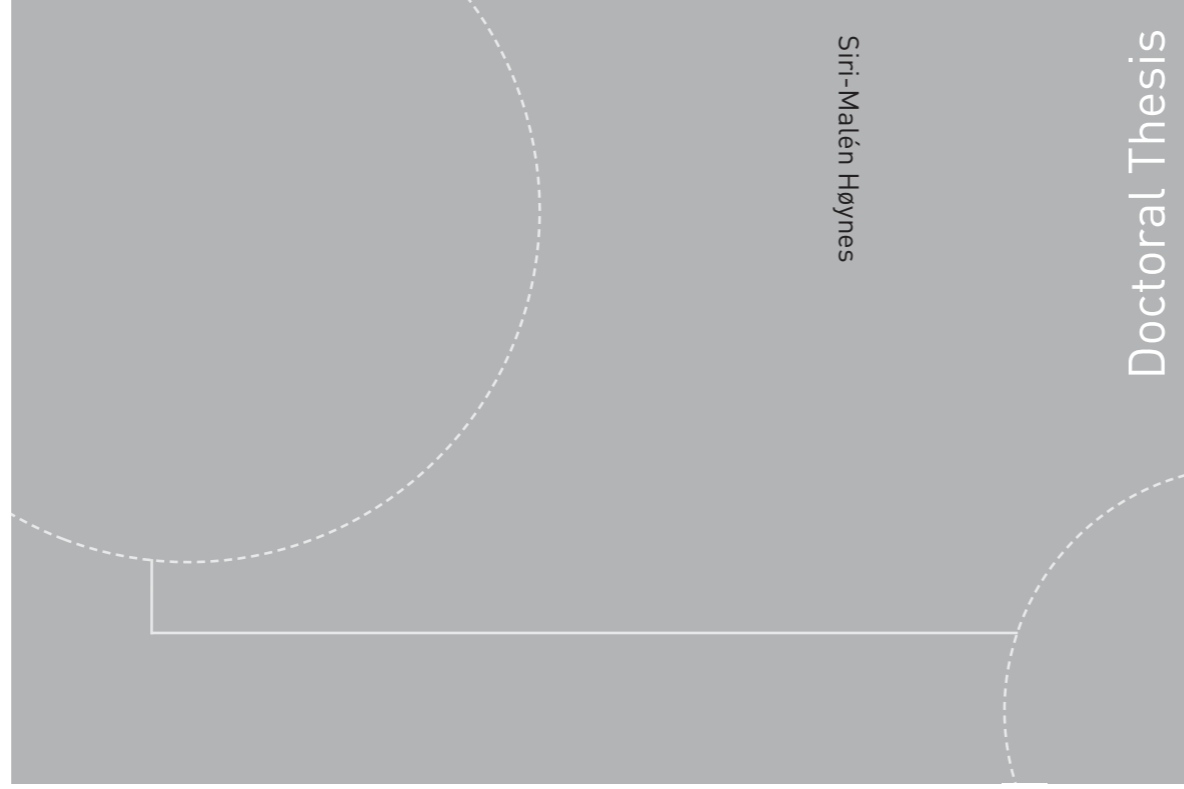


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Siri-Malén Høyenes

**Bratteli diagrams associated to
Toeplitz flows and expansive Cantor
minimal systems.**

Siri-Malén Høynes

Bratteli diagrams associated to Toeplitz flows and expansive Cantor minimal systems.

Thesis for the degree of Philosophiae Doctor

Trondheim, August 2014

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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (PhD) at the Norwegian University of Science and Technology (NTNU).

Firstly I would like to thank my co-advisor Christian Skau, who has been my de facto advisor during my PhD studies. Both for encouraging me to pursue a PhD and for his constant encouragement and valuable advice throughout my studies. He has pointed out interesting problems to look at and during many fruitful discussions he has shared his vast knowledge about this topic with me. I would also like to thank my advisor Toke Meier Carlsen for helpful advice.

During the first half of 2012 I had the pleasure of spending some months at the University of Victoria and the University of Ottawa. I would like to thank Professor Ian Putnam and Professor Thierry Giordano for hosting me and for helpful advice and discussions during my stay there. I would also like to thank Professor David Handelman for fruitful discussions during the stay at the University of Ottawa.

I want to thank my friends for interesting discussions and good company both during breaks at work, after work and during the numerous conferences I've traveled to during my studies. A special thanks to my family and boyfriend for their continued loving support and encouragement throughout my studies.

Trondheim
August 2014

Siri-Malén Høyenes

Outline of thesis

The thesis consists of an Introduction and the two papers:

Paper I

Toeplitz flows and their ordered K-theory. Accepted for publication in *Ergod. Th. and Dynam. Syst.*, 2014.

Paper II

Finite-rank Bratteli-Vershik diagrams are expansive – a new proof. Submitted to *Colloquium Mathematicum* for publication, 2014.

Introduction

This thesis consists of an Introduction and the two papers:

1. *Toeplitz flows and their ordered K-theory*
2. *Finite-rank Bratteli-Vershik diagrams are expansive – a new proof.*

Paper 1 studies the important class of dynamical systems called Toeplitz flows in terms of (ordered) Bratteli diagrams and K-theory. The thrust of the paper is to give a definite characterization of Toeplitz flows in terms of their K^0 -groups, which, in fact, simultaneously characterizes these dynamical systems in terms of their orbit structure.

Paper 2 contains a new proof of a result by T. Downarowicz and A. Maass [DM], while at the same time addressing the open problem they raise about a certain lower bound. We are able to find a significantly lower bound than they found, which we conjecture is optimal. The result proved in [DM] was used in paper 1 to exhibit examples of (ordered) Bratteli diagrams and compute the associated dimension groups, where the corresponding Bratteli-Vershik systems are Toeplitz flows. We found the proof in [DM] very hard to follow. However, the importance of the result proved in [DM] motivated us to find a more accessible proof.

The Introduction will give the background and set the stage, so to say, for the two papers. We will give a survey of topological dynamical systems, more specifically, Cantor minimal systems, and (ordered) Bratteli diagrams associated to these. The two papers are mainly treating special kinds of Cantor minimal systems, namely shift systems, also called symbolic dynamical systems. In particular, we will survey the family of Toeplitz flows, the type of shift systems that are the focus of paper 1. An important tool used in both papers to prove our results are Bratteli diagrams. In addition to explain what a Bratteli diagram is and how one can associate an ordered abelian group (dimension group) to it, we will explain the link between ordered Bratteli diagrams and Cantor minimal systems.

1 Topological dynamical systems

By a *topological dynamical system* we will mean a pair (X, G) , where X is a compact metrizable space and G is a countable (discrete) group acting as homeomorphisms on X . In this thesis we will use the term *dynamical system* to mean a compact metric space X together with a homeomorphism $T: X \rightarrow X$, and this dynamical system will be denoted by (X, T) . This induces in a natural way a \mathbb{Z} -action on X . The orbit of $x \in X$ under this action is $\{T^n x \mid n \in \mathbb{Z}\}$ and will be denoted by $\text{orbit}_T(x)$. If all the orbits are dense in X we say that (X, T)

is a *minimal system*. It is a simple observation that (X, T) is minimal if and only if $T(A) = A$ for some closed $A \subseteq X$ implies that $A = X$ or \emptyset . In this thesis we shall often consider minimal dynamical systems where X is a Cantor set, i.e. a totally disconnected compact space with no isolated points. In this case we say that (X, T) is a *Cantor minimal system*.

We will denote the natural numbers $\{1, 2, 3, \dots\}$ by \mathbb{N} , the integers by \mathbb{Z} , the rational numbers by \mathbb{Q} , the real numbers by \mathbb{R} . Also, let $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r \geq 0\}$, $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$. \mathbb{Z}_n will denote the group $\{0, 1, \dots, n-1\}$ where addition is done modulo n . When A and B are two sets $A - B$ will mean all elements in A except elements in $A \cap B$.

By a *map between two dynamical systems* (X, T) and (Y, S) we will mean a continuous map $\pi: X \rightarrow Y$ which satisfies $S(\pi(x)) = \pi(Tx)$, $\forall x \in X$. (Observe that this implies that $S^n(\pi(x)) = \pi(T^n x)$ for all $n \in \mathbb{Z}$.) If this map in addition is surjective we say that (X, T) is an *extension* of (Y, S) and that (Y, S) is a *factor* of (X, T) , and we call π a *factor map*. Sometimes we will use the notation $\pi: (X, T) \rightarrow (Y, S)$. Two dynamical systems are *conjugate* if the map between them is a bijection, and then we write $(X, T) \cong (Y, S)$. We say that (X, T) is *flip conjugate* to (Y, S) if $(X, T) \cong (Y, S)$ or $(X, T) \cong (Y, S^{-1})$. The dynamical systems (X, T) and (Y, S) are *orbit equivalent* if there exists a homeomorphism $F: X \rightarrow Y$ such that $F(\text{orbit}_T(x)) = \text{orbit}_S(F(x))$ for all $x \in X$. We call F an *orbit map*.

Remark 1.1. Clearly flip conjugacy implies orbit equivalence. It is a fact that if X (and hence Y) is a connected space then orbit equivalence between (X, T) and (Y, S) implies flip conjugacy. This follows by a simple argument using a result of Sierpiński, cf. [Ku, Theorem 6, Ch. V, §47, III]. This has the consequence that the study of orbit equivalence is only interesting as it pertains to *Cantor minimal systems* (X, T) , i.e. X is a Cantor set on which T acts minimally. The K -theoretic invariant we are going to introduce is an invariant for orbit equivalence, and so we will assume henceforth that our dynamical systems are Cantor minimal, even though some of the subsequent definitions apply to more general systems.

Let (X, T) , (Y, S) , both (X, T) and (Y, S) being Cantor minimal systems, be orbit equivalent with orbit map F . For each $x \in X$ there exists a unique integer $n(x)$ (respectively, $m(x)$) such that $F(Tx) = S^{n(x)}(F(x))$, $F(T^{m(x)}x) = S(F(x))$. We call $m, n: X \rightarrow \mathbb{Z}$ the orbit cocycles associated to the orbit map F .

Definition 1.2. We say that (X, T) and (Y, S) are strong orbit equivalent if there exists an orbit map $F: X \rightarrow Y$ such that each of the two associated orbit cocycles $m, n: X \rightarrow \mathbb{Z}$ have at most one point of discontinuity [GPS, Definition 1.3]

1.1 Entropy

If we have a continuous map $T: X \rightarrow X$ where (X, \mathcal{B}, μ) is a measure space and $\mu \in M(X, T)$, where $M(X, T)$ denotes the set of measure preserving transformations (i.e. $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}(X)$) we have the notion of *measure-theoretic entropy* $h_\mu(T)$ [Wa, Chapter 4]. For topological dynamical systems we also have a similar notion, *topological entropy*. We will define this concept as it was originally defined using open covers, and present a results describing the connection between the two notions of entropy. To do so, we need to introduce some concepts. If α and β are two open (finite) covers of X , then so is the *join* of α and β , $\alpha \vee \beta$, which consists of sets $A \cap B$ where $A \in \alpha$ and $B \in \beta$. If any set $A \in \alpha$ is a subset of a set in β then we say that α is a *refinement* of β and we write $\beta \prec \alpha$.

Definition 1.3. If α is an open cover of X we define the entropy of α to be $H(\alpha) = \log N(\alpha)$, where $N(\alpha)$ is the number of sets in a subcover of α with the smallest cardinality.

Definition 1.4. If α is an open cover of X and $T: X \rightarrow X$ is a continuous map then we define the *entropy of T relative to α* to be

$$h(T, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} T^{-i} \alpha \right).$$

Here $\bigvee_{i=0}^{n-1} T^{-i} \alpha$ means the join $\alpha \vee T^{-1} \alpha \vee \dots \vee T^{-(n-1)} \alpha$. (Notice that the continuity of T ensures that $T^{-i} \alpha$ is an open cover of X for all i .)

This limit does exists [Wa, Theorem 7.1].

Definition 1.5. The *topological entropy* of T is defined to be

$$h(T) = \sup_{\alpha} h(T, \alpha),$$

where α is an open cover of X .

Topological entropy is a conjugacy invariant [Wa, Theorem 7.2], and is a non-negative real number, possibly $+\infty$. The following result, which we state here without a proof, shows the close connection between the notion of measure-theoretic entropy and topological entropy.

Theorem 1.6. *Let $T: X \rightarrow X$ be a continuous map of a compact metric space X . Then $h(T) = \sup \{h_\mu(T) \mid \mu \in M(X, T)\}$.*

For a proof, see [Wa, Theorem 8.6].

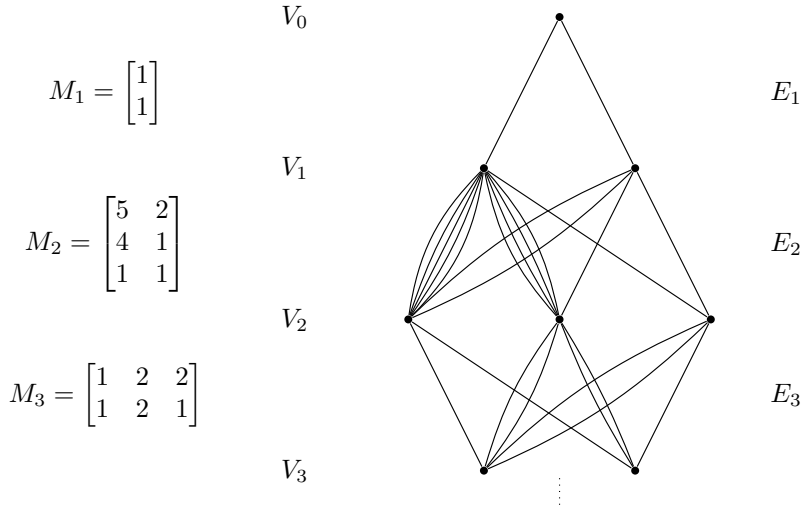


Figure 1: An example of a Bratteli diagram

2 Bratteli diagrams

A Bratteli diagram (V, E) consists of a set of vertices $V = \sqcup_{n=0}^{\infty} V_n$ and a set of edges $E = \sqcup_{n=1}^{\infty} E_n$, where the V_n 's and the E_n 's are finite disjoint sets and where $V_0 = \{v_0\}$ is a one-point set. The edges in E_n connect vertices in V_{n-1} with vertices in V_n . If e connects $v \in V_{n-1}$ with $u \in V_n$ we write $s(e) = v$ and $r(e) = u$, where $s: E_n \rightarrow V_{n-1}$ and $r: E_n \rightarrow V_n$ are the source and range maps, respectively. We will assume that $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and that $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$. A Bratteli diagram can be given a diagrammatic presentation with V_n the vertices at level n and E_n the edges between V_{n-1} and V_n . If $|V_{n-1}| = t_{n-1}$ and $|V_n| = t_n$ then the edge set E_n is described by a $t_n \times t_{n-1}$ incidence matrix $M_n = (m_{ij}^n)$, where m_{ij}^n is the number of edges connecting $v_i^n \in V_n$ with $v_j^{n-1} \in V_{n-1}$ (see Figure 1). In such a diagrammatic presentation of the Bratteli diagram we can also illustrate the source and range maps as seen in Figure 2. Let $k, l \in \mathbb{Z}^+$ with $k < l$ and let $E_{k+1} \circ E_k \circ \cdots \circ E_l$ denote all the paths from V_k to V_l . Specifically, $E_{k+1} \circ E_k \circ \cdots \circ E_l = \{(e_{k+1}, \dots, e_l) \mid e_i \in E_i, i = k+1, \dots, l; r(e_i) = s(e_{i+1}), i = k+1, \dots, l-1\}$.

We define $r((e_{k+1}, \dots, e_l)) = r(e_l)$ and $s((e_{k+1}, \dots, e_l)) = s(e_{k+1})$. Notice that the corresponding incidence matrix is the product $M_l M_{l-1} \cdots M_{k+1}$ of the incidence matrices.

Definition 2.1. Given a Bratteli diagram (V, E) and a sequence $0 = m_0 < m_1 <$

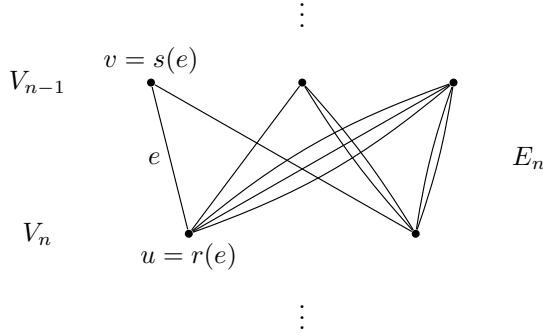


Figure 2: Here $t_{n-1} = 3$, $t_n = 2$, and the source of the edge $e \in E_n$ is $s(e) = v \in V_{n-1}$ and the range, $r(e)$ is $u \in V_n$.

$m_2 < \dots$ in \mathbb{Z}^+ , we define the *telescoping* of (V, E) to $\{m_n\}$ as (V', E') , where $V'_n = V_{m_n}$ and $E'_n = E_{m_{n-1}+1} \circ \dots \circ E_{m_n}$, and the source and the range maps are as above.

Definition 2.2. We say that the Bratteli diagram (V, E) is *simple* if there exists a telescoping of (V, E) such that the resulting Bratteli diagram (V', E') has full connection between all consecutive levels, i.e. the entries of all the incidence matrices are non-zero.

Given a Bratteli diagram (V, E) we define the infinite path space associated to (V, E) , namely

$$X_{(V,E)} = \{(e_1, e_2, \dots) \mid e_i \in E_i, r(e_i) = s(e_{i+1}); \forall i \geq 1\}.$$

Clearly $X_{(V,E)} \subseteq \prod_{n=1}^{\infty} E_n$, and we give $X_{(V,E)}$ the relative topology, $\prod_{n=1}^{\infty} E_n$ having the product topology. Loosely speaking this means that two paths in $X_{(V,E)}$ are close if the initial parts of the two paths agree on a long initial stretch. Also, $X_{(V,E)}$ is a closed subset of $\prod_{n=1}^{\infty} E_n$, and is compact.

On $X_{(V,E)}$ we can define the metric d by $d(x, y) = \frac{1}{n}$ if $x = (e_1, e_2, \dots, e_{n-1}, e_n, \dots)$ and $y = (e_1, e_2, \dots, e_{n-1}, e'_n, \dots)$ and $e_n \neq e'_n$. This metric will be compatible with the topology on $X_{(V,E)}$. Let $p = (e_1, e_2, \dots, e_n) \in E_1 \circ \dots \circ E_n$ be a finite path starting at $v_0 \in V_0$. We define the *cylinder set* $U(p) = \{(f_1, f_2, \dots) \in X_{(V,E)} \mid f_i = e_i, i = 1, 2, \dots, n\}$. The collection of cylinder sets is a basis for the topology on $X_{(V,E)}$. The cylinder sets are clopen (i.e. closed and open) sets, and so $X_{(V,E)}$ is a compact, totally disconnected metric space – metric because the collection of cylinder sets is countable. If (V, E) is simple then $X_{(V,E)}$

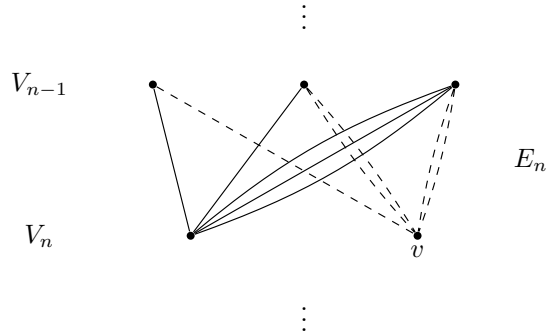


Figure 3: For $v \in V_n$ the edges in $r^{-1}(v) \subset E_n$ are the dashed edges.

has no isolated points, and so $X_{(V,E)}$ is a Cantor set. (We will in the sequel disregard the trivial case where $|X_{(V,E)}|$ is finite.)

Let $P_n = E_1 \circ \dots \circ E_n$ be the set of finite paths of length n (starting at the top vertex). We define the truncation map $\tau_n: X_{(V,E)} \rightarrow P_n$ by $\tau_n((e_1, e_2, \dots)) = (e_1, e_2, \dots, e_n)$. If $m \geq n$ we have the obvious truncation map $\tau_{m,n}: P_m \rightarrow P_n$.

There is an obvious notion of isomorphism between Bratteli diagrams (V, E) and (V', E') ; namely, there exists a pair of bijections between V and V' preserving the gradings and intertwining the respective source and range maps. Let \sim denote the equivalence relation on Bratteli diagrams generated by isomorphism and telescoping. One can show that $(V, E) \sim (V', E')$ iff there exists a Bratteli diagram (W, F) such that telescoping (W, F) to odd levels $0 < 1 < 3 < \dots$ yields a diagram isomorphic to some telescoping of (V, E) , and telescoping (W, F) to even levels $0 < 2 < 4 < \dots$ yields a diagram isomorphic to some telescoping of (V', E') .

2.1 Ordered Bratteli diagrams and the Bratteli-Vershik model

An *ordered Bratteli diagram* (V, E, \geq) is a Bratteli diagram (V, E) together with a partial order \geq in E so that edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. In other words, we have a linear order on each set $r^{-1}(v)$, $v \in V \setminus V_0$ (see Figure 3). We let E_{\min} and E_{\max} , respectively, denote the minimal and maximal edges in the partially ordered set E .

Note that if (V, E, \geq) is an ordered Bratteli diagram and $k < l$ in \mathbb{Z}^+ , then the set $E_{k+1} \circ E_{k+2} \circ \dots \circ E_l$ of paths from V_k to V_l with the same range can be

given an induced (lexicographic) order as follows:

$$(e_{k+1} \circ e_{k+2} \circ \cdots \circ e_l) > (f_{k+1} \circ f_{k+2} \circ \cdots \circ f_l)$$

if for some i with $k+1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. If (V', E') is a telescoping of (V, E) then, with this induced order from (V, E, \geq) , we get again an ordered Bratteli diagram (V', E', \geq) .

Definition 2.3. We say that the ordered Bratteli diagram (V, E, \geq) , where (V, E) is a simple Bratteli diagram, is *properly ordered* if there exists a unique min path $x_{\min} = (e_1, e_2, \dots)$ and a unique max path $x_{\max} = (f_1, f_2, \dots)$ in $X_{(V, E)}$. (That is, $e_i \in E_{\min}$ and $f_i \in E_{\max}$ for all $i = 1, 2, \dots$)

Let (V, E) be a properly ordered Bratteli diagram, and let $X_{(V, E)}$ be the path space associated to (V, E) . Then $X_{(V, E)}$ is a Cantor set. Let $T_{(V, E)}$ be the *lexicographic map* on $X_{(V, E)}$, i.e. if $x = (e_1, e_2, \dots) \in X_{(V, E)}$ and $x \neq x_{\max}$ then $T_{(V, E)}x$ is the successor of x in the lexicographic ordering. Specifically, let k be the smallest natural number so that $e_k \notin E_{\max}$. Let f_k be the successor of e_k (and so $r(e_k) = r(f_k)$). Let $(f_1, f_2, \dots, f_{k-1})$ be the unique least element in $E_1 \circ E_2 \circ \cdots \circ E_{k-1}$ from $s(f_k) \in V_{k-1}$ to the top vertex $v_0 \in V_0$. Then $T_{(V, E)}((e_1, e_2, \dots)) = (f_1, f_2, \dots, f_k, e_{k+1}, e_{k+2}, \dots)$. We define $T_{(V, E)}x_{\max} = x_{\min}$. Then it is easy to check that $T_{(V, E)}$ is a minimal homeomorphism on $X_{(V, E)}$. We note that if $x \neq x_{\max}$ then x and $T_{(V, E)}x$ are *cofinal*, i.e. the edges making up x and $T_{(V, E)}x$, respectively, agree from a certain level on. We will call the Cantor minimal system $(X_{(V, E)}, T_{(V, E)})$ a *Bratteli-Vershik system*. There is an obvious way to telescope a properly ordered Bratteli diagram, getting another properly ordered Bratteli diagram, such that the associated Bratteli-Vershik systems are conjugate – the map implementing the conjugacy is the obvious one. By telescoping we may assume without loss of generality that the properly ordered Bratteli diagram has the property that at each level all the minimal edges (respectively the maximal edges) have the same source.

Theorem 2.4 ([HPS]). *Let (X, T) be a Cantor minimal system. Then there exists a properly ordered Bratteli diagram (V, E, \geq) such that the associated Bratteli-Vershik system $(X_{(V, E)}, T_{(V, E)})$ is conjugate to (X, T) .*

Proof sketch. Let $x_0 \in X$ and let $\{U_n\}_{n \in \mathbb{Z}_+}$ be a decreasing sequence of clopen sets of X such that $U_0 = X$ and $U_n \searrow \{x_0\}$. For each U_n we construct a finite number of towers “built” over U_n in the following way:

For $n \in \mathbb{N}$, let $\lambda_n: U_n \rightarrow \mathbb{N}$ (where \mathbb{N} is given the discrete topology) be defined by $\lambda_n(y) = \inf \{m \in \mathbb{N} \mid T^m y \in U_n\}$. Since we chose U_n to be clopen, λ_n will be continuous, so $\lambda_n(U_n) \subseteq \mathbb{N}$ is compact, i.e.

$$\lambda_n(U_n) = \{m_1, m_2, \dots, m_{k_n}\}.$$

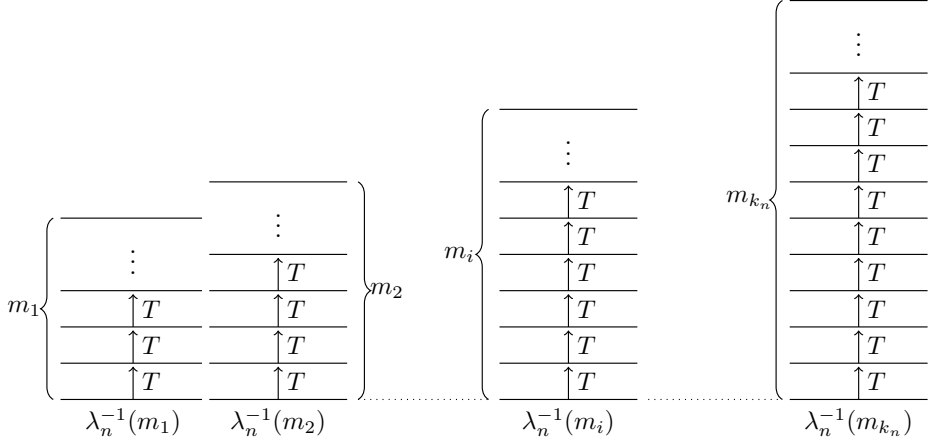


Figure 4: k towers of height m_i , $i = 1, 2, \dots, k_n$ (here $m_1 < m_2 < \dots < m_{k_n}$).

The collection $\{\lambda_n^{-1}(m_1), \lambda_n^{-1}(m_2), \dots, \lambda_n^{-1}(m_{k_n})\}$ will be a clopen partition of U_n . By construction $T^j(\lambda_n^{-1}(m_i)) \not\subset U_n$ when $0 < j < m_i$, and $T^{m_i}(\lambda_n^{-1}(m_i)) \subset U_n$ for all $i = 0, 1, \dots, k_n$. Thus we get k_n towers over the subset U_n where T maps one subset onto the next subset in the tower as seen in Figure 4. T applied to the top level of any tower is somewhere in U_n since $T^{m_i}(\lambda_n^{-1}(m_i)) \subset U_n$ for $i = 1, 2, \dots, k_n$. For convenience we label the sets such that $U_n(i, j)$ is the j -th floor in the i -th tower (the ground floor is the zero-th floor), i.e. $U_n(i, j) = T^j(\lambda_n^{-1}(m_i))$. By construction,

$$\tilde{X} = \bigcup_{i=1}^{k_n} \bigcup_{j=0}^{m_i-1} U_n(i, j)$$

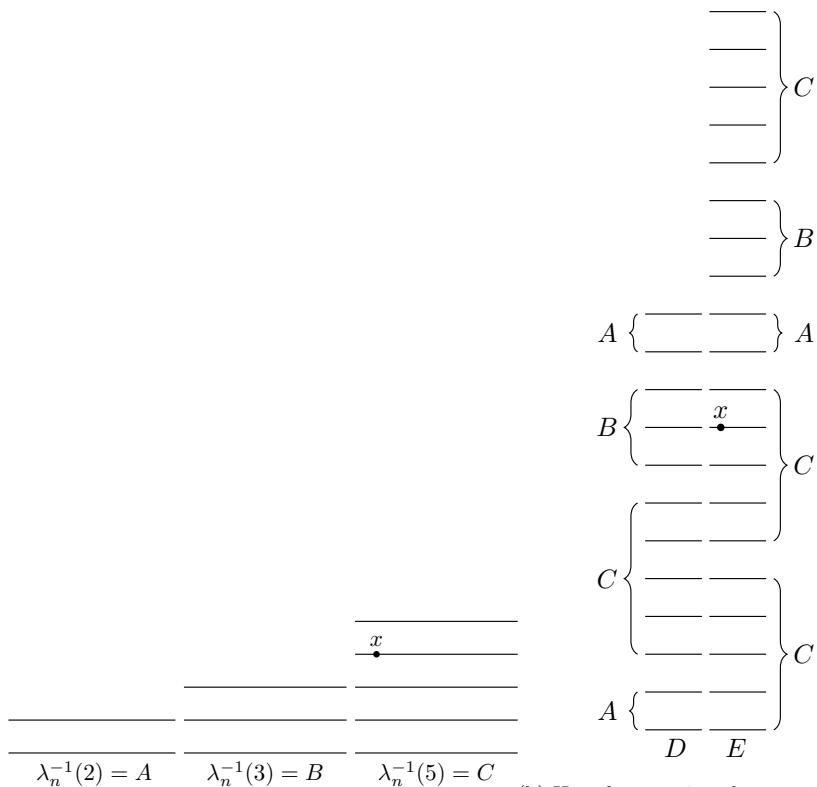
is closed, T -invariant and contains the non-empty set U_n , so $\tilde{X} = X$. T being a homeomorphism ensures that $Y(i, j) \cap Y(i', j') = \emptyset$ when $(i, j) \neq (i', j')$ so $\{U_n(i, j) \mid i = 1, 2, \dots, k_n, j = 0, 1, \dots, m_i - 1\}$ is a clopen partition of X . These towers may be vertically subdivided, giving rise to more towers (some of them of the same height), such that we obtain the following scenario: The clopen partitions $\{\mathcal{P}_n\}_{n \in \mathbb{Z}^+}$ of X that the towers associated to the various U_n 's generate are nested, $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \dots$, and the union of the \mathcal{P}_n 's is a basis for the topology of X . We build the properly ordered Bratteli diagram (V, E, \geq) by letting the vertices V_n at level n correspond to the various towers built over U_n , so if we have k_n different towers built over U_n then we will have $|V_n| = k_n$. The edges between levels $n - 1$ and n and their ordering is determined by the order in which the towers at level n traverse the towers at level $n - 1$.

To illustrate what we mean by this, we look at an example. Assume that over

U_n we get 3 towers, as seen in Figure 5a, and over the set U_{n+1} we get 2 towers, as seen in Figure 5b. (We omit indicating the T -map.) We construct towers over U_{n+1} , where the various ground floors will be subsets of the various ground floors in Figure 5a. Specifically, the subdivision mentioned above is done in such a way that $U_{n+1}(1, 0) \subseteq U_n(i, 0)$ and $U_{n+1}(2, 0) \subseteq U_n(j, 0)$, for some $i, j \in \{1, 2, 3\}$. Say $i = 1$ and $j = 3$ (in general we can also have $i = j$). Let us now look at the tower with floor $U_{n+1}(1, 0)$. Since $U_{n+1}(1, 0) \subseteq U_n(1, 0)$ we must also have $U_{n+1}(1, 1) \subseteq U_n(1, 1)$. By the construction of the towers over U_n we know that $T(U_{n+1}(1, 1)) \subset U_n$, if we also have $T(U_{n+1}(1, 1)) \subset U_{n+1}$ then the first tower is complete, so assume that this is not the case. Then $T(U_{n+1}(1, 1)) \subset U_n(i, 0)$ for some $i \in \{1, 2, 3\}$, say in this case $i = 3$. Then by the same argument as above, we will have to traverse the tower over $U_n(3, 0)$ before we will again end up back in U_n . We continue this process until we end up back in U_{n+1} , and in each step we will end up traversing one of the towers in the tower construction over U_n as seen in Figure 5b. In this example, level n and $n + 1$ of the corresponding Bratteli diagram would be as in Figure 6. We call the diagram we get by this construction (V, E) . Given a point $x \in X$ we can associate an infinite path in $X_{(V, E)}$. The point x seen in Figure 5 can be associated (uniquely) to an edge in E_{n+1} in the following way. In the tower construction over U_n , x lies in tower C and in the tower construction over U_{n+1} , x lies in tower E , hence x shall be associated to an edge between C and E . Since x lies in the part of the tower that corresponds to the second traverse of a tower from the previous construction, edge number 2 is the correct one. Doing this at every level gives the infinite path that corresponds to x . Now we have that $x_0 \in U_n$ for all n , so x_0 will correspond to the unique min path x_{\min} in the ordered Bratteli diagram (V, E) . It is easy to see that Tx will correspond to the “next path” in the lexicographic order. \square

Let (V, E, \geq) be a properly ordered Bratteli diagram, and let $(X_{(V, E)}, T_{(V, E)})$ be the associated Bratteli-Vershik system. For each $k \in \mathbb{N}$ let P_k as above denote the paths from V_0 to V_k , i.e. the paths from $v_0 \in V_0$ to some $v \in V_k$. We define the map $\pi_k : X_{(V, E)} \rightarrow P_k^{\mathbb{Z}}$ by $\pi_k(x) = \left(\tau_k(T_{(V, E)}^n x) \right)_{n=-\infty}^{\infty}$, where $\tau_k : X_{(V, E)} \rightarrow P_k$ is the truncation map. Let S_k denote the shift map on $P_k^{\mathbb{Z}}$. Then the following diagram commutes

$$\begin{array}{ccc}
 X_{(V, E)} & \xrightarrow{T_{(V, E)}} & X_{(V, E)} \\
 \pi_k \downarrow & & \downarrow \pi_k \\
 X_k & \xrightarrow{S_k} & X_k
 \end{array}$$



(a) Here $k_n = 3$ and $m_1 = 2, m_2 = 3$ and $m_3 = 5$. (b) Here $k_{n+1} = 2$ and $m_1 = 12, m_2 = 20$ and the traversing of the towers over U_n in Figure 5a is indicated. $D = \lambda_{n+1}^{-1}(12)$ and $E = \lambda_{n+1}^{-1}(20)$

Figure 5: The towers over U_n and U_{n+1} , respectively.

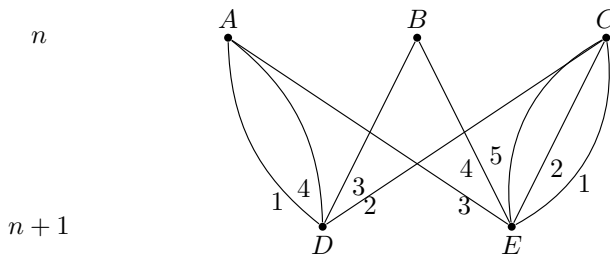


Figure 6: The ordered edges between levels n and $n + 1$ of the Bratteli diagram associated to the tower construction in Figure 5.

where $X_k = \pi_k(X_{(V,E)})$. One observes that π_k is a continuous map, and so X_k is a compact shift-invariant subset of $P_k^{\mathbb{Z}}$. So (X_k, S_k) is a factor of $(X_{(V,E)}, T_{(V,E)})$. For $k > l$ there is an obvious factor map $\pi_{k,l}: X_k \rightarrow X_l$ such that the following diagram commutes

$$\begin{array}{ccc} X_k & \xrightarrow{S_k} & X_k \\ \pi_{k,l} \downarrow & & \downarrow \pi_{k,l} \\ X_l & \xrightarrow{S_l} & X_l \end{array} .$$

If $m > k > l$ we have $\pi_{m,l} = \pi_{k,l} \circ \pi_{m,k}$. We set $\pi_{k,k} = \text{id}|_{X_k}$. It is now easy to show that $(X_{(V,E)}, T_{(V,E)})$ is the inverse limit $\varprojlim_{k \in \mathbb{N}} (X_k, S_k)$. In fact, let $(X, T) = \varprojlim_{k \in \mathbb{N}} (X_k, S_k)$. Then

$$X = \left\{ x = (x_k) \in \prod_{k \in \mathbb{N}} X_k \mid \pi_{n,m}(x_n) = x_m \text{ for all } m, n \in \mathbb{N} \text{ with } m \leq n \right\}$$

and $Tx = T((x_k)) = (S_k x_k)$. There are natural factor maps $\Theta: (X, T) \rightarrow (X_k, S_k)$. One easily sees that $(X, T) \cong (X_{(V,E)}, T_{(V,E)})$.

Definition 2.5. Let (V, E) be a Bratteli diagram. If there exists $K < \infty$ such that $|V_n| \leq K$ for all $n \geq 1$ then we say that (V, E) has *finite rank*. By telescoping we may in this case actually assume that $|V_n|$ has the same value ($\leq K$) for all $n \geq 1$. If the Bratteli diagram (V, E) is such that $|V_n|$ has the same value ($\leq K$) for all $n \geq 1$ and in addition the incidence matrices M_n for all $n \geq 2$ are equal we say that the Bratteli diagram is *stationary*. If we have an ordered Bratteli diagram (V, E, \geq) such that the Bratteli diagram (V, E) is stationary and the ordering is the same at all levels (beyond level 1) we say that (V, E, \geq) is a *stationary ordered Bratteli diagram*.

3 Expansive systems, shift systems and equicontinuous systems.

If $X = \prod_{\infty} \Lambda = \Lambda^{\mathbb{Z}}$, Λ a finite alphabet, is given the product topology it is easy to see that X is a Cantor set. The shift operator, $S: X \rightarrow X$, defined by $(Sx)(n) = x(n+1)$ is a homeomorphism on X , called the *full shift* (or Bernoulli shift) on Λ . If $Y \subseteq X$ is closed and S -invariant then (Y, S) is called a *subshift*. (By slight abuse of notation we will most of the time denote the restriction $S|_Y$ of S

to Y again by S .) If furthermore $S|_Y$ is minimal, the subshift $(Y, S) (= (Y, S|_Y))$ is a Cantor minimal system. (We will assume that Y is infinite, and thus (Y, S) is not a periodic (equivalently, finite) system.)

A dynamical system (X, T) , where the topology on X is given by the metric d is said to be *expansive* if there exists a constant $\delta > 0$ such that for all $x, y \in X$, $x \neq y$, there exists $n \in \mathbb{Z}$ such that $d(T^n x, T^n y) > \delta$. Note that the constant δ does *not* depend on the choice of x and y , and we say that the expansive system (X, T) has *expansive constant* δ . A subshift is easily seen to be expansive. In fact, all expansive Cantor minimal systems are conjugate to a subshift as we will see in Proposition 3.7. This justifies using the term *symbolic systems* about expansive Cantor minimal systems. We will now show that the notion of expansiveness does not depend on the metric d , but only on the topology of X following the proof given in [Wa, §5.6]. In order to prove this we need to introduced the following concept.

Definition 3.1. Let (X, T) be a dynamical system where X is a compact metric space. A finite open cover α of X is a *generator* for T if for every bisequence $\{A_i\}_{i=-\infty}^{\infty}$ with $A_i \in \alpha$ the set $\bigcap_{i=-\infty}^{\infty} T^{-i} \overline{A_i}$ contains at most one point. If we instead have that $\bigcap_{i=-\infty}^{\infty} T^{-i} A_i$ contains at most one point for every bisequence we say that α is a *weak generator* for T .

If a finite open cover α is a generator for T , then it is obviously also a weak generator for T . But in fact, if T has a weak generator then it also has a generator.

Proposition 3.2. T has a weak generator iff T has a generator.

Proof. Assume $\beta = \{B_1, \dots, B_k\}$ is a weak generator for T . Let $\delta > 0$ be the Lebesgue number of β (i.e. each subset of X of diameter less than or equal to δ lies in B_i for some $i = 1, \dots, k$). Let α be a finite open cover of X where all elements $A_i \in \alpha$ have $\text{diam}(A_i) \leq \delta$, (i.e. $d(x, y) \leq \delta$ for all $x, y \in A_i$), and hence $\text{diam}(\overline{A_i}) \leq \delta$. Now let $\{A_i\}_{i=-\infty}^{\infty}$ be a bisequence with $A_i \in \alpha$. Since $\text{diam}(\overline{A_i}) \leq \delta$, there is $B_{n_i} \in \beta$ such that $\overline{A_i} \subseteq B_{n_i}$. So $\bigcap_{i=-\infty}^{\infty} T^{-i} \overline{A_i} \subseteq \bigcap_{i=-\infty}^{\infty} T^{-i} B_{n_i}$. Since β is a weak generator, $\bigcap_{i=-\infty}^{\infty} T^{-i} \overline{A_i}$ must also contain at most one point, so α is a generator for T . \square

Theorem 3.3. Let T be a homomorphism on a compact, metric space (X, d) . Then T is expansive iff T has a generator iff T has a weak generator.

Proof. Assume T is expansive with expansive constant $\delta > 0$. Let α be a finite open cover consisting of balls of radius at most $\frac{\delta}{2}$. Assume $\{A_n\}_{n=-\infty}^{\infty}$ is a bisequence from α and $x, y \in \bigcap_{n=-\infty}^{\infty} T^{-n} \overline{A_n}$. Then for all $n \in \mathbb{Z}$ we have $d(T^n x, T^n y) \leq \delta$, so since T is expansive this must mean that $x = y$ and so α is a generator for T .

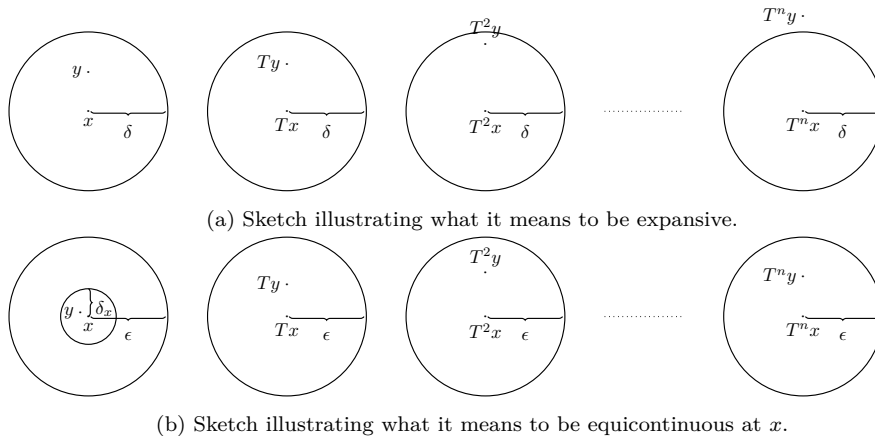


Figure 7: In the expansive system we can start with x and y arbitrarily close together, and still some T^n will map them at least δ apart. In the equicontinuous system, if we start with a y sufficiently close to x they will be mapped to points which are less than ϵ apart.

Conversely assume α is a weak generator for T . Let $\delta > 0$ be the Lebesgue number of α , and let $x, y \in X$. If $d(T^n x, T^n y) \leq \delta$ there is some $A_n \in \alpha$ such that $x, y \in A_n$. Now assume that $d(T^n x, T^n y) \leq \delta$ for all $n \in \mathbb{Z}$, then we get a bisequence $\{A_n\}_{n=-\infty}^{\infty}$ where $A_n \in \alpha$ such that $T^n x, T^n y \in A_n$. Then $x, y \in \bigcap_{n=-\infty}^{\infty} T^{-n} A_n$, so $x = y$ since α is a weak generator, so T is expansive. This together with Proposition 3.2 finishes the proof. \square

Corollary 3.4. *The concept of expansiveness of the dynamical system (X, T) is independent of the metric d (compatible with the topology on X).*

Proof. The concept of a generator is independent of which metric we put on X and as seen in Theorem 3.3 being expansive is equivalent to having a generator. The only thing depending on the metric is the expansive constant. \square

Definition 3.5. A dynamical system (X, T) , where (X, d) is a metric system, is *equicontinuous at $x \in X$* if given $\epsilon > 0$ there exists $\delta_x > 0$ s.t. $d(x, y) < \delta_x \Rightarrow d(T^n x, T^n y) < \epsilon, \forall n \in \mathbb{Z}$. If (X, T) is equicontinuous at every point $x \in X$ then we say that (X, T) is *equicontinuous*.

Remark 3.6. The notions of expansive systems and equicontinuous systems are diametrically opposite. This is illustrated in Figure 7.

Recall that (X_k, S_k) is the subshift of $P_k^{\mathbb{Z}}$ (where P_k is the set of all finite paths from $v_0 \in V_0$ to vertices in V_k) defined in section 2.1.

Proposition 3.7. *Assume $(X_{(V,E)}, T_{(V,E)})$ is expansive. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X_k, S_k) by the map $\pi_k: X_{(V,E)} \rightarrow X_k$.*

Proof. Since the π_k 's are factor maps, all we need to show is that there exists k_0 such that π_k is injective for all $k \geq k_0$. Recall that $(X_{(V,E)}, T_{(V,E)})$ being expansive means that there exists $\delta > 0$ such that given $x \neq y$ there exists $n_0 \in \mathbb{Z}$ such that $d(T_{(V,E)}^{n_0}x, T_{(V,E)}^{n_0}y) > \delta$, where d is some metric on $X_{(V,E)}$ compatible with the topology. Choose k_0 such that we have $d(x', y') < \delta$ for all x', y' that agree on the k_0 first edges. Now assume that $\pi_k(x) = \pi_k(y)$ for some $k \geq k_0$. By the definition of π_k this means that, for all $n \in \mathbb{Z}$, $\tau_k(T_{(V,E)}^n x) = \tau_k(T_{(V,E)}^n y)$, and so $d(T_{(V,E)}^n x, T_{(V,E)}^n y) < \delta$ for all $n \in \mathbb{Z}$ because of our choice of k_0 . This contradicts that $d(T_{(V,E)}^{n_0}x, T_{(V,E)}^{n_0}y) > \delta$. Hence π_k is injective for all $k \geq k_0$, proving the proposition. \square

Generally for a topological dynamical system (X, T) we have that $0 \leq h(T) \leq \infty$, but for expansive system we have the following result [Wa, Corollary 7.11.1].

Theorem 3.8. *An expansive homeomorphism has finite topological entropy.*

It is noteworthy that within the family of Cantor minimal systems \mathcal{C} expansiveness is a generic property. This means that the set of expansive systems is a dense \mathcal{G}_δ -set in \mathcal{C} , where \mathcal{C} is given an appropriate topology making it a Polish space (i.e. a complete metric space). For details, cf. [H].

3.1 Substitution systems

Let Λ be a finite alphabet, the elements of Λ are called *letters*, we let Λ^+ denote the set of finite words over this alphabet and $\Lambda^* = \Lambda^+ \cup \{e\}$, where e is the empty word. Elements of $\Lambda^{\mathbb{Z}}$ are called *sequences* over the alphabet Λ . It is obvious what we mean by a subword of a word in Λ^+ (or of a sequence $\Lambda^{\mathbb{Z}}$). The set of all subwords of a sequence $x \in \Lambda^{\mathbb{Z}}$ is the language $\mathcal{L}(x)$ associated to x . If we pick a point $x \in \Lambda^{\mathbb{Z}}$ then $\text{orbit}_T(x)$ will be a closed T -invariant subspace of $\Lambda^{\mathbb{Z}}$, so $(\text{orbit}_T(x), T)$ is a subshift. Here T denotes the subshift. This subshift will be minimal if and only if x is uniformly recurrent, i.e. all finite subwords of x will appear with bounded gaps in x [F, Theorems 1.15, 1.17 and Proposition 1.22]. By a *substitution* on the alphabet Λ we will mean a map $\theta: \Lambda \rightarrow \Lambda^+$. This can be extended to maps $\theta: \Lambda^+ \rightarrow \Lambda^+$ and $\theta: \Lambda^{\mathbb{Z}} \rightarrow \Lambda^{\mathbb{Z}}$ by concatenation, again denoting these by θ . Similarly, we can define $\theta^n: \Lambda \rightarrow \Lambda^+$ by concatenation. For

the substitution θ we define the language $\mathcal{L}(\theta)$ to be the set of words which are subwords of $\theta^n(\eta)$ for some $\eta \in \Lambda$ and for some $n \geq 1$.

We will henceforth only consider *primitive* substitutions $\theta: \Lambda \rightarrow \Lambda^+$, i.e. there exists $n > 0$ such that for all $a, b \in \Lambda$ we have that b is a subword of $\theta^n(a)$. (To avoid trivial cases we will assume that there exists $a \in \Lambda$ such that $\lim_{n \rightarrow \infty} |\theta^n(a)| \rightarrow \infty$, where $|\theta^n(a)|$ denotes the length of $\theta^n(a)$.)

Definition 3.9. Let X_θ be the set of $x \in \Lambda^{\mathbb{Z}}$ such that every finite subword of x is in $\mathcal{L}(\theta)$. Let T_θ be the shift on $\Lambda^{\mathbb{Z}}$ restricted to X_θ . X_θ is a closed subset of $\Lambda^{\mathbb{Z}}$ which is closed under the shift, so (X_θ, T_θ) is a subshift which we will call the *substitution dynamical system* associated to θ .

The following is a well-known result, cf. [Qu].

Theorem 3.10. *Every substitution dynamical system is minimal and uniquely ergodic.*

We are not interested in finite substitution systems (X_θ, T_θ) , i.e. $|X_\theta| < \infty$. So we will assume that θ is *aperiodic*, i.e. $|X_\theta| = \infty$. (There is an algorithm which decides whether a given substitution is aperiodic or not.)

Theorem 3.11 ([DHS]). *The family of Bratteli-Vershik systems associated with stationary, properly ordered Bratteli diagrams is (up to isomorphism) the disjoint union of the family of aperiodic substitution minimal systems and the family of stationary odometer systems. Furthermore, the correspondence in question is given by an explicit and algorithmic effective construction. The same is true of the computation of the dimension group associated with a substitution minimal system.*

Substitution minimal systems are countable up to conjugation. This stands in stark contrast to the next class of symbolic systems we will describe, namely Toeplitz flows, which are uncountable.

3.2 Toeplitz flows

Toeplitz flows were introduced by Jacobs and Keane in 1969 [JK]. In 1984 Susan Williams made a thorough analysis of Toeplitz flows [W]. Her constructions are important for our work, so we will give a survey of her analysis.

Definition 3.12. Let $\eta \in \Lambda^{\mathbb{Z}}$ be $\eta = (\eta(n))_{n \in \mathbb{Z}}$ where Λ is a finite alphabet. Then we define

$$\begin{aligned} \text{Per}_p(\eta, \sigma) &= \{n \in \mathbb{Z} \mid \eta(n + mp) = \sigma, \forall m \in \mathbb{Z}\}, \sigma \in \Lambda, \\ \text{Per}_p(\eta) &= \cup_{\sigma \in \Lambda} \text{Per}_p(\eta, \sigma). \end{aligned}$$

If all entries in η has some period, then we say that η is a *Toeplitz sequence*, i.e. η is Toeplitz if

$$\text{Aper}(\eta) = \mathbb{Z} - (\cup_{p \in \mathbb{N}} \text{Per}_p(\eta)) = \emptyset.$$

Observation 3.13. *If $p|q$, then we obviously have $\text{Per}_p(\eta) - q = \text{Per}_p(\eta)$.*

Definition 3.14. We say that p is an *essential period* of η if

$$\text{Per}_p(\eta) - q = \text{Per}_p(\eta) \Rightarrow p|q.$$

In particular, this means that $\text{Per}_p(\eta) - q \neq \text{Per}_p(\eta)$ for all $0 < q < p$ when p is an essential period of η .

Definition 3.15. Let $\eta \in \Lambda^{\mathbb{Z}}$ be a Toeplitz sequence, then the dynamical system $(\overline{\mathcal{O}(\eta)}, S)$, where $\mathcal{O}(\eta) = \text{orbit}_S(\eta) (= \{S^n \eta \mid n \in \mathbb{Z}\})$ and S is the shift operator restricted to $\mathcal{O}(\eta)$, is called a *Toeplitz system*.

Observation 3.16. $(\overline{\mathcal{O}(\eta)}, S)$ is a *minimal system*. (One shows namely easily that η is uniformly recurrent, and so by what we said above, the system will be minimal.)

A small subclass of Toeplitz flows can be constructed as substitution minimal systems. For example, the Feigenbaum sequence which is the fixed point of the substitution

$$\theta = \begin{cases} 0 \mapsto 11 \\ 1 \mapsto 10 \end{cases}$$

will give a substitution minimal system that is also a Toeplitz flow (the Feigenbaum sequence will be a Toeplitz sequence).

Definition 3.17. Let $\omega \in \overline{\mathcal{O}(\eta)}$. The *p-skeleton* of ω is

$$S_p(\omega)(k) = \begin{cases} \omega(k) & k \in \text{Per}_p(\omega) \\ * & k \notin \text{Per}_p(\omega) \end{cases}$$

Observe that if $\omega \in \Lambda^{\mathbb{Z}}$, then $S_p(\omega) \in (\Lambda \cup \{*\})^{\mathbb{Z}}$.

Definition 3.18. $(p_i)_{i \in \mathbb{N}}$, is said to be a *periodic structure* for a non-periodic Toeplitz sequence η if

1. for all i , p_i is an essential period for η
2. $p_i | p_{i+1}$ for all i
3. $\cup_i^{\infty} \text{Per}_{p_i}(\eta) = \mathbb{Z}$

Observation 3.19. *If $(p_i)_{i \in \mathbb{N}}$ is a periodic structure for η then*

$$\lim_{i \rightarrow \infty} S_{p_i}(\eta) = \eta$$

since $\text{Aper}(\eta) = \emptyset$

The following lemma is due to S. Williams [W], and is presented here with slight modifications (dropping the assumption that p is an essential period in points (i), (iii) and (iv)). We will in the sequel be a little imprecise with notation for the sake of brevity. This should not cause any misunderstanding. We will for example write $n \in \mathbb{Z}_p (= \mathbb{Z}/p\mathbb{Z})$ for $n \in \mathbb{Z}$ instead of the more precise $n + p\mathbb{Z} \in \mathbb{Z}_p$.

Lemma 3.20. *For $p \in \{2, 3, \dots\}$ and $n \in \mathbb{Z}_p$ define $\overline{A_n^p} = \overline{\{S^{n+mp}\eta \mid m \in \mathbb{Z}\}} \subset \overline{\mathcal{O}(\eta)}$. (Since $\overline{A_{i+p}^p} = \overline{A_i^p}$ we only need to consider $n \in \mathbb{Z}_p$.) We have*

$$(i) \ \omega \in \overline{A_n^p} \Rightarrow S_p(\omega) = S_p(S^n\eta).$$

(ii) *If p is an essential period of η , then $\{\overline{A_n^p} \mid n \in \mathbb{Z}_p\}$ is a partition of $\overline{\mathcal{O}(\eta)}$ of clopen sets in $\overline{\mathcal{O}(\eta)}$. In this case the implication in (i) is actually an equivalence.*

$$(iii) \ \overline{A_n^p} \supset \overline{A_m^q} \text{ when } p|q \text{ and } m \equiv n \pmod{p}.$$

$$(iv) \ S\overline{A_n^p} = \overline{A_{n+1}^p}.$$

Proof. (i) If $\omega \in \overline{A_n^p}$ then we see easily that $S_p(\omega) = S_p(S^n\eta)$. Since $\omega \in \overline{\{S^{n+mp}\eta \mid m \in \mathbb{Z}\}}$ we can not have that ω differs from $S^n\eta$ on elements that are a part of the p -skeleton of $S^n\eta$, the only possibility is that the p -skeleton of ω have additional entries. Assume that $\omega \in \overline{A_n^p}$. Assume now that $S_p(\omega) \neq S_p(S^n\eta)$, hence there must be $k \in \mathbb{Z}$ such that $S_p(\omega) = \sigma \in \Lambda$ and $S_p(S^n\eta)(k) = *$. Since $k \notin \text{Per}_p(\eta, \sigma)$ there must be $m_0 \in \mathbb{Z}$ such that $S^n\eta(k + m_0p) = \tau \neq \sigma$. In addition, there must be q such that $p|q$ such that $k + m_0p \in \text{Per}_q(\eta, \tau)$ since η is a Toeplitz sequence. Now, for any $m \geq 0$ we can find $\alpha \in \mathbb{Z}_{\frac{q}{p}}$ such that $q|(m + \alpha)p$.

$$\begin{aligned} S^{n+mp}\eta(k + m_0p + \alpha p) &= S^n\eta(k + m_0p + (\alpha + m)p) = \tau \neq \sigma \\ &= \omega(k + (m_0 + \alpha)p) = \omega(k + m_0p + \alpha p). \end{aligned}$$

So $d(\omega, S^{n+mp}\eta) \geq 2^{-k-m_0p-q}$ so ω can not be in $\overline{A_n^p}$.

(ii) If $x \in \mathcal{O}(\eta)$, then there exists $m \in \mathbb{Z}$ such that $x = S^m\eta$. Since $m = m' + lp$ for some $m' \in \mathbb{Z}_p$ and $l \in \mathbb{Z}$, $x = S^{m'+lp}\eta \in \overline{A_{m'}^p}$. Thus

$$\mathcal{O}(\eta) = \bigcup_{n=0}^{p-1} \overline{A_n^p}, \text{ so } \overline{\mathcal{O}(\eta)} = \bigcup_{n=0}^{p-1} \overline{A_n^p}.$$

Assume p is an essential period for η , that $n \neq m, n, m \in \mathbb{Z}_p$ and that $0 \leq m < n$. If this union is disjoint the sets $\overline{A_n^p}$ are open, so it remains to show that this union is disjoint. Assume ad absurdum that there exists $\omega \in \overline{A_n^p} \cap \overline{A_m^p}$. (i) then implies that $S_p(\omega) = S_p(S^n\eta) = S_p(S^m\eta)$. Since $\text{Per}_p(S^n\eta, \sigma) = \text{Per}_p(\eta, \sigma) - n$ and $\text{Per}_p(S^m\eta, \sigma) = \text{Per}_p(\eta, \sigma) - m$ for all $\sigma \in \Lambda$ this equality implies that $\text{Per}_p(\eta, \sigma) - (n - m) = \text{Per}_p(\eta, \sigma) \forall \sigma \in \Lambda$ so

$$\text{Per}_p(\eta) - (n - m) = \text{Per}_p(\eta), 0 < n - m < p$$

which contradicts that p is an essential period. Thus $\overline{A_n^p} \cap \overline{A_m^p} = \emptyset$. That we get an equivalence in (i) follows since $\omega \in \overline{A_m^p}$ for some $m, 0 \leq m \leq p$. Then $S_p(S^m\eta) = S_p(\omega) = S_p(S^n\eta)$ by (i), and since p is now an essential period we must have $n = m$.

(iii) If $p|q$ then $q = \alpha p$ for some $\alpha \in \mathbb{N}$ and $m \equiv n \pmod{p}$ gives $m = n + \beta p$ for some $\beta \in \mathbb{Z}$. Obviously

$$\begin{aligned} A_m^q &= \{S^{m+kq}\eta \mid k \in \mathbb{Z}\} = \{S^{n+(\beta+k\alpha)p}\eta \mid k \in \mathbb{Z}\} \\ &\subseteq \{S^{n+lp}\eta \mid l \in \mathbb{Z}\} = A_n^p, \end{aligned}$$

so $\overline{A_m^q} \subseteq \overline{A_n^p}$.

(iv) $A_n^p = \{S^{n+lp}\eta \mid l \in \mathbb{Z}\}$ so

$$SA_n^p = \{S^{1+n+lp}\eta \mid l \in \mathbb{Z}\} = A_{n+1}^p \pmod{p}.$$

S is a homeomorphism, so we also get $S\overline{A_n^p} = \overline{A_{n+1}^p} \pmod{p}$. □

Definition 3.21. Let $(p_i)_{i=1}^\infty$ be a periodic structure for the Toeplitz sequence η . For each $i \in \mathbb{N}$ define $\phi_i: \mathbb{Z}_{p_{i+1}} \rightarrow \mathbb{Z}_{p_i}$ by $\phi_i(n) = n \pmod{p_i}$, where $n \in \mathbb{Z}_{p_{i+1}}$.

$$\mathbb{Z}_{p_1} \xleftarrow{\phi_1} \mathbb{Z}_{p_2} \xleftarrow{\phi_2} \dots \mathbb{Z}_{p_{i-1}} \xleftarrow{\phi_{i-1}} \mathbb{Z}_{p_i} \xleftarrow{\phi_i} \mathbb{Z}_{p_{i+1}} \dots$$

We then define the *odometer group* $G_{\mathbf{p}}$ associated to the periodic structure $\mathbf{p} = (p_i)_{i=1}^\infty$ to be the inverse limit

$$G_{\mathbf{p}} = \varprojlim_i (\mathbb{Z}_{p_i}, \phi_i).$$

Remark 3.22. The group $G_{\mathbf{p}}$ is naturally isomorphic to the \mathfrak{a} -adic group

$$G_{\mathfrak{a}} = \prod_{i=1}^{\infty} \{0, 1, \dots, a_i - 1\}, a_i = \frac{p_i}{p_{i-1}}$$

where we set $p_0 = 1$ and the addition is defined by 'carry over' to the right. We refer to [HR, Chapter II] for background information on \mathfrak{a} -adic groups.

Definition 3.23. Let $\widehat{1} = (1, 0, 0, \dots) \in G_{\mathfrak{a}}$ and let $\rho_{\widehat{1}}$ be the rotation by $\widehat{1}$ on $G_{\mathfrak{a}}$. That is, $\rho_{\widehat{1}}(x) = x + \widehat{1}$, where $x \in G_{\mathfrak{a}}$. We call $(G_{\mathfrak{a}}, \rho_{\widehat{1}})$ an *odometer system* (or just an odometer for short). In particular, it is a Cantor minimal system.

Remark 3.24. By the natural isomorphism between $G_{\mathfrak{a}}$ and $G_{\mathfrak{p}}$ it is easy to see that the rotation $\rho_{\widehat{1}}$ on $G_{\mathfrak{a}}$ corresponds to adding $(1, 1, 1, \dots)$ in $\prod_{k \in \mathbb{N}} \mathbb{Z}_{p_k}$ (appropriately moded out, cf. the description prior to Definition 2.5) to get the conjugate rotation on $G_{\mathfrak{p}}$.

Remark 3.25. It is a fact that the family consisting of compact groups G that are both monothetic (i.e. contains a dense copy of \mathbb{Z} , which of course implies that G is abelian) and Cantor (as a topological space), coincides with the family of \mathfrak{a} -adic groups. It is also noteworthy that all minimal rotations (in particular rotation by $\widehat{1}$) on such groups are conjugate. This is a consequence of the fact that the dual group of an \mathfrak{a} -adic group is a torsion group, cf. [HR, Chapter VI].

4 Dimension groups

Recall that an ordered (torsion free) abelian group G with positive cone G^+ satisfies

- (i) $G^+ - G^+ = G$
- (ii) $G^+ + G^+ \subseteq G^+$
- (iii) $G^+ \cap (-G^+) = \{0\}$

Definition 4.1. (G, G^+) is a *dimension group* if G is a countable, ordered abelian group with positive cone G^+ such that

1. G is unperforated ($a \in G, n \in \mathbb{N}, na \in G^+ \Rightarrow a \in G^+$)
2. G has the Riesz interpolation property ($a_i \leq b_j, (i, j = 1, 2) \Rightarrow \exists c \in G$ such that $a_i \leq c \leq b_j, (i, j = 1, 2)$).

An *order unit* for the dimension group (G, G^+) is an element $u \in G^+ \setminus \{0\}$ such that for any $x \in G$ there exists an $n \in \mathbb{N}$ such that $x \leq nu$ (we use the convention that $a \leq b$ if $b - a \in G^+$). A dimension group may have more than one order unit. If we pick a distinguished order unit u we write (G, G^+, u) , or sometimes just (G, u) , to indicate that.

Notation We write $(G_1, G_1^+, u_1) \cong (G_2, G_2^+, u_2)$ to mean that there exists a group isomorphism $\phi: G_1 \rightarrow G_2$ such that $\phi(G_1^+) = G_2^+$ and $\phi(u_1) = u_2$. For short we say that G_1 and G_2 are *order-isomorphic* by a map respecting order units.

Definition 4.2. The dimension group (G, G^+) is *simple* if it contains no non-trivial order ideal J , i.e. $J = J^+ - J^+$ where $J^+ = J \cap G^+$ and $0 \leq a \leq b \in J \Rightarrow a \in J$. This is equivalent to that all non-zero elements in G^+ are order units.

Definition 4.3. Let G be a simple dimension group and let $u \in G^+ - \{0\}$. We say that $a \in G$ is infinitesimal if $-\epsilon u \leq a \leq \epsilon u$ for all $0 < \epsilon \in \mathbb{Q}$. (If $\epsilon = \frac{p}{q}$, $p, q \in \mathbb{N}$, then $a \leq \epsilon u$ means that $qa \leq pu$.) An equivalent definition is: $a \in G$ is infinitesimal if $p(a) = 0$ for all $p \in S_u(G)$, where $S_u(G)$ is introduced below. (It is evident that the infinitesimal elements do not depend on the particular order unit u .) The collection of infinitesimal elements of G form a subgroup, the *infinitesimal subgroup* of G , which we denote by $\text{Inf}G$.

Remark 4.4. The quotient group $G/\text{Inf}G$ has a natural induced ordering, i.e. $[a] > 0$ if $a >$, where $[\]$ denotes the quotient map. It is then easy to see that $G/\text{Inf}G$ becomes a simple dimension group with no infinitesimal elements except 0. If G has distinguished order unit u then $G/\text{Inf}G$ inherits the distinguished order unit $[u]$. Note that an order isomorphism $\alpha: G_1 \rightarrow G_2$, i.e. $\alpha(G_1^+) = G_2^+$, where G_1 and G_2 are dimension groups, maps $\text{Inf}G_1$ onto $\text{Inf}G_2$.

Definition 4.5. (Elliott) Let (V, E) be a Bratteli diagram where for each n V_n denotes the nodes at level n and M_n is the incidence matrix between levels $n - 1$ and n . Then we define $(K_0(V, E), K_0(V, E)^+)$ to be the direct limit of

$$(\mathbb{Z}^{|V_0|}, (\mathbb{Z}^+)^{|V_0|}) \xrightarrow{M_1} (\mathbb{Z}^{|V_1|}, (\mathbb{Z}^+)^{|V_1|}) \xrightarrow{M_2} (\mathbb{Z}^{|V_2|}, (\mathbb{Z}^+)^{|V_2|}) \xrightarrow{M_3} \dots$$

(Recall that $|V_0| = 1$.) The positive cone $K_0(V, E)^+$ is given by the induced ordering, where each $(\mathbb{Z}^{|V_n|}, (\mathbb{Z}^+)^{|V_n|})$ has the simplicial ordering. The map $M_n: (\mathbb{Z}^{|V_{n-1}|}, (\mathbb{Z}^+)^{|V_{n-1}|}) \rightarrow (\mathbb{Z}^{|V_n|}, (\mathbb{Z}^+)^{|V_n|})$ is given by the matrix multiplication of M_n with $\mathbb{Z}^{|V_{n-1}|}$ column vectors.

Proposition 4.6. $(K_0(V, E), K_0(V, E)^+, \mathbf{1})$, where $\mathbf{1}$ is represented by $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$ is a dimension group with canonical order unit.

Theorem 4.7 ([EHS]). All dimension groups (G, G^+, u) with a distinguished order unit u arise as described in Proposition 4.6, i.e. there exists a Bratteli diagram (V, E) such that $(G, G^+, u) \cong (K_0(V, E), K_0(V, E)^+, \mathbf{1})$. Also, (G, G^+) is a simple dimension group if and only if (V, E) is a simple Bratteli diagram.

We will not give a presentation of (simple) dimension groups in terms of Cantor minimal systems.

Definition 4.8. Let (X, T) be a Cantor minimal system and let

$$\partial_T: \mathcal{C}(X, \mathbb{Z}) \rightarrow \mathcal{C}(X, \mathbb{Z})$$

be defined by $\partial_T(f) = f - f \circ T^{-1}$. Then

$$K^0(X, T) \stackrel{\text{def}}{=} \mathcal{C}(X, \mathbb{Z}) / \partial_T(\mathcal{C}(X, \mathbb{Z})).$$

Let $K^0(X, T)^+ = \{[f] \in K^0(X, T) \mid [f] = [g], \text{ and } g \geq 0\}$ and let $\mathbf{1}$ denote $[1]$. ($[\]$ denotes the quotient map.)

Theorem 4.9 ([HPS]). *($K^0(X, T), K^0(X, T)^+$) is a simple dimension group. If (V, E) is a Bratteli-Vershik model for the Cantor minimal system (X, T) then $(K^0(X, T), K^0(X, T)^+, \mathbf{1}) \cong (K_0(V, E), K_0(V, E)^+, \mathbf{1})$.*

Definition 4.10. A compact convex set K is a Choquet simplex if every point $x \in K$ is the barycenter of a unique positive and normalized boundary measure μ_x , i.e. $\mu_x(\partial_e K) = 1$ where $\partial_e K$ denotes the extreme boundary of K . So $a(x) = \int_K a d\mu_x$ for all $a \in \text{Aff}(K)$. Recall that $a \in \text{Aff}(K)$ if $a: K \rightarrow \mathbb{R}$ is continuous and $a(\lambda x + (1 - \lambda)y) = \lambda a(x) + (1 - \lambda)a(y)$ for all $x, y \in K$ and $0 \leq \lambda \leq 1$. We observe that the finite dimensional simplices coincide with the standard simplices.

Let (G, G^+, u) be a dimension group with a distinguished order unit $u \in G^+$ and let $p: G \rightarrow \mathbb{R}$ a homomorphism. If p is positive (i.e. $p(G^+) \geq 0$) and $p(u) = 1$ we say that p is a *state*. The set of all states on (G, G^+, u) is denoted $S_u(G)$ and may be given a natural topology making it a compact convex set. In fact, $S_u(G)$ is naturally embedded in \mathbb{R}^G as a convex set. With \mathbb{R}^G given the product topology, we give $S_u(G)$ the relative topology.

We define the map $\theta: G \rightarrow \text{Aff}(S_u(G))$ by $\theta(a) = \hat{a}$, where $\hat{a}(p) = p(a)$, $p \in S_u(G)$. In particular, $\theta(u) = 1$. Clearly θ is an additive map. We have the following important result.

Theorem 4.11 ([E, Theorems 4.2 and 4.4]). *Suppose (G, G^+, u) is a simple (noncyclic) dimension group with distinguished order unit u . Then $S_u(G)$ is a Choquet simplex and $\theta: G \rightarrow \text{Aff}(S_u(G))$ determines the order on G in the sense that $G^+ = \{a \in G \mid \theta(a) = \hat{a} \text{ is a strictly positive function on } S_u(G)\} \cup \{0\}$. Furthermore, $\theta(G)$ is a dense (in the uniform topology) additive subgroup of $\text{Aff}(S_u(G))$. Also, $\ker(\theta) = \text{Inf}G$.*

Corollary 4.12. *Let (G, G^+, u) be as in Theorem 4.11 such that $\text{Inf}G = \{0\}$. Then (G, G^+, u) is order isomorphic to a countable dense additive subgroup of $\text{Aff}(K)$ for some Choquet simplex K mapping u to the constant function 1. (Here $\text{Aff}(K)$ is given the strict ordering.)*

The following theorem is a fairly straightforward corollary of Theorem 2.4.

Theorem 4.13 ([HPS]). *Let (X, T) be a Cantor minimal system. All (noncyclic) simple dimension groups (G, G^+, u) with distinguished order unit u are order*

isomorphic to $(K^0(X, T), K^0(X, T)^+, \mathbb{1})$ for some Cantor minimal system (X, T) . Furthermore, the state space $S_{\mathbb{1}}(K^0(X, T))$ may be identified in an obvious way with the Choquet simplex $M(X, T)$ of T -invariant probability measures; in fact, these two Choquet simplices are affinely homeomorphic. In particular, $[f] \in K^0(X, T)^+ - \{0\}$ if and only if $\int_X f d\mu > 0$ for all $\mu \in M(X, T)$.

Theorem 4.14 ([GPS]). *The Cantor minimal systems (X, T) and (Y, S) are strong orbit equivalent if and only if $(K^0(X, T), K^0(X, T)^+, \mathbb{1}) \cong (K^0(Y, S), K^0(Y, S)^+, \mathbb{1})$.*

Theorem 4.15 ([GPS]). *The Cantor minimal systems (X, T) and (Y, S) are orbit equivalent if and only if $(\widetilde{K^0(X, T)}, \widetilde{K^0(X, T)^+}, \widetilde{\mathbb{1}}) \cong (\widetilde{K^0(Y, S)}, \widetilde{K^0(Y, S)^+}, \widetilde{\mathbb{1}})$, where*

$$\widetilde{K^0(X, T)} = K^0(X, T)/\text{Inf}(K^0(X, T)), \quad \widetilde{K^0(Y, S)} = K^0(Y, S)/\text{Inf}(K^0(Y, S)),$$

and the positive cones and order units are induced by the quotient maps.

Remark 4.16. The idea behind introducing $K^0(X, T)$ with an ordering for a Cantor minimal system (X, T) comes from (non-commutative) operator algebra theory. In fact, one can show that $K^0(X, T)$ as defined above is order-isomorphic to the K_0 -group of the associated C^* -crossed product $\mathcal{C}(X) \rtimes_T \mathbb{Z}$ by a map respecting distinguished order units. The latter K_0 -group comes with a natural order. It turns out that the ordered K_0 -group of $\mathcal{C}(X) \rtimes_T \mathbb{Z}$, with its natural order unit corresponding to the unit element, is a complete isomorphism invariant, and so it follows that (X, T) is strong orbit equivalent to (Y, S) if and only if $\mathcal{C}(X) \rtimes_T \mathbb{Z}$ is $*$ -isomorphic to $\mathcal{C}(Y) \rtimes_S \mathbb{Z}$ [GPS].

4.1 Rational subgroups

Definition 4.17. The rational subgroup $\mathbb{Q}(G, u)$ of the simple dimension group $G = (G, G^+, u)$ with order unit u is defined by

$$\mathbb{Q}(G, u) = \{g \in G \mid ng = mu, n \in \mathbb{N}, m \in \mathbb{Z}\}$$

where $\mathbb{Q}(G, u)^+ = \mathbb{Q}(G, u) \cap G^+$.

Proposition 4.18. *The group $\mathbb{Q}(G, u)$ is order isomorphic to a subgroup of \mathbb{Q} .*

Proof. Let $\phi: \mathbb{Q}(G, u) \rightarrow \mathbb{Q}$ be defined by $\phi(g) = \frac{m}{n}$, where $n \in \mathbb{N}, m \in \mathbb{Z}$, such that $ng = mu$. The map is well-defined. In fact, if $ng = mu$ and $n'g = m'u$, then we get by multiplying with n' and n respectively: $n'n'g = n'm'u$ and $nn'g = nm'u$. Subtracting we get $(n'm - nm')u = 0$. Since G is torsion free we get $n'm - nm' = 0$, and so $\frac{m}{n} = \frac{m'}{n'}$.

We need to show that ϕ is one-to-one and that ϕ is a group homomorphism. Assume $\phi(g_1) = \frac{m_1}{n_1}$ and $\phi(g_2) = \frac{m_2}{n_2}$, that is $n_1g_1 = m_1u$ and $n_2g_2 = m_2u$. Then we also have (by multiplying with n_1 and n_2 respectively) $n_2n_1g_1 = n_2m_1u$ and $n_1n_2g_2 = n_1m_2u$. Adding this we get $n_2n_1g_1 + n_1n_2g_2 = n_1n_2(g_1 + g_2) = (n_2m_1 + n_1m_2)u$ and $\phi(g_1 + g_2) = \frac{(n_2m_1 + n_1m_2)}{n_1n_2} = \frac{m_1}{n_1} + \frac{m_2}{n_2} = \phi(g_1) + \phi(g_2)$. Assume $\phi(g) = 0 = \frac{0}{1}$. Then $g = 0$ and so ϕ is one-to-one. So $\phi: \mathbb{Q}(G, u) \rightarrow \phi(\mathbb{Q}(G, u)) \subset \mathbb{Q}$ is an isomorphism. Clearly $\phi(g) \geq 0 \Leftrightarrow g \geq 0$, and so ϕ is an order isomorphism. \square

Proposition 4.19. $G/\mathbb{Q}(G, u)$ is torsion free.

Proof. Assume $g \in G$ and $ng \in \mathbb{Q}(G, u)$. Since $ng \in \mathbb{Q}(G, u)$ there exists $r \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that

$$r(ng) = mu.$$

But then $(rn)g = mu$ and so $g \in \mathbb{Q}(G, u)$. This proves that $G/\mathbb{Q}(G, u)$ is torsion free. \square

It is straightforward to check the following proposition.

Proposition 4.20. $(G_1, G_1^+, u_1) \cong (G_2, G_2^+, u_2) \Rightarrow \mathbb{Q}(G_1, u_1) \cong \mathbb{Q}(G_2, u_2)$.

A consequence of this is that since by Theorem 4.13 we have $(K^0(X, T), K^0(X, T)^+, \mathbb{1}) \cong (K_0(V, E), K_0(V, E)^+, [\mathbb{1}])$, we get $\mathbb{Q}(K^0(X, T), \mathbb{1}) \cong \mathbb{Q}(K_0(V, E), [\mathbb{1}])$.

5 Factors of dynamical systems

We shall now look more closely at factors of dynamical systems. Since we are only interested in Cantor minimal systems, we will restrict our attention to dynamical systems with a \mathbb{Z} -action.

Definition 5.1. Let (Y, S) be a factor of (X, T) , and assume that (Y, S) is equicontinuous (cf. Definition 3.5). We say that (Y, S) is the *maximal equicontinuous factor* of (X, T) if *all* other equicontinuous factors of (X, T) are also factors of (Y, S) .

Proposition 5.2. *If (X, T) , where (X, d) is a metric space, is equicontinuous then there exists an isometric metric $\bar{d}: X \times X \rightarrow \mathbb{R}^+$, i.e. $\bar{d}(Tx, Ty) = \bar{d}(x, y)$, $\forall x, y \in X$, which is equivalent with d , i.e. generates the same topology on X .*

Proof. Define $\bar{d}: X \times X \rightarrow \mathbb{R}^+$ by

$$\bar{d}(x, y) = \sup_{n \in \mathbb{Z}} \{d(T^n x, T^n y)\}.$$

We can easily see that $\bar{d}(x, y) = \bar{d}(y, x)$ and $\bar{d}(x, y) = 0 \Leftrightarrow x = y$. The triangle inequality also holds since for all $n \in \mathbb{Z}$

$$\begin{aligned} d(T^n x, T^n z) &\leq d(T^n x, T^n y) + d(T^n y, T^n z) \\ &\leq \bar{d}(x, y) + \bar{d}(y, z). \end{aligned}$$

So $\bar{d}(x, z) = \sup_{n \in \mathbb{Z}} d(T^n x, T^n z) \leq \bar{d}(x, y) + \bar{d}(y, z)$ and \bar{d} does define a metric on X . Observe that by construction $\bar{d}(Tx, Ty) = \bar{d}(x, y)$ for all $x, y \in X$. Since $d(x, y) \leq \bar{d}(x, y)$ for all $x, y \in X$ the map $\text{id}: (X, \bar{d}) \rightarrow (X, d)$ is continuous. So we need only to check that the map $\text{id}: (X, d) \rightarrow (X, \bar{d})$ is continuous. Let $B(x, r) = \{w \in X \mid d(w, x) < r\}$ denote the open ball with centre x and d -radius r . Let $\epsilon > 0$. By equicontinuity there is a $\delta_z > 0$ for every $z \in X$ such that $y \in B(z, \delta_z) \Rightarrow d(T^n y, T^n z) < \epsilon/2$ for all $n \in \mathbb{Z}$. $\{B(z, \delta_z) \mid z \in X\}$ will be an open covering of X . By Lebesgue's covering lemma, there is $\delta > 0$ such that any ball of radius δ is contained in $B(z, \delta_z)$ for some $z \in X$.

So let $x, y \in X$ such that $d(x, y) < \delta$. Then there is a $z \in X$ such that $\{x, y\} \subset B(z, \delta_z)$, and so $d(T^n x, T^n y) \leq d(T^n x, T^n z) + d(T^n z, T^n y) < \epsilon/2 + \epsilon/2 = \epsilon$. So $\text{id}: (X, d) \rightarrow (X, \bar{d})$ is continuous. \square

Proposition 5.3. *Assume that (X, T) is an equicontinuous minimal dynamical system. Then X can be given a group structure such that X is a compact abelian group and T is rotation by an element in X .*

Proof. Since (X, T) is minimal, $\{T^n x_0 \mid n \in \mathbb{Z}\}$ is dense in X for all $x_0 \in X$. We will define group multiplication and inverse on the dense subset $\text{orbit}_T(x_0) = \{T^n x_0 \mid n \in \mathbb{Z}\}$ for some x_0 . We show that these operations are uniformly continuous and then we can extend them continuously to all of X . We define multiplication $\text{orbit}_T(x_0) \times \text{orbit}_T(x_0) \rightarrow \text{orbit}_T(x_0)$ by $(T^n x_0, T^m x_0) \mapsto T^{n+m} x_0$. The inverse of an element $T^n x_0$ we define to be $T^{-n} x_0$. By the previous proposition we can assume that the metric d on X is isometric.

Given $\epsilon > 0$ choose $\delta = \epsilon/2$. Then for $n, n', m, m' \in \mathbb{Z}$ with $d(T^n x_0, T^{n'} x_0) < \delta$ and $d(T^m x_0, T^{m'} x_0) < \delta$ we get

$$\begin{aligned} d(T^{n+m} x_0, T^{n'+m'} x_0) &\leq d(T^{n+m} x_0, T^{n'+m} x_0) + d(T^{n'+m} x_0, T^{n'+m'} x_0) \\ &\leq d(T^m(T^n x_0), T^m(T^{n'} x_0)) + d(T^{n'}(T^m x_0), T^{n'}(T^{m'} x_0)) \\ &= d(T^n x_0, T^{n'} x_0) + d(T^m x_0, T^{m'} x_0) \\ &= 2\delta = \epsilon. \end{aligned}$$

For the inverse map, notice that

$$\begin{aligned} d(T^{-n} x_0, T^{-n'} x_0) &= d(T^n(T^{-n} x_0), T^n(T^{-n'} x_0)) = d(x_0, T^{n-n'} x_0) \\ &= d(T^{n'} x_0, T^{n'}(T^{n-n'} x_0)) = d(T^{n'} x_0, T^n x_0) < \epsilon, \end{aligned}$$

so both the inverse map and the group multiplication is uniformly continuous and thus can be extended to all of X . X will obviously be an abelian group and we observe that x_0 is the identity element.

Applying T can now be seen as simply multiplying with an element in X . In fact, if $x \in \text{orbit}_T(x_0)$ then $x = T^n x_0$ for some $n \in \mathbb{Z}$ so

$$Tx = T(T^n x_0) = T^{n+1} x_0 = (Tx_0) \cdot (T^n x_0)$$

because of the way we defined multiplication in X . If $x \in X$ then there is some Cauchy sequence $\{x_k\}$ in $\text{orbit}_T(x_0)$ converging to x , say $x_n = T^{n_k} x_0$, and $Tx = \lim_{n \rightarrow \infty} T(T^{n_k} x_0)$. By continuity of the multiplication

$$T^{n_k} x_0 \rightarrow x \Rightarrow T(T^{n_k} x_0) = (Tx_0) \cdot (T^{n_k} x_0) \rightarrow (Tx_0) \cdot x,$$

hence $Tx = (Tx_0) \cdot x$, i.e. applying T to x is the same as rotating by the element Tx_0 . \square

Theorem 5.4. *Let (X, T) be a minimal dynamical system. Then there exists a maximal equicontinuous factor $\pi: (X, T) \rightarrow (Y, S)$ such that for any equicontinuous factor $\phi: (X, T) \rightarrow (Z, R)$ there is a unique factor map $\psi: (Y, S) \rightarrow (Z, R)$ such that $\psi \circ \pi = \phi$. The system (Y, S) is unique up to conjugacy.*

For a proof, see [K, Th. 2.44].

Definition 5.5. We say that the factor map $\pi: (X, T) \rightarrow (Y, S)$ is *almost one-to-one* if there exists $y \in Y$ such that $\pi^{-1}(y)$ is a singleton set.

It might be surprising that we are interested in a property of a single point of the dynamical system. But knowing that the factor map is one-to-one at a point x_0 actually implies that there is a dense \mathcal{G}_δ set of points where π is one-to-one. To see this we first notice that if $\pi^{-1}(\pi(x_0)) = \{x_0\}$ then $\pi^{-1}(\pi(x)) = \{x\}$ for all $x \in \text{orbit}_T(x_0)$. In fact, let $x = T^n x_0$ for some $n \in \mathbb{Z}$ and set $\pi(x_0) = y_0$. Since $\pi(x) = \pi(T^n x_0) = S^n \pi(x_0) = S^n y_0$, we get that $x \in \pi^{-1}(S^n y_0)$. Let $z \in \pi^{-1}(S^n y_0)$. Then $\pi(T^{-n} z) = S^{-n} \pi(z) = S^{-n} S^n y_0 = y_0$. Hence $T^{-n} z = x_0$, and so $z = T^n x_0 = x$. Hence $\pi^{-1}(\pi(x)) = \{x\}$, proving our assertion. So for minimal systems the set of points where π is almost one-to-one is dense.

Recall that a \mathcal{G}_δ set is a countable intersection of open sets. Let d be a metric on X and let $B(x, \frac{1}{n})$ denote the open ball around x with radius $\frac{1}{n}$. Define

$$A(x, n) = \left\{ y \in Y \mid \pi^{-1}(y) \subseteq B\left(x, \frac{1}{n}\right) \right\} (\subseteq Y).$$

We claim that $A(x, n)$ is open. In fact, let $y \in A(x, n)$, i.e. $\pi^{-1}(y) \subseteq B(x, \frac{1}{n})$. We must show that there exists a neighborhood around y which lie entirely

inside $A(x, n)$. Assume this is not the case. Then we can find $y_n \rightarrow y$ such that $\pi^{-1}(y_n) \not\subseteq B(x, \frac{1}{n})$. Hence there exists $x_n \in X$, $\pi(x_n) = y_n$, such that $d(x_n, x) \geq \frac{1}{n}$. By choosing a subsequence we may assume that $x_n \rightarrow x_0$. Then $d(x_0, x) \geq \frac{1}{n}$. Also, $y_n = \pi(x_n) \rightarrow \pi(x_0) = y$, and so $x_0 \in \pi^{-1}(y)$, which is a contradiction. This finishes the proof that $A(x, n)$ is open. Then $A_n = \bigcup_{x \in X} A(x, n) \subseteq Y$ is also open. We now claim that $B = \pi^{-1}(\bigcap_{n=1}^{\infty} A_n) = (\bigcap_{n=1}^{\infty} \pi^{-1}(A_n))$ equals the set $\{x \in X \mid \pi^{-1}(\pi(x)) = \{x\}\}$. Assume now that $x_1, x_2 \in B$ and $\pi(x_1) = \pi(x_2) = y$. Then $y \in \bigcap_{n=1}^{\infty} A_n$, and so $y \in \bigcup_{x \in X} A(x, n)$ for all $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$ there exists an $x^{(n)} \in X$ such that $y \in A(x^{(n)}, n)$. Since $x_1, x_2 \in \pi^{-1}(y) \subseteq B(x^{(n)}, \frac{1}{n})$ we get $d(x_1, x^{(n)}) < \frac{1}{n}$ and $d(x_2, x^{(n)}) < \frac{1}{n}$. By the triangle inequality we have $d(x_1, x_2) < \frac{2}{n}$. Since this holds for all n we must have $x_1 = x_2$, so $B \subseteq \{x \in X \mid \pi^{-1}(\pi(x)) = \{x\}\}$. Now assume that $z \in \{x \in X \mid \pi^{-1}(\pi(x)) = \{x\}\}$. Then $\pi^{-1}(\pi(z)) = \{z\}$ and so obviously $\pi^{-1}(\pi(z)) \subseteq B(z, \frac{1}{n})$ for all n . Thus $\pi(z) \in A(z, n)$ for all n . Hence $\pi(z) \in \bigcap_{n=1}^{\infty} A_n$ and so $z \in B$. Thus $\{x \in X \mid \pi^{-1}(\pi(x)) = \{x\}\} \subseteq B$. This shows that π is one-to-one on a dense \mathcal{G}_δ subset of X .

The following Theorem is due to Paul [P], and will be helpful in identifying when a factor is the maximal equicontinuous factor of a dynamical system.

Theorem 5.6. *Let (X, T) be a minimal dynamical system. If (Z, R) is a factor where the factor map $\pi: X \rightarrow Z$ is almost one-to-one, and (Y, S) is a group rotation, then (Y, S) is the maximal equicontinuous factor of (X, T) .*

Proof. Assume (Y, S) is the maximal equicontinuous factor of (X, T) with factor map $\pi_2: (X, T) \rightarrow (Y, S)$. Then there exists a factor map $\pi_1: (Y, S) \rightarrow (Z, R)$ such that the following diagram commutes.

$$\begin{array}{ccc}
 X & \xrightarrow{T} & X \\
 \pi_2 \downarrow & \circ & \downarrow \pi_2 \\
 \pi \circ Y & \xrightarrow{S} & Y \circ \pi \\
 \pi_1 \downarrow & \circ & \downarrow \pi_1 \\
 Z & \xrightarrow{R} & Z
 \end{array}$$

If we can show that π_1 is an isomorphism, then (Z, R) is in fact the maximal equicontinuous factor. Since π is almost one-to-one, i.e. there exists $x' \in X$ such

that $\pi^{-1}(\pi(x')) = \{x'\}$, we get

$$\pi_1^{-1}(\pi(x')) = \pi_1^{-1}(\pi_1(\pi_2(x'))) = \{\pi_2(x')\}.$$

If we show that $|\pi_1^{-1}(z)|$ is constant for all $z \in Z$ we consequently get that $|\pi_1^{-1}(z)| = 1$ for all $z \in Z$, and so π_1 is a conjugation.

Since (Z, R) is a group rotation, Z has a group structure and there is an element $z_0 \in Z$ such that $Rz = z_0 \cdot z$. (We write the group operation multiplicatively.) Clearly $R^n z = z_0^n \cdot z$ for $n \in \mathbb{Z}$. There exists an $x_0 \in X$ such that $\pi(x_0) = z_0$. Consider the element $y_0 = \pi_2(x_0) \in Y$. Since (Y, S) is an equicontinuous system Y can be given a group structure as described in Proposition 5.3 such that y_0 is the identity element and $Sy = (Sy_0) \cdot y$. Since $\pi = \pi_1 \circ \pi_2$ we have $\pi_1(y_0) = z_0$. Define $\tilde{\pi}_1: Y \rightarrow Z$ by $\tilde{\pi}_1(y) = \pi_1(y) \cdot z_0^{-1}$. Since π_1 is onto we get that $\tilde{\pi}_1$ is onto. We get $\tilde{\pi}_1(S^n y_0) = \pi_1(S^n y_0) \cdot z_0^{-1} = R^n(\pi_1(y_0)) \cdot z_0^{-1} = R^n(z_0) \cdot z_0^{-1} = z_0^n$. We then get

$$\begin{aligned} \tilde{\pi}_1(S^n y_0 \cdot S^m y_0) &= \tilde{\pi}_1(S^{n+m} y_0) = z_0^{n+m} = z_0^n \cdot z_0^m \\ &= \tilde{\pi}_1(S^n y_0) \cdot \tilde{\pi}_1(S^m y_0) \end{aligned}$$

so $\tilde{\pi}_1$ is a group homomorphism on $\text{orbit}_S(y_0)$. Since the systems are minimal and $\text{orbit}_S(y_0)$ is a dense subgroup of Y , $\tilde{\pi}_1$ extends continuously to a group homomorphism $\tilde{\pi}_1: Y \rightarrow Z$. Clearly $|\pi_1^{-1}(z)| = |\tilde{\pi}_1^{-1}(z)|$ for $z \in Z$. Since $\tilde{\pi}_1$ is a group homomorphism we must have that $|\tilde{\pi}_1^{-1}(z)|$ is the same for all $z \in Z$. Combining this with the fact that there exists a $z \in Z$ such that $|\pi_1^{-1}(z)| = 1$, we get that π_1 is one-to-one, and hence a conjugation, thus finishing the proof. \square

The following theorem is due to S. Williams [W].

Theorem 5.7. *($G_{\mathbf{p}}, \rho_{\hat{1}}$) is the maximal equicontinuous factor of $(\overline{\mathcal{O}(\eta)}, S)$ if $\mathbf{p} = (p_i)_{i \in \mathbb{N}}$ is a periodic structure for the Toeplitz sequence η . (We will identify $G_{\mathbf{p}}$ with $G_{\mathbf{a}}$ as explained in Remark 3.22.)*

Proof. We need to define a factor map $\pi: \overline{\mathcal{O}(\eta)} \rightarrow G_{\mathbf{p}}$ which is continuous, surjective and such that

$$\begin{array}{ccc} \overline{\mathcal{O}(\eta)} & \xrightarrow{S} & \overline{\mathcal{O}(\eta)} \\ \pi \downarrow & & \downarrow \pi \\ G_{\mathbf{p}} & \xrightarrow{\rho_{\hat{1}}} & G_{\mathbf{p}} \end{array}$$

commutes. We also need to show that we can apply Theorem 5.6 to show that $(G_{\mathbf{p}}, \rho_{\widehat{\Gamma}})$ is in fact the maximal equicontinuous factor of $(\overline{\mathcal{O}(\eta)}, S)$.

Recall that $A_n^p = \{S^{n+mp}\eta \mid m \in \mathbb{Z}\}$ and that a $g \in G_{\mathbf{p}}$ is of the form $g = (n_i)_{i=1}^{\infty}$ where $n_j \equiv n_i \pmod{p_i}$ when $i < j$.

Given $g \in G_{\mathbf{p}}$ define the set

$$A_g = \bigcap_{i=1}^{\infty} \overline{A_{n_i}^{p_i}}.$$

By (iii) in Lemma 3.20 $\overline{A_{n_1}^{p_1}} \supseteq \overline{A_{n_2}^{p_2}} \supseteq \cdots$ so for all $N \in \mathbb{N}$

$$\bigcap_{i=1}^N \overline{A_{n_i}^{p_i}} = \overline{A_{n_N}^{p_N}} \neq \emptyset,$$

thus $A_g \neq \emptyset$. Each $\overline{A_{n_i}^{p_i}}$ is clopen, so A_g will be closed.

Define $\pi: \overline{\mathcal{O}(\eta)} \rightarrow G_{\mathbf{p}}$ by $\pi^{-1}(g) = A_g$. For this to be a well defined function, we need $A_{g_1} \cap A_{g_2} = \emptyset$ when $g_1 \neq g_2$ and that any $x \in \overline{\mathcal{O}(\eta)}$ is in a A_g for some g , i.e. we need $\{A_g \mid g \in G_{\mathbf{p}}\}$ to be a partition of $\overline{\mathcal{O}(\eta)}$. Let $x \in \overline{\mathcal{O}(\eta)}$, then for each $i \in \mathbb{N}$ there exists an n_i such that $x \in \overline{A_{n_i}^{p_i}}$ since $\{\overline{A_{n_i}^{p_i}} \mid 0 \leq n_i < p_i\}$ is a partition of $\overline{\mathcal{O}(\eta)}$ for each $i \in \mathbb{N}$ (by (ii) in Lemma 3.20). Let $g = (n_i)_{i=1}^{\infty}$, then $x \in A_g$.

Now assume $g \neq g'$ where $g' = (m_i)_{i=1}^{\infty}$ and $g = (n_i)_{i=1}^{\infty}$. Then there is at least one $j \in \mathbb{N}$ such that $n_j \neq m_j$. Again by (ii) in Lemma 3.20 this implies that $\overline{A_{m_j}^{p_j}} \cap \overline{A_{n_j}^{p_j}} = \emptyset$ so $A_g \cap A_{g'} = \emptyset$ and $\{A_g \mid g \in G_{\mathbf{p}}\}$ is a partition of $\overline{\mathcal{O}(\eta)}$ as desired.

By construction π is surjective, but we need to show that it is also continuous. The collection $\{B_m^j \mid j \in \mathbb{N}, 0 \leq m < p_j\}$ of cylinder sets $B_m^j = \{(n_i)_{i=1}^{\infty} \in G_{\mathbf{p}} \mid n_j = m\}$ is a basis for $G_{\mathbf{p}}$.

$$\pi^{-1}(B_m^j) = \bigcup_{g \in B_m^j} A_g = \bigcup_{g \in B_m^j} \bigcap_{i=1}^{\infty} \overline{A_{n_i}^{p_i}} = \overline{A_m^{p_j}},$$

which is open in $\overline{\mathcal{O}(\eta)}$ so π is continuous.

By (iv) in Lemma 3.20 we get

$$SA_g = S \bigcap_{i=1}^{\infty} \overline{A_{n_i}^{p_i}} = \bigcap_{i=1}^{\infty} S \overline{A_{n_i}^{p_i}} = \bigcap_{i=1}^{\infty} \overline{A_{n_i+1 \pmod{p_i}}^{p_i}} = A_{g+\widehat{1}},$$

so $\pi \circ S(A_g) = \pi(SA_g) = g + \widehat{1} = \rho_{\widehat{\Gamma}}(g) = \rho_{\widehat{\Gamma}}(\pi(A_g)) = \rho_{\widehat{\Gamma}} \circ \pi(A_g)$. The diagram commutes and π is in fact a factor map.

We have that $\rho_{\widehat{1}}$ is a rotation on $G_{\mathbf{p}}$. If we in addition have that π is almost one-to-one, then $(G_{\mathbf{p}}, \rho_{\widehat{1}})$ is the maximal equicontinuous factor by Theorem 5.6.

Assume $\omega \in \pi^{-1}(\pi(\eta))$. By the definition of A_n^p it is easy to see that $\eta \in A_0^p$ for all p , so $\pi(\eta) = (0, 0, 0, \dots) = \widehat{0}$. So $\omega \in \pi^{-1}(\pi(\eta))$ means that $\omega \in A_{\widehat{0}}$, i.e. $\omega \in \overline{A_0^{p_i}}$ for all $i \in \mathbb{N}$. By (i) in Lemma 3.20, this implies that $S_{p_i}(\omega) = S_{p_i}(S^0\eta) = S_{p_i}(\eta), \forall i$, thus since $\eta = \lim_{i \rightarrow \infty} S_{p_i}(\eta)$, ω must equal η so $\pi^{-1}(\pi(\eta)) = \{\eta\}$ and π is almost one-to-one. \square

Remark 5.8. We have shown that the maximal equicontinuous factor of a Toeplitz flow is an odometer. This was first proved by Paul [P]. The converse is partially true: An expansive almost one-to-one extension of an odometer system is a Toeplitz flow [MP]. In paper 1 we identify for which K^0 groups it is possible to find such an expansive almost one-to-one extension. We will present some results which give some restrictions as to which K^0 -groups we could expect this to work for.

In order to prove the next theorem, we will need the following results.

Lemma 5.9 (Gottschalk-Hedlund, [GH]). *Let (X, T) be a minimal dynamical system and $f \in \mathcal{C}(X)$. Then the following are equivalent:*

- $f = g - g \circ T^{-1}$ for a $g \in \mathcal{C}(X)$
- There exists $x_0 \in X$ such that

$$\sup_n \left| \sum_{i=0}^{n-1} f(T^i x_0) \right| < \infty.$$

The following proposition is well known among the experts, but we know no source to cite for a proof. Therefore we will present a proof here.

Proposition 5.10. *If (Y, S) is a factor of (X, T) , with factor map $\pi: X \rightarrow Y$ then $\pi_*: M(X, T) \rightarrow M(Y, S)$, where $\pi_*(\mu) = \mu \circ \pi^{-1}$, is a continuous (in the w^* -topology) affine surjection.*

Proof. The only nontrivial thing to prove is the surjection assertion. Let $\mu \in M(Y, S)$. Since we may embed $\mathcal{C}(Y)$ in $\mathcal{C}(X)$ by the map $f \rightarrow f \circ \pi$, we get by Hahn-Banach that there exists a probability measure ν on X such that $\pi_*(\nu) = \mu$. We claim that $\nu \circ T^{-1} = T_*\nu \in (\pi_*)^{-1}(\mu)$, where $(T_*\nu)(A) = \nu(T^{-1}(A))$ for A a Borel set in X . We must show that $\pi_*(T_*\nu) = \mu$. For B a Borel set in Y we get:

$$\begin{aligned} (\pi_*(T_*\nu))(B) &= (T_*\nu)(\pi^{-1}(B)) = \nu(T^{-1}(\pi^{-1}(B))) \\ &= \nu(\pi^{-1}(S^{-1}(B))) = \mu(S^{-1}(B)) = \mu(B) \end{aligned}$$

since $\pi \circ T = S \circ \pi$. This proves the claim. Notice that we also get that $\nu \circ T^{-i} \in (\pi_*)^{-1}(\mu)$ for all $i \in \mathbb{N}$.

Clearly $(\pi_*)^{-1}(\mu)$ is a convex, w^* -compact subset of $P(X)$, the set of probability measures on X . Let $\nu \in (\pi_*)^{-1}(\mu)$, and define $\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} \nu \circ T^{-i} \in (\pi_*)^{-1}(\mu)$. A subsequence $(\nu_{n_k})_k$ will converge in the w^* -topology to some $\theta \in (\pi_*)^{-1}(\mu)$. Now $\nu_{n_k} - \nu_{n_k} \circ T^{-1} = \frac{1}{n_k}(\nu - \nu \circ T^{-n_k-1}) \rightarrow 0$. Hence $\theta = \theta \circ T^{-1}$, and so $\theta \in M(X, T)$. Since $\pi_*(\theta) = \mu$ we are done. \square

Theorem 5.11 (Glasner–Weiss [GW]). *If the minimal dynamical system (Y, S) is a factor of the minimal system (X, T) with factor map $\pi: X \rightarrow Y$, then*

$$\pi^* : K^0(Y, S) \hookrightarrow K^0(X, T)$$

is an order-embedding (where $\pi^([h]) = [h \circ \pi]$).*

This result was originally proved in [GW, Prop. 3.1], but we will give a proof here.

Proof. The first thing we need to show, is that π^* is a well-defined injective map. From $\pi: X \rightarrow Y$ we get the map $\mathcal{C}(Y) \rightarrow \mathcal{C}(X)$, defined by

$$h \mapsto h \circ \pi.$$

and composing this with the quotient map we get

$$\theta: \mathcal{C}(Y) \rightarrow \mathcal{C}(X)/\partial_T \mathcal{C}(X)$$

where $\ker(\theta) = \{h \in \mathcal{C}(Y) \mid h \circ \pi \in \partial_T \mathcal{C}(X)\}$, i.e. h is in $\ker(\theta)$ iff there is a $g \in \mathcal{C}(X)$ such that $h \circ \pi = g - g \circ T^{-1}$. For π^* to be a well-defined injection we need to prove that $\ker(\theta) = \partial_S \mathcal{C}(Y)$. First assume $h \in \partial_S \mathcal{C}(Y)$, so $h = l - l \circ S^{-1}$ for some $l \in \mathcal{C}(Y)$. Then

$$h \circ \pi = (l - l \circ S^{-1}) \circ \pi = l \circ \pi - l \circ S^{-1} \circ \pi = (l \circ \pi) - (l \circ \pi) \circ T^{-1} \in \partial_T \mathcal{C}(X),$$

so $h \in \ker(\theta)$.

Assume now that $h \in \ker(\theta)$, so $h \circ \pi = g - g \circ T^{-1}$ for a $g \in \mathcal{C}(X)$. To show that $h = l - l \circ S^{-1}$ for some l we only need to find $y_0 \in Y$ such that

$$\sup_n \left| \sum_{i=0}^{n-1} h(S^i y_0) \right| < \infty \tag{1}$$

by Lemma 5.9. But since we know that $h \circ \pi = g - g \circ T^{-1}$ we get, from the same lemma that there exists x_0 such that

$$\sup_n \left| \sum_{i=0}^{n-1} h \circ \pi(T^i x_0) \right| < \infty.$$

But $h \circ \pi(T^i x_0) = h \circ S^i(\pi(x_0))$ so if we choose $y_0 = \pi(x_0)$ then (1) is satisfied and so $h \in \ker(\theta)$. So we have proved that π^* is a well-defined injection.

To show that π^* is an order embedding we need to show that $\pi^*([h]) > 0$ in $K^0(X, T)$ if and only if $[h] > 0$ in $K^0(Y, S)$, where $h \in \mathcal{C}(Y)$. If $[h] > 0$ then there exists $g \in \mathcal{C}(Y)$, $g(y) > 0$ for all $y \in Y$, such that $[h] = [g]$. Then $\pi^*([h]) = \pi^*([g]) = [g \circ \pi]$ which clearly is > 0 in $K^0(X, T)$. Conversely, assume $[h \circ \pi] = \pi^*([h]) > 0$ in $K^0(X, T)$. By Theorem 4.13 this means that $\int_X h \circ \pi d\nu > 0$ for all $\nu \in M(X, T)$. Now $\int_X h \circ \pi d\nu = \int_Y h d\pi^*(\nu)$. By Proposition 5.10 π_* maps $M(X, T)$ onto $M(Y, S)$, and so we get $\int_Y h d\mu > 0$ for all $\mu \in M(Y, S)$. By Theorem 4.13 this implies that $[h] > 0$ in $K^0(Y, S)$, and the proof is complete. \square

Proposition 5.12. *If π in Theorem 5.11 in addition is almost one-to-one, then the quotient*

$$K^0(X, T)/\pi^*(K^0(Y, S))$$

is torsion free.

Remark 5.13. A sketch of the proof of this is given in Lemma 4.2 in paper 1. Proposition 5.12 is one of many tools to obtain the results in paper 1.

References

- [D] T. Downarowicz. Survey of odometers and Toeplitz flows. *Algebraic and Topological Dynamics (Kolyada, Manin, Ward eds), Contemporary Math.*, **385** (2005), 7–38.
- [DHS] F. Durand, B. Host and C.F. Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergod. Th. and Dynam. Sys.*, **19** (1999), 995–1035.
- [DM] T. Downarowicz and A. Maass. Finite-rank Bratteli-Vershik diagrams are expansive. *Ergod. Th. and Dynam. Sys.*, **28** (2008), 739–747.
- [E] E. Effros. *Dimensions and C^* -algebras*. Number 46 in CBMS Regional Conference Series in Mathematics, 1981.
- [EHS] E. Effros, D. Handelman and C. Shen. Dimension groups and their affine representations. *Amer. J. Math.*, **102** (1980), 385–407.
- [F] H. Furstenberg. Poincaré Recurrence and Number Theory. *Bull. Amer. Math. Soc. (N.S.)*, **5** (1981), no. 3, 211–234.
- [GH] W. H. Gottschalk and G. A. Hedlund. *Topological Dynamics, vol. 36*. Amer. Math. Soc. Colloq. Publ, Providence, R.I., 1955.
- [GPS] T. Giordano, I.F. Putnam and C. Skau. Topological orbit equivalence and C^* -crossed products. *J. Reine Angew. Math.*, **469** (1995), 51–111.
- [GW] E. Glasner and B. Weiss. Weak orbit equivalence of Cantor minimal systems. *Int. J. Math.*, **6** (1995), 569–579.
- [H] M. Hochman. Genericity in topological dynamics. *Ergod. Th. and Dynam. Sys.*, **28** (2008), 125–165.
- [HPS] R.H. Herman, I.F. Putnam and C.F. Skau. Ordered Bratteli diagrams, dimension groups, and topological dynamics. *Int. J. Math.*, **3** (1992), 827–864.
- [HR] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis I*. Grundle Math. Wiss. **115** Springer Verlag, 1963.
- [JK] K. Jacobs and M. Keane. 0–1-sequences of Toeplitz type. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **13** (1969), 123–131.
- [K] P. Kůrka. *Topological and Symbolic Dynamics*. Cours Spécialisés **11**, Société Mathématique de France, 2003.

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- [Ku] K. Kuratowski. *Topology, Vol. II*. Academic Press, New York, 1968.
- [MP] N. G. Markley, and M. E. Paul. Almost automorphic symbolic minimal sets without unique ergodicity. *Israel J. Math.*, **34** (1979), 259–272.
- [P] M. E. Paul. Construction of almost automorphic symbolic minimal flows. *General Topology Appl.*, **6** (1976), 45–56.
- [Qu] M. Queffélec. *Substitution Dynamical Systems (Lecture Notes in Mathematics, 1294)*. Springer, 1987.
- [W] S. Williams. Toeplitz minimal flows which are not uniquely ergodic. *Z. Wahrs. verw. Gebiete.*, **67** (1984), 95–107.
- [Wa] P. Walters. *An Introduction to Ergodic Theory*. Graduate Texts in Mathematics **79**, Springer-Verlag, 1982.

Paper I

Toeplitz flows and their ordered K-theory

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TOEPLITZ FLOWS AND THEIR ORDERED K-THEORY

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Abstract

To a Toeplitz flow (X, T) we associate an ordered K^0 -group, denoted $K^0(X, T)$, which is order isomorphic to the K^0 -group of the associated (non-commutative) C^* -crossed product $C(X) \rtimes_T \mathbb{Z}$. However, $K^0(X, T)$ can be defined in purely dynamical terms, and it turns out to be a complete invariant for (strong) orbit equivalence. We characterize the K^0 -groups that arise from Toeplitz flows (X, T) as exactly those simple dimension groups (G, G^+) that contain a noncyclic subgroup H of rank one that intersects G^+ nontrivially. Furthermore, the Bratteli diagram realization of (G, G^+) can be chosen to have the ERS-property, i.e. the incidence matrices of the Bratteli diagram have equal row sums. We also prove that for any Choquet simplex K there exists an uncountable family of pairwise non-orbit equivalent Toeplitz flows (X, T) such that the set of T -invariant probability measures $M(X, T)$ is affinely homeomorphic to K , where the entropy $h(T)$ may be prescribed beforehand. Furthermore, the analogous result is true if we substitute strong orbit equivalence for orbit equivalence, but in that case we can actually prescribe both the entropy and the maximal equicontinuous factor of (X, T) . Finally, we present some interesting concrete examples of dimension groups associated to Toeplitz flows.

1 Introduction.

Toeplitz flows have been extensively studied, both as topological and measure-theoretic dynamical systems, since they were first introduced by Jacobs and Keane in 1969 [JK]. (Note: In spite of the word “flow”, which would indicate an \mathbb{R} -action, Toeplitz flows are dynamical systems with a \mathbb{Z} -action generated by a single homeomorphism.) In this paper we will exclusively study Toeplitz flows as topological dynamical systems. A Toeplitz flow is a special minimal subshift on a finite alphabet Λ (in particular, it is a *symbolic system*) defined in terms of a so-called *Toeplitz sequence* on Λ , the latter being associated to certain arithmetic progressions. (Incidentally, the reason the name “Toeplitz” has been attached to these systems is that Toeplitz in 1924 [T] gave an explicit construction of almost periodic functions (in the Bohr sense) on \mathbb{R} , where he as a device used arithmetic progressions.) Toeplitz flows are closely related to the well understood odometer systems, and may in a certain sense be seen as the simplest Cantor minimal systems beyond the odometer systems. Yet, surprisingly, Toeplitz

flows exhibit a richness of properties that the odometer systems do not have. As examples of this we mention that any Choquet simplex can be realized as the set of invariant probability measures for some Toeplitz flow. Furthermore, for every $0 \leq t < \infty$ there exists a Toeplitz flow (in fact, an uncountable family of pairwise non-isomorphic Toeplitz flows) whose topological entropy is equal to t .

An entirely new approach to study Toeplitz flows, and, for that matter, Cantor minimal systems in general – thereby providing powerful new tools – came from an unsuspected source : non-commutative C^* -algebras. In fact, by studying the so-called C^* -crossed product associated to a Cantor minimal system a complete isomorphism invariant turns out to be a special ordered abelian group, a so-called dimension group. This group is the K^0 -group of the crossed product and it comes with a natural ordering. It turns out that the K^0 -group can be defined purely in dynamical terms, and that it is a complete invariant for (strong) orbit equivalence. This invariant is completely independent of other invariants traditionally used to study dynamical systems, like spectral invariants and entropy. Furthermore, the invariant is in many cases effectively computable. We mention two papers that illustrate this. The first is the paper by Durand, Host and Skau [DHS] where substitutional dynamical systems are studied using the K^0 -approach. The other is a paper by Gjerde and Johansen [GJ] where Toeplitz flows are treated. In the latter paper a “clean” conceptual proof – as seen from the theory of dimension groups – is given of the Choquet simplex realization result alluded to above. (The original proof of that result is due to Downarowicz [D], after preliminary results had been obtained by Williams [W].) Furthermore, the dimension groups (i.e. the K^0 -groups), associated to Toeplitz flows are described in their paper in terms of Bratteli diagrams with what they call the EPN-property (which is the same as the ERS-property – the term we prefer to use). In this paper we will both extend their results considerably, and we will give a more satisfactory description of the K^0 -groups. In fact, the main thrust of this paper is to give the definite characterization of the K^0 -groups associated to Toeplitz flows. This is done in an intrinsic way, meaning that we give a characterization in terms of the group itself, while the Bratteli diagram realizations of the groups in question play an important, but auxiliary role. By our characterization we can easily exhibit concrete examples of such groups, some of these will be described in Section 5. Furthermore, our results underscore in a striking manner that the K^0 -group of a Toeplitz flow (X, T) does not “see” the entropy $h(T)$ of T , these two entities are independent. (See the Remark after Theorem 2.4.)

We remark that Toeplitz flows in contrast to substitution minimal systems are “unstable”. By that we mean that whereas an induced system of a substitution system is again a substitution system, and a Cantor factor of a substitution system is either again a substitution system or an odometer, this is not true for Toeplitz

flows in general. (Cf. [DD].) There is a “tiny” overlap between Toeplitz flows and substitution minimal systems. In fact, some – but not all – substitution minimal systems associated to (primitive, aperiodic) substitutions of constant length are Toeplitz flows. (Cf. [M, Theorem 6.03].)

2 Main results

We formulate our main results, referring to Section 3 for definitions of some of the terms occurring in the statements of the theorems below.

Theorem 2.1. *Let $0 \leq t < \infty$. The following two sets are equal (up to order isomorphisms):*

- (i) $\{(K^0(X, T), K^0(X, T)^+ \mid (X, T) \text{ Toeplitz flow, } h(T) = t\}$
- (ii) $\{(G, G^+) \mid G \text{ simple dimension group containing a noncyclic subgroup } H \text{ of rank one such that } H \cap G^+ \neq \{0\}\}$

Theorem 2.2. *Let (G, G^+, u) be a simple dimension group with order unit u , and assume the rational (and hence rank one) subgroup $\mathbb{Q}(G, u)$ is noncyclic. Let $0 \leq t < \infty$. There exists a Toeplitz flow (X, T) such that*

- (i) $(G, G^+, u) \cong (K^0(X, T), K^0(X, T)^+, \mathbf{1})$
- (ii) *The entropy $h(T)$ of T equals t .*
- (iii) *The set of (continuous) eigenvalues of T is $\{e^{2\pi i s} \mid s \in \mathbb{Q}(G, u)\}$*

Theorem 2.3. *Let $0 \leq t < \infty$. The following three sets of simple dimension groups with order units are equal (up to order isomorphisms preserving order units):*

$$\begin{aligned} \mathcal{T}_t &= \{(K^0(X, T), K^0(X, T)^+, \mathbf{1}) \mid (X, T) \text{ Toeplitz flow, } h(T) = t\} \\ \mathcal{G} &= \{(G, G^+, u) \mid G \text{ simple dimension group with } \mathbb{Q}(G, u) \text{ noncyclic}\} \\ \mathcal{B} &= \left\{ (K_0(V, E), K_0(V, E)^+, [\mathbf{1}]) \mid \begin{array}{l} (V, E) \text{ simple Bratteli diagram} \\ \text{with the ERS property} \end{array} \right\}. \end{aligned}$$

Theorem 2.4. *Let $0 \leq t < \infty$. For any Choquet simplex K there exists an uncountable family of pairwise non-orbit equivalent Toeplitz flows $\mathcal{A} = \{(X, T)\}$, such that for each $(X, T) \in \mathcal{A}$:*

- (i) $K \cong M(X, T)$, i.e. K is affinely homeomorphic to the T -invariant probability measures on X .

(ii) $h(T) = t$.

Now let (Y, S) be any odometer and let K be any Choquet simplex. There exists an uncountable family of pairwise non-strong orbit equivalent Toeplitz flows $\tilde{\mathcal{A}} = (\tilde{X}, \tilde{T})$, such that for each $(\tilde{X}, \tilde{T}) \in \tilde{\mathcal{A}}$:

(i) $K \cong M(\tilde{X}, \tilde{T})$

(ii) $h(\tilde{T}) = t$

(iii) The maximal equicontinuous factor of (\tilde{X}, \tilde{T}) is (Y, S) .

Remark. As we alluded to above, Theorem 2.4 yields in a striking way as a corollary that for Toeplitz flows orbit structure and entropy are independent entities.

Our result Theorem 2.2(i) has an interesting consequence which we will briefly describe. In [GMPS] the following remarkable result is proved (cf. [GMPS, Theorem 2.5]):

Let d be a natural number. A minimal \mathbb{Z}^d -action on the Cantor set is orbit equivalent to a minimal \mathbb{Z} -action (i.e. to a Cantor minimal system (X, T)). A complete invariant for orbit equivalence is a simple dimension group with order unit (G, G^+, u) , such that $\text{Inf}G = \{0\}$. Specifically, G is given by

$$G = \mathcal{C}(X, \mathbb{Z}) / \left\{ f \in \mathcal{C}(X, \mathbb{Z}) \mid \int_X f d\mu = 0, \forall \mu \in M(X, \mathbb{Z}^d) \right\}.$$

(Here $M(X, \mathbb{Z}^d)$ denotes the set of invariant probability measures under the \mathbb{Z}^d -action.) G^+ is given by the induced ordering from $\mathcal{C}(X, \mathbb{Z})$ and u corresponds to the constant function $1 \in \mathcal{C}(X, \mathbb{Z})$.

Remark. An open problem raised in [GMPS] is: what is the range of the invariant (G, G^+, u) when $d > 1$? (For $d = 1$ it is known that the range is all simple dimension groups with order unit (G, G^+, u) such that $\text{Inf}G = \{0\}$, cf. [HPS, Theorem 5.4].)

In [CP] the following theorem is proved (cf. [CP, Theorem B]):

Let (X, T) be a Toeplitz flow and let $d \geq 1$. There exists a Toeplitz \mathbb{Z}^d -subshift (and so, in particular, a minimal \mathbb{Z}^d -action on the Cantor set) which is orbit equivalent to (X, T) .

Combining Theorem 2.2(i) with the two results cited above we get the following interesting result.

Proposition 2.5. *Let d be a natural number. The range of the orbit invariant for minimal \mathbb{Z}^d -actions on the Cantor set contains all simple dimension groups with order unit (G, G^+, u) , where $\text{Inf}G = \{0\}$, such that $\mathbb{Q}(G, u)$ is noncyclic.*

3 Basic concepts and definitions and key background results.

3.1 Dynamical systems

Throughout this paper we will use the term dynamical system to mean a compact metric space X together with a homeomorphism $T: X \rightarrow X$, and we will denote this by (X, T) . This induces in a natural way a \mathbb{Z} -action on X . The orbit of $x \in X$ under this action is $\{T^n x \mid n \in \mathbb{Z}\}$ and will be denoted by $\text{orbit}_T(x)$. If all the orbits are dense in X we say that (X, T) is a *minimal system*. It is a simple observation that (X, T) is minimal if and only if $TA = A$ for some closed $A \subseteq X$ implies that $A = X$ or \emptyset .

(We will denote the natural numbers $\{1, 2, 3, \dots\}$ by \mathbb{N} , the integers by \mathbb{Z} , the rational numbers by \mathbb{Q} , the real numbers by \mathbb{R} . Also, let $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r \geq 0\}$, $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$.)

Definition 3.1. We say that a dynamical system (Y, S) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, S) if there exists a continuous surjection $\pi: X \rightarrow Y$ which satisfies $S(\pi(x)) = \pi(Tx)$, $\forall x \in X$. We call π a *factor map*. If π is a bijection then we say that (X, T) and (Y, S) are *conjugate*, and we write $(X, T) \cong (Y, S)$. We say that (X, T) is *flip conjugate* to (Y, S) if $(X, T) \cong (Y, S)$ or $(X, T) \cong (Y, S^{-1})$.

Definition 3.2. The dynamical systems (X, T) and (Y, S) are *orbit equivalent* if there exists a homeomorphism $F: X \rightarrow Y$ such that $F(\text{orbit}_T(x)) = \text{orbit}_S(F(x))$ for all $x \in X$. We call F an *orbit map*.

Remark. Clearly flip conjugacy implies orbit equivalence. One can show that if X (and hence Y) is a connected space then orbit equivalence between (X, T) and (Y, S) implies flip conjugacy. (This follows by a simple argument using a result of Sierpiński, cf. [K, Theorem 6, Ch. V, §47, III].) This has the consequence that the study of orbit equivalence is only interesting as it pertains to *Cantor minimal systems* (X, T) , i.e. X is a Cantor set on which T acts minimally. The K -theoretic invariant we are going to introduce is an invariant for orbit equivalence, and so we will assume henceforth that our dynamical systems are Cantor minimal, even though some of the subsequent definitions apply to more general systems.

Let (X, T) , (Y, S) and F be as in Definition 3.2, both (X, T) and (Y, S) being Cantor minimal systems. For each $x \in X$ there exists a unique integer $n(x)$ (respectively, $m(x)$) such that $F(Tx) = S^{n(x)}(F(x))$, $F(T^{m(x)}x) = S(F(x))$. We call $m, n: X \rightarrow \mathbb{Z}$ the orbit cocycles associated to the orbit map F .

Definition 3.3. We say that (X, T) and (Y, S) are *strong orbit equivalent* if

there exists an orbit map $F: X \rightarrow Y$ such that each of the two associated orbit cocycles $m, n: X \rightarrow \mathbb{Z}$ have at most one point of discontinuity.

Remark. Boyle (cf. [BT]) proved that if the orbit cocycles are continuous for all $x \in X$, then (X, T) and (Y, S) are flip conjugate. So strong orbit equivalence is in a sense the mildest weakening possible of flip conjugacy.

We shall need the concept of *induced transformation* and *Kakutani equivalence*.

Definition 3.4. Let (X, T) be a Cantor minimal system and let A be a clopen subset of X (hence A is again a Cantor set). Let $T_A: A \rightarrow A$ be the *first return map*, i.e. $T_A(x) = T^{r_A(x)}x$, where $r_A(x) = \min \{n \in \mathbb{N} \mid T_A^n x \in A\}$. We say that (A, T_A) , which is again Cantor minimal, is the induced system of (X, T) with respect to A .

Definition 3.5. The Cantor minimal systems (X, T) and (Y, S) are *Kakutani equivalent* if (up to conjugacy) they have a common induced system. (One can show that this is an equivalence relation on the family of Cantor minimal systems.)

Definition 3.6. Let (X, T) and (Y, S) be (Cantor) minimal systems, (Y, S) being a factor of (X, T) by a map $\pi: X \rightarrow Y$. If there exists a point $x \in X$ such that $\pi^{-1}(\pi(x)) = \{x\}$ we say that π is almost one-to-one and we say that (X, T) is an almost one-to-one extension of (Y, S) .

Remark. One can show that the subset B of X consisting of point z in X satisfying the above condition is a dense \mathcal{G}_δ -set in X . In fact, define $A(x, n) = \{y \in Y \mid \pi^{-1}(y) \subseteq B(x, 2^{-n})\}$, where $B(x, 2^{-n})$ is the open ball around x with radius 2^{-n} . One shows that $A(x, n)$ is an open subset of Y . Define $A_n = \bigcup_{v \in X} A(v, n)$. Then A_n is an open subset of Y . A simple argument yields that the subset B in question is equal to $\pi^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} \pi^{-1}(A_n)$, which is a dense \mathcal{G}_δ -set.

3.2 Toeplitz flows

Definition 3.7. (X, T) is *expansive* if there exists $\delta > 0$ such that if $x \neq y$ then $\sup_n d(T^n x, T^n y) > \delta$, where d is a metric that gives the topology of X . (Expansiveness is independent of the metric as long as the metric gives the topology of X .)

Let $\Lambda = \{a_1, a_2, \dots, a_n\}$, $n \geq 2$, be a finite alphabet and let $Z = \Lambda^{\mathbb{Z}}$ be the set of all bi-infinite sequences of symbols from Λ with Z given the product topology – thus Z is a Cantor set. Let $S: Z \rightarrow Z$ denote the shift map, $S: (x_n) \rightarrow (x_{n+1})$. If X is a closed subset of Z such that $S(X) = X$, we say that (X, S) is a *subshift*, where we denote the restriction of S to X again by S . Subshifts are easily seen

to be expansive. We state the following well-known fact as a proposition. (Cf. [Wa, Theorem 5.24].)

Proposition 3.8. *Let (X, T) be a Cantor minimal system. Then (X, T) is conjugate to a minimal subshift on a finite alphabet if and only if (X, T) is expansive.*

As a general reference on Toeplitz flows we refer to [W]. (Cf. also [D].)

Definition 3.9. Let $\eta = (\eta(n))_{n \in \mathbb{Z}} \in \Lambda^{\mathbb{Z}}$, where Λ is a finite alphabet. Then we define for $\sigma \in \Lambda$, $p \in \mathbb{N}$

$$\text{Per}_p(\eta, \sigma) = \{n \in \mathbb{Z} \mid \eta(n + mp) = \sigma, \forall m \in \mathbb{Z}\}.$$

Let

$$\text{Per}_p(\eta) = \bigcup_{\sigma \in \Lambda} \text{Per}_p(\eta, \sigma).$$

We say that η is a *Toeplitz sequence* if $\bigcup_{p \in \mathbb{N}} \text{Per}_p(\eta) = \mathbb{Z}$.

By the *p-skeleton* of η we will mean the part of η which is periodic with period p ; more precisely, we define the *p-skeleton* to be the sequence obtained from η by replacing $\eta(n)$ with a new symbol $*$ for all $n \notin \text{Per}_p(\eta)$. We say that p is an *essential period* of η if the *p-skeleton* of η is not periodic with any smaller period. The least common multiple of two essential periods is again an essential period, a fact which is easily verified.

Definition 3.10. Assume that a Toeplitz sequence η is non-periodic. A *periodic structure* for η is a strictly increasing sequence $(p_i)_{i \in \mathbb{N}}$ such that p_i is an essential period of η for all i , $p_i \mid p_{i+1}$ and $\bigcup_{i=1}^{\infty} \text{Per}_{p_i}(\eta) = \mathbb{Z}$. (A periodic structure always exists for a (non-periodic) Toeplitz sequence, cf. [W].)

Definition 3.11. Let $\eta \in \Lambda^{\mathbb{Z}}$ be a Toeplitz sequence. The dynamical system $(\overline{\mathcal{O}(\eta)}, S)$ is called a *Toeplitz flow*, where $\mathcal{O}(\eta) = \text{orbit}_S(\eta)$ and S is the shift map.

Periodic sequences are Toeplitz sequences. Every Toeplitz sequence is almost periodic, i.e. every word occurring in η appears with bounded gap between successive occurrences. Hence $(\overline{\mathcal{O}(\eta)}, S)$ is minimal. In the sequel we will only consider non-periodic Toeplitz flows, and so Toeplitz flows are expansive Cantor minimal systems.

Let $(p_i)_{i \in \mathbb{N}}$ be a periodic structure for the (non-periodic) Toeplitz sequence η , and denote the associated Toeplitz flow by (Y, S) . Let $(G_{\mathfrak{a}}, \rho_{\widehat{\mathfrak{a}}})$ denote the *odometer* (also called *adding machine*) associated to the \mathfrak{a} -adic group

$$G_{\mathfrak{a}} = \prod_{i=1}^{\infty} \left\{ 0, 1, \dots, \frac{p_i}{p_{i-1}} - 1 \right\},$$

where $\mathbf{a} = \left\{ \frac{p_i}{p_{i-1}} \right\}_{i \in \mathbb{N}}$ (we set $p_0 = 1$) and where $\rho_{\widehat{\mathbb{T}}}(x) = x + \widehat{1}$, where $\widehat{1} = (1, 0, 0, \dots)$. We note that $G_{\mathbf{a}}$ is naturally isomorphic to the inverse limit group

$$\mathbb{Z}/p_1\mathbb{Z} \xleftarrow{\phi_1} \mathbb{Z}/p_2\mathbb{Z} \xleftarrow{\phi_2} \mathbb{Z}/p_3\mathbb{Z} \xleftarrow{\phi_3} \dots$$

where $\phi_i(n)$ is the residue of n modulo p_i . It is a fact that the family consisting of compact groups G that are both monothetic (i.e. contains a dense copy of \mathbb{Z} , which of course implies that G is abelian) and Cantor (as a topological space), coincides with the family of \mathbf{a} -adic groups. It is also noteworthy that all minimal rotations (in particular rotation by $\widehat{1}$) on such groups are conjugate. This is a consequence of the fact that the dual group of an \mathbf{a} -adic group is a torsion group. If $\mathbf{a} = \{p\}_{i \in \mathbb{N}}$, where p is a prime, then $G_{\mathbf{a}}$ is the p -adic integers. (We refer to [HR, Vol 1] for background information on \mathbf{a} -adic groups.)

Recall that every (minimal) dynamical system has a *maximal equicontinuous factor*, the latter being, by a well-known theorem, conjugate to a minimal rotation on a compact abelian group G , and so, in particular, is uniquely ergodic. Specifically, let (G, ρ_g) be the maximal equicontinuous factor of (X, T) , where $\rho_g: G \rightarrow G$ is rotation by g , i.e. $\rho_g(x) = x + g$ for $x \in G$, and $\Phi: X \rightarrow G$ is the factor map. Let (H, ρ_h) be another minimal group rotation factor of (X, T) where $\Psi: X \rightarrow H$ is the factor map. Then (H, ρ_h) is a factor of (G, ρ_g) by a factor map $\pi: G \rightarrow H$ such that $\Psi = \pi \circ \Phi$.

One can detect the maximal equicontinuous factor (G, ρ_g) of (X, T) by determining the (continuous) eigenvalues Γ of T , i.e.

$$\Gamma = \{ \lambda \in \mathbb{T} \mid f \circ T = \lambda f \text{ for some } 0 \neq f \in \mathcal{C}(X) \},$$

where \mathbb{T} is the unit circle. Now Γ is a countable and discrete subgroup of the discrete circle group \mathbb{T}_d , and $G = \widehat{\Gamma}$ is the compact dual group with $g \in G$ being the character on Γ defined by $g(\gamma) = \gamma$, $\forall \gamma \in \Gamma$. Incidentally, the dual group $\widehat{G_{\mathbf{a}}}$ of the odometer group $G_{\mathbf{a}}$ has only torsion elements and is equal to the following subgroup of \mathbb{T}_d , namely

$$\widehat{G_{\mathbf{a}}} = \left\{ e^{\frac{2\pi i l}{a_1 a_2 \dots a_k}} \mid \mathbf{a} = \{a_i\}_{i \in \mathbb{N}}, k \in \mathbb{N}, l \in \mathbb{Z} \right\}.$$

If (X, T) is a Toeplitz flow with a periodic structure $(p_i)_{i \in \mathbb{N}}$, then the maximal equicontinuous factor is the odometer system associated to $\mathbf{a} = \left(\frac{p_i}{p_{i-1}} \right)_{i \in \mathbb{N}}$. Furthermore, the factor map $\pi: X \rightarrow G_{\mathbf{a}}$ is an almost one-to-one map. The converse is also true. We state two theorems that establish this fact.

Theorem 3.12 ([P]). *Let (X, T) be a minimal almost one-to-one extension of a (minimal) group rotation system (G, ρ_g) . Then (G, ρ_g) is the maximal equicontinuous factor of (X, T) .*

Theorem 3.13 ([MP]). *An expansive Cantor minimal system (X, T) is a Toeplitz flow if and only if (X, T) is an almost one-to-one extension of an odometer system $(G_{\mathbf{a}}, \rho_{\overline{\Gamma}})$. Furthermore, $(G_{\mathbf{a}}, \rho_{\overline{\Gamma}})$ is the maximal equicontinuous factor of (X, T) .*

A more detailed analysis of the situation is found in [W]. We summarize by listing some salient points from [W] which is relevant for this paper:

Let η be a (non-periodic) Toeplitz sequence and let $(\overline{\mathcal{O}(\eta)}, S)$ be the associated Toeplitz flow. Let $(p_i)_{i \in \mathbb{N}}$ be a periodic structure for η . Define $A_n^i = \{S^m \eta \mid m \equiv n \pmod{p_i}\}$.

1. $\overline{A_n^i}$ is the set of all $\omega \in \overline{\mathcal{O}(\eta)}$ with the same p_i -skeleton as $S^n(\eta)$.
2. $\{\overline{A_n^i} \mid n \in \mathbb{Z}/p_i\mathbb{Z}\}$ is a partition of $\overline{\mathcal{O}(\eta)}$ into clopen sets.
3. $\overline{A_n^i} \supset \overline{A_m^j}$ for $i < j$ and $m \equiv n \pmod{p_i}$.
4. $S\overline{A_n^i} = \overline{A_{n+1}^i}$.
5. $\omega, \theta \in \bigcap_{i=1}^{\infty} \overline{A_{n_i}^i}$ (where $n_i \equiv n_j \pmod{p_i}$ for $j \geq i$) if and only if ω and θ have the same p_i -skeleton for all $i \in \mathbb{N}$. In particular, $\bigcap_{i=1}^{\infty} \overline{A_{n_i}^i} = \{\omega\}$ if and only if ω is a Toeplitz sequence. This implies that $\pi^{-1}(\pi(\omega)) = \{\omega\}$ if and only if ω is a Toeplitz sequence. Here $\pi: \overline{\mathcal{O}(\eta)} \rightarrow G_{\mathbf{a}}$ is the factor map, $(G_{\mathbf{a}}, \rho_{\overline{\Gamma}})$ being the maximal equicontinuous factor of $(\overline{\mathcal{O}(\eta)}, S)$.

Since $\text{Per}_{p_i}(\eta)$ is periodic it has density in \mathbb{Z} given by

$$d_i = \frac{1}{p_i} |\{n \in \mathbb{Z}/p_i\mathbb{Z} \mid n \in \text{Per}_{p_i}(\eta)\}|$$

where $|E|$ denotes the number of elements in the set E . The d_i are increasing, and we set $d = \lim_{i \rightarrow \infty} d_i$.

Definition 3.14. The Toeplitz sequence η is *regular* if $d = 1$.

Notation. Let (X, T) be a dynamical system. We denote by $M(X, T)$ the set of T -invariant probability measures on X . We say that (X, T) is *uniquely ergodic* if $M(X, T)$ is a singleton.

Fact. If (X, T) is a minimal dynamical system, then $M(X, T)$ is a Choquet simplex (where $M(X, T)$ is given the w^* -topology).

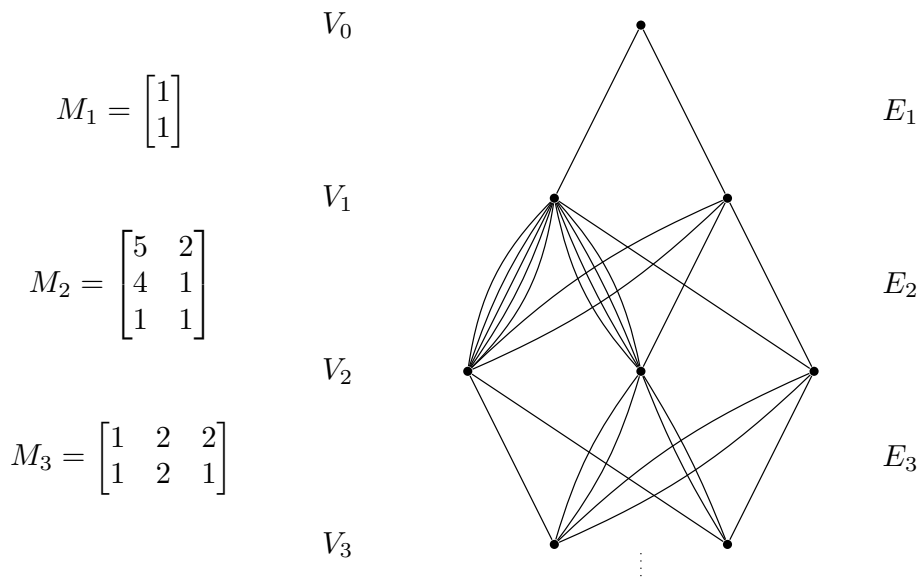


Figure 1: An example of a Bratteli diagram

Theorem 3.15 ([JK]). *If $(\overline{\mathcal{O}(\eta)}, S)$ is a regular Toeplitz flow (i.e. η is a regular Toeplitz sequence), then it is uniquely ergodic and the (topological) entropy $h(S)$ is zero.*

We will return to Toeplitz flows and their properties after we have introduced the (ordered) K-theory associated to such systems.

3.3 Bratteli diagrams and dimension groups.

(As general references for the material in this section we refer to [E], [HPS] and [GPS1].)

3.3.1 Bratteli diagrams

A Bratteli diagram (V, E) consists of a set of vertices $V = \sqcup_{n=0}^{\infty} V_n$ and a set of edges $E = \sqcup_{n=1}^{\infty} E_n$, where the V_n 's and the E_n 's are finite disjoint sets and where $V_0 = \{v_0\}$ is a one-point set. The edges in E_n connect vertices in V_{n-1} with vertices in V_n . If e connects $v \in V_{n-1}$ with $u \in V_n$ we write $s(e) = v$ and $r(e) = u$, where $s: E_n \rightarrow V_{n-1}$ and $r: E_n \rightarrow V_n$ are the source and range maps, respectively. We will assume that $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and that $r^{-1}(v) \neq \emptyset$

for all $v \in V \setminus V_0$. A Bratteli diagram can be given a diagrammatic presentation with V_n the vertices at level n and E_n the edges between V_{n-1} and V_n . If $|V_{n-1}| = t_{n-1}$ and $|V_n| = t_n$ then the edge set E_n is described by a $t_n \times t_{n-1}$ incidence matrix $M_n = (m_{ij}^n)$, where m_{ij}^n is the number of edges connecting $v_i^n \in V_n$ with $v_j^{n-1} \in V_{n-1}$ (see Figure 1). Let $k, l \in \mathbb{Z}^+$ with $k < l$ and let $E_{k+1} \circ E_k \circ \cdots \circ E_l$ denote all the paths from V_k to V_l . Specifically, $E_{k+1} \circ E_k \circ \cdots \circ E_l = \{(e_{k+1}, \dots, e_l) \mid e_i \in E_i, i = k+1, \dots, l; r(e_i) = s(e_{i+1}), i = k+1, \dots, l-1\}$. We define $r((e_{k+1}, \dots, e_l)) = r(e_l)$ and $s((e_{k+1}, \dots, e_l)) = s(e_{k+1})$. Notice that the corresponding incidence matrix is the product $M_l M_{l-1} \cdots M_{k+1}$ of the incidence matrices.

Definition 3.16. The Bratteli diagram (V, E) with incidence matrices $(M_n)_{n=1}^\infty$ has the ERS-property (ERS = Equal Row Sum) if the row sums of the incidence matrices are constant. Let the constant row sum of M_n be r_n . We associate the supernatural number $\prod_{n=1}^\infty r_n$ to (V, E) . (See the comments after Definition 3.23.)

Definition 3.17. Given a Bratteli diagram (V, E) and a sequence $0 = m_0 < m_1 < m_2 < \cdots$ in \mathbb{Z}^+ , we define the *telescoping* of (V, E) to $\{m_n\}$ as (V', E') , where $V'_n = V_{m_n}$ and $E'_n = E_{m_{n-1}+1} \circ \cdots \circ E_{m_n}$, and the source and the range maps are as above.

Remark. Observe that the ERS-property is preserved under telescoping, and that is also the case for the associated supernatural number.

Definition 3.18. We say that the Bratteli diagram (V, E) is *simple* if there exists a telescoping of (V, E) such that the resulting Bratteli diagram (V', E') has full connection between all consecutive levels, i.e. the entries of all the incidence matrices are non-zero.

Given a Bratteli diagram (V, E) we define the infinite path space associated to (V, E) , namely

$$X_{(V,E)} = \{(e_1, e_2, \dots) \mid e_i \in E_i, r(e_i) = s(e_{i+1}); \quad \forall i \geq 1\}.$$

Clearly $X_{(V,E)} \subseteq \prod_{n=1}^\infty E_n$, and we give $X_{(V,E)}$ the relative topology, $\prod_{n=1}^\infty E_n$ having the product topology. Loosely speaking this means that two paths in $X_{(V,E)}$ are close if the initial parts of the two paths agree on a long initial stretch. Also, $X_{(V,E)}$ is a closed subset of $\prod_{n=1}^\infty E_n$, and is compact.

Let $p = (e_1, e_2, \dots, e_n) \in E_1 \circ \cdots \circ E_n$ be a finite path starting at $v_0 \in V_0$. We define the *cylinder set* $U(p) = \{(f_1, f_2, \dots) \in X_{(V,E)} \mid f_i = e_i, i = 1, 2, \dots, n\}$. The collection of cylinder sets is a basis for the topology on $X_{(V,E)}$. The cylinder sets are clopen sets, and so $X_{(V,E)}$ is a compact, totally disconnected metric space

– metric because the collection of cylinder sets is countable. If (V, E) is simple then $X_{(V,E)}$ has no isolated points, and so $X_{(V,E)}$ is a Cantor set. (We will in the sequel disregard the trivial case where $|X_{(V,E)}|$ is finite.)

Let $P_n = E_1 \circ \dots \circ E_n$ be the set of finite paths of length n (starting at the top vertex). We define the truncation map $\tau_n: X_{(V,E)} \rightarrow P_n$ by $\tau_n((e_1, e_2, \dots)) = (e_1, e_2, \dots, e_n)$. If $m \geq n$ we have the obvious truncation map $\tau_{m,n}: P_m \rightarrow P_n$.

There is an obvious notion of isomorphism between Bratteli diagrams (V, E) and (V', E') ; namely, there exists a pair of bijections between V and V' preserving the gradings and intertwining the respective source and range maps. Let \sim denote the equivalence relation on Bratteli diagrams generated by isomorphism and telescoping. One can show that $(V, E) \sim (V', E')$ iff there exists a Bratteli diagram (W, F) such that telescoping (W, F) to odd levels $0 < 1 < 3 < \dots$ yields a diagram isomorphic to some telescoping of (V, E) , and telescoping (W, F) to even levels $0 < 2 < 4 < \dots$ yields a diagram isomorphic to some telescoping of (V', E') .

3.3.2 Dimension groups

By an ordered group we shall mean a countable abelian group G together with a subset G^+ , called the positive cone, such that

1. $G^+ - G^+ = G$
2. $G^+ \cap (-G^+) = \{0\}$
3. $G^+ + G^+ \subset G^+$

We shall write $a \leq b$ if $b - a \in G^+$. We say that an ordered group is *unperforated* if $a \in G$ and $na \in G^+$ for some $a \in G$ and $n \in \mathbb{N}$ implies that $a \in G^+$. Observe that an unperforated group is torsion free. By an *order unit* for (G, G^+) we mean an element $u \in G^+$ such that for every $a \in G$, $a \leq nu$ for some $n \in \mathbb{N}$.

Definition 3.19. A *dimension group* (G, G^+, u) with distinguished order unit u is an unperforated ordered group (G, G^+) satisfying the *Riesz interpolation property*, i.e. given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ ($i, j = 1, 2$), there exists $c \in G$ with $a_i \leq c \leq b_j$ ($i, j = 1, 2$).

We write $(G_1, G_1^+, u_1) \cong (G_2, G_2^+, u_2)$ if there exists an order isomorphism $\phi: G_1 \rightarrow G_2$, i.e. ϕ is a group isomorphism such that $\phi(G_1^+) = G_2^+$, and $\phi(u_1) = u_2$.

To a Bratteli diagram (V, E) we can associate an ordered group, which we will denote by $K_0(V, E)$ (because of its connection to (ordered) K-theory). Let

$V = \sqcup_{n=0}^{\infty} V_n$ and let $(M_n)_{n=1}^{\infty}$ be the incidence matrices. Then we have a system of simplicially ordered groups and positive maps

$$(\mathbb{Z} =) \mathbb{Z}^{|V_0|} \xrightarrow{M_1} \mathbb{Z}^{|V_1|} \xrightarrow{M_2} \mathbb{Z}^{|V_2|} \longrightarrow \dots$$

where the positive homomorphism $M_n: \mathbb{Z}^{|V_{n-1}|} \rightarrow \mathbb{Z}^{|V_n|}$ is given by matrix multiplication with the incidence matrix M_n . ($\mathbb{Z}^{|V_n|}$ is a column vector, and an element in $\mathbb{Z}^{|V_n|}$ is positive if all its entries are non-negative.) By definition $K_0(V, E)$ is the inductive limit of the system above, and $K_0(V, E)$ is given the induced order. We denote the positive cone by $K_0(V, E)^+$. $K_0(V, E)$ has a distinguished *order unit*, namely the element $[\mathbf{1}]$ in $K_0(V, E)^+$ corresponding to the element $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$. The triple $(K_0(V, E), K_0(V, E)^+, [\mathbf{1}])$ denotes the countable, ordered abelian group $K_0(V, E)$ with positive cone $K_0(V, E)^+$ and distinguished order unit $[\mathbf{1}]$. We will sometimes in the sequel for short only write $K_0(V, E)$, the ordering and the order unit being implicitly understood.

One can show that $(V, E) \cong (V', E')$ if and only if $K_0(V, E)$ is order isomorphic to $K_0(V', E')$ by a map sending the distinguished order unit of $K_0(V, E)$ to the distinguished order unit of $K_0(V', E')$.

It is straightforward to verify that $(K_0(V, E), K_0(V, E)^+)$ is a dimension group. The converse however, is not obvious and we state it as a theorem formulated in such a way that it suits our purpose.

Theorem 3.20 ([EHS]). *Let (G, G^+, u) be a dimension group with distinguished order unit u . Then there exists a Bratteli diagram (V, E) such that $(G, G^+, u) \cong (K_0(V, E), K_0(V, E)^+, [\mathbf{1}])$.*

A dimension group (G, G^+) is *simple* if it contains no non-trivial order ideals. An order ideal is a subgroup J such that $J = J^+ - J^+$ (where $J^+ = J \cap G^+$) and $0 \leq a \leq b \in J$ implies $a \in J$. It is easily seen that (G, G^+) is a simple dimension group if and only if every $a \in G^+ \setminus \{0\}$ is an order unit. In the sequel we will exclusively work with noncyclic (i.e. $G \not\cong \mathbb{Z}$) simple dimension groups (G, G^+) .

Let (G, G^+, u) be a simple dimension group with distinguished order unit u . We say that a homomorphism $p: G \rightarrow \mathbb{R}$ is a *state* if p is positive (i.e. $p(G^+) \geq 0$) and $p(u) = 1$. Denote the collection of all states on (G, G^+, u) by $S_u(G)$. Now $S_u(G)$ is a convex compact subset of the locally convex space \mathbb{R}^G with the product topology. In fact, one can show that $S_u(G)$ is a Choquet simplex. It is a fact that $S_u(G)$ determines the order on G . In fact,

$$G^+ = \{a \in G \mid p(a) > 0, \forall p \in S_u(G)\} \cup \{0\}.$$

Definition 3.21. Let (G, G^+) be a simple dimension group and let $u \in G^+ \setminus \{0\}$. We say that $a \in G$ is *infinitesimal* if $-\epsilon u \leq a \leq \epsilon u$ for all $0 < \epsilon \in \mathbb{Q}^+$. (If

$\epsilon = \frac{p}{q}$, $p, q \in \mathbb{N}$, then $a \leq \epsilon u$ means that $qa \leq pu$. It is easy to see that the definition does not depend upon the particular order unit u .) An equivalent definition is: $a \in G$ is infinitesimal if $p(a) = 0$ for all $p \in S_u(G)$. The collection of infinitesimal elements of G form a subgroup, *the infinitesimal subgroup of G* , which we denote by $\text{Inf}(G)$.

Remark The quotient group $G/\text{Inf}(G)$ is again a simple dimension group in the induced order, and the infinitesimal subgroup of $G/\text{Inf}(G)$ is trivial. Also, an order unit for G maps to an order unit for $G/\text{Inf}(G)$.

The following theorem summarizes some facts that are relevant for our situation, and the proof can be found in [E].

Theorem 3.22. *Let (G, G^+, u) be a simple (noncyclic) dimension group with distinguished order unit u . The map $\theta: G \rightarrow \text{Aff}(S_u(G))$ from G to the additive group of continuous affine functions on the Choquet simplex $S_u(G)$ defined by $\theta(g)(p) = p(g)$ is a strict order preserving map (i.e. $\theta(g)(p) > 0$ for all $p \in S_u(G)$ implies $g > 0$). Furthermore, $\text{Im}(\theta)$ is dense in $\text{Aff}(S_u(G))$ in the uniform norm and contains the constant function 1, and $\ker(\theta) = \text{Inf}(G)$.*

Conversely, suppose K is a Choquet simplex and H is a countable dense subgroup of $\text{Aff}(K)$, and that $\theta: G \rightarrow H$ is a homomorphism of a torsion free countable abelian group G onto H . Then letting

$$G^+ = \{g \in G \mid \theta(g)(p) > 0 \text{ all } p \in K\} \cup \{0\}$$

we get that (G, G^+) is a simple dimension group such that $\text{Inf}G = \ker(\theta)$. In particular, if $G = H$ (with θ the identity map) and G contains the constant function 1, then $\text{Inf}(G) = \{0\}$ and $S_1(G)$ is affinely homeomorphic to K by the map sending $k \in K$ to $\hat{k}: G \rightarrow \mathbb{R}$, where $\hat{k}(g) = g(k)$, $g \in G$.

Definition 3.23. By a *rational group H* we shall mean a (additive) subgroup of \mathbb{Q} that contains \mathbb{Z} . We say that H is a *noncyclic rational group* if H is not isomorphic to \mathbb{Z} . Clearly $(H, H^+, 1)$ is a simple dimension group with distinguished order unit 1, where $H^+ = H \cap \mathbb{Q}^+$.

Since rational dimension groups are going to play such an important role in this paper we will give a short description of them (Cf. [F, Chapter XIII, Section 85]). First of all they are exactly the countable torsion-free groups of rank one. Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots$ be a *supernatural number*, where p_1, p_2, p_3, \dots are the primes $2, 3, 5, \dots$ listed in increasing order, and $0 \leq k_i \leq \infty$ for each i . (If $(r_n)_{n=1}^\infty$ is a sequence of natural numbers, then we get a supernatural number $\prod_{n=1}^\infty r_n$ in an obvious way by factoring each r_n into a product of primes.) Clearly $n \in \mathbb{N}$ if and only if $k_i < \infty$ for all i , and $k_i = 0$ for all but finitely many i 's. If $m = p_1^{l_1} p_2^{l_2} p_3^{l_3} \cdots$ is another supernatural number we multiply n and m as

$nm = p_1^{k_1+l_1} p_2^{k_2+l_2} p_3^{k_3+l_3} \dots$, where $k_i + l_i = \infty$ if either k_i or l_i are equal to ∞ . We say that m divides n (notation $m|n$) if $l_i \leq k_i$ for all i . For $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots$ let

$$G(n) = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}, b|n \right\}.$$

Then $G(n)$ is a rational group and all rational groups are of this form. Furthermore, $G(n)$ is isomorphic to $G(m)$ if and only if there exists $a, b \in \mathbb{N}$ such that $an = bm$. In particular, all groups $G(n)$, where $n \in \mathbb{N}$, are isomorphic to $G(1) = \mathbb{Z}$. We note that $G(n)$ is p -divisible (i.e. for every $a \in G$ there exists $x \in E$ such that $px = a$) for some prime p if and only if p occurs with infinite multiplicity in the factorization of n . The group $G(n)$ can be made into a dimension group in exactly two ways, namely by letting the positive cone $G(n)^+$ be $G(n) \cap \mathbb{Q}^+$ or $-G(n) \cap \mathbb{Q}^+$, respectively. If $\mathbf{a} = (a_1, a_2, \dots)$ is a sequence of natural numbers with $a_i \geq 2$ for all i , we associate the noncyclic rational group $\left\{ \frac{m}{a_1 a_2 \dots a_k} \mid m \in \mathbb{Z}, k \in \mathbb{N} \right\}$. Clearly this is the same as the group $G(n)$, where n is the supernatural number $\prod_{i=1}^{\infty} a_i$. (Here the factorization of $\prod_{i=1}^{\infty} a_i$ into products of primes is obviously understood.)

Definition 3.24. Let (G, G^+, u) be a simple dimension group with order unit u . We define the rational subgroup of G , denoted $\mathbb{Q}(G, G^+, u)$ (or $\mathbb{Q}(G, u)$ for short), to be

$$\mathbb{Q}(G, u) = \{g \in G \mid ng = mu \text{ for some } n \in \mathbb{N}, m \in \mathbb{Z}\}.$$

Proposition 3.25. *Let $\mathbb{Q}(G, u)$ be as in Definition 3.24. The map $\Gamma: \mathbb{Q}(G, u) \rightarrow \mathbb{Q}$ defined by $\Gamma(g) = \frac{m}{n}$ if $ng = mu$, is an injective order homomorphism sending u to $1 \in \mathbb{Z}$, where $\mathbb{Q}(G, u)^+ = \mathbb{Q}(G, u) \cap G^+$. Thus $\mathbb{Q}(G, u)$ is order isomorphic to a rational group. Furthermore, $G/\mathbb{Q}(G, u)$, as an abstract group, is torsion free.*

Proof. The map Γ is well-defined. In fact, if $ng = mu$ and $n'g = m'u$, then we get by multiplying with n' and n respectively: $nn'g = n'mu$ and $nn'g = nm'u$. Subtracting we get: $(n'm - nm')u = 0$. Since G is torsion free we get $n'm - nm' = 0$, and so $\frac{m}{n} = \frac{m'}{n'}$. Similarly we show that Γ is a group homomorphism sending u to 1. If $\Gamma(g) = 0$, then $ng = 0$ for some $n \in \mathbb{N}$, and so $g = 0$ by torsion-freeness of G . So Γ is injective. It is straightforward to show that $\Gamma(g) \geq 0 \Leftrightarrow g \geq 0$, and so Γ is an order isomorphism onto its image. Hence $\mathbb{Q}(G, u)$ is order isomorphic to a rational group.

To show that $G/\mathbb{Q}(G, u)$ is torsion free, let $k\bar{g} = 0$ for some $k \in \mathbb{N}$, where \bar{g} is the image of $g \in G$ under the quotient map. This implies that $kg \in \mathbb{Q}(G, u)$, and so there exist $n \in \mathbb{N}, m \in \mathbb{Z}$ such that $nkg = mu$. Hence $g \in \mathbb{Q}(G, u)$, and so $\bar{g} = 0$, thus proving that $G/\mathbb{Q}(G, u)$ is torsion free. \square

Remark. Let $(G_1, G_1^+, u_1) \cong (G_2, G_2^+, u_2)$ by a map $\phi: G_1 \rightarrow G_2$. Then it is easily seen that $\phi(\mathbb{Q}(G_1, u_1)) = \mathbb{Q}(G_2, u_2)$. So, loosely speaking, we have that isomorphic dimension groups with distinguished order units have the same rational subgroups.

The notion of rational subgroup of a dimension group with distinguished order unit depends heavily upon the choice of the order unit as the following example shows.

Example 3.26. Let H be a noncyclic subgroup of \mathbb{Q} containing \mathbb{Z} . Let $G = H \oplus \mathbb{Z}$ with $G^+ = \{(h, k) \mid h > 0, k \in \mathbb{Z}\} \cup \{(0, 0)\}$. Then (G, G^+) is a simple dimension group with $\text{Inf}(G) = 0 \oplus \mathbb{Z}$. If we choose the order unit $u = (1, 0)$ for G , then one shows easily that $\mathbb{Q}(G, u) = H \oplus 0 \cong H$. However, if we choose the order unit $\tilde{u} = (1, 1)$ for G , then $\mathbb{Q}(G, \tilde{u}) = \{(k, k) \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$.

3.3.3 Ordered Bratteli diagram and the Bratteli-Vershik model

An *ordered Bratteli diagram* (V, E, \geq) is a Bratteli diagram (V, E) together with a partial order \geq in E so that edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. In other words, we have a linear order on each set $r^{-1}(v)$, $v \in V \setminus V_0$. We let E_{\min} and E_{\max} , respectively, denote the minimal and maximal edges of the partially ordered set E .

Note that if (V, E, \geq) is an ordered Bratteli diagram and $k < l$ in \mathbb{Z}^+ , then the set $E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l$ of paths from V_k to V_l with the same range can be given an induced (lexicographic) order as follows:

$$(e_{k+1} \circ e_{k+2} \circ \cdots \circ e_l) > (f_{k+1} \circ f_{k+2} \circ \cdots \circ f_l)$$

if for some i with $k+1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. If (V', E') is a telescoping of (V, E) then, with this induced order from (V, E, \geq) , we get again an ordered Bratteli diagram (V', E', \geq) .

Definition 3.27. We say that the ordered Bratteli diagram (V, E, \geq) , where (V, E) is a simple Bratteli diagram, is *properly ordered* if there exists a unique min path $x_{\min} = (e_1, e_2, \dots)$ and a unique max path $x_{\max} = (f_1, f_2, \dots)$ in $X_{(V, E)}$. (That is, $e_i \in E_{\min}$ and $f_i \in E_{\max}$ for all $i = 1, 2, \dots$)

Let (V, E) be a properly ordered Bratteli diagram, and let $X_{(V, E)}$ be the path space associated to (V, E) . Then $X_{(V, E)}$ is a Cantor set. Let $T_{(V, E)}$ be the *lexicographic map* on $X_{(V, E)}$, i.e. if $x = (e_1, e_2, \dots) \in X_{(V, E)}$ and $x \neq x_{\max}$ then $T_{(V, E)}x$ is the successor of x in the lexicographic ordering. Specifically, let k be the smallest natural number so that $e_k \notin E_{\max}$. Let f_k be the successor of e_k (and so $r(e_k) = r(f_k)$). Let $(f_1, f_2, \dots, f_{k-1})$ be the unique least element in $E_1 \circ E_2 \circ \cdots \circ E_{k-1}$ from $s(f_k) \in V_{k-1}$ to the top vertex $v_0 \in V_0$. Then $T_{(V, E)}((e_1, e_2, \dots)) = (f_1, f_2, \dots, f_k, e_{k+1}, e_{k+2}, \dots)$. We define $T_{(V, E)}x_{\max} = x_{\min}$. Then it is easy to check that $T_{(V, E)}$ is a minimal homeomorphism on $X_{(V, E)}$. We note that if $x \neq x_{\max}$ then x and $T_{(V, E)}x$ are *cofinal*, i.e. the edges making up x and $T_{(V, E)}x$, respectively, agree from a certain level on. We will call the Cantor minimal system

$(X_{(V,E)}, T_{(V,E)})$ a *Bratteli-Vershik system*. There is an obvious way to telescope a properly ordered Bratteli diagram, getting another properly ordered Bratteli diagram, such that the associated Bratteli-Vershik systems are conjugate – the map implementing the conjugacy is the obvious one. By telescoping we may assume without loss of generality that the properly ordered Bratteli diagram has the property that at each level all the minimal edges (respectively the maximal edges) have the same source.

Theorem 3.28 ([HPS]). *Let (X, T) be a Cantor minimal system. Then there exists a properly ordered Bratteli diagram (V, E, \geq) such that the associated Bratteli-Vershik system $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X, T) .*

Proof sketch. Let $x_0 \in X$ and let $\{U_n\}_{n \in \mathbb{Z}^+}$ be a decreasing sequence of clopen sets of X such that $U_0 = X$ and $U_n \searrow \{x_0\}$. For each U_n we construct a finite number of towers “built” over U_n . These are determined by the map $\lambda_n : U_n \rightarrow \mathbb{N}$, $\lambda_n(y) = \inf\{m \in \mathbb{N} \mid T^m y \in U_n\}$. If $\lambda_n(U_n) = \{m_1, m_2, \dots, m_{k_n}\}$, then we get at first k_n towers of height m_1, m_2, \dots, m_{k_n} , respectively. These may be vertically subdivided, giving rise to more towers (some of them of the same height), such that we obtain the following scenario: The clopen partitions $\{\mathcal{P}_n\}_{n \in \mathbb{Z}^+}$ of X that the towers associated to the various U_n ’s generate are nested, $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \dots$, and the union of the \mathcal{P}_n ’s is a basis for the topology of X . We build the properly ordered Bratteli diagram (V, E, \geq) by letting the vertices V_n at level n correspond to the various towers built over U_n . The ordering of the edges between levels $n-1$ and n is determined by the order in which the towers at level n traverse the towers at level $n-1$.

Remark. The simplest Bratteli-Vershik model (V, E, \geq) for the odometer $(G_{\mathbf{a}}, T)$ associated to $\mathbf{a} = (a_i)_{i \in \mathbb{N}}$ is obtained by letting $V_n = 1$ for all n , and the number of edges between V_{n-1} and V_n be a_n .

Definition 3.29. Let (X, T) be a Cantor minimal system. Define the (additive) *coboundary map* $\partial_T : \mathcal{C}(X, \mathbb{Z}) \rightarrow \mathcal{C}(X, \mathbb{Z})$ by $\partial_T f = f - f \circ T^{-1}$. Define

$$K^0(X, T) = \mathcal{C}(X, \mathbb{Z}) / \partial_T(\mathcal{C}(X, \mathbb{Z}))$$

and give $K^0(X, T)$ the induced order, i.e.

$$K^0(X, T)^+ = \{[f] \in K^0(X, T) \mid [f] = [g], \text{ for some } g \geq 0\},$$

where $[f]$ denotes the class of $f \in \mathcal{C}(X, \mathbb{Z})$. Let $\mathbf{1}(= [1])$ denote the distinguished order unit corresponding to the constant function 1 on X . We will in the sequel sometimes for short only write $K^0(X, T)$, the ordering and the order unit being implicitly understood.

Remark. Let $(G_{\mathbf{a}}, \rho_{\mathbf{1}})$ be the odometer associated to $\mathbf{a} = (a_i)_{i \in \mathbb{N}}$. Then $K^0(G_{\mathbf{a}}, \rho_{\mathbf{1}})$ is order isomorphic to the rational group associated to \mathbf{a} , namely

$$\left\{ \frac{m}{a_1 a_2 \cdots a_n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \right\},$$

and so, in particular, $\mathbb{Q}(K^0(X, T), \mathbf{1}) \cong \mathbb{Q}(K_0(V, E), [\mathbf{1}])$ by a map preserving the canonical order units.

The following result, which is implicit in [GPS1, Section 2], is highly relevant for this paper and we present a proof due to B. Host (cf. [O, Theorem 2.2]).

Proposition 3.30. *Let (X, T) be a Cantor minimal system and let p be a natural number greater than 1. The following are equivalent:*

- (i) $\frac{1}{p} \in \mathbb{Q}(K^0(X, T), \mathbf{1})$
- (ii) $\frac{2\pi i}{p}$ is a continuous eigenvalue for T , i.e. $\exists f \in \mathcal{C}(X)$, $f \neq 0$, such that $f \circ T = e^{\frac{2\pi i}{p}} f$.

Proof. (i) \Rightarrow (ii): For any set E , let 1_E denote the characteristic function of E . By condition (i) there exist continuous functions $f, g: X \rightarrow \mathbb{Z}$ such that $pf - 1_X = g - g \circ T^{-1}$, where f is non-zero (and hence g is non-zero). Let $n \in \text{Im}(g)$ and let $A = g^{-1}(\{n\})$. Then A is a non-empty clopen set. For $x \in A$ let $r(x)$ be the first return time of x to A , i.e. $r(x)$ is the smallest positive integer such that $T^{r(x)}x \in A$. Since T is minimal the function $r: A \rightarrow \mathbb{Z}$ is uniformly bounded and $\bigcup_{x \in A} \bigcup_{j=1}^{r(x)} T^j x = X$. We get:

$$\begin{aligned} p \left(\sum_{j=1}^{r(x)} f(T^j x) \right) - r(x) &= \sum_{j=1}^{r(x)} (pf(T^j x) - 1_X(T^j x)) = \sum_{j=1}^{r(x)} (g(T^j x) - g(T^{j-1} x)) \\ &= g(T^{r(x)} x) - g(x). \end{aligned}$$

If $x \in A$, the right hand side is 0, and so the left hand side yields that $r(x)$ must be a (positive) multiple of p . Let $\tilde{A} = \bigcup_{l \geq 0} T^{lp}(A)$, and so, in particular, A is an open set. Then for all $x \in A$ the first return time to \tilde{A} is p . Then $X = \bigsqcup_{j=0}^{p-1} T^j(\tilde{A})$ (disjoint union), and so, in particular, $\tilde{A}, T(\tilde{A}), \dots, T^{p-1}(\tilde{A})$, are clopen sets. Define $f \in \mathcal{C}(X)$ by

$$f(x) = e^{\frac{j2\pi i}{p}} \text{ if } x \in T^j(\tilde{A}).$$

Then one easily sees that $f \circ T = e^{\frac{2\pi i}{p}} f$, and so $\frac{2\pi i}{p}$ is a continuous eigenvalue for T .

(ii) \Rightarrow (i): Let $\frac{2\pi i}{p}$ be a continuous eigenvalue for T , and let $f \in \mathcal{C}(X)$ be a non-zero eigenfunction, i.e. $f \circ T = e^{\frac{2\pi i}{p}} f$. We may assume that $|f(x)| = 1$ for all $x \in X$ and that $A = f^{-1}(\{1\})$ is non-empty. Now clearly $A, T(A), \dots, T^{p-1}(A)$ are disjoint closed sets, and $T\left(\bigcup_{j=0}^{p-1} T^j(A)\right) = \bigcup_{j=0}^{p-1} T^j(A)$, and so by minimality of T , we have $\bigcup_{j=0}^{p-1} T^j(A) = X$. Hence A is clopen, and so $1_A \in \mathcal{C}(X)$. Now

$$\begin{aligned} p1_A - 1_X &= (1_A - 1_A) + (1_A - 1_{T(A)}) + \cdots + (1_A - 1_{T^{p-1}(A)}) \\ &= 0 + (1_A - 1_A \circ T^{-1}) + \cdots + (1_A - 1_A \circ T^{-(p-1)}). \end{aligned}$$

Also

$$\begin{aligned} 1_A - 1_A \circ T^{-j} &= (1_A - 1_A \circ T^{-1}) + (1_A \circ T^{-1} - (1_A \circ T^{-1}) \circ T^{-1}) \\ &\quad + \cdots + (1_A \circ T^{-(j-1)} - (1_A \circ T^{-(j-1)}) \circ T^{-1}) \end{aligned}$$

is a coboundary, and so $p1_A - 1_X$ is a coboundary. This shows that $\frac{1}{p} \in \mathbb{Q}(K^0(X, T), \mathbf{1})$. \square

Combining Proposition 3.30 with Theorems 3.12 and 3.13 (and the remarks just preceding these theorems), we get the following result:

Proposition 3.31. *Let (X, T) be a Cantor minimal system. Then $\mathbb{Q}(K^0(X, T), \mathbf{1})$ completely determines the maximal equicontinuous Cantor factor of (X, T) . Specifically, the maximal equicontinuous Cantor factor is the odometer $(G_{\mathbf{a}}, \rho_{\mathbf{1}})$ associated to the \mathbf{a} -adic number $\mathbf{a} = (a_1, a_2, \dots)$, where $\mathbb{Q}(K^0(X, T), \mathbf{1}) \cong K^0(G_{\mathbf{a}}, \rho_{\mathbf{1}})$. (The latter group is described in the Remark preceding 3.30.)*

The following theorem is a fairly straightforward corollary of Theorem 3.28.

Theorem 3.32 ([HPS]). *Let (X, T) be a Cantor minimal system. Then $(K^0(X, T), K^0(X, T)^+)$ is a simple dimension group, and all (noncyclic) simple dimension groups (G, G^+, u) with distinguished order unit u are order isomorphic to $(K^0(X, T), K^0(X, T)^+, \mathbf{1})$ for some Cantor minimal system (X, T) . In particular, if (V, E, \geq) is a properly ordered Bratteli diagram such that (X, T) is conjugate to $(X_{(V, E)}, T_{(V, E)})$, then $K^0(X, T) \cong K_0(V, E)$ (as ordered groups with canonical order units), and so, in particular, $\mathbb{Q}(K^0(X, T), \mathbf{1}) \cong \mathbb{Q}(K_0(V, E), [\mathbf{1}])$. Furthermore, the state space $S_{\mathbf{1}}(K^0(X, T))$ may be identified in an obvious way with the Choquet simplex $M(X, T)$ of T -invariant probability measures; in fact, these two Choquet simplices are affinely homeomorphic.*

Remark. Changing the order unit corresponds dynamically to considering induced systems. (Cf. Definitions 3.4 and 3.5.) In fact, let (V, E, \geq) be a simply ordered Bratteli diagram, and let (V', E', \geq) be the resulting simply ordered Bratteli

diagram after we have made a finite change to (V, E, \geq) . (By a finite change we mean adding and/or removing a finite number of edges and then making arbitrary choices of linear orderings of the edges meeting at the same vertex for a finite number of vertices.) Then $(X_{(V,E)}, T_{(V,E)})$ is Kakutani equivalent to $(X_{(V',E')}, T_{(V',E')})$, which can be seen as an immediate consequence of how the Vershik map is defined. Furthermore, if (X, T) is Cantor minimal with associated dimension group $K^0(X, T)$ with distinguished order unit, then choosing a new order unit, say u , there exists a Cantor minimal system (Y, S) which is Kakutani equivalent to (X, T) such that $(K^0(Y, S), K^0(Y, S)^+, \mathbb{1}) \cong (K^0(X, T), K^0(X, T)^+, u)$. In fact, (Y, S) is obtained from (X, T) by making a finite change to the Bratteli-Vershik model for (X, T) .

Theorem 3.33 ([GPS1]). *The Cantor minimal systems (X, T) and (Y, S) are strong orbit equivalent if and only if $(K^0(X, T), K^0(X, T)^+, \mathbb{1}) \cong (K^0(Y, S), K^0(Y, S)^+, \mathbb{1})$.*

Remark. The idea behind introducing $K^0(X, T)$ with an ordering for a Cantor minimal system (X, T) comes from (non-commutative) operator algebra theory. In fact, one can show that $K^0(X, T)$ as defined above is abstractly isomorphic to the K_0 -group of the associated C^* -crossed product $\mathcal{C}(X) \rtimes_T \mathbb{Z}$. The latter K_0 -group comes with a natural order, which translates to the order we introduced on $K^0(X, T)$. It turns out that the ordered K_0 -group of $\mathcal{C}(X) \rtimes_T \mathbb{Z}$, with its natural scaling corresponding to the unit element, is a complete isomorphism invariant, and so it follows that (X, T) is strong orbit equivalent to (Y, S) if and only if $\mathcal{C}(X) \rtimes_T \mathbb{Z}$ is $*$ -isomorphic to $\mathcal{C}(Y) \rtimes_S \mathbb{Z}$ [GPS1].

Theorem 3.34 ([GPS1]). *The Cantor minimal systems (X, T) and (Y, S) are orbit equivalent if and only if $(\widetilde{K^0(X, T)}, \widetilde{K^0(X, T)^+}, \widetilde{\mathbb{1}}) \cong (\widetilde{K^0(Y, S)}, \widetilde{K^0(Y, S)^+}, \widetilde{\mathbb{1}})$, where*

$\widetilde{K^0(X, T)} = K^0(X, T)/\text{Inf}(K^0(X, T))$, $\widetilde{K^0(Y, S)} = K^0(Y, S)/\text{Inf}(K^0(Y, S))$, and the positive cones and order units are induced by the quotient maps.

Let (V, E, \geq) be a properly ordered Bratteli diagram, and let $(X_{(V,E)}, T_{(V,E)})$ be the associated Bratteli-Vershik system. For each $k \in \mathbb{N}$ let P_k as above denote the paths from V_0 to V_k , i.e. the paths from $v_0 \in V_0$ to some $v \in V_k$. For $x \in X_{(V,E)}$ we associate the bi-infinite sequence $\pi_k(x) = \left(\tau_k(T_{(V,E)}^n x) \right)_{n=-\infty}^{\infty} \in P_k^{\mathbb{Z}}$ over the finite alphabet P_k , where $\tau_k : X_{(V,E)} \rightarrow P_k$ is the truncation map. Let S_k denote the shift map on $P_k^{\mathbb{Z}}$. Then the following diagram commutes

$$\begin{array}{ccc}
X_{(V,E)} & \xrightarrow{T_{(V,E)}} & X_{(V,E)} \\
\pi_k \downarrow & & \downarrow \pi_k \\
X_k & \xrightarrow{S_k} & X_k
\end{array}$$

where $X_k = \pi_k(X_{(V,E)})$. One observes that π_k is a continuous map, and so X_k is a compact shift-invariant subset of $P_k^{\mathbb{Z}}$. So (X_k, S_k) is a factor of $(X_{(V,E)}, T_{(V,E)})$. For $k > l$ there is an obvious factor map $\pi_{k,l}: X_k \rightarrow X_l$, and one can show that $(X_{(V,E)}, T_{(V,E)})$ is the inverse limit of the system $\{(X_k, S_k)\}_{k \in \mathbb{N}}$. All the systems (X_k, S_k) are clearly expansive. One has the following result which will be important for us. Even though the result is well known by people familiar with (ordered) Bratteli diagrams, no proof has been written down as far as we know. So we will present a proof here.

Proposition 3.35. *Assume $(X_{(V,E)}, T_{(V,E)})$ is expansive. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X_k, S_k) by the map $\pi_k: X_{(V,E)} \rightarrow X_k$.*

Proof. Since the π_k 's are factor maps, all we need to show is that there exists k_0 such that π_k is injective for all $k \geq k_0$. Recall that $(X_{(V,E)}, T_{(V,E)})$ being expansive means that there exists $\delta > 0$ such that given $x \neq y$ there exists $n_0 \in \mathbb{Z}$ such that $d(T_{(V,E)}^{n_0}x, T_{(V,E)}^{n_0}y) > \delta$, where d is some metric on $X_{(V,E)}$ compatible with the topology. Choose k_0 such that $d(x, y) < \delta$ if x and y agree (at least) on the k_0 first edges. Now assume that $\pi_k(x) = \pi_k(y)$ for some $k \geq k_0$. By the definition of π_k this means that, for all $n \in \mathbb{Z}$, $\tau_k(T_{(V,E)}^n x) = \tau_k(T_{(V,E)}^n y)$, and so $d(T_{(V,E)}^n x, T_{(V,E)}^n y) < \delta$ for all $n \in \mathbb{Z}$ because of our choice of k_0 . This contradicts that $d(T_{(V,E)}^{n_0}x, T_{(V,E)}^{n_0}y) > \delta$. Hence π_k is injective for all $k \geq k_0$, proving the proposition. \square

3.4 Order embeddings of dimension groups associated to factor maps.

We have seen that Toeplitz flows can be characterized by being (expansive) almost one-to-one Cantor minimal extensions of odometer systems (cf. Theorem 3.13). The following theorem will therefore be important for us since it relates the K^0 -groups of the extension and the factor, respectively, of Cantor minimal systems.

Theorem 3.36 ([GW, Proposition 3.1]). *Let (X, T) and (Y, S) be Cantor minimal systems such that (X, T) is an extension of (Y, S) by the factor map $\pi: X \rightarrow Y$.*

Then $\pi^*: K^0(Y, S) \rightarrow K^0(X, T)$ defined by $\pi^*([h]) = [h \circ \pi]$ is an order embedding, i.e. $[h] \geq 0$ if and only if $\pi^*([h]) \geq 0$ for $h \in \mathcal{C}(Y, \mathbb{Z})$. (Here $[h]$ and $[h \circ \pi]$ denote the class of h and $h \circ \pi$ in $K^0(Y, S)$ and $K^0(X, T)$, respectively.)

Remark. The proof of Theorem 3.36 has two ingredients. The first is the use of the Gottschalk-Hedlund lemma [GH], which in our context says that $g \in \mathcal{C}(X, \mathbb{Z})$ is a coboundary, i.e. $g = f - f \circ T^{-1}$ for some $f \in \mathcal{C}(X, \mathbb{Z})$, if and only if $\sup_n \left| \sum_{i=0}^{n-1} g(T^i x_0) \right| < \infty$ for some $x_0 \in X$. This will establish that π^* is well-defined. The second ingredient, which gives the order embedding, is applying Theorem 3.22 (cf. also Theorem 3.32) together with the fact that π induces a surjective map of $M(X, T)$ onto $M(Y, S)$.

4 Proofs of the main results

The proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3 stated in Section 2 will rest heavily upon results obtained earlier by Gjerde-Johansen [GJ] and Sugisaki [S1], [S2], [S3], where the paper [S3] is extending and being inspired, so to speak, by an analogous result proved by Giordano, Putnam, Skau in [GPS2]. The proofs in [S1], [S2] and [S3] are rather technical, involving very clever manipulations of Bratteli diagrams. Looking carefully at crucial steps in the proofs of the main theorems in [S2] and [S3], in particular, we could deduce more specific properties of the Bratteli diagrams that appear, starting with our basic setting. This, combined with the Bratteli-Vershik model for Toeplitz flows established in [GJ] and the embedding result of [GW], will, loosely speaking, give the proofs of our three first theorems. (The proof of the fourth theorem, Theorem 2.4, requires a somewhat different approach.) However, this should not be construed as saying that our theorems are simply corollaries of these earlier results. But one could perhaps say that the results we obtain represent the culmination of the study of (topological) Toeplitz flows from the perspective of orbit equivalence and/or K^0 -groups.

Lemma 4.1. *The Bratteli-Vershik model associated to a properly ordered Bratteli diagram (V, E, \geq) with the ERS-property has the odometer corresponding to the supernatural number of (V, E) (cf. Definition 3.16) as a factor. The factor map π is almost one-to-one. If the Bratteli-Vershik system $(X_{(V,E)}, T_{(V,E)})$ is expansive, then for some k we get that $\pi_k(x_{\min})$ is a Toeplitz sequence and $\pi_k: X_{(V,E)} \rightarrow X_k$ is a conjugacy map between $(X_{(V,E)}, T_{(V,E)})$ and (X_k, S_k) (cf. Proposition 3.35). In particular, $(X_{(V,E)}, T_{(V,E)})$ is a Toeplitz flow.*

Proof. We will give the proof by appealing to Figure 2, where we have drawn on the left a particular Bratteli diagram (V, E) with the ERS-property. Let the incidence matrices be $(M_n)_{n=1}^\infty$, say, with r_n equal to the (constant) row sum of M_n . We “collapse” (V, E) in an obvious way to create the diagram (W, F) on the right in

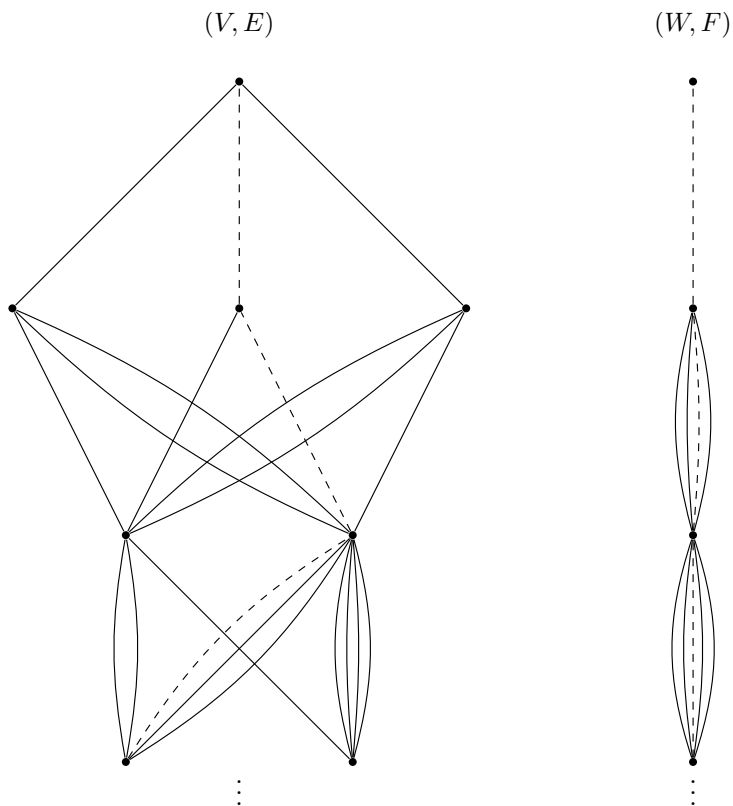


Figure 2: Assuming the edges in both (V, E) and (W, F) are ordered left to right, the factor map π will take the dashed path in (V, E) to the dashed path in (W, F) .

Figure 2, where $|W_n| = 1$ for all n and the number of edges between W_{n-1} and W_n is r_n . Let them be linearly ordered as $f_1^{(n)} < f_2^{(n)} < \dots < f_{r_n}^{(n)}$. The factor map $\pi: X_{(V,E)} \rightarrow X_{(W,F)}$ is defined by reading off the labels, so to say. That is, let $x = (e_n)_{n=1}^\infty \in X_{(V,E)}$ and let e_n be the k_n 'th edge in the linear ordering of $r^{-1}(r(e_n))$. We then map e_n to $f_{k_n}^{(n)}$, thus getting $\pi(x) \in X_{(W,F)}$. It is easy to see that π is a continuous map that intertwines $T_{(V,E)}$ and $T_{(W,F)}$. Also, $\pi^{-1}\{\pi(x_{\min})\} = \{x_{\min}\}$, where x_{\min} is the unique minimal path of $X_{(V,E)}$, and so π is almost one-to-one. The map π is onto since the image of π is a compact, hence closed, subset of $X_{(W,F)}$, and π clearly maps orbit $_{T_{(V,E)}}(x_{\min})$ to a dense subset of $X_{(W,F)}$. The factor $(X_{(W,F)}, T_{(W,F)})$ is by construction an odometer with the properties stated. If $(X_{(V,E)}, T_{(V,E)})$ is expansive, then for some $k \in \mathbb{N}$, $\pi_k: X_{(V,E)} \rightarrow X_k$ is a conjugacy map by Proposition 3.35. As pointed out in Section 3.3.3 we may assume that at each level all the minimal edges (respectively the maximal edges) have the same source. This scenario is obtained by telescoping the original diagram, and all essential properties are preserved by this operation. The ERS-property of (V, E) implies that $\pi_k(x_{\min})$ is a Toeplitz sequence, a fact that is easily shown; we omit the details. So $(X_{(V,E)}, T_{(V,E)})$ is a Toeplitz flow (which, incidentally, also is a consequence of Theorem 3.13). \square

Remark. Lemma 4.1 is the easy part of Theorem 8 of [GJ]. Conversely, if one starts with a Toeplitz flow (X, T) one can construct a properly ordered Bratteli diagram (V, E, \geq) such that (V, E) has the ERS-property and $(X, T) \cong (X_{(V,E)}, T_{(V,E)})$. This is achieved by merging the structure of Toeplitz flows as described in Section 3.2 with the construction described in the proof sketch of Theorem 3.28. (See [GJ, Theorem 8].) In [S1] it is proved that for any ordered Bratteli diagram (V, E, \geq) such that (V, E) has the ERS-property, there exists another properly ordered Bratteli diagram (V', E', \geq) such that (V', E') has the ERS-property, with $K_0(V', E') \cong K_0(V, E)$ (as ordered groups with distinguished order units) and $(X_{(V',E')}, T_{(V',E')})$ is Toeplitz. By Theorem 3.32, $K^0(X, T) \cong K^0(X_{(V',E')}, T_{(V',E')})$, and so (X, T) is strong orbit equivalent to $(X_{(V',E')}, T_{(V',E')})$ by Theorem 3.33.

Lemma 4.2. *Let $\pi: (X, T) \rightarrow (Y, S)$ be an almost one-to-one factor map between Cantor minimal systems. Then*

$$K^0(X, T)/\pi^*(K^0(Y, S))$$

is torsion free.

Proof. We will give a heuristic argument in order to highlight the basic idea behind the proof, avoiding the somewhat messy details that tend to obscure the understanding. (For a more detailed proof, see [S3, Theorem 3.1].) Since π is a factor map, we know from Theorem 3.36 that $\pi^*: K^0(Y, S) \rightarrow K^0(X, T)$ is an

order embedding where $\pi^*([h]) = [h \circ \pi]$. (Here $h \in \mathcal{C}(Y, \mathbb{Z})$ and $[h]$ and $[h \circ \pi]$ denote the classes of h and $h \circ \pi$ in $K^0(Y, S)$ and $K^0(X, T)$, respectively.) We will construct specific Bratteli-Vershik models for (X, T) and (Y, S) , respectively. Let $x_0 \in X$ be such that $\pi^{-1}(\pi(x_0)) = x_0$. Let $\{U_n\}_{n \in \mathbb{Z}^+}$ be a decreasing sequence of clopen sets in Y such that $U_0 = Y$ and $U_n \searrow \{y_0\}$, where $y_0 = \pi(x_0)$. We now proceed as described in the proof sketch of Theorem 3.28. Let $\{\mathcal{P}_n\}_{n \in \mathbb{Z}^+}$ be the nested sequence of clopen partitions of Y associated to the tower constructions built over the various U_n 's, such that the union of the \mathcal{P}_n 's is a basis for the topology of Y . Let (W, F, \geq) be the resulting properly ordered Bratteli diagram, from which we get a Bratteli-Vershik model for (Y, S) . Consider the clopen sets $\widetilde{U}_n = \pi^{-1}(U_n)$, $n \in \mathbb{Z}^+$. Clearly $\{\widetilde{U}_n\}_{n \in \mathbb{Z}^+}$ is a descending sequence of clopen subsets of X such that $\widetilde{U}_0 = X$, $\widetilde{U}_n \searrow \{x_0\}$. Proceeding again as described in the proof sketch of Theorem 3.28 we get a simply ordered Bratteli diagram (V, E, \geq) , from which we get a Bratteli-Vershik model for (X, T) . Now $K^0(X, T) \cong K_0(V, E)$ and $K^0(Y, S) \cong K_0(W, F)$ as ordered groups with canonical order units. We note that the functions $\lambda_n: U_n \rightarrow \mathbb{N}$ and $\widetilde{\lambda}_n: \widetilde{U}_n \rightarrow \mathbb{N}$ defined by $\lambda_n(u) = \inf\{m \in \mathbb{N} \mid S^m u \in U_n\}$ and $\widetilde{\lambda}_n(\widetilde{u}) = \inf\{m \in \mathbb{N} \mid T^m \widetilde{u} \in \widetilde{U}_n\}$, respectively, are related by $\lambda_n(u) = \widetilde{\lambda}_n(\widetilde{u})$ if $\widetilde{u} \in \pi^{-1}(u)$. This has implications for how $K_0(W, F)$ is embedded in $K_0(V, E)$. Loosely speaking, each $w \in W_n$ (which corresponds to a tower T_w over U_n) is split into a finite number of vertices v_1, v_2, \dots, v_l in V_n (this corresponds to the tower T_w being subdivided into l towers $T_{v_1}, T_{v_2}, \dots, T_{v_l}$ over \widetilde{U}_n , each of the same height as T_w). The factor map $\pi: X_{(V, E)} \rightarrow X_{(W, F)}$ is a kind of “collapsing” map similarly to the one exhibited in Figure 2. While the scenario exhibited in Figure 2 is very simple, it does illustrate the essential point. In fact, the single vertices at levels 1, 2 and 3 of (W, F) in Figure 2 split into three, two and two vertices, respectively, at levels 1, 2 and 3 of (V, E) . The image of the group element of $K_0(W, F)$ that is -7 , say, at level 2 of (W, F) in Figure 2 is represented by $(-7, -7)$ at level 2 of (V, E) . In general, a group element in $K_0(W, F)$ which is represented as being $b \in \mathbb{Z}$ at $w \in W_n$, and zero at the other vertices in W_n , is mapped to the group element in $K_0(V, E)$ that is represented by being b at each of the vertices in V_n associated to w , and zero elsewhere. This extends by linearity in an obvious way to any element in $K_0(W, F)$ that are represented as a vector at level n . This “locally constancy” property, which is preserved at the higher levels under the canonical mappings of the Bratteli diagram, is what completely characterizes the embedding of $K_0(W, F)$ into $K_0(V, E)$. This clearly implies that $K_0(V, E)/\pi^*(K_0(W, F))$ is torsion free. If namely $g \in K_0(V, E)$ is such that $kg \in \pi^*(K_0(W, F))$ for some $k \in \mathbb{N}$, then kg is represented by some “locally constant” vector at level n of (V, E) , say. But then clearly g is also represented by a “locally constant” vector (at some higher level of (V, E) than n , perhaps), and hence g lies in $\pi^*(K_0(W, F))$. \square

Remark. The converse of Lemma 4.2 is not true. In a recent paper by Glasner and Host they construct Cantor minimal systems (Y, S) and (X, T) such that (X, T) is an extension of (Y, S) by a map $\pi: (X, T) \rightarrow (Y, S)$ which is not an almost one-to-one extension, and $K^0(X, T)/\pi^*(K^0(Y, S))$ is non-zero and torsion free [GHo, Appendix C]. In fact, their example can be adjusted to make (Y, S) an odometer and (X, T) to be expansive.

Lemma 4.3. *Let (V, E, \geq) be a simple Bratteli diagram with the ERS-property. Let M_n be the incidence matrix between levels $n - 1$ and n , and let r_n be the (constant) row sum of M_n . Let m be the supernatural number $\prod_{n=1}^{\infty} r_n$. Let $H = \mathbb{Q}(K_0(V, E), [\mathbf{1}])$ where $[\mathbf{1}]$ is the canonical order unit of $K_0(V, E)$. Then H is order isomorphic to the rational group $G(m)$ by a map sending $[\mathbf{1}]$ to $1 \in G(m)$. Furthermore, H is represented in an obvious way by constant vectors at each level of (V, E) , i.e. vectors of the form $(a, a, a, \dots, a)^{\text{tr}} \in \mathbb{Z}^{|V_n|}$ for each n (tr denotes the transpose).*

Proof. The proof is an immediate consequence of the fact that

$$M_n(1, 1, \dots, 1)^{\text{tr}} = (r_n, r_n, \dots, r_n)^{\text{tr}}$$

where $(1, 1, \dots, 1)^{\text{tr}} \in \mathbb{Z}^{|V_{n-1}|}$, $(r_n, r_n, \dots, r_n)^{\text{tr}} \in \mathbb{Z}^{|V_n|}$, which yields

$$M_n M_{n-1} \cdots M_1(1) = \left(\prod_{k=1}^n r_k, \prod_{k=1}^n r_k, \dots, \prod_{k=1}^n r_k \right)^{\text{tr}} \in \mathbb{Z}^{|V_n|}.$$

□

Remark. We will say that (V, E) is an ERS realization of $G \cong K_0(V, E)$ with respect to a subdimension group $H \subseteq G$ if H is embedded in $K_0(V, E)$ as in Lemma 4.3.

Lemma 4.4. *Let $(J, J^+, 1)$ be a noncyclic rational group (cf. Definition 3.23), and let (Y, S) be an odometer such that $K^0(Y, S) \cong J$ (as ordered groups with distinguished order units). Let $\iota: J \rightarrow G$ be an order embedding of J into a simple dimension group (G, G^+, u) preserving the order units, such that $G/\iota(J)$ is torsion free. There exists a properly ordered Bratteli diagram (V, E, \geq) such that:*

- (i) (V, E) has the ERS-property
- (ii) $K^0(X_{(V, E)}, T_{(V, E)}) \cong G$ (as ordered groups with distinguished order units).
- (iii) $(X_{(V, E)}, T_{(V, E)})$ is an almost one-to-one extension of (Y, S) .
- (iv) (Y, S) is the maximal equicontinuous factor of (X, T) .

(v) $(J, J^+, 1) \cong \mathbb{Q}(K_0(V, E), [1])$.

Proof. Assertions (ii) and (iii) are the main result of [S3], namely Theorem 1.1 (see also Corollary 1.2). Since the properly ordered Bratteli diagram associated to (Y, S) is very special – having a single vertex at each level – the almost one-to-one extension of (Y, S) constructed in the proof of Theorem 1.1 in [S3], which is obtained by constructing a properly ordered Bratteli diagram (V, E, \geq) based on the one associated to (Y, S) , will have property (i). (For details, cf. Remark 3.2 and Proposition 3.3 of [S3].) The assertion (iv) is a consequence of Theorem 3.12. The assertion (v) follows from (iii) and Lemma 4.3. In fact, $K^0(Y, S)$ embeds into $K_0(V, E)$ as constant vectors at each level of (V, E) . (Cf. the proof of Lemma 4.2, keeping in mind that in our case the properly ordered Bratteli diagram (W, F, \geq) appearing there and being associated to (Y, S) , has one vertex at each level.) \square

Lemma 4.5. *Let (V, E, \geq) be a properly ordered Bratteli diagram such that (V, E) has the ERS-property. Let $0 \leq t < \infty$. There exists a properly ordered Bratteli diagram $(\tilde{V}, \tilde{E}, \geq)$ such that:*

- (i) (\tilde{V}, \tilde{E}) has the ERS-property
- (ii) $K_0(\tilde{V}, \tilde{E}) \cong K_0(V, E)$ (as ordered groups with distinguished order units).
- (iii) $(X_{(\tilde{V}, \tilde{E})}, T_{(\tilde{V}, \tilde{E})})$ is expansive.
- (iv) $h(T_{(\tilde{V}, \tilde{E})}) = t$.

Proof. The assertions (ii), (iii) and (iv) are the main result (Theorem 1.1) of [S2]. (Note that by Theorems 3.32, 3.33 and Proposition 3.8, respectively, strong orbit equivalence is related to K_0 -groups, and expansiveness is related to subshifts, respectively.) Now in the proof of Theorem 1.1 of [S2] various simple Bratteli diagrams are constructed, modifying the given Bratteli diagram (V, E) . However, each modification preserves the ERS-property of the original Bratteli diagram (V, E) . (For details, cf. Propositions 4.2, 4.4 and Sections 5.1, 5.3 and 5.4 of [S2].) So the properly ordered Bratteli diagram $(\tilde{V}, \tilde{E}, \geq)$ that eventually arises in the proof of Theorem 1.1 of [S2] will have all the properties listed in Lemma 4.5. \square

Proof of Theorem 2.1. Let (X, T) be a Toeplitz flow (with $h(T) = t$). By Theorem 3.13, (X, T) is an almost one-to-one extension of an odometer (Y, S) , the factor map being $\pi: X \rightarrow Y$. By Theorem 3.36, $\pi^*: K^0(Y, S) \rightarrow K^0(X, T)$ is an order embedding sending the distinguished order unit of $K^0(Y, S)$ to the one in $K^0(X, T)$. Set $G = K^0(X, T)$, $H = \pi^*(K^0(Y, S))$. Then H is a noncyclic subgroup of G of rank one such that $H \cap G^+ \neq \{0\}$.

Conversely, assume that (G, G^+) is a simple dimension group containing the noncyclic subgroup H such that $H \cap G^+ \neq \{0\}$. Let $u \in H \cap G^+$ be any non-zero

element. We consider the simple dimension group (G, G^+, u) with distinguished order unit u . Now H is a subgroup of the rational subgroup $\mathbb{Q}(G, u)$ of G . In fact, if $h \in H$ then there exists $m, n \in \mathbb{Z}$ such that $nh = mu$, since H is of rank one. (We may assume without loss of generality that $n \in \mathbb{N}$.) In particular, we get that $\mathbb{Q}(G, u)$ is noncyclic. Now we apply Lemma 4.4 with $J = \mathbb{Q}(G, u)$ and $\iota: J \rightarrow G$ the inclusion map. We keep the notation of Lemma 4.4. Apply Lemma 4.5 to the properly ordered Bratteli diagram (V, E, \geq) constructed in Lemma 4.4 to get $(\tilde{V}, \tilde{E}, \geq)$ with the properties listed in Lemma 4.5. Set $X = X_{(\tilde{V}, \tilde{E})}$ and $T = T_{(\tilde{V}, \tilde{E})}$. Then $K^0(X, T) \cong K_0(\tilde{V}, \tilde{E}) \cong K_0(V, E) \cong G$ as ordered groups with distinguished order units. This completes the proof of Theorem 2.1. \square

Proof of Theorem 2.2. Set $J = \mathbb{Q}(G, u)$. By assumption J is a noncyclic rational group (hence of rank one). The inclusion map $\iota: J \rightarrow G$ is an order embedding, and by Proposition 3.25, G/J is torsion free. Let (Y, S) be an odometer system such that $K^0(Y, S) \cong J$ (as ordered groups with distinguished order units). Applying Lemma 4.4 we get a properly ordered Bratteli diagram (V, E, \geq) satisfying the properties listed in Lemma 4.4. Now apply Lemma 4.5 to (V, E, \geq) to get $(\tilde{V}, \tilde{E}, \geq)$ satisfying the properties listed in Lemma 4.5. Let $(X, T) = (X_{(\tilde{V}, \tilde{E})}, T_{(\tilde{V}, \tilde{E})})$. By Lemma 4.1 we get that (X, T) is a Toeplitz flow. Finally, invoking Proposition 3.30, we get that (X, T) satisfies all the properties listed in Theorem 2.2. (Recall that by Remark to Proposition 3.25 we have $(\mathbb{Q}(G, u) \cong \mathbb{Q}(K_0(V, E), [\mathbf{1}]) \cong \mathbb{Q}(K_0(\tilde{V}, \tilde{E}), [\mathbf{1}]))$.) \square

Proof of Theorem 2.3. By Theorem 2.2 we get the inclusion $\mathcal{G} \subseteq \mathcal{T}_t$. The inclusion $\mathcal{T}_t \subseteq \mathcal{G}$ is a consequence of Theorem 3.13 and Proposition 3.31. Hence we have $\mathcal{G} = \mathcal{T}_t$.

The inclusion $\mathcal{B} \subseteq \mathcal{G}$ follows from Lemma 4.3. The inclusion $\mathcal{T}_t \subseteq \mathcal{B}$ follows from the main result, Theorem 8, of [G.J]. Altogether this implies that $\mathcal{T}_t = \mathcal{G} = \mathcal{B}$. \square

Proof of Theorem 2.4. We prove the first part: Let H be a dense rational subgroup of \mathbb{Q} . Let x_0 be any point in K , and let \mathcal{E} be a countable set in $\text{Aff}(K)$, the continuous, real-valued affine functions on K , such that \mathcal{E} is a dense (in the uniform topology) subset of $\{a \in \text{Aff}(K) \mid a(x_0) = 0\}$. Let G be the (countable) additive subgroup of $\text{Aff}(K)$ generated by H and \mathcal{E} , where we identify every element in H with a constant affine function. Then $(G, G^+, 1)$ is a simple dimension group with order unit the constant function 1, and G^+ is the obvious positive cone; furthermore, $\text{Inf}(G) = \{0\}$, and the state space $S_1(G)$ is affinely homeomorphic with K (cf. Theorem 3.22). It is a simple matter to check that $\mathbb{Q}(G, 1) = H$. By Theorem 2.2 there exists a Toeplitz flow (X, T) such that $(G, G^+, 1) \cong (K^0(X, T), K^0(X, T)^+, \mathbf{1})$ and $h(T) = t$. By Theorem 3.32 we get that $M(X, T)$ is affinely homeomorphic to $S_1(G)$, and so to K . Now if $(G_1, G_1^+, 1) \cong (G_2, G_2^+, 1)$, where

G_1 and G_2 are constructed as G above, then $\mathbb{Q}(G_1, 1) \cong \mathbb{Q}(G_2, 1)$ as ordered groups by a positive map sending 1 to 1. Clearly one can choose an uncountable family of non-isomorphic groups H of the type described above. Hence there exists an uncountable family of non-isomorphic dimension groups $(G, G^+, 1)$ of the type constructed above. The corresponding uncountable family of Toeplitz flows (X, T) are then pairwise non-orbit equivalent by Theorem 3.34.

We prove the second part: Let (Y, S) be associated to the \mathfrak{a} -adic group $G_{\mathfrak{a}}$, where $\mathfrak{a} = (a_1, a_2, \dots)$. Choose H to be the rational group associated to \mathfrak{a} , i.e. $H = \left\{ \frac{m}{a_1 a_2 \dots a_n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \right\}$. As above we identify H with constant affine functions on K in an obvious way. Let \mathcal{E} be as above, and let G be generated by H and \mathcal{E} as above. Let N be any countable torsion-free abelian group, and let $\tilde{G} = G \oplus N$, with $\tilde{G}^+ = G^+ \oplus 0$ and order unit $u = 1 \oplus 0$. It is easily seen that $\mathbb{Q}(\tilde{G}, u) \cong \mathbb{Q}(G, 1) \cong H$ as ordered groups with distinguished order units. Clearly $\text{Inf}(\tilde{G}) = 0 \oplus N \cong N$. Also, $S_u(\tilde{G}) \cong S_1(G) \cong K$. By Theorem 2.2 there exists a Toeplitz flow (\tilde{X}, \tilde{T}) such that $K^0(\tilde{X}, \tilde{T}) \cong \tilde{G}$ (as ordered groups with distinguished order units) and $h(\tilde{T}) = t$. By Proposition 3.31 we get that the maximal equicontinuous factor of (\tilde{X}, \tilde{T}) is (Y, S) , and by Theorem 3.32 we get that $M(\tilde{X}, \tilde{T}) \cong K$. Clearly there exists an uncountable family of pairwise non-isomorphic groups N of the above type. The associated simple dimension groups $\tilde{G} = G \oplus N$ are then pairwise non-isomorphic. (Note that if $(G_1, G_1^+, u_1) \cong (G_2, G_2^+, u_2)$, then $\text{Inf}G_1 \cong \text{Inf}G_2$.) The corresponding Toeplitz flows (\tilde{X}, \tilde{T}) are then pairwise non-strong orbit equivalent by Theorem 3.33. This finishes the proof of Theorem 2.4. \square

5 Examples

We will first state some results that – combined with our theorems – will provide a rich source of examples of Toeplitz flows. We first need a definition.

Definition 5.1. Let (V, E) be a simple Bratteli diagram such that $|V_n| \leq l < \infty$ for all n . We say that (V, E) is of finite rank. If $|V_n| = k$ for all $n = 1, 2, \dots$, we say that (V, E) is of rank k .

Remark. If (V, E) is of finite rank we may telescope (V, E) to get a new Bratteli diagram of rank some k . It is easily seen that if (V, E) is of rank k , then the dimension group $K_0(V, E)$ has rank $\leq k$. Furthermore, the state space $S_{[1]}(K_0(V, E))$ of $K_0(V, E)$ has at most k extreme points, and so is a (finite-dimensional) m -simplex for some $m \leq k$.

Theorem 5.2 ([DM]). *Let (V, E, \geq) be a properly ordered Bratteli diagram such that (V, E) is of finite rank. Then $(X_{(V,E)}, T_{(V,E)})$ is either expansive or it is an odometer.*

Theorem 5.3. *Let (V, E, \geq) be a properly ordered Bratteli diagram such that (V, E) has finite rank. Then the entropy of $(X_{(V,E)}, T_{(V,E)})$ is zero.*

We will prove Theorem 5.3 at the end of this section. Combining Theorem 5.2 and Theorem 5.3 with Lemma 4.1 we get the following result.

Theorem 5.4. *Let (V, E, \geq) be a properly ordered Bratteli diagram with the following two properties:*

- (i) (V, E) is of finite rank (and so $K_0(V, E)$ is of finite rank).
- (ii) (V, E) has the ERS property (and so $\mathbb{Q}(K_0(V, E), [1])$ is noncyclic).

Then $(X_{(V,E)}, T_{(V,E)})$ is either an odometer or a Toeplitz flow of entropy zero. In particular, if $K_0(V, E)$ is not a (noncyclic) rational group, then $(X_{(V,E)}, T_{(V,E)})$ is a Toeplitz flow.

There is a partial converse to Theorem 5.4 due to Handelmann [H, Theorem 8.5]. (Incidentally, the paper [H] treats a more general situation under the assumption that the dimension groups in question have a unique state.) We formulate his result using our terminology and notation.

Theorem 5.5. *Let (G, G^+, u) be a simple dimension group with order unit u . Assume $\mathbb{Q}(G, u)$ is noncyclic and that $S_u(G)$ is a one-point set. (We say that G has a unique state, the order unit being understood.) Assume $\text{rank}(G) = k$. There exists a simple Bratteli diagram (V, E) of rank at most $k + 1$ with the ERS property, such that $(G, G^+, u) \cong (K_0(V, E), K_0(V, E)^+, [1])$ and (V, E) is an ERS realization of G with respect to $\mathbb{Q}(G, u)$. (Cf. Remark after Lemma 4.3.) Giving (V, E) a proper ordering the associated Bratteli-Vershik system is either an odometer or a uniquely ergodic Toeplitz flow of zero entropy.*

Remark. In particular, the scenario described in Theorem 5.5 occurs when $G = H \oplus \mathbb{Z}^m$, where H is a noncyclic rational group, and $G^+ = H^+ \oplus 0$, with $u = 1 \oplus 0$. Clearly $\text{Inf}(G) = 0 \oplus \mathbb{Z}^m \cong \mathbb{Z}^m$, and $\mathbb{Q}(G, u) \cong H$ (as ordered groups with order unit u and 1 , respectively). Clearly G has a unique state and $\text{rank}(G) = m + 1$. So by Theorem 5.5 there exists an ERS realization (V, E) of G with respect to H such that (V, E) has rank at most $m + 2$.

Another example where Theorem 5.5 applies is $G = \mathbb{Q} + \mathbb{Q}\alpha \subseteq \mathbb{R}$, where α is an irrational number, and G inherits the ordering from \mathbb{R} , i.e. $G^+ = G \cap \mathbb{R}^+$, the order unit being 1 . Then $\mathbb{Q}(G, 1) = \mathbb{Q}$ and $\text{rank}(G) = 2$. So by Theorem 5.5, G has an ERS realization (V, E) with respect to \mathbb{Q} , with (V, E) of rank at most 3. (In fact, in this particular case it suffices with a rank equal to 2, cf. Example 5.6.)

We will give explicit examples below illustrating some of these scenarios.

Example 5.6. Let $G = \mathbb{Q} + \mathbb{Q}\alpha \subseteq \mathbb{R}$ with G^+ and order unit 1 as above, and so $\mathbb{Q}(G, 1) = \mathbb{Q}$. We may assume without loss of generality that $0 < \alpha < \frac{1}{2}$. Let

$$\alpha = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}}$$

be the continued fraction expansion of α . Then $\mathbb{Q} + \mathbb{Q}\alpha$ is order isomorphic to the inductive limit

$$\mathbb{Q} \xrightarrow{A_1} \mathbb{Q}^2 \xrightarrow{A_2} \mathbb{Q}^2 \xrightarrow{A_3} \mathbb{Q}^2 \xrightarrow{A_4} \dots \tag{*}$$

where the incidence matrices are

$$A_1 = \begin{bmatrix} a_0 \\ 1 \end{bmatrix}, A_i = \begin{bmatrix} a_{i-1} & 1 \\ 1 & 0 \end{bmatrix}, i \geq 2,$$

and the order unit is the canonical one, i.e. represented by $1 \in \mathbb{Q}$. This follows from the fact that the dimension group $\mathbb{Z} + \mathbb{Z}\alpha \subseteq \mathbb{R}$ is represented by (*) with the \mathbb{Q} 's replaced by \mathbb{Z} 's. (Cf. [Sk, 3.3, Example (ii)].) We will assume $a_0 = 1$ and so $A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (We make this assumption just for convenience; nothing essential is changed by this, but the ensuing construction becomes more streamlined.) We will show that the inductive limit in (*) is order isomorphic to the inductive limit

$$\mathbb{Z} \xrightarrow{B_1} \mathbb{Z}^2 \xrightarrow{B_2} \mathbb{Z}^2 \xrightarrow{B_3} \mathbb{Z}^2 \xrightarrow{B_4} \dots \tag{**}$$

where the incidence matrices B_n have equal row sums – hence the associated simple Bratteli diagram has the ERS property. The B_n 's are obtained from the A_n 's by the following procedure, where we have adapted the construction in the proof of Theorem 11 in [GJ] to our setting: Let $A_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$. Then

$J'_2 A_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where J'_2 is the diagonal 2×2 matrix $\text{diag}(\alpha_1^{-1}, \alpha_2^{-1})$. Let m_2 be the least common multiple of the denominators of the entries of $J'_2 A_2$, and let $k_2 = 2m_2$. Set $J_2 = k_2 J'_2$. Then all entries of $J_2 A_2$ are in \mathbb{N} and are divisible by 2. Furthermore, $J_2 A_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \end{bmatrix}$. Assume we have constructed J_2, \dots, J_{n-1} . Let J'_n

be the appropriate diagonal matrix over \mathbb{Q}^+ such that $J'_n A_n J_{n-1}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let m_n be the least common multiple of the denominators of the entries of $J'_n A_n J_{n-1}^{-1}$, and let $k_n = nm_n$. Set $J_n = k_n J'_n$. Then all entries of $J_n A_n J_{n-1}^{-1}$ are in \mathbb{N} and are divisible by n . Furthermore, $J_n A_n J_{n-1}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k_n \\ k_n \end{bmatrix}$, where we observe that n is a divisor of k_n . Set $B_n = J_n A_n J_{n-1}^{-1}$. Setting $J_0 = \text{id}$, $J_1 = \text{id}$, we get the commutative diagram

$$\begin{array}{ccccccc}
G : & & \mathbb{Q} & \xrightarrow{A_1} & \mathbb{Q}^2 & \xrightarrow{A_2} & \mathbb{Q}^2 & \xrightarrow{A_3} & \mathbb{Q}^2 & \xrightarrow{A_4} & \dots \\
& & \downarrow J_0 = \text{id} & & \downarrow J_1 = \text{id} & & \downarrow J_2 & & \downarrow J_3 & & \\
H : & & \mathbb{Q} & \xrightarrow{B_1} & \mathbb{Q}^2 & \xrightarrow{B_2} & \mathbb{Q}^2 & \xrightarrow{B_3} & \mathbb{Q}^2 & \xrightarrow{B_4} & \dots
\end{array}$$

which establishes an order isomorphism between the two associated inductive limits G and H (respecting the distinguished order units). Now H is order isomorphic to the dimension group associated to $(**)$ since the latter is a divisible group (because n is a divisor of each entry of B_n). We conclude that $G = \mathbb{Q} + \mathbb{Q}\alpha \cong K_0(V, E)$ (as ordered groups with distinguished order units), where (V, E) is the Bratteli diagram associated to $(**)$. Since $B_n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k_n \\ k_n \end{bmatrix}$, we see that (V, E) has the ERS property. Furthermore, since n divides k_n we observe that (V, E) is a rank 2 ERS realization of G with respect to $\mathbb{Q}(G, 1) (= \mathbb{Q})$.

Finally, any proper ordering of (V, E) will yield a Bratteli-Vershik (BV-) system which is a uniquely ergodic Toeplitz system of entropy zero. An intriguing question is if by considering all proper orderings there arises an uncountable family of pairwise non-conjugate BV-systems? These BV-systems are of course all strong orbit equivalent, having the same K^0 -group $\mathbb{Q} + \mathbb{Q}\alpha$.

Example 5.7. *2-symmetric Bratteli diagrams (V, E) and their associated simple dimension groups.* Let $|V_n| = 2$ for all $n \geq 1$ and let the incidence matrix M_n between levels $n - 1$ and n be 2-symmetric, i.e. of the form

$$M_n = \begin{bmatrix} l_n & k_n \\ k_n & l_n \end{bmatrix}, \text{ where } 1 \leq k_n < l_n$$

for all $n \geq 2$, and $M_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (See Figure 3.) So $\text{rank}(V, E) = 2$ and (V, E) has the ERS-property (as well as the ECS property, i.e. equal column sums). (The 2-symmetric Bratteli diagrams were studied in [FM] in connection with classifying the symmetries of UHF C^* -algebras. Cf. also [B, Chapter III, Section 7.7.4].) It will be convenient to write for $n \geq 2$:

$$l_n = \frac{q_n + r_n}{2}, \quad k_n = \frac{q_n - r_n}{2}, \quad \text{where } q_n \text{ and } r_n \text{ have the same parity,}$$

and so $q_n = l_n + k_n$, $r_n = l_n - k_n$. By a simple computation we get that

$$M_n M_{n-1} \cdots M_2 = \frac{1}{2} \begin{bmatrix} s_n + t_n & s_n - t_n \\ s_n - t_n & s_n + t_n \end{bmatrix}, \quad \text{where } s_n = \prod_{i=2}^n q_i, \quad t_n = \prod_{i=2}^n r_i.$$

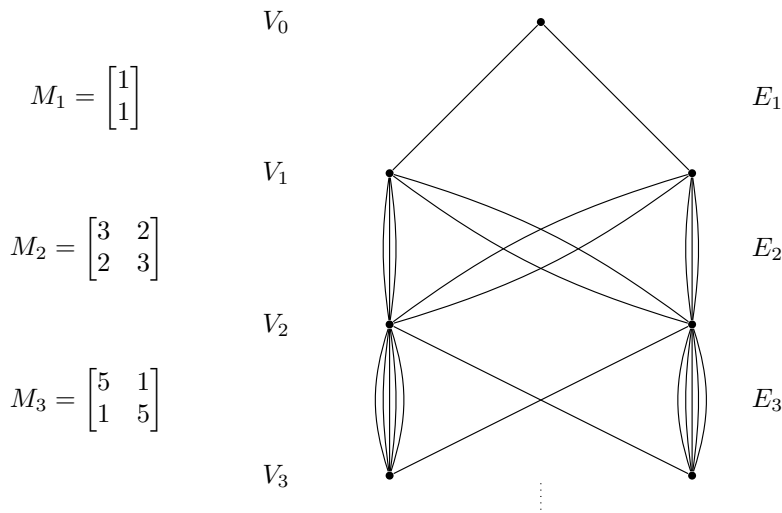


Figure 3: A 2-symmetric Bratteli diagram

We note that s_n and t_n have the same parity and that

$$\frac{s_n}{t_n} = \prod_{i=2}^n \frac{q_i}{r_i} \nearrow \prod_{i=2}^{\infty} \frac{q_i}{r_i} = \alpha.$$

Two scenarios can occur. Firstly, if $\alpha < \infty$ then the state space $S_{[\mathbf{1}]}(K_0(V, E))$ of $K_0(V, E)$ has two extreme points. Specifically,

$$K_0(V, E) = \left\{ \left(\frac{k}{2s_n}, \frac{l}{2t_n} \right) \mid k, l \in \mathbb{Z} \text{ with the same parity, } n \in \mathbb{N} \right\} \subseteq \mathbb{Q}^2 \subseteq \mathbb{R}^2.$$

$K_0(V, E)^+$ consists of those elements in $K_0(V, E)$ that lie inside the cone determined by the two half-lines with slopes α and $-\alpha$ respectively. The canonical order unit $[\mathbf{1}]$ of $K_0(V, E)$ is represented by $(1, 0)$ ($= (\frac{2}{2}, 0)$) and the two extreme states are determined by projecting orthogonally to the two lines with slopes α and $-\alpha$, respectively. We note that $K_0(V, E)$ does not split as a direct sum of two (noncyclic) rational groups, although it "almost" does. (More on that when describing the unique state case below.) Furthermore, $\mathbb{Q}(K_0(V, E), [\mathbf{1}])$ is the subgroup of $K_0(V, E)$ represented by

$$\left\{ \left(\frac{k}{2s_n}, 0 \right) \mid k \in \mathbb{Z} \text{ even, } n \in \mathbb{N} \right\}$$

and so is order isomorphic to the noncyclic rational group $G(m)$, where $m = \prod_{i=1}^{\infty} q_i$.

The second scenario occurs when $\alpha = \infty$. Then $S_{[1]}(K_0(V, E)) = \{\tau\}$ is a one-point set. The unique state τ is intimately related to the Perron-Frobenius (P-F) eigenvalues and eigenvectors of the incidence matrices $\{M_n\}_{n=2}^{\infty}$. In fact, q_n is the P-F eigenvalue of M_n with left (right) eigenvector $(1, 1)$ ($(1, 1)^{\text{tr}}$). The other eigenvalue is r_n with left (right) eigenvector $(1, -1)$ ($(1, -1)^{\text{tr}}$). The state τ is determined by

$$\tau(v_n) = \frac{1}{q_1 q_2 \cdots q_n} (1, 1) \vec{v}_n$$

where v_n denotes the element in $K_0(V, E)$ represented by the column vector $\vec{v}_n \in \mathbb{Z}^2$ at level n of the Bratteli diagram, and where we for normalization purposes have set $q_1 = 2$ so that $\tau([1]) = 1$. In particular we get that

$$\tau(K_0(V, E)) = \left\{ \frac{m}{q_1 q_2 \cdots q_n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \right\} = Q (\subseteq \mathbb{Q}).$$

Furthermore, $\tau(H) = \left\{ \frac{m}{q_1 q_2 \cdots q_n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \right\}$ where $H = \mathbb{Q}(K_0(V, E), [1])$, since H is represented as "constant" vectors at each level of (V, E) , cf. Lemma 4.3. We notice that $\tau(H) = 2\tau(K_0(V, E)) = 2Q$. Now we focus on the interesting special case where $r_n = 1$, for all $n \geq 2$. Hence $l_n = k_n + 1$, and q_n is odd for all $n \geq 2$. In particular, $Q = \tau(K_0(V, E))$ is not 2-divisible. Now $\ker(\tau) = \text{Inf}(K_0(V, E))$ and $K_0(V, E)^+ = \tau^{-1}(\mathbb{R}^+ - \{0\})$. We observe that $\ker(\tau) \cong \mathbb{Z}$ since $\tau(v_n) = 0$ for all $n \geq 1$, where $\vec{v}_n = (1, -1)^{\text{tr}}$ (cf. the notation above), and $M_n(1, -1)^{\text{tr}} = (1, -1)^{\text{tr}}$ for all $n \geq 2$. Thus we get the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(V, E) \xrightarrow{\tau} Q \longrightarrow 0$$

However, there is no splitting $K_0(V, E) = Q \oplus \mathbb{Z}$, with τ the projection onto Q , the reason being that Q is not 2-divisible. In fact, if such a splitting occurred then clearly Q would be equal to the rational subgroup H . This would imply that $\tau(H) = Q$ and, combining this with $\tau(H) = 2Q$, we would get that Q is 2-divisible which is a contradiction. (For more details, cf. [H, Section 8].)

Remark. By our results we know that all proper orderings of a 2-symmetric Bratteli diagram yield Bratteli-Vershik maps that are Toeplitz flows. One can show that when $r_n = 1$ ($n \geq 2$) and (V, E) is given the left-right ordering (meaning that we order the edges ranging at the same vertex from left to right), then the Toeplitz sequence that corresponds to the (unique) minimal path is regular (cf. Definition 3.14).

Proof of Theorem 5.3 By telescoping we may assume that $k = |V_1| = |V_2| = \dots (= \text{rank}(V, E))$, where $V = \{V_0, V_1, V_2, \dots\}$, $E = \{E_1, E_2, \dots\}$. By Theorem 5.2 we may assume that $(X_{(V,E)}, T_{(V,E)})$ is expansive, since if not it would be an odometer which has zero entropy. This implies in particular that we may assume that $k \geq 2$. By telescoping the initial part of (V, E) we may assume that $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X_1, S_1) (and hence conjugate to (X_n, S_n) for all $n \geq 1$) by the map $\pi_1: X_{(V,E)} \rightarrow X_1 (= \pi_1(X_{(V,E)}) \subseteq E_1^{\mathbb{Z}} (= P_1^{\mathbb{Z}}))$ defined by $\pi_1(x) = \left(\tau_1(T_{(V,E)}^n x)\right)_{n=-\infty}^{\infty}$, where $\tau_1: X_{(V,E)} \rightarrow E_1 (= P_1)$ is the truncation map. (Cf. Proposition 3.35 and the description of key constructions associated to (V, E, \geq) , as well as notation and terminology preceding the proposition.) For $v \in V_n$ we consider all the paths, say $\{p_1, p_2, \dots, p_l\}$ from $v_0 \in V_0$ to v , which will be a subset of P_n . Let those paths have the ordering $p_1 < p_2 < \dots < p_l$, say, in the induced lexicographic ordering. Then $w(v) = \tau_1(p_1)\tau_1(p_2) \cdots \tau_1(p_l)$ will be a word over E_1 . Let $W_n = \{w(v) \mid v \in V_n\}$. Notice that $|W_n| = k$ for all n . Define (X_{W_n}, S_{W_n}) to be the subshift of the full shift on $E_1^{\mathbb{Z}}$, where X_{W_n} is the set of all bisequences formed by concatenation of words in W_n . Now we observe that $E_1^{\mathbb{Z}} \supseteq X_{W_1} \supseteq X_{W_2} \supseteq \dots \supseteq X_1$. In fact, this is an immediate consequence of how the Vershik map is defined.

Recall that the entropy of a subshift (X, T) is $h(T) = \lim_{q \rightarrow \infty} \frac{1}{q} \log |B_q(X)|$, where $B_q(X)$ is the set of words of length q occurring in X . (Cf. [Wa, Theorem 7.13].) Clearly $h(S_1) \leq h(S_{W_n})$ for all n , and so it suffices to show that $h(S_{W_n}) \rightarrow 0$ as $n \rightarrow \infty$. Let l_n be the length of the shortest word in W_n . (Clearly $l_n \rightarrow \infty$ as $n \rightarrow \infty$.) Assume $w \in X_{W_n}$ is a subword of length m of a concatenation of s words from W_n . Then we easily see that

$$s \leq \left\lceil \frac{m}{l_n} \right\rceil + 1$$

where $\lceil x \rceil$ is the smallest integer which is larger or equal to x . There is at most k^s different ways to concatenate s words in W_n , and so we get

$$|B_m(W_n)| \leq \sum_{s=0}^{\lceil \frac{m}{l_n} \rceil + 1} k^s = \frac{k^{\lceil \frac{m}{l_n} \rceil + 2} - 1}{k - 1} \leq k^{\lceil \frac{m}{l_n} \rceil + 2} \leq k^{\frac{m}{l_n} + 3}.$$

This implies that $\frac{1}{n} \log |B_m(W_n)| \leq \frac{m+3}{l_n} \log k \rightarrow 0$ as $n \rightarrow \infty$. Hence $h(S_{W_n}) \rightarrow 0$ as $n \rightarrow \infty$, thus completing the proof. \square

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References

- [B] B. Blackadar. *K-theory for Operator Algebras*. Math. Sci. Research Institute Publ. **5**, Springer-Verlag, 1986.
- [BT] M. Boyle and J. Tomiyama. Bounded topological orbit equivalence and C^* -algebras. *J. Math. Soc. Japan*, **50**, (1998), no. 2, 317–329.
- [CP] M. I. Cortez and S. Petite. Invariant measures and orbit equivalence for generalized Toeplitz subshifts. <http://arxiv.org/abs/1106.4280>.
- [D] T. Downarowicz. Survey of odometers and Toeplitz flows. *Algebraic and Topological Dynamics (Kolyada, Manin, Ward eds)*, *Contemporary Math.*, **385** (2005), 7–38.
- [DD] T. Downarowicz and F. Durand. Factors of Toeplitz flows and other almost 1-1 extensions over group rotations. *Math. Scand.*, **90** (2002), 57–72.
- [DGS] M. Denker, C. Grillenberger and K. Sigmund. *Ergodic Theory on Compact Spaces*. Number 527 in Lecture Notes in Math. Springer, 1976.
- [DHS] F. Durand, B. Host and C.F. Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergod. Th. and Dynam. Sys.*, **19** (1999), 995–1035.
- [DM] T. Downarowicz and A. Maass. Finite-rank Bratteli-Vershik diagrams are expansive. *Ergod. Th. and Dynam. Sys.*, **28** (2008), 739–747.
- [E] E. Effros. *Dimensions and C^* -algebras*. Number 46 in CBMS Regional Conference Series in Mathematics, 1981.
- [EHS] E. Effros, D. Handelman and C. Shen. Dimension groups and their affine representations. *Amer. J. Math.*, **102** (1980), 385–407.
- [F] L. Fuchs. *Infinite Abelian Groups, Volume II*. Academic Press, New York, 1973.
- [FM] T. Fack and O. Marechal. Sur la classification des symetries des C^* -algèbres UHF. *Can. J. Math.*, **31** (1979), 496–523.
- [GH] W. H. Gottschalk and G. A. Hedlund. *Topological Dynamics, vol. 36*. Amer. Math. Soc. Colloq. Publ, Providence, R.I., 1955.
- [GHo] E. Glasner and B. Host. Extensions of Cantor minimal systems and dimension groups. <http://arxiv.org/abs/1111.0506>. To be published in *J. Reine Angew. Math.*

- [GJ] R. Gjerde and Ø. Johansen. Bratteli-Vershik models for Cantor minimal systems: application to Toeplitz flows. *Ergod. Th. and Dynam. Sys.*, **20** (2000), 1687–1710.
- [GMPS] T. Giordano, H. Matui, I.F. Putnam and C. Skau. Orbit equivalence for Cantor minimal \mathbb{Z}^d -systems. *Invent. Math.*, **179** (2010) no. 1, 119–158.
- [GPS1] T. Giordano, I.F. Putnam and C. Skau. Topological orbit equivalence and C^* -crossed products. *J. Reine Angew. Math.*, **469** (1995), 51–111.
- [GPS2] T. Giordano, I.F. Putnam and C. Skau. K-theory and asymptotic index for certain almost one-to-one factors. *Math. Scand.*, **89** (2001), 297–319.
- [GW] E. Glasner and B. Weiss. Weak orbit equivalence of Cantor minimal systems. *Int. J. Math.*, **6** (1995), 569–579.
- [H] D. Handelman. Equal column sum and equal row sum dimension group realizations <http://arxiv.org/abs/1301.2799>
- [HPS] R.H. Herman, I.F. Putnam and C.F. Skau. Ordered Bratteli diagrams, dimension groups, and topological dynamics. *Int. J. Math.*, **3** (1992), 827–864.
- [HR] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis I*. Grundle Math. Wiss. **115** Springer Verlag, 1963.
- [JK] K. Jacobs and M. Keane. 0–1-sequences of Toeplitz type. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **13** (1969), 123–131.
- [K] K. Kuratowski. *Topology, Vol. II*. Academic Press, New York, 1968.
- [M] J. C. Martin. Substitution minimal flows. *Amer. J. Math.*, **93** (1971), 503–526.
- [MP] N. G. Markley, and M. E. Paul. Almost automorphic symbolic minimal sets without unique ergodicity. *Israel J. Math.*, **34** (1979), 259–272.
- [O] N.S. Ormes. Strong orbit realization for minimal homeomorphisms. *Journal d'Analyse Mathématique*, **71** (1997), 103–133.
- [P] M. E. Paul. Construction of almost automorphic symbolic minimal flows. *General Topology Appl.*, **6** (1976), 45–56.
- [S1] F. Sugisaki. Toeplitz flows, ordered Bratteli diagrams and strong orbit equivalence. *Ergod. Th. and Dynam. Sys.*, **21** (2001), 1867–1881.

-
- [S2] F. Sugisaki. On the subshift within a strong orbit equivalence class for minimal homeomorphisms. *Ergod. Th. and Dynam. Sys.*, **27** (2007), 971–990.
- [S3] F. Sugisaki. Almost one-to-one extensions of Cantor minimal systems and order embeddings of simple dimension groups. *Münster J. Math.*, **4** (2011), 141–169.
- [Sk] C.F. Skau. Ordered K-theory and minimal symbolic dynamical systems. *Colloq. Math.*, **84/85** (2000), 203–227.
- [T] O. Toeplitz. Beispiele zur Theorie der fastperiodischen Funktionen. *Math. Ann.*, **98** (1928), 281–295.
- [W] S. Williams. Toeplitz minimal flows which are not uniquely ergodic. *Z. Wahrs. verw. Gebiete.*, **67** (1984), 95–107.
- [Wa] P. Walters. *An Introduction to Ergodic Theory*. Graduate Texts in Mathematics **79**, Springer-Verlag, 1982.

Paper II

Finite-rank Bratteli-Vershik diagrams are
expansive – a new proof

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