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Theory and Applications of Conservation Laws

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Abstract

This thesis examines the properties, applications and usefulness of the different conservation laws in everyday life from our homes to the industry.

We investigate some mathematical derivations of linear and non-linear partial differential equations which are used as models to solve problems for instance in the applications of oil recovery process were we produce a model to find the amount of water that passes through the production well. Also investigations are made on derivations of the shallow water wave equations in one-dimension in which case emphasis is placed on important assumptions which are used to produce a simplified model which can be solved analytically.

An overview of the different conservation laws are used in order to get the right models or equations. Emphasis was also placed on the different techniques used to solve the characteristic equations depending on the nature and direction of its characteristic speed. There are different ways of finding the solutions of the conservation differential equations but this thesis specifically is concentrated on the two types of solutions, that is, the shock and rarefaction solutions which are obtain at different characteristic speeds. Entropy conditions were also studied in order to get a phyically admissible weak solutions which allows a shock profile.

A numerical scheme was employed to find an approximate solution to the shallow water wave equations. The numerical approach used in the thesis is based on the Lax–Friedrichs scheme which is built on differential equations by difference methods.

Also further work will be needed for instance to look at two or three dimesional shallow water equation as well as it can be extended to look at applications of conservation in a two dimensional or network systems of cars.

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Contents

Chapter 1

Introduction

In the introductory part of the thesis, we will discuss some pertaining areas concerning deriving the Conservation principles, one-dimensional conservation laws, first order Partial Differential Equations (PDEs) and method of characteristics. And later on as we progress, we will point out the main direction of the research. This brief introduction leads us into conserving and tracking some mechanical systems. We shall consider a group of nonlinear partial differential equations like the hyberbolic conservation equation. These equations give a basic understanding of continuum mechanical systems which are very useful in diverse ways such as in the conservation of mass, conservation of momentum and conservation in energy. The hyberbolic conservation laws are differential equations of the form

$$u_t + (f(u))_x = 0, \tag{1.1}$$

which can then be written in Cauchy Riemann form as

$$\frac{\partial}{\partial t}u(x,t) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(u(x,t)) = 0, \text{ such that } u(t=0) = u_0, \tag{1.2}$$

where $u_t = \frac{\partial u}{\partial t}$,$f = (f_1, f_2, \ldots, f_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathrm{R^n}$, and $u(t_0) = u_0$ is the initial condition at $t(0) = t_0 = 0$. Practically t normally denotes the time variable, while x describes the spatial variation in m space dimensions. u is unknown while f is known, a given function which can be vector valued in which case we have a system of equations. This project puts more emphasis on theoretical forms of conservation laws which are applied in flow of oil in a petroleum reservoir, waves breaking at a shore, traffic modelling and in gas dynamics.

1.1 Some Basic concepts of Conservational Principles

Let us consider some quantity that we describe by a density $\varphi(x;t)$. The amount within a given closed region R of space may be expressed by the volume integral,

$$M(t) = \int_R \varphi(x;t)\mathrm{d}V \tag{1.3}$$

Where we can think of the density as the amount per unit volume, sometimes it is also defined as the amount per unit area or the amount per unit length depending on the material medium. We also consider the situation in continuum mechanics were we have a material variable which follows the flow passively and material volume that is, situation were the particle in question is fixed in space. Some of the material variables include mass density, energy density and momentum density.

In connection to motion of the material variable we have the flux of a material which is bascically about transport or flow. Flux is the amount transported through a surface per unit time. The flux has direction and can easily be described as a vector field $\boldsymbol{j}(x;t)$ where $\boldsymbol{j}$ is the transport direction, and $|\boldsymbol{j}|$ is the magnitude of the amount per area and time unit. Let consider a frame and denote $\boldsymbol{n}$ is a normal vector which is defined as the direction of the flux. This indicates that nothing passes through the frame if $\boldsymbol{n}$ is orthogonal to $\boldsymbol{j}$. Typically, we formulate a general equation by considering the amount $\mathrm{d}M$ passing through $\mathrm{d}\sigma$ in time $\mathrm{d}t$, that is,

$$\mathrm{d}M = \boldsymbol{j}\cdot\boldsymbol{n}\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{1.4}$$

The total amount flowing out through a surface Σ in space with normal vector n per unit of time is now given by the surface integral

$$\phi_\Sigma = \frac{\mathrm{d}M}{\mathrm{d}t} = \int_\Sigma \boldsymbol{j}\cdot\boldsymbol{n}\,\mathrm{d}\sigma. \tag{1.5}$$

Using the *Divergence Theorem*, it is possible to rewrite the the surface integrals over a closed surface ∂R of volume R.

$$\int_{\partial R} \boldsymbol{j}\cdot\boldsymbol{n}\,d\sigma = \int_R \nabla\cdot\boldsymbol{j}\,dV. \tag{1.6}$$

Lemma 1.1. *If the density, $\varphi(x,t)$ is a material variable passively following a continuous flow with a velocity vector $\boldsymbol{v}$, then the flux can be written as $\boldsymbol{j} = \varphi\boldsymbol{v}$.*

Proof. Consider the volume $=$ area $\times$ height. That is,

$$dV = \boldsymbol{v}\cdot\boldsymbol{n}\,dt\,d\sigma.$$

Also the amount from (1.4) and (1.5) can be written as $dM = \varphi dV = \varphi\boldsymbol{v}\cdot\boldsymbol{n}\,dt\,d\sigma$. This implies that

$$\phi_\Sigma = \frac{\mathrm{d}M}{\mathrm{d}t} = \int_\Sigma \varphi\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}\sigma, \tag{1.7}$$

Comparing (1.7) with (1.6), we have $\boldsymbol{j} = \varphi \boldsymbol{v}$. □

We also consider the production density given by

$$q(x,t) = \frac{\text{production amount}}{\text{volume} \times \text{time}},$$

where $Q_R(t) = \int_R q(x,t)\,dV$ is the production rate.

Let us consider a material of control volume which is geometrically closed with imaginary region R and boundary ∂R in space. Also consider that the material has density $\rho(x,t)$ moving with a flux $\boldsymbol{j}(x,t)$ and has an outward unit normal, $\boldsymbol{n}$ depending on x and t. Now we can observe that the rate of change of the total amount in R can be written as

$$\frac{\text{change of total amount in R}}{\text{time}} = \frac{\text{net amount entering through } \partial R}{\text{time}} + \frac{\text{production in R}}{\text{time}},$$

however we can write

$$\frac{\text{change of total amount in R}}{\text{time}} = -\text{flux through } \partial R + \frac{\text{production in R}}{\text{time}}.$$

This implies

$$\frac{d}{dt}\int_R \rho(x,t)\,dV = -\int_{\partial R} \boldsymbol{j}(x,t,\rho)\cdot \boldsymbol{n}\,d\sigma + \int_R q(x,t)\,dV, \tag{1.8}$$

which represent the *conservation law* in integral form. Let us assume ρ is continuously differentiable, $\boldsymbol{j}$ and q are continuous and (1.8) is well defined. Then we can move the total derivative with respect to t under the integral sign, and using the divergence theorem we have,

$$\frac{d}{dt}\int_R \rho\,dV = \int_R \frac{\partial \rho}{\partial t}\,dV \tag{1.9}$$

$$\int_{\partial R} \boldsymbol{j}\cdot \boldsymbol{n}\,d\sigma = \int_R \nabla\cdot \boldsymbol{j}\,dV \tag{1.10}$$

Lemma 1.2. *If ρ, $\boldsymbol{j}$ and q are sufficiently smooth functions of x and t for all domain $R \subset R_0$ in which the domain is simply connected then*

$$\frac{\partial \rho}{\partial t} + \nabla\cdot \boldsymbol{j} - q = 0 \tag{1.11}$$

Proof. Let $R \subset R_0$, then $\frac{d}{dt}\int_B \rho dV = -\int_{\partial B} \boldsymbol{j}\cdot \boldsymbol{n} d\sigma + \int_B q dV$ since ρ and $\boldsymbol{j}$ are smooth. Now from (1.9) and (1.10), we have

$$\int_R \left(\frac{\partial \rho}{\partial t} + \nabla\cdot \boldsymbol{j} - q\right) dV = 0. \tag{1.12}$$

Now divide (1.16) by $\int_R dV$ and consider any arbitrary $x \in R_0$, and let $R = B_r(x) \subset R_0$ be an open ball centered on x with radius, r for small values of r ($r \ll 1$). Suppose φ is continuous, then

$$\frac{\int_{B(x,r)} \varphi(y)\, dV}{\int_{B(x,r)} dV} \to \varphi(x) \text{ as } r \to 0.$$

Hence we obtain the conservation law in differential form written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} - q = 0 \qquad \square$$

1.1.1 CONSERVATION LAW IN ONE DIMENSIONAL SPACE

Let us consider the conservation law in one dimensional space and suppose we have a finite line segment on the x-axis of closed interval $[a, b]$. Then we can write

$$\frac{d}{dt}\int_a^b \rho\, dx = -(j(b) - j(a)) + \int_a^b q\, dx. \tag{1.13}$$

Note that the boundary unit vector, $n = -1$ in a and $n = 1$ in b. Let $a = x$, $b = x + \delta x$ and divide by δx sending $\delta x \to 0$ so that we have

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} j(x, t, \rho(x, t)) + q. \tag{1.14}$$

We can simplify the composite functions in (1.14) by using chain rule to get

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial j}{\partial x} + \frac{\partial j}{\partial \rho} \cdot \frac{\partial \rho}{\partial x}\right) + q. \tag{1.15}$$

Hence we obtain the one dimensional space conservation law in differential form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial \rho} \cdot \frac{\partial \rho}{\partial x} = -\frac{\partial j}{\partial x} + q \tag{1.16}$$

METHOD OF CHARACTERISTICS

The method of characteristic propagates information along the characteristic paths. Since (1.16) is a partial differential equation, solving it directly would be difficult but we can solve the equation by transforming the partial differential equation to an ordinary differential equation. Then we solve the ordinary differential equation by the method of characteristics along side with some initial or boundary values which is quite easy to solve. Once we obtain a solution, then we invert or transform the solution to its original form.

Consider a general non-homogeneous differential equation for functions of two variables x and t written as

$$\begin{cases} u_t + a(x, t, u)u_x = b(x, t, u), & t > 0 \\ u(x, 0) = u_0(x), & t = 0. \end{cases} \tag{1.17}$$

Assume $a(x,t,u)$ is non-zero and observe that we can compare (1.16) with (1.17) if we let $u = \rho$, $a = \frac{\partial j}{\partial \rho}$ and $b = -\frac{\partial j}{\partial x} + q$. Equation of this type is said to be *quasi-linear* since the highest derivative occurs linearly. The idea now is to find $(x(t), z(t))$ such that $z(t) = u(x(t),t)$, where $u(x,t)$ is the solution of (1.17). Now differentiating $z(t) = u(x(t),t)$ with respect to t and comparing the equation with equation (1.17) we have that

$$\dot{z} = u_t(x(t),t) + \dot{x}\, u_x(x(t),t) = b(x(t),t,u(x(t),t)) \tag{1.18}$$

So $z(t) = u(x(t),t)$ we get the following ordinary differential equation if

$$\begin{cases} \dot{x} = a(x,t,z), & x(0) = x_0 \\ \dot{z} = b(x,t,z), & z(0) = u(x,0) = u_0(x_0), \end{cases}. \tag{1.19}$$

and this gives us the characteristic equation for (1.17).

1.2 Example

Let us focus on some of the basic examples of hyperbolic conservation laws and how we solve partial differential equation by method of characteristics. A simple example of using conservation law is the one-dimensional non-homogeneous Burgers' equation, which is a simple non-linear-equation given by,

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 2, & \text{for } t > 0, \\ u(x,0) = u_0(x_0). \end{cases} \tag{1.20}$$

The conservation law gives rise to the equation we have above which expresses that u is conserved with flux density given by $\frac{u^2}{2}$. An equation of this form is said to be conservative because it expresses conservation most directly. We shall transform the equation to the one we know already, however this form of transformation will be discussed further in the next chapter. We write (1.20) in the quasilinear form as

$$\begin{aligned} u_t + uu_x = 2, &\qquad \text{for } t > 0, \\ u(x,0) = u_0(x_0), & \end{aligned}$$

we obtain the characteristics equation by letting $z(t) = u(x(t),t)$, $\dot{x} = u$ and

$$\begin{aligned} \dot{x} = z(t), &\qquad x(0) = 0, \\ \dot{z} = 2, &\qquad z(0) = u_0(x_0). \end{aligned} \tag{1.21}$$

Solving the above ordinary differential equation you can check that

$$\begin{aligned} x(t) &= t^2 + u_0(x_0)t + x_0 \\ z(t) &= 2t + u_0(x_0) \end{aligned} \tag{1.22}$$

Hence, $u(x(t), t) = u(t^2 + u_0(x_0)t + x_0, t) = 2t + u_0(x_0)$ solves the differential equation and solutions of this nature is normally a weak solution.
Sometimes solutions of the characteristics equation can be used to find explicit solutions of conservation laws. Problems of these nature will be discussed further. One requires a tool or a method to find the possible weak solution which solves the Riemann problems. In order to proceed, we turn to discuss entropy conditions.

1.3 Entropy Conditions

In the example above, we observe that the conservation laws may have several weak solutions and that some mechanism is required to pick out the right ones. But since the weak solutions may lack uniquenesss, additional assumptions must be imposed to select the (physically) relevant ones [1]. Before we proceed to find the solutions of the Riemann problems, one requires a technique to choose the correct possibly several weak solutions. Therefore, we introduce and discussion entropy conditions. There are different forms of entropy conditions in conservation laws which include the *travelling wave entropy condition*, *Lax entropy condition* and *Kružkov entropy condition*. But we shall discuss the travelling wave entropy condition in this section and later on look at the Lax entropy condition. Lets consider a quasi-linear equation of the form

$$u_t + [f(u)]_x = 0, \tag{1.23}$$

whose solutions, $u = u(x, t)$ and assume that f is sufficiently smooth. We replace (1.23) with

$$u_t^\varepsilon + [f(u^\varepsilon)]_x = \varepsilon\, u_{xx}^\varepsilon, \tag{1.24}$$

as $\varepsilon \to 0$. This equation is known as the *viscous regularization*, where the conservation law (1.23) is replaced by (1.24). Note that the right-hand side of (1.24) models the effect of viscosity or diffusion. The idea is to look for solutions of the conservation law that are limits of the regularized equation when $\varepsilon \to 0$. In order words since (1.24) has some diffusion, the conservation law will represent a limit model when the diffusion is small. Now considering (1.24), we assume ε is positive for the equation to be wellposed [5]. Supposing that (1.23) has a solution of the form

$$u(x,t) = \begin{cases} u_l, & \text{for } x < st, \\ u_r, & \text{for } x \geq st, \end{cases} \tag{1.25}$$

where u_l and u_r are constant states moving with speed s on each discontinuity. Suppose $u(x,t)$ is the pointwise limit almost everywhere of some $u^\varepsilon(x,t) = U((x-st)/\varepsilon)$ as $\varepsilon \to 0$. Then $u(x,t)$ is said to satisfy a *travelling wave entropy condition.* Put $u^\varepsilon(x,t) = U((x-st)/\varepsilon)$ into (1.23) so that

$$\frac{d}{dt}U(\xi) + \frac{d}{dx}f(U(\xi)) = \varepsilon\frac{d}{dx}\left(\frac{d}{dx}U(\xi)\right), \tag{1.26}$$

where $U = U(\xi)$ and $\xi = (x-st)/\varepsilon$. Note that u^ε solves (1.24) in the classical sense, since U is smooth. Simplifying further we have

$$-s\dot{U} + \frac{d}{d\xi}f(U) = \ddot{U}, \tag{1.27}$$

where $\dot{U} = dU/d\xi$. We integrate with respect to ξ to get

$$\dot{U} = -sU + f(u) + C, \tag{1.28}$$

where C is a constant of integration. We can observe that depending on whether $x - st$ is positive or negative, ξ will turn to plus or minus infinity as $\varepsilon \to 0$. That is, if u should be the limit of u^ε then

$$\lim_{\varepsilon\to 0} u^\varepsilon = \lim_{\varepsilon\to 0} U(\xi) = \begin{cases} \lim_{\xi\to-\infty} U(\xi), \\ \lim_{\xi\to+\infty} U(\xi), \end{cases} = u(x,t) \text{ in } (1.25).$$

From the above analysis, it follows that

$$\lim_{\xi\to\pm\infty} \dot{U}(\xi) = 0. \tag{1.29}$$

Now put (1.29) into (1.28) to get the constant of integration C to be

$$C = su_l - f(u_l) = su_r - f(u_r), \tag{1.30}$$

which indeed gives us the Rankine–Hugoniot condition

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l}. \tag{1.31}$$

Therefore, the travelling wave U must satisfy the following boundary value problem:

$$\dot{U} = -s(U - u_r) + (f(U) - f(u_r)), \quad \text{with} \quad U(\pm\infty) = \begin{cases} u_r, \\ u_l. \end{cases} \tag{1.32}$$

From the Rankine–Hugoniot condition, we observe that both u_l and u_r are fixed points for this equation and that any isolated discontinuity satisfies the travelling wave entropy condition if (1.32) holds locally across the discontinuity.

1.4 Outline of the Thesis

Conservation laws play an important role in formulating a mathematical model for many physical systems. Modelling these physical systems follow some basic steps which includes identifying the appropriate conservation laws (example mass, momentum and energy) and their corresponding densities and fluxes. We write their corresponding equations using conservation principles and then close the system of equations by proposing an appropriate relationships between the fluxes and the densities.

Chapter 2 presents a Riemann problem with two main possible ways of solving it, that is, rarefaction wave solution and shock solution. It also emphasises some boundary value problems associated with the Riemann problem.

Chapter 3 investigates a branch of continuum mechanics called fluid mechanics where we study the motion of fluids by finding fundamental equations which can be used to describe its motion. These fundamental equations are built on conservation laws or principles. In chapter 3, we observe the motion of particles in a control volume which can either be described using Eulerian or Lagrangian description. Also, we derive some hyperbolic equations by using the Reynold's transport theorem.

In Chapter 4, we derive the shallow water wave equations for an ideal incompressible and inviscid fluid. Here in this chapter, important assumptions are made together with the conservation of mass and momentum in order to derive the shallow water wave equations. The shallow water equations is a two dimensional non-linear hyperbolic system of equations. It also gives a derivation of a three dimensional system for an ideal gas. The chapter also gives the rarefaction and shock solutions for the shallow water waves equations. A discussion of the entropy condition is used as well as the solutions of the Riemann problem for the shallow water wave equations is obtained. Due to the non-linearity nature of the shallow water wave equations, a numerical scheme is used to solve the problem by modifying the solution of the dam break problem which shares some similar properties and characteristics with the shallow water wave equations. The numerical techniques used to simulate the shallow water flow is the finite difference methods with the Lax–Friedrichs scheme.

Finally, in chapter 5, we apply the theory of conservation laws in oil recovery process. Discusions as well as observations are made in order to explain the mechanism involved in both primary and secondary oil recovery process.

Chapter 6 presents the conclusions of the thesis.

Chapter 2

Riemann problems and boundary conditions

In chapter 1, we focused on the exterior part of the derivation and the use of conservation princples. In this section, we observe some Riemann problems for the conservation law. In this chapter, we show how, subject to certain conditions, there exits a unique solution to the general value problem. We shall organize some well followed and precise methods on how solutions resulting from conservation laws can be constructed and this is the solution of the Riemann problem. However we have observed that conservation laws may have weak solutions and this we shall see it as we progress on the project. We shall consider two main methods of solving the Riemann problem using rarefaction wave and shock solution. We will then explain some initial and boundary value problems and how the structure is laid down.

2.1 The Riemann Problem

In differential formulation for the conservation laws, the Riemann problem is generally of the form

$$\begin{cases} \rho_t + (j(\rho))_x = \rho_t + j'(p)\rho_x = 0, \quad t > 0 \\ \rho(x,0) = \begin{cases} \rho_l, & x < 0 \\ \rho_r, & x > 0. \end{cases} \end{cases} \tag{2.1}$$

That is the Riemann problem is an inital value problem for the conservation law with a piecewise constant initial condition. Letting $z(t) = \rho(x(t),t)$ gives us the characteristic equation

$$\begin{aligned} \dot{x} &= j'(z), \\ \dot{z} &= 0 \end{aligned} \tag{2.2}$$

and this gives the solution

$$\begin{aligned} z &= \begin{cases} \rho_l, & x_0 < 0 \\ \rho_r, & x_0 > 0 \end{cases} \\ x &= x_0 + t \cdot \begin{cases} j'(\rho_l), & x_0 < 0 \\ j'(\rho_r), & x_0 > 0. \end{cases} \end{aligned} \tag{2.3}$$

Note that $\rho_l \neq \rho_r$ and both are constants. Also, $j'(\rho)$ represents the characteristics speed or kinematic speed. There are three cases solutions to the Riemann problem where we want to determine $\rho(x,t)$ for $-\infty < x < \infty$ and $t > 0$ but we will consider and elaborate more on two important ones.

Now let $c(\rho_l)$ and $c(\rho_r)$ be two characteristics speed starting from the origin whose solutions lies within the regions of ρ_l and ρ_r. If $c(\rho_l) = c(\rho_r)$, then the characteristics speed will be parallel for both positive and negative x_0. Then the solution is called a *contact discontinuity*. The discontinuous occur along the contact of two parts,that is, along the characteristic line $x = c(\rho_l)\,t = c(\rho_r)\,t$. But the conservation law is only satisfied if

$$(\rho_l - \rho_r)c(\rho_l) + j(\rho_l) - j(\rho_r) = 0$$

and this doesn't need to be the case even if $c(\rho_l) = c(\rho_r)$. The solution of the conservation law becomes complicated if this extra condition is not met. The equation above is known as the *Rankine–Hugoniot conditon* (refer to discussion page 12).

2.1.1 RAREFACTION WAVE

Here we consider the situation were the characteristics speeds are different from eachother, that is, if $c(\rho_l) < c(\rho_r)$ where $\rho_l < \rho_r$. If $c(\rho)$ is *monotonically increasing* when ρ goes from ρ_l to ρ_r, the solution creates a gap and this is term as a *dead sector*. This dead sector is solved by filling it with a *rarefaction* or *expansion wave*. Here, the characteristics have to start at the origin. Solutions to the rarefaction wave can be generated in the following way by considering solutions of the form:

$$\rho(x,t) = \begin{cases} \rho_l, & x < tj'(\rho_l) \\ \phi(\frac{x}{t}), & tj'(\rho_l) < x < tj'(\rho_r) \\ \rho_r, & x < tj'(\rho_r). \end{cases} \tag{2.4}$$

Now let us put $\rho(x,t) = \phi(\frac{x}{t})$ into equation (2.1) and assume that $\phi \in C^2$ with finitely many inflection points. Since both the equation and the initial data are invariant, it can be transformed that is $x \mapsto kx$ and $t \mapsto kt$, which clearly look for solutions of the form $\rho(x,t) = \phi\left(\frac{x}{t}\right)$. We get the following equation

$$\begin{aligned} \partial_t \phi\left(\frac{x}{t}\right) + \partial_x \phi\left(\frac{x}{t}\right) &= 0, \\ \phi' \cdot \left(-\frac{x}{t^2} + j' \cdot \frac{1}{t}\right) &= 0 \end{aligned} \tag{2.5}$$

and assuming $\phi'\left(\frac{x}{t}\right) \neq 0$, we obtain

$$j'(\phi) = \frac{x}{t}. \tag{2.6}$$

Suppose j' is strictly monotonic, we can invert (2.6) to obtain the solution $\phi = (j^{-1})(\frac{x}{t})$. We then replace j' by a monotonic function on the interval between ρ_l and ρ_r. Here, j develops a lower convex enclosure or envelope for ϕ in the the interval $[\rho_l, \rho_r]$. The lower convex envelope is defined to be the largest convex function that is smaller than or equal to ϕ in the the interval

$[\rho_l, \rho_r]$. Then we solve the evolving equation to get what $\phi\left(\frac{x}{t}\right)$ represents. Consider the following example:

$$\rho_t + \left(\frac{1}{3}\rho^3\right)_x = 0 \tag{2.7}$$

with its intial boundary data

$$\rho(x,0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Solving this by the method characteristics we have the characteristic equation:

$$x = x_0 + t\rho(x_0, 0).$$

The characteristic speed is $c(\rho) = \rho^2$ with

$$c(\rho) = \begin{cases} 0, & \text{for } \rho = 0 \\ 1, & \text{for } \rho = 1. \end{cases}$$

Since the characteristic speed at 0 is smaller than the characteristic speed at 1 that is, $c(\rho = 0) < c(\rho = 1)$. Then the characteristic speeds will form a dead sector and we fill this dead sector or gap with a rarefaction fan or wave. Therefore, from (2.7) we have $j'(\rho) = \rho^2$. This implies that

$$\phi(x,t) = \sqrt{\frac{x}{t}}$$

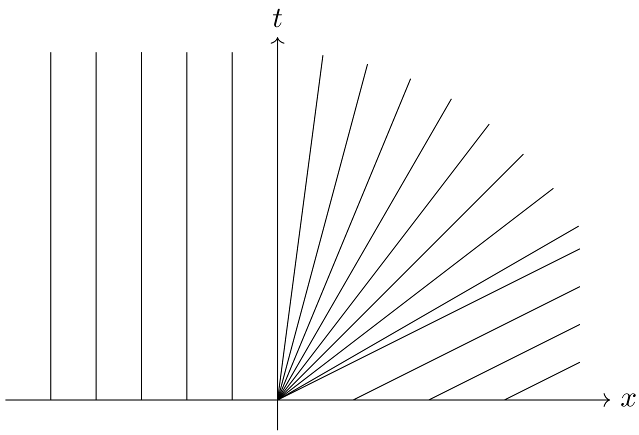


Figure 2.1: Characteristic diagram.

Hence, one can observe from the characteristic diagram, the behaviour and pattern of the characteristic solution and check that the solution is

$$\rho(x,t) = \begin{cases} 0, & x < 0 \\ \sqrt{\frac{x}{t}}, & 0 < x < t \\ 1, & x < t. \end{cases}$$

2.1.2 SHOCK SOLUTION

Consider the case where we have the reverse that is $c(\rho_l) > c(\rho_r)$, the characteristics speed are different. Then the characteristics speed will cross or collide and this is what we term as the shock and the solution obtain from the collision is called the shock solution. Even though the solution outside the collision area can be found using the method of characteristics, it is sometimes of no use in the situation where the characteristic collide. In order to solve the problem, we go back to the original conservation law and we introduce a discontinuity $x = s(t)$, called the shock. To find the shock speed, we consider the interval $[a, b]$ such that it includes the discontinuity. So we consider a control volume enclosing a discontinuity in the density and remember that the change of content in the interval $[a, b]$ during a time interval δt must satisfy the conservation law in integral form. We can show this in the following way:

$$\rho(x,t) = \begin{cases} \rho_l, & x < s(t) \\ \rho_r, & x > s(t). \end{cases} \tag{2.8}$$

Consider conservation law in integral form

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = -\big(j(\rho(b,t)) - j(\rho(a,t))\big). \tag{2.9}$$

Consider the discontinuity $x = s(t)$ and $a \leq s(t) \leq b$. Equivalently,

$$\frac{d}{dt}\left(\int_a^{s(t)} \rho_l dx + \int_{s(t)}^b \rho_r dx\right) = j(\rho_l) - j(\rho_r).$$

This implies

$$\frac{d}{dt}\big((s(t) - a)\rho_l - (b - s(t))\rho_r\big) = j(\rho_l) - j(\rho_r).$$

Simplifying the above equation, we obtain the *Rankine–Hugoniot condition.* That is,

$$\dot{s}(t)(\rho_l - \rho_r) = j(\rho_l) - j(\rho_r). \tag{2.10}$$

Now to find $s(t)$ we solve the following equation:

$$\dot{s}(t) = \frac{j(\rho_l) - j(\rho_r)}{\rho_l - \rho_r}. \tag{2.11}$$

Note that here $s(0) = s_0 = 0$.

Let us consider a classical example of how we use the shock solution:

$$\rho_t + \rho\rho_x = \rho_t + \frac{1}{2}(\rho^2)_x = 0. \tag{2.12}$$

With the initial data

$$\rho(x,0) = \rho_0(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0. \end{cases}$$

Solving this by the method of characteristics we have the characteristic equation:

$$x = x_0 + t\rho(x_0, 0). \tag{2.13}$$

The characteristic speed is $c(\rho) = \rho$ and

$$c(\rho) = \begin{cases} 1, & \text{for } \rho = 1 \\ 0, & \text{for } \rho = 0. \end{cases}$$

We can observe that the characteristic speed at 1 is greater than the characteristic speed at 0, that is, $c(\rho = 1) > c(\rho = 0)$. When this happens, the characteristic speeds will cross or collide with each other and this problem is then solved with a shock solution. That is, from (2.12) we solve the problem using:

$$\dot{s}(t) = \frac{\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2}{1 - 0} = \frac{1}{2}.$$

This implies that $s(t) = \frac{1}{2}t$ where $s(0) = 0$.

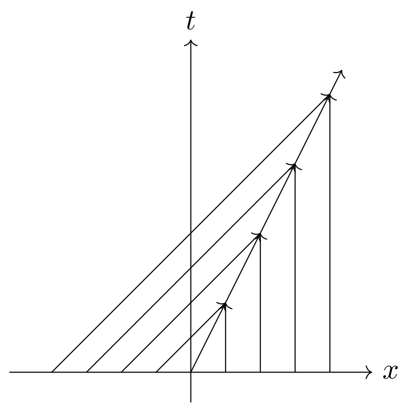


Figure 2.2: Characteristic diagram.

Hence, one can observe from the characteristic diagram, the behaviour and pattern of the characteristic solution and check that the shock solution is of the form

$$\rho(x,t) = \begin{cases} 1, & x < \frac{1}{2}t \\ 0, & x > \frac{1}{2}t. \end{cases} \tag{2.14}$$

2.2 Boundary value problems

In the begining part of the chapter you might suspect how boundary condition for hyperbolic problems are closely related to the theory of characteristics. We have previously looked at some of the problems and also discussed certain instances were we have been observing some boundary conditions attached to the differential equations but mostly we used the boundary condition to aid in finding an absolute solution as seen in the examples above. Let us consider the boundary value problem, that is;

$$\begin{cases} f_t + (j(f))_x = 0, & x > a,\, t > 0 \\ f(a,t) = b, & x = a,\, t > 0 \\ f(x,0) = f_0(x), & x > a,\, t = 0. \end{cases} \tag{2.15}$$

The above equation gives us an initial condition as well as a boundary within which we can obtain or solve the problem. Here the characteristics start on the boundary and the characteristics equation is of the form:

$$\begin{aligned} z(t) &= f(x(t),t) \\ \dot{x}(t) &= j'(z) = c(z) \\ \dot{z}(t) &= 0. \end{aligned} \tag{2.16}$$

And the initial condition is of the form:

$$x(t_0) = x_0; z(t_0) = f(x(t_0),t_0) = f(x_0,t_0). \tag{2.17}$$

By solving the differential eqaution in (2.16) we obtain the following solution

$$\begin{aligned} x =& x_0 + (t-t_0)c(f(x_0,t_0)) \\ z =& f(x_0,t_0). \end{aligned} \tag{2.18}$$

From (2.15) and (2.18) we can see that the solution can be written in two cases which is of the form:

$$\begin{aligned} &x_0 > a,\;\; t_0 = 0, \text{ where } f(x_0,t_0) = f_0(x_0) \\ &x = x_0 + tc(f(x_0)) \\ &z = f(x_0) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} x_0 &= a,\ \ t_0 > 0, \text{ where } f(x_0, t_0) = b \\ x &= x_0 + t\, c(b) \\ z &= b. \end{aligned} \tag{2.20}$$

Here, the boundary condition leads to two possibilities in which we can have the solution, that is, inflow and outflow of boundary.

(i) When the characteristics speed at b is greater than or equal to 0, that is $c(b) = j'(b) \geq 0$ we obtain what is called inflow. This is where the characteristic's at $x = a$ go into the domain $x \geq a$. When this occurs, the solutions are obtained by doing the usual method of characteristics to obtain a shock or rarefaction.

(ii) When the characteristics speed at b is less than 0, that is $c(b) = j'(b) \ < 0$ we obtain what is called outflow. This is where the characteristic's at $x = a$ leaves the domain $x \geq a$. When this occurs, the solution does not see boundary condition at $x = a$, hence the boundary condition is ignored at this stage. As usual we obtain the solution by using the method of characteristics to get a shock or rarefaction solution.

The boundary condition normally posed here are Dirichlet boundary condition which we have seen mostly, however in certain instances we are given the flux condition but it is still possible to solve the problem. Consider the equation

$$\begin{cases} h_t + (j(h))_x = 0, & x > a,\, t > 0, \\ j(h) = 0, & x = a,\, t > 0, \\ h(x, 0) = h_0(x), & x > a,\, t = 0. \end{cases} \tag{2.21}$$

And the idea is to convert (2.21) to Dirichlet boundary condition by solving

$$j(h) = 0 \tag{2.22}$$

for h. But if h in (2.22) has many solutions, we select only solutions giving inflow at the boundary. Now let us consider the case where we apply the knowledge of boundary conditions.
Suppose we have a traffic light at $x = 0$ where the light is green when $t < 0$ and it is red when $t \geq 0$. Let us assume that for $t \leq 0$, $\rho(x, t) = \frac{1}{2}$. Also consider the differential equation which models the density of cars as a function of x and t in the form

$$\rho_t + \frac{\partial}{\partial x} j(\rho) = 0, \tag{2.23}$$

where the car flux is $j(\rho) = \rho(1 - \rho)$ and $\rho(x(t), t)$ is the density of cars. We have two cases where we obtain our solution.
Case 1 for $t > 0$
Now note that when the traffic light is red, it implies there is no flux at $x = 0$ and the same thing applies to the case when the traffic light is green. Here two boundary value problems arises

for $x < 0$ and $x > 0$. At the boundary condition we have

$$j(\rho) = \rho(1 - \rho) = 0 \tag{2.24}$$

which implies that $\rho = 0$ or $\rho = 1$. We can now choose our inflow values by finding the kinematic speed and check whether it gives an inflow or outflow boundaries. The kinematics speed is

$$c(\rho) = j'(\rho) = 1 - 2\rho = \begin{cases} 1 & \text{for } \rho = 0 \\ -1 & \text{for } \rho = 1 \end{cases} \tag{2.25}$$

and this implies that

$$\begin{aligned} &\rho = 1 \text{ gives inflow boundaries in } x < 0, \\ &\rho = 0 \text{ gives outflow boundaries in } x < 0, \\ &\rho = 0 \text{ gives inflow boundaries in } x > 0 \text{ and} \\ &\rho = 1 \text{ gives outflow boundaries in } x > 0. \end{aligned}$$

Note that this is done properly by some reasonable choices. Let us solve case 1 by rewriting (2.23) as

$$\begin{cases} \rho_t + (j(\rho))_x = 0, & x < 0,\, t > 0 \\ \rho = 1, & x = 0,\, t > 0 \\ \rho = \frac{1}{2}, & x < 0,\, t = 0. \end{cases} \tag{2.26}$$

By using the method of characteristics and (2.25) we obtain,

$$x(t) = x_0 + (t - t_0)\, c\,(\rho(x_0, t_0)),$$

which implies that

$$x(t) = \begin{cases} x_0, & \text{for } x_0 < 0,\, t_0 = 0 \\ t - t_0, & \text{for } x_0 = 0,\, t_0 > 0. \end{cases} \tag{2.27}$$

Here you can check that the characteristics collide hence we get a shock solution. By using Rankine-Hugonoit condition, we have

$$\dot{s}(t) = \frac{j(\frac{1}{2}) - j(1)}{\frac{1}{2} - 1} = -\frac{1}{2}.$$

This implies that $s(t) = -\frac{1}{2}t$ for $s(0) = 0$, we obtain the solution for the car density as:

$$\rho(x,t) = \begin{cases} 1, & \text{for } -\frac{1}{2}t < x < 0 \\ \frac{1}{2}, & \text{for } x < -\frac{1}{2}t. \end{cases} \tag{2.28}$$

Case 2 for $t > 0$

In this situation we consider the case were the boundary values arises when $x > 0$. Now finding the solution is similar to what we did earlier on, so we can verify for

$$\begin{cases} \rho_t + (j(\rho))_x = 0, & \text{for } x > 0,\, t > 0 \\ \rho = 0, & \text{for } x = 0,\, t > 0 \\ \rho = \frac{1}{2}, & \text{for } x > 0,\, t = 0, \end{cases}$$

the solution is of the form:

$$\rho(x,t) = \begin{cases} 1, & \text{for } 0 < x < \frac{1}{2}t \\ \frac{1}{2}, & \text{for } x > \frac{1}{2}t. \end{cases} \tag{2.29}$$

Chapter 3

Conservation laws of continuum mechanics in hyperbolic systems

Continuum mechanics is a section of mechanics in which we study and analyze the kinematics and mechanical behaviour of materials. Continuum mechanics are modelled as a continuous mass which deals with the physical properties of solids and liquids (fluids). This physical properties however are independent of any particular coordinate in which they are observed. But in this chapter we will focus more on properties and conservation equations on fluids such as the the shallow water wave equations.

3.1 Motivation

Fluid mechanics is a branch of continuum mecahanics in which we study the movement of fluids. Most aspects of fluid mechanics can be used to describe and model a large number of complex systems such as river flooding, dam breaks and movements of gases through pipeline. The fundamental equations governing fluid in motion are built on the principles of conservation law of mass, momentum and energy. Many variations of these equations can be constructed from different kinds of assumptions regarding the nature of the fluid, such as its viscosity, compressibility, the dimensions and properties of the domain in which the fluid is situated. However in practice, it is highly necessary to make limiting assumptions, that are acceptable for a particular application, in order to obtain approximate analytical and numerical solutions which describes the flow.

3.2 Eulerian and Lagrangian Description

Most of the things surrounding us are made of anything that has mass and can occupy space (has volume), that is the theory of matter. Anything that is made of matter has four states namely solid, liquid, gas and plasma; these states and their characteristics are explained by the laws of physics. The building blocks behind these states of matter is acknowledged in continuum and fluid mechanics. These laws of physics may measure materials on the move or stationary but however in the beginning chapter we saw that materials like liquid and gases, where the continuum is always moving are considered as a material variable while those that are fixed or measured based on their stationary position are said to be control volume. The control volume contains different mass particles at different times in a fixed space. Transport in continuous medium in a fixed position can be described in the *Eulerian* way by considering the flow velocity $v = v(x,t)$ at each point x. Let R_0 be a control volume, then the conservation of density, φ in R_0 is given by

$$\frac{d}{dt}\int_{R_0} \varphi(x,t)\,dV + \int_{\partial R_0} \varphi(x,t)\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma = \int_{R_0} q(x,t)\,dV. \tag{3.1}$$

While the material is said to have a *Lagrangian* description if the coordinates follow the mass particles with respect to time, with velocity defined as: $\dot{x}(t) = v(x(t),t)$. Note that it is much easier to use the Lagrangian form as compared to Eulerian form since we take measurement of the material medium as we are on the move as seen in Newton's laws of motion and in relative motion of two objects on the move. Whenever a portion of the material medium follows the mass in $R(0)$ at any time this forms a *material region*, $R(t)$. Using the flow of a differential equation or dynamical system, let $\dot{x} = v(x,t)$ so that $x(t_0) = x_0$ and consider $x(t) = \Phi(x_0,t_0,t)$. Then $\Phi(x_0,t_0,t)$ satisfies

$$\frac{\partial}{\partial t}\Phi(x_0,t_0,t) = v(\Phi(x_0,t_0,t),t). \tag{3.2}$$

Then we can write the material region as $R(t) = \{\Phi(x_0,0,t),\ x_0 \in \mathbb{R}\}$ which we can also be defined as $R(t) = \{x(t) : \dot{x}(t) = v(x(t),t),\ x(0) = x_0 \text{ for all } x_0 \in R(0)\}$. The conservation of density, $\varphi(x,t)$ as a material variable in $R(t)$ can be written as

$$\frac{d}{dt}\int_{R(t)} \varphi(x,t)\,dV + 0 = \int_{R(t)} q(x,t)\,dV. \tag{3.3}$$

The material region changes its orientation in space as time elapses coinciding with the control volume at one instant of time. In this chapter we shall observe some important conservation laws in continuum and fluid mechanics such as conservation of mass, conservation of momentum, conservation of energy and Newtonian fluids, and how useful they can be applied in mechanics. We will also see some conversions, analysis and derivation of important hyperbolic equations in fluid mechanics. These hyperbolic equations can be derived by using *Reynold's Transport Theorem.*

3.2.1 Reynold's Transport Theorem

From the begining chapter we saw that the Divergence theorem was used to derive the conservation of density in Eulerian and Lagrangian form. This however forms a family in which the Reynold's Transport Theorem can be derived. The Reynold's Transport Theorem uses the divergence theorem to combine both the Eulerian and Lagrangian form. Let us consider a one-dimensional case by looking at the theorem below.

Theorem 3.1. *In one-dimension case, suppose* $F(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$ *where* $f(x,t)$ *is smooth and continous, then we can write*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx\bigg|_{t=0} = \frac{d}{dt}\int_{a(0)}^{b(0)} f(x,t)\,dx\bigg|_{t=0} + f(b(0),0)\,\frac{\partial b}{\partial t}(0) - f(a(0),0)\,\frac{\partial a}{\partial t}(0) \tag{3.4}$$

This is a simple result which can proved using the Fundamental Theorem of Calculus (FTC) and the Chain Rule.

Theorem 3.2. *In an* n *multi-dimensional case for* $n > 1$*, suppose that the region,* $R(t)$ *as defined earlier in this chapter as* $R(t) = \{\boldsymbol{x}(t) : \dot{\boldsymbol{x}}(t) = v(\boldsymbol{x}(t),t), \boldsymbol{x}(0) = \boldsymbol{x}_0 \text{ for all } \boldsymbol{x}_0 \in R(0)\}$ *is enclosed by a moving boundary,* $\partial R(t)$ *is a nice region. Assume also that all the points on the boundary are marked so that we can track them as time elapses and also as the points moves along the boundary at any time,t, the material region will have a uniform velocity* $v(\boldsymbol{x}(t),t)$*, where* $\boldsymbol{x}(t) \in \partial R(t)$ *which is assumed to be sufficiently nice function. Then we may formulate Reynolds transport theorem for integrals of a function* $f(\boldsymbol{x}(t),t)$*(assume the function is nice and smooth) over the moving region* $R(t)$ *as follows:*

$$\frac{d}{dt}\int_{R(t)} f(\boldsymbol{x},t)\,dV\bigg|_{t=0} = \frac{d}{dt}\int_{R(0)} f(\boldsymbol{x},t)\,dV\bigg|_{t=0} + \int_{\partial R_0} f(\boldsymbol{x},0)\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma. \tag{3.5}$$

3.2.2 Mass conservation

Remember in the beginning part of the introduction we saw some conservation principles being the main projection in continuum mechanic; conservation of mass principle.
Here in this chapter we combine the conservation of mass with the Reynold's transport theorem. Now let us replace $f(\boldsymbol{x}(t),t) = \rho(\boldsymbol{x}(t),t)$ and without sources and sinks, the mass within a material region $R(t)$ will be a constant. This is written as

$$\frac{d}{dt}\int_{R(t)} \rho(\boldsymbol{x}(t),t)\,dV = \frac{d}{dt}\int_{R(0)} \rho(\boldsymbol{x},t)\,dV\bigg|_{t=0} + \int_{\partial R(0)} \rho(\boldsymbol{x},0)\boldsymbol{v}(\boldsymbol{x},0)\cdot\boldsymbol{n}(\boldsymbol{x},0)\,d\sigma = 0. \tag{3.6}$$

Now for any time, t and let us fix any arbitrary control volume, R then we can write

$$\frac{d}{dt}\int_R \rho\,dV + \int_{\partial R} \rho\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma = 0. \tag{3.7}$$

This is the conservation law in *integral form* when there is no production (that is, there is no sources or sinks) within the region, R. Suppose however there exists a production $q(\boldsymbol{x}, t)$, then we have the usual conservation of mass written as:

$$\frac{d}{dt}\int_R \rho\, dV + \int_{\partial R} \rho \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \int_R q(\boldsymbol{x}, t)\, dV. \tag{3.8}$$

In differential form we have $\rho_t + \bigtriangledown \cdot (\rho \boldsymbol{v}) = q$.
Recall for a stationary flow where ρ, $\boldsymbol{v}$ and q are independent of time, t. Then equivalently $\int_\partial \rho \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \int_R q(\boldsymbol{x}, t)\, dV \iff \bigtriangledown \cdot (\rho \boldsymbol{v}) = q$. Also for an incompressible flow: $\rho = constant$, then equivalently $\nabla \cdot \boldsymbol{v} = 0$ where $q = 0$.

3.2.3 Conservation of momentum

Momentum in physics is simply the product of a body's mass by it's velocity. This is very important in continuum mechanics. In the introduction we saw different kinds of densities and one of them was *momentum density*. In continuum mechanics, we define *momentum density* as momentum per unit volume, that is, $p = \rho \boldsymbol{v}$. By conservation principles, the conservation of momentum yields production and this is caused as a result of Newton's Second Law of Motion, stating that; the rate of change of momentum is directly proportional to the force applied and takes place in the direction of the force simply put, force is the product of a body's mass times it's acceleration with respect to space and time. Let us consider a material region, $R(t)$ containing a fixed collection of mass particle. Now applying Newton's Second Law to the material region, $R(t)$ we have

$$\text{change of momentum of body mass} = \sum \text{forces applied to the body mass}$$

Symbolically, this means that:

$$\frac{d}{dt}\int_{R(t)} \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)\, dV = \sum \boldsymbol{F}(t) \tag{3.9}$$

where $\sum \boldsymbol{F}(t)$ is the total force acting on a material region. Now applying Reynold's transport theorem at time, $t = 0$ (combination of Lagrangian form, $R(t)$ and Eulerian form, $R(0)$) and considering a fixed control volume, R the conservation of momentum can be written as

$$\frac{d}{dt}\int_R \rho \boldsymbol{v}\, dV + \int_{\partial R} (\rho \boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{n})\, d\sigma = \sum \boldsymbol{F}$$

Forces

However, the forces acting on the mass can be differentiated between *body forces* (also known as *mass forces*) and surface forces.

(i) The body forces in general can be written as

$$\boldsymbol{F}_b(t) = \int_R \boldsymbol{f}_b(\boldsymbol{x}, t)\, dV.$$

Let us take a look at some common body forces:

a) Gravitational force written in integral form as

$$\boldsymbol{F}_g(t) = -\int_R \rho(\boldsymbol{x}, t)\boldsymbol{g}\, dV.$$

b) Electromagnetic forces

c) Coriolis force: suppose a material region, R is fixed rotation coordinate system (for instance, the atmosphere over Norway), this force arise as a result of the control volume being fixed on the earth's atmospere and thus it is rotating with the earth. The Coriolis force is given by

$$\boldsymbol{F}_c = \int_R \rho(\boldsymbol{\Omega} \times \boldsymbol{v})\, dV.$$

Another form of rotational force is the $centripetal\ force$ which is caused by a body going through circular motion. The centripetal force cause a body to move in a circular path under a fixed control volume, R. The centripetal force is given by

$$\boldsymbol{F}_{cp} = \int_R \rho(-\Omega \times (\Omega \times \boldsymbol{r}))\, dV,$$

where Ω is the angular velocity and $\boldsymbol{r}$ is the radius of the circular path from the centre of the object.

(ii) Another type of force is the surface or contact forces which in general can be written as

$$\boldsymbol{F}_s(t) = \int_R \boldsymbol{f}_s(\boldsymbol{x}, t) d\sigma.$$

Let us consider some surface forces:

a) Normal (Pressure) force: The Pressure force acts in the opposite direction of the normal force applied to it.

b) Tangential ($shear\ stress$): Shear stress is simply defined as the force applied to a body per unit area. It is a force that acts along a surface or acts perpendicularly on a surface, examples are the $viscous\ drag$ which is caused whenever a spherical ball is vertically fired into a viscous fluid and the shear deformation caused when a solid cylindrical container sinks into the sea depth as a result of low pressure stored inside the solid container, the solid gets crushed or deformed as the pressure at the bottom part of the sea increases. Now the force acting on a surface, $d\sigma$ can be given by

$$d\boldsymbol{F}_s = \boldsymbol{T} \cdot \boldsymbol{n}\, d\sigma,$$

where $\boldsymbol{n}$ is the outward unit normal and $\boldsymbol{T}$ is the stress tensor (in mechanics the stress tensor is represented by a symmetric 3×3 matrix) and the momentum conservation law can be written in integral form as

$$\frac{d}{dt}\int_R \rho\boldsymbol{v}\, dV + \int_{\partial R} (\rho\boldsymbol{v})\boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \int_R \rho\boldsymbol{f}_b dV + \int_{\partial R} \boldsymbol{T} \cdot \boldsymbol{n}\, d\sigma. \tag{3.10}$$

The equation above can be written in $\mathbb{R}^3$ as

$$\frac{d}{dt}\int_R \rho v_i\, dV + \int_{\partial R} (\rho v_i)\boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \int_R \rho f_{b,i}\, dV + \int_{\partial R} \boldsymbol{t}_i \cdot \boldsymbol{n}\, d\sigma \quad \text{for } i = 1, 2, 3 \tag{3.11}$$

By using divergence theorem, we can rewrite (3.11) in differential form as

$$\frac{\partial}{\partial t}(\rho v_i) + \nabla \cdot (\rho v_i \boldsymbol{v}) = f_{b,i} + \nabla \cdot \boldsymbol{t}_i \quad \text{for } i = 1, 2, 3, \tag{3.12}$$

where $\boldsymbol{t}_i$ is the i'th row of T.

3.2.4 Conservation of energy

Energy is simply the ability to do work. In mechanics, energy is one of the basic quantitative properties which describe a physical system or state of an object. Energy can be converted, transferred or transformed among a number of forms that may exists and can be measured in different forms. The energy of a physical system can be increased or decreased when it is transformed into or out of a system. The internal energy stored in a physical system is called *specific energy*. The specific energy of a system is the amount of energy per unit mass and it sometimes referred as energy density. The total specific energy can be calculated by simple addition when it is composed of multiple non-interacting parts or has a multiple distinct forms of energy. The forms of energies associated with the internal energy of a system includes kinetic energy of a moving object, potential energy of an object by virtue of its position or orientation such as gravitation and radiant energy carried by light, chemical energy and other electromagnetic radiation. Mathematically, the total energy of a system with respect to time, t is given by $E(t) = \int_{R(t)} e\rho \, dV$, where $e = e(x,t)$ is the energy function in space, x and time, t which combines the kinetic energy and internal energy of a system. The specific energy, $e = \frac{1}{2}|\boldsymbol{v}|^2 + u$ and note that the specific energy is a material variable. By the first law of thermodynamics, the time rate of total energy of a system is given by

$$\frac{dE(t)}{dt} = \frac{d}{dt}\int_{R(t)} e\rho \, dV = \frac{dQ}{dt} - \frac{dW}{dt} \tag{3.13}$$

where $\frac{dQ}{dt}$ is the added heat and $\frac{dW}{dt}$ is the work done by the system. We will consider some work done by the physical system in different forms per unit time; this gives us *power* which by definition is the amount of energy used or consumed in a system per unit time. Let consider

(a) Work against the mass (body) forces:

$$\frac{dW_b}{dt} = -\int_R f_b \cdot \boldsymbol{v} \, dV.$$

(b) Work against the surface (contact) forces:

$$\frac{dW_s}{dt} = -\int_{\partial R} (\boldsymbol{T} \cdot \boldsymbol{n}) \cdot \boldsymbol{v} \, dV.$$

(c) Other work performed by the system: $\frac{dW_t}{dt}$

The conservation of energy gives us:

$$\frac{d}{dt}\int_R e\rho \, dV + \int_{\partial R} e\rho\boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma = \frac{dQ}{dt} - \frac{dW_t}{dt} + \int_R \boldsymbol{f}_b \cdot \boldsymbol{v} \, dV + \int_{\partial R} (\boldsymbol{T} \cdot \boldsymbol{n}) \cdot \boldsymbol{v} \, dV. \tag{3.14}$$

Note that we can write the conservation of energy in differential form using Divergence Theroem, provided that the nice and smoothness conditions are fulfill and we have seen a couple of how the differential forms are derived from the integral form from the beginning of this chapter.

Chapter 4

Shallow Water Equations

Let us consider a system of differential equation which are made up of vectors in $\mathbb{R}^n$ given by

$$u_t + f(u)_x = 0, \tag{4.1}$$

where $u = u(x,t) = (u_1, u_2, ..., u_n) \in C^2$ is the solution of the differential equation in $\mathbb{R}^n$ and $f = f(u) = (f_1, f_2, ..., f_n) \in C^2$ is the rate of flow of a material per unit area which is known as flux in $\mathbb{R}^n$. We will consider some basic concepts for systems: fundamental characteristics of shock waves and rarefaction waves. We also look at situations or conditions which helps in selecting the right solutions for a given system differential equations. The above results will not only give us the required solution but it will help us to know that a system of differential equation is well-posed for the Riemann problem for systems of hyperbolic conservation laws with little or no variation in the initial data. To start dealing with the fundamental properties of systems of hyperbolic conservation laws, we begin by discussing the shallow water wave equations. The shallow water wave equations are used to represent the flow of water in a region in which the horizontal dimension of the water body, say the wave length, greatly exceeds the depth. These equations arises from the basic equations of fluid mechanics for an inviscid and incompressible fluid. Generally a two-dimensional approximation of the flow is used which assumes negligible vertical acceleration and a hydrostatic pressure distribution under which pressure increases linearly with depth. If the flow is characterised by some vertical acceleration then it is necessary to relax the hydrostatic pressure assumption. If a non-hydrostatic distribution is assumed, variations of the shallow water wave equations changes to the well-known *"Boussinesq equations"* [7]. The Shallow water flows are characterized by flow regions with a free surface, an impermeable bottom topography, such as a sea floor, and horizontal velocity that dominates the flow field. The flow may be generated by gravitational forces associated with sloped beds, wind that creates stress on the free surface of the fluid or an applied pressure gradient, such as tidal influence which causes a disturbance in the impermeable bottom, say from an earthquake. The model for shallow water waves is a bit similar to that of flood waves water equation caused by dam breaks. We will use the shallow water model as both a motivation and an example in which all fundamental quantities will be computed. We now starts with some important and interesting derivation of the equations governing shallow water waves in one dimension.

4.1 Derivation of the Shallow-Water Equations

To derive the shallow water equations, we shall consider a one dimensional channel water flowing along the x-axis with an ideal incompressible and inviscid fluid with constant density ρ, and assume that the bottom of the water channel is horizontal. In the shallow water wave approximation, we assume that the velocity of the water $v = v(x,t)$ is a function of time t, and the position horizontally along the channel measured with respect to the x-axis. We also assume that there is no vertical motion in the water and the distance from the surface of the water to the bottom part of the water is denoted by $h = h(x,t)$. To model the water flow, we use conservation of mass and conservation of momentum. We consider the conservation of mass of a system but we first define the control volume as $R = \{(x,z) : x_0 \leq x \leq x_0 + \Delta x, 0 \leq z \leq h(x,t)\}$. The rate of change of mass is equal to the change or difference in out flow and in flow of fluid within the control volume R is given by

$$\frac{d}{dt}\int_R \rho\, dV + \int_{\partial R} \rho \boldsymbol{v}\cdot\boldsymbol{n}\, dV = 0, \quad \text{where } dV = dxdz. \tag{4.2}$$

Furthermore we can simplify again to get

$$\frac{d}{dt}\int_{x_0}^{x+\Delta x}\int_0^{h(x,t)} \rho\, dx\, dz + \int_{R\cap\{x=x_0+\Delta x\}} \rho \boldsymbol{v}\cdot\boldsymbol{n}\, d\sigma - \int_{R\cap\{x=x_0\}} \rho \boldsymbol{v}\cdot\boldsymbol{n}\, d\sigma = 0$$

which implies that

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x}\int_0^{h(x,t)} \rho\, dx\, dz + \int_0^{h(x_0+\Delta x,t)} \rho v(x_0+\Delta x) dz - \int_0^{h(x_0,t)} \rho v(x_0+\Delta x) dz = 0.$$

Assuming smoothness of the functions and domains involved, we can rewrite the right-hand side in the form

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x}\int_0^{h(x,t)} \rho\, dx\, dz + \int_{x_0}^{x_0+\Delta x} [\rho\, v(x_0+\Delta x) - \rho v(x_0)] dx = 0. \tag{4.3}$$

Remember we assume an incompressible fluid were ρ is constant, so using this assumption, we have

$$\rho\left(\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} h(x,t) dx + \int_{x_0}^{x_0+\Delta x} [v(x_0+\Delta x) - \rho v(x_0)] dx\right) = 0.$$

Dividing the equation above by $\rho\Delta x$ and letting $\Delta x \to 0$, we get

$$\frac{dh}{dt} + \frac{\partial}{\partial x}(vh) = h_t + (vh)_x = 0. \tag{4.4}$$

Since this equation has two unknown functions, h and v, we cannot solve the equation immediately, so we have to introduce another equation by using the conservation of momentum. Now to derive the second equation describing the conservation of momentum, consider that the fluid is in hydrostatic pressure or balance, that is, the pressure exactly balances the effect of gravity and introduce the pressure $P = P(x, y, t)$ and assume a small change of the form, $[x_0, x_0 + \Delta x] \times [y, y + \Delta y]$ in the fluid. What has been described above can be written as

$$(P(\tilde{x}, y + \Delta y, t) - P(\tilde{x}, y, t))\Delta x = -\Delta x \rho g \Delta y$$

for some $\tilde{x} \in [x_0, x_0 + \Delta x]$, where g is the acceleration due to gravity. Now dividing by $\Delta x \Delta y$ and letting $x_0, x_0 + \Delta\ x$ go to x as Δy goes to 0 we get

$$\frac{dP}{dy}(x, y, t) = -\rho g.$$

Now integrating over y and using the control volume we have above, we find that

$$P(x, y, t) = \rho g(h(x, t) - y). \tag{4.5}$$

This equation however balances the pressure to 0 at the fluid surface. Now we study the fluid along the x-direction between x_0 and $x_0 + \Delta x$, and compute the change of momentum for this part of the fluid and we find that

$$\frac{d}{dt}\int_R \rho \boldsymbol{v} dV + \int_{\partial R} \rho \boldsymbol{v}(\boldsymbol{v} \times \boldsymbol{n}) d\sigma = \sum \boldsymbol{F} \cdot \boldsymbol{e_x}$$

but we assume that force on the right is 0, that is $\sum \boldsymbol{F} \cdot \boldsymbol{e_x} = 0$. The total derivative in the first part of the equation above indicates the situation where we consider the overall parts of the fluid channels but in the equation written below one will realize that we are using partial derivations of the beginning part of the equation since we are considering the conservation in the x-diection.

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_0}^{x_0+\Delta x}\int_0^{h(x,t)} \rho v dx dy = \int_0^{h(x_0+\Delta x,t)} P(x_0 + \Delta x, y, t)dy - \int_0^{h(x_0,t)} P(x_0, y, t)dy + \\ \int_0^{h(x_0+\Delta x,t)} \rho v(x_0 + \Delta x, y, t)^2 dy - \int_0^{h(x_0,t)} \rho v(x_0 + \Delta x, y, t)^2 dy = 0.\end{aligned}$$

We shall rewrite the equation above as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_0}^{x_0+\Delta x} \rho v h(x, t)dx + \rho g \left(h(x_0 + \Delta x, t)^2 - \frac{1}{2}h(x_0 + \Delta x, t)^2\right) \\ -\rho g \left(h(x_0, t)^2 - \frac{1}{2}h(x_0, t)^2\right) + \int_{x_0}^{x_0+\Delta x} \frac{\partial}{\partial x}(\rho v^2 h)dx = 0.\end{aligned}$$

Furthermore, simplifying again we find that

$$\frac{\partial}{\partial t}\int_{x_0}^{x_0+\Delta x} \rho v h(x, t)dx + \rho g \int_{x_0}^{x_0+\Delta x} \frac{\partial}{\partial x}\left(\frac{1}{2}h^2\right) dx - \int_{x_0}^{x_0+\Delta x} \frac{\partial}{\partial x}(\rho v^2 h)dx.$$

Now dividing by $\Delta x \rho$ and taking $\Delta x \to 0$ we get the differential formulation as

$$(vh)_t + \left(v^2 h + \frac{1}{2}gh^2\right)_x = 0. \tag{4.6}$$

By using dimensional analysis, we scale the acceleration due to gravity, g to 1 such that g has order 1, therefore we obtain the form

$$(vh)_t + \left(v^2h + \frac{1}{2}h^2\right)_x = 0$$

Hence, we obtain the following differential systems of conservation laws:

$$h_t + (vh)_x = 0, \quad (vh)_t + \left(v^2h + \frac{1}{2}gh^2\right)_x = 0, \tag{4.7}$$

where h is the height or depth of the fluid and v is the velocity of the fluid. Let us introduce a variable q defined by

$$q = vh, \tag{4.8}$$

where q is known as the *kinematic viscosity* of the fluid. Rewriting (4.7) we find that

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{h^2}{2} \end{pmatrix}_x = 0. \tag{4.9}$$

The combined equations above is often called the *shallow water equations* or *Saint-Venant equations* and this constitute what is termed as a *hyperbolic system.*

4.1.1 THE CASE OF A THREE DIMENSIONAL SYSTEM FOR AN IDEAL GAS

Before we continue analyzing the properties and nature of solutions of the system of differential conservation laws, let us look at some of its usefulness and motivation in modelling gas dynamics using Euler equations. First, consider the flow of an ideal gas in a long, thin horizontal pipeline. Also, assume that the velocity v depends on x and t only, and that there is no movement in the $y-$ and $z-$direction. We also assume that the temperature T is constant for simplicity and we will ignore viscosity. Now for an ideal gas, the pressure p is linked to the density ρ by the ideal gas law $p = \rho CT$, where C is a constant depending on the molar mass of the gas. Let us use $x \in [x_0, x_0 + \Delta x]$ as the control volume at time t to derive the conservation laws. Assume that there is no production of mass of gases in the pepeline and so the conservation of mass in integral form is

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} \rho(x,t)dx + \rho(x_0+\Delta x,t)v(x_0+\Delta x,t) - \rho(x_0,t)v(x_0,t) = 0.$$

Now assume the only force acting on the gas is the pressure force, and so the conservation of momentum can be expressed as

$$\begin{aligned}\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} v(x,t)\rho(x,t)dx + \rho(x_0+\Delta x,t)v(x_0+\Delta x,t)^2 \\ -\rho(x_0,t)v(x_0,t)^2 + p(x_0+\Delta x,t) - p(x_0,t) = 0.\end{aligned}$$

We now divide both sides of the equations by Δx and then let $\Delta x \to 0$ to get the differential form of the equations. If we use the fact that $\lim_{\delta\to 0}\frac{1}{\delta}\int_y^{y+\delta} f(z)dz = f(y)$, and also assuming that we can change the order of taking the limit and differentiation we find that

$$\begin{aligned}\rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + (\rho v^2 + p)_x &= 0.\end{aligned} \tag{4.10}$$

The assumptions of the ideal gas and constant temperature gives

$$\rho_t + (\rho v)_x = 0, \quad (\rho v)_t + (\rho v^2 + \rho CT)_x = 0. \tag{4.11}$$

Suppose we increase the pressure difference we will observe that the temperature will no longer be constant. In that case if we are to include a variable temperature then we will get one more unknown variable in which we will need to get another equation to supplement it. We have already used the conservation of mass and momentum so we have to use the conservation of energy. The energy density is given by $E = \frac{1}{2}\rho v^2 + \rho e$, where e is the specific internal energy. Let's assume that the specific internal energy of the ideal gas is given by $e = \hat{c}\rho T$, where $\hat{c}$ is the specific heat capacity.

The work done by the gas in the control volume on the surroundings are due to the the pressure. The first law of thermodynamics asserts that the rate of change in the total energy equals to the net flow of energy into the domain minus net work done by the system. This means that

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} E(x,t)dx + E(x_0+\Delta x,t)v(x_0+\Delta x,t) - E(x_0,t)v(x_0,t)$$
$$+p(x_0+\Delta x,t)v(x_0+\Delta x,t) - p(x_0,t)v(x_0,t) = 0.$$

We find the differential form of the equation if we divide the equation by Δx and then let $\Delta x \to 0$. Therefore we find the differential form of conservation law of energy as

$$E_t + (Ev + pv)_x = 0, \tag{4.12}$$

where $p = \frac{E}{\rho} - \frac{1}{2}v^2$. Finally combining the differential equations of the conservation of laws of mass, momentum and energy, that is, (4.11) and (4.12) gives the system

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ Ev + pv \end{pmatrix}_x = 0. \tag{4.13}$$

Ealier on we considered the system of conserved differential equation given by

$$u_t + f(u)_x = 0.$$

We observed f as a vector-valued function so we will make some assumptions that explains alot of its properties in the situation of a scalar case carried to the case of systems. In order to have finite speed propagation, which depicts the characteristics of hyperbolic equations, is observed by assuming the Jocobian of f, denoted by df and has n real eigenvalues. This notion using spectral theorem for the eigenvalue $\lambda_j(u)$ can be be summarised in the form

$$Df(u)m_j(u) = \lambda_j(u)m_j(u), \quad \lambda_j(u) \in \mathbb{R}, \quad j = 1, 2, \cdots, n. \tag{4.14}$$

One can solve the above equation using spectral decomposition or by elementary method using the case where the determinant of the evolving characteristic polynomial equation is equated to zero to solve for the eigenvalues. We can find the eigenvalues and order them as

$$\lambda_1(u) \leq \lambda_2(u) \leq \cdots \leq \lambda_n(u). \tag{4.15}$$

We then find the corresponding eigenvectors $m_j(u)$ by putting the eigenvalues back into (4.14) to solve for the eigenvectors. A system in which we have a full set of eigenvectors with real eigenvalues is called *hyperbolic*, and if all the eigenvalues are different, then we say that the system is *strictly hyperbolic*.
Now lets go back to the shallow-water model to see whether the system is hyperbolic. So let rewrite (4.9) as

$$u_t + (f(h,q))_x = 0 \tag{4.16}$$

where $u = (h,q)$, $f(h,q) = \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{h^2}{2} \end{pmatrix}$ and one can verify that the Jacobian of f is $Df = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{h^2} + h & \frac{2q}{h} \end{pmatrix}$.

We solve for the eigenvalues by letting $\det(\lambda I - Df) = 0$, where I is a 2×2 identity matrix, so we find that

$$\begin{vmatrix} \lambda & -1 \\ \frac{q^2}{h^2} - h & \lambda - \frac{2q}{h} \end{vmatrix} = 0.$$

Expanding and simplifying we get

$$\left(\lambda - \frac{q}{h}\right)^2 - h = 0.$$

We can easily find that

$$\lambda_-(u) = \frac{q}{h} - \sqrt{h} < \frac{q}{h} + \sqrt{h} = \lambda_+(u). \tag{4.17}$$

Simply note that $\lambda_\pm(u) = \frac{q}{h} \pm \sqrt{h}$ implies

$$Df - \lambda_\pm(u)I = \begin{pmatrix} -\frac{q}{h} \mp \sqrt{h} & 1 \\ -\frac{q^2}{h^2} + h & \frac{q}{h} \mp \sqrt{h} \end{pmatrix} \tag{4.18}$$

and so the eigenspace corresponding to $\lambda_\pm(u)$ is the null space of the matrix, and that is spanned by the vector

$$\begin{pmatrix} 1 \\ \frac{q}{h} \pm \sqrt{h} \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_\pm(u) \end{pmatrix}. \tag{4.19}$$

Clearly we observe that the eigenvalues and their corresponding eigenvectors are given by

$$\lambda_\pm(u) = \frac{q}{h} \pm \sqrt{h} \quad \text{and} \quad m_\pm(u) = \begin{pmatrix} 1 \\ \frac{q}{h} \pm \sqrt{h} \end{pmatrix}. \tag{4.20}$$

Since all the eigenvalues are real and distinct, and the systems exhibits a full set eigenvectors, hence the system is said to be strictly hyperbolic.

Now we check for linearity and non-linearity in the system, since it need to determine how we solve the systems. A system is said to be linearly degenerate if $m_\pm(u) \cdot \nabla\lambda_\pm = 0$ and a system is strictly non-linear if $m_\pm(u) \cdot \nabla\lambda_\pm \neq 0$. Note that this method however is a situation which plays a role in normalization process for the eigenvectors. Let us consider the the shallow water waves and remember we got $u = (h,q)$ and $f(h,q) = \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{h^2}{2} \end{pmatrix}$ with eigenvalues $\lambda_\pm(u) = \frac{q}{h} \pm \sqrt{h}$ and corresponding eigenvectors $m_\pm(u) = \begin{pmatrix} 1 \\ \frac{q}{h} \pm \sqrt{h} \end{pmatrix}$. This implies that

$$m_\pm(u) \cdot \nabla\lambda_\pm = \begin{pmatrix} 1 \\ \frac{q}{h} \pm \sqrt{h} \end{pmatrix}^T \cdot \begin{pmatrix} -\frac{q}{h^2} \pm \frac{1}{2\sqrt{h}} \\ \frac{1}{h} \end{pmatrix}.$$

Simplifying further we can check that

$$m_{\pm}(u) \cdot \nabla\lambda_{\pm} = -\frac{q}{h^2} \pm \frac{1}{2\sqrt{h}} + \frac{q}{h^2} \pm \frac{1}{\sqrt{h}} = \pm\frac{3}{2\sqrt{h}} \neq 0, \tag{4.21}$$

for $h > 0$. Therefore we can see that the shallow water waves are strictly non-linear for the system of hyperbolic differential equation.

4.2 RAREFACTION SOLUTIONS FOR THE SHALLOW WATER EQUATIONS

Now for consistency in our results, we normalize the eigenvectors and we normally use $m_{\pm}(u) \cdot \nabla\lambda_{\pm} = 1$ to normalize the eigenvectors. From now onwards, we will reconstruct the eigenvectors by normalizing it to get

$$m_{\pm}(u) = \pm\frac{2}{3}\sqrt{h}\begin{pmatrix} 1 \\ \frac{q}{h} \pm \sqrt{h} \end{pmatrix} \tag{4.22}$$

so that $m_{\pm}(u) \cdot \nabla\lambda_{\pm} = 1$ holds. We now check for rarefraction waves in the solution. To do this we let

$$\begin{aligned} \dot{h} &= 1 \\ \dot{q} &= \frac{q}{h} \pm \sqrt{h} \end{aligned} \tag{4.23}$$

which is not normalized while

$$\begin{aligned} \dot{h} &= \pm\frac{2}{3}\sqrt{h} \\ \dot{q} &= \pm\frac{2}{3}\sqrt{h}\left(\frac{q}{h} \pm \sqrt{h}\right) \end{aligned} \tag{4.24}$$

is the normalized form. But obviously when you simplify both the normalized and the non-normalized form of the eigenvectors we obtain

$$\frac{dq}{dh} = \frac{q}{h} \pm \sqrt{h}, \tag{4.25}$$

which is a non-homogeneous differential equation. Now we split (4.25) into

$$\frac{dq}{dh} = \lambda_{-} = \frac{q}{h} - \sqrt{h} \quad \text{and} \quad \frac{dq}{dh} = \lambda_{+} = \frac{q}{h} + \sqrt{h}. \tag{4.26}$$

From (4.25) we can check that

$$\frac{d}{dh}\left(\frac{q}{h}\right) = \frac{1}{h}\left(\frac{dq}{dh} - \frac{q}{h}\right) = \pm\frac{1}{\sqrt{h}}. \tag{4.27}$$

We integrate (4.27) so that

$$\int d\left(\frac{q}{h}\right) = \int \pm\frac{1}{\sqrt{h}}dh,$$

where A is a constant of integration.
This implies that

$$\frac{q}{h} = \pm 2\sqrt{h} + A.$$

We can find A by assuming h_l, q_l and v_l are fixed such that $h = h_l$, $q = q_l$ and $v = v_l$. That is,

$$\frac{q_l}{h_l} = \pm 2\sqrt{h_l} + A,$$

therefore

$$A = \frac{q_l}{h_l} \mp 2\sqrt{h_l}. \tag{4.28}$$

Note change of signs, that is $\pm$ becomes $\mp$ when it crosses the equal sign. So we obtain

$$q = q_l \frac{h}{h_l} - 2h(\sqrt{h} - \sqrt{h_l}) \tag{4.29}$$

and

$$q = q_l \frac{h}{h_l} + 2h(\sqrt{h} - \sqrt{h_l}) \tag{4.30}$$

Also we can observe from (4.17) that λ_- is increasing strictly along the rarefraction wave, which means we have to use $0 < h \leq h_l$ in (4.29) and similarly we have to use $h \geq h_l$ in (4.30). Hence we obtain our final result of the solution of the rarefaction waves which expressed in terms of h as

$$\begin{aligned} R_1: \quad q &= q_l \frac{h}{h_l} - 2h(\sqrt{h} - \sqrt{h_l}) \quad \text{for } 0 < h \leq h_l, \\ R_2: \quad q &= q_l \frac{h}{h_l} + 2h(\sqrt{h} - \sqrt{h_l}) \quad \text{for } h \geq h_l. \end{aligned} \tag{4.31}$$

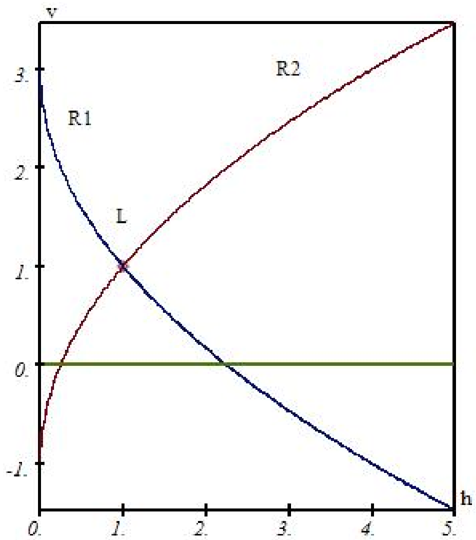
Alternatively, we can rewrite the above equation in terms of v and h by using the substitution $v = q/h$ so that

$$\begin{aligned} R_1: \quad v &= v_l - 2(\sqrt{h} - \sqrt{h_l}) \quad \text{for } 0 < h \leq h_l, \\ R_2: \quad v &= v_l + 2(\sqrt{h} - \sqrt{h_l}) \quad \text{for } h \geq h_l. \end{aligned} \tag{4.32}$$

So we can observe the nature of the rarefaction curves from figure below.

4.3 SHOCK SOLUTIONS FOR THE SHALLOW WATER EQUATIONS

For the shock solutions lets recall from the previous chapters were we realize that the solutions obtain from the differential form of the hyperbolic conservation equations are mostly weak solutions which may either be in implicit or explicit form. We saw that the use of Rankine–Hugoniot condition applies to situations where the characteristics coincide or collide with eachother. But our main focus in this part is to characterize the nature of solutions of the Rankine–Hugoniot relation. Assume the left state u_l is fixed and consider u_r as the right side that satisfies the



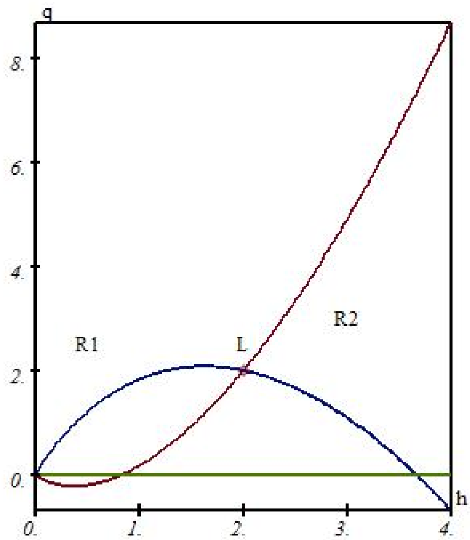


Figure 4.1: A graph of v against h and q against h is plotted in order to observe the pattern and behaviour of the rarefaction curves.

Rankine–Hugoniot condition, that is,

$$s(u_l - u_r) = f(u_l) - f(u_r), \tag{4.33}$$

where s is the shock speed. For a given left u_l, we can form a set, which we will call the *Hugoniot locus* and write $H(u_l)$ as

$$H(u_l) := \{u_r \mid \exists s \in \mathbb{R} \text{ such that } s(u_l - u_r) = f(u_l) - f(u_r)\}. \tag{4.34}$$

Now considering the Shallow water equation we can compute the Hugoniot locus by writing $u_l = \begin{pmatrix} h_l \\ q_l \end{pmatrix}$ and $u_r = \begin{pmatrix} h_r \\ q_r \end{pmatrix}$ as the left and right state respectively, where h_l and h_r are both positive and greater than 0 by assumption. Then we can write the solution of the flow as

$$u(x,t) = \begin{pmatrix} h(x,t) \\ q(x,t) \end{pmatrix} = \begin{cases} u_l = \begin{pmatrix} h_l \\ q_l \end{pmatrix} & \text{for } x < st, \\ u_r = \begin{pmatrix} h_r \\ q_r \end{pmatrix} & \text{for } x > st. \end{cases} \tag{4.35}$$

From the Shallow water equation we have

$$\begin{aligned} s(h_l - h_r) &= q_l - q_r, \\ s(q_l - q_r) &= \left(\frac{q_l^2}{h_l} + \frac{h_l^2}{2}\right) - \left(\frac{q_r^2}{h_r} + \frac{h_r^2}{2}\right), \end{aligned} \tag{4.36}$$

we then eliminate s in the equations above to obtain

$$\frac{q_l - q_r}{\left(\frac{q_l^2}{h_l} + \frac{h_l^2}{2}\right) - \left(\frac{q_r^2}{h_r} + \frac{h_r^2}{2}\right)} = \frac{h_l - h_r}{q_l - q_r}. \tag{4.37}$$

We now do some background simplification of (4.37) to get

$$(q_l - q_r)^2 = (h_l - h_r)\left[\left(\frac{q_l^2}{h_l} - \frac{q_r^2}{h_r}\right) + \frac{1}{2}\left(h_l^2 - h_r^2\right)\right], \tag{4.38}$$

and then introduce the variable $v = \frac{q}{h}$ given previously by (4.8) so that

$$(v_l h_l - v_r h_r)^2 = (h_l - h_r)\left[\left(v_l^2 h_l - v_r^2 h_r\right) + \frac{1}{2}\left(h_l^2 - h_r^2\right)\right].$$

Expanding and simplifying, we get a quadratic eqaution in v_r^2, that is,

$$v_r^2 - 2v_l v_r - \frac{1}{2}(h_l^{-1} + h_r^{-1})(h_l - h_r)^2 + v_l^2 = 0. \tag{4.39}$$

with the solution

$$v_r = v_l \pm \frac{1}{\sqrt{2}}(h_l - h_r)\sqrt{h_l^{-1} + h_r^{-1}}, \tag{4.40}$$

or alternatively,

$$q_r = v_r h_r = q_l \frac{h_r}{h_l} \pm \frac{h_r}{\sqrt{2}}(h_l - h_r)\sqrt{h_l^{-1} + h_r^{-1}}. \tag{4.41}$$

We now substitute what was obtain in (4.40) or (4.41) into (4.36) to get formulas for the corresponding shock speeds since that is what we are interested to find. In other words, we write

$$s(h_l - h_r) = q_l - q_r$$

as

$$s = \frac{q_l - q_r}{h_l - h_r} = \frac{v_l h_l - v_r h_r}{h_l - h_r},$$

which implies

$$s = \frac{v_l h_l - v_r h_l + v_r h_l - v_r h_r}{h_l - h_r} = \frac{(v_l - v_r)h_l + v_r(h_l - h_r)}{h_l - h_r},$$

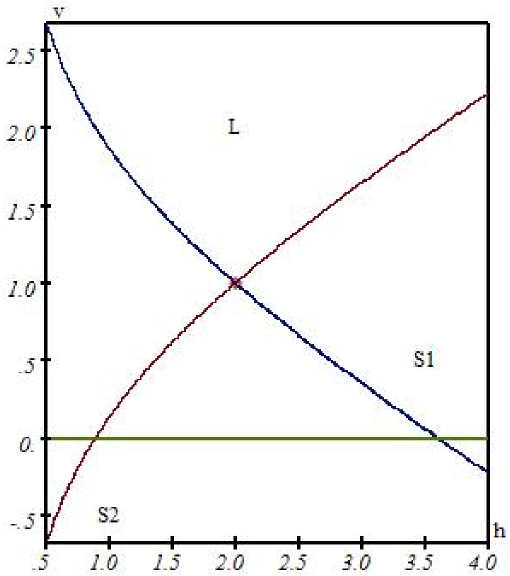
and finally we get the shock speed

$$s = v_r + \frac{v_l - v_r}{h_l - h_r} h_l = v_r \pm \frac{1}{\sqrt{2}}\sqrt{h_l^{-1} + h_r^{-1}}, \tag{4.42}$$

or

$$s = v_l \pm \frac{h_r}{\sqrt{2}}\sqrt{h_l^{-1} + h_r^{-1}}. \tag{4.43}$$

Here we see that for a given a left state u_l, there are two distinct curves on which the Rankine–



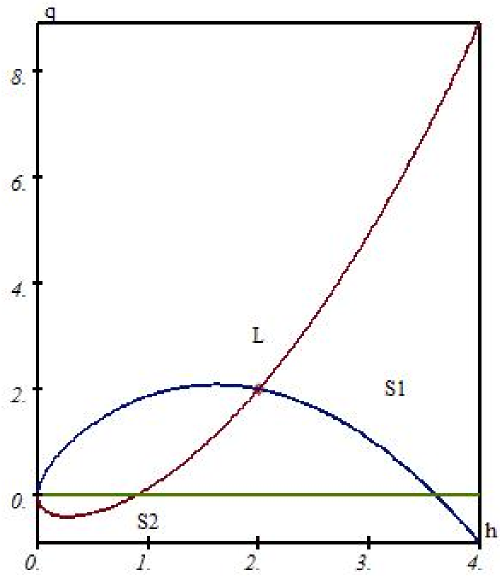


Figure 4.2: For simplicity of the plots we repalce q_r, v_r h_r with q, v and h respectively in both (4.40) and (4.41). In other words, we plot the shock curves, $v = v_l \pm \frac{1}{\sqrt{2}}(h_l - h)\sqrt{h_l^{-1} + h^{-1}}$ and $q = q_l \frac{h}{h_l} \pm \frac{h}{\sqrt{2}}(h_l - h)\sqrt{h_l^{-1} + h^{-1}}$ in order to visualize the pattern and nature of the curves.

Hugoniot condition holds, that is,

$$H_1(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} + \frac{h}{\sqrt{2}}(h_l - h)\sqrt{h_l^{-1} + h^{-1}} \end{pmatrix} \middle| \; h > 0 \right\}, \tag{4.44}$$

and

$$H_2(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} - \frac{h}{\sqrt{2}}(h_l - h)\sqrt{h_l^{-1} + h^{-1}} \end{pmatrix} \middle| \; h > 0 \right\}. \tag{4.45}$$

Hence, the Hugoniot locus becomes

$$H(u_l) := H_1(u_l) \cup H_2(u_l).$$

So we can observe the nature of the Hugoniot locus from the figure below where a graph of v against h and q against h is plotted.

4.4 ENTROPY CONDITION

In chapter 1, we observed the entropy condition for the scalar conservation law where we discussed the travelling wave entropy condition. In this section, we will select the curves that satisfies the entropy condition. Since we have obtained the Hugoniot loci for shallow water waves, we can then proceed to determine the curves which give admissible shocks. Remember from (1.32) that a solution between two fixed states u_l and u_r with speed s given by (1.25) gives a viscous profile if $u(x,t)$ is the limit of $u^\varepsilon(x,t) = U((x-st)/\varepsilon) = U(\xi)$ as $\varepsilon \to 0$ with $\xi = (x-st)/\varepsilon$, which satisfies

$$u_t^\varepsilon + [f(u^\varepsilon)]_x = \varepsilon\, u_{xx}^\varepsilon. \tag{4.46}$$

Using $\lim_{\varepsilon\to 0} U(\xi) = u_l$ for $\xi < 0$ and integrating the equation above with respect to ξ we get

$$\dot{U} = -s(U - u_l) + f(U) - f(u_l). \tag{4.47}$$

We study the solutions of $H_1(u_l)$ where we consider (h,v) variables. So differentiating $q = vh$ with respect to ξ we get

$$\dot{q} = \dot{v}h + v\dot{h}. \tag{4.48}$$

Using (4.48) and (4.47), we can simplify further to find that there is a viscous profile in (h,q) if and only if (h,v) satisfies

$$\begin{pmatrix} \dot{h} \\ \dot{v} \end{pmatrix} = G(h,v) := \begin{pmatrix} vh - v_l h_l - s(h-h_l) \\ (v-v_l)(vl - s)\frac{h_l}{h} + \frac{h^2-h_l^2}{2h} \end{pmatrix} = \begin{pmatrix} vh - v_l h_l - s(h-h_l) \\ (v-v_l)\frac{h_r h_l \delta}{h} + \frac{h^2-h_l^2}{2h} \end{pmatrix}, \tag{4.49}$$

where $\delta = \frac{\sqrt{h_l^{-1}+h_r^{-1}}}{\sqrt{2}}$ and $v_l - s = h_r\delta$.

We study (4.49) at the initial point by finding its Jacobian which is given by

$$DG = \begin{pmatrix} v-s & h \\ \frac{h^2+h_l^2}{2h^2} - (v-v_l)\frac{h_r h_l \delta}{h^2} & \frac{h_r h_l \delta}{h} \end{pmatrix}. \tag{4.50}$$

The eigenvalues of DG are $-s + v \pm \sqrt{h} = -s + \lambda_i(u)$, where for $i = 1$ we have $\lambda_1(u) = v - \sqrt{h}$ and for $i = 2$ we have $\lambda_2(u) = v + \sqrt{h}$.

The left state u_l gives

$$\begin{pmatrix} h_r\delta & h_l \\ 1 & h_r\delta \end{pmatrix} \tag{4.51}$$

and its eigenvalues are $h_r\delta \pm \sqrt{h_l}$. The eigenvalues are positive when $h_r > h_l$; which makes (h_l, v_l) a point source. Also at the right state we have

$$\begin{pmatrix} h_l\delta & h_r \\ 1 & h_l\delta \end{pmatrix} \tag{4.52}$$

and its eigenvalues are $h_l\delta \pm \sqrt{h_r}$. It can be observed that (h_r, v_r) is a saddle point since one of the eigenvalues is positive and one is negative when $h_r > h_l$.

Hence we can write the above result as

$$\lambda_1(u_r) < s < \lambda_1(u_l), \quad s < \lambda_2(u_r). \tag{4.53}$$

(4.53) is called the Lax inequalities and any shock satisfying these inequalities is a *Lax 1-shock* or a *slow Lax shock*.

Let $h_l > h_r$ and consider the Riemann problem with initial and weak solution given by

$$\begin{pmatrix} h(x,0) \\ v(x,0) \end{pmatrix} = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < 0, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}}\sqrt{h_l^{-1} + h_r^{-1}} \end{pmatrix} & \text{for } x \geq 0, \end{cases} \tag{4.54}$$

and

$$\begin{pmatrix} h(x,t) \\ v(x,t) \end{pmatrix} = \begin{cases} \begin{pmatrix} h_l \\ 0 \end{pmatrix} & \text{for } x < st, \\ \begin{pmatrix} h_r \\ \frac{h_l - h_r}{\sqrt{2}}\sqrt{h_l^{-1} + h_r^{-1}} \end{pmatrix} & \text{for } x \geq st, \end{cases} \tag{4.55}$$

respectively with speed $s = -\frac{h_r}{\sqrt{2}}\sqrt{h_l^{-1} + h_r^{-1}}$, where we have a higher water bank at rest to the left and a lower water bank to right. Here we observe that fluid from the lower water bank moves away from the higer water bank thereby pushing the higher water bank. Here you will realize a smaller wave h_r on the right side of Figure 4.3 moving a bigger wave h_l on the left state, that is, $h_l > h_r$ and this is interpreted as a travelling wave. However this phenonemom is not normal since it violates the *law of momentum conservation* as well as *Newton's second law of motion*. The only way a smaller wave will move a bigger wave is when the smaller wave is moving with a very high speed but in our case h_r is starting with initial velocity 0 hence it not physical for such situation to happen. Therefore is unreasonable for this to occur hence solution obtained here is known as *unphysical solution*.

But if we consider the other way round were $h_l < h_r$ then we will observe that the higher water bank is moving into the lower water bank thereby causing the higher water bank to push the lower water bank rather. Also you can observe from Figure 4.4 that h_r is bigger than h_l. Here, the bigger wave on the right state moves into the smaller wave h_l on the left state. This is what we expect to get since it is reasonable for a bigger wave pushing a smaller wave. The solution obtained in this case is known as *physical solution*.

Now we notice that for $h_l > h_r$, the eigenvalues at the left state u_l has different sign, that is, one of the eignenvalues is either negative or positive. This makes u_l a saddle point. Whiles at the right state both eigenvalues are positive, and thus u_r is a point source. Therefore we can write the above result as

$$\lambda_2(u_r) < s < \lambda_2(u_l), \quad s > \lambda_2(u_l). \tag{4.56}$$

The inequalities in (4.56) is called the *Lax 2-shock* or a *fast Lax shock*. Note that (4.53) and (4.56) are known as the *Lax entropy conditions*.

Now for general systems, we can define the Lax entropy conditions for the shallow water equations as:

Definition 4.1. A shock solution

$$u(x,t) = \begin{cases} u_l, & \text{for } x < st, \\ u_r, & \text{for } x \geq st, \end{cases} \tag{4.57}$$

is said to be a Lax i-shock if the shock speed s satisfies the inequalities

$$\lambda_{i-1}(u_l) < s < \lambda_i(u_l), \quad \lambda_i(u_r) < s < \lambda_{i+1}(u_r), \tag{4.58}$$

where $\lambda_0 = -\infty$ and $\lambda_{n+1} = +\infty$. □

Note that in general the Lax condition is equivalent to the existence of viscous profiles only for weak shocks. And even then, it is not easy to prove. The usual proof relies on the *Conley index*. Hence a weak shock is said to have a viscous profile if and only if the Lax entropy conditions are satisfied. Such shocks are term as *admissible* and we denote the part of the Hugoniot Locus where the Lax i conditions are satisfied by S_i. So for the shallow water wave equations we have the following curves, that is;

$$S_1(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} + \frac{h}{\sqrt{2}}(h_l - h)\sqrt{h_l^{-1} + h^{-1}} \end{pmatrix} \middle| \; h \geq h_l \right\}, \tag{4.59}$$

and

$$S_2(u_l) := \left\{ \begin{pmatrix} h \\ q_l \frac{h}{h_l} - \frac{h}{\sqrt{2}}(h_l - h)\sqrt{h_l^{-1} + h^{-1}} \end{pmatrix} \middle| \; h \leq h_l \right\}, \tag{4.60}$$

which are similar to what we had in (4.44) and (4.45) but there is some differences between them namely: their inequalities are opposite so that S_1 and R_1 extend in opposite directions from the left state (and they join smoothly there) and similarly with S_2 and R_2.

Now without hesitation, we proceed to find a solution for the Riemann problem.

4.5 SOLUTION TO THE RIEMANN PROBLEM

We find the solution to the Riemann problem by assuming that the left state u_l is fixed, and we construct the solution for the Riemann problem for all possible right states u_r .This can also be done by fixing u_r and considering the space of all left states u_l. Here we combine the properties of the rarefaction waves and the shock waves from the previous sections by defining the *wave curves*

$$W_i(u_l) := R_i(u_l) \cup S_j(u_l) \quad \text{for } i = 1, 2. \tag{4.61}$$

Suppose the left state u_l is given, then for each u_r we have to find one middle state u_m such that $u_m \in W_1(u_l)$ and $u_r \in W_2(u_m)$. It is important to consider the *backward* second wave curve $W_2^*(u_r)$ consisting of middle states u_m which can be linked to u_r on the right with a *fast* wave. Then the Riemann problem will have a unique solution if and only if there is a unique intersection between $W_1(u_l)$ and $W_2^*(u_r)$. In this situation, it is obvious that the intersection will be the middle state. The curve $W_1(u_l)$ is given by

$$v = \begin{cases} v_l - 2(\sqrt{h} - \sqrt{h_l}) & \text{for } 0 \leq h \leq h_l, \\ v_l - \frac{h-h_l}{\sqrt{2}}\sqrt{h_l^{-1} + h^{-1}} & \text{for } h \geq h_l. \end{cases} \tag{4.62}$$

Let define $W_2^*(u_r)$ as

$$v = \begin{cases} v_r + 2(\sqrt{h} - \sqrt{h_r}) & \text{for } 0 \leq h \leq h_r, \\ v_r + \frac{h-h_r}{\sqrt{2}}\sqrt{h_r^{-1} + h^{-1}} & \text{for } h \geq h_r \end{cases} \tag{4.63}$$

which is easily obtained by combining R_2 of (4.32) and S_2 of (4.60). We observe $W_1(u_l)$ is strictly decreasing starting from $v_l + 2\sqrt{h_l}$ at $h = 0$ and it is unbounded since $h > h_l$. Whiles $W_2^*(u_r)$ is strictly increasing with minimum $v_r - 2\sqrt{h_r}$ at $h = 0$ and it is also unbounded for $h > h_r$.

From the above analysis on $W_1(u_l)$ and $W_2^*(u_r)$, we can conclude that the Riemann problem for the shallow water wave equation has a unique solution in the region where

$$v_l + 2\sqrt{h_l} \geq v_r - 2\sqrt{h_r}. \tag{4.64}$$

Next we find the middle state u_m which depends on the type of wave curves that intersect by making the following four cases.

(a) Let the $u_m \in S_1(u_l)$ and $u_r \in R_2(u_m)$ such that $u_m = (h_m, q_m) = (h_m, h_m v_m)$ (we know these terms from the previous sections). So we get

$$v_m = v_l - \frac{1}{\sqrt{2}}(h_m - h_l)\sqrt{h_l^{-1} + h_m^{-1}}, \quad v_r = v_m + 2(\sqrt{h_r} - \sqrt{h_m}).$$

We simplify both equations to get

$$v_r - v_l = 2(\sqrt{h_r} - \sqrt{h_m}) - \frac{(h_m - h_l)}{\sqrt{2}}\sqrt{h_l^{-1} + h_m^{-1}}. \tag{4.65}$$

(b) We consider the case where $u_m \in R_1(u_l)$ and $u_r \in S_2(u_m)$. So by combining the two curves we get

$$v_r - v_l = \frac{(h_r - h_m)}{\sqrt{2}}\sqrt{h_l^{-1} + h_m^{-1}} - 2(\sqrt{h_m} - \sqrt{h_l}). \tag{4.66}$$

(c) Also consider the situation where $u_m \in S_1(u_l)$ and $u_r \in S_2(u_m)$. When we combine these regions we get

$$v_r - v_l = \frac{(h_r - h_m)}{\sqrt{2}}\sqrt{h_r^{-1} + h_m^{-1}} - \frac{(h_m - h_l)}{\sqrt{2}}\sqrt{h_l^{-1} + h_m^{-1}}. \tag{4.67}$$

(d) Finally lets consider the case where $u_m \in R_1(u_l)$ and $u_r \in R_2(u_m)$. Then the middle state u_m can be found to be

$$v_m = v_l - 2(\sqrt{h_m} - \sqrt{h_l}), \quad v_r = v_m + 2(\sqrt{h_r} - \sqrt{h_m}),$$

which by solving both equations for h_m we get

$$\sqrt{h_m} = \frac{2(\sqrt{h_r} + \sqrt{h_l}) - (v_r - v_l)}{4}. \tag{4.68}$$

From (4.64), we can see that it consistent with (4.68) and that Riemann problem has a unique solution consisting of a slow wave followed by a fast wave if the right states satisfies the following:

$$u_r \in \{u \in \langle 0, \infty\rangle \times \mathbb{R} : 2(\sqrt{h_r} + \sqrt{h_l}) \geq (v_r - v_l)\}. \tag{4.69}$$

To sum up the solution of the Riemann problem around the left state, we let $w_i(x/t; h_m, h_l)$ be the solution of the Riemann problem for $u_m \in W_i(u_l)$. We shall consider the notation $\sigma_i^{\pm}$ to represent the fastest and slowest wave speed in order to simplify the description of the full solution. So for $i = 1$ and $h_r < h_l$, w_1 is a rarefaction wave solution with slowest speed $\sigma_i^- = \lambda_i(u_l)$ and fastest speed $\sigma_i^+ = \lambda_i(u_r)$. A similar situation also exists for the shock wave solution. Hence we can write the solution for the Riemann problem as

$$u(x,t) = \begin{cases} u_l & \text{for } x \leq \sigma_1^- t, \\ w_1(x/t; u_m, u_l) & \text{for } \sigma_1^- t \leq x \leq \sigma_1^+ t, \\ u_m & \text{for } \sigma_1^+ t \leq x \leq \sigma_2^- t, \\ w_2(x/t; u_r, u_m) & \text{for } \sigma_2^- t \leq x \leq \sigma_2^+ t, \\ u_r & \text{for } x \geq \sigma_2^+ t, \end{cases} \tag{4.70}$$

This is a local solution for the Riemann problem but we can also find a global solution for the Riemann problem.

Let $E = \{(h,v) \mid h \in [0,\infty\rangle, v \in \mathbb{R}\}$ and lets keep the same solution in the region where we have already constructed a solution. Therefore it is left to construct in the region

$$u_r \in D := \{u \in E \mid 2(\sqrt{h_r} + \sqrt{h_l}) \leq (v_r - v_l)\} \cup \{h = 0\}. \tag{4.71}$$

Now using a slow rarefaction wave, lets link the left state u_l with the middle state u_m on $h = 0$. Now using R_1 of (4.32), we get the middle state

$$v_m = v_l + 2\sqrt{h_l}. \tag{4.72}$$

From this state we jump to the point v^* at $h^* = 0$ such that the fast rarefaction wave starting at v^* hits the right state u_r. Similarly using R_2 of (4.32), we observe that $v^* = v_r - 2\sqrt{h_r}$. Hence summing up all these results we get the global solution of the Riemann problem for the shallow water wave equations as

$$u(x,t) = \begin{cases} u_l & \text{for } x \leq \lambda_1(u_l)t, \\ R_1(x/t; u_l) & \text{for } \lambda_1(u_l)t \leq x \leq (\sqrt{h_l} + v_l)t, \\ u_m & \text{for } (\sqrt{h_l} + v_l)t \leq x \leq v^*t, \\ R_2(x/t; (0, v^*)) & \text{for } v^*t \leq x \leq \lambda_2(u_r)t, \\ u_r & \text{for } x \geq \lambda_2(u_r)t, \end{cases} \tag{4.73}$$

where $u_l = \begin{pmatrix} h_l \\ v_l \end{pmatrix}$ and $u_r = \begin{pmatrix} h_r \\ v_r \end{pmatrix}$. This however solves the Riemann problem for the shallow water wave equations.

We can observe that finding the analytic solution for the shallow water wave equations is a bit complicated. We therefore proceed to find a numerical solution to the shallow water problem by using the solution to the dam break problem which was used by Helge Holden and Nils Henrik Risebro [5].

4.6 NUMERICAL SOLUTIONS

The scheme used for the numerical simulation of the system of PDEs is the Lax–Friedrichs scheme, which in this case is explicit; that is, there is no need to solve a system of simultaneous equations at each time step. The Lax–Friedrichs scheme for hyperbolic conservation laws is given by:

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{1}{2}\lambda\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right) \tag{4.74}$$

where $\lambda = \frac{k}{h}$ with k being the step size in time increment and h is the step size in space increment for the scheme. Translating (4.74) into our system of equations in (4.7) yields the following

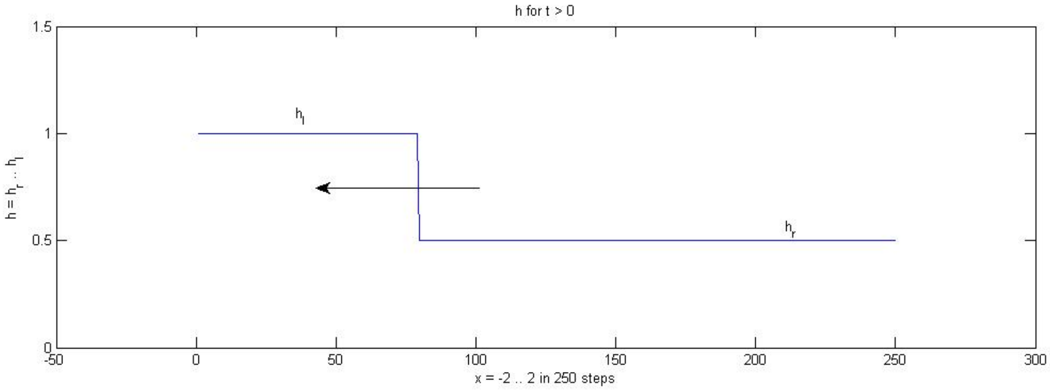


Figure 4.3: Unphysical solution.

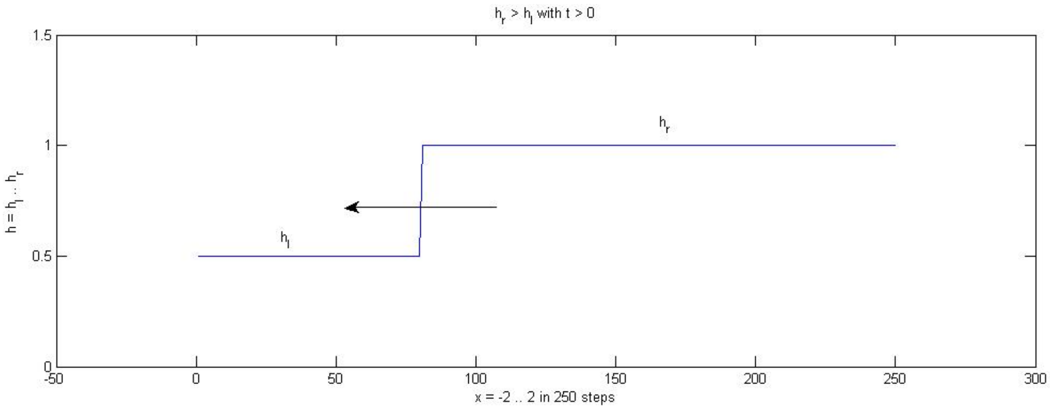


Figure 4.4: physical solution.

equations:

$$h_j^{n+1} = \frac{1}{2}\left(h_{j+1}^n + h_{j-1}^n\right) - \frac{\lambda}{2}\left(q_{j+1}^n - q_{j-1}^n\right) \tag{4.75}$$

and

$$q_j^{n+1} = \frac{1}{2}\left(q_{j+1}^n + q_{j-1}^n\right) - \frac{\lambda}{2}\left[\frac{2q_j^n}{h_j^n}(q_{j+1}^n - q_{j-1}^n) - \left(\frac{q_j^n}{h_j^n}\right)^2(h_{j+1}^n - h_{j-1}^n) + h_j^n(h_{j+1}^n - h_{j-1}^n)\right] \tag{4.76}$$

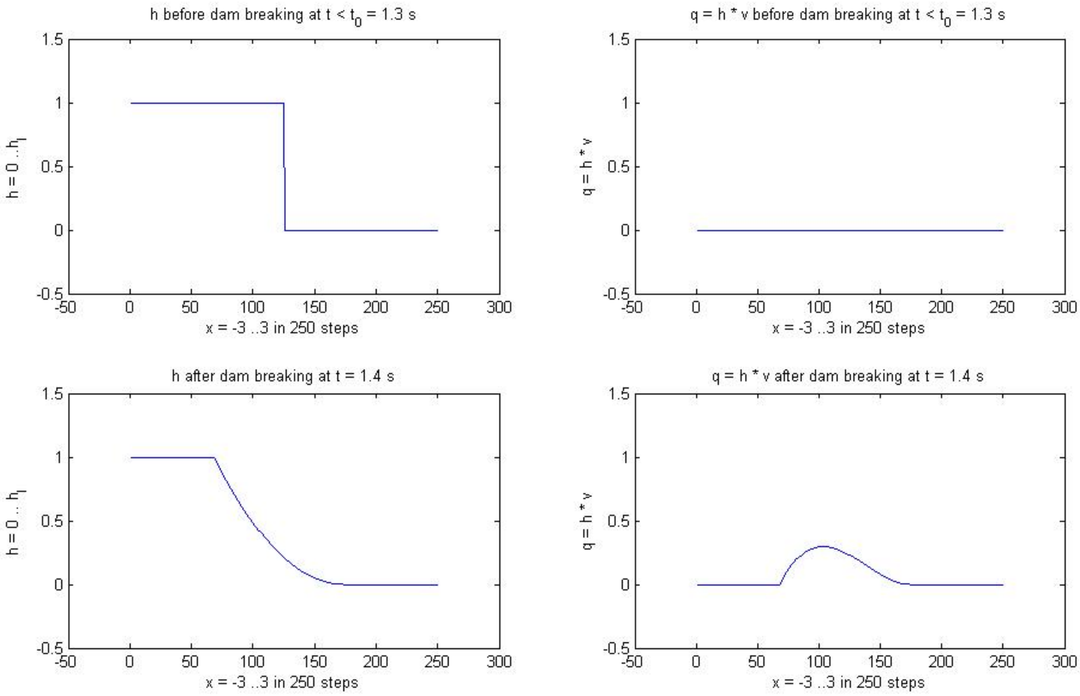


Figure 4.5: Analytical solution of the dam break in matlab.

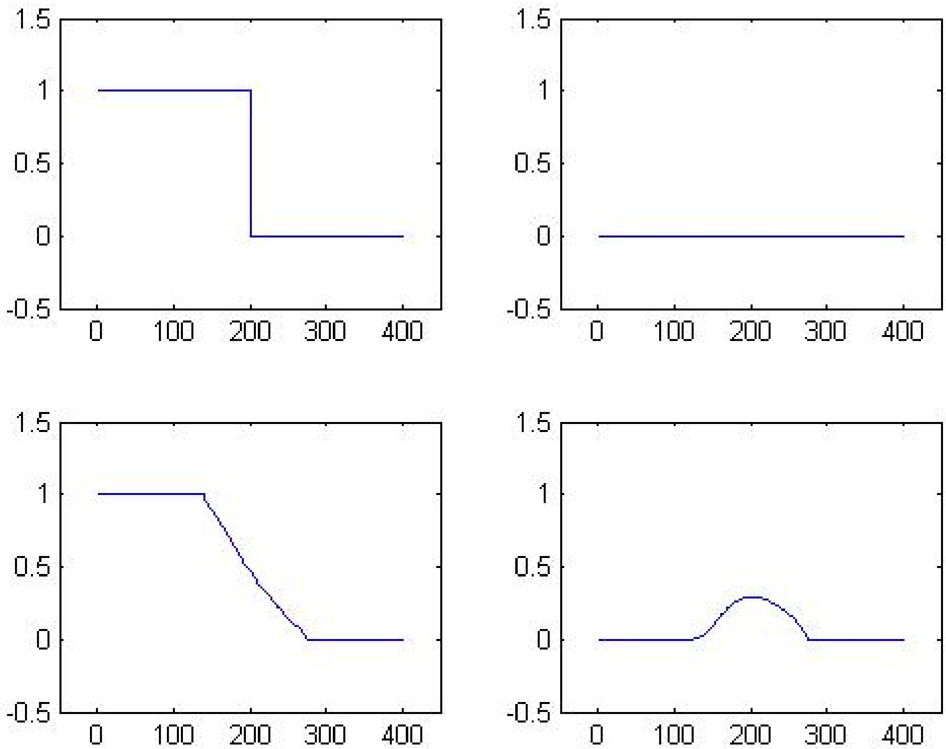


Figure 4.6: Numerical plots of the numerical solution of the dam break in matlab, where h is plotted on the left and q on the right. The vertical and horizontal axis is the same as in Figure 4.5 except the number of spatial steps.

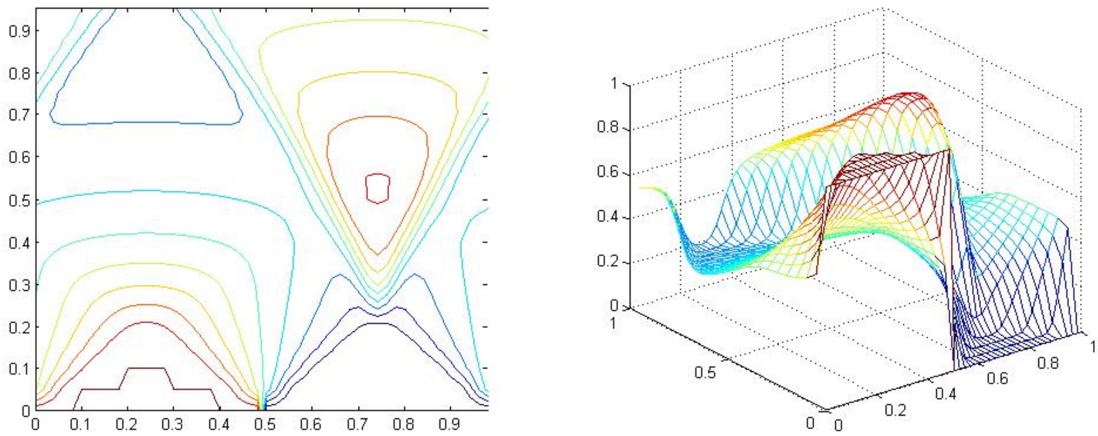


Figure 4.7: Contour plot and the surface plot with periodic boundary conditions.

Chapter 5

Application of conservation laws in oil recovery process

In this section, we will use the conservation laws and some techniques to construct a simple model or equation for oil recovery in a reservoir. We start by considering a porous reservoir, where the reservoir have pores which are connected so that the fluids may move through the media. The media with pores can be sand, clay, soil or rocks and the pores can be filled with fluid such as air, water or oil.

5.1 Mechanism involved in primary oil recovery process

Rocks in the reservoir provides resistance which opposes the fluid flow and this is attributed to permeability. This gives the impression that permeability is resistance to fluid flow, but it's opposite: higher permeability means less resistance. The fluid flows in the pores of the rock; in general, higher porosity and bigger pores lead to higher permeability. The rock is compressed and the volume of oil in the reservoir increases and this is as a result of geological, biological or chemical processes which builds up pressure in the reservoir. As a result of all these processes, a portion of the oil can be recovered by drilling a well in the reservoir. During drilling of the well, pressure at the bottom increases causing oil to spill out through the well. The mechanism involved here is by openning a well so that as oil is being collected, pressure will decrease in the reservoir, the rock will expand and presses the oil out of the reservoir through the well. The flow of the oil will stop when hydrostatic pressure is attained, that is, a situation whereby the pressure at the bottom becomes equal to the atmospheric pressure. This process of oil recovery is known as the *primary oil recovery process*. Note that the fluid flow (applies to both water and oil) is described by Darcy's law which is given by

$$u = -\frac{1}{\mu}\boldsymbol{K}(\nabla P + \rho g e_z), \tag{5.1}$$

where u is the volume flux per unit area, μ is the viscosity, $\boldsymbol{K} = \boldsymbol{K}(x, y, z)$ is the permeability matrix which depends mostly on the types of rock we dealing with and it is usually a function of spatial coordiantes, ρ is the density, g is the acceleration due to gravity which is constant, P is the pressure and $e_z = [0,0,1]^T$ is the unit vector in the direction of the depth of the reservoir. But the issue now is we are not interested in the primary oil recovering but we are interested in collecting the oil left in the reservoir. But usually in the industrial practice, the amount of oil that actually remains is very large in volume. Hence we consider a *secondary oil recovery process* where water is injected into the reservoir to restore a pressure gradient which can displace the oil which is left after the primary oil recovery process.

5.2 Mechanism involved in secondary oil recovery process

We inject water at the left end of the reservoir which is meant to push the oil out but we shall encounter a problem here since water and oil have different viscosities and mobilities. Also water travel faster than oil so we set up a model which solves this problem by considering two phases, that is, water and oil phase. Consider a small scale reservoir containing two wells, where we let one well be J which allows the injection of water and the other well be L which allows for extration of oil. The injection well works by maintaining a pressure gradient during which oil is extracted. Since we have different viscosities between oil and water, we divide the reservoir into two phases, with $b = w$ to represent water and $b = o$ to represent oil. We consider the saturation, S_b which represents the portion of pore volume occupied by the phase b: where the saturation of water is denoted by S_w and the saturation of oil is denoted S_o.
We use Darcy's law, which is given by

$$\boldsymbol{u}_b^* = -\frac{s_b}{\mu_b} k \nabla P, \tag{5.2}$$

which holds for each phase $b \in \{w, o\}$. In (5.2), u_b^* is the volume flux per unit area, μ_b is the viscosity is phase b, k is permeability which is assumed to be constant, and P is the pressure. In (5.2) we only consider the volume flux of flow of the phase a, and since fractions of water and oil fill up the pore volume, we assume that $S_w + S_o = 1$. Hence by assuming a *linear relation* between the flux and the amount of phase b we have a factor S_b in the Darcy's law. We consider the conservation of water and oil given by

$$\frac{d}{dt^*}\int_R \rho_b \phi s_b \,\mathrm{d}V + \int_{\partial R} \rho_b \boldsymbol{u}_b^* \cdot \hat{\boldsymbol{n}} \,\mathrm{d}\sigma = 0, \tag{5.3}$$

which holds for each phase, b where ρ_b represent the mass density of phase $b \in \{w, o\}$, ϕ is the porosity of the reservoir and R is the material region.

Note that in the conservation eqaution, the rigth-hand side of the equation is zero since we assume there is no production. Using divergence theorem we get the differential form of the conservation law as

$$\frac{\partial}{\partial t^*}(\rho_b \phi s_b) + \nabla \cdot (\rho_b \boldsymbol{u}_b^*) = 0, \quad b \in \{w, o\},$$

or, using (5.2) we obtain

$$\frac{\partial}{\partial t^*}(\rho_b \phi s_b) - \nabla \cdot (\frac{\rho_b s_b}{\mu_b} k \nabla P) = 0, \quad b \in \{w, o\}. \tag{5.4}$$

Let us assume incompressibility of both liquids and the reservoir, then we obtain the system of equations on one space dimension:

$$\phi \frac{\partial s_w}{\partial t^*} - \frac{\partial}{\partial x^*}\left(\frac{k s_w}{\mu_w}\frac{\partial p}{\partial x^*}\right) = 0 \tag{5.5}$$

$$\phi \frac{\partial s_o}{\partial t^*} - \frac{\partial}{\partial x^*}\left(\frac{k s_o}{\mu_o}\frac{\partial p}{\partial x^*}\right) = 0 \tag{5.6}$$

$$s_w + s_o = 1. \tag{5.7}$$

Define the total flux of both water and oil as $u^* = u_w^* + u_o^*$. Then from the one-dimensional version of (5.2) and(5.5) to (5.7) we get that

$$\begin{aligned}\frac{\partial u^*}{\partial x^*} &= \frac{\partial u_w^*}{\partial x^*} + \frac{\partial u_o^*}{\partial x^*} = \frac{\partial}{\partial x^*}\left(-\frac{k s_w}{\mu_w}\frac{\partial p}{\partial x^*}\right) + \frac{\partial}{\partial x^*}\left(-\frac{k s_o}{\mu_o}\frac{\partial p}{\partial x^*}\right)\\ &= -\phi\frac{\partial s_w}{\partial t^*} - \phi\frac{\partial s_o}{\partial t^*} = -\phi\frac{\partial}{\partial t^*}(s_w + s_o) = 0,\end{aligned}$$

and thus u^* is constant in space, say $u^*(x^*, t^*) = C(t^*)$ for some constant function C. Let us assume that the phases for water and oil is initially divided into the negative and positive real lines, respectively. Furthermore, we require that $\lim_{x^* \to -\infty} u^*(x^*, t^*) = \bar{u} \geq 0$ for a constant $\bar{u}$, that is, water is injected from the left at a constant rate.

This however gives us

$$\bar{u} = \lim_{x^* \to -\infty} u^*(x^*, t^*) = \lim_{x^* \to -\infty} C(t^*) = C(t^*),$$

hence $u^*(x^*, t^*) = \bar{u}$ is constant for all x^* and t^*. Now using Darcy's law (5.2), we have that

$$\bar{u} = u^* = -\left(\frac{s_w}{\mu_w} + \frac{s_o}{\mu_o}\right) k \frac{\partial p}{\partial x^*},$$

such that

$$k\frac{\partial p}{\partial x^*} = -\frac{\mu_o}{\frac{\mu_o}{\mu_w} s_w + s_o}\bar{u}.$$

Furthermore, define the fractional flow

$$f(s_w) = \frac{\mu_o}{\mu_w}\left(\frac{s_w}{\frac{\mu_o}{\mu_w} s_w + s_o}\right) = \frac{\mu_r s_w}{\mu_r s_w + s_o} = \frac{\mu_r s_w}{1 + (\mu_r - 1) s_w}, \tag{5.8}$$

where the viscosity ratio is defined as $\mu_r = \frac{\mu_o}{\mu_w}$ and $\mu_r > 1$.

Note that f in reality is only a function of s_w because of (5.7). This gives us

$$k\frac{s_w}{\mu_w}\frac{\partial p}{\partial x^*} = -f(s_w)\bar{u},$$

which is inserted into (5.5) to get

$$\phi \frac{\partial s_w}{\partial t^*} + \frac{\partial}{\partial x^*}\left(f(s_w)\bar{u}\right) = 0. \tag{5.9}$$

Let the injection well be located at $x_J = 0$ and also consider the production well to be located at $x_L > 0$. The water is injected at a constant rate from the left, so that $s_w(0, t^*) = 1$ for all times. Furthermore, lets assume we have no water in the reservoir initially, when this happens we can see that $s_w(x^*, 0) = 0$ for all $x^* \in (x_J, x_L]$. Note that the equation in (5.9) is unscaled, so putting the unscaled equation in (5.9) along side with some boundary values, we have

$$\begin{cases} \phi \dfrac{\partial s_w}{\partial t^*} + \dfrac{\partial}{\partial x^*}\left(f(s_w)\bar{u}\right) = 0 & 0 < x^* < x_L,\ t^* > 0 \\ s_w(0, t^*) = 1 & x^* = 0,\ t \geq 0 \\ s_w(x^*, 0) = 0 & 0 < x^* \leq x_L,\ t^* = 0. \end{cases}$$

Using dimensional analysis here, we scale (5.9) by obvious scaling with $x^* = x_L x$ and setting $t^* = Tt$, we find

$$\frac{\partial s_w}{\partial t} + \frac{\bar{u}T}{\phi x_L}\frac{\partial}{\partial x}\left(f(s_w)\right) = 0.$$

Also balancing further we get $T = \phi x_L/\bar{u}$, such that if we define $s(x, t) = s_w(x_L x, Tt)$ we obtain the scaled equation

$$\begin{cases} \dfrac{\partial s}{\partial t} + \dfrac{\partial}{\partial x}\left(f(s)\right) = 0 & 0 < x < 1,\ t > 0 \\ s(0, t) = 1 & x = 0,\ t \geq 0 \\ s(x, 0) = 0 & 0 < x \leq 1,\ t = 0, \end{cases} \tag{5.10}$$

which is an hyperbolic boundary value problem, where the PDE is known as the *Buckley–Leverett equation.*

Note that the model presented here ignores the effect of capillary pressure, which gives a pressure difference between the water and the oil at the same spot in the reservoir. The net effect is to make the fractional flow function S-shape. Because of the inflection point, the solution of the Riemann problem is a bit more complicated.

Now let A^* denote the amount of water that goes into the production well before all the oil initially contained in the reservoir has been extracted. We also introduce a scaling for the flux, that is $u^* = \bar{u}U$, then we have that $U(x, t) = 1$ by our previous results. Also $A^* = Aa$, where $A = T\bar{u}$ is the natural scaling for the amount of water, because

$$A^* = \int_{t_1^*}^{t_2^*} u^*(x_L, t^*)\,\mathrm{d}t^* = \bar{u}(t_2^* - t_1^*), \tag{5.11}$$

where $t_1^* = \min\{t^* > 0 : s_w(x_L, t^*) > 0\}$ and $t_2^* = \min\{t^* > 0 : s_w(x_L, t^*) = 1\}$. Now we analyze the model further to find $t_1 = \min\{t > 0 : s(1, t) > 0\}$, and $a = t_2 - t_1$, where $t_2 = \min\{t > 0 : s(1, t) = 1\}$.

5.3 ANALYSIS OF THE MODEL

We analyze the model given above by using the method of characteristics to solve (5.10). Let $z(t) = s(x(t), t)$ be the solution of the hyperbolic equation. Therefore we find the characteristic equations as

$$\begin{aligned} \dot{z} &= 0, \qquad z(t_0) = s(x(t_0), t_0) = s(x_0, t_0) \\ \dot{x} &= f'(z), \quad x(t_0) = x_0, \end{aligned}$$

with solutions

$$\begin{aligned} x(t) &= x_0 + (t - t_0) f'(s(x_0, t_0)) \\ z(t) &= s(x_0, t_0). \end{aligned} \tag{5.12}$$

As a result of the Dirichlet boundary conditions we have that, there exists two cases for the characteristics equations which is given as

$$x(t) = \begin{cases} x_0 + t f'(0), \text{ for } x_0 > 0 \text{ and } t_0 = 0 \text{ such that } s(x_0, t_0) = 0, \\ (t - t_0) f'(1), \text{ for } x_0 = 0,\ t_0 > 0 \text{ such that } s(x_0, t_0) = 1. \end{cases}$$

Calculating the characteristic or kinematic speed we have

$$f'(s) = \frac{\mu_r}{[1 + (\mu_r - 1)s]^2}, \tag{5.13}$$

where $f'(0) = \mu_r$ and $f'(1) = 1/\mu_r$. Since $f'(0) > 0$ and $f'(1) > 0$, we get inflow at the the boundary $x = 0$, and the characteristic lines are given by

$$x(t) = x_0 + \mu_r t \quad \text{for} \quad x_0 > 0, \qquad \text{and} \qquad x(t) = (t - t_0)/\mu_r \quad \text{for} \quad x_0 = 0, t_0 > 0.$$

The characterstic diagram is plotted in Figure 5.1. Since both $f'(0)$ and $f'(1)$ are greater than zero for $\mu_r > 1$, we get a dead sector or a gap between the characteristics lines and this kind of characteristic behaviour is seen in previous chapters in this thesis. Now we solve the problem of the dead sector by filling the gap created with rarefaction wave or fan and the solution is given by letting $s(x, t) = \varphi(x/t)$, where φ is such that the differential equation in (5.10) holds. This gives us the criterion that $f'(\varphi(x/t)) = x/t$, and we obtain

$$\varphi(x/t) = \frac{1}{\mu_r - 1}\left(\sqrt{\frac{\mu_r}{x/t}} - 1\right). \tag{5.14}$$

Also you will realize that we have ignored the negative square root since $s \in [0, 1]$. Therefore we find the rarefaction wave solution as

$$s(x, t) = \begin{cases} 1 & \text{if } x < t/\mu_r \\ \varphi(x/t) & \text{if } t/\mu_r < x < t\mu_r \\ 0 & \text{if } x > t\mu_r \end{cases} \tag{5.15}$$

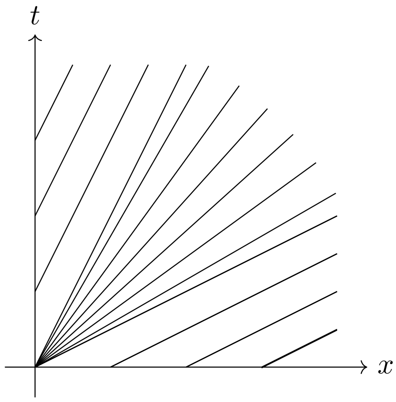


Figure 5.1: Characteristic diagram.

From our scaling, we obtain $t_1 = 1/\mu_r$ and $t_2 = \mu_r$, and hence the time it takes for the water to reach the production well is given by

$$t_1^* = \frac{\phi x_L}{\bar{u}\mu_r}, \tag{5.16}$$

and the amount of water that goes through L before all the oil initially contained between J and L has been recovered is given by

$$A^* = T\bar{u} \cdot a = \phi x_L \left(\mu_r - \frac{1}{\mu_r}\right) = \phi x_L \mu_r \left(1 - \frac{1}{\mu_r^2}\right).$$

From the above analysis, we can observe that when μ_r increases, t_1^* also reduces. Hence, highly viscous oil will cause water to enter the production well quickly. Furthermore, the amount of water that goes through the well A^* is approximately the same as $\phi x_L \mu_r$ when μ_r is large, so the amount of water depends almost linearly on the viscosity ratio. Therefore, highly viscous oil will cause lots of water to be extracted in the production well, which in this situation will be problematic.

Moreover, t_1^* depends linearly on $1/\bar{u}$, and therefore too high volume flux will cause water to enter the production well quickly. A similar result also holds for increased porosity ϕ.

We can also observe that both t_1^* and A^* depends linearly on x_L. Therefore as a result of the above consequences of our analysis, if an oil production company wants to stop the production before water enters the production well, x_L it should be chosen to be large. However, if the oil company is able to perform a separation process for the oil and water after extraction, although it is expensive, then x_L should be chosen small.

Chapter 6

Conclusion

In this thesis, we study and discuss the use of conservation principles to simplify physical problems by formulating simple assumptions which makes derivation of the conservation model more simpler to solve. We began by deriving a one dimensional equation by assuming that no production is made by the system. The one-dimensional equation is a conservation equation in differential form. The differential form of the conservation law is solved by method of characteristics which transforms the partial differential equation to an ordinary differential equation. The solutions obtained through this method is normally a weak solution which can also be used to find explicit solutions of the conservation laws. We realized that one requires a method or a tool to find the possible weak solution which solves the problem and this leads to the discussion of entropy conditions.

We identified three different forms of the entropy conditions in which the travelling wave and Lax entropy condition was discussed. In the travelling wave entropy condition, we assumed there is some diffusion in the conservation equation in which it is represented as a limit model when the diffusion is small. Using the travelling wave equation, we can deduce the Rankine–Hugoniot condition for some fixed points. Any solution that satisfies this condition is a weak solution. So by the travelling wave entropy condition, any isolated discontinuity must satisfy the entropy condition before its solution will hold. With these in mind, we can proceed to find an appropriate solution for the Riemann problem.

This thesis presents some Riemann problems with boundary conditions. The Riemann problem produced to two forms of solutions which depends on the nature and direction ot its speed; the solutions are rarefaction and shock solution.

Important assumptions were made in order to derive the shallow water wave equations. These assumptions includes; assuming a one dimensional channel flowing along the horizontal axis with an ideal incompressible and inviscid fluid, assuming that there is no vertical motion in the water, and the pressure distribution is hydrostatic. The derived shallow water equations is a non-linear hyperbolic system. The non-linearity of the shallow water wave equations makes finding analytic solutions difficult, and these solutions are restricted to a small set of problems. The equations often accept discontinuous solutions even when the the initial conditions are smooth. However solutions of these nature do not satisfy the partial differential equation form of the shallow water equations and so the integral form of the equations must be considered. We then use the integral form to find the Rankine–Hugoniot condition which hold whenever there is a shock. Also the

Rankine–Hugoniot condition does not guarantee a unique solution and so the physical admissible weak solutions must satisfy the Lax entropy condition.

A numerical scheme is implemented to give an approximate solution to shallow water wave equations. Lax–Friedrichs scheme of finite difference method is used to simulate the problem. The scheme is fast and explicit so in this case there is no need to solve simultanoeus equations at each time step. Getting the matlab codes to work wasn't easy but we manage to solve the shallow water problem by considering a dam break problem instead.

The thesis also present an industrial application to the use of conservation laws to model the flow of oil from a reservoir. The analysis and result of the model gives an idealized start up in the extraction of oil. This thesis employed most of the procedures by Helge Holden and Nils Henrik Risebro [5] and attempt was made to simplify, modify and understand some of the techniques used in analyzing and solving problems especially the shallow water wave equations. The thesis produced accurate results in the numerical solutions and I believe this results can be extended further to solve more complex models or problems.

In view of this, I hope we have demonstrated different mathematical formulations for the theory of conservation laws and its applications in continuum mechanics.

Chapter 7

SOURCE CODE FOR THE NUMERICAL SOLUTIONS

The codes presented below gives an intuition into the process required to prepare and implement both analytical and numerical solution.

To set up the numerical codes for both the physical and unphysical solution of the shallow water wave equations, we begin by setting the number of temporal (time) and spatial steps using the following commands

```
N = 500;
Nx = N/2 - 1;
dx = 1/Nx;
dt = 1/(N);
p = dt/dx;
```

Here, Omega is the spatial interval, h_l and h_r are the heights at the left and right state respectively. This forms a boundary condition and it is presented as

```
Omega = linspace(-2 + dx ,2 - dx , Nx - 1); %% Omega in (-2,2)
dist = abs(Omega(1) - Omega(2));
time = dt;
hr = 1.0; %% hl < hr gives physical solution,
hl = 0.5;  % hr < hl gives unphysical solution
```

We initialize the vectors at $t = 0$ which forms an initial condition (that is, $u(x, t = 0)$). Note that u is a solution of h and q.

```
%% Solutions at t = 0
ht0_lt = ones((Nx - 1)/2, 1) * hl;     % X < 0
ht0_leq = ones((Nx - 1)/2, 1) * hr;   % X > 0
qt0_lt = zeros((Nx - 1)/2, 1);         % X < 0 ...
                                          qt0_leq = ones((Nx - 1)/2, 1) * ......
       % X > 0
    (hl - hr)/sqrt(2) * sqrt(1/hr + 1/hl) * hr;
%% Shock speed and conditions below and above the shock speed*t
s = - hr * sqrt(1/hr + 1/hl) / sqrt(2); %% shock speed
%%
h_0 = hl; h_L = hr;   %%%% u(x,t) on boundary of Omega {-2, 2}, t > 0
q_0 = hl * 0;
```

```
q_L = hr * (hl - hr) / sqrt(2) * sqrt(1/hr + 1/hl);
Zh = [ h_0; ht0_lt; ht0_leq; h_L ];
Zq = [ q_0; qt0_lt; qt0_leq; q_L ];
Z0 = [Zh; Zq]; % initial vector
Zhnew = zeros(Nx - 1,1); Zqnew = zeros(Nx - 1,1);
```

We make space for the memory in mathlab. The first vector obtain at $t = 0$ is stored in the sequence of solutions (SOL). This is presented as

```
%% Allocate memory for solution and start the algorithm..
SOL = zeros(N,20*N); %% 2*(Nx + 1) = N;
SOL(:,1) = Z0;
tic %% time algorithm ..
```

Now we get one outer loop for the temporal steps and one inner loop for the spatial steps. Also for each time steps there is $Nx - 1$ spatial steps. The only spatial steps we don't solve for in the loop is the boundary points which are added at the end of the spatial loop. Then each temporal step is added to the global solution which is executed in the following process

```
for j = 2:20*N;
time = time + dt;
for i = 2:Nx;
if (i == 2)
    dist = Omega(1);
else
 dist = dist + ddist;
end
 Z(1:3) = Zh(i-1:i+1);
 Z(4:6) = Zq(i-1:i+1);
if (dist < s*time)
      [x] = hl;
      [y] = 0;
elseif (dist ≥ s*time)
      [x] = hr;
      [y] = -s * (hl - hr);
end
 Zhnew(i - 1) = x;
Zqnew(i - 1) = y;
end
Zh = [h_0; Zhnew; h_L];
Zq = [q_0; Zqnew; q_L];
SOL(:,j) = [Zh;Zq];
end
toc %% time algorithm ..
```

Then we run the solution. But to extract h and q solutions for plotting, we enter the following commands in mathlab after running the code.

```
h = SOL(1:Nx+1,: )
h = SOL(Nx+2:end,: )
```

Note that each of these are solutions at each temporal step starting at $t = 0$.

We also set up the codes for the analytical solution of the shallow water wave equations by using the dam break problem. The following commands are a bit similar to the codes above so the description of the codes is similar to what we had previously.

```
N = 500;
Nx = N/2 - 1;
dt = 1/N;
Omega = linspace(-3 , 3 , Nx + 1); %% Omega in [-3,3]
dist = Omega(1);
dx = abs(abs(Omega(1))-abs(Omega(2)));
p = dt/dx;
time = dt;
hr = 0.0; hl = 1.0; %% hl < hr
%% Shock speed and conditions below and above the shock speed*t
s = sqrt(hl); %% shock speed
hst_lt = ones((Nx - 1)/2, 1) * hl; hst_leq = zeros((Nx - 1)/2, 1);
qst_lt = zeros((Nx - 1)/2, 1); qst_leq = zeros((Nx - 1)/2, 1);
%%
ht0_lt = [hst_lt; hst_leq];
qt0_lt = [qst_lt; qst_leq];
h_0 = hl;
h_L = 0;
q_0 = 0;
q_L = 0;
Zh = [ h_0; ht0_lt; h_L ];
Zq = [ q_0; qt0_lt; q_L ];
ufun = @(x,t) 1/9 * (2*s - x./t).^2;
qfun = @(x,t) 2/27 * (2*s - x./t).^2 .* (s + x./t);
t0 = 1.3; % time lapse before dam breaking
Z0 = [Zh; Zq];
%% Allocate memory for solution and start the algorithm ..
SOL = zeros(N,5*N); %% 2*(Nx + 1) = N;
SOL(:,1) = Z0;
tic %% time algorithm ..
for j = 2:5*N;
time = time + dt;
Zhnew = zeros(Nx - 1,1);
Zqnew = zeros(Nx - 1,1);
%%
if(time > t0);
k = 2;
while((Omega(1) + k*dx) < -s*time);
    k = k + 1;
end
k1 = k;
temp1 = Omega(1) + k1*dx;
k = k1 + 1;
while(((temp1 + (k + 1)*dx) ≤ 2*s*time) && (k < (Nx - 1)));
```

```
    k = k + 1;
end
k2 = k;
Omeganew = linspace(-s*time,2*s*time,k2 - k1 + 1);
Wunew = ufun(Omeganew(:),time);
Wvnew = qfun(Omeganew(:),time);
end
%% Before time t0
if (time ≤ t0)
Zhnew(1:124) = hl;
Zqnew(1:124) = 0;
Zhnew(125:end) = 0;
Zqnew(125:end) = 0;
%% After time t0
elseif( time > t0 )
Zhnew(1:(k1-1)) = hl;
Zqnew(1:(k1-1)) = 0;
   Zhnew(k1:k2) = Wunew;
   Zqnew(k1:k2) = Wvnew;
Zhnew(k2 + 1:end) = 0;
Zqnew(k2 + 1:end) = 0;
end
Zh = [h_0; Zhnew; h_L];
Zq = [q_0; Zqnew; q_L];
SOL(:,j) = [Zh;Zq];
end
toc %% time algorithm ..
```

Furthermore, we also set up the numerical codes for the numerical solution of the shallow water wave equations by using the dam break problem.

```
%% Shallow water code using Lax-Friedrichs scheme
% We use periodic boundary conditions!
tic
%% Initial data and simulation parameters
M = 100; % number of grid points in the x direction
N = 200; % number of grid points in the t direction
dx= 0.01;
dt= 0.005;
% h0 and v0 are initial data, must be M by 1 arrays
h0=[ones(M/2,1);zeros(M/2,1)];
v0=zeros(M,1);
% h, q will hold the solution
h = zeros(M,N);
q = zeros(M,N);
halflambda = dt/(2*dx);
% insert the initial data
h(:,1) = h0;
q(:,1) = h0 .* v0;
for n = 1:(N-1)
    hm = circshift(h(:,n),1);  % hm(i)=h(i-1,n)
```

```
    hp = circshift(h(:,n),-1); % hp(i)=h(i+1,n)
    qm = circshift(q(:,n),1);  % qm(i)=q(i-1,n)
    qp = circshift(q(:,n),-1); % qp(i)=q(i+1,n)
    fq = q(:,n).^2 ./ (0.00001+h(:,n)) + 0.5 * h(:,n).^2;
    % note above kludge to avoid dividing by zero
    % these two implement Lax-Friedrichs:
    h(:,n+1) = 0.5*(hp+hm)-halflambda*(qp-qm);
    q(:,n+1) = 0.5*(qp+qm)-halflambda*(circshift(fq,-1)-circshift(fq,1));
end
toc
%% Finally, plot the result
x=linspace(0,(M-1)*dx,M)'*ones(1,N);
t=ones(M,1)*linspace(0,(N-1)*dt,N);
% a mesh with too many points is hard to plot
% include only grid points Dx apart in the x direction
% and Dt apart in the t direction
% Dx and Dt must be positive integers
% make them bigger if you make M, N bigger up above
Dx=2;
Dt=10;
mesh(x(1:Dx:end,1:Dt:end),t(1:Dx:end,1:Dt:end),h(1:Dx:end,1:Dt:end));
% There are probably better ways to plot this for inclusion in the thesis
```

The impletation of these codes were made possible by following some of the criteria used by Helge Holden and Nils Henrik Risebro [5].

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