R)\|u\|_{W^{1, q}\left(B_{R}\right)} \tag{6.2}
\end{equation*}
$$

Next let $q:=\frac{p}{p-1}$ and we argue as in the proof of Theorem 1.5 in [45] to show that $f \in \bigcap_{1<p<\infty} W_{l o c}^{1, p}\left(\mathbb{R}^{d}\right)$.
Let $|h|<R / 6, k=1, \cdots, d$, and $\eta \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ be such that $\operatorname{supp} \eta \subset B_{2 R / 3}$ and $\eta \equiv 1$ in $B_{R / 2}$. Since $f \in L^{\infty}$, the difference quotient

$$
\Delta_{h}^{k} f(x):=\frac{f\left(x+h e_{k}\right)-f(x)}{h} \in L^{p}\left(B_{R}\right)
$$

and hence

$$
\left|\eta \Delta_{h}^{k} f\right|^{p-1} \operatorname{sign}\left(\Delta_{h}^{k} f\right) \in L^{q}\left(B_{R}\right)
$$

By the classical results (see Theorem 9.15 in [28]), there is a unique strong solution in $W_{0}^{1, q}\left(B_{R}\right) \cap W^{2, q}\left(B_{R}\right)$ for the following Dirichlet problem

$$
\frac{1}{2} a_{i j} \partial_{i} \partial_{j} v_{h}=\left|\eta \Delta_{h}^{k} f\right|^{p-1} \operatorname{sign}\left(\Delta_{h}^{k} f\right) \quad \text { in } B_{R}, \quad v_{h}=0 \quad \text { in } \partial B_{R}
$$

such that

$$
\begin{equation*}
\left\|v_{h}\right\|_{W^{2, q}\left(B_{R}\right)} \leq C\left\|\left|\eta \Delta_{h}^{k} f\right|^{p-1}\right\|_{L^{q}\left(B_{R}\right)} \leq C\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{R}\right)}^{p-1} \tag{6.3}
\end{equation*}
$$

By a density argument, (6.1) holds for all elements $u \in W^{2, q}\left(\mathbb{R}^{d}\right)$ with compact support. Hence we choose $u:=\eta \Delta_{-h}^{k} v_{h}$ and follow the arguments in [45] for the local terms to get

$$
\begin{aligned}
\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{R}\right)}^{p} \leq & \frac{1}{4}\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{R}\right)}^{p}+C \\
& +\left\|f \int_{|z|<1} \operatorname{div}_{x}(p(\cdot, z) u) \nu(d z)\right\|_{L^{1}\left(B_{R}\right)}+\left\|f \tilde{I}_{r} u\right\|_{L^{1}\left(B_{R}\right)}
\end{aligned}
$$

By Hölder's and Young's inequalities, (6.3), (H2), and $\int_{|z|<1}|z| \nu(d z)<\infty$, we have

$$
\begin{aligned}
& \int_{B_{R}}\left|f(x) \int_{|z|<1} \operatorname{div}(p(x, z) u(x)) \nu(d z)\right| d x \\
& \leq C \int_{|z|<1}|z| \nu(d z)\|f\|_{L^{p}\left(B_{R}\right)}\left\|v_{h}\right\|_{W^{1, q}\left(B_{R}\right)} \\
& \leq C\|f\|_{L^{p}\left(B_{R}\right)}\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{R}\right)}^{p-1} \\
& \leq \frac{1}{4}\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{R}\right)}^{p}+C .
\end{aligned}
$$

Similarly, with the aid of (6.2),

$$
\begin{aligned}
& \int_{B_{R}}\left|f(x) \tilde{I}_{r} u(x)\right| d x \leq\|f\|_{L^{p}\left(B_{R}\right)}\left\|v_{h}\right\|_{\left.W^{1, q}\left(B_{R}\right)\right)} \\
& \leq C\|f\|_{L^{p}\left(B_{R}\right)}\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{R}\right)}^{p-1} \leq \frac{1}{4}\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{R}\right)}^{p}+C .
\end{aligned}
$$

Combining the above estimates we get that $\left\|\eta \Delta_{h}^{k} v_{h}\right\|_{L^{p}\left(B_{R}\right)}^{p} \leq 4 C$. Thus by definitions of $\eta$ and $v_{h}$,

$$
\left\|\Delta_{h}^{k} f\right\|_{L^{p}\left(B_{R / 2}\right)} \leq 4 C, \text { for all }|h| \in(0,1 / 6)
$$

By the property of difference quotients, (cf. e.g. Theorem 5.8.3 in [24]), we have that the weak derivative $\partial_{k} f$ exists in $B_{R / 2}$ and $\left\|\partial_{k} f\right\|_{L^{p}\left(B_{R / 2}\right)}<\infty$. Since $R>0$ is arbitrary, $f \in \bigcap_{1<p<\infty} W_{\text {loc }}^{1, p}\left(\mathbb{R}^{d}\right)$.
With this extra regularity we can further rearrange (6.1) as

$$
\int_{\mathbb{R}^{d}} f(x) \sum_{i, j=1}^{d} \partial_{i}\left(\frac{1}{2} a_{i j}(x) \partial_{j} u(x)\right) d x=\int_{\mathbb{R}^{d}} u\left[g+G-\tilde{I}_{r}^{*} f\right]
$$

where

$$
G(x):=\sum_{i=1}^{d} \partial_{i} f(x)\left(\frac{1}{2} \sum_{j=1}^{d} \partial_{j} a_{i j}-b_{i}(x)\right)+\int_{|z|<1} D f(x) p(x, z) \nu(d z)
$$

By the regularity of $f,(\mathrm{H} 2)$ and (HE) with $s=1$, and Lemma 2.3.6 in [26] again we observe that $g, G$, and $\tilde{I}_{r}^{*} f \in L_{l o c}^{p}\left(\mathbb{R}^{d}\right)$ for all $1<p<\infty$. Hence by Proposition 1.1 in [45] we have $f \in \bigcap_{1<p<\infty} W_{\text {loc }}^{2, p}\left(\mathbb{R}^{d}\right)$.

Now we will prove part (b), and we start with the following estimate.
Lemma 6.1. Assume (H2) with constant $K>0$. Then

$$
\left|q\left(x_{1}, z\right)-q\left(x_{2}, z\right)\right| \leq 2 K|z|\left|x_{1}-x_{2}\right|, \quad \text { for all } x_{1}, x_{2} \in \mathbb{R}^{d} \text { and }|z|<1 / 2 K
$$

Proof. By definition

$$
\begin{aligned}
& \left|q\left(x_{1}, z\right)-q\left(x_{2}, z\right)\right|=\left|p\left(y\left(x_{1}, z\right), z\right)-p\left(y\left(x_{2}, z\right), z\right)\right| \\
& =\left|p\left(x_{1}-q\left(x_{1}, z\right), z\right)-p\left(x_{2}-q\left(x_{2}, z\right), z\right)\right| \\
& \leq K|z|\left|x_{1}-x_{2}+q\left(x_{2}, z\right)-q\left(x_{1}, z\right)\right| \\
& \leq K|z|\left|x_{1}-x_{2}\right|+\frac{1}{2}\left|q\left(x_{2}, z\right)-q\left(x_{1}, z\right)\right| .
\end{aligned}
$$

Hence $\left|q\left(x_{1}, z\right)-q\left(x_{2}, z\right)\right| \leq 2 K|z|\left|x_{1}-x_{2}\right|$.
Proof of Proposition 2.3 (b). The first two steps below mainly follow the same ideas of proving Theorem 1.5 in [45] and Theorem 1.3 in [44] respectively.
Step I. $W_{l o c}^{1, p}$ regularity for $p \in(1, \infty)$.
Let $R>1$ and $B_{R}(0)$ be the ball of radius $R$ centered at the origin. For any $x_{0} \in B_{R}(0)$, denote $B_{r}$ the ball of radius $r>0$ centered at $x_{0}$.
According to (H2) and Lemma 4.2, for this $R>0$ there exists $r_{R}>0$ such that

$$
\sup _{|x|<R,|z|<r_{R}}|p(x, z)| \leq 1 / 6 \quad \text { and } \quad \sup _{|x|<R,|z|<r_{R}}|q(x, z)| \leq 1 / 6 .
$$

This is the principal part of the operator $I_{r}$. The outer part of $I_{r}$,

$$
\int_{r_{R} \leq|z|<r}[f(x+p(x, z))-f(x)-p(x, z) D f(x)] \nu(d z)
$$

is a bounded operator in $L^{p}\left(\mathbb{R}^{d}\right)$ (cf. Lemma 2.3.6 in [26]) plus a first-order differential operator. So they can be absorbed into other lower order terms. Hence without loss of generality, one can suppose $r_{R}=r$.
Let $p>d$ be so large that the conjugate index $q:=\frac{p}{p-1}<\frac{d}{s-1}$. Let $\eta \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ be a nonnegative truncation function such that $\operatorname{supp} \eta \subset B_{2 / 3}$ and $\eta \equiv 1$ in $B_{1 / 2}$. Then for $|h| \in(0,1 / 6)$ and $k=1, \ldots, d$, difference quotient

$$
\Delta_{h}^{k} f(x):=\frac{f\left(x+h e_{k}\right)-f(x)}{h} \in L^{p}\left(B_{1}\right),
$$

and hence

$$
\left|\eta \Delta_{h}^{k} f\right|^{p-1} \operatorname{sign}\left(\Delta_{h}^{k} f\right) \in L^{q}\left(B_{1}\right)
$$

Since $\int_{|z|<1}|z|^{s} \nu(d z)<\infty$, we can take $\gamma_{1}=s$ to be the boundary order of the operator $I_{r}^{*}$ defined in Theorem 3.1.22 in [26]. By this theorem there exists a unique strong solution of the Dirichlet problem $v_{h} \in W_{0}^{1, q}\left(B_{1}\right) \cap W^{2, q}\left(B_{1}\right)$ :

$$
\begin{equation*}
\left(A_{r}+I_{r}\right) v_{h}=\left|\eta \Delta_{h}^{k} f\right|^{p-1} \operatorname{sign}\left(\Delta_{h}^{k} f\right) \quad \text { in } B_{1}, \quad v_{h}=0 \quad \text { in } B_{1}^{c} \tag{6.4}
\end{equation*}
$$

satisfying the estimate

$$
\begin{equation*}
\left\|v_{h}\right\|_{W^{2, q}\left(B_{1}\right)} \leq C\left\|\left|\eta \Delta_{h}^{k} f\right|^{p-1}\right\|_{L^{q}\left(B_{1}\right)}=C\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{1}\right)}^{p-1} \tag{6.5}
\end{equation*}
$$

Next we denote

$$
I_{r}\left(\eta \Delta_{-h}^{k} v_{h}\right)(x)=\Delta_{-h}^{k}\left(\eta(x) I_{r} v_{h}(x)\right)+R(x, h)
$$

The commutator $R(x, h)$ will be explicitly computed in Step III where we will also establish the following estimate

$$
\begin{equation*}
\|R(\cdot, h)\|_{L^{q}\left(B_{1}\right)} \leq C\left\|v_{h}\right\|_{W^{2, q}\left(B_{1}\right)} \tag{6.6}
\end{equation*}
$$

where $C>0$ is a constant independent of $h$.
We now continue as in the proof of Theorem 1.5 in [45]. First, observe that

$$
\begin{equation*}
\left(A_{r}+I_{r}\right)\left(\eta \Delta_{-h}^{k} v_{h}\right)=\Delta_{-h}^{k}\left(\eta\left(A_{r}+I_{r}\right) v_{h}\right)+\tilde{R}(x, h), \tag{6.7}
\end{equation*}
$$

where the commutator $\tilde{R}(x, h)$, consisting of $R(x, h)$ and commutator of the local terms, belongs to $L_{l o c}^{q}\left(\mathbb{R}^{d}\right)$ satisfying (see [45])

$$
\|\tilde{R}(x, h)\|_{L^{q}\left(B_{1}\right)} \leq C\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{1}\right)}^{p-1}
$$

Since $f \in L_{l o c}^{p}$, it follows from a density arguments that equation (2.1) holds for all functions in $W^{2, q}\left(\mathbb{R}^{d}\right)$ with compact support. Hence we take $u:=\eta v_{h}$ in (2.1), use (6.7) and (6.4) consecutively to obtain

$$
\begin{aligned}
& \int_{B_{1}} g \eta \Delta_{-h}^{k} v_{h}=\int_{B_{1}} f\left(A_{r}+I_{r}\right)\left(\eta \Delta_{-h}^{k} v_{h}\right) \\
& =\int_{B_{1}}\left[\left(\Delta_{h}^{k} f\right) \eta\left(A_{r}+I_{r}\right) v_{h}+f \tilde{R}\right]=\int_{B_{1}}\left[\left|\eta \Delta_{h}^{k} f\right|^{p}+f \tilde{R}\right]
\end{aligned}
$$

By Hölder's inequality, (6.6), (6.5), and Young's inequality, we have further that

$$
\begin{aligned}
\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{1}\right)}^{p} & \leq \int_{B_{1}}\left(|f \tilde{R}|+\left|g \eta \Delta_{-h}^{k} v_{h}\right|\right) \\
& \leq\|f\|_{L^{p}\left(B_{1}\right)}\|\tilde{R}\|_{L^{q}\left(B_{1}\right)}+\|g\|_{L^{p}\left(B_{1}\right)}\left\|\eta \Delta_{-h}^{k} v_{h}\right\|_{L^{q}\left(B_{1}\right)} \\
& \leq C\left(\|f\|_{L^{p}\left(B_{1}\right)}+\|g\|_{L^{p}\left(B_{1}\right)}\right)\left\|v_{h}\right\|_{W^{2, q}\left(B_{1}\right)} \\
& \leq C^{\prime}\left(\|f\|_{L^{p}\left(B_{1}\right)}+\|g\|_{L^{p}\left(B_{1}\right)}\right)\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{1}\right)}^{p-1} \\
& \leq \frac{1}{2}\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{1}\right)}^{p}+C^{\prime \prime} .
\end{aligned}
$$

Therefore $\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{1}\right)}^{p} \leq 2 C^{\prime \prime}$, uniformly for $h \rightarrow 0$. By the property of difference quotient (cf. e.g. Theorem 5.8.3 in [24]), $\partial_{k} f$ exists and belongs to $L^{p}\left(B_{1 / 2}\right)$. Since $k$ and $x_{0} \in B_{R}(0)$ were arbitrarily chosen, we can use the finite covering theorem to conclude that $f \in W^{1, p}\left(B_{R}(0)\right)$.
Step II. $W_{l o c}^{2, p}$ regularity for $p \in\left(1, \frac{d}{s-1}\right)$.
By the previous step, $f$ is now a weak solution of equation (2.1). For a rigorous interpretation of the non-local operator in the weak sense, one can refer to Section 3.2 in [26]. Here we formally rewrite the equation as

$$
\begin{equation*}
\frac{1}{2} \sum_{i, j=1}^{d} \partial_{i}\left(a_{i j} \partial_{j} f\right)+f+I_{r}^{*} f=g+R^{\prime} \tag{6.8}
\end{equation*}
$$

where $R^{\prime}$ is a local term belonging to $L_{l o c}^{p}\left(\mathbb{R}^{d}\right)$ for all $p \in(1, \infty)$. Especially, multiplying on both sides of (6.8) the truncation function $\eta$ defined in the previous step, we observe $v:=\eta f \in W_{0}^{1, p}\left(B_{1}\right)$ is a weak solution of the following equation

$$
\begin{equation*}
\frac{1}{2} \sum_{i, j=1}^{d} \partial_{i}\left(a_{i j} \partial_{j} v\right)+v+I_{r}^{*} v=\eta g+R^{\prime \prime} \tag{6.9}
\end{equation*}
$$

where $R^{\prime \prime}$ consists of local terms, local commutators which are in $L^{p}\left(B_{1}\right)$ for all $p \in(1, \infty)$, and a non-local commutator

$$
\begin{aligned}
& \int_{|z|<r}[(\eta(x+p(x, z))-\eta(x)-p(x, z) D \eta(x)) f(x+p(x, z)) \\
& \quad+p(x, z) D \eta(x)(f(x+p(x, z))-f(x))] \nu(d z)
\end{aligned}
$$

which also belongs to $L^{p}\left(B_{1}\right)$ for all $p \in(1, \infty)$. This commutator is well-defined for $f \in W_{l o c}^{1, p}$ and it is derived by considering a truncation of the Lévy measure by $0<\varepsilon \leq|z|<r$ and passing $\varepsilon$ to $0+$, where the resulting truncated operators are well-defined for functions in $W^{1, p}\left(\mathbb{R}^{d}\right)$ with compact support.
By Theorem 3.2.3 in [26], the weak maximum principle holds for the operator $\frac{1}{2} \sum_{i, j=1}^{d} \partial_{i}\left(a_{i j} \partial_{j} \cdot\right)+i d+I_{r}^{*}$ in (6.9). Moreover $w \equiv 1$ is a weak subsolution (cf. (3.2.24) in Section 3.2 of [26]) for the same operator. Hence Theorem 3.2.5 in [26] can be applied to (6.9) and it guarantees the equation (6.9) has a unique weak solution in $W_{0}^{1, p}\left(B_{1}\right)$ for $p \in(1, d /(s-1))$. Moreover, according to Theorem 3.1.22 in [26], equation (6.9) also admits a strong solution in $W_{0}^{1, p}\left(B_{1}\right) \cap W^{2, p}\left(B_{1}\right)$ for $p \in(1, d /(s-1))$. Therefore $v$ is a strong solution and $f \in W^{2, p}\left(B_{1 / 2}\right)$. With the finite covering theorem, we can show $f \in W^{2, p}\left(B_{R}(0)\right)$.
Step III. Finally, we prove (6.6). For simplicity, we will write $h e_{k}=h$ when there is no ambiguity. After a long but elementary computation we write $R(x, h)=$

$$
\begin{aligned}
& \frac{1}{-h} \int_{|z|<r} \sum_{i=1}^{8} R_{i}(x, z, h) \nu(d z) \text { with } \\
& \sum_{i=1}^{8} R_{i}(x, z, h) \\
& :=(\eta(x)-\eta(x-h)) \\
& \quad \cdot\left[v_{h}(x-h-q(x, z)) m(x, z)-v_{h}(x-h)+D_{x}\left[p(x, z) v_{h}(x-h)\right]\right] \\
& +\left[\eta(x-q(x, z)) m(x, z)-\eta(x)+D_{x}[p(x, z) \eta(x)]\right] \\
& \quad \cdot\left(v_{h}(x-q(x, z)-h)-v_{h}(x-q(x, z))\right) \\
& +p(x) D \eta(x)\left[v_{h}(x-h)-v_{h}(x)-v_{h}(x-q(x, z)-h)+v_{h}(x-q(x, z))\right] \\
& +\eta(x)\left[v_{h}(x-q(x, z))-v_{h}(x-q(x, z)-h)\right]\left(m(x, z)+D_{x} \cdot p(x, z)-1\right) \\
& + \\
& \quad \eta(x-h)\left[v_{h}(x-q(x-h, z)-h)-v_{h}(x-h)+q(x-h, z) D v_{h}(x-h)\right] \\
& +(m(x, z)-m(x-h, z)) \\
& +\eta(x-h) D v_{h}(x) \\
& \quad \cdot[p(x, z)-q(x, z) m(x, z)-(p(x-h, z)-q(x-h, z) m(x-h, z))] \\
& +\eta(x-h) v_{h}(x)\left[m(x, z)+D_{x} \cdot p(x, z)-1-\left(m(x-h, z)+D_{x} \cdot p(x-h, z)-1\right)\right] \\
& +\eta(x-h)\left[v_{h}(x-q(x, z)-h)-v_{h}(x-q(x-h, z)-h)\right. \\
& \left.\quad+q(x, z) D v_{h}(x-h)-q(x-h, z) D v_{h}(x-h)\right] m(x, z) .
\end{aligned}
$$

Recall that $\operatorname{supp} \eta \subset B_{2 / 3},|h| \in(0,1 / 6), \sup _{|x|<R,|z|<r}|p(x, z)| \leq 1 / 6$ and $\sup _{|x|<R,|z|<r}|q(x, z)| \leq 1 / 6$. In particular $x+p(x, z)-h \in B_{1}$ for $x \in B_{2 / 3}$.
We will show that for $i=1, \cdots, 8$,

$$
\begin{equation*}
\left\|\frac{1}{-h} \int_{|z|<r} R_{i}(\cdot, z, h) \nu(d z)\right\|_{L^{q}\left(B_{1}\right)} \leq C\left\|v_{h}\right\|_{W^{2, q}\left(B_{1}\right)} . \tag{6.10}
\end{equation*}
$$

For $R_{1},|\eta(x)-\eta(x-h)| \leq C|h|$ and observe

$$
\begin{aligned}
& v_{h}(x-h-q(x, z)) m(x, z)-v_{h}(x-h)+D_{x}\left[p(x, z) v_{h}(x-h)\right] \\
& =\left(v_{h}(x-h-q(x, z))-v_{h}(x-h)+q(x, z) D v_{h}(x-h)\right) m(x, z) \\
& \quad+D v_{h}(x-h)(p(x, z)-q(x, z) m(x, z)) \\
& \quad+v_{h}(x-h)\left(m(x, z)+D_{x} p(x, z)-1\right)
\end{aligned}
$$

The first term equals

$$
\int_{0}^{1}(1-\theta) q^{T}(x, z)\left(D^{2} v_{h}\right)(x-h-\theta q(x, z)) q(x, z) d \theta m(x, z) .
$$

Then the rest of arguments for $R_{1}$ follows from (6.5) and the proof of Proposition 3.2 (a).

The analysis of $R_{2}$ is the same as for $R_{1}$, with the roles of $\eta$ and $v_{h}$ exchanged.
For $R_{3}$, estimate (6.10) follows from the observation

$$
R_{3}(x, z, h)=h(p(x, z) D \eta(x))^{T} \int_{0}^{1} \int_{0}^{1}\left(D^{2} v_{h}\right)(x-\theta q(x, z)-\xi h) q(x, z) d \xi d \theta
$$

When $i=4$, we obtain (6.10) from the proof of Proposition 3.2 (a) and the fact

$$
v_{h}(x-q(x, z))-v_{h}(x-q(x, z)-h)=\int_{0}^{1} h e_{k}\left(D v_{h}\right)(x-q(x, z)-\theta h) d \theta
$$

For $R_{5}$, it suffices to show $|m(x, z)-m(x-h, z)| \leq C_{R}|z||h|$ for all $|x| \leq R$. Indeed, with the same notations as in Section 4 , denote $M(x, z):=\left(D_{y} p\right)(y(x, z), z)$. Then

$$
\begin{aligned}
& m(x, z)-m(x-h, z) \\
& =\frac{1}{1+\operatorname{tr} M(x, z)+P(x, z)}-\frac{1}{1+\operatorname{tr} M(x-h, z)+P(x-h, z)} \\
& =\frac{\operatorname{tr} M(x-h, z)-\operatorname{tr} M(x, z)+P(x-h, z)-P(x, z)}{(1+\operatorname{tr} M(x, z)+P(x, z))(1+\operatorname{tr} M(x-h, z)+P(x-h, z))}
\end{aligned}
$$

By the global Lipschitz condition on $p(x, z)$ for $x$, the denominator of the last fraction is bounded away from 0 uniformly for all $x \in \mathbb{R}^{d}$ and all $|z|<r$. Moreover by (H2')

$$
\begin{aligned}
& |\operatorname{tr} M(x-h, z)-\operatorname{tr} M(x, z)| \\
& \leq \sum_{k=1}^{d}\left|\left(\partial_{y_{k}} p\right)(y(x-h, z), z)-\left(\partial_{y_{k}} p\right)(y(x, z), z)\right| \\
& =\sum_{k=1}^{d}\left|\left(\partial_{y_{k}} p\right)(x-h-q(x-h, z), z)-\left(\partial_{y_{k}} p\right)(x-q(x, z), z)\right| \\
& \leq C|z||h| .
\end{aligned}
$$

Similarly, we have $|P(x-h, z)-P(x, z)| \leq C|z|^{2}|h|$, since $P(x, z)$ are sum of products of at least two $x$-derivatives of $p(x, z)$ by the definition of the Jacobian.

Now we analyze $R_{6}$. First we compute

$$
\begin{aligned}
& p(x, z)-q(x, z) m(x, z)-(p(x-h, z)-q(x-h, z) m(x-h, z)) \\
& =p(x, z)-q(x, z)+q(x, z)(1-m(x, z)) \\
& \quad-p(x-h, z)-q(x-h, z)+q(x-h, z)(1-m(x-h, z)) \\
& \quad \int_{0}^{1} \int_{0}^{1} D_{x}^{2} p(x-\theta q(x, z)+\xi(h+\theta(q(x, z)-q(x-h, z)))) \\
& \quad \cdot(h+\theta(q(x, z)-q(x-h, z))) q(x, z) d \xi d \theta \\
& \quad+\int_{0}^{1}(q(x, z)-q(x-h, z)) D_{x} p(x-h+\theta q(x-h, z)) d \theta \\
& \quad+(q(x, z)-q(x-h, z))(1-m(x, z)) \\
& \quad+q(x-h, z)(m(x, z)-m(x-h, z))
\end{aligned}
$$

Then according to $\left|D_{x}^{2} p(x, z)\right| \leq C_{R}|z|$ from (H2'), Lemmas 6.1, 4.2 and 4.6, we know the first three terms are bounded by $C_{R}|z|^{2}|h|$ for all $|x| \leq R$.
In view of Lemma (4.2), for the last term, it follows from $|m(x, z)-m(x-h, z)| \leq$ $C_{R}|z||h|$ for all $|x| \leq R$, given in the analysis for $R_{5}$.
Now we turn to $R_{7}$. Note that

$$
\begin{aligned}
& m(x, z)+D_{x} \cdot p(x, z)-1-\left[m(x-h, z)+D_{x} \cdot p(x-h, z)-1\right] \\
& =m(x, z)+\left(\operatorname{div}_{y} p\right)(y(x, z), z)-1-\left[m(x-h, z)+\left(\operatorname{div}_{y} p\right)(y(x-h, z), z)-1\right] \\
& \quad+\left[\operatorname{div}_{x} p(x, z)-\left(\operatorname{div}_{y} p\right)(y(x, z), z)\right]-\left[\operatorname{div}_{x} p(x-h, z)-\left(\operatorname{div}_{y} p\right)(y(x-h, z), z)\right] \\
& =: R_{7}^{\prime}+R_{7}^{\prime \prime}
\end{aligned}
$$

Similar to $R_{6}$, with the estimate $\left|D_{x}^{2} p(x, z)\right| \leq C_{R}|z|$ from (H2'),

$$
R_{7}^{\prime}=\frac{-P+P \operatorname{tr} M+(\operatorname{tr} M)^{2}}{1+\operatorname{tr} M+P}(x, z)-\frac{-P+P \operatorname{tr} M+(\operatorname{tr} M)^{2}}{1+\operatorname{tr} M+P}(x-h, z)
$$

can be estimated in the same manner and it is bounded by $C_{R}|z|^{2}|h|$. To illustrate, we treat one typical term in the above difference.

$$
\begin{aligned}
& \left(\partial_{i} p\right)(y(x-h, z), z)\left(\partial_{j} p\right)(y(x-h, z), z)-\left(\partial_{i} p\right)(y(x, z), z)\left(\partial_{j} p\right)(y(x, z), z) \\
& =\left(\partial_{i} p\right)(y(x-h, z), z)\left(\left(\partial_{j} p\right)(y(x-h, z), z)-\left(\partial_{j} p\right)(y(x, z), z)\right) \\
& \quad+\left(\partial_{j} p\right)(y(x, z), z)\left(\left(\partial_{i} p\right)(y(x-h, z), z)-\left(\partial_{i} p\right)(y(x, z), z)\right)
\end{aligned}
$$

By (H2') and Lemma 4.1,

$$
\left|\left(\partial_{i} p\right)(y(x, z), z)\right| \leq C(1+|y(x, z)|)|z| \leq 2 C(1+|x|)|z| \leq C_{R}|z|
$$

We also observe that

$$
\begin{aligned}
& \left(\partial_{j} p\right)(y(x-h, z), z)-\left(\partial_{j} p\right)(y(x, z), z) \\
& =\int_{0}^{1}\left(h e_{k}-q(x-h, z)+q(x, z)\right) \\
& \quad \cdot\left(D_{x} \partial_{j} p\right)\left(x-q(x, z)-\theta\left(h e_{k}-q(x-h, z)+q(x, z)\right), z\right) d \theta
\end{aligned}
$$

Then the estimate (6.10) will follow from (H2') and Lemma 6.1.

Next, we compute that

$$
\begin{aligned}
R_{7}^{\prime \prime}= & \int_{0}^{1}(q(x-h, z)-q(x, z))\left(D_{x}\left(\operatorname{div}_{x} p\right)\right)(x-\theta q(x, z), z) d \theta \\
& +\int_{0}^{1} \int_{0}^{1} q(x-h, z)(h+\theta(q(x-h, z)-q(x, z))) \\
& \cdot\left(D_{x}^{2}\left(\operatorname{div}_{x} p\right)\right)(x-h-\theta q(x-h, z)+\xi(h+\theta(q(x-h, z)-q(x, z))), z) d \xi
\end{aligned}
$$

By (H2'), we obtain $\left|R_{7}^{\prime \prime}(x, z, h)\right| \leq C|z|^{2}|h|$ for all $|z|<r$.
For $R_{8}$, observe that

$$
\begin{aligned}
R_{8}=\eta(x-h) & \int_{0}^{1} \int_{0}^{1} \theta|q(x, z)-q(x-h, z)|^{2} \\
& \cdot\left(D^{2} v_{h}\right)(x-\xi \theta(q(x, z)-q(x-h, z)-h)) d \xi d \theta m(x, z)
\end{aligned}
$$

Therefore by Lemma 6.1, $\left\|\frac{1}{-h} \int_{|z|<r} R_{8}(\cdot, z, h) \nu(d z)\right\|_{L^{q}\left(B_{1}\right)} \leq C\left\|\eta \Delta_{h}^{k} f\right\|_{L^{p}\left(B_{1}\right)}^{p-1}$ too. The proof is complete.

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