On the accuracy of gradient estimation in extremum-seeking control using small perturbations ⋆

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Abstract

In many extremum-seeking control methods, perturbations are added to the parameter signals to estimate derivatives of the objective function (that is, the steady-state parameter-to-performance map) in order to optimize the steady-state performance of the plant using derivative-based algorithms. However, large perturbations are often undesirable or inapplicable due to practical constraints and a high cost of operation. Yet, many extremum-seeking control algorithms rely solely on perturbations to estimate all required derivatives. The corresponding derivative estimates, especially the Hessian and higher-order derivatives, may be qualitatively poor if the perturbations are small. In this work, we investigate the use of the nominal parameter signals in addition to the perturbations to improve the accuracy of the gradient estimate. In turn, a more accurate gradient estimate may result in a faster convergence and may allow for a higher tuning-gain selection. In addition, we show that, if accurate curvature information of the objective function is available via estimation or a priori knowledge, it may be used to further enhance the accuracy of the gradient estimate.

Key words: Extremum-seeking control; Gradient estimation; Performance optimization; Stability; Nonlinear systems.

1 Introduction

Extremum-seeking control is a collection of adaptive-control methods that optimize the steady-state performance of a plant in real time [1, 13, 19, 26]. By defining performance (or performance cost) as the output of a cost function of tunable plant parameters and measurable performance indicators, often no explicit knowledge of the plant dynamics is required. The steady-state relation between the parameters and the performance is commonly assumed to be given by a static input-to-output map [14, 27], where the extremum of the map corresponds to the optimal steady-state performance. We refer to this map as the objective function. Many extremum-seeking control methods rely on extracting derivative information of the objective function from the parameters and performance signals of the plant [19, 20]. Subsequently, these derivatives are utilized by gradient-based [14, 19, 27] or Newton-based [7, 16, 19] algorithms to steer the plant parameters towards the extremum of the objective function, thereby optimizing the steady-state plant performance. The majority of extremum-seeking control methods utilize perturbations to ensure that the parameter signals are sufficiently rich to estimate the derivatives of the objective function. The derivative estimates are obtained by correlating the perturbations and the time signal of the plant performance [2, 5, 14, 21, 27]. The nominal part of the parameter signals is often ignored. The true values of the derivatives are commonly not obtained, because the performance of the plant is unequal to the steady-state performance due to plant dynamics and measurement noise.

To keep the dynamic transients of the performance signal small, the controller is generally chosen to be slower than the dynamics of the plant [14, 17]. For a limited class of plants, high-amplitude high-frequency perturbations can be used to overpower the original plant dynamics, enforcing an arbitrarily fast convergence upon the plant [4, 15, 23, 29]. However, contrary to these highly invasive methods, in practice, one often wishes to keep the disruption of nominal operation to a minimum to keep operational costs low, the response of the plant predictable, and state and output values within predefined limits. This can be achieved by using small perturbations. The use of small perturbations results in relatively little perturbation-related content in the time signal of the plant performance, which may lead to poor estimates of the derivatives of the objective function, especially in the

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presence of measurement noise. This is particularly true for the estimate of the Hessian and higher-order derivatives of the objective function. Therefore, gradient-based algorithms may be preferred over Newton-based ones in this case. Additionally, the contribution of the nominal part of the parameter signals to the performance signal of the plant is relatively large if the perturbations are small. With the help of an observer, the nominal part of the parameter signals may be included in the estimation process to increase the accuracy of the derivative estimates [3, 6, 9, 22]. Although not specifically identified as such, the effect of the nominal part of the parameter signals on the produced derivative estimates is investigated in [6, 8] by a comparison of various extremum-seeking control methods. However, due to the significant differences between the used methods, it is unclear if the observed results are due to the use of the nominal part of the parameter signals or due to any other structural difference. Moreover, because extremum-seeking control is highly dependent on tuning, the obtained performance of any extremum-seeking method is for a large extent determined by the tuning capabilities of the user.

In this work, we introduce an extremum-seeking controller for which the contribution of the nominal part of the parameter signals to the gradient estimate can be isolated. Therefore, the effect of incorporating the nominal parameters in the estimation process can be investigated using a single controller, which largely eliminates the challenges that affect the comparisons in [6, 8]. In addition, we show that curvature information of the objective function, if available, may further enhance the accuracy of the gradient estimate. The results in this work may be regarded as an extension of the results in [10] in which the nominal parameters and curvature information are not utilized for gradient estimation.

This work is organized as follows. After introducing the extremum-seeking problem in Section 2, our controller is presented in Section 3. A stability analysis of the closed-loop optimization scheme is provided in Section 4. In Section 5, we study, with the help of simulation examples, the effects of incorporating in the gradient estimate the nominal parameter signals and curvature information of the objective function. The conclusion of this work is presented in Section 6. The sets of real numbers, positive real numbers and nonnegative real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$, respectively. The sets of natural numbers (nonnegative integers) and positive integers are given by $\mathbb{N}$ and $\mathbb{N}_{>0}$. The Euclidean norm is denoted by $\|\cdot\|$. We write the identity matrix and zero matrix as $\mathbf{I}$ and $\mathbf{0}$, respectively.

2 Formulation of the extremum-seeking problem

Consider the following multi-input-single-output nonlinear system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ y(t) &= h(\mathbf{x}(t), \mathbf{u}(t)) + d(t), \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$ is the state, $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$ is the input, $y \in \mathbb{R}$ is the output, $d \in \mathbb{R}$ is a disturbance and $t \in \mathbb{R}_{\geq 0}$ is the time. The dimensions of the state and the input are given by $n_\mathbf{x}, n_\mathbf{u}, \in \mathbb{N}_{>0}$, respectively. In the context of extremum-seeking control, the system can be regarded as a cascade of the plant and the cost function that quantifies the performance of the plant (see [11] for example), where the input $\mathbf{u}$ is a vector of tunable plant parameters and the output y is the output of the cost function, which we call the performance measurement. The output of the function h is the output of the cost function in the absence of measurement noise. The disturbance d represents the contribution of measurement noise to the output of the cost function. The state $\mathbf{x}$, the disturbance d and the functions $\mathbf{f}$ and h are unknown. Therefore, the relation between the parameters and the performance of the plant is unknown.

We make several assumptions with respect to the input-to-output behavior of the system in order to optimize the steady-state performance of the plant. First, we assume that, for each constant vector of plant parameters $\mathbf{u}$, there exists a constant steady-state solution of the system denoted by $\mathbf{x} = \mathbf{X}(\mathbf{u})$.

Assumption 1 *There exists a twice continuously differentiable map $\mathbf{X} : \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}^{n_\mathbf{u}}$ and a constant $L_\mathbf{X} \in \mathbb{R}_{>0}$ such that*

$$\mathbf{0} = \mathbf{f}(\mathbf{X}(\mathbf{u}), \mathbf{u}), \qquad \left\| \frac{d\mathbf{X}}{d\mathbf{u}}(\mathbf{u}) \right\| \leq L_\mathbf{X} \tag{2}$$

for all $\mathbf{u} \in \mathscr{U}$.

In addition, we assume that the steady-state solution $\mathbf{x} = \mathbf{X}(\mathbf{u})$ is unique and exponentially stable for constant inputs.

Assumption 2 *There exist constants $\mu_\mathbf{x}, \nu_\mathbf{x} \in \mathbb{R}_{>0}$ such that, for each constant $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$, the solutions of the system satisfy*

$$\|\tilde{\mathbf{x}}(t)\| \leq \mu_\mathbf{x} \|\tilde{\mathbf{x}}(t_0)\| e^{-\nu_\mathbf{x}(t-t_0)}, \tag{3}$$

with $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{X}(\mathbf{u})$, for all $t \geq t_0 \geq 0$ and all $\mathbf{x}(t_0) \in \mathbb{R}^{n_\mathbf{x}}$.

The disturbance-free steady-state relation between constant plant parameters and the plant performance can now be expressed by the static input-to-output map

$$F(\mathbf{u}) = h(\mathbf{X}(\mathbf{u}), \mathbf{u}). \tag{4}$$

We refer to the map F as the objective function. We assume that the cost function is designed such that there exists a unique minimum of the objective function that corresponds to the optimal steady-state plant performance. This is formulated as follows.

Assumption 3 *The objective function $F : \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}$ is twice continuously differentiable and contains a unique minimum. Let $\mathbf{u}^*$ denote the corresponding minimizer. There exist constants $L_{F1}, L_{F2} \in \mathbb{R}_{>0}$ such that*

$$\frac{dF}{d\mathbf{u}}(\mathbf{u})(\mathbf{u} - \mathbf{u}^*) \geq L_{F_1} \|\mathbf{u} - \mathbf{u}^*\|^2, \qquad \left\| \frac{d^2F}{d\mathbf{u} d\mathbf{u}^T}(\mathbf{u}) \right\| \leq L_{F2} \tag{5}$$

for all $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$.

Although the exact formulation may vary, assumptions on the existence and the attractiveness of the steady-state solution of the system, and the existence of an extremum [1] of the objective function are common in extremum-seeking control [9, 14, 26, 27]. To guarantee the soundness of the stability analysis in Section 4, we assume in addition that the following bounds on the derivatives of the functions $\mathbf{f}$ and h hold.

Assumption 4 *The function $\mathbf{f}: \mathbb{R}^{n_\mathbf{x}} \times \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}^{n_\mathbf{x}}$ and $h: \mathbb{R}^{n_\mathbf{x}} \times \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}$ are twice continuously differentiable. Moreover, there exist constants $L_{\mathbf{fx}}, L_{\mathbf{fu}}, L_{h\mathbf{x}}, L_{h\mathbf{u}} \in \mathbb{R}_{>0}$ such that*

$$\begin{aligned} \left\| \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x},\mathbf{u}) \right\| &\leq L_{\mathbf{fx}}, & \left\| \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x},\mathbf{u}) \right\| &\leq L_{\mathbf{fu}}, \\ \left\| \frac{\partial^2 h}{\partial \mathbf{x} \partial \mathbf{x}^T}(\mathbf{x},\mathbf{u}) \right\| &\leq L_{h\mathbf{x}}, & \left\| \frac{\partial^2 h}{\partial \mathbf{x} \partial \mathbf{u}^T}(\mathbf{x},\mathbf{u}) \right\| &\leq L_{h\mathbf{u}} \end{aligned} \tag{6}$$

for all $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$ and all $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$.

Remark 5 *In practice, it is sufficient to assume that the bounds in Assumptions 1-4 hold for all $\mathbf{x}$ and all $\mathbf{u}$ in the operating region of the plant, which is generally bounded. For a bounded operating region, the existence of the upper bounds on the derivatives of the functions $\mathbf{X}$, F, $\mathbf{f}$ and h in Assumptions 1, 3 and 4 follows directly form the twice continuous differentiability of the respective functions.*

We note that $\mathbf{X}$, F and $\mathbf{u}^*$ are unknown because the functions $\mathbf{f}$ and h are unknown. Nonetheless, we present an extremum-seeking controller that optimizes the steady-state plant performance by regulating $\mathbf{u}$ towards $\mathbf{u}^*$.

3 Controller design

We start our controller design by introducing a perturbation to the plant-parameter signals:

$$\mathbf{u}(t) = \hat{\mathbf{u}}(t) + \alpha_\omega(t)\omega(t), \tag{7}$$

where $\hat{\mathbf{u}} \in \mathbb{R}^{n_\mathbf{u}}$ is the nominal value of the plant parameters, and where the vector of perturbations $\omega \in \mathbb{R}^{n_\mathbf{u}}$ is given by

$$\omega(t) = [\omega_1(t),\ \omega_2(t),\ \ldots,\ \omega_{n_\mathbf{u}}(t)]^T, \tag{8}$$

with

$$\omega_i(t) = \begin{cases} \sin\left(\frac{i+1}{2}\int_0^t \eta_\omega(\tau)d\tau\right), & \text{if } i \text{ is odd}, \\ \cos\left(\frac{i}{2}\int_0^t \eta_\omega(\tau)d\tau\right), & \text{if } i \text{ is even} \end{cases} \tag{9}$$

for $i = 1,2,\ldots,n_\mathbf{u}$. The tuning parameters $\alpha_\omega, \eta_\omega \in \mathbb{R}_{>0}$ determine the amplitude and frequency of the perturbations and satisfy the differential equations

$$\dot{\alpha}_\omega(t) = -g_\alpha(t)\alpha_\omega(t), \quad \dot{\eta}_\omega(t) = -g_\omega(t)\eta_\omega(t), \tag{10}$$

where $g_\alpha, g_\omega \in \mathbb{R}_{\geq 0}$ are time-varying gains. They are constant if the gains g_α and g_ω are zero. The use of constant tuning parameters leads often to practical convergence with respect to the optimal steady-state performance of the plant; see [13, 14, 23, 27] for example. For strictly positive values of g_α and g_ω, α_ω and η_ω decay to zero as time goes to infinity, which allows us to obtain asymptotic convergence to the optimal steady-state performance as in [10, 18, 25, 28]. Because extremum-seeking control methods are almost entirely based on measurements, asymptotic or practical converge can only be guaranteed under certain noise conditions. As pointed out in [10], a sufficient condition for asymptotic and practical convergence is that the zero-mean component of the disturbance d and the perturbations of the controller are uncorrelated. We make the following assumption.

Assumption 6 *The disturbance $d: \mathbb{R}_{\geq 0} \to \mathbb{R}$ is integrable. Define the mean*

$$b_d = \lim_{T\to\infty} \frac{1}{T}\int_0^T d(t)dt \tag{11}$$

and the zero-mean component

$$\tilde{d}(t) = d(t) - b_d. \tag{12}$$

The perturbation vector and the zero-mean component of the disturbance d are uncorrelated; that is,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \omega(t)\tilde{d}(t)dt = 0. \tag{13}$$

Moreover, there exist constants $q_d, q_{\omega d} \in \mathbb{R}_{\geq 0}$ such that

$$\left| \int_0^t \tilde{d}(\tau)d\tau \right| \leq q_d, \quad \left\| \int_0^t \omega(\tau)\tilde{d}(\tau)d\tau \right\| \leq q_{\omega d} \tag{14}$$

for all $t \geq 0$.

Uncorrelation of the perturbation vector and the disturbance is also assumed in [1, 26]. Alternatively, asymptotic and practical convergence can be obtained if the disturbance satisfies certain stochastic properties, as proved in [25].

Similar to [10], the following general model of the input-to-output behavior of the plant is obtained from (1), (4), (7), (10), (12) and Taylor's theorem:

$$\begin{aligned} \dot{m}_1(t) &= \eta_\mathbf{m}(t)\mathbf{Q}_1^T(t)\mathbf{m}_2(t) - \alpha_\omega^2(t)\mathbf{Q}_1^T(t)\mathbf{w}(t) \\ \dot{\mathbf{m}}_2(t) &= -g_\alpha(t)\mathbf{m}_2(t) + \alpha_\omega^2(t)\mathbf{w}(t) \\ y(t) &= m_1(t) + (\mathbf{Q}_1(t) + \omega(t))^T\,\mathbf{m}_2(t) \\ &\quad + \alpha_\omega^2(t)v(t) + z(t) + \tilde{d}(t), \end{aligned} \tag{15}$$

[1] The extremum is a minimum in this case.

with state

$$m_1(t) = F(\hat{\mathbf{u}}(t)) - \alpha_\omega(t)\frac{dF}{d\mathbf{u}}(\hat{\mathbf{u}}(t))\mathbf{Q}_1(t) + b_d,$$
$$\mathbf{m}_2(t) = \alpha_\omega(t)\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}(t)) \tag{16}$$

and disturbances

$$\mathbf{w}(t) = \frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}}(t))\frac{\dot{\hat{\mathbf{u}}}(t)}{\alpha_\omega(t)},$$
$$v(t) = \omega^T(t)\int_0^1 (1-s)\frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}}(t) + s\alpha_\omega(t)\omega(t))ds\,\omega(t),$$
$$z(t) = h(\mathbf{x}(t),\mathbf{u}(t)) - h(\mathbf{X}(\mathbf{u}(t)),\mathbf{u}(t)), \tag{17}$$

where $\mathbf{Q}_1 \in \mathbb{R}^{n_\mathbf{u}}$ is a known function of time. In particular, $\mathbf{Q}_1$ is the solution of the differential equation

$$\dot{\mathbf{Q}}_1(t) = -\eta_\mathbf{m}(t)\mathbf{Q}_1(t) + g_\alpha(t)\mathbf{Q}_1(t) + \frac{\dot{\hat{\mathbf{u}}}(t)}{\alpha_\omega(t)}, \tag{18}$$

where $\eta_\mathbf{m} \in \mathbb{R}_{>0}$ is a tuning parameter that satisfies

$$\dot{\eta}_\mathbf{m}(t) = -g_\mathbf{m}(t)\eta_\mathbf{m}(t), \tag{19}$$

with time-varying gain $g_\mathbf{m} \in \mathbb{R}_{\geq 0}$. Note that the state $\mathbf{m}_2$ in (16) is equal to the gradient of the objective function scaled by α_ω. Hence, an estimate of the gradient of the objective function can be obtained from an estimate of the state of the model (15). We introduce the following observer:

$$\begin{aligned}
\dot{\hat{m}}_1(t) &= -\alpha_\omega^2(t)\mathbf{Q}_1^T(t)\hat{\mathbf{w}}(t) \\
&\quad + \eta_\mathbf{m}(t)\left(y(t) - \hat{m}_1(t) - \alpha_\omega^2(t)\hat{v}(t)\right) \\
\dot{\hat{\mathbf{m}}}_2(t) &= -g_\alpha(t)\hat{\mathbf{m}}_2(t) + \alpha_\omega^2(t)\hat{\mathbf{w}}(t) \\
&\quad + \eta_\mathbf{m}(t)\mathbf{Q}_2(t)(\mathbf{Q}_1(t)+\omega(t))\big(y(t) - \hat{m}_1(t) \\
&\quad - (\mathbf{Q}_1(t)+\omega(t))^T\hat{\mathbf{m}}_2(t) - \alpha_\omega^2(t)\hat{v}(t)\big) \\
\dot{\mathbf{Q}}_2(t) &= \eta_\mathbf{m}(t)\mathbf{Q}_2(t) - 2g_\alpha(t)\mathbf{Q}_2(t) \\
&\quad - \eta_\mathbf{m}(t)\mathbf{Q}_2(t)\left(\mathbf{Q}_1(t)\mathbf{Q}_1^T(t) + \frac{1}{2}\mathbf{I}\right)\mathbf{Q}_2(t)
\end{aligned} \tag{20}$$

where $\hat{m}_1$ and $\hat{\mathbf{m}}_2$ are estimates of m_1 and $\mathbf{m}_2$, respectively, and where $\mathbf{Q}_2 \in \mathbb{R}^{n_\mathbf{u}\times n_\mathbf{u}}$ is a symmetric positive-definite matrix. The signals $\hat{\mathbf{w}}$ and $\hat{v}$ are estimates of the disturbances $\mathbf{w}$ and v in (17), respectively. The disturbances $\mathbf{w}$ and v depend on the Hessian of the objective function. It is possible to estimate the Hessian with the approaches in [7, 16, 20]. However, the accuracy of the resulting estimate may be relatively poor if the perturbation amplitude is small. Nonetheless, if a reasonably accurate estimate $\mathbf{H} : \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}^{n_\mathbf{u}\times n_\mathbf{u}}$ of the Hessian of the objective function is available due to estimation or a priori knowledge, we may choose

$$\hat{\mathbf{w}}(t) = \mathbf{H}(\hat{\mathbf{u}}(t))\frac{\dot{\hat{\mathbf{u}}}(t)}{\alpha_\omega(t)}, \quad \hat{v}(t) = \frac{1}{2}\omega^T(t)\mathbf{H}(\hat{\mathbf{u}}(t))\omega(t) \tag{21}$$

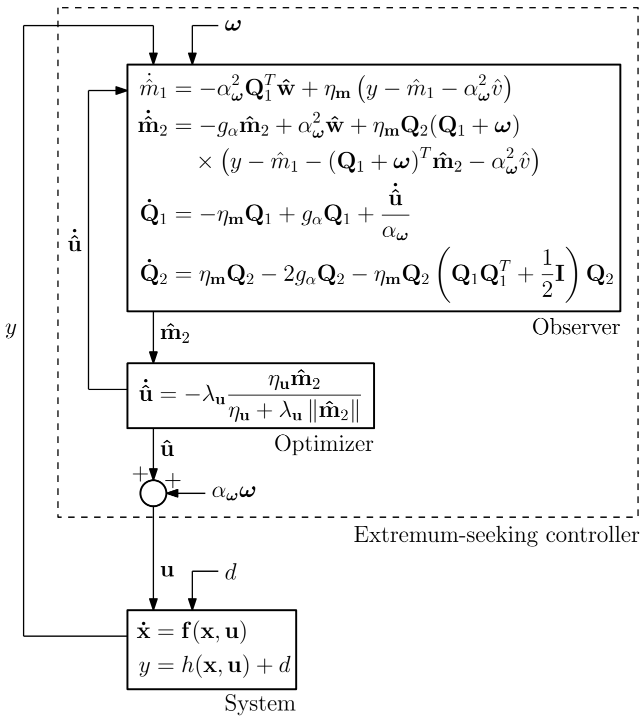

Fig. 1. Closed-loop system of plant and extremum-seeking controller.

to enhance the accuracy of the state estimate. In line with the boundedness of the Hessian of the objective function (see Assumption 3), we assume that $\mathbf{H}$ is bounded; there exists a constant $L_\mathbf{H} \in \mathbb{R}_{>0}$ such that

$$\|\mathbf{H}(\hat{\mathbf{u}})\| \leq L_\mathbf{H} \tag{22}$$

for all $\hat{\mathbf{u}} \in \mathbb{R}^{n_\mathbf{u}}$. Without any knowledge of the Hessian, we may choose $\hat{\mathbf{w}}(t) = \mathbf{0}$ and $\hat{v}(t) = 0$ instead.

We define the following optimizer to steer the nominal part of the plant parameters in the gradient-descent direction towards the minimum of the objective function:

$$\dot{\hat{\mathbf{u}}}(t) = -\lambda_\mathbf{u}(t)\frac{\eta_\mathbf{u}(t)\hat{\mathbf{m}}_2(t)}{\eta_\mathbf{u}(t) + \lambda_\mathbf{u}(t)\|\hat{\mathbf{m}}_2(t)\|}, \tag{23}$$

where the tuning parameters $\eta_\mathbf{u}, \lambda_\mathbf{u} \in \mathbb{R}_{>0}$ are defined by the differential equations

$$\dot{\eta}_\mathbf{u}(t) = -g_\mathbf{u}(t)\eta_\mathbf{u}(t), \quad \dot{\lambda}_\mathbf{u}(t) = -g_\lambda(t)\lambda_\mathbf{u}(t), \tag{24}$$

with time-varying gains $g_\mathbf{u}, g_\lambda \in \mathbb{R}_{\geq 0}$. By normalizing the adaptation gain in (23), we prevent the solutions of the closed-loop system of plant and controller from having a finite escape time. The interconnection of the plant in (1) and the controller in (7), (18), (20), (23) is depicted in Figure 1.

4 Stability analysis

We briefly study the stability of the closed-loop system in (1), (7), (18), (20) and (23) to identify suitable operating conditions.

Theorem 7 *Let the gains g_α, g_ω, $g_{\mathbf{m}}$, $g_{\mathbf{u}}$ and g_λ in* (10), (19) *and* (24) *be given by*

$$g_\alpha(t) = \frac{r_\alpha}{r_0+t}, \quad g_\omega(t) = \frac{r_\omega}{r_0+t}, \quad g_{\mathbf{m}}(t) = \frac{r_{\mathbf{m}}}{r_0+t}, \\ g_{\mathbf{u}}(t) = \frac{r_{\mathbf{u}}}{r_0+t}, \quad g_\lambda(t) = \frac{r_\lambda}{r_0+t}, \tag{25}$$

where $r_0 \in \mathbb{R}_{>0}$ and $r_\alpha, r_\omega, r_{\mathbf{m}}, r_\lambda, r_{\mathbf{u}} \in \mathbb{R}_{\geq 0}$ are constants that are chosen such that

$$r_\alpha \leq r_{\mathbf{m}}, \qquad r_\omega \leq r_{\mathbf{m}}, \\ r_\alpha + r_{\mathbf{m}} \leq r_{\mathbf{u}} \leq 1, \quad r_{\mathbf{m}} \leq r_\alpha + r_\lambda \leq 1. \tag{26}$$

Moreover, let $\alpha_\omega(0), \eta_\omega(0), \eta_{\mathbf{m}}(0), \eta_{\mathbf{u}}(0), \lambda_{\mathbf{u}}(0) \in \mathbb{R}_{>0}$. Under Assumptions 1-4 and 6, there exist sufficiently small constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{R}_{>0}$ and sufficiently large constants $c_1, c_2, c_3 \in \mathbb{R}_{>0}$ such that, if there exists a time $t_1 \geq 0$ such that

$$\eta_\omega(t) \leq \varepsilon_1, \qquad \eta_{\mathbf{m}}(t) \leq \eta_\omega(t)\varepsilon_2, \\ \eta_{\mathbf{u}}(t) \leq \alpha_\omega(t)\eta_{\mathbf{m}}(t)\varepsilon_3, \quad \alpha_\omega(t)\lambda_{\mathbf{u}}(t) \leq \eta_{\mathbf{m}}(t)\varepsilon_4 \tag{27}$$

for all $t \geq t_1$, then the solutions of the closed-loop system in (1), (7), (18), (20) *and* (23) *are bounded for all $t \geq 0$, all $\mathbf{x}(0) \in \mathbb{R}^{n_{\mathbf{x}}}$, all $\mathbf{Q}_1(0) \in \mathbb{R}^{n_{\mathbf{u}}}$, all symmetric positive-definite $\mathbf{Q}_2(0) \in \mathbb{R}^{n_{\mathbf{u}} \times n_{\mathbf{u}}}$, all $\hat{m}_1(0) \in \mathbb{R}$, all $\hat{\mathbf{m}}_2(0) \in \mathbb{R}^{n_{\mathbf{u}}}$ and all $\hat{\mathbf{u}}(0) \in \mathbb{R}^{n_{\mathbf{u}}}$. In addition,*

$$\limsup_{t\to\infty} \|\hat{\mathbf{u}}(t) - \mathbf{u}^*\| \\ \leq \limsup_{t\to\infty} \max\left\{\alpha_\omega(t)c_1, \eta_\omega(t)c_2, \frac{\eta_{\mathbf{m}}(t)}{\alpha_\omega(t)}c_3(q_d + q_{\omega d})\right\}. \tag{28}$$

Proof. See Section 4.1. □

By choosing the gains g_α, g_ω, $g_{\mathbf{m}}$, $g_{\mathbf{u}}$ and g_λ as in (25), the tuning parameters α_ω, η_ω, $\eta_{\mathbf{m}}$, $\eta_{\mathbf{u}}$ and $\lambda_{\mathbf{u}}$ can be written as

$$\alpha_\omega(t) = \frac{r_0^{r_\alpha}\alpha_\omega(0)}{(r_0+t)^{r_\alpha}}, \quad \eta_\omega(t) = \frac{r_0^{r_\omega}\eta_\omega(0)}{(r_0+t)^{r_\omega}}, \\ \eta_{\mathbf{m}}(t) = \frac{r_0^{r_{\mathbf{m}}}\eta_{\mathbf{m}}(0)}{(r_0+t)^{r_{\mathbf{m}}}}, \quad \eta_{\mathbf{u}}(t) = \frac{r_0^{r_{\mathbf{u}}}\eta_{\mathbf{u}}(0)}{(r_0+t)^{r_{\mathbf{u}}}}, \\ \lambda_{\mathbf{u}}(t) = \frac{r_0^{r_\lambda}\lambda_{\mathbf{u}}(0)}{(r_0+t)^{r_\lambda}} \tag{29}$$

for all $t \geq 0$. The tuning parameters are constant if r_α, r_ω, $r_{\mathbf{m}}$, r_λ and $r_{\mathbf{u}}$ are zero. For constant tuning parameters, convergence to an arbitrarily small region can be guaranteed under suitable tuning (see (28)), which implies practical convergence. Alternatively, if we choose r_α, r_ω, $r_{\mathbf{m}}$, r_λ and $r_{\mathbf{u}}$ such that $0 < r_\alpha < r_{\mathbf{m}}$, $0 < r_\omega < r_{\mathbf{m}}$, $r_\alpha + r_{\mathbf{m}} < r_{\mathbf{u}} \leq 1$ and $r_{\mathbf{m}} < r_\alpha + r_\lambda \leq 1$ as in [10, Corollary 14], then the tuning parameters α_ω, η_ω, $\eta_{\mathbf{m}}$, $\eta_{\mathbf{u}}$ and $\lambda_{\mathbf{u}}$ decay to zero as time elapses. Moreover, there always exists a time $t_1 \geq 0$ such that the inequalities in (27) are satisfied for all $t \geq t_1$. Hence, tuning of the initial parameter values is not necessary. In addition, asymptotic convergence to the optimal steady-state performance is obtained as the right-hand side of (28) reduces to zero.

The rapidness of the perturbations, the observer and the optimizer depends on the tuning parameters α_ω, η_ω, $\eta_{\mathbf{m}}$, $\eta_{\mathbf{u}}$ and $\lambda_{\mathbf{u}}$. Similar to [10, 14, 17], we obtain from (27) that different time scales can be assigned to components of the extremum-seeking scheme for $t \geq t_1$ if the constants ε_1, ε_2, ε_3 and ε_4 are sufficiently small:

- fast – the plant;
- medium fast – the perturbations of the controller;
- medium slow – the observer of the controller;
- slow – the optimizer of the controller.

To avoid unwanted drift of the state variables $\mathbf{x}$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\hat{m}_1$, $\hat{\mathbf{m}}_2$ and $\hat{\mathbf{u}}$, the tuning parameters of the controller should preferably be selected such that the inequalities in (27) hold for $t_1 = 0$.

4.1 Proof of Theorem 7

The proof of Theorem 7 largely follows the same lines as the proof of [10, Theorem 7]. For notational convenience, we introduce the following coordinate transformation:

$$\begin{aligned}
\tilde{\mathbf{x}}(t) &= \mathbf{x}(t) - \mathbf{X}(\mathbf{u}(t)), \\
\tilde{m}_1(t) &= \hat{m}_1(t) - m_1(t) - \eta_{\mathbf{m}}(t)k_1(t) - \frac{\eta_{\mathbf{m}}(t)}{\eta_\omega(t)}\mathbf{l}_1^T(t)\mathbf{m}_2(t), \\
\tilde{\mathbf{m}}_2(t) &= \frac{\eta_{\mathbf{m}}(t)}{\eta_\omega(t)}\mathbf{Q}_2(t)\mathbf{l}_1(t)\left(\hat{m}_1(t) - m_1(t) - \eta_{\mathbf{m}}(t)k_1(t)\right) \\
&\quad + \left(\mathbf{I} + \frac{\eta_{\mathbf{m}}(t)}{\eta_\omega(t)}\mathbf{Q}_2(t)\left(\mathbf{Q}_1(t)\mathbf{l}_1^T(t) + \mathbf{l}_1(t)\mathbf{Q}_1^T(t) + \mathbf{l}_2(t)\right)\right) \\
&\quad \times \left(\hat{\mathbf{m}}_2(t) - \mathbf{m}_2(t) - \eta_{\mathbf{m}}(t)\mathbf{Q}_2(t)\left(\mathbf{Q}_1(t)k_1(t) + \mathbf{k}_2(t)\right)\right), \\
\tilde{\mathbf{Q}}_2(t) &= \mathbf{Q}_2^{-1}(t) - \frac{1}{2}\mathbf{I}, \\
\tilde{\mathbf{u}}(t) &= \hat{\mathbf{u}}(t) - \mathbf{u}^*,
\end{aligned} \tag{30}$$

with

$$k_1(t) = \int_0^t \tilde{d}(\tau)d\tau, \\ \mathbf{k}_2(t) = \int_0^t \omega(\tau)\tilde{d}(\tau)d\tau \tag{31}$$

and

$$\mathbf{l}_1(t) = \int_0^t \eta_\omega(\tau)\omega(\tau)d\tau, \\ \mathbf{l}_2(t) = \int_0^t \eta_\omega(\tau)\left(\omega(\tau)\omega^T(\tau) - \frac{1}{2}\mathbf{I}\right)d\tau. \tag{32}$$

Using this transformation, the convergence of the solutions of the closed-loop scheme in (1), (7), (18), (20) and (23) may be divided into four stages:

- $0 \leq t < t_1$: the tuning parameters converge to the bounds in (27) (the state variables of the closed-loop system may drift);
- $t_1 \leq t < t_2$: the variable $\mathbf{Q}_1$ converges to a region of the origin and remains there (the variable $\tilde{\mathbf{Q}}_2$ may drift);
- $t_2 \leq t < t_3$: the variable $\tilde{\mathbf{Q}}_2$ converges to a region of the origin and remains there;
- $t_1 \leq t < t_3$: the variable $\tilde{\mathbf{x}}$ converges to a region of the origin and remains there (the variables $\tilde{\mathbf{u}}$, $\tilde{m}_1$ and $\tilde{\mathbf{m}}_2$ may drift);
- $t \geq t_3$: the variables $\tilde{\mathbf{u}}$, $\tilde{m}_1$ and $\tilde{\mathbf{m}}_2$ converge to a region of the origin and remain there.

To prove Theorem 7, we first derive bounds on the variables $\mathbf{Q}_1$, $\tilde{\mathbf{Q}}_2$ and $\tilde{\mathbf{x}}$ in coherence with the first three stages.

Lemma 8 *Under the conditions of Theorem 7, the solutions of $\mathbf{Q}_1$ are bounded for all $t \geq 0$ and all $\mathbf{Q}_1(0) \in \mathbb{R}^{n_\mathbf{u}}$. Moreover, there exists a time $t_2 \geq t_1$ such that*

$$\|\mathbf{Q}_1(t)\| \leq \frac{1}{4} \tag{33}$$

for all $t \geq t_2$.

Proof. See Appendix A. □

We use the bound on the solutions of $\mathbf{Q}_1$ in Lemma 8 to derive a bound on the solutions of $\tilde{\mathbf{Q}}_2$.

Lemma 9 *Under the conditions of Theorem 7, the solutions of $\tilde{\mathbf{Q}}_2$ are bounded for all $t \geq 0$ and all $\tilde{\mathbf{Q}}_2(0) \in \mathbb{R}^{n_\mathbf{u} \times n_\mathbf{u}}$ for which $\mathbf{Q}_2(0)$ is symmetric and positive definite. Moreover, there exists a time $t_3 \geq t_2$ such that*

$$\|\tilde{\mathbf{Q}}_2(t)\| \leq \frac{1}{4} \tag{34}$$

for all $t \geq t_3$.

Proof. See Appendix B. □

The bound on the solutions of $\tilde{\mathbf{x}}$ is given next.

Lemma 10 *Under the conditions of Theorem 7, the solutions of $\tilde{\mathbf{x}}$ are bounded for all $t \geq 0$ and all $\tilde{\mathbf{x}}(0) \in \mathbb{R}^{n_\mathbf{x}}$. Moreover, there exists a constant $c_\mathbf{x} \in \mathbb{R}_{>0}$ and a time $t_3 \geq t_1$ such that*

$$\|\tilde{\mathbf{x}}(t)\| \leq \alpha_\omega(t)\eta_\omega(t)c_\mathbf{x} \tag{35}$$

for all $t \geq t_3$.

Proof. The proof is analogous to that of [10, Lemma 8]. □

The bounds in Lemmas 8-10 are subsequently used to prove the existence of ISS-Lyapunov functions (see [24], for example) for the variables $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$ for $t \geq t_3$.

Lemma 11 *Under the conditions of Theorem 7, there exists a time $t_3 \geq t_2$ such that the solutions of $\tilde{m}_1$ and $\tilde{\mathbf{m}}_2$ are bounded for all $0 \leq t \leq t_3$, all $\tilde{m}_1(0) \in \mathbb{R}$ and all $\tilde{\mathbf{m}}_2(0) \in \mathbb{R}^{n_\mathbf{u}}$. Moreover, there exist a function $V_\mathbf{m} : \mathbb{R} \times \mathbb{R}^{n_\mathbf{u}} \times \mathbb{R}^{n_\mathbf{u} \times n_\mathbf{u}} \to \mathbb{R}_{\geq 0}$ and constants $\gamma_{\mathbf{m}1}, \gamma_{\mathbf{m}2}, \ldots, \gamma_{\mathbf{m}5}, c_{\mathbf{m}1}, c_{\mathbf{m}2}, \ldots, c_{\mathbf{m}6} \in \mathbb{R}_{>0}$ such that*

$$\begin{aligned} \max\left\{\gamma_{\mathbf{m}1}|\tilde{m}_1(t)|^2, \gamma_{\mathbf{m}2}\|\tilde{\mathbf{m}}_2(t)\|^2\right\} &\leq W_\mathbf{m}(t) \\ &\leq \max\{\gamma_{\mathbf{m}3}|\tilde{m}_1(t)|^2, \gamma_{\mathbf{m}4}\|\tilde{\mathbf{m}}_2(t)\|^2\} \end{aligned} \tag{36}$$

for all $t \geq t_3$, where we applied the shorthand notation $W_\mathbf{m}(t) = V_\mathbf{m}(\tilde{m}_1(t), \tilde{\mathbf{m}}_2(t), \mathbf{Q}_2(t))$. In addition, we have that

$$\dot{W}_\mathbf{m}(t) \leq -\eta_\mathbf{m}(t)\gamma_{\mathbf{m}5}W_\mathbf{m}(t) \tag{37}$$

whenever

$$\begin{aligned} W_\mathbf{m}(t) \geq \max\Big\{ & \alpha_\omega^4(t)c_{\mathbf{m}1}, \alpha_\omega^2(t)\eta_\omega^2(t)c_{\mathbf{m}2}, \\ & \alpha_\omega^2(t)\eta_\omega^2(t)c_{\mathbf{m}3}\|\tilde{\mathbf{u}}(t)\|^2, \frac{\alpha_\omega^2(t)\eta_\mathbf{m}^2(t)}{\eta_\omega^2(t)}c_{\mathbf{m}4}\|\tilde{\mathbf{u}}(t)\|^2, \\ & \frac{\alpha_\omega^4(t)\lambda_\mathbf{u}^2(t)}{\eta_\mathbf{m}^2(t)}c_{\mathbf{m}5}\|\tilde{\mathbf{u}}(t)\|^2, \eta_\mathbf{m}^2(t)c_{\mathbf{m}6}(q_d + q_\omega)^2\Big\} \end{aligned} \tag{38}$$

for all $t \geq t_3$.

Proof. See Appendix C. □

Lemma 12 *Under the conditions of Theorem 7, there exists a time $t_3 \geq t_2$ such that the solutions of $\tilde{\mathbf{u}}$ are bounded for all $0 \leq t \leq t_3$ and all $\tilde{\mathbf{u}}(0) \in \mathbb{R}^{n_\mathbf{u}}$. Moreover, there exist a function $V_\mathbf{u} : \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}_{\geq 0}$ and constants $\gamma_{\mathbf{u}1}, \gamma_{\mathbf{u}2}, \gamma_{\mathbf{u}3}, \gamma_{\mathbf{u}4}, c_{\mathbf{u}1}, c_{\mathbf{u}2} \in \mathbb{R}_{>0}$ such that*

$$\gamma_{\mathbf{u}1}\|\tilde{\mathbf{u}}(t)\|^2 \leq V_\mathbf{u}(\tilde{\mathbf{u}}(t)) \leq \gamma_{\mathbf{u}2}\|\tilde{\mathbf{u}}(t)\|^2 \tag{39}$$

for any $t \geq t_3$. In addition, we have that

$$\begin{aligned} \dot{V}_\mathbf{u}(\tilde{\mathbf{u}}(t)) \leq -\min\Big\{ & \alpha_\omega(t)\lambda_\mathbf{u}(t)\gamma_{\mathbf{u}3}V_\mathbf{u}(\tilde{\mathbf{u}}(t)), \\ & \eta_\mathbf{u}(t)\gamma_{\mathbf{u}4}\sqrt{V_\mathbf{u}(\tilde{\mathbf{u}}(t))}\Big\} \end{aligned} \tag{40}$$

whenever

$$V_\mathbf{u}(\tilde{\mathbf{u}}(t)) \geq \max\left\{\frac{c_{\mathbf{u}1}}{\alpha_\omega^2(t)}\|\tilde{\mathbf{m}}_2(t)\|^2, \frac{\eta_\mathbf{m}^2(t)}{\alpha_\omega^2(t)}c_{\mathbf{u}2}(q_d + q_{\omega d})^2\right\} \tag{41}$$

for any $t \geq t_3$.

Proof. The proof follows the same lines as the proof of [10, Lemma 11]. □

Similar to [10], we introduce the Lyapunov-function candidate

$$V(\tilde{m}_1, \tilde{\mathbf{m}}_2, \tilde{\mathbf{u}}, \mathbf{Q}_2, \alpha_\omega) = \max\left\{ V_{\mathbf{u}}(\tilde{\mathbf{u}}), \frac{1}{\alpha_\omega^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q}_2) \right\} \tag{42}$$

to prove that the solutions of $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$ remain bounded for all $t \geq t_3$; see Lemma 13.

Lemma 13 *Under the conditions of Theorem 7, there exist constants $\gamma_{V1}, \gamma_{V2}, \gamma_{V3}, c_{V1}, c_{V2}, c_{V3} \in \mathbb{R}_{>0}$ such that the solutions of $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$ are bounded for all $t \geq t_3$, all $\tilde{m}_1(t_3) \in \mathbb{R}$, $\tilde{\mathbf{m}}_2(t_3) \in \mathbb{R}^{n_\mathbf{u}}$ and all $\tilde{\mathbf{u}}(t_3) \in \mathbb{R}^{n_\mathbf{u}}$. Moreover, the solutions of $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$ satisfy*

$$\limsup_{t\to\infty} \max\left\{ \frac{\gamma_{V1}}{\alpha_\omega(t)}|\tilde{m}_1(t)|, \frac{\gamma_{V2}}{\alpha_\omega(t)}\|\tilde{\mathbf{m}}_2(t)\|, \gamma_{V3}\|\tilde{\mathbf{u}}(t)\| \right\} \leq \limsup_{t\to\infty} \max\left\{ \alpha_\omega(t) c_{V1}, \eta_\omega(t) c_{V2}, \frac{\eta_\mathbf{m}(t)}{\alpha_\omega(t)} c_{V3}(q_d + q_{\omega d}) \right\}. \tag{43}$$

Proof. The proof follows the same steps as the proof of [10, Lemma 12] □

Combining Lemmas 8-13 and the coordinate transformation in (30) completes the proof of Theorem 7 [2].

5 Simulation comparisons

The contribution of the nominal part of the parameter signals to the estimate of the gradient of the objective function can be attributed to the vector $\mathbf{Q}_1$; see (18). This contribution may be removed by setting $\mathbf{Q}_1$ to zero. Similarly, the use of curvature information of the objective function in the gradient estimation process is solely linked to the disturbance estimates $\hat{\mathbf{w}}$ and $\hat{v}$ in (21). Setting $\hat{\mathbf{w}}$ and $\hat{v}$ to zero (that is, setting $\mathbf{H}(\hat{\mathbf{u}}) = \mathbf{0}$) eliminates the use of curvature information. We present the following two examples to illustrate the effects of the nominal part of the parameter signals and the use of curvature information on the performance of the extremum-seeking controller in Section 3.

5.1 Example 1: nominal part of the parameter signals

Let us consider the system:

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t)x_2^2(t) + (1+u(t))^2 \\ \dot{x}_2(t) &= -x_2(t) + \frac{1}{1+u^2(t)} \\ y(t) &= x_1(t) + 1, \end{aligned} \tag{44}$$

[2] The boundedness of $\tilde{p}_1$ and $\tilde{\mathbf{p}}_2$ in (C.1) is used to prove the boundedness of $\hat{m}_1$ and $\hat{\mathbf{m}}_2$ for $0 \leq t \leq t_3$; see Appendix C.

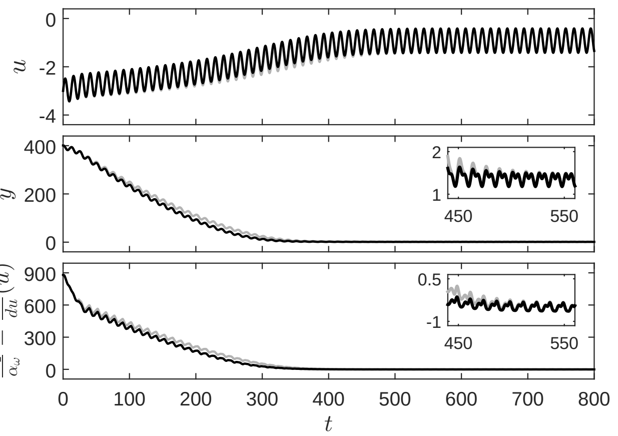

Fig. 2. Plant parameter u, performance measurement y and gradient estimation error $\frac{\hat{m}_2}{\alpha_\omega} - \frac{dF}{du}(\hat{u})$ as a function of time for $\mathbf{Q}_1$ in (18) (black) and $\mathbf{Q}_1 = \mathbf{0}$ (gray), and $\alpha_\omega = 0.5$.

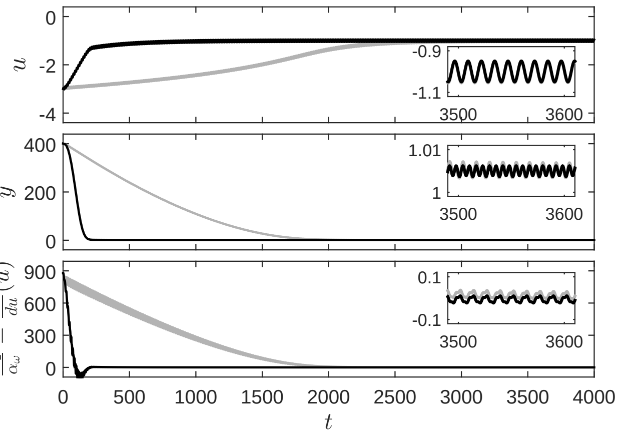

Fig. 3. Plant parameter u, performance measurement y and gradient estimation error $\frac{\hat{m}_2}{\alpha_\omega} - \frac{dF}{du}(\hat{u})$ as a function of time for $\mathbf{Q}_1$ in (18) (black) and $\mathbf{Q}_1 = \mathbf{0}$ (gray), and $\alpha_\omega = 0.05$.

with state $\mathbf{x} = [x_1, x_2]^T$ and objective function $F(u) = (1+u)^2(1+u^2)^2 + 1$. To investigate the effect of the nominal part of the parameter signal on the performance of the controller, consider the constant tuning parameters $\alpha_\omega = 0.5$, $\eta_\omega = 0.5$, $\eta_\mathbf{m} = 0.3$, $\eta_\mathbf{u} = 0.01$ and $\lambda_\mathbf{u} = 0.03$ ($r_\alpha = r_\omega = r_\mathbf{m} = r_\mathbf{u} = r_\lambda = 0$). Moreover, let $\hat{\mathbf{w}} = \mathbf{0}$ and $\hat{v} = 0$. The parameter signal, the performance measurement and the gradient-estimation error with and without the use of the nominal part of the parameter signal (that is, $\mathbf{Q}_1$ in (18) and $\mathbf{Q}_1 = \mathbf{0}$, respectively) are depicted in Fig. 2. The parameter u converges towards its performance-optimizing value $u^* = -1$ for both cases. Resultantly, the performance measurements converge towards the minimum $F(u^*) = 1$. For this relatively large perturbation amplitude $\alpha_\omega = 0.5$, the contribution of the nominal part of the parameter signal is minor. Because the differences between the gradient estimates are small, we observe in Fig. 2 that the results for both controllers are comparable. Now, if we decrease the perturbation amplitude to $\alpha_\omega = 0.05$, the parameter signal is no longer dominated by the perturbations. Fig. 3 displays that the obtained gradient estimate is more accurate if the nominal part of the parameter signal is used by the gradient estimator in addition to the perturbations. As a result, a much faster convergence to the steady-state optimum is obtained, which matches the observations in [6, 8]. If the plant remains close to steady state, adding the nominal part of the parameter to the estimation process generally has a positive effect on the gradient estimate and allows for a higher gain selection. However, additional simulations (not presented here) indicate that incorporating the nominal part of the parameter signals may result in a worse performance if the plant is not close to steady state, which is commonly the case for the high-amplitude high-frequency methods in [4, 15, 23, 29], for example.

5.2 Example 2: curvature information of the objective function

Curvature information of the objective function acts as feedforward for the estimation of the gradient of the objective function. It may especially improve the gradient estimate if the time scales of the observer and the optimizer of the controller are close (see Section 4), in which case the observer has relatively little time to correct its estimate based on the feedback provided by the performance measurement. To illustrate this, consider the system:

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + u_1(t) - u_2(t)\\ \dot{x}_2(t) &= -2x_2(t) + 4x_1(t)u_1(t)\\ \dot{x}_3(t) &= -x_3(t) + u_2(t) - 3\\ y(t) &= x_2(t) + 2x_3(t)u_2(t),\end{aligned} \tag{45}$$

with state $\mathbf{x} = [x_1, x_2, x_3]^T$, input $\mathbf{u} = [u_1, u_2]^T$ and objective function $F(\mathbf{u}) = 2u_1^2 + 2u_2^2 - 2u_1u_2 - 6u_2$. We select $r_0 = 50$, $r_\alpha = r_\omega = 0.45$, $r_\mathbf{m} = 0.5$, $r_\mathbf{u} = 1$, $r_\lambda = 0.1$, $\alpha_\omega(0) = 0.2$, $\eta_\omega(0) = 0.5$, $\eta_\mathbf{m}(0) = 0.04$, $\eta_\mathbf{u}(0) = 0.03$ and $\lambda_\mathbf{u}(0) = 10$. We consider the cases $\mathbf{H}(\hat{\mathbf{u}}) = \frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}})$ and $\mathbf{H}(\hat{\mathbf{u}}) = \mathbf{0}$. The trajectories of the plant parameters are illustrated in Fig. 4. Fig. 5 displays the corresponding distance to the optimal values $\mathbf{u}^* = [1,2]^T$, the performance measurement and the Euclidean norm of the gradient estimation error. With curvature information (that is, $\mathbf{H}(\hat{\mathbf{u}}) = \frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}})$), the plant parameters converge to the performance-optimizing values $\mathbf{u}^*$ as time goes to infinity, while the gradient estimation error remains relatively small throughout the optimization process. Without curvature information (that is, $\mathbf{H}(\hat{\mathbf{u}}) = \mathbf{0}$), the response to changes in the direction of the gradient is slow and the plant parameters overshoot the performance-optimal values several times before settling, leading to a slower convergence. The overshoot may be prevented by decreasing the initial optimizer gains (that is, the values of $\eta_\mathbf{u}(0)$ and $\lambda_\mathbf{u}(0)$). However, lowering the optimizer gains hampers the overall convergence speed of extremum-seeking scheme. Although curvature information may be used to enhance the estimate of the gradient of the objective function, doing so requires that the estimate of the Hessian of the objective function is reasonably accurate. Additional simulations show that a bad estimate of the Hessian may impair the gradient estimate instead.

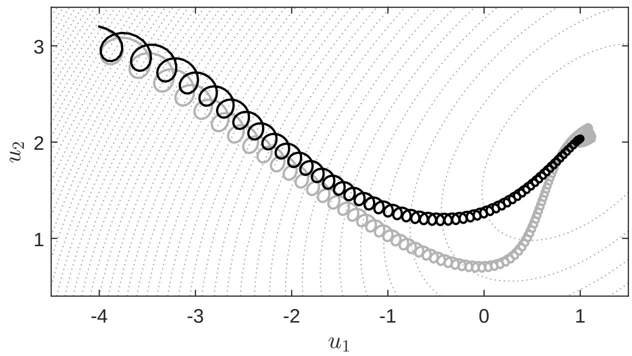

Fig. 4. Trajectories of the plant parameters $\mathbf{u} = [u_1, u_2]^T$ for $\mathbf{H}(\hat{\mathbf{u}}) = \frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}})$ (black) and $\mathbf{H}(\hat{\mathbf{u}}) = \mathbf{0}$ (gray).

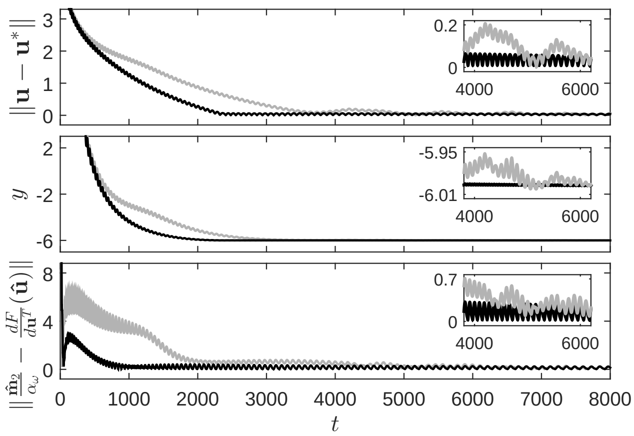

Fig. 5. Euclidean norm of parameter error $\|\mathbf{u} - \mathbf{u}^*\|$, performance measurement y and Euclidean norm of gradient-estimation error $\|\frac{\hat{\mathbf{m}}_2}{\alpha_\omega} - \frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}})\|$ as a function of time for $\mathbf{H}(\hat{\mathbf{u}}) = \frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}})$ (black) and $\mathbf{H}(\hat{\mathbf{u}}) = \mathbf{0}$ (gray).

6 Conclusion

We have presented an extremum-seeking controller for which the contribution of the nominal part of the parameter signals to the gradient estimate of the objective function can be isolated to study its influence. Simulations display that including the nominal part of the parameter signals in the estimation process helps to improve the accuracy of the gradient estimate if the perturbation-related content of the plant-parameter signals is low, and if the state of the plant remains close to steady state. In turn, a more accurate estimate may enhance the convergence speed and may allow for a higher gain selection. In addition, we have shown that incorporating curvature information of the objective function may further improve the accuracy of the gradient estimate, especially if the time scales of the observer and optimizer of the controller are close.

Appendices

A Proof of Lemma 8

Consider the Lyapunov-function candidate

$$V_{\mathbf{Q}1}(\mathbf{Q}_1) = \|\mathbf{Q}_1\|^2. \tag{A.1}$$

From (18) and (A.1), it follows that

$$\dot{V}_{\mathbf{Q}1}(\mathbf{Q}_1) = -2\eta_{\mathbf{m}}\|\mathbf{Q}_1\|^2 + 2g_\alpha\|\mathbf{Q}_1\|^2 + 2\mathbf{Q}_1^T\frac{\dot{\hat{\mathbf{u}}}}{\alpha_\omega}. \tag{A.2}$$

It follows from (23) that $\|\dot{\hat{\mathbf{u}}}\| \leq \eta_{\mathbf{u}}$. Also, from (10), (24) and (25)-(27), we have that there exists a constant $L_g \in \mathbb{R}_{>0}$ such that $g_\alpha(t) \leq \alpha_\omega(t)\lambda_{\mathbf{u}}(t)L_g$ for all $t \geq 0$. From Young's inequality, (A.1), (A.2), $\|\dot{\hat{\mathbf{u}}}\| \leq \eta_{\mathbf{u}}$ and $g_\alpha \leq \alpha_\omega\lambda_{\mathbf{u}}L_g$, we obtain

$$\dot{V}_{\mathbf{Q}1}(\mathbf{Q}_1) \leq -\eta_{\mathbf{m}}\left(1 - \frac{2\alpha_\omega\lambda_{\mathbf{u}}}{\eta_{\mathbf{m}}}L_g\right)V_{\mathbf{Q}1}(\mathbf{Q}_1) + \frac{\eta_{\mathbf{u}}^2}{\alpha_\omega^2\eta_{\mathbf{m}}}. \tag{A.3}$$

Using the comparison lemma [12, Lemma 3.4], (A.1) and (A.3), it is not difficult to show that, for any initial condition $\mathbf{Q}_1(0) \in \mathbb{R}^{n_{\mathbf{u}}}$, $\mathbf{Q}_1(t)$ remains bounded for all $0 \leq t \leq t_1$. We get from (A.3) that

$$\dot{V}_{\mathbf{Q}1}(\mathbf{Q}_1) \leq -\eta_{\mathbf{m}}(1 - 2L_g\varepsilon_4)V_{\mathbf{Q}1}(\mathbf{Q}_1) + \eta_{\mathbf{m}}\varepsilon_3^2 \tag{A.4}$$

for all $t \geq t_1$, all $\eta_{\mathbf{u}} \leq \alpha_\omega\eta_{\mathbf{m}}\varepsilon_3$ and all $\alpha_\omega\lambda_{\mathbf{u}} \leq \eta_{\mathbf{m}}\varepsilon_4$. Applying the comparison lemma [12, Lemma 3.4] yields

$$V_{\mathbf{Q}1}(\mathbf{Q}_1(t)) \leq \max\left\{2V_{\mathbf{Q}1}(\mathbf{Q}_1(t_1))e^{-\int_{t_1}^t \frac{1}{2}\eta_{\mathbf{m}}(\tau)d\tau}, \frac{1}{16}\right\} \tag{A.5}$$

for all $t \geq t_1$ and sufficiently small values of ε_3 and ε_4. Because $\mathbf{Q}_1(t_1)$ is bounded, we obtain from (A.1) and (A.5) that $\mathbf{Q}_1(t)$ is bounded for all $t \geq t_1$. Moreover, it follows from (A.1) and (A.5) that the inequality in (33) holds for all $t \geq t_2$, where $t_2 \geq t_1$ is sufficiently large.

B Proof of Lemma 9

We define the Lyapunov-function candidate

$$V_{\mathbf{Q}2}(\tilde{\mathbf{Q}}_2) = \operatorname{tr}\left(\tilde{\mathbf{Q}}_2^2\right), \tag{B.1}$$

which can be bounded by

$$\|\tilde{\mathbf{Q}}_2\|^2 \leq V_{\mathbf{Q}2}(\tilde{\mathbf{Q}}_2) \leq n_{\mathbf{u}}\|\tilde{\mathbf{Q}}_2\|^2. \tag{B.2}$$

From (20) and (30), it follows that its time derivative is given by

$$\begin{aligned}\dot{V}_{\mathbf{Q}2}(\tilde{\mathbf{Q}}_2) = &-2\eta_{\mathbf{m}}\operatorname{tr}\left(\tilde{\mathbf{Q}}_2^2\right) + 4g_\alpha\operatorname{tr}\left(\tilde{\mathbf{Q}}_2^2\right)\\ &+ 2g_\alpha\operatorname{tr}\left(\tilde{\mathbf{Q}}_2\right) + 2\eta_{\mathbf{m}}\operatorname{tr}\left(\tilde{\mathbf{Q}}_2\mathbf{Q}_1\mathbf{Q}_1^T\right).\end{aligned} \tag{B.3}$$

As in the proof of Lemma 8, we note that there exists a constant $L_g \in \mathbb{R}_{>0}$ such that $g_\alpha(t) \leq \alpha_\omega(t)\lambda_{\mathbf{u}}(t)L_g$ for all $t \geq 0$. Using Young's inequality, (B.1) and (B.3), we get

$$\begin{aligned}\dot{V}_{\mathbf{Q}2}(\tilde{\mathbf{Q}}_2) \leq &-\eta_{\mathbf{m}}\left(1 - \frac{5\alpha_\omega\lambda_{\mathbf{u}}}{\eta_{\mathbf{m}}}L_g\right)V_{\mathbf{Q}2}(\tilde{\mathbf{Q}}_2)\\ &+ \alpha_\omega\lambda_{\mathbf{u}}L_g n_{\mathbf{u}} + \eta_{\mathbf{m}}\|\mathbf{Q}_1\|^4.\end{aligned} \tag{B.4}$$

Because $\mathbf{Q}_1$ is bounded for all $t \geq 0$ (see Lemma 8), from the comparison lemma [12, Lemma 3.4], (B.2) and (B.4), it follows that, for any initial condition $\tilde{\mathbf{Q}}_2(0) \in \mathbb{R}^{n_{\mathbf{u}} \times n_{\mathbf{u}}}$, $\tilde{\mathbf{Q}}_2(t)$ remains bounded for all $0 \leq t \leq t_2$. Similarly, from the comparison lemma [12, Lemma 3.4], (B.2), (B.4) and the bound on $\mathbf{Q}_1$ for $t \geq t_2$ in Lemma 8, we obtain

$$V_{\mathbf{Q}2}(\tilde{\mathbf{Q}}_2(t)) \leq \max\left\{2V_{\mathbf{Q}2}(\tilde{\mathbf{Q}}_2(t_2))e^{-\int_{t_2}^t \frac{1}{2}\eta_{\mathbf{m}}(\tau)d\tau}, \frac{1}{16}\right\} \tag{B.5}$$

for all for $t \geq t_2$ and all $\alpha_\omega\lambda_{\mathbf{u}} \leq \eta_{\mathbf{m}}\varepsilon_4$, where ε_4 is sufficiently small. Because $\tilde{\mathbf{Q}}_2(t_2)$ is bounded, it follows from (B.2) and (B.5) that $\tilde{\mathbf{Q}}_2(t)$ is bounded for all $t \geq t_2$. Moreover, we obtain from (B.2) and (B.5) that the inequality in (34) holds for all $t \geq t_3$, where $t_3 \geq t_2$ is a sufficiently large constant.

C Proof of Lemma 11

To show that the solutions of $\tilde{m}_1(t)$ and $\tilde{\mathbf{m}}_2(t)$ are bounded for all $0 \leq t \leq t_3$, consider the variables

$$\begin{aligned}\tilde{p}_1 &= \hat{m}_1 - m_1 - \eta_{\mathbf{m}}k_1,\\ \tilde{\mathbf{p}}_2 &= \hat{\mathbf{m}}_2 - \mathbf{m}_2 - \eta_{\mathbf{m}}\mathbf{Q}_2(\mathbf{Q}_1k_1 + \mathbf{k}_2).\end{aligned} \tag{C.1}$$

The time derivatives of $\tilde{p}_1$ and $\tilde{\mathbf{p}}_2$ are given by

$$\begin{aligned}\dot{\tilde{p}}_1 = &-\eta_{\mathbf{m}}\tilde{p}_1 - \alpha_\omega^2\mathbf{Q}_1^T\tilde{\mathbf{w}} - \alpha_\omega^2\eta_{\mathbf{m}}\tilde{v} + \eta_{\mathbf{m}}\omega^T\mathbf{m}_2\\ &+ \eta_{\mathbf{m}}z + (g_\alpha - \eta_{\mathbf{m}})\eta_{\mathbf{m}}k_1,\\ \dot{\tilde{\mathbf{p}}}_2 = &-g_\alpha\tilde{\mathbf{p}}_2 + \alpha_\omega^2\tilde{\mathbf{w}} - \eta_{\mathbf{m}}\mathbf{Q}_2\dot{\mathbf{Q}}_1 - \eta_{\mathbf{m}}\mathbf{Q}_2(\mathbf{Q}_1 + \omega)\Big(p_1 + k_1\\ &+ (\mathbf{Q}_1 + \omega)^T(\mathbf{p}_2 + \eta_{\mathbf{m}}\mathbf{Q}_2(\mathbf{Q}_1k_1 + \mathbf{k}_2)) + \alpha_\omega^2\tilde{v} - z\Big)\\ &+ (g_{\mathbf{m}} - g_\alpha)\eta_{\mathbf{m}}\mathbf{Q}_2(\mathbf{Q}_1k_1 + \mathbf{k}_2) - \eta_{\mathbf{m}}\dot{\mathbf{Q}}_2(\mathbf{Q}_1k_1 + \mathbf{k}_2),\end{aligned} \tag{C.2}$$

with $\tilde{\mathbf{w}} = \hat{\mathbf{w}} - \mathbf{w}$ and $\tilde{v} = \hat{v} - v$. Similar to the proof of [10, Lemma 10], from Lemmas 8-10, the conditions in Theorem 7, and the definitions and assumptions in this work, we obtain that all signals in the right-hand sides of (C.2), except $\tilde{p}_1$ and $\tilde{\mathbf{p}}_2$, are bounded on the compact time interval $[0, t_3]$. Moreover, because $\tilde{p}_1$ and $\tilde{\mathbf{p}}_2$ appear linearly in the right-hand sides of (C.2), we conclude from (C.2) that $\tilde{p}_1(t)$ and $\tilde{\mathbf{p}}_2(t)$ are bounded for all $0 \leq t \leq t_3$. This can be formally proved using a Lyapunov approach with candidate function $\tilde{p}_1^2 + \tilde{\mathbf{p}}_2^T\mathbf{Q}_2^{-1}\tilde{\mathbf{p}}_2$. The proof is omitted for brevity. Subsequently, it follows from (30) and the boundedness of $\tilde{p}_1$, $\tilde{\mathbf{p}}_2$, $\eta_{\mathbf{m}}$, $\frac{\eta_{\mathbf{m}}}{\eta_\omega}$, $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{m}_2$ and $\mathbf{Q}_2$ that $\tilde{m}_1(t)$ and $\tilde{\mathbf{m}}_2(t)$ are also bounded for all $0 \leq t \leq t_3$.

The prove the remaining part of the lemma, we introduce the Lyapunov-function candidate

$$V_{\mathbf{m}}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}_2)=\tilde{m}_1^2+\tilde{\mathbf{m}}_2^T\mathbf{Q}_2^{-1}\tilde{\mathbf{m}}_2. \qquad \text{(C.3)}$$

Following similar steps as in the proof of [10, Lemma 9], it is not difficult to show that, for any symmetric positive-definite $\mathbf{Q}_2(0)\in\mathbb{R}^{n_\mathbf{u}\times n_\mathbf{u}}$, $\mathbf{Q}_2^{-1}(t)$ remains positive definite and bounded for all $t\geq 0$. In addition, it follows from (30) and Lemma 9 that

$$\frac{1}{4}\mathbf{I}\preceq\mathbf{Q}_2^{-1}(t)\preceq\frac{3}{4}\mathbf{I} \qquad \text{(C.4)}$$

for all $t\geq t_3$. Therefore, the function $V_\mathbf{m}$ can be bounded by

$$\begin{aligned}&\max\left\{|\tilde{m}_1|^2,\frac{1}{4}\|\tilde{\mathbf{m}}_2\|^2\right\}\leq V_\mathbf{m}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}_2)\\&\quad\leq\max\left\{2|\tilde{m}_1|^2,\frac{3}{2}\|\tilde{\mathbf{m}}_2\|^2\right\},\end{aligned} \qquad \text{(C.5)}$$

for all $t\geq t_3$. The time derivative of the function $V_\mathbf{m}$ is given by

$$\begin{aligned}\dot{V}_\mathbf{m}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}_2)&=-\eta_\mathbf{m}\tilde{m}_1^2-\eta_\mathbf{m}\tilde{\mathbf{m}}_2^T\mathbf{Q}_2^{-1}\tilde{\mathbf{m}}_2-\frac{\eta_\mathbf{m}}{2}\|\tilde{\mathbf{m}}_2\|^2\\&-\eta_\mathbf{m}\left(\tilde{m}_1+\mathbf{Q}_1^T\tilde{\mathbf{m}}_2\right)^2+2\eta_\mathbf{m}\tilde{m}_1e_1+2\eta_\mathbf{m}\tilde{\mathbf{m}}_2^T\mathbf{Q}_2^{-1}\mathbf{e}_2,\end{aligned} \qquad \text{(C.6)}$$

where e_1 and $\mathbf{e}_2$ are defined in (C.5), with $\mathbf{J}_1=\mathbf{Q}_1\mathbf{l}_1^T+\mathbf{l}_1\mathbf{Q}_1^T+\mathbf{l}_2$, $\mathbf{J}_2=\mathbf{I}+\frac{\eta_\mathbf{m}}{\eta_\omega}\mathbf{Q}_2\mathbf{J}_1$, $\mathbf{J}_3=\frac{1}{\eta_\mathbf{m}}\left(\dot{\mathbf{Q}}_2-(g_\mathbf{m}-g_\omega)\mathbf{Q}_2\right)$, $\mathbf{J}_4=\mathbf{Q}_2\left(\mathbf{J}_1\tilde{\mathbf{m}}_2+\mathbf{l}_1\left(\tilde{m}_1+\frac{\eta_\mathbf{m}}{\eta_\omega}\mathbf{l}_1^T\mathbf{m}_2\right)\right)$ and $\mathbf{J}_5=\mathbf{Q}_1k_1+\mathbf{k}_2$. From Young's inequality, (C.3) and (C.6), we get

$$\begin{aligned}\dot{V}_\mathbf{m}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}_2)&\leq-\frac{\eta_\mathbf{m}}{2}V_\mathbf{m}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}_2)\\&\quad+4\eta_\mathbf{m}e_1^2+4\eta_\mathbf{m}\|\mathbf{Q}_2^{-1}\|\|\mathbf{e}_2\|^2.\end{aligned} \qquad \text{(C.8)}$$

Similar to the proof of Lemma 8, we note that there exists a constant $L_g\in\mathbb{R}_{>0}$ such that $g_\alpha(t)\leq\alpha_\omega(t)\lambda_\mathbf{u}(t)L_g$, $g_\omega(t)\leq\alpha_\omega(t)\lambda_\mathbf{u}(t)L_g$, $g_\mathbf{m}(t)\leq\alpha_\omega(t)\lambda_\mathbf{u}(t)L_g$ for all $t\geq 0$. By applying these bounds and similar bounds to those in the proof of [10, Lemma 10], we obtain from Lemmas 8-10, the conditions in Theorem 7, and the definitions and assumptions in this work that there exist constants $L_{\mathbf{e}i1},L_{\mathbf{e}i2},\ldots,L_{\mathbf{e}i7}\in\mathbb{R}_{>0}$ such that

$$\begin{aligned}\|\mathbf{e}_i\|&\leq\frac{\alpha_\omega\lambda_\mathbf{u}}{\eta_\mathbf{m}}L_{\mathbf{e}i1}V_\mathbf{m}^{\frac{1}{2}}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}_2)+\alpha_\omega^2L_{\mathbf{e}i2}+\alpha_\omega\eta_\omega L_{\mathbf{e}i3}\\&\quad+\alpha_\omega\eta_\omega L_{\mathbf{e}i4}\|\tilde{\mathbf{u}}\|+\frac{\alpha_\omega\eta_\mathbf{m}}{\eta_\omega}L_{\mathbf{e}i5}\|\tilde{\mathbf{u}}\|+\frac{\alpha_\omega^2\lambda_\mathbf{u}}{\eta_\mathbf{m}}L_{\mathbf{e}i6}\|\tilde{\mathbf{u}}\|\\&\quad+\eta_\mathbf{m}L_{\mathbf{e}i7}(q_d+q_{\omega d})\end{aligned} \qquad \text{(C.9)}$$

for $i\in\{1,2\}$, all $t\geq t_3$, all $\eta_\omega\leq\varepsilon_1$, all $\eta_\mathbf{m}\leq\eta_\omega\varepsilon_2$, all $\eta_\mathbf{u}\leq\alpha_\omega\eta_\mathbf{m}\varepsilon_3$ and all $\alpha_\omega\lambda_\mathbf{u}\leq\eta_\mathbf{m}\varepsilon_4$. where we assume without loss of generality that ε_2 is sufficiently small such that $\mathbf{J}_2^{-1}$ is well-defined and bounded for all $t\geq t_3$ and all $\eta_\mathbf{m}\leq\eta_\omega\varepsilon_2$. Now, assuming that ε_4 is sufficiently small, we obtain from (C.4), (C.8) and (C.9) that there exist constants $c_{\mathbf{m}1},c_{\mathbf{m}2},\ldots,c_{\mathbf{m}6}\in\mathbb{R}_{>0}$ such that

$$\begin{aligned}\dot{V}_\mathbf{m}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}_2)&\leq-\frac{\eta_\mathbf{m}}{4}V_\mathbf{m}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}_2)\\&+\frac{\eta_\mathbf{m}}{8}\max\Bigg\{\alpha_\omega^4c_{\mathbf{m}1},\alpha_\omega^2\eta_\omega^2c_{\mathbf{m}2},\alpha_\omega^2\eta_\omega^2c_{\mathbf{m}3}\|\tilde{\mathbf{u}}\|^2,\\&\frac{\alpha_\omega^2\eta_\mathbf{m}^2}{\eta_\omega^2}c_{\mathbf{m}4}\|\tilde{\mathbf{u}}\|^2,\frac{\alpha_\omega^4\lambda_\mathbf{u}^2}{\eta_\mathbf{m}^2}c_{\mathbf{m}5}\|\tilde{\mathbf{u}}\|^2,\eta_\mathbf{m}^2c_{\mathbf{m}6}(q_d+q_\omega)^2\Bigg\}\end{aligned} \qquad \text{(C.10)}$$

for all $t\geq t_3$, all $\eta_\omega\leq\varepsilon_1$, all $\eta_\mathbf{m}\leq\eta_\omega\varepsilon_2$, all $\eta_\mathbf{u}\leq\alpha_\omega\eta_\mathbf{m}\varepsilon_3$ and all $\alpha_\omega\lambda_\mathbf{u}\leq\eta_\mathbf{m}\varepsilon_4$. The bounds in (36)-(38) follow from (C.5) and (C.10) (with $\gamma_{\mathbf{m}5}=\frac{\eta_\mathbf{m}}{8}$).

References

[1] K. B. Ariyur and M. Krstić. *Real-time optimization by extremum-seeking control*. Wiley-Interscience, Hoboken, NJ, 2003.

[2] K. T. Atta, A. Johansson, and T. Gustafsson. Extremum seeking control based on phasor estimation. *Systems & Control Letters*, 85:37–45, 2015.

[3] D. W. Clark and K. R. Godfrey. Simultaneous estimation of first and second derivatives of a cost function. *Electronic Letters*, 2(9):338–339, 1966.

[4] H.-B. Dürr, M. S. Stanković, C. Ebenbauer, and K. H. Johansson. Lie bracket approximation of extremum seeking systems. *Automatica*, 49(6):1538–1552, 2013.

[5] V. W. Eveleigh. *Adaptive control and optimization techniques*. McGraw-Hill, London, 1967.

[6] G. Gelbert, J. P. Moeck, O. Paschereit, and R. King. Advanced algorithms for gradient estimation in one- and two-parameter extremum seeking controllers. *Journal of Process Control*, 22(4):700–709, 2012.

[7] A. Ghaffari, M. Krstić, and D. Nešić. Multivariable Newton-based extremum seeking. *Automatica*, 48(8):1759–1767, 2012.

[8] M. Guay and D. J. Burns. A comparison of extremum seeking algorithms applied to vapor compression system optimization. In *Proceedings of the 2014 American Control Conference (ACC)*, pages 1076–1081, Portland, Oregon, USA, June 4-6, 2014.

[9] M. Guay and D. Dochain. A time-varying extremum-seeking control approach. *Automatica*, 51:356–363, 2015.

[10] M. Haring and T. A. Johansen. Asymptotic stability of perturbation-based extremum-seeking control for nonlinear plants. *IEEE Transactions on Automatic Control*, 62(5):2302–2317, 2017.

[11] M. Haring, N. van de Wouw, and D. Nešić. Extremum-seeking control for nonlinear systems with periodic steady-state outputs. *Automatica*, 49(6):1883–1891, 2013.

[12] H. K. Khalil. *Nonlinear systems, 3rd edition*. Prentice Hall, Upper Saddle River, NJ, 2002.

$$
\begin{aligned}
e_1 &= -\frac{\alpha_\omega^2}{\eta_{\mathbf{m}}}\mathbf{Q}_1^T\tilde{\mathbf{w}} - \alpha_\omega^2\tilde{v} + z + (g_{\mathbf{m}} - \eta_{\mathbf{m}})k_1 + (g_{\mathbf{m}} - g_\omega - \eta_{\mathbf{m}})\frac{1}{\eta_\omega}\mathbf{l}_1^T\mathbf{m}_2 - \frac{1}{\eta_\omega}\mathbf{l}_1^T\dot{\mathbf{m}}_2, \\
\mathbf{e}_2 &= \frac{\eta_{\mathbf{m}}}{\eta_\omega}\left(\mathbf{J}_3\mathbf{J}_1 + \frac{1}{\eta_{\mathbf{m}}}\mathbf{Q}_2\left(\dot{\mathbf{Q}}_1\mathbf{l}_1^T + \mathbf{l}_1\dot{\mathbf{Q}}_1^T\right)\right)\left(\tilde{\mathbf{m}}_2 - \frac{\eta_{\mathbf{m}}}{\eta_\omega}\mathbf{J}_2^{-1}\mathbf{J}_4\right) - \frac{\eta_{\mathbf{m}}^2}{\eta_\omega^2}\mathbf{Q}_2\left(\mathbf{Q}_1\omega^T + \omega\mathbf{Q}_1^T + \omega\omega^T - \frac{1}{2}\mathbf{I}\right)\mathbf{J}_2^{-1}\mathbf{J}_4 \\
&\quad + \frac{\eta_{\mathbf{m}}}{\eta_\omega}\mathbf{J}_3\mathbf{l}_1\left(\tilde{m}_1 + \frac{\eta_{\mathbf{m}}}{\eta_\omega}\mathbf{l}_1^T\mathbf{m}_2\right) + \frac{\eta_{\mathbf{m}}}{\eta_\omega}\mathbf{Q}_2\left(\omega\mathbf{l}_1^T + \mathbf{l}_1\omega^T\right)\mathbf{m}_2 + \frac{\eta_{\mathbf{m}}}{\eta_\omega}\mathbf{Q}_2\mathbf{l}_1\left(-\tilde{m}_1 + e_1 - (g_{\mathbf{m}} - g_\omega)\frac{1}{\eta_\omega}\mathbf{l}_1^T\mathbf{m}_2 + \frac{1}{\eta_\omega}\mathbf{l}_1^T\dot{\mathbf{m}}_2\right) \\
&\quad + \frac{g_\alpha}{\eta_\omega}\mathbf{J}_4 - (g_\alpha - g_{\mathbf{m}})\mathbf{J}_2\mathbf{Q}_2\mathbf{J}_5 - \mathbf{J}_2\dot{\mathbf{Q}}_2\mathbf{J}_5 - \mathbf{J}_2\mathbf{Q}_2\dot{\mathbf{Q}}_1k_1 - \alpha_\omega^2\mathbf{J}_2\tilde{\mathbf{w}} + \mathbf{J}_2\mathbf{Q}_2\left(\mathbf{Q}_1 + \omega\right)\left(z - \alpha_\omega^2\tilde{v} - \eta_{\mathbf{m}}k_1 + \frac{\eta_{\mathbf{m}}}{\eta_\omega}\mathbf{l}_1^T\mathbf{m}_2\right) \\
&\quad + \frac{\eta_{\mathbf{m}}}{\eta_\omega}\mathbf{J}_2\mathbf{Q}_2\left(\mathbf{Q}_1 + \omega\right)\left(\mathbf{Q}_1 + \omega\right)^T\left(\mathbf{J}_2^{-1}\mathbf{J}_4 - \eta_{\mathbf{m}}\mathbf{Q}_2\mathbf{J}_5\right) - \frac{\eta_{\mathbf{m}}}{\eta_\omega}\mathbf{Q}_2\mathbf{J}_1\left(\frac{g_\alpha}{\eta_{\mathbf{m}}}\tilde{\mathbf{m}}_2 + \mathbf{Q}_2\left(\mathbf{Q}_1 + \omega\right)\left(\tilde{m}_1 + \left(\mathbf{Q}_1 + \omega\right)^T\tilde{\mathbf{m}}_2\right)\right)
\end{aligned}
\tag{C.7}
$$

[13] S. Z. Khong, D. Nešić, Y. Tan, and C. Manzie. Unified frameworks for sampled-data extremum seeking control: global optimisation and multi-unit systems. *Automatica*, 49(9):2720–2733, 2013.

[14] M. Krstić and H.-H. Wang. Stability of extremum seeking feedback for general nonlinear dynamic systems. *Automatica*, 36(4):595–601, 2000.

[15] W. H. Moase and C. Manzie. Fast extremum-seeking for Wiener-Hammerstein plants. *Automatica*, 48(10):2433–2443, 2012.

[16] W. H. Moase, C. Manzie, and M. J. Brear. Newton-like extremum-seeking for the control of thermoacoustic instability. *IEEE Transactions on Automatic Control*, 55(9):2094–2105, 2010.

[17] W. H. Moase, Y. Tan, D. Nešić, and C. Manzie. Non-local stability of a multi-variable extremum-seeking scheme. In *Proceedings of the 2011 Australian Control Conference*, pages 38–43, Melbourne, Australia, November 10-11, 2011.

[18] S. J. Moura and Y. A. Chang. Lyapunov-based switched extremum seeking for photovoltaic power maximization. *Control Engineering Practice*, 21(7):971–980, 2013.

[19] D. Nešić, Y. Tan, C. Manzie, A. Mohammadi, and W. Moase. A unifying framework for analysis and design of extremum seeking controllers. In *Proceedings of the 24th Chinese Control and Decision Conference*, Taiyuan, May 23-25, 2012.

[20] D. Nešić, Y. Tan, W. H. Moase, and C. Manzie. A unifiying approach to extremum seeking: adaptive schemes based on estimation of derivatives. In *Proceedings of the 49th IEEE Conference on Decision and Control*, pages 4625–4630, Atlanta, GA, USA, December 15-17, 2010.

[21] J. I. Poveda and A. R. Teel. A framework for a class of hybrid extremum seeking controllers with dynamic inclusions. *Automatica*, 76:113–126, 2017.

[22] J. J. Ryan and J. L. Speyer. Peak-seeking control using gradient and Hessian estimates. In *Proceedings of the 2010 American Control Conference*, Baltimore, MD, USA, June 30-July 2, 2010.

[23] A. Scheinker and M. Krstić. *Model-free stabilization by extremum seeking*. SpringerBriefs in Electrical and Computer Engineering. Springer, 2017.

[24] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995.

[25] M. S. Stanković and D. M. Stipanović. Extremum seeking under stochastic noise and applications to mobile sensors. *Automatica*, 46(8):1243–1251, 2010.

[26] Y. Tan, W. H. Moase, C. Manzie, D. Nešić, and I. M. Y. Mareels. Extremum seeking from 1922 to 2010. In *Proceedings of the 29th Chinese Control Conference*, Beijing, China, July 29-31, 2010.

[27] Y. Tan, D. Nešić, and I. M. Y. Mareels. On non-local stability properties of extremum seeking control. *Automatica*, 42(6):889–903, 2006.

[28] L. Wang, S. Chen, and K. Ma. On stability and application of extremum seeking control without steady-state oscillation. *Automatica*, 68:18–26, 2016.

[29] C. Zhang, A. Siranosian, and M. Krstić. Extremum seeking for moderately unstable systems and for autonomous vehicle target tracking witwith position measurements. *Automatica*, 43(10):1832–1839, 2007.