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Two results in Harmonic Analysis and PDEs

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NTNU
Norwegian University of
Science and Technology
Faculty of Information Technology,
Mathematics and Electrical Engineering
Department of Mathematical Sciences

NTNU – Trondheim
Norwegian University of
Science and Technology

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Norwegian University of
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Thesis for the degree of Philosophiae Doctor

Trondheim, May 2014

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Preface

This doctoral thesis consists of two articles:

Qifan Li, *Local Well-Posedness for the Periodic Korteweg-de Vries equation in Analytic Gevrey Classes*, Communications on Pure and Applied Analysis, 1097-1109, Issue 3, May 2012.

Verena Bögelein and Qifan Li, *Very weak solutions of degenerate parabolic systems with non-standard $p(x,t)$-growth*, Nonlinear Analysis: Theory, Methods Applications, 190-225, Volume 98, March 2014. (The second author did the major part of the work.)

In these papers we use Harmonic analysis technique to study the partial differential equations. In the first paper we use multilinear analysis to study the nonlinear term of KdV equations. In the second paper, we apply the technique of Whitney extensions and strong maximal functions to construct a proper test functions to prove the higher integrability of the very weak solutions of degenerate parabolic systems with non-standard $p(x,t)$-growth.

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I would like to express my deep appreciation and sincere gratitude to my supervisor Professor Peter Lindqvist for his advise and to Professor Verena Bögelein for her collaboration in the writing of the second paper.

I would also like to thank Professor Henrik Kalisch for the discussion of the analyticity solution of dispersive equations.

Finally, I would like to offer my deep gratitute to the Institut Mittag-Leffler who provided financial support and a pleasant working atmosphere during the Fall of 2013.

Trondheim,

March 3rd, 2014

Qifan Li

Summary and Conclusions

In the first paper, we study local well-posedness of the Cauchy problem for the generalized periodic Korteweg-deVries equation (GKdV)

$$\begin{cases} \partial_t u + \partial^3_{xxx} u + u^k \partial_x u = 0 \qquad u: \ \mathbb{T} \times [0, T] \to \mathbb{R} \\ u(x,0) = u_0(x) \quad x \in \mathbb{T} \end{cases} \tag{1}$$

We prove that, in the case $s \geq 1$ and $k \geq 1$, for initial data $u_0(x)$ in $G^{\sigma,s}$, $\sigma > 0$, there exists a small positive time T, such that the initial-value problem (1) is well-posed in the space $C([0,T], G^{\sigma,s})$.

In the second paper, we consider the degenerate parabolic systems whose model is the parabolic $p(x,t)$ -Laplacian system,

$$\partial_t u - \operatorname{div}\big(|Du|^{p(x,t)-2} Du\big) = \operatorname{div}\big(|F|^{p(x,t)-2} F\big)$$

in the degenerate range, i.e. $p(x,t) \geq 2$. We show that any very weak solution $u : \Omega \times (0,T) \to \mathbb{R}^N$ with $|Du|^{p(\cdot)(1-\varepsilon)} \in L^1$ belongs to the natural energy space, i.e. $|Du|^{p(\cdot)} \in L^1_{\text{loc}}$, provided $\epsilon > 0$ is small enough.

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LOCAL WELL-POSEDNESS FOR THE PERIODIC KORTEWEG-DE VRIES EQUATION IN ANALYTIC GEVREY CLASSES

Qifan Li

Department of Mathematics
University of Bergen
5008 Bergen, Norway

(Communicated by Igor Kukavica)

Abstract. Motivated by the work of Grujić and Kalisch, [Z. Grujić and H. Kalisch, *Local well-posedness of the generalized Korteweg-de Vries equation in spaces of analytic functions*, Differential and Integral Equations **15** (2002) 1325–1334], we prove the local well-posedness for the periodic KdV equation in spaces of periodic functions analytic on a strip around the real axis without shrinking the width of the strip in time.

1 Introduction

This paper studies the local well-posedness of the Cauchy problem for the generalized periodic Korteweg-deVries equation (GKdV)

$$\begin{cases} \partial_t u + \partial^3_{xxx} u + u^k \partial_x u = 0 & u : \ \mathbb{T} \times [0,T] \to \mathbb{R} \\ u(x,0) = u_0(x) & x \in \mathbb{T} \end{cases} \tag{1}$$

with initial data $u_0(x)$ in a class of periodic functions analytic in a symmetric strip around the real axis. The number k is taken to be a positive integer and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the torus. For $\sigma > 0$, $s \in \mathbb{R}$, denote Gevrey classes $G^{\sigma,s}$ to be the subset of $L^2(\mathbb{T})$ such that

$$\|u_0\|^2_{G^{\sigma,s}} = \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} e^{2\sigma\langle n\rangle} |\widehat{u_0}(n)|^2 < \infty$$

where $\langle n\rangle := 1 + |n|$ and $\widehat{u_0}(n)$ denotes the Fourier transform of u_0 on torus.

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In [18], Kato and Masuda introduced a method of obtaining spatial analyticity of solution for a large class of semi-linear evolution equations, and the research on Gevrey regularity for the solution of the semi-linear equations goes back to the work of Foias and Temam [10]. Further results concerning periodic solutions of Navier-Stokes equations in Gevrey spaces have been obtained by Biswas [1]. We refer to [2, 12] for the study of Kuramoto-Sivashinsky equation. For a treatment of a more general case of nonlinear parabolic equations, we refer the reader to [9]. Also, a number of authors have obtained solutions in Gevrey spaces without strong regularizing effects. Here we mention the recent work of Kukavica and Vicol on the three-dimensional Euler equations [21], and a body of work concerning KdV-like equations (see, for example, Hayashi [14, 15], Bouard et al. [5], Grujić and Kalisch [13], Bona et al. [4]). As explained in [3, 16, 17], analyticity of solution of the KdV equation plays an essential role in the numerical study of the equation.

The example constructed in [11] shows that the solution of GKdV equation with an appropriate analytic data may not be analytic in the time variable t. So, we must restrict our attention to the spatial analyticity of the solution of GKdV. Grujić and Kalisch [13] proved local well-posedness of non-periodic GKdV for a strip without shrinking the width of the strip in time. It is of interest to know whether it is possible to establish the same result for the periodic case.

Kato's smoothing effect was shown to be useful in the proof of the main theorem in [13]. However, this technique cannot be used in dealing with GKdV with periodic boundary data. Our approach is in the spirit of [8, Theorem 1] and the proof relies on the Bourgain's bilinear estimate [6], multilinear estimate in [22] and linear estimates in [7, 8]. In addition, the proof reveals some new aspects in the estimation of the time-cutoff function which are essential in the proof of the main nonlinear estimate which is given in Lemma 3.2.

Denote by $C([0,T], G^{\sigma,s})$ the space of continuous functions from the time interval $[0,T]$ into $G^{\sigma,s}$. We will prove the following theorem.

Theorem 1.1. *Let $s \geq 1$ and $k \geq 1$. For initial data in $G^{\sigma,s}$, $\sigma > 0$, there exists a small positive time T, such that the initial-value problem (1) is well-posed in the space $C([0,T], G^{\sigma,s})$.*

The paper is organized as follows. In Section 2, we set up notations and terminologies and deal with linear estimates. Section 3 is devoted to the study of bilinear estimates, and Section 4 provides a proof of the multilinear estimate. In Section 5, Theorem 1.1 is proved via a contraction argument.

2 Some Notations and Linear Estimates

Throughout this paper, $A \lesssim B$ denotes the estimate $A \le CB$, where the constant $C > 0$ possibly depending on s, k and *independent* of σ. We say that $A \approx B$, if $A \lesssim B$ and $B \lesssim A$. We also denote by $A \ll B$ the estimate $A \lesssim \frac{1}{K} B$ for a large constant $K > 0$. The Lebesgue classes on the integer set and real line are denoted by l^p and L^q respectively, while the following notation is used to denote the $l^p - L^q$ space-time norms: $\|f(n,\lambda)\|_{l^p_n L^q_\lambda} = \| \|f(n,\lambda)\|_{L^q_\lambda} \|_{l^p_n}$.

Let $u(x,t)$ be a function defined on the cylinder $\mathbb{T} \times \mathbb{R}$ and $s,\ b \in \mathbb{R}$. The space-time Fourier transform of $u(x,t)$ is defined by

$$\hat{u}(n,\lambda) = \int_{\mathbb{R}} \int_{\mathbb{T}} u(x,t) e^{-2\pi i \lambda t - 2\pi i n x} dx dt,$$

where $n \in \mathbb{Z}$. We denote by $\mathscr{F}_t[u(x,t)]$ the partial Fourier transform of u in variable t and by $\mathscr{F}_x[u(x,t)]$ the partial Fourier transform in variable x. We define the $X^{s,b} = X^{s,b}_{\tau=\xi^3}(\mathbb{T} \times \mathbb{R})$ norm of $u(x,t)$ by

$$\|u\|_{X^{s,b}} = \left\| \langle \lambda - n^3 \rangle^b \langle n \rangle^s \hat{u}(n,\lambda) \right\|_{l^2_n L^2_\lambda},$$

where $\langle \cdot \rangle := 1 + |\cdot|$. This norm was introduced by Bourgain [6] and the space-time symbol is adapted to the linear part of KdV equation.

The low-regularity study of (1) is usually considered in spaces $X^{s,\frac{1}{2}}$ (see [6, 8, 22]). In order to overcome difficulty in persistence property in this case, authors [8] and [22] introduced the function space $Y^{s,b}$ to be the subset of $X^{s,b}$ such that

$$\|u\|_{Y^{s,b}} = \|u\|_{X^{s,b}} + \|\langle n \rangle^s \hat{u}(n,\lambda)\|_{l^2_n L^1_\lambda} < \infty.$$

It is indicated in [13] that we have to introduce another family of function spaces which are adapted to the study of Gevrey regularity. For $\sigma \ge 0$, define $X^{\sigma,s,b}$ norm of $u(x,t)$ by

$$\|u\|_{X^{\sigma,s,b}} = \left\| \langle \lambda - n^3 \rangle^b \langle n \rangle^s e^{\sigma \langle n \rangle} \hat{u}(n,\lambda) \right\|_{l^2_n L^2_\lambda}.$$

We shall use the space $Y^{\sigma,s,b}$ which equipped with the norm

$$\|u\|_{Y^{\sigma,s,b}} = \|u\|_{X^{\sigma,s,b}} + \left\| e^{\sigma \langle n \rangle} \langle n \rangle^s \hat{u}(n,\lambda) \right\|_{l^2_n L^1_\lambda}.$$

By the Riemann-Lebesgue lemma, the Fourier transform of an L^1 function is continuous and bounded, and we have the embedding property

$$Y^{\sigma,s,b} \subset C([0,T], G^{\sigma,s}) \subset L^\infty([0,T], G^{\sigma,s}). \tag{2}$$

We will also need the space $Z^{\sigma,s,b}$ with the norm defined by

$$\|u\|_{Z^{\sigma,s,b}} = \|u\|_{X^{\sigma,s,-b}} + \left\| \frac{e^{\sigma \langle n \rangle} \langle n \rangle^s}{\langle \lambda - n^3 \rangle} \hat{u}(n,\lambda) \right\|_{l^2_n L^1_\lambda}.$$

Consider initial value problem of the Airy equation on $\mathbb{T}$:

$$\begin{cases} \partial_t w + \partial^3_{xxx} w = 0 \\ \qquad w(x,0) = w_0(x), \ x \in \mathbb{T}. \end{cases} \tag{3}$$

The explicit solution of the initial value problem (3) can be expressed in terms of the semigroup $S(t)$ via Fourier transform,

$$w(x,t) = S(t)w_0 = c\sum_{n\in\mathbb{Z}} e^{2\pi i(xn+tn^3)}\widehat{w_0}(n).$$

We shall establish linear estimates for the propagator $S(t)$. Let $\psi(t)$ be a bump function supported in $[-2,2]$ and equal to one on $[-1,1]$. Denote by $0<\delta<1$ a small constant which need to be determined later.

Lemma 2.1. *We have*

$$\|\psi(t/\delta)S(t)u_0\|_{Y^{\sigma,s,\frac{1}{2}}} \lesssim \|u_0\|_{G^{\sigma,s}}$$

for all $s\in\mathbb{R}$ and $\sigma\geq 0$.

Proof. Let us first write $\widehat{\psi(t/\delta)S(t)}u_0(n,\lambda) = \widehat{u_0}(n)\delta\hat{\psi}(\delta(\lambda-n^3))$. By the definition of $X^{\sigma,s,b}$,

$$\|\psi(t/\delta)S(t)u_0\|^2_{X^{\sigma,s,\frac{1}{2}}} = \sum_n e^{2\sigma\langle n\rangle}\langle n\rangle^{2s}|\widehat{u_0}(n)|^2\int_{\mathbb{R}}\langle\lambda\rangle\delta^2|\hat{\psi}(\delta\lambda)|^2 d\lambda.$$

Since $\int_{\mathbb{R}}\langle\lambda\rangle\delta^2|\hat{\psi}(\delta\lambda)|^2 d\lambda \lesssim 1+\delta$, we get $\|\psi(t/\delta)S(t)u_0\|_{X^{\sigma,s,\frac{1}{2}}} \lesssim \|u_0\|_{G^{\sigma,s}}$. On the other hand, we see at once that $\left\|e^{\sigma\langle n\rangle}\langle n\rangle^s\widehat{\psi(t/\delta)S(t)}u_0\right\|^2_{l^2_nL^1_\lambda} \lesssim \|u_0\|^2_{G^{\sigma,s}}$, which completes the proof. □

Having established Lemma 2.1, we repeat the proof of [8, Lemma 3.1], and we get Lemma 2.2.

Lemma 2.2. *We have*

$$\left\|\psi(t/\delta)\int_0^t S(t-t')F(t')dt'\right\|_{Y^{\sigma,s,\frac{1}{2}}} \lesssim \|F\|_{Z^{\sigma,s,\frac{1}{2}}}$$

for all $s\in\mathbb{R}$, $\sigma\geq 0$ and test functions F on $\mathbb{T}\times\mathbb{R}$.

We also need to estimate the cutoff function $\psi(t/\delta)u$ in the space $X^{\sigma,s,\frac{1}{2}}$. We present a proof in a spirit of [20, Lemma 3.2].

Lemma 2.3. *Let $\sigma\geq 0$. We have*

$$\|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} \lesssim \|u\|_{Y^{\sigma,s,\frac{1}{2}}}$$

for all $s\in\mathbb{R}$ and $\sigma\geq 0$.

Proof. By the definition of $Y^{\sigma,s}$, the proof is reduced to showing that, if $a = n^3$ then

$$\int_{\mathbb{R}} |\hat{u}*_\lambda(\delta\hat{\psi}(\delta\lambda))(l)|^2\langle l-a\rangle dl \lesssim \int_{\mathbb{R}} |\hat{u}(n,\lambda)|^2\langle\lambda-a\rangle d\lambda + \|\hat{u}(n,\lambda)\|_{L^1_\lambda}^2 \quad (4)$$

where $*_\lambda$ is the convolution in variable λ.

According to the proof of [20, Lemma 3.2], we have

$$\begin{aligned}&\int_{\mathbb{R}} |\hat{u} *_\lambda (\delta\hat{\psi}(\delta\lambda))(l)|^2\langle l-a\rangle dl\\ &\lesssim \int_{\mathbb{R}} |e^{2\pi i a t}\mathscr{F}_x[u](n,t)\partial_t^{\frac{1}{2}}\psi(\delta^{-1}t)|^2 dt + \int_{\mathbb{R}} |\hat{u}(n,\lambda)|^2|\lambda-a|d\lambda\end{aligned}$$

and

$$\int_{\mathbb{R}} |\hat{u} *_\lambda (\delta\hat{\psi}(\delta\lambda))(l)|^2 dl \lesssim \int_{\mathbb{R}} |\hat{u}(n,\lambda)|^2 d\lambda.$$

By the Plancherel theorem and the Young inequality,

$$\begin{aligned}\int_{\mathbb{R}} |e^{2\pi i a t}\mathscr{F}_x[u](n,t)\partial_t^{\frac{1}{2}}\psi(\delta^{-1}t)|^2 dt &= \left\|\widehat{e^{2\pi i n^3 t}}u(n,\lambda) *_\lambda \widehat{\partial_t^{\frac{1}{2}}}\psi(\delta^{-1}t)(\lambda)\right\|_{L^2_\lambda}^2\\ &\le \left\|\hat{u}(n,\lambda-n^3)\right\|_{L^1_\lambda}^2 \left\|\lambda^{\frac{1}{2}}\delta\hat{\psi}(\delta\lambda)\right\|_{L^2_\lambda}^2\\ &\lesssim \|\hat{u}(n,\lambda)\|_{L^1_\lambda}^2,\end{aligned}$$

which shows (4), and the proof of Lemma 2.3 is completed. □

3 Bilinear Estimates

The bilinear estimate is a standard technique in dealing with nonlinear term in the equation. This kind of technique has been used and developed by many authors (See, for instance [6, 13, 19, 23]).

Lemma 3.1. *Let $s \ge 0$, $\sigma \ge 0$, and suppose the functions u, v satisfy $\int_{\mathbb{T}} u dx = 0$ and $\int_{\mathbb{T}} v dx = 0$. Assume that $\|v\|_{Y^{\sigma,s,\frac{1}{2}}} < \infty$ and $\|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} < \infty$. Then*

$$\left\|\psi(t/\delta)^2\partial_x(uv)\right\|_{X^{\sigma,s,-\frac{1}{2}}} \lesssim \delta^{\frac{1}{12}}\|v\|_{Y^{\sigma,s,\frac{1}{2}}}\|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}}.$$

Proof. The main idea of the proof is due to Bourgain [6, page 221].

Since $\int_{\mathbb{T}} u = 0$ and $\int_{\mathbb{T}} v = 0$, we write

$$\begin{aligned}f(n,\lambda) &= \langle\lambda-n^3\rangle^{\frac{1}{2}}|n|^s e^{\sigma\langle n\rangle}|\widehat{\psi(t/\delta)u}(n,\lambda)|,\\ g(n,\lambda) &= \langle\lambda-n^3\rangle^{\frac{1}{2}}|n|^s e^{\sigma\langle n\rangle}|\widehat{\psi(t/\delta)v}(n,\lambda)|.\end{aligned}$$

Let $h(n,\lambda)\in l_n^2L_\lambda^2$ and $\|h\|_{l_n^2L_\lambda^2}\le 1$, we introduce a trilinear form:

$$\Lambda(f,g,h)=\sum_{n\neq 0}\sum_{\substack{n_1\neq 0,\\ n_1\neq n}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{\sigma\langle n\rangle}e^{-\sigma\langle n-n_1\rangle}e^{-\sigma\langle n_1\rangle}h(n,\lambda)f(n_1,\lambda_1)}{\langle\lambda-n^3\rangle^{\frac12}\langle\lambda_1-n_1^3\rangle^{\frac12}} \times\frac{g(n-n_1,\lambda-\lambda_1)|n|^{s+1}|n_1|^{-s}|n-n_1|^{-s}}{\langle\lambda-\lambda_1-(n-n_1)^3\rangle^{\frac12}}d\lambda d\lambda_1.$$

Thus we need only to estimate $\Lambda(f,g,h)$.

Since $|n|\lesssim|n_1||n-n_1|$ and $e^{\sigma|n|}e^{-\sigma|n-n_1|}e^{-\sigma|n_1|}\le 1$, we obtain

$$|\Lambda(f,g,h)|\lesssim\sum_{n\neq 0}\sum_{\substack{n_1\neq 0,\\ n_1\neq n}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|f(n_1,\lambda_1)g(n-n_1,\lambda-\lambda_1)h(n,\lambda)||n|d\lambda_1}{\langle\lambda-n^3\rangle^{\frac12}\langle\lambda_1-n_1^3\rangle^{\frac12}\langle\lambda-\lambda_1-(n-n_1)^3\rangle^{\frac12}}.$$

From resonance identity $n^3=(n-n_1)^3+n_1^3+3nn_1(n-n_1)$, we get

$$\max\left\{|\lambda-\lambda_1-(n-n_1)^3|,\ |\lambda_1-n_1^3|,\ |\lambda-n^3|\right\}\ge|n||n_1||n-n_1|. \tag{5}$$

As pointed out in [6, Theorem 7.41], we have

$$|\Lambda(f,g,h)|\lesssim\|FG\|_{L_x^2L_t^2}\|h\|_{l_n^2L_\lambda^2}\qquad\text{if}\quad|\lambda-n^3|\gtrsim n^2,$$
$$|\Lambda(f,g,h)|\lesssim\|G\|_{L_x^4L_t^4}\|H\|_{L_x^4L_t^4}\|f\|_{l_n^2L_\lambda^2}\qquad\text{if}\quad|\lambda_1-n_1^3|\gtrsim n^2,$$

where $\hat F(n,\lambda)=f(n,\lambda)\langle\lambda-n^3\rangle^{-\frac12}$, $\hat G(n,\lambda)=g(n,\lambda)\langle\lambda-n^3\rangle^{-\frac12}$ and $\hat H(n,\lambda)=h(n,\lambda)\langle\lambda-n^3\rangle^{-\frac12}$. Let us focus on the first of the above cases. Recalling that $\|h\|_{l_n^2L_\lambda^2}\le 1$ by assumption, and using Cauchy-Schwarz, it appears that we have to estimate the terms $\|F\|_{L_x^4L_t^4}$ and $\|G\|_{L_x^4L_t^4}$. Recalling the Strichartz estimate [6, Proposition 7.15]

$$\|F\|_{L_x^4L_t^4}\lesssim\|F\|_{X^{0,\frac13}} \tag{6}$$

it becomes plain that the terms $\|F\|_{X^{0,\frac13}}$ and $\|G\|_{X^{0,\frac13}}$ have to be controlled. To this end, define a a square-integrable function

$$\theta(x,t)=|\partial_x|^se^{\sigma(I+|\partial_x|)}u(x,t)=\mathscr{F}_x^{-1}\left[|n|^se^{\sigma\langle n\rangle}\mathscr{F}_xu\right](x,t)$$

where I denotes the identity operator. We also set

$$\hat\vartheta(n,\lambda)=|n|^se^{\sigma\langle n\rangle}\widehat{\psi(t/\delta)}u(n,\lambda)=\mathscr{F}_t\left[\psi(t/\delta)(\mathscr{F}_x\theta)(n,t)\right](\lambda).$$

Using the Strichartz estimate (6) for the function $\psi(t/\delta)\theta$ yields

$$\begin{aligned}\sum_{n\neq 0}\int_{\mathbb{R}}|\hat\vartheta(n,\lambda)|^2d\lambda&=\int_{\mathbb{R}}\int_{\mathbb{T}}\chi_{[-3,3]}(t/\delta)\left|\psi(t/\delta)\theta(x,t)\right|^2dxdt\\&\lesssim\delta^{\frac12}\|\psi(t/\delta)\theta\|_{L_x^4L_t^4}^2\lesssim\delta^{\frac12}\|\psi(t/\delta)\theta\|_{X^{0,\frac13}}^2\\&=\delta^{\frac12}\|\psi(t/\delta)u\|_{X^{\sigma,s,\frac13}}^2.\end{aligned}\tag{7}$$

By Hölder's inequality and (7), we get

$$\begin{aligned}\|F\|^2_{X^{0,\frac{1}{3}}} &= \sum_{n\neq 0}\int_{\mathbb{R}} \langle \lambda - n^3\rangle^{-\frac{1}{3}} f(n,\lambda)^2 d\lambda \\ &= \sum_{n\neq 0}\int_{\mathbb{R}} \left(|\widehat{\psi(t/\delta)u}(n,\lambda)|^2 |n|^{2s} e^{2\sigma\langle n\rangle}\right)^{\frac{1}{3}} f(n,\lambda)^{\frac{4}{3}} d\lambda \qquad (8) \\ &\leq \delta^{\frac{1}{6}} \|\psi(t/\delta)u\|^2_{X^{\sigma,s,\frac{1}{2}}} .\end{aligned}$$

Making use of the argument above, we deduce

$$\|G\|_{X^{0,\frac{1}{3}}} \lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)v\|_{X^{\sigma,s,\frac{1}{2}}} \lesssim \delta^{\frac{1}{12}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}}$$

from Lemma 2.3. Thus the estimate in the case $|\lambda - n^3| \gtrsim n^2$ may be continued as follows.

$$\|FG\|_{L^2_x L^2_t} \leq \|F\|_{L^4_x L^4_t} \|G\|_{L^4_x L^4_t} \lesssim \delta^{\frac{1}{12}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} . \qquad (9)$$

For the case when $|\lambda_1 - n_1^3| \gtrsim n^2$, we use the Strichartz estimate (6) to find $\|H\|_{L^4_x L^4_t} \lesssim \|h\|_{l^2_n L^2_\lambda} \leq 1$. Recalling the definition of $f(n,\lambda)$, a similar argument yields as in the previous case yields

$$\|G\|_{L^4_x L^4_t} \|f\|_{l^2_n L^2_\lambda} \lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}} \qquad (10)$$

Finally, interchanging f and g, we obtain

$$|\Lambda(f,g,h)| \lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}} \qquad (11)$$

for the case $|\lambda - \lambda_1 - (n-n_1)^3| \gtrsim n^2$ by symmetry. Now based on (9)-(11), we have

$$\begin{aligned}\left\|\partial_x\left(\psi(t/\delta)^2 uv\right)\right\|_{X^{\sigma,s,-\frac{1}{2}}} &= \sup_{\|h\|_{l^2_n L^2_\lambda}\leq 1} |\Lambda(f,g,h)| \\ &\lesssim \delta^{\frac{1}{12}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} .\end{aligned}$$

□

Remark 1. Note we have actually proved that

$$\|\partial_x(uv)\|_{X^{\sigma,s,-\frac{1}{2}}} \lesssim \|u\|_{X^{\sigma,s,\frac{1}{2}}} \|v\|_{X^{\sigma,s,\frac{1}{2}}} \qquad (12)$$

for $s \geq 0$ and $\sigma \geq 0$.

The bilinear estimate for periodic KdV equation in Sobolev spaces with negative indices has been studied by Kenig, Ponce and Vega [19]. As the counterexample shows in [19, Theorem 1.4], the boundedness of the quadratic term fails for Sobolev indices below $-\frac{1}{2}$.

Corollary 1. *For functions u, v satisfying $\int_{\mathbb{T}} u = 0$, $\int_{\mathbb{T}} v = 0$, the estimate (12) holds for $s \geq -\frac{1}{2}$.*

Proof. According to the above remark, we only need to consider the case $-\frac{1}{2} \le s \le 0$. Let $\rho = -s \ge 0$, we follow the definition of multiplier bounds which was introduced by Tao [23]. It remains to show that

$$\left\| \frac{e^{\sigma\langle n\rangle} e^{-\sigma\langle n-n_1\rangle} e^{-\sigma\langle n_1\rangle} |n|^{1-\rho}|n_1|^{\rho}|n-n_1|^{\rho}}{\langle \lambda - n^3\rangle^{\frac{1}{2}} \langle \lambda_1 - n_1^3\rangle^{\frac{1}{2}} \langle \lambda - \lambda_1 - (n-n_1)^3\rangle^{\frac{1}{2}}} \right\|_{[3,\mathbb{Z}\times\mathbb{R}]} \lesssim 1.$$

Since $e^{\sigma|n|}e^{-\sigma|n-n_1|}e^{-\sigma|n_1|} \le 1$, the comparison principle [23, Lemma 3.1] reduce this estimate to

$$\left\| \frac{|n|^{1-\rho}|n_1|^{\rho}|n-n_1|^{\rho}}{\langle \lambda - n^3\rangle^{\frac{1}{2}} \langle \lambda_1 - n_1^3\rangle^{\frac{1}{2}} \langle \lambda - \lambda_1 - (n-n_1)^3\rangle^{\frac{1}{2}}} \right\|_{[3,\mathbb{Z}\times\mathbb{R}]} \lesssim 1,$$

which has been proved by Kenig, Ponce and Vega [19, Theorem 1.2]. □

In order to estimate the bilinear term in space of $Z^{\sigma,s,\frac{1}{2}}$, it will necessary to analyze the proof of [8, Proposition 1]. We will prove the following result in analogy with discussions in [8, Proposition 1].

Lemma 3.2. *Let $s \ge \frac{1}{2}$, $\sigma \ge 0$, $\int_{\mathbb{T}} u dx = 0$, $\int_{\mathbb{T}} v dx = 0$ and $0 \le \kappa \ll 1$. Assume that $\int_{\mathbb{T}} uv dx = 0$, $\|v\|_{Y^{\sigma,s-1,\frac{1}{2}}} < \infty$ and $\|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} < \infty$. Then*

$$\left\| \frac{\langle n\rangle^s e^{\sigma\langle n\rangle} \widehat{\psi(t/\delta)^2 uv}(n,\lambda)}{\langle \lambda - n^3\rangle^{1-\kappa}} \right\|_{l_n^2 L_\lambda^1} \lesssim \delta^{\frac{1}{200}} \|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} \|v\|_{Y^{\sigma,s-1,\frac{1}{2}}}.$$

Proof. Since $\int_{\mathbb{T}} uv = 0$, the quantity $\langle n\rangle^s$ can be replaced with $|n|^s$ in the left hand side of the estimate. Let square-integrable functions u_1 and u_2 be defined by

$$\widehat{u_1}(n,\lambda) = \langle \lambda - n^3\rangle^{\frac{1}{2}} |n|^{s-1} e^{\sigma\langle n\rangle} \widehat{\psi(t/\delta) v}(n,\lambda)$$
$$\widehat{u_2}(n,\lambda) = \langle \lambda - n^3\rangle^{\frac{1}{2}} |n|^{s-1} e^{\sigma\langle n\rangle} \widehat{\psi(t/\delta) u}(n,\lambda).$$

Since $e^{\sigma|n|}\le e^{\sigma|n_1|}e^{\sigma|n-n_1|}$ and $|n|^{s-\frac12}\le|n-n_1|^{s-\frac12}|n_1|^{s-\frac12}$, we obtain

$$\begin{aligned}
&\frac{|n|^s e^{\sigma\langle n\rangle}|\widehat{\psi(t/\delta)^2uv}(n,\lambda)|}{\langle\lambda-n^3\rangle^{1-\kappa}}\\
&\le\sum_{\substack{n_1\neq0,\\ n_1\neq n}}\int_{\mathbb{R}}\frac{|\widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)|d\lambda_1}{\langle\lambda-n^3\rangle^{1-\kappa}\langle\lambda_1-n_1^3\rangle^{\frac12}\langle\lambda-\lambda_1-(n-n_1)^3\rangle^{\frac12}}\\
&\qquad\times\frac{e^{\sigma\langle n\rangle}|n|^s}{e^{\sigma\langle n_1\rangle}e^{\sigma\langle n-n_1\rangle}|n_1|^{s-1}|n-n_1|^{s-1}}\\
&\le\sum_{\substack{n_1\neq0,\\ n_1\neq n}}\int_{\mathbb{R}}\frac{|\widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)|d\lambda_1}{\langle\lambda-n^3\rangle^{1-\kappa}\langle\lambda_1-n_1^3\rangle^{\frac12}\langle\lambda-\lambda_1-(n-n_1)^3\rangle^{\frac12}}\\
&\qquad\times|n|^{\frac12}|n_1|^{\frac12}|n-n_1|^{\frac12}\\
&:=S(n,\lambda).
\end{aligned}$$

To estimate $\|S(n,\lambda)\|_{l^2_nL^1_\lambda}$ we note that the resonance relation (5) enables us to distinguish three cases once again.

If $|\lambda-\lambda_1-(n-n_1)^3|\ge|n||n_1||n-n_1|$, $S(n,\lambda)$ can be dominated by

$$S(n,\lambda)\le\sum_{\substack{n_1\neq0,\\ n_1\neq n}}\int_{\mathbb{R}}\frac{|\widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)|d\lambda_1}{\langle\lambda-n^3\rangle^{\frac23-\kappa}\langle\lambda-n^3\rangle^{\frac13}\langle\lambda_1-n_1^3\rangle^{\frac12}}.$$

Taking first the L^1_λ-norm, using the Cauchy-Schwarz inequality, and recognizing that $\int_{\mathbb{R}}|\langle\lambda-n^3\rangle^{-\frac23+\kappa}|^2d\lambda$ is finite, it follows from duality that

$$\begin{aligned}\|S(n,\lambda)\|_{l^2_nL^1_\lambda}\lesssim\sup_{\|\widehat{u_3}\|_{l^2_nL^2_\lambda}\le1}\sum_{n,n_1}\int_{\mathbb{R}^2}&\widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)\langle\lambda_1-n_1^3\rangle^{-\frac12}\\
&\times\widehat{u_3}(n,\lambda)\langle\lambda-n^3\rangle^{-\frac13}d\lambda_1d\lambda.\end{aligned}\tag{13}$$

Now define $\widehat{u_2'}(n_1,\lambda_1)=\widehat{u_2}(n_1,\lambda_1)\langle\lambda_1-n_1^3\rangle^{-\frac12}$ and $\widehat{u_3'}(n,\lambda)=\overline{\widehat{u_3}(n,\lambda)\langle\lambda-n^3\rangle^{-\frac13}}$. Note that from (6) and (8), we gain the estimates

$$\|u_2'\|_{L^4_xL^4_t}\lesssim\|u_2'\|_{X^{0,\frac13}}\lesssim\delta^{\frac1{12}}\|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac12}}\tag{14}$$

and

$$\|u_3'\|_{L^4_xL^4_t}\lesssim\|u_3'\|_{X^{0,\frac13}}=\|\widehat{u_3}\|_{l^2_nL^2_\lambda}.\tag{15}$$

Thus, using Parseval's relation, (14)-(15) and Lemma 2.3, the estimate takes the form

$$\begin{aligned}\|S(n,\lambda)\|_{l^2_n L^1_\lambda} &\lesssim \sup_{\|\widehat{u_3}\|_{l^2_n L^2_\lambda}\le 1} \int_{\mathbb{T}\times\mathbb{R}} u_1 u_2' u_3' dt dx \\ &\lesssim \sup_{\|\widehat{u_3}\|_{l^2_n L^2_\lambda}\le 1} \|u_1\|_{L^2_x L^2_t}\|u_2'\|_{L^4_x L^4_t}\|u_3'\|_{L^4_x L^4_t} \\ &\lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} \|v\|_{Y^{\sigma,s-1,\frac{1}{2}}}.\end{aligned} \tag{16}$$

By symmetry, we also have

$$\|S(n,\lambda)\|_{l^2_n L^1_\lambda} \lesssim \delta^{\frac{1}{12}}\|v\|_{Y^{\sigma,s-1,\frac{1}{2}}}\|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} \tag{17}$$

for the case $|\lambda_1 - n_1^3| \ge |n||n_1||n-n_1|$.

We now turn to the remaining case $|\lambda - n^3| \ge |n||n_1||n-n_1|$. This will be split into three subcases. Suppose first that we also have

$$|\lambda-\lambda_1-(n-n_1)^3| \gtrsim (\delta|n||n-n_1||n_1|)^{\frac{1}{100}}.$$

Let $\widehat{u_1'}(n-n_1,\lambda-\lambda_1) = \widehat{u_1}(n-n_1,\lambda-\lambda_1)\langle\lambda-\lambda_1-(n-n_1)^3\rangle^{-\frac{1}{3}}$, and let $\widehat{u_2'}(n_1,\lambda_1) = \widehat{u_2}(n_1,\lambda_1)\langle\lambda_1-n_1^3\rangle^{-\frac{1}{2}}$ as before. Then we deduce that

$$\begin{aligned}S(n,\lambda) &\le \sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \int_{\mathbb{R}} \frac{|n|^{\frac{1}{2}}|n_1|^{\frac{1}{2}}|n-n_1|^{\frac{1}{2}}}{\langle\lambda-n^3\rangle^{1-\kappa}} \frac{\widehat{u_1'}(n-n_1,\lambda-\lambda_1)\widehat{u_2'}(n_1,\lambda_1)}{\langle\lambda-\lambda_1-(n-n_1)^3\rangle^{\frac{1}{6}}} d\lambda_1 \\ &\le \sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \int_{\mathbb{R}} \frac{|n|^{\frac{1}{2}}|n_1|^{\frac{1}{2}}|n-n_1|^{\frac{1}{2}}}{\langle\lambda-n^3\rangle^{1-\kappa}} \frac{\widehat{u_1'}(n-n_1,\lambda-\lambda_1)\widehat{u_2'}(n_1,\lambda_1)}{(\delta|n|\;|n_1|\;|n-n_1|)^{\frac{1}{600}}} d\lambda_1,\end{aligned}$$

and the estimate continues as

$$\begin{aligned}\|S(n,\lambda)\|_{l^2_n L^1_\lambda} \lesssim& \delta^{-\frac{1}{600}} \left\| \langle\lambda-n^3\rangle^{-\frac{1}{2}-\frac{1}{600}+\kappa} \sum_{n_1}\int_{\mathbb{R}} \widehat{u_1'}(n-n_1,\lambda-\lambda_1)\widehat{u_2'}(n_1,\lambda_1)d\lambda_1 \right\|_{l^2_n L^1_\lambda} \\ \lesssim& \delta^{-\frac{1}{600}}\|u_1'u_2'\|_{L^2_x L^2_t} \lesssim \delta^{-\frac{1}{600}}\|u_1'\|_{L^4_x L^4_t}\|u_2'\|_{L^4_x L^4_t}\end{aligned}$$

by using the Cauchy-Schwarz inequality, and the Plancherel theorem in the same way as in the previous case. It follows from (14) and (15) that

$$\|S(n,\lambda)\|_{l^2_n L^1_\lambda} \lesssim \delta^{\frac{1}{12}-\frac{1}{600}}\|v\|_{Y^{\sigma,s-1,\frac{1}{2}}}\|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}}. \tag{18}$$

Similarly, for the second subcase $|\lambda_1 - n_1^3| \gtrsim (\delta|n||n-n_1||n_1|)^{\frac{1}{100}}$, the argument above can be repeated, and (18) holds, as well.

We proceed to consider the third subcase where

$$\max\left\{|\lambda-\lambda_1-(n-n_1)^3|,\ |\lambda_1-n_1^3|\right\} \ll (\delta|n||n_1||n-n_1|)^{\frac{1}{100}}.$$

Since δ is taken to be a small number, we have $|\lambda - n^3| \approx |n||n_1||n-n_1|$. Therefore, it is plain that $\|S(n,\lambda)\|_{L^1_\lambda}$ can be majorized by

$$\sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \int_{\mathcal{A}_2}\int_{\mathcal{A}_1} (|n||n_1||n-n_1|)^{\kappa-\frac{1}{2}}\, \widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)d\lambda_1 d\lambda,$$

where the domain of integration is given by

$$\mathcal{A}_1(n,n_1,\lambda) = \{\lambda_1\in\mathbb{R} \ : \ |\lambda-\lambda_1-(n-n_1)^3| \leq (\delta|n||n_1||n-n_1|)^{\frac{1}{100}}\}$$

and

$$\mathcal{A}_2(n,n_1) = \{\lambda_1\in\mathbb{R} \ : \ |\lambda_1-n_1^3| \leq (\delta|n||n_1||n-n_1|)^{\frac{1}{100}}\}$$

Using the Cauchy-Schwarz inequality in each integral, the last expression is dominated by

$$\delta^{\frac{1}{200}+\frac{1}{200}} \sum_{\substack{n_1\neq 0,\\ n_1\neq n}} (|n||n_1||n-n_1|)^{\kappa-\frac{1}{2}+\frac{1}{200}+\frac{1}{200}} \|\widehat{u_1}(n-n_1,\lambda)\|_{L^2_\lambda}\|\widehat{u_2}(n_1,\lambda)\|_{L^2_\lambda}.$$

Now since $|n||n_1||n-n_1|$ takes only nonzero integer values, we may write

$$\begin{aligned}
&\|S(n,\lambda)\|_{l^2_n L^1_\lambda}\\
\lesssim&\delta^{\frac{1}{100}}\left\|\sum_{n_1}\langle nn_1n-n_1\rangle^{\kappa-\frac{1}{2}+\frac{1}{100}}\|\widehat{u_1}(n-n_1,\lambda)\|_{L^2_\lambda}\|\widehat{u_2}(n_1,\lambda)\|_{L^2_\lambda}\right\|_{l^2_n} \qquad (19)\\
\lesssim&\delta^{\frac{1}{100}}\|\widehat{u_1}(n,\lambda)\|_{l^2_nL^2_\lambda}\|\widehat{u_2}(n,\lambda)\|_{l^2_nL^2_\lambda}.
\end{aligned}$$

Now recalling the definition of $\widehat{u_1}$ and $\widehat{u_2}$, it becomes clear that the estimated can be concluded in the same way as the previous cases. For more details of the last step we refer the reader to [8, page 200]. Combining estimates (16)-(19), we finish the proof of the lemma. □

4 A Multilinear Estimate

We shall use the multilinear estimate in a variant of [22, Lemma 4.2].

Lemma 4.1. *If $k\geq 1$, $s\geq 1$ and $\sigma\geq 0$, then*

$$\left\|\psi(t/\delta)\prod_{i=1}^k u_i\right\|_{X^{\sigma,s-1,\frac{1}{2}}} \lesssim \prod_{i=1}^k \|u_i\|_{Y^{\sigma,s,\frac{1}{2}}}.$$

Proof. Denote $h(n,\lambda)\in l^2_nL^2_\lambda$ and $\|h\|_{l^2_nL^2_\lambda}\leq 1$. We let $\epsilon>0$ be a sufficiently small number, it follows that

$$\left\|\lambda^{\frac{1}{2}-\epsilon}\delta\hat{\psi}(\delta\lambda)\right\|_{L^2_\lambda}\lesssim\delta^\epsilon\lesssim 1, \quad \left\|\lambda^{-\epsilon}\delta\hat{\psi}(\delta\lambda)\right\|_{L^1_\lambda}\lesssim\delta^\epsilon\lesssim 1. \qquad (20)$$

Since $s \geq 1 > \frac{1}{2}$, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\left\|e^{\sigma\langle n\rangle}\hat{u}(n,\lambda)\right\|_{l_n^1L_\lambda^1} &= \sum_n \langle n\rangle^{-s}\langle n\rangle^{s}e^{\sigma\langle n\rangle}\int_{\mathbb{R}}|\hat{u}(n,\lambda)|d\lambda \\ &\lesssim \left\|e^{\sigma\langle n\rangle}\langle n\rangle^{s}\hat{u}(n,\lambda)\right\|_{l_n^2L_\lambda^1} \leq \|u\|_{Y^{\sigma,s,\frac{1}{2}}}.\end{aligned} \tag{21}$$

We will only prove the Lemma 4.1 for $k \geq 3$, since the situation will be simpler when we deal with the case $k = 1$ and $k = 2$. We let $v_3 = \prod_{i=3}^{k} u_i$. Since $e^{\sigma|n|} \leq e^{\sigma|n-n_3|}e^{\sigma|n_3-n_4|}\cdots e^{\sigma|n_{k-1}|}$, it follows from (21) and the Young inequality,

$$\begin{aligned}&\left\|e^{\sigma\langle n\rangle}\widehat{v_3}\right\|_{l_n^1L_\lambda^1} \\ &= \sum_{n,n_3,\ldots,n_{k-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^{k-3}} e^{\sigma\langle n\rangle}|\widehat{u_3}(n-n_3,\lambda-\lambda_3)||\widehat{u_4}(n_3-n_4,\lambda_3-\lambda_4)| \times \cdots \\ &\quad \times |\widehat{u_{k-1}}(n_{k-2}-n_{k-1},\lambda_{k-2}-\lambda_{k-1})||\widehat{u_k}(n_{k-1},\lambda_{k-1})|d\lambda_3\cdots d\lambda_{k-1}d\lambda \\ &\qquad \leq \left\|e^{\sigma\langle n\rangle}\widehat{u_3} * \cdots * e^{\sigma\langle n\rangle}\widehat{u_k}\right\|_{l_n^1L_\lambda^1} \\ &\qquad \leq \prod_{i=3}^{k}\left\|e^{\sigma\langle n\rangle}\widehat{u_i}\right\|_{l_n^1L_\lambda^1} \leq \prod_{i=3}^{k}\|u_i\|_{Y^{\sigma,s,\frac{1}{2}}}.\end{aligned} \tag{22}$$

The multilinear form $\Lambda(h,u_1,u_2,v_3)$ is defined by

$$\begin{aligned}\Lambda(h,u_1,u_2,v_3) = \sum_{n,n_1,n_2}\int_{\mathbb{R}^4} & e^{\sigma\langle n\rangle}\langle\lambda-n^3\rangle^{\frac{1}{2}-\epsilon}\langle n\rangle^{s-1+2\epsilon}|h(n,\lambda)| \\ & \times |\widehat{u_1}(n-n_1,\lambda-\lambda_1)||\widehat{u_2}(n_1-n_2,\lambda_1-\lambda_2)| \\ & \times |\widehat{v_3}(n_2,\lambda_2-\lambda_3)||\delta\hat{\psi}(\delta\lambda_3)|d\lambda_1d\lambda_2d\lambda_3d\lambda\end{aligned}$$

and, consequently,

$$\left\|\psi(t/\delta)\prod_{i=1}^{k}u_i\right\|_{X^{\sigma,s-1+2\epsilon,\frac{1}{2}-\epsilon}} = \sup_{\|h(n,\lambda)\|_{l_n^2L_\lambda^2}\leq 1}\Lambda(h,u_1,u_2,v_3).$$

Let u_1', u_2' and v_3' be square integrable functions such that

$$\widehat{u_1'} = e^{\sigma\langle n\rangle}\widehat{u_1}, \quad \widehat{u_2'} = e^{\sigma\langle n\rangle}\widehat{u_2}, \quad \text{and} \quad \widehat{v_3'} = e^{\sigma\langle n\rangle}\widehat{v_3}.$$

Since $e^{\sigma|n|} \leq e^{\sigma|n-n_1|}e^{\sigma|n_1-n_2|}e^{\sigma|n_2|}$, we have

$$\begin{aligned}&\Lambda(h,u_1,u_2,v_3) \\ &\leq \sum_{n,n_1,n_2}\int_{\mathbb{R}^4}\langle\lambda-n^3\rangle^{\frac{1}{2}-\epsilon}\langle n\rangle^{s-1+2\epsilon}|h(n,\lambda)||\widehat{u_1'}(n-n_1,\lambda-\lambda_1)| \\ &\quad \times |\widehat{u_2'}(n_1-n_2,\lambda_1-\lambda_2)||\widehat{v_3'}(n_2,\lambda_2-\lambda_3)||\delta\hat{\psi}(\delta\lambda_3)|d\lambda_1d\lambda_2d\lambda_3d\lambda\end{aligned} \tag{23}$$

We denote by $\Lambda'(h, u_1', u_2', v_3')$ the right hand side of (23). As in the proof of [22, Lemma 4.2], estimate (20) gives

$$\Lambda'(h, u_1', u_2', v_3') \lesssim \|u_1'\|_{Y^{s,\frac{1}{2}}} \|u_2'\|_{Y^{s,\frac{1}{2}}} \|v_3'\|_{l_n^1 L_\lambda^1}.$$

Combining this estimate with (22) and (23), we get

$$\left\| \psi(t/\delta) \prod_{i=1}^{k} u_i \right\|_{X^{\sigma,s-1,\frac{1}{2}-\epsilon}} \lesssim \prod_{i=1}^{k} \|u_i\|_{Y^{\sigma,s,\frac{1}{2}}}.$$

The Lemma 4.1 follows for $k \geq 3$ by letting $\epsilon \to 0$ and the Fatou lemma. □

5 Proof of Theorem 1.1

It is indicated in [22] and [8] that up to a gauge transform, we can rewrite (1) as follows:

$$\begin{cases} \partial_t u + \partial_{xxx}^3 u + \mathbf{P}(\mathbf{P}(u^k)\partial_x u) = 0 \\ \qquad\qquad u(x,0) = u_0(x), \qquad x \in \mathbb{T}, \end{cases} \tag{24}$$

where $\mathbf{P}$ is the projection operator defined by $\mathbf{P}(u) = u - \int_{\mathbb{T}} u(x,t)dx$. The well-posedness problem of (1) is reduced to consider the initial value problem (24).

Since we have the embedding property (2), it is necessary to use the contraction principle on function space $Y^{\sigma,s,\frac{1}{2}}$. Let $r = \|u_0\|_{G^{\sigma,s}} < \infty$. By Lemma 2.1, there exists a constant $c_1 > 0$ such that

$$\|\psi(t/\delta)S(t)u_0\|_{Y^{\sigma,s,\frac{1}{2}}} \leq c_1 \|u_0\|_{G^{\sigma,s}}.$$

We aim to show that the integral operator

$$\Gamma(u) = \psi(t/\delta)S(t)u_0 - \psi(t/\delta)\int_0^t S(t-t')\psi^2(t'/\delta)\mathbf{P}\left(\mathbf{P}(u^k)\partial_x u\right) dt'$$

is a contraction map on the set $\mathfrak{B} = \{\|u\|_{Y^{\sigma,s,\frac{1}{2}}} \leq 2c_1 r\}$.

It is easy to check that $\partial_x u = \mathbf{P}(\partial_x u)$, $\mathbf{P}\partial_x = \partial_x \mathbf{P}$ and $\|\partial_x v\|_{Y^{\sigma,s-1,\frac{1}{2}}} \approx \|v\|_{Y^{\sigma,s,\frac{1}{2}}}$ for $v \in Y^{\sigma,s,\frac{1}{2}}$ and $\int_{\mathbb{T}} v(x,t)dx = 0$. It follows from Lemma 3.1 and Lemma 4.1 that

$$\begin{aligned} \left\|\psi(t/\delta)^2 \mathbf{P}\left[\mathbf{P}(u^k)\partial_x u\right]\right\|_{X^{\sigma,s,-\frac{1}{2}}} &\approx \left\|\psi(t/\delta)^2 \partial_x \left[\mathbf{P}(u^k)\mathbf{P}(\partial_x u)\right]\right\|_{X^{\sigma,s-1,-\frac{1}{2}}} \\ &\lesssim \delta^{\frac{1}{200}} \|u\|_{Y^{\sigma,s,\frac{1}{2}}} \left\|\psi(t/\delta) u^k\right\|_{X^{\sigma,s-1,\frac{1}{2}}} \\ &\lesssim \delta^{\frac{1}{200}} \|u\|_{Y^{\sigma,s,\frac{1}{2}}}^{k+1}. \end{aligned}$$

On the other hand, by Lemma 2.3, Lemma 3.2 with $\kappa = 0$, and Lemma 4.1,

$$\left\| \frac{\langle n\rangle^s e^{\sigma\langle n\rangle} \widehat{\mathbf{P}(\psi(t/\delta)u^k)\mathbf{P}(\psi(t/\delta)\partial_x u)}(n,\lambda)}{\langle \lambda - n^3\rangle} \right\|_{l^2_n L^1_\lambda} \lesssim \delta^{\frac{1}{200}} \left\|\partial_x u\right\|_{Y^{\sigma,s-1,\frac12}} \left\|\psi(t/\delta)u^k\right\|_{X^{\sigma,s-1,\frac12}}$$
$$\lesssim \delta^{\frac{1}{200}} \|u\|^{k+1}_{Y^{\sigma,s,\frac12}}.$$

Therefore, we have

$$\left\|\psi(t/\delta)^2 \mathbf{P}\left(\mathbf{P}(u^k)\partial_x u\right)\right\|_{Z^{\sigma,s,\frac12}} \lesssim \delta^{\frac{1}{200}} \|u\|^{k+1}_{Y^{\sigma,s,\frac12}}.$$

Combining this estimate with Lemma 2.2, we deduce that there exists a constant $c_2 > 0$ such that

$$\|\Gamma(u)\|_{Y^{\sigma,s,\frac12}} \leq c_1 \|u_0\|_{G^{\sigma,s}} + c_2 \delta^{\frac{1}{200}} \|u\|^{k+1}_{Y^{\sigma,s,\frac12}}.$$

If we take

$$T < \delta < \left(\frac{1}{2^{k+1} c_2 (c_1 r)^k}\right)^{200}$$

then $\Gamma(\mathfrak{B}) \subset \mathfrak{B}$.

We are now in a position to verify that Γ is a contraction. By a similar argument as above, it is not hard to show that

$$\|\Gamma(u) - \Gamma(v)\|_{Y^{\sigma,s,\frac12}} \lesssim \delta^{\frac{1}{200}} \sum_{k-1\leq l\leq k} \|\psi(t/\delta) P_l(u,v)\|_{X^{\sigma,s-1,\frac12}} \|u - v\|_{Y^{\sigma,s,\frac12}},$$

where $P_l(u,v)$ is a homogeneous polynomial of degree l. Since $u, v \in \mathfrak{B}$, there exists a constant $c_3 > 0$ by Lemma 4.1, such that

$$\|\Gamma(u) - \Gamma(v)\|_{Y^{\sigma,s,\frac12}} \leq c_3 \delta^{\frac{1}{200}} r^k \|u - v\|_{Y^{\sigma,s,\frac12}}.$$

If we set

$$T < \delta < \min\left\{ \left(\frac{1}{2^{k+1} c_2 (c_1 r)^k}\right)^{200}, \left(\frac{1}{2 r^k c_3}\right)^{200} \right\},$$

then Γ is a contraction on $\mathfrak{B}$. It follows that Γ has a unique fixed point u in $\mathfrak{B}$ and u solves the initial value problem (1).

To prove continuous dependence on the initial data, suppose u and $\bar{u}$ are solutions corresponding to initial data u_0 and $\bar{u}_0$. Following the argument above, we arrive at

$$\|u - \bar{u}\|_{Y^{\sigma,s,\frac12}} \leq c \|u_0 - \bar{u}_0\|_{G^{\sigma,s}} + \frac{1}{2} \|u - \bar{u}\|_{Y^{\sigma,s,\frac12}}.$$

Combining this inequality with (2), continuous dependence in $C([0,T], G^{\sigma,s})$ of the solution on the initial data in $G^{\sigma,s}$ is immediate, as shown by the estimate

$$\|u-\bar{u}\|_{L^\infty([0,T],G^{\sigma,s})} \le c\|u-\bar{u}\|_{Y^{\sigma,s,\frac{1}{2}}} \le c\|u_0-\bar{u}_0\|_{G^{\sigma,s}}.$$

Remark 2. If we consider the integral operator

$$\Phi(u) = \psi(t)S(t)u_0 - \psi(t)\int_0^t S(t-t')\psi^2(t')\mathbf{P}\left[\mathbf{P}(u^k)\partial_x u\right]dt',$$

from a similar contraction argument and Corollary 1, it is a simple matter to establish the following corollary.

Corollary 2. *Let $s \ge \frac{1}{2}$ when $k=1$ and $s \ge 1$ when $k \ge 2$. The initial-value problem (1) is well-posed in the space $C([0,1],G^{\sigma,s})$ if initial data in $G^{\sigma,s}$, $\sigma > 0$ is sufficiently small.*

Remark 3. Similarly as in the proof of [13, Lemma 6], we can prove the uniqueness of the solution (1) in $C([0,T],G^{\sigma,s})$ when $s > \frac{3}{2}$.

In fact, if $s > \frac{3}{2}$, from Hölder inequality,

$$\begin{aligned}\|\partial_x u\|_{L^\infty_x L^\infty_t} &= \sup_{0\le t\le T}\|\partial_x u\|_{L^\infty_x} \\ &\le \sup_{0\le t\le T}\left\|ne^{\sigma\langle n\rangle}\mathscr{F}_x u(n,t)\right\|_{l^1_n} \lesssim \sup_{0\le t\le T}\|u(\cdot,t)\|_{G^{\sigma,s}} < \infty.\end{aligned} \tag{25}$$

Suppose u and v are solutions to (1) in $C([0,T],G^{\sigma,s})$ with $u(x,0) = v(x,0)$. Let $e = u - v$. Using the fact $ee_{xxx} = \partial_x(ee_{xx}) - \frac{1}{2}\partial_x(e_x^2)$, we get the estimate

$$\frac{d}{dt}\|e(\cdot,t)\|^2_{L^2(\mathbb{T})} \le cP(u,u_x,v,v_x)\|e(\cdot,t)\|^2_{L^2(\mathbb{T})}$$

where $P(u,u_x,v,v_x)$ is a polynomial with respect to u, u_x, v and v_x. From (25) and Gronwall's inequality, we know that $e \equiv 0$.

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REFERENCES

[1] A. Biswas, *Local existence and Gevrey regularity of 3-D Navier-Stokes equations with l_p initial data*, J. Differential Equations, **215** (2005), 429–447.

[2] A. Biswas and D. Swanson, *Existence and generalized Gevrey regularity of solutions to the Kuramoto-Sivashinsky equation in $\mathbb{R}^n$*, J. Differential Equations, **240** (2007), 145–163.

[3] M. Bjørkavåg and H. Kalisch, *Exponential Convergence of a Spectral Projection of the KdV Equation*, Physics Letters A, **365** (2007), 278–283.
[4] J. L. Bona, Z. Grujić and H. Kalisch, *Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized KdV-equation*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **22** (2005), 783–797.
[5] A. de Bouard, N. Hayashi and K. Kato, *Gevrey regularizing effect for the (generalized) Korteweg-de Vries equation and nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **6** (1995), 673–725.
[6] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, Parts II, Geometric Funct. Anal., **3** (1993), 209–262.
[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc., **16** (2003), 705–749.
[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao. *Multilinear estimates for periodic KdV equations and applications*, J. Functional Anal., **211** (2004), 173–218.
[9] A. B. Ferrari and E. S. Titi, *Gevrey regularity for nonlinear analytic parabolic equations*, Comm. Partial Differential Equations, **23** (1998), 1–16.
[10] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Functional Anal., **87** (1989), 359–369.
[11] J. Gorsky and A. A. Himonas, *Construction of non-analytic solutions for the generalized KdV equation*, J. Math. Anal. Appl., **303** (2005), 522–529.
[12] Z. Grujić, *Spatial analyticity on the global attractor for the Kuramoto-Sivashinsky equation*, J. Dynam. Differential Equations, **12** (2000), 217–227.
[13] Z. Grujić and H. Kalisch, *Local well-posedness of the generalized Korteweg-de Vries equation in spaces of analytic functions*, Differential Integral Equations, **15** (2002), 1325–1334.
[14] N. Hayashi, *Analyticity of solutions of the Korteweg-de Vries equation*, SIAM J. Math. Anal., **22** (1991), 1738–1743.
[15] H. Hayashi, *Solutions of the (generalized) Korteweg-de Vries equation in the Bergman and Szegö spaces on a sector*, Duke Math. J., **62** (1991), 575–591.
[16] H. Kalisch, *Rapid convergence of a Galerkin projection of the KdV equation*, C. R. Math. Acad. Sci. Paris, **341** (2005), 457–460.
[17] H. Kalisch and X. Raynaud, *On the rate of convergence of a collocation projection of the KdV equation*, M2AN Math. Model. Numer. Anal., **41** (2007), 95–110.
[18] T. Kato and K. Masuda, *Nonlinear evolution equations and analyticity I*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **3** (1986), 455–467.
[19] C. E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equations*, J. Amer. Math. Soc., **9** (1996), 573–603.
[20] C. E. Kenig, G. Ponce and L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J., **71** (1993), 1–21.
[21] I. Kukavica and V. Vicol, *On the radius of analyticity of solutions to the three-dimensional Euler equations*, Proc. Amer. Math. Soc., **137** (2009), 669–677.
[22] G. Staffilani, *On solutions for periodic generalized KdV equations.* Internat. Math. Res. Notices, **18** (1997), 899–917.

[23] T. Tao, *Multilinear weighted convolution of L^2 functions, and applications to non-linear dispersive equations*, Amer. J. Math., **123** (2001), 890–908.

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E-mail address: qifan_li@yahoo.com

VERY WEAK SOLUTIONS OF DEGENERATE PARABOLIC SYSTEMS WITH NON-STANDARD $p(x,t)$-GROWTH

VERENA BÖGELEIN AND QIFAN LI

Abstract. We study higher integrability of very weak solutions to certain degenerate parabolic systems whose model is the parabolic $p(x,t)$ -Laplacian system,

$$\partial_t u - \operatorname{div}(|Du|^{p(x,t)-2}Du) = \operatorname{div}(|F|^{p(x,t)-2}F).$$

Under natural assumptions on the exponent function $p\colon \Omega\times(0,T)\to[2,\infty)$, we prove that any very weak solution $u:\Omega\times(0,T)\to\mathbb{R}^N$ with $|Du|^{p(\cdot)(1-\varepsilon)}\in L^1$ belongs to the natural energy space, i.e. $|Du|^{p(\cdot)}\in L^1_{\mathrm{loc}}$, provided $\varepsilon>0$ is small enough.

Keywords: Higher integrability; Gehring's lemma; parabolic p-Laplacean; non-standard growth condition; degenerate parabolic systems

1. Introduction

The reverse Hölder inequality for the solutions of elliptic systems was first studied by Meyer [24]. In principle the argument of the proof is based on Caccioppoli's inequality and an application of Gehring's lemma [15]. Furthermore, Lewis [22] and Iwaniec and Sbordone [18] independently introduced a definition of *very weak* solutions for elliptic systems, that is solutions which do not belong to the natural energy space. Actually, the very weak solutions belong to a slightly larger Sobolev space than the natural one. However, in [18, 22] it has been proved that this kind of solutions are indeed the weak solutions, provided the deficit is now too large. This result was extended in [23] to the degenerate elliptic systems with a Muckenhoupt weight. The treatment of the higher integrability for weak and very weak solutions to elliptic equations with non-standard $p(x)$-growth goes back to Zhikov [25], Bögelein and Zatorska-Goldstein [3]. The treatment of the time dependent parabolic case is much more difficult. The higher integrability of weak solutions to parabolic p-Laplacian type systems has been established by Kinnunen and Lewis [19]; see also [5, 4] for the case of higher order systems. The treatment of very weak solutions is much more delicate, because the solution itself cannot be used as a testing function. Using a subtle and involved construction of a testing function by Whitney cylinders, Kinnunen and Lewis [20] succeeded to prove the higher integrability of very weak solutions to parabolic systems of p-Laplacian type. The case of higher order systems was subsequently treated by Bögelein [6, 4]. Recently, Zhikov and Pastukhova [26] and independently Bögelein and Duzaar [7] proved the higher integrability of weak solutions to parabolic systems with non-standard $p(x,t)$-growth whose model is the parabolic $p(x,t)$-Laplacian system

$$\partial_t u - \operatorname{div}(|Du|^{p(x,t)-2}Du) = \operatorname{div}(|F|^{p(x,t)-2}F)$$

(see also [1] for the scalar case). Motivated by this work, we will study the very weak solutions to this kind of parabolic systems in this paper. Our main result states that any very weak solution is indeed a weak solution, provided the deficit in integrability is not too large. This problem was suggested as an open problem in the field of differential equations with non-standard growth in the overview article [17].

Our proof is in the spirit of [20]. Since the solution multiplied by a cut-off function cannot be used as a testing function in the weak formulation of the system, we have to construct a suitable testing function. This is achieved by a parabolic Lipschitz truncation argument. The major difficulty in our proof stems from the fact that the usual Poincaré inequality cannot be used in the case of variable exponent Lebesgue spaces. Instead, we have to use delicate localization arguments in order to get control on the lower order terms. More precisely, we first use a mixed type maximal function containing first and zero order terms. Subsequently we prove suitable bounds for the lower order terms on larger cylinders. We also remark that unlike to the elliptic case in which the Hardy-Littlewood maximal function plays an important role in the proof, (see [22]), the proof for the parabolic case should use strong maximal functions instead (see [20, 4, 6]). Unfortunately, as pointed out by Kopaliani [21], the strong maximal functions are not bounded in $L^{p(\cdot)}$ unless $p(\cdot) \equiv$ constant. Furthermore, we have to work with a "non-standard version" of the intrinsic geometry invented by DiBenedetto and Friedman [11, 12]; see also the monograph [10]. As a consequence, we have to use the modified parabolic distance $d_z(z_1, z_2) := \max\{\, |x_1 - x_2|, \sqrt{\lambda^{(p(z)-2)/p(z)}|t_1 - t_2|}\,\}$ defined in [7]. Contrary to the case of standard p-growth, the distance d_z depends on the point $z \in \mathbb{R}^{n+1}$ and cannot be considered as a metric space.

This paper is organized as follows. We state the main result in § 2. In § 3 we provide some preliminary material, while in § 4 we explain the construction of the testing function. In particular, we prove the existence of a Whitney type decomposition of the super level set of a certain strong maximal function. Subsequently, in § 5 we provide certain Poincaré type inequalities for constant integrability exponents. § 6 is devoted to the proof of the Caccioppoli inequality. First, we prove suitable estimates for the testing function constructed in § 4 and then we show the Lipschitz continuity of the testing function. Thereby, we use the integral characterization of Hölder spaces (see [9]) instead of pointwise estimates. This idea has been used in [8, 13] in a different context. The proof of the Caccioppoli inequality is given in § 6.4. Subsequently, § 7 is intended to prove estimates for the lower order terms which play a crucial role in the next section. In § 8, we prove the reverse Hölder inequality under an additional assumption.

Finally, in §9, we finish the proof of the higher integrability of very weak solutions.

2. Statement of the Main Result

In the following, Ω will denote a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and $\Omega_T := \Omega \times (0,T) \subset \mathbb{R}^{n+1}$, $T > 0$ is the associated space-time cylinder. We denote by Du the differentiation with respect to the space variables, while $\partial_t u$ stands for the time derivative. Points in $\mathbb{R}^{n+1}$ will be denoted by $z = (x,t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We shall use parabolic cylinders of the form $Q_\varrho(z_0) = B_\varrho(x_0) \times (t_0 - \varrho^2, t_0 + \varrho^2)$, where $B_\varrho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| \leq \varrho\}$ denotes the ball of radius ϱ with center x_0 in $\mathbb{R}^n$. We consider degenerate parabolic systems of the type

$$\partial_t u - \operatorname{div} A(z, Du) = B(z, Du), \tag{2.1}$$

where the vector fields $A, B : \Omega_T \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfy the following non-standard $p(z)$-growth and ellipticity conditions:

$$\begin{aligned} |A(z,\xi)| &\leq L(1 + |\xi| + |F|)^{p(z)-1} \\ |B(z,\xi)| &\leq L(1 + |\xi| + |F|)^{p(z)-1} \\ \langle A(z,\xi), \xi \rangle &\geq \nu |\xi|^{p(z)} - |F|^{p(z)} \end{aligned} \tag{2.2}$$

for any $z \in \Omega_T$ and $\xi \in \mathbb{R}^{nN}$. Here, $F \colon \Omega_T \to \mathbb{R}^{nN}$ with $|F|^{p(\cdot)} \in L^1(\Omega_T)$ and $0 < \nu < L$ are fixed structural parameters. For the exponent function $p : \Omega_T \to [2, \infty)$ we assume that it is continuous with a moduls of continuity $\omega : \Omega_T \to [0,1]$. More precisely, we assume that

$$2 \leq p(z) \leq \gamma_2 < \infty \quad \text{and} \quad |p(z_1) - p(z_2)| \leq \omega(d_P(z_1, z_2)), \tag{2.3}$$

holds for any $z, z_1, z_2 \in \Omega_T$ and some $\gamma_2 > 2$. For a brief discussion on the lower bound on $p(\cdot)$ we refer to Remark 9.1. Since our estimates are of local nature, it is not restrictive to assume an upper bound for $p(\cdot)$. For simplicity, we only consider the degenerate case where $p(\cdot) \geq 2$. As usual, the parabolic distance d_P is given by

$$d_P(z_1, z_2) := \max\left\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\right\} \qquad \text{for } z_1 = (x_1, t_1) \text{ and } z_2 = (x_2, t_2) \in \mathbb{R}^{n+1}.$$

The modulus of continuity ω is assumed to be a concave, non-decreasing function satisfying the following weak logarithmic continuity condition:

$$\sup_{0 \leq \varrho \leq 1} \omega(\varrho) \log\left(\frac{1}{\varrho}\right) < L < \infty. \tag{2.4}$$

The spaces $L^p(\Omega, R^N)$ and $W^{1,p}(\Omega, R^N)$ are the usual Lebesgue and Sobolev spaces. Moreover, for a variable exponent $p(\cdot)$, we denote by $L^{p(\cdot)}(\Omega_T, \mathbb{R}^k)$, $k \in \mathbb{N}$, the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega_T, \mathbb{R}^k) := \left\{ v \in L^1(\Omega_T, \mathbb{R}^k) : \int_{\Omega_T} |v|^{p(\cdot)} dz < \infty \right\}.$$

For more details on variable exponent Lebesgue and Sobolev spaces we refer the reader to [14]. We now can give the definition of a very weak solution to (2.1).

Definition 2.1. Let $\varepsilon \in (0,1)$. We say that $u \in L^2(\Omega_T, \mathbb{R}^N)$ is a *very weak solution to the parabolic system* (2.1) *with deficit* ε if and only if

$$u \in L^{p(\cdot)(1-\varepsilon)}(\Omega_T, \mathbb{R}^N) \quad \text{and} \quad Du \in L^{p(\cdot)(1-\varepsilon)}(\Omega_T, \mathbb{R}^{Nn})$$

and

$$\int_{\Omega_T} u \cdot \partial_t \varphi - \langle A(z, Du), D\varphi \rangle \, dz = - \int_{\Omega_T} B(z, Du) \cdot \varphi \, dz \tag{2.5}$$

holds, whenever $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$.

The following theorem is our main result.

Theorem 2.2. *Let $p : \Omega_T \to [2, \gamma_2]$ satisfy (2.3) and (2.4). Then there exists a constant $\varepsilon_0 = \varepsilon_0(n, N, L, \gamma_2) > 0$ such that the following holds: Whenever $u \in L^2(\Omega_T, \mathbb{R}^N) \cap L^{p(\cdot)(1-\varepsilon)}(\Omega_T, \mathbb{R}^N)$ and $|Du| \in L^{p(\cdot)(1-\varepsilon)}(\Omega_T)$ with some $\varepsilon \in (0, \varepsilon_0]$ is a very weak solution to the parabolic system (2.1) under the assumptions (2.2) and $F \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^{nN})$, then we have*

$$|Du| \in L^{p(\cdot)}_{\mathrm{loc}}(\Omega_T).$$

Moreover, for $M \geq 1$ there exists a radius $r_0 = r_0(n, N, L, \gamma_2, M)$ such that there holds: If

$$\int_{\Omega_T} (|u| + |Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)} \, dz \leq M \tag{2.6}$$

such that for any parabolic cylinder $Q_{2r}(z_0) \subseteq \Omega_T$ with $r \in (0, r_0]$ there holds

$$\fint_{Q_r(z_0)} |Du|^{p(\cdot)} dz \leq c \left(\fint_{Q_{2r}(z_0)} (|Du| + |F|)^{p(\cdot)(1-\varepsilon)} dz \right)^{1+\frac{\varepsilon p_0}{2-\varepsilon p_0}} + c \fint_{Q_{2r}(z_0)} (|F| + 1)^{p(\cdot)} \, dz, \tag{2.7}$$

where $c = c(n, N, L, \gamma_2) > 0$ and $p_0 = p(z_0)$.

3. Preliminary material and notation

For a point $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and parameters $\varrho > 0$, $\lambda \geq 1$, we define the scaled cylinder $Q_\varrho^{(\lambda)}(z_0)$ by

$$Q_\varrho^{(\lambda)}(z_0) := B_\varrho(x_0) \times \Lambda_\varrho^{(\lambda)}(z_0), \text{ where } \Lambda_\varrho^{(\lambda)}(z_0) := (t_0 - \lambda^{(2-p_0)/p_0} \varrho^2, t_0 + \lambda^{(2-p_0)/p_0} \varrho^2),$$

and $p_0 := p(z_0)$. For $\alpha > 0$, we write $\alpha Q_\varrho^{(\lambda)}(z_0)$ for the scaled cylinder $Q_{\alpha\varrho}^{(\lambda)}(z_0)$. Moreover, for a function $f \in L^1(\mathbb{R}^{n+1}, \mathbb{R}^k)$, $k \in \mathbb{N}$ we define its strong maximal function by

$$M(f)(z) := \sup \left\{ \fint_Q |f| d\tilde{z} : \; z \in Q, \; Q \text{ is parabolic cylinder} \right\}.$$

Here, by parabolic cylinder we mean that Q is a cylinder of the form $B \times \Lambda$ where B is a ball in $\mathbb{R}^n$ and $\Lambda \subset \mathbb{R}$ is an interval. To simplify the notations, we write f_G instead of $⨍_G f dz$ for any subset $G \subset \mathbb{R}^{n+1}$. We will use the following iteration lemma, which is a standard tool and can be found in [16].

We can reformulate the parabolic system (2.5) in its Steklov form as follows:

(3.1)
$$\int_\Omega \partial_t [u]_h(\cdot,t)\varphi + \langle [A(z,Du)]_h, D\varphi\rangle(\cdot,t)dx = -\int_\Omega \langle [B(z,Du)]_h, \varphi\rangle(\cdot,t)dx$$

for all $\varphi \in C_0^\infty(\Omega,\mathbb{R}^N)$ and a.e. $t \in (0,T)$. For the proof of (3.1) and the definition of the Steklov averages $[\cdot]_h$, we refer the reader for instance to [4, Chapter 8.2].

The following iteration lemma is a standard tool in order to reabsorb certain terms from the right hand side into the left.

Lemma 3.1. *Let $\theta \in (0,1)$, $C_1, C_2 \geq 0$ and $\beta > 0$. Then there exists a constant $c = c(\theta,\beta) > 0$ such that there holds: For any non-negative bounded function $\phi\colon [r,\varrho] \to \mathbb{R}_+$ satisfying*

$$\phi(s) \leq \theta\phi(t) + C_1(t-s)^{-\beta} + C_2 \qquad \text{for all } 0 < r \leq s < t \leq \varrho,$$

we have

$$\phi(r) \leq c\left[C_1(\varrho - r)^{-\beta} + C_2\right].$$

Next, we state Gagliardo-Nirenberg's inequality in a form which shall be convenient for our purposes later.

Lemma 3.2. *Let $B_\varrho(x_0) \subset \mathbb{R}^n$ with $0 < \varrho \leq 1$, $1 \leq \sigma, q, r < \infty$ and $\theta \in (0,1)$ such that $-n/\sigma \leq \theta(1 - n/q) - (1-\theta)n/r$. Then there exists a constant $c = c(\sigma, n)$ such that for any $u \in W^{1,q}(B_\varrho(x_0))$ there holds:*

$$⨍_{B_\varrho(x_0)} \left|\frac{u}{\varrho}\right|^\sigma dx \leq c\left(⨍_{B_\varrho(x_0)} \left|\frac{u}{\varrho}\right|^q + |Du|^q dx\right)^{\theta\sigma/q} \left(⨍_{B_\varrho(x_0)} \left|\frac{u}{\varrho}\right|^r dx\right)^{(1-\theta)\sigma/r}.$$

4. Construction of the test function

In this section, we will construct a suitable testing function for the weak form (2.5) of the parabolic system. To this aim we fix a cylinder $Q_\varrho^{(\lambda)}(z_0)$ with $0 < \varrho \leq 1$, $\lambda \geq 1$ and $Q_{32\varrho}^{(\lambda)}(z_0) \subset \Omega_T$. Letting ϱ_1 and ϱ_2 be two fixed numbers such that $\varrho \leq \varrho_1 < \varrho_2 \leq 16\varrho$, we set

$$Q^{(0)} := Q_\varrho^{(\lambda)}(z_0), \quad Q^{(1)} := Q_{\varrho_1}^{(\lambda)}(z_0), \quad Q^{(2)} := Q_{\varrho_1^+}^{(\lambda)}(z_0), \quad Q^{(3)} := Q_{\varrho_2^-}^{(\lambda)}(z_0),$$

$$Q^{(4)} := Q_{\varrho_2}^{(\lambda)}(z_0), \quad Q^{(5)} := Q_{16\varrho}^{(\lambda)}(z_0), \quad Q^{(6)} := Q_{32\varrho}^{(\lambda)}(z_0),$$

where $\varrho_1^+ = \varrho_1 + \frac{1}{3}(\varrho_2 - \varrho_1)$ and $\varrho_2^- = \varrho_1 + \frac{2}{3}(\varrho_2 - \varrho_1)$. We note that $Q^{(0)} \subset Q^{(1)} \subset Q^{(2)} \subset Q^{(3)} \subset Q^{(4)} \subset Q^{(5)} \subset Q^{(6)}$. In the following, we will write $B^{(k)}$ for the projection of $Q^{(k)}$ in x direction and $\Lambda^{(k)}$ for the projection of $Q^{(k)}$

in t direction for $k \in \{0, \cdots, 6\}$. Denoting by p_1 and p_2 the minimum and maximum of $p(\cdot)$ over the cylinder $Q^{(6)}$, i.e.

$$p_1 = \inf_{Q^{(6)}} p(\cdot) \quad \text{and} \quad p_2 = \sup_{Q^{(6)}} p(\cdot),$$

and taking into account that $p(\cdot) \geq 2$ and that ω is concave, we find that

$$p_2 - p_1 \leq \omega\Big(\max\Big\{64\varrho, \sqrt{\lambda^{(2-p_0)/p_0}(64\varrho)^2}\Big\}\Big) \leq \omega(64\varrho) \leq 64\omega(\varrho). \tag{4.1}$$

Therefore, by the concavity of ω and assumption (2.4), we have that

$$\varrho^{-(p_2-p_1)} \leq \varrho^{-64\omega(\varrho)} = \exp\big[64\omega(\varrho)\log\tfrac{1}{\varrho}\big] \leq e^{64L} \tag{4.2}$$

Next, we fix constants $\tilde{q}$ and ε such that

$$1 < \tilde{q} < \frac{\gamma_2}{\gamma_2 - 1} \quad \text{and} \quad 0 < \varepsilon < 1 - \frac{1}{\tilde{q}} \leq \frac{1}{\gamma_2}.$$

Throughout this section, we shall assume that

(4.3)
$$\lambda^{1-\varepsilon} \leq \fint_{Q^{(0)}} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \quad \text{and} \quad \fint_{Q^{(5)}} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \leq \lambda^{1-\varepsilon}.$$

holds true. Then, writing $p_0 = p(z_0)$ as usual and using the fact that $|Q^{(0)}| = c(n)\varrho^{2+n}\lambda^{(2-p_0)/p_0}$ and assumption (2.6), we find that

$$\lambda^{\frac{2}{p_0}-\varepsilon} \leq \frac{1}{c(n)\varrho^{n+2}} \int_{Q^{(0)}} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \leq \frac{c(n)M}{\varrho^{n+2}}.$$

Since $\lambda \geq 1$ and $\frac{2}{p_0} - \varepsilon \geq \frac{1}{p_0}$, this leads to the following upper bound for λ:

$$\lambda \leq \left(\frac{c(n)M}{\varrho^{n+2}}\right)^{p_0}. \tag{4.4}$$

This together with (4.1) implies that for $\varrho > 0$ with $\varrho \leq \varrho_0 \leq \frac{1}{64M}$, there holds

$$\lambda^{(p_2-p_1)/p_0} \leq c(n)\,\varrho^{-(n+2)(p_2-p_1)} M^{p_2-p_1} \leq c(n)\, e^{64(n+3)L}. \tag{4.5}$$

Moreover, we choose ϱ_0 so small that $\omega(4\varrho_0) \leq 1 - \gamma_2(1 - \frac{1}{\tilde{q}})$. Then, we have for $\varrho \in (0, \varrho_0]$ that

$$p_2 - 1 \leq p_1 + \omega(4\varrho) - 1 \leq p_1 + \omega(4\varrho_0) - 1 \leq p_1 - \gamma_2(1 - \tfrac{1}{\tilde{q}}) \leq \tfrac{p_1}{\tilde{q}}. \tag{4.6}$$

For $\lambda_1 \geq 1$ we denote the lower level set of the maximal function

$$M_{Q^{(4)}}(z) := M\Big[\Big(\tfrac{1}{\varrho}|u - u_{Q^{(1)}}| + |Du| + |F| + 1\Big)^{p(\cdot)/\tilde{q}} \chi_{Q^{(4)}}\Big](z)^{\tilde{q}(1-\varepsilon)}$$

by

$$E(\lambda_1) := \{z \in Q^{(4)} : M_{Q^{(4)}}(z) \leq \lambda_1^{1-\varepsilon}\}.$$

If $E(\lambda_1) = \emptyset$, we have by the boundedness of the strong maximal function that

$$\lambda_1^{1-\varepsilon}|Q^{(4)}| \leq \int_{Q^{(4)}} M\Big[\Big(\tfrac{1}{\varrho}|u - u_{Q^{(1)}}| + |Du| + |F| + 1\Big)^{p(\cdot)/\tilde{q}} \chi_{Q^{(4)}}\Big]^{\tilde{q}(1-\varepsilon)} dz$$

$$\leq c \int_{Q^{(4)}} \Big(\tfrac{1}{\varrho}|u - u_{Q^{(1)}}| + |Du| + |F| + 1\Big)^{p(\cdot)(1-\varepsilon)}\, dz$$
$$\leq c\, \tilde{\lambda}^{1-\varepsilon}|Q^{(4)}| =: c_E^{1-\varepsilon}\, \tilde{\lambda}^{1-\varepsilon}|Q^{(4)}|,$$

where $c_E = c_E(n, \gamma_2)$ and

$$\tilde{\lambda} := \lambda + \bigg(⨍_{Q^{(4)}} \Big(\tfrac{1}{\varrho}|u - u_{Q^{(1)}}|\Big)^{p(\cdot)(1-\varepsilon)}\, dz\bigg)^{\frac{1}{1-\varepsilon}}. \tag{4.7}$$

For $\lambda_1 \leq c_E\tilde{\lambda}$, this leads to a contradiction. Therefore, we conclude that $E(\lambda_1)$ is nonempty for $\lambda_1 > c_E\tilde{\lambda}$. We note that the set $E(\lambda_1)$ is bounded and closed. Therefore, for any fixed point $z \in Q^{(4)} \setminus E(\lambda_1)$, there exists a neighbourhood $\tilde{Q}$ such that $\tilde{Q} \subset Q^{(4)} \setminus E(\lambda_1)$. This motivates us to establish the following Lemma.

Lemma 4.1. *Let $\lambda \geq 1$, $\lambda_1 \geq c_E\lambda$ and $\alpha \geq 1$, $z \in Q^{(4)} \setminus E(\lambda_1)$ and define*

$$r_z := d_z(z, E(\lambda_1)) \quad where \quad d_z(z_1, z_2) := \max\Big\{ |x_1 - x_2|,\ \sqrt{\lambda_1^{(p(z)-2)/p(z)}|t_1 - t_2|} \Big\}.$$

Then for any $z_1, z_2 \in Q^{(4)} \cap \alpha Q_{r_z}^{(\lambda_1)}(z)$ we have

$$|p(z_1) - p(z_2)| \leq 32 \max\{\alpha, 1\}\, \omega(\min\{r_z, \varrho\}) \quad and \quad \lambda_1^{|p(z_1)-p(z_2)|} \leq c_\alpha,$$

where the constant c_α depends on n, L, γ_2 and α.

Proof. We first observe that since $p(\cdot) \geq 2$ and $\lambda, \lambda_1 \geq 1$ we have

$$|p(z_2) - p(z_1)| \leq \omega(\min\{2\alpha r_z, 32\varrho\}) \leq 32 \max\{\alpha, 1\}\, \omega(\min\{r_z, \varrho\}),$$

where we also used the concavity of ω. This proves the first assertion of the lemma. Since $Q^{(4)} \cap Q_{r_z}^{(\lambda_1)}(z) \subset Q^{(4)} \setminus E(\lambda_1)$, we use Chebyshev inequality and the boundedness of the strong maximal functions to obtain

$$|Q^{(4)} \cap Q_{r_z}^{(\lambda_1)}(z)| \leq |Q^{(4)} \setminus E(\lambda_1)| \leq \frac{1}{\lambda_1^{1-\varepsilon}} \int_{Q^{(4)}} M_{Q^{(4)}}\, dz$$
$$\leq \frac{c}{\lambda_1^{1-\varepsilon}} \int_{Q^{(4)}} \Big(\tfrac{1}{\varrho}|u - u_{Q^{(1)}}| + |Du| + |F| + 1\Big)^{p(\cdot)(1-\varepsilon)}\, dz =: \frac{c\,\mathcal{A}}{\lambda_1^{1-\varepsilon}},$$

with the obvious meaning of $\mathcal{A}$ and a constant $c = c(n, \gamma_2)$. This implies the following upper bound for λ_1:

$$\lambda_1^{1-\varepsilon} \leq \frac{c\,\mathcal{A}}{|Q^{(4)} \cap \alpha Q_{r_z}^{(\lambda_1)}(z)|}. \tag{4.8}$$

To estimate the lower bound for $|Q^{(4)} \cap \alpha Q_{r_z}^{(\lambda_1)}(z)|$, we recall that $z \in Q^{(4)} \setminus E(\lambda_1)$ implies

$$|Q^{(4)} \cap \alpha Q_{r_z}^{(\lambda_1)}(z)| \geq c\, \min\{r_z^n, \varrho^n\} \min\Big\{\lambda_1^{\frac{2-p(z)}{p(z)}} r_z^2, \lambda^{\frac{2-p(z)}{p(z)}}\varrho^2\Big\} \geq c\, \lambda_1^{\frac{2}{\gamma_2}-1} \min\{r_z^{n+2}, \varrho^{n+2}\}.$$

Together with (4.8) this shows that

$$\lambda_1 \leq \bigg[c\,\mathcal{A} \max\bigg\{\frac{1}{r_z^{2+n}}, \frac{1}{\varrho^{2+n}}\bigg\}\bigg]^{\frac{\gamma_2}{2-\varepsilon\gamma_2}}, \tag{4.9}$$

where c depends on n and α. Next, we estimate $\mathcal{A}$ as follows:

$$\begin{aligned}\mathcal{A} &\le c\,\varrho^{-\gamma_2(1-\varepsilon)}\left[\int_{Q^{(4)}}(|u|+|Du|+|F|+1)^{p(\cdot)(1-\varepsilon)}\,dz+|Q^{(1)}|(|u_{Q^{(1)}}|+1)^{p_2(1-\varepsilon)}\right]\\ &\le c\,\varrho^{-\gamma_2}\left[M+|Q^{(1)}|^{-\frac{p_2-p_1}{p_1}}M^{\frac{p_2}{p_1}}+|Q^{(1)}|\right]\le c(n,L,\gamma_2)\,\varrho^{-\gamma_2}M^2,\end{aligned}$$

where we have used that $|Q^{(1)}|\le c(n)$, $\frac{p_2}{p_1}\le 2$ which we can always assume and $|Q^{(1)}|^{-\frac{p_2-p_1}{p_1}}\le c(n,L)$ which follows from (4.2) and (4.5). Inserting the bound for $\mathcal{A}$ into (4.9) and taking into account that $\varrho^{-(p_2-p_1)}\le e^{4L}$ and $M^{p_2-p_1}\le e^L$ (see (4.2) and (4.5)) we end up with

$$\lambda_1^{|p(z_1)-p(z_2)|}\le c(n,\gamma_2,L,\alpha).$$

This finishes the proof of the lemma. □

To construct our test function, we need the following version of the Whitney decomposition theorem for non-uniformly parabolic cylinders.

Lemma 4.2. *There exist Whitney-type cylinders $\{Q_i\}_{i=1}^\infty$ with $Q_i\equiv B_i\times\Lambda_i := Q_{r_i}^{(\lambda_1)}(z_i)$, having the following properties:*

(i) $Q^{(4)}\setminus E(\lambda_1)=\bigcup_{i=1}^\infty Q^{(4)}\cap Q_i$,

(ii) *In each point of $Q^{(4)}\setminus E(\lambda_1)$ intersect at most $c(n,L,\gamma_2)$ of the cylinders $2Q_i$.*

(iii) *There exists a constant $c=c(n,L,\gamma_2)$ such that for any Whitney cylinders Q_i and Q_j with $2Q_i\cap 2Q_j\neq\emptyset$, there holds*

$$|B_i|\le c\,|B_j|\le c\,|B_i| \quad\textit{and}\quad |\Lambda_i|\le c\,|\Lambda_j|\le c\,|\Lambda_i|.$$

(iv) *There exists a constant $\hat c=\hat c(n,L,\gamma_2)$ such that for all $i\in\mathbb{N}$ there holds*

$$\hat cQ_i\subset\mathbb{R}^{n+1}\setminus E(\lambda_1) \quad\textit{and}\quad 2\hat cQ_i\cap E(\lambda_1)\neq\emptyset,$$

(v) *For the constant $\hat c$ from (iv) there holds: $2Q_i\cap 2Q_j\neq\emptyset$ implies $2Q_i\subset\hat cQ_j$.*

Proof. By $\chi=\chi(n,L)\ge 5$ we denote the constant from [7, (7.1)]. We fix a point $z\in Q^{(4)}\setminus E(\lambda_1)$ and let d_z be the corresponding parabolic metric which was defined in Lemma 4.1. We set $r_z=\frac{\delta}{\chi}d_z(z,E(\lambda_1))$ where $\delta\in(0,1/4)$ will be fixed at the end of the proof. Then, $\mathcal{F}=\{Q_{r_z}^{(\lambda_1)}(z)\}_{z\in Q^{(4)}\setminus E(\lambda_1)}$ is a covering of the set $Q^{(4)}\setminus E(\lambda_1)$. From [7, Lemma 7.1] applied with λ_1 instead of λ (note that instead of verifying assumption [7, (7.1)] we can use Lemma 4.1 with $\alpha=1$ in order to bound the terms coming from the difference $p(z_1)-p(z_2)$ in the proof of [7, Lemma 7.1]) we infer the existence of a countable sub-collection $\mathcal{G}=\{Q_{r_{z_i}}^{(\lambda_1)}(z_i)\}_{i=1}^\infty\subseteq\mathcal{F}$ of disjoint parabolic cylinders such that

$$Q^{(4)}\setminus E(\lambda_1)\subset\bigcup_{i=1}^\infty\chi Q_{r_{z_i}}^{(\lambda_1)}(z_i)\cap Q^{(4)}. \tag{4.10}$$

For $i \in \mathbb{N}$, we now set $r_i := \chi r_{z_i} = \delta d_{z_i}(z_i, E(\lambda_1))$ and define the parabolic cylinders

$$Q_i \equiv B_i \times \Lambda_i := Q_{r_i}^{(\lambda_1)}(z_i)$$

In the following we will verify that statements (i)-(iv) are true for the parabolic cylinders $\{Q_i\}_{i=1}^{\infty}$.

Statement (i) directly follows from (4.10) and the fact that $r_i < d_{z_i}(z_i, E(\lambda_1))$, so that $Q_i \subset \mathbb{R}^{n+1} \backslash E(\lambda_1)$ for any $i \in \mathbb{N}$. Now, we come to the proof of (iii). For any parabolic cylinder Q_i, we set $p_0^i = p(z_i)$ and $p_1^i = \min\{p(z) : z \in Q^{(4)} \cap 2Q_i\}$ and $p_2^i = \max\{p(z) : z \in Q^{(4)} \cap 2Q_i\}$. Since $2Q_i \subset \mathbb{R}^{n+1} \backslash E(\lambda_1)$ (note that $2r_i < d_{z_i}(z_i, E(\lambda_1))$), the application of Lemma 4.1 with the choice $\alpha = 1$ ensures that $\lambda_1^{p_2^i - p_1^i} \le c(n, L, \gamma_2)$. We now consider $i, j \in \mathbb{N}$ such that $2Q_i \cap 2Q_j \neq \emptyset$ and show that the two parabolic distances d_{z_i} and d_{z_j} are equivalent. Since $2Q_i \cap 2Q_j \neq \emptyset$, there exists a point $\bar{z} \in 2Q_i \cap 2Q_j$ and therefore, we have

$$\lambda_1^{p_0^j - p_0^i} = \lambda_1^{p_0^j - p(\bar{z})} \lambda_1^{p(\bar{z}) - p_0^i} \le \lambda_1^{p_2^j - p_1^j} \lambda_1^{p_2^i - p_1^i} \le c(n, L, \gamma_2).$$

This allows us to estimate the distance with respect to d_{z_i} of two arbitrary points $\hat{z} = (\hat{x}, \hat{t})$ and $z' = (x', t')$ in $\mathbb{R}^{n+1}$ by

$$d_{z_i}(\hat{z}, z') = \max\left\{|\hat{x} - x'|, \lambda_1^{(p_0^j - p_0^i)/p_0^j p_0^i} \sqrt{\lambda_1^{(2-p_0^j)/p_0^j} |\hat{t} - t'|}\right\} \le c(n, L, \gamma_2)\, d_{z_j}(\hat{z}, z').$$

We now let $\widehat{z_j}$ be a point in $E(\lambda_1)$ such that $d_{z_j}(z_j, \widehat{z_j}) = d_{z_j}(z_j, E(\lambda_1))$ and $\bar{z} \in 2Q_i \cap 2Q_j$ as before. With these choices we see that

$$\begin{aligned} r_i = \delta d_{z_i}(z_i, E(\lambda_1)) &\le \delta d_{z_i}(z_i, \widehat{z_j}) \le \delta \left(d_{z_i}(z_i, \bar{z}) + d_{z_i}(\bar{z}, \widehat{z_j}) + d_{z_i}(\bar{z}, \widehat{z_j})\right) \\ &\le \delta \left(2r_i + c d_{z_j}(\bar{z}, \widehat{z_j}) + c d_{z_j}(\bar{z}, \widehat{z_j})\right) \le \delta \left(2r_i + 2c r_j + \tfrac{c}{\delta} r_j\right) \le \tfrac{1}{2} r_i + 2c r_j, \end{aligned}$$

which implies that $r_i \le c(n, L, \gamma_2) r_j$. Since we can interchange i and j in the preceding argument, this proves the first claim in (iii). Furthermore, we find that

$$|\Lambda_i| = 2\lambda_1^{(2-p_0^i)/p_0^i} r_i^2 = \lambda_1^{2(p_0^j - p_0^i)/p_0^j p_0^i} \lambda_1^{(2-p_0^j)/p_0^j} r_i^2 \le c\, \lambda_1^{(2-p_0^j)/p_0^j} r_j^2 \le c(n, L, \gamma_2)\, |\Lambda_j|,$$

which proves the second claim in (iii). Next, we will show the statement (v). We consider $i, j \in \mathbb{N}$ such that $2Q_i \cap 2Q_j \neq \emptyset$. Then, there exists $\bar{z} \in 2Q_i \cap 2Q_j$ and hence for any $\tilde{z}_i \in 2Q_i$ there holds

$$\begin{aligned} d_{z_j}(z_j, \tilde{z}_i) &\le d_{z_j}(z_j, \bar{z}) + d_{z_j}(\bar{z}, z_i) + d_{z_j}(z_i, \tilde{z}_i) \\ &\le 2r_j + c d_{z_i}(\bar{z}, z_i) + c d_{z_i}(z_i, \tilde{z}_i) \le 2r_j + 4c r_i \le \hat{c}(n, L, \gamma_2)\, r_j, \end{aligned}$$

where we have used $d_{z_j} \le c d_{z_i}$ and $r_i \le c r_j$ from the proof of (iii). Therefore, we know that $2Q_i \subset \hat{c} Q_j$ as claimed in (v). At this point, we perform the choice of δ. Choosing $\delta = \delta(n, L, \gamma_2) = 1/\hat{c}$, we have that $\hat{c} Q_i \subset \mathbb{R}^{n+1} \subset E(\lambda_1)$. Now, we observe that $\alpha Q_i \cap E(\lambda_1) \neq \emptyset$ if $\alpha > \hat{c}$ and $\alpha Q_i \subset Q^{(4)} \backslash E(\lambda_1)$ if $\alpha < \hat{c}$. This proves (iv). Finally, we come to the proof of statement (ii). Here, we consider $z \in Q^{(4)} \backslash E(\lambda_1)$ and denote $I_z := \{i \in \mathbb{N} : z \in 2Q_i\}$. Let $j \in I_z$. Then, by (iii) we have for any $k \in I_z$ that $|B_k| \ge c^{-1} |B_i|$ and therefore

$\inf_{k\in I_z}\varrho_k > 0$. This ensures that I_z is finite and therefore there exists an element $i_0 \in I_z$ such that $\min_{i\in I_z}|Q_i| = |Q_{i_0}|$. Moreover, by (v) we know that $Q_i \subset \hat{c}Q_{i_0}$ for any $i \in I_z$. Taking into account that the cylinders $\chi^{-1}Q_i$ are disjoint, we have for the cardinality of the set I_z that $\#I_z\,|\chi^{-1}Q_{i_0}| \le |\bar{c}Q_{i_0}|$ and hence $\#I_z \le (\bar{c}\chi)^{n+2}$. This proves (ii) and therefore the proof of the Lemma is complete. □

Subordinate to the cylinders Q_i, we can construct a partition of unity as stated in the following lemma.

Lemma 4.3. *There exists a partition of unity* $\{\psi_i\}_{i=1}^\infty$ *on* $\mathbb{R}^{n+1}\setminus E(\lambda_1)$, *i.e.* $\sum_{i=1}^\infty \psi_i \equiv 1$ *on* $\mathbb{R}^{n+1}\setminus E(\lambda_1)$ *having following properties,*

$$\begin{cases} \psi_i \in C_0^\infty(2Q_i), \quad 0 \le \psi_i \le 1, \quad \text{and} \quad \psi_i \ge c \quad \text{on } Q_i, \\ |\partial_t\psi_i| \le c\lambda_1^{(p_0^i-2)/p_0^i} r_i^{-2}, \quad |D\psi_i| \le c r_i^{-1}, \end{cases}$$

where c only depends on n, L, γ_2. □

For $i \in \mathbb{N}$, we define $I(i) := \{j \in \mathbb{N} : \operatorname{supp}\psi_j \cap \operatorname{supp}\psi_i \ne \emptyset\}$ and by $\#I(i)$ we denote the number of elements in $I(i)$. From Lemma 4.2 (ii), we know that $\#I(i) \le c(n, L, \gamma_2)$ for any $i \in \mathbb{N}$. Furthermore, for $i \in \mathbb{N}$ we define the enlarged cylinder

$$\widehat{Q}_i := \hat{c}Q_i \equiv Q_{\hat{r}_i}^{(\lambda_1)}(z_i),$$

where $\hat{r}_i := \hat{c}r_i$ and $\hat{c} = \hat{c}(n, L, \gamma_2)$ denotes the constant from Lemma 4.2 (v). Then, by Lemma 4.2 (v) we see that $\bigcup_{j\in I(i)} \operatorname{supp}\psi_j \subset \bigcup_{j\in I(i)} 2Q_j \subset \widehat{Q}_i$. We now define the function $v(z) \equiv v(x,t) := \eta(x)\zeta(t)[u - u_{Q^{(1)}}]$, where $\eta \in C_0^\infty(B^{(3)})$, $\zeta \in C_0^\infty(\Lambda^{(3)})$ are cutoff functions satisfying

$$\begin{cases} \eta \equiv 1 \text{ in } B^{(2)}, \quad 0 \le \eta \le 1, \ |D\eta| \le c(\varrho_2 - \varrho_1)^{-1} \\ \zeta \equiv 1 \text{ in } \Lambda^{(2)}, \quad 0 \le \zeta \le 1, \ |\partial_t\zeta| \le c\lambda^{(p_0-2)/p_0}(\varrho_2^2 - \varrho_1^2)^{-1}. \end{cases}$$

It follows that $\operatorname{supp}(\eta\zeta) \subset Q^{(3)}$. Then, for Q_i and ψ_i as in Lemmas 4.2 and 4.3 we define the test function

$$\tilde{v}(z) \equiv \tilde{v}(x,t) := \begin{cases} v(z), & \text{for } z \in E(\lambda_1), \\ \displaystyle\sum_{i=1}^\infty v_{Q_i\cap Q^{(4)}}\psi_i(z), & \text{for } z \in \mathbb{R}^{n+1} \setminus E(\lambda_1). \end{cases} \tag{4.11}$$

Note that $v_{Q_i\cap Q^{(4)}} \ne 0$ implies that $Q_i \cap Q^{(3)} \ne \emptyset$ and consequently $\operatorname{supp}\psi_i \cap Q^{(3)} \ne \emptyset$. For this reason we are mainly interested in getting estimates on such cylinders. Before, we have to introduce some more notation. We set

$$S_1 := \left\{t \in \mathbb{R}^1 : |t - t_0| \le \lambda^{(2-p_0)/p_0}\left(\varrho_1 + \tfrac{1}{9}(\varrho_2 - \varrho_1)\right)^2\right\}$$

and

$$S_2 := \left\{t \in \mathbb{R}^1 : |t - t_0| \le \lambda^{(2-p_0)/p_0}\left(\varrho_1 + \tfrac{2}{9}(\varrho_2 - \varrho_1)\right)^2\right\}.$$

and note that $\Lambda^{(1)} \subset S_1 \subset S_2 \subset \Lambda^{(2)}$. Furthermore, we need to consider the set

$$\Theta := \{i \in \mathbb{N} : \operatorname{supp} \psi_i \cap S_1 \neq \emptyset\}$$

and we decompose the set Θ as follows:

$$\Theta_1 := \{i \in \Theta : \widehat{Q}_i \subset \mathbb{R}^n \times S_2\} \quad \text{and} \quad \Theta_2 \equiv \Theta \backslash \Theta_1.$$

We find that if $i \in \Theta_1$ and $4r_i \leq \varrho_2 - \varrho_1$ then $Q_i \subset Q^{(4)}$. While if $i \in \Theta_2$ then there holds $\lambda_1^{(2-p_0^i)/p_0^i} r_i^2 \geq \hat{c}^{-2} s$, where $s = \lambda^{(2-p_0)/p_0}(\varrho_2 - \varrho_1)\varrho$.

5. Poincaré type inequalities

Since we have to derive estimates on intersections of parabolic cylinders, we will formulate Poincaré type estimates for very general types of sets. The first one can be deduced from [6, Lemma 4.1, Lemma 4.2] and [7, Lemma 5.1].

Lemma 5.1. *Let u be a very weak solution to* (2.1) *with* (2.2) *and deficit $\varepsilon > 0$. Suppose that $\tilde{\Omega} \Subset \Omega$ is a convex open set such that $B_\varrho(y) \subset \tilde{\Omega} \subset B_{\alpha\varrho}(y)$ for some $y \in \mathbb{R}^n$, $0 < \varrho \leq 1$ and $\alpha > 1$ and $T_1, T_2 \subset (0, T)$ are two intervals. Then for $1 \leq \theta \leq \inf_{z \in \tilde{\Omega} \times (T_1 \cup T_2)} p(z)(1-\varepsilon)$, there holds*

$$\fint_{\tilde{\Omega} \times T_1} |u - (u)_{\tilde{\Omega} \times T_2}|^\theta dz \leq c\,\varrho^\theta \left(\fint_{\tilde{\Omega} \times T_1} |Du|^\theta dz + \fint_{\tilde{\Omega} \times T_2} |Du|^\theta dz \right) + c\,\varrho^{-\theta} \left(\int_{T_1 \cup T_2} \fint_{\tilde{\Omega}} (1 + |Du| + |F|)^{p(\cdot)-1} dz \right)^\theta$$

where the constant c depends only on n, N, L, γ_2 and α.

Corollary 5.2. *Let $M \geq 1$ and ≥ 1 be fixed. Then there exists $\varrho_0 = \varrho_0(n, \gamma_1, L, , M)$ such that the following holds: Assume that u is a very weak solution to* (2.1) *with* (2.2) *and deficit $0 < \varepsilon < \frac{1}{2\gamma_2}$ satisfying* (2.6). *Suppose that on the parabolic cylinder $Q_{\varrho,s}(z_0) = B_\varrho(x_0) \times (t_0 - s, t_0 + s)$ with $0 < \varrho \leq \varrho_0$, $0 < s \leq \lambda^{(2-p_0)/p_0}\varrho^2$, $\lambda_1 \geq c_E\lambda$ and $Q_{\varrho,s}(z_0) \Subset Q^{(6)}$, there holds*

$$\fint_{Q_{\varrho,s}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \leq \lambda^{1-\varepsilon}. \tag{5.1}$$

Then for any $1 \leq \theta \leq \inf_{z \in Q_{\varrho,s}(z_0)} p(z)(1-\varepsilon)$, we have

$$\fint_{Q_{\varrho,s}(z_0)} |u - (u)_{Q_{\varrho,s}(z_0)}|^\theta dz \leq c\varrho^\theta \lambda^{\theta/p_0}, \tag{5.2}$$

where the constant c depends only on n, N, L, γ_2, and .

Proof. We apply Lemma 5.1, with $T_1 = T_2 = (t_0 - s, t_0 + s)$ and $\tilde{\Omega} = B_\varrho(x_0)$ to obtain

$$\fint_{Q_{\varrho,s}(z_0)} |u - (u)_{Q_{\varrho,s}(z_0)}|^\theta dz \leq c\,\varrho^\theta \fint_{Q_{\varrho,s}(z_0)} |Du|^\theta\, dz$$

$$
\begin{aligned}
&+ c\,\varrho^\theta \left(\lambda^{(2-p_0)/p_0} \fint_{Q_{\varrho,s}(z_0)} (|Du| + |F| + 1)^{p(\cdot)-1}\, dz \right)^\theta \\
&:= I_1 + I_2.
\end{aligned}
$$

We denote by $\tilde p_1$ and $\tilde p_2$ the infimum and supremum of $p(\cdot)$ over the intrinsic cylinder $Q_{\varrho,s}(z_0)$. As in (4.5), using (5.1), we deduce that $\lambda^{(\tilde p_2-\tilde p_1)/p_0} \le c(n)e^{2(n+3)L}$. We use Hölder's inequality, (5.1) and (5.2) to obtain

$$
I_1 \le c\,\varrho^\theta \left(\fint_{Q_{\varrho,s}(z_0)} (|Du| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \right)^{\theta/\tilde p_1(1-\varepsilon)} \le c\,\varrho^\theta \lambda^{\frac{\theta}{\tilde p_1}} \le c(n,,L)\,\varrho^\theta \lambda^{\frac{\theta}{p_0}}.
$$

To estimate I_2, we impose a bound for $\varrho \le \varrho_0$ such that $\omega(4\varrho_0) < \frac{1}{\gamma_2}$. Since $\varepsilon < \frac{1}{2\gamma_2}$ by assumption, we conclude that

$$
\frac{1}{\gamma_2} + \tilde p_1 \varepsilon < \frac{1}{\gamma_2} + \frac{1}{2} \le 1, \quad \text{which implies} \quad \tilde p_2 - \tilde p_1 \le \omega(4\varrho_0) < \frac{1}{\gamma_2} < 1 - p_1\varepsilon.
$$

We use the estimate above and Hölder's inequality to conclude that

$$
\begin{aligned}
I_2 &\le c\,\varrho^\theta \left(\lambda^{\frac{2-p_0}{p_0}} \left(\fint_{Q_{\varrho,s}(z_0)} (|Du| + |F| + 1)^{p(\cdot)\frac{\tilde p_2-1}{\tilde p_1}}\, dz \right) \right)^\theta \\
&\le c\,\varrho^\theta \left(\lambda^{\frac{2-p_0}{p_0}} \left(\fint_{Q_{\varrho,s}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \right)^{\frac{\tilde p_2-1}{\tilde p_1(1-\varepsilon)}} \right)^\theta \\
&\le c\,\varrho^\theta \left(\lambda^{\frac{2-p_0}{p_0}} \lambda^{\frac{\tilde p_2-1}{\tilde p_1}} \right)^\theta \le c\,\varrho^\theta \lambda^{\frac{\theta}{p_0}} \lambda^{\frac{\theta(\tilde p_1-p_0)}{p_0\tilde p_1} + \frac{\theta(\tilde p_2-\tilde p_1)}{\tilde p_1}} \le c\,\varrho^\theta \lambda^{\frac{\theta}{p_0}},
\end{aligned}
$$

since $\lambda \ge 1$, $p_0 > 1$ and $\lambda^{(p_2-\tilde p_1)/p_0} \le c(n)e^{2(n+3)L}$. This proves the Corollary. □

6. Caccioppoli type inequality

We now state the Caccioppoli inequality for very weak solutions to (2.1). However, the proof of the Caccioppoli inequality will be one of the main difficulties in proving the higher integrability for very weak solutions. Since the solution itself is not an admissible testing function, we will use the function $\tilde v$, constructed in (4.11) instead. However, it is quite delicate to prove the necessary estimates for $\tilde v$. This will be achieved in § 6.1. To simplify the notation, we denote

$$
\mu \equiv \mu(\varrho, \varrho_1, \varrho_2) := \left(\frac{\varrho}{\varrho_2 - \varrho_1} \right)^\beta
$$

for some constant β that only depends on $n, N, , L$. The precise value of β may change from line to line. Now, we state our Caccioppoli type inequality as follows:

Theorem 6.1. *Let $M \ge 1$. Then there exist $\varepsilon = \varepsilon(n, N, L, \gamma_2)$ and $\varrho_0 = \varrho_0(n, M, \gamma_1, L)$ such that the following holds: Suppose that u is a very weak solution to the parabolic system (2.5) and let the assumptions of Theorem*

2.2 be satisfied. Finally, assume that for some parabolic cylinder $Q := Q_\varrho^{(\lambda)}(z_0)$ *with* $32Q = Q_{32\varrho}^{(\lambda)}(z_0) \subset \Omega_T$ *with* $0 < 32\varrho \le \varrho_0$ *the following intrinsic coupling holds:*

$$\lambda^{1-\varepsilon} \le \fint_Q (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \tag{6.1}$$

and

$$\fint_{16Q} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \le \lambda^{1-\varepsilon}. \tag{6.2}$$

Then, specifically, for $\varrho_1 = \varrho$ *and* $\varrho_2 = 16\varrho$ *we have*

$$\begin{aligned}\fint_Q |Du|^{p(z)(1-\varepsilon)} dz + \sup_{t\in\Lambda} \fint_{B\times\{t\}} |u - u_Q|^2 m_{16Q}^{-\varepsilon}\, dx &\le c \fint_{2Q} \left|\frac{u - u_Q}{\varrho}\right|^{p(\cdot)(1-\varepsilon)} dz + c\lambda^{\frac{p_0-2}{p_0}-\varepsilon} \fint_{2Q} \left|\frac{u - u_Q}{\varrho}\right|^2 dz \\ &\quad + c \fint_{2Q} (1 + |F|)^{p(\cdot)(1-\varepsilon)} dz,\end{aligned} \tag{6.3}$$

where $m_{16Q}(z) = \max\{(c_E\tilde\lambda)^{1/(1-\varepsilon)}, M_{16Q}(z)\}$ *and* $\tilde\lambda$ *is defined in* (4.7). *Moreover, for* $\varrho \le \varrho_1 < \varrho_2 \le 16\varrho$*, there holds*

$$\begin{aligned}\sup_{t\in\Lambda_{\varrho_1}^{(\lambda)}(t_0)} \fint_{B_{\varrho_1}(x_0)\times\{t\}} |u - u_{Q_{\varrho_1}^{(\lambda)}(z_0)}|^2 m_{16Q}^{-\varepsilon} dx &\le c\mu \fint_{Q_{\varrho_2}^{(\lambda)}(z_0)} \left|\frac{u - u_{Q_{\varrho_1}^{(\lambda)}(z_0)}}{\varrho}\right|^{p(\cdot)(1-\varepsilon)} dz \\ &\quad + c\mu\lambda^{\frac{p_0-2}{p_0}-\varepsilon} \fint_{Q_{\varrho_2}^{(\lambda)}(z_0)} \left|\frac{u - u_{Q_{\varrho_1}^{(\lambda)}(z_0)}}{\varrho}\right|^2 dz + c\mu\lambda^{1-\varepsilon}|Q|.\end{aligned} \tag{6.4}$$

In any case, the constants c depend only on n, ν*, L and* γ_2*.*

We remark that the arbitrariness of the parameters ϱ_1 and ϱ_2 in (6.4) is only needed in the proof of the estimates for the lower order terms in § 7.

6.1. **Estimates for the test functions.** To start with, we state a simple geometric lemma without a proof.

Lemma 6.2. *Let* $Q \subset \mathbb{R}^{n+1}$ *be any parabolic cylinder and* $\tilde Q \subset \mathbb{R}^{n+1}$ *a parabolic cylinder centered at some point* $z \in Q$*. Then, for any* $\alpha > 1$ *there holds*

$$|Q \cap \alpha\tilde Q| \le c(n)\,\alpha^{n+2}|Q \cap \tilde Q|.$$

We remark that from the proof of Lemma 4.2, each Whitney type parabolic cylinder Q_i is centered in $Q^{(4)}$. This enables us to use Lemma 6.2 with $(\tilde Q, Q)$ replaced by $(Q_i, Q^{(4)})$. Next, we wish to investigate uniform estimates for the mean values of $|Du| + \frac{1}{\varrho}|u - u_{Q^{(1)}}| + |F| + 1$ on the Whitney cylinders. This is the result of the following lemma.

Lemma 6.3. *Let $\lambda_1 \geq c_E\tilde{\lambda}$ and $Q_i \subset \mathbb{R}^{n+1}$ be a parabolic cylinder of Whitney type. Then for $1 \leq \theta \leq p_1/\tilde{q}$ we have*

$$\fint_{Q^{(4)}\cap Q_i} \Big(|Du| + \tfrac{1}{\varrho}|u - u_{Q^{(1)}}| + |F| + 1\Big)^\theta dz \leq c\lambda_1^{\theta/p_0^i}$$

for some constant c depending only on n, N, L, θ, γ_2, and $\tilde{q}$.

Proof. To simplify the notation, we write $\mathfrak{U} := |Du| + \frac{1}{\varrho}|u - u_{Q^{(1)}}| + |F| + 1$. It is easily seen that there exists $\hat{z} \in E(\lambda_1)$ and a parabolic cylinder Q' with $Q' \cap E(\lambda_1) \neq \emptyset$ and $\hat{z} \in Q'$ such that $Q^{(4)} \cap 4Q_i \subset Q'$ and $|Q'| \leq c(n)|Q^{(4)} \cap 4Q_i|$. Therefore we conclude that

$$\fint_{Q^{(4)}\cap 4Q_i} \mathfrak{U}^{p(\cdot)/\tilde{q}}\, dz \leq c \fint_{Q'} \mathfrak{U}^{p(\cdot)/\tilde{q}}\, dz \leq cM_{Q^{(4)}}(\hat{z})^{1/(\tilde{q}(1-\varepsilon))} \leq c\lambda_1^{1/\tilde{q}}.$$

Next, we use Lemma 6.2, Hölder's inequality and the estimate above to obtain

$$\fint_{Q^{(4)}\cap Q_i} \mathfrak{U}^\theta\, dz \leq c\left(\fint_{Q^{(4)}\cap 4Q_i} \mathfrak{U}^{p(\cdot)/\tilde{q}} dz\right)^{\theta\tilde{q}/p_1^i} \leq c\lambda_1^{\theta/p_1^i} \leq c\lambda_1^{\theta/p_0^i},$$

where we used Lemma 4.1 for the last estimate. This proves the Lemma. □

Lemma 6.4. *Let $\lambda_1 \geq c_E\tilde{\lambda}$ and $Q_i \subset \mathbb{R}^{n+1}$ be a parabolic cylinder of Whitney type. Then for $1 \leq \theta \leq p_1/\tilde{q}$ there holds,*

$$\fint_{Q^{(4)}\cap Q_i} |v|^\theta dz \leq c\varrho^\theta \lambda_1^{\theta/p_0^i} \quad \text{and} \quad \fint_{Q^{(4)}\cap Q_i} |Dv|^\theta dz \leq c\left(\tfrac{\varrho}{\varrho_2-\varrho_1}\right)^\theta \lambda_1^{\theta/p_0^i}.$$

Proof. From the definition of v and Lemma 6.3 we immediately deduce the first estimate. To get the estimate for Dv, we first compute

$$|Dv| \leq \zeta\eta|Du| + \zeta|u - u_{Q^{(1)}}||D\eta| \leq |Du| + \tfrac{c}{\varrho_2-\varrho_1}|u - u_{Q^{(1)}}|,$$

where we used that $|D\eta| \leq c/(\varrho_2 - \varrho_1)$. At this point, Lemma 6.3 immediately yields the second estimate of the lemma. □

Lemma 6.5. *Let $\lambda_1 \geq c_E\tilde{\lambda}$ and $Q_i \subset \mathbb{R}^{n+1}$ be a parabolic cylinder of Whitney type with $Q^{(3)} \cap Q_i \neq \emptyset$ and $i \in \Theta_1$. Then for $1 \leq \theta \leq p_1/\tilde{q}$ there holds,*

$$\fint_{Q^{(4)}\cap Q_i} |v - v_{Q^{(4)}\cap Q_i}|^\theta dz \leq c\mu \min\{r_i, \varrho\}^\theta \lambda_1^{\theta/p_0^i},$$

where the constant c depends only on n, N, L, and γ_2.

Proof. Initially, we observe that $i \in \Theta_1$ yields $\Lambda_i \subset \Lambda^{(2)}$ and there exists a point $y \in B_i \cap B^{(4)}$ such that $B_{c_1 \min\{r_i,\varrho\}}(y) \subset B_i \cap B^{(4)} \subset B_{c_2 \min\{r_i,\varrho\}}(y)$ for some constants $0 < c_1 < c_2$. From the proof of [6, Lemma 5.11] we can construct a weight function $\tilde{\eta} \in C_0^\infty(B_i \cap B^{(4)})$ satisfying $\tilde{\eta} \geq 0$, $\int_{\mathbb{R}^n} \tilde{\eta} dx = 1$ and $|D\tilde{\eta}| \leq c\max\{r_i^{-(1+n)}, \varrho^{-(1+n)}\}$. As in [6, Lemma 5.11] we find that

$$\fint_{Q^{(4)}\cap Q_i} |v - v_{Q^{(4)}\cap Q_i}|^\theta dz \leq c \fint_{Q^{(4)}\cap Q_i} |v - v_{\tilde{\eta}}|^\theta dz + c \max_{t_1,t_2\in\Lambda_i} \left|v_{\tilde{\eta}}(t_2) - v_{\tilde{\eta}}(t_1)\right|^\theta,$$

where $v_{\tilde{\eta}}(t) := \int_{R^n}(v\tilde{\eta})(\cdot,t)\,dx$. We now apply Poincaré's inequality slice-wise to the function $v(\cdot,t)$ to obtain

$$\fint_{Q^{(4)}\cap Q_i} |v - v_{Q^{(4)}\cap Q_i}|^\theta dz \le c\min\{r_i,\varrho\}^\theta \fint_{Q^{(4)}\cap Q_i} |Dv|^\theta dz + c \sup_{t_1,t_2\in\Lambda_i} \left|v_{\tilde{\eta}}(t_2) - v_{\tilde{\eta}}(t_1)\right|^\theta .$$

Observe that we have $\zeta(t) = 1$ for any $t \in \Lambda_i \subset \Lambda^{(2)}$ and $v(x,t) = [u(x,t) - u_{Q^{(1)}}]\eta(x)$. Then we use the Steklov form (3.1) of the parabolic system with $\varphi = \eta\tilde{\eta}$, and obtain for $h > 0$ the following estimate:

$$\begin{aligned}\left|([u]_h)_{\eta\tilde{\eta}}(t_2) - ([u]_h)_{\eta\tilde{\eta}}(t_1)\right| &= \left|\int_{t_1}^{t_2}\int_{B^{(3)}\cap B_i} [u]_h \eta\tilde{\eta}\,dxdt\right| \\ &\le \int_{t_1}^{t_2}\int_{B^{(3)}\cap B_i} |\langle [A(z,Du)]_h, D(\eta\tilde{\eta})\rangle| + |\langle [B(z,Du)]_h, \eta\tilde{\eta}\rangle|\,dxdt.\end{aligned}$$

Letting $h \downarrow 0$ and using assumption (2.2) we find that

$$\left|v_{\tilde{\eta}}(t_2) - v_{\tilde{\eta}}(t_1)\right| \le c(1 + \|D(\eta\tilde{\eta})\|_{L^\infty}) \int_{Q^{(3)}\cap Q_i} (1 + |Du| + |F|)^{p(\cdot)-1}\,dz.$$

To estimate the right hand side of the above estimate, we will distinguish between the cases $r_i \ge \varrho$ and $r_i < \varrho$.

In the case $r_i \ge \varrho$, we see that $|D(\eta\tilde{\eta})| \le c|D\eta| + c|D\tilde{\eta}| \le c\mu\varrho^{-(1+n)}$. Using also Hölder's inequality and (4.5) it follows that

$$\begin{aligned}\left|v_{\tilde{\eta}}(t_2) - v_{\tilde{\eta}}(t_1)\right| &\le c\mu\varrho^{-(1+n)}|Q^{(5)}| \left(\fint_{Q^{(5)}} (1 + |Du| + |F|)^{p(\cdot)(1-\varepsilon)}\,dz\right)^{\frac{p_2-1}{p_1(1-\varepsilon)}} \\ &\le c\mu\varrho^{-(1+n)}\varrho^{2+n}\lambda^{(2-p_0)/p_0}\lambda^{(p_2-1)/p_1} \le c\mu\varrho\lambda^{1/p_0} \le c\mu\varrho\lambda^{1/p_0^i}.\end{aligned}$$

In the case $r_i < \varrho$ we have $|D(\eta\tilde{\eta})| \le c\mu r_i^{-(1+n)}$. Using this information and applying Lemma 6.3 with the choice $\theta = p_2^i - 1$ we find that

$$\begin{aligned}\left|v_{\tilde{\eta}}(t_2) - v_{\tilde{\eta}}(t_1)\right| &\le c\mu r_i^{-(1+n)} \int_{Q^{(3)}\cap Q_i} (1 + |Du| + |F|)^{p_2^i-1}\,dz \\ &\le c\,\mu r_i^{-(1+n)}\lambda_1^{(p_2^i-1)/p_0^i}|Q_i| \le c\mu r_i\lambda_1^{1/p_0^i}\lambda_1^{(p_1^i-p_0^i)/(p_0^i p_1^i)} \le c\mu r_i\lambda_1^{1/p_0^i},\end{aligned}$$

which proves the desired estimate. □

Remark 6.6. From Lemma 6.4 we conclude that

$$\left|v_{Q_i\cap Q^{(4)}}\right| \le \fint_{Q_i\cap Q^{(4)}} |v|dz \le c\mu\varrho\lambda_1^{1/p_0^i} \tag{6.5}$$

and furthermore for any $i \in \mathbb{N}$, we have

(6.6)

$$\|\tilde{v}\|_{L^\infty(2Q_i)} = \left\|\sum_{j\in I(i)}^\infty v_{Q_j\cap Q^{(4)}}\psi_j\right\|_{L^\infty} \le \sup_{j\in I(i)}\left|v_{Q_j\cap Q^{(4)}}\right| \le c\mu \sup_{j\in I(i)} \varrho\lambda_1^{1/p_0^j} \le c\mu\varrho\lambda_1^{1/p_0^i},$$

where we have used Lemma 4.1 for the last estimate. For a fixed $i \in \mathbb{N}$, we know by Lemma 4.2 (v) that $Q_i \subset \widehat{Q}_j$ for any $j \in I(i)$ and therefore $\widehat{Q}_i = \hat{c}Q_i \subset \hat{c}\widehat{Q}_j = \hat{c}^2 Q_j$. From Lemma 6.2, we get

$$|Q^{(4)} \cap \widehat{Q}_i| \le |Q^{(4)} \cap \hat{c}^2 Q_j| \le c(n)\hat{c}^{2n+4}|Q^{(4)} \cap Q_j|. \tag{6.7}$$

Let $\hat{p}_1^i$ and $\hat{p}_2^i$ be the infimum and supremum of $p(\cdot)$ over the intrinsic cylinder $\widehat{Q}_i$. We use Lemma 4.1 with (α, r_z) replaced by $(\hat{c}, r_i)$ to obtain $\lambda_1^{\hat{p}_2^i - \hat{p}_1^i} \le c(n, L, \gamma_2)$. For $i \in \Theta_1$, we now apply the proof of Lemma 6.5 again with p_1^i and p_2^i replaced by $\hat{p}_1^i$ and $\hat{p}_2^i$, to obtain

$$⨍_{Q^{(4)}\cap\widehat{Q}_i} |v - v_{Q^{(4)}\cap\widehat{Q}_i}| dz \le c\mu \min\{r_i, \varrho\}\lambda_1^{1/p_0^i}. \tag{6.8}$$

We now use (6.7) and (6.8) to deduce

$$|v_{Q^{(4)}\cap Q_j} - v_{Q^{(4)}\cap Q_i}| \le c\frac{|Q^{(4)} \cap \widehat{Q}_i|}{|Q^{(4)} \cap Q_j|} ⨍_{Q^{(4)}\cap\widehat{Q}_i} |v - v_{Q^{(4)}\cap\widehat{Q}_i}| dz \le c\mu \min\{r_i, \varrho\}\lambda_1^{1/p_0^i}. \tag{6.9}$$

Lemma 6.7. *Let $\lambda_1 \ge c_E\tilde{\lambda}$ and $Q_i \subset \mathbb{R}^{n+1}$ be a parabolic cylinder of Whitney type with $Q^{(3)} \cap Q_i \neq \emptyset$. Then for $z \in Q^{(4)} \cap 2Q_i$ we can bound $D\tilde{v}$ as follows: In the case $i \in \Theta_1$, or $i \in \Theta_2$ and $\varrho \le r_i$ there holds*

$$\varrho^{-1}|\tilde{v}(z)| + |D\tilde{v}(z)| \le c\mu\lambda_1^{1/p_0^i}. \tag{6.10}$$

In the case $i \in \Theta_1$ we have for any $\delta \in (0, 1)$ that

$$r_i^{-1}|\tilde{v}(z)| + |D\tilde{v}(z)| \le \tfrac{c\mu}{\delta}\lambda_1^{1/p_0^i} + c\delta r_i^{-2}\lambda_1^{-1/p_0^i}|v_{Q^{(4)}\cap Q_i}|^2. \tag{6.11}$$

In the case $i \in \Theta_2$ there holds with $s = \lambda^{(2-p_0)/p_0}(\varrho_2 - \varrho_1)\varrho$ that

$$r_i^{-1}|\tilde{v}(z)| + |D\tilde{v}(z)| \le c\lambda_1^{1/p_0^i} + cs^{-1}\lambda_1^{(1-p_0^i)/p_0^i} ⨍_{Q^{(4)}\cap\widehat{Q}_i} |v|^2 dz. \tag{6.12}$$

Moreover, we can bound the time-derivative $\partial_t\tilde{v}$ as follows: In the case $i \in \Theta_1$ there holds

$$|\partial_t\tilde{v}(z)| \le c\mu\lambda_1^{(p_0^i-1)/p_0^i} r_i^{-2}\min\{r_i, \varrho\}. \tag{6.13}$$

In the case $i \in \Theta_2$ we have

$$|\partial_t\tilde{v}(z)| \le c\mu s^{-1}\varrho\lambda_1^{1/p_0^i}. \tag{6.14}$$

In any cases the constant c depends only on n, N, L, and γ_2.

Proof. Let us first prove (6.10). For $z \in Q^{(4)} \cap Q_i$, we note that $\sum_{j\in I(i)} \psi_j(z) \equiv 1$ and this implies $\sum_{j\in I(i)} D\psi_j(z) \equiv 0$. In the case $i \in \Theta_1$, we apply (6.9) and Lemma 4.2 (iii) to infer that

$$|D\tilde{v}(z)| = \Big|\sum_{j\in I(i)} [v_{Q_j\cap Q^{(4)}} - v_{Q_i\cap Q^{(4)}}]D\psi_j(z)\Big| \le c\mu r_i^{-1}\min\{r_i, \varrho\}\lambda_1^{1/p_1^i} \le c\mu\lambda_1^{1/p_0^i}.$$

While in the case $i \in \Theta_2$ and $\varrho \le r_i$ we use (6.5) to obtain

$$|D\tilde{v}(z)| \le \sum_{j\in I(i)} |v_{Q_j\cap Q^{(4)}}||D\psi_j(z)| \le c\mu\varrho r_i^{-1}\lambda_1^{1/p_1^i} \le c\mu\lambda_1^{1/p_0^i}.$$

The estimates above together with (6.6) yield (6.10). To estimate (6.11) and (6.13), we only need to consider the case $i \in \Theta_1$. We conclude from (6.9) for $\delta \in (0,1)$ that

$$\begin{aligned}|\tilde{v}(z)| &= \Big|\sum_{j\in I(i)} [v_{Q_j\cap Q^{(4)}} - v_{Q_i\cap Q^{(4)}}]\psi_j(z)\Big| + |v_{Q_i\cap Q^{(4)}}| \\ &\le \tfrac{c\mu}{\delta} r_i\lambda_1^{1/p_1^i} + c\delta\lambda_1^{-1/p_1^i} r_i^{-1}|v_{Q_i\cap Q^{(4)}}|^2,\end{aligned}$$

where we have used Young's inequality and the fact that $\mu \ge c$. The estimate above and (6.10) imply (6.11). Since $\sum_{j\in I(i)} \partial_t\psi_j(z) \equiv 0$, we now use Lemma 4.2 (iii), Lemma 4.3 and (6.9) again to find that

$$|\partial_t\tilde{v}(z)| = \Big|\sum_{j\in I(i)} [v_{Q_j\cap Q^{(4)}} - v_{Q_i\cap Q^{(4)}}]\partial_t\psi_j(z)\Big| \le c\mu\lambda_1^{(p_0^i-1)/p_0^i} r_i^{-2}\min\{r_i,\varrho\},$$

which proves (6.13). We now turn our attention to the case when $i \in \Theta_2$. From (6.7) and Young's inequality, we infer that

$$|v_{Q_i\cap Q^{(4)}}| \le \frac{|Q^{(4)}\cap \widehat{Q}_i|}{|Q^{(4)}\cap Q_j|} \fint_{\widehat{Q}_i\cap Q^{(4)}} |v|dz \le c\fint_{\widehat{Q}_i\cap Q^{(4)}} |v|dz \le cr_i\lambda_1^{1/p_0^i} + r_i^{-1}\lambda_1^{-1/p_0^i}\fint_{\widehat{Q}_i\cap Q^{(4)}} |v|^2dz.$$

Since $i \in \Theta_2$, we see that $\lambda_1^{(2-p_0^i)/p_0^i} r_i^2 \ge cs$ and $|\partial_t\psi_j(z)| \le c\lambda_1^{(p_0^i-2)/p_0^i} r_i^{-2} \le cs^{-1}$. From Lemma 4.1, we conclude that

$$\begin{aligned}r_i^{-1}|\tilde{v}(z)| + |D\tilde{v}(z)| &\le \sum_{j\in I(i)} r_j^{-1}|v_{Q_j\cap Q^{(4)}}||\psi_j(z)| + \sum_{j\in I(i)} |v_{Q_j\cap Q^{(4)}}||D\psi_j(z)| \\ &\le c\lambda_1^{1/p_0^i} + cs^{-1}\lambda_1^{(1-p_0^i)/p_0^i}\fint_{\widehat{Q}_i\cap Q^{(4)}} |v|^2dz,\end{aligned}$$

which proves (6.12). Next, we use (6.5) to obtain

$$|\partial_t\tilde{v}(z)| \le \sum_{j\in I(i)} |v_{Q_j\cap Q^{(4)}}||\partial_t\psi_j(z)| \le c\mu s^{-1}\varrho\lambda_1^{1/p_0^i}.$$

This proves (6.14) and the proof of Lemma 6.7 is complete. □

Lemma 6.8. *Suppose that $\lambda_1 \ge c_E\tilde{\lambda}$ and $Q_i \subset \mathbb{R}^{n+1}$ is a parabolic cylinder of Whitney type with $Q^{(3)} \cap Q_i \neq \emptyset$. Then, there holds:*

$$\int_{Q^{(4)}\setminus E(\lambda_1)} |\tilde{v}|^2dz \le c\int_{Q^{(3)}\setminus E(\lambda_1)} |v|^2dz. \tag{6.15}$$

In the case $i \in \Theta_1$, we have

$$\fint_{Q^{(4)}\cap Q_i} |\tilde{v} - v|dz \le c\mu\min\{r_i,\varrho\}\lambda_1^{1/p_0^i}. \tag{6.16}$$

In the case $i \in \Theta$, there holds

$$\int_{B^{(4)}\times S_1} |\partial_t \tilde{v} \cdot (\tilde{v} - v)| dz \le c\mu\lambda_1 |Q^{(4)} \backslash E(\lambda_1)| + \frac{c}{s} \int_{Q^{(4)}} |v|^2 dz, \tag{6.17}$$

where $s = \lambda^{(2-p_0)/p_0}(\varrho_2 - \varrho_1)\varrho$. In any cases the constant c depends only on n, N, L and γ_2.

Proof. We begin with the proof of (6.15). Since $Q^{(4)}\backslash E(\lambda_1) = \bigcup_{i=1}^{\infty}(Q^{(4)} \cap \widehat{Q}_i)$ and the collection $\{\widehat{Q}_i\}_{i=1}^{\infty}$ has a finite overlap, we infer that

$$\begin{aligned}
\int_{Q^{(4)}\backslash E(\lambda_1)} |\tilde{v}|^2 dz &\le c \sum_{i=1}^{\infty} \sum_{j \in I(i)} \frac{|Q^{(4)} \cap \widehat{Q}_i|}{|Q^{(4)} \cap Q_j|} \int_{Q_j \cap Q^{(4)}} |v|^2 dz \\
&\le c \sum_{i=1}^{\infty} \int_{\widehat{Q}_i \cap Q^{(4)}} |v|^2 dz \\
&\le c \int_{Q^{(3)}\backslash E(\lambda_1)} |v|^2 dz,
\end{aligned}$$

since $\operatorname{supp} v \subset Q^{(3)}$ and this proves (6.15). Next, we turn our attention to the case when $i \in \Theta_1$. From Lemma 6.5 and (6.9), we obtain

$$\begin{aligned}
\fint_{Q_i \cap Q^{(4)}} |\tilde{v} - v|^\theta dz &\le \fint_{Q_i \cap Q^{(4)}} |v - v_{Q_i \cap Q^{(4)}}|^\theta dz + \sum_{j \in I(i)} \|\psi_j\|_{L^\infty}^\theta |v_{Q_j \cap Q^{(4)}} - v_{Q_i \cap Q^{(4)}}|^\theta \\
&\le c\mu \min\{r_i, \varrho\}^\theta \lambda_1^{\theta/p_1^i},
\end{aligned}$$

which proves (6.16). Finally we consider the proof of (6.17). We write the left hand side of (6.17) as follows:

$$\begin{aligned}
\int_{B^{(4)}\times S_1} |\partial_t \tilde{v} \cdot (\tilde{v} - v)| dz &\le \sum_{i \in \Theta_1} \int_{Q^{(4)} \cap Q_i} |\partial_t \tilde{v} \cdot (\tilde{v} - v)| dz + \sum_{i \in \Theta_2} \int_{Q^{(4)} \cap Q_i} |\partial_t \tilde{v} \cdot (\tilde{v} - v)| dz \\
&:= L_1 + L_2.
\end{aligned}$$

We now apply (6.13) and (6.16) to get the estimate for L_1:

$$L_1 \le c\mu \sum_{i \in \Theta_1} |Q^{(4)} \cap Q_i| r_i \lambda_1^{1/p_1^i} \lambda_1^{(p_0^i - 1)/p_0^i} r_i^{-1} \le c\mu\lambda_1 |Q^{(4)} \backslash E(\lambda_1)|,$$

since $\lambda_1^{(p_0^i - p_1^i)/p_0^i p_1^i} \le c$. To estimate L_2, we observe that

$$\int_{Q^{(4)} \cap Q_i} |\tilde{v}| dz \le c \sum_{j \in I(i)} \frac{|Q^{(4)} \cap \widehat{Q}_i|}{|Q_j \cap Q^{(4)}|} \int_{\widehat{Q}_i \cap Q^{(4)}} |v| dz \le c \int_{\widehat{Q}_i \cap Q^{(4)}} |v| dz.$$

Since $i \in \Theta_2$, there holds

$$|\partial_t \tilde{v}| \le c\lambda_1^{(p_0^i - 2)/p_0^i} r_i^{-2} \fint_{Q^{(4)} \cap \widehat{Q}_i} |v| dz \le c s^{-1} \fint_{Q^{(4)} \cap \widehat{Q}_i} |v| dz.$$

Therefore, we can bound L_2 from above as follows:

$$
\begin{aligned}
L_2 &\le cs^{-1} \sum_{i\in\Theta_2} \left(\fint_{Q^{(4)}\cap\widehat{Q}_i} |v| dz \right) \left(\int_{Q^{(4)}\cap Q_i} |v| dz + \int_{Q^{(4)}\cap Q_i} |\tilde{v}| dz \right) \\
&\le cs^{-1} \sum_{i\in\Theta_2} |Q^{(4)} \cap \widehat{Q}_i| \left(\fint_{Q^{(4)}\cap\widehat{Q}_i} |v| dz \right)^2 \\
&\le cs^{-1} \int_{Q^{(3)}} |v|^2 dz,
\end{aligned}
$$

which proves (6.17) and the proof of Lemma 6.8 is complete. □

6.2. **Estimates on the super-level set $Q^{(4)} \backslash E(\lambda_1)$.** Recalling that for $i \in \mathbb{N}$ the enlarged cylinder $\widehat{Q}_i = \hat{c}Q_i \equiv Q^{(\lambda_1)}_{\hat{r}_i}(z_i)$ has the radius $\hat{r}_i = \hat{c}r_i$ where $\hat{c} = \hat{c}(n, L, \gamma_2)$ denotes the constant from Lemma 4.2 (v).

Lemma 6.9. *Let $\lambda_1 \ge c_E\tilde{\lambda}$ and $Q_i \subset \mathbb{R}^{n+1}$ be a parabolic cylinder of Whitney type with $Q^{(3)} \cap Q_i \ne \emptyset$ with $i \in \Theta$. Then, with $s = \lambda^{(2-p_0)/p_0}(\varrho_2 - \varrho_1)\varrho$, there holds:*

(6.18)

$$
\int_{Q^{(3)}\cap 2Q_i} (|Du| + |F| + 1)^{p(z)-1} \big[(\varrho^{-1}\delta_1(i) + \max\{\varrho^{-1}, r_i^{-1}\}\delta_2(i))|\tilde{v}| + |D\tilde{v}| \big] dz \\
\le c\mu\lambda_1 |Q^{(4)} \cap Q_i| + c\delta_2(i) s^{-1} \int_{Q^{(3)}\cap\widehat{Q}_i} |v|^2 dz.
$$

and in the case $\hat{r}_i < (\varrho_2 - \varrho_1)/12$ we have for any $\varepsilon_1 \in (0, 1)$ that

(6.19)

$$
\int_{Q^{(3)}\cap 2Q_i} (|Du| + |F| + 1)^{p(z)-1} \big[r_i^{-1} |\tilde{v}| + |D\tilde{v}| \big] dz \\
\le \tfrac{c\mu}{\varepsilon_1} \lambda_1 |Q^{(4)} \cap Q_i| + \varepsilon_1 \delta_1(i) |B_i| |v_{Q^{(4)}\cap Q_i}|^2 + c\delta_2(i) s^{-1} \int_{Q^{(3)}\cap\widehat{Q}_i} |v|^2 dz.
$$

Here $\delta_1(i) \equiv 1$ if $i \in \Theta_1$ and $\delta_1(i) \equiv 0$ otherwise, and $\delta_2(i) \equiv 1$ if $i \in \Theta_2$ and $\delta_2(i) \equiv 0$ otherwise.

Proof. We first note that Lemma 6.3 also holds with $2Q_i$ instead of Q_i with a larger constant, i.e.

$$
\fint_{Q^{(4)}\cap 2Q_i} (1 + |Du| + |F|)^{p(z)-1} \, dz \le c\lambda_1^{(p_2^i - 1)/p_0^i} \le c\lambda_1^{(p_0^i - 1)/p_0^i}.
$$

In the case $i \in \Theta_1$ or $i \in \Theta_2$ and $\varrho \le r_i$, we use the last estimate, Lemma 6.2 and Lemma 6.7 (6.10) to obtain

$$
\int_{Q^{(3)}\cap 2Q_i} (|Du| + |F| + 1)^{p(z)-1} \big[\varrho^{-1} |\tilde{v}| + |D\tilde{v}| \big] dz \le c\mu\lambda_1 |Q^{(4)} \cap Q_i|.
$$

In the case $i \in \Theta_2$ and $\varrho \geq r_i$, we apply estimate (6.12) from Lemma 6.7 and Lemma 6.2 to get $\varrho^{-1} \leq r_i^{-1}$ and therefore

$$\begin{aligned}\int_{Q^{(3)}\cap 2Q_i} &(|Du| + |F| + 1)^{p(z)-1}[r_i^{-1}|\tilde{v}| + |D\tilde{v}|]dz \\ &\leq c\lambda_1 \lambda_1^{\frac{p_1^i - p_0^i}{p_0^i p_1^i}} |Q^{(4)} \cap (2Q_i)| + cs^{-1}\lambda_1^{\frac{p_1^i - p_0^i}{p_0^i p_1^i}} |Q^{(4)} \cap (2Q_i)| \fint_{\widehat{Q}_i \cap Q^{(4)}} |v|^2 dz \\ &\leq c\mu\lambda_1 |Q^{(4)} \cap Q_i| + cs^{-1} \int_{\widehat{Q}_i \cap Q^{(4)}} |v|^2 dz,\end{aligned}$$

since $\lambda_1 \geq 1$ and $p_1^i \leq p_0^i$. This proves (6.18) and (6.19) in the case $i \in \Theta_2$. It now remains to consider the case $\hat{r}_i < (\varrho_2 - \varrho_1)/12$ and $i \in \Theta_1$. We infer that $\widehat{Q}_i \subset Q^{(4)}$ in this case. From Lemma 6.2 and estimate (6.11) from Lemma 6.7 we find that

$$\begin{aligned}\int_{(2Q_i)\cap Q^{(3)}} &(|Du| + |F| + 1)^{p(z)-1}[r_i^{-1}|\tilde{v}| + |D\tilde{v}_h|]dz \\ &\leq \tfrac{c\mu}{\varepsilon_1}\lambda_1 |Q^{(4)} \cap (2Q_i)| + c\varepsilon_1 r_i^{-2} \lambda_1^{\frac{p_0^i - 2}{p_0^i}} \lambda_1^{\frac{p_0^i - p_1^i}{p_0^i p_1^i}} |Q_i| |v_{Q_j \cap Q^{(4)}}|^2 \\ &\leq \tfrac{c\mu}{\varepsilon_1}\lambda_1 |Q^{(4)} \cap Q_i| + c\varepsilon_1 |B_i| |v_{Q_j \cap Q^{(4)}}|^2,\end{aligned}$$

since $|Q_i| = |B_i| \times r_i^2 \lambda_1^{(2-p_0^i)/p_0^i}$ and the proof of Lemma 6.9 is complete. □

Remark 6.10. Under the assumptions in Lemma 6.9, we conclude that

$$\begin{aligned}\int_{Q^{(4)}\setminus E(\lambda_1)} &(|Du| + |F| + 1)^{p(z)-1}[(\varrho_2 - \varrho_1)^{-1}|\tilde{v}| + |D\tilde{v}|]dz \\ &\leq \mu \sum_{i=1}^{\infty} \int_{(2Q_i)\cap Q^{(3)}} (|Du| + |F| + 1)^{p(z)-1}[\varrho^{-1}|\tilde{v}| + |D\tilde{v}|]dz \\ &\leq c\mu\lambda_1 |Q^{(4)} \setminus E(\lambda_1)| + cs^{-1} \int_{Q^{(4)}\setminus E(\lambda_1)} |v|^2 dz.\end{aligned}$$

Lemma 6.11. *Let $\lambda_1 \geq c_E \tilde{\lambda}$. Then for any $i \in \Theta_1$, $\varepsilon_1 \in (0,1)$ and a.e. $t \in S_1$ there holds*

$$\left| \int_{B^{(4)}\times\{t\}} (v - v_{Q_i \cap Q^{(4)}})\tilde{v}\psi_i dx \right| \leq \tfrac{c\mu}{\varepsilon_1}\lambda_1 |Q^{(4)} \cap Q_i| + \mu\varepsilon_1 \delta_3(i) |B_i| |v_{Q_i \cap Q^{(4)}}|^2 \tag{6.20}$$

where $\delta_3(i) \equiv 1$ if $\hat{r}_i < (\varrho_2 - \varrho_1)/12$ and $\delta_3(i) \equiv 0$ otherwise. When $i \in \Theta_2$ then for a.e. $t \in S_1$ we have

$$\left| \int_{B^{(4)}\times\{t\}} v\tilde{v}\psi_i \, dx \right| \leq c\mu\lambda_1 |Q^{(4)} \cap Q_i| + cs^{-1} \int_{Q^{(3)}\cap\widehat{Q}_i} |u - u_{Q^{(1)}}|^2 dz. \tag{6.21}$$

Moreover, for a.e. $t \in S_1$ we have

$$\int_{(B^{(4)}\setminus E_t(\lambda_1))\times\{t\}} (|v|^2 - |v - \tilde{v}|^2)dx \geq -c\mu\lambda_1 |Q^{(4)} \setminus E(\lambda_1)| - c\mu s^{-1} \int_{Q^{(4)}} |u - u_{Q^{(1)}}|^2 dz. \tag{6.22}$$

Proof. From the definition of $\tilde v$, we find that $\tilde v = \sum_{j\in I(i)} (v)_{Q^{(4)}\cap Q_j}\psi_j \in C^\infty(2Q_i)$. This allows us to take the function $\varphi \equiv \eta\zeta\tilde v\psi_i$ as a test function in the Steklov formulation (3.1) of the parabolic system and we get

$$\int_{t_i-\lambda_1^{(2-p_0^i)/p_0^i} r_i^2}^{t} \int_{\mathbb{R}^n} \Big[\partial_t[u]_h\eta\zeta\tilde v\psi_i + \langle [A(z,Du)]_h, D(\eta\zeta\tilde v\psi_i)\rangle + \langle [B(z,Du)]_h, \eta\zeta\tilde v\psi_i\rangle\Big] dxd\tau = 0$$

for any $t \in S_1$. Let a be a constant which will be chosen later. Noting that $\psi_i(\cdot, t_i - \lambda_1^{(2-p_0^i)/p_0^i} r_i^2) \equiv 0$, we use integration by parts to obtain

$$\begin{aligned}\int_{t_i-\lambda_1^{(2-p_0^i)/p_0^i} r_i^2}^{t} \int_{\mathbb{R}^n} \partial_t[u]_h\eta\zeta\tilde v\psi_i dxd\tau \\ = \int_{\mathbb{R}^n\times\{t\}} [u-a]_h\eta\zeta\tilde v\psi_i dx - \int_{t_i-\lambda_1^{(2-p_0^i)/p_0^i} r_i^2}^{t} \int_{\mathbb{R}^n} [u-a]_h\eta\partial_t(\zeta\tilde v\psi_i) dxd\tau.\end{aligned}$$

We insert this in the previous equation and pass to the limit $h \to 0$. Then we apply the growth condition (2.2) and recall that $\operatorname{supp}(\eta\zeta) \subset Q^{(3)}$ to get

(6.23)
$$\begin{aligned}\left|\int_{B^{(4)}\times\{t\}} (u-a)\eta\zeta\tilde v\psi_i dx\right| &\le c\int_{Q^{(3)}} (1+|Du|+|F|)^{p(\cdot)-1}(|D(\eta\tilde v\psi_i)| + |\eta\tilde v\psi_i|)\zeta dz \\ &\quad + c\int_{Q^{(3)}} |u-a||\partial_t(\zeta\tilde v\psi_i)|\eta dz =: I + II.\end{aligned}$$

We begin with the estimate for I. Noting that $|D\eta| \le c(\varrho_2-\varrho_1)^{-1}$ and $|D\psi_i| \le cr_i^{-1}$ we see that $|D(\eta\tilde v\psi_i)| \le c|D\tilde v| + c\max\{\varrho^{-1}, r_i^{-1}\}|\tilde v|$. This implies for I that

$$I \le c\int_{Q^{(3)}\cap(2Q_i)} (1+|Du|+|F|)^{p(z)-1}(|D\tilde v| + \max\{\varrho^{-1}, r_i^{-1}\}|\tilde v|)\, dz. \tag{6.24}$$

We are now in a position to show (6.21), where we consider $i \in \Theta_2$. We use estimate (6.18) of Lemma 6.9 to infer that

$$I \le c\mu\lambda_1|Q^{(4)} \cap Q_i| + cs^{-1}\int_{Q^{(3)}\cap\widehat{Q}_i} |v_h|^2 dz.$$

To estimate II, we note that $|\partial_t\psi_i| \le c\lambda_1^{(p_0^i-2)/p_0^i} r_i^{-2} \le cs^{-1}$ and

$$\begin{aligned}|\partial_t(\zeta\tilde v\psi_i)| &\le c|\partial_t\tilde v| + c|\tilde v|(|\partial_t\zeta| + |\partial_t\psi_i|) \le c|\partial_t\tilde v| + cs^{-1}|\tilde v| \\ &\le cs^{-1}\sum_{j\in I(i)} \fint_{Q_j\cap Q^{(4)}} |v|dz \le cs^{-1}\fint_{\widehat{Q}_i\cap Q^{(4)}} |v|dz \\ &\le cs^{-1}\fint_{\widehat{Q}_i\cap Q^{(4)}} |u-u_{Q^{(1)}}|\chi_{Q^{(3)}} dz.\end{aligned}$$

We now choose $a = u_{Q^{(1)}}$ and subsequently use Höder's inequality to obtain

$$II \le cs^{-1}|\widehat{Q}_i \cap Q^{(3)}|\left(\fint_{\widehat{Q}_i\cap Q^{(4)}} |u-u_{Q^{(1)}}|\chi_{Q^{(3)}} dz\right)^2 \le cs^{-1}\fint_{\widehat{Q}_i\cap Q^{(3)}} |u-u_{Q^{(1)}}|^2 dz.$$

This proves (6.21). We now come to the proof of (6.20), where we consider $i \in \Theta_1$. In the case $\hat r_i \ge (\varrho_2-\varrho_1)/12$, we infer from (6.24) and estimate (6.18) from Lemma 6.9 that

$$\begin{aligned} I &\le c \int_{Q^{(3)}\cap(2Q_i)} (1+|Du|+|F|)^{p(z)-1}(|D\tilde v| + \max\{\varrho^{-1},(\varrho_2-\varrho_1)^{-1}\}|\tilde v|)dz \\ &\le c\mu \int_{Q^{(3)}\cap(2Q_i)} (1+|Du|+|F|)^{p(z)-1}(|D\tilde v| + \varrho^{-1}|\tilde v|)dz \\ &\le c\mu\lambda_1|Q^{(4)}\cap Q_i|. \end{aligned}$$

In the case $\hat r_i < (\varrho_2-\varrho_1)/12$, we use (6.24) and estimate (6.19) from Lemma 6.9 to obtain for $\varepsilon_1 \in (0,1)$ that

$$I \le \tfrac{c\mu}{\varepsilon_1}\lambda_1|Q^{(4)}\cap Q_i| + \varepsilon_1\delta_3(i)|B_i||v_{Q_i\cap Q^{(4)}}|^2, \tag{6.25}$$

where $\delta_3(i)\equiv 1$ in the case $\hat r_i < (\varrho_2-\varrho_1)/12$ and $\delta_3(i)\equiv 0$ otherwise. We now turn our attention to the estimate of II and start with the case $\hat r_i \ge (\varrho_2-\varrho_1)/12$. In (6.23) we choose $a = u_{Q^{(1)}}$. Since $i\in\Theta_1$, we see that $\zeta\equiv 1$ on $\operatorname{supp}\psi_i$ and

$$II \le c\int_{Q^{(3)}} |u-u_{Q^{(1)}}||\partial_t(\psi_i\tilde v)|dz \le c\int_{Q^{(3)}\cap(2Q_i)} |u-u_{Q^{(1)}}|\left[|\partial_t\tilde v| + \lambda_1^{(p_0^i-2)/p_0^i} r_i^{-2}|\tilde v|\right]dz.$$

We now apply estimate (6.13) from Lemma 6.7 to obtain

$$|\partial_t\tilde v| \le c\mu\lambda_1^{(p_0^i-1)/p_0^i} r_i^{-1} \le c\mu\lambda_1^{(p_0^i-1)/p_0^i}(\varrho_2-\varrho_1)^{-1} \le c\mu\lambda_1^{(p_0^i-1)/p_0^i}\varrho^{-1}.$$

Combining this with estimate (6.5) from Remark 6.6 we find that

$$\begin{aligned} II &\le c\int_{Q^{(3)}\cap(2Q_i)} |u-u_{Q^{(1)}}|\left[\mu\lambda_1^{(p_0^i-1)/p_0^i}\varrho^{-1} + \mu\varrho\lambda_1^{(p_0^i-p_1^i)/p_1^ip_0^i}\lambda_1^{(p_0^i-1)/p_0^i}(\varrho_2-\varrho_1)^{-2}\right]dz \\ &\le c\mu\lambda_1^{(p_0^i-1)/p_0^i}\varrho^{-1}|Q^{(4)}\cap(2Q_i)| \fint_{Q^{(4)}\cap(2Q_i)} |u-u_{Q^{(1)}}|\chi_{Q^{(3)}}dz \\ &\le c\mu\lambda_1|Q^{(4)}\cap Q_i|, \end{aligned}$$

where we have used Lemma 6.2 and Lemma 6.4. This implies that for $i\in\Theta_1$ and $\hat r_i \ge (\varrho_2-\varrho_1)/12$, we have

$$\left|\int_{B^{(4)}} v\tilde v\psi_i(\cdot,t)dx\right| = \left|\int_{B^{(4)}} (u-u_{Q^{(1)}})\eta\zeta\tilde v\psi_i(\cdot,t)dx\right| \le I + II \le c\mu\lambda_1|Q^{(4)}\cap Q_i|.$$

Next, since $i\in\Theta_1$ we find that $Q^{(4)}\cap 2Q_i = [B^{(4)}\cap(2B_i)]\times 2\Lambda_i$ and therefore

$$\begin{aligned} |B^{(4)}\cap(2B_i)| &= 2|\Lambda_i|^{-1}|Q^{(4)}\cap 2Q_i| \\ &\le c\lambda_1^{(p_0^i-2)/p_0^i}(\varrho_2-\varrho_1)^{-2}|Q^{(4)}\cap 2Q_i| \le c\mu\lambda_1^{(p_0^i-2)/p_0^i}\varrho^{-2}|Q^{(4)}\cap Q_i|. \end{aligned}$$

Joining the estimate above with estimate (6.5) from Remark 6.6 and (6.6), we conclude

$$\begin{aligned} \left|\int_{B^{(4)}} v_{Q_i\cap Q^{(4)}}\tilde v\psi_i(\cdot,t)dx\right| &\le c\mu\varrho^2\lambda_1^{2/p_1^i}|B^{(4)}\cap(2B_i)| \\ &\le c\mu\lambda_1\lambda_1^{2(p_0^i-p_1^i)/p_1^ip_0^i}|Q^{(4)}\cap Q_i| \le c\mu\lambda_1|Q^{(4)}\cap Q_i|, \end{aligned}$$

which proves (6.20) in the case $i \in \Theta_1$ and $\hat{r}_i \geq (\varrho_2 - \varrho_1)/12$. We now turn our attention to the case when $i \in \Theta_1$ and $\hat{r}_i \leq (\varrho_2 - \varrho_1)/12$. We first observe that $\widehat{Q}_i \subset Q^{(4)}$ and we choose $a = u_{\widehat{Q}_i}$ in this case. For $z \in (2Q_i) \cap Q^{(4)}$ we infer from estimate (6.11) of Lemma 6.7 and (6.13) that $\zeta(z) \equiv 1$ and

$$|\partial_t(\zeta\psi_i\tilde{v})(z)| \leq c|\partial_t\psi_i| \left[\tfrac{\mu}{\varepsilon_1}\lambda_1^{1/p_1^i} r_i + \varepsilon_1\lambda_1^{-1/p_1^i} r_i^{-1}|v_{Q_i\cap Q^{(4)}}|^2\right] + c\mu\lambda_1^{(p_0^i-1)/p_0^i} r_i^{-1}$$
$$\leq \tfrac{c\mu}{\varepsilon_1}\lambda_1^{(p_0^i-1)/p_0^i} r_i^{-1} + c|\Lambda_i|^{-1}\varepsilon_1\lambda_1^{-1/p_0^i} r_i^{-1}|v_{Q_i\cap Q^{(4)}}|^2.$$

To proceed further, we use Lemma 5.1 with $(\tilde{\Omega}, T_1, T_2, \theta)$ replaced by $(\tilde{B}_i, \tilde{T}_i, \tilde{T}_i, 1)$, Lemma 4.1 and Hölder's inequality to find that

$$\fint_{\widehat{Q}_i} |u - u_{\widehat{Q}_i}| dz \leq cr_i \fint_{\widehat{Q}_i} |Du| dz + c\lambda_1^{(2-p_0^i)/p_0^i} r_i \fint_{\widehat{Q}_i} (1 + |Du| + |F|)^{p(\cdot)-1} dz$$
$$\leq cr_i\lambda_1^{1/p_0^i}\lambda_1^{(p_0^i-\hat{p}_1^i)/p_0^i\hat{p}_1^i} + cr_i\lambda_1^{1/p_0^i}\lambda_1^{(\hat{p}_2^i-\hat{p}_1^i)/\hat{p}_1^i}\lambda_1^{(\hat{p}_1^i-p_0^i)/p_0^i\hat{p}_1^i} \leq cr_i\lambda_1^{1/p_0^i}.$$

Combining the estimates above we infer that

$$II = \int_{Q^{(3)}} |u - u_{\widehat{Q}_i}||\partial_t(\zeta\tilde{v}\psi_i)|\eta dz \leq c\mu\varepsilon_1^{-1}\lambda_1|\widehat{Q}_i| + c\varepsilon_1|\Lambda_i|^{-1}|\widehat{Q}_i||v_{Q_i\cap Q^{(4)}}|^2$$
$$\leq \tfrac{c\mu}{\varepsilon_1}\lambda_1|Q^{(4)} \cap Q_i| + c\varepsilon_1|B_i||v_{Q_i\cap Q^{(4)}}|^2.$$

Together this estimate and (6.25) yield that

$$\left|\int_{B^{(4)}\times\{t\}} (u - u_{\widehat{Q}_i})\eta\zeta\tilde{v}\psi_i dx\right| \leq c\mu\varepsilon_1^{-1}\lambda_1|Q^{(4)} \cap Q_i| + \varepsilon_1|B_i||v_{Q_i\cap Q^{(4)}}|^2.$$

Since $i \in \Theta_1$, we have $\zeta \equiv 1$ and $v = \eta(x)[u - u_{Q^{(1)}}]$ on $\operatorname{supp}\psi_i$. In order to prove (6.20), we note that

$$(v - v_{Q_i\cap Q^{(4)}})\tilde{v}\psi_i = (u - u_{\widehat{Q}_i})\eta\tilde{v}\psi_i + ((u_{\widehat{Q}_i} - u_{Q^{(1)}})\eta - v_{Q_i})\tilde{v}\psi_i.$$

Letting $U(x) := (u_{\widehat{Q}_i} - u_{Q^{(1)}})\eta(x) - v_{Q_i}$, we compute that for $x \in 2B_i$ that

$$|U(x)| \leq |U(x) - U_{B_i}| + |U_{B_i}| \leq cr_i \sup_{2B_i} |\nabla U| + |U_{B_i}| := U_1 + U_2.$$

Since $\widehat{Q}_i \cap E(\lambda_1) \neq \emptyset$ by Lemma 4.2 (iv), there exists $\tilde{z} \in 2\widehat{Q}_i \cap E(\lambda_1)$ and therefore, we have

$$U_1 \leq cr_i|D\eta||u_{\widehat{Q}_i} - u_{Q^{(1)}}| \leq cr_i(\varrho_2 - \varrho_1)^{-1}\varrho \fint_{\widehat{Q}_i} \varrho^{-1}|u - u_{Q^{(1)}}| dz$$
$$\leq c\mu r_i \left[\fint_{2\widehat{Q}_i} \left(\varrho^{-1}|u - u_{Q^{(1)}}| + 1\right)^{p(\cdot)/\tilde{q}} \chi_{Q^{(4)}} dz\right]^{\tilde{q}/\hat{p}_1^i}$$
$$\leq c\mu r_i M_{Q^{(4)}}(\tilde{z})^{\frac{1}{\hat{p}_1^i(1-\varepsilon)}} \leq c\mu r_i\lambda_1^{(p_0^i-\hat{p}_1^i)/p_0^i\hat{p}_1^i}\lambda_1^{1/p_0^i} \leq c\mu r_i\lambda_1^{1/p_0^i}.$$

To estimate U_2, observe that $u_{Q^{(1)}}(\eta)_{B_i} = \fint_{Q_i} u_{Q^{(1)}}\eta dz$ and therefore

$$U_2 = \left|[(u_{\widehat{Q}_i} - u_{Q^{(1)}})\eta - ((u - u_{Q^{(1)}})\eta)_{Q_i}]_{B_i}\right| \leq \fint_{\widehat{Q}_i} |u - u_{\widehat{Q}_i}|\eta dz \leq cr_i\lambda_1^{1/p_0^i}.$$

The estimates above together with estimate (6.11) from Lemma 6.7 yield that

$$\begin{aligned}\left|\int_{B^{(4)}\times\{t\}} U\eta\tilde{v}\psi_i dx\right| &\le c(U_1+U_2)|B_i|\max_{Q^{(4)}\cap(2Q_i)}|\tilde{v}|\\ &\le c\mu r_i\lambda_1^{1/p_0^i}\left[r_i\varepsilon_1^{-1}\lambda_1^{1/p_0^i}\lambda_1^{(p_0^i-p_1^i)/p_1^ip_0^i}|B_i|+\varepsilon_1\lambda_1^{-1/p_1^i}r_i^{-1}|B_i||v_{Q_i\cap Q^{(4)}}|^2\right]\\ &\le c\mu\varepsilon_1^{-1}\lambda_1|Q^{(4)}\cap Q_i|+\mu\varepsilon_1|B_i||v_{Q_i\cap Q^{(4)}}|^2,\end{aligned}$$

since $\widehat{Q}_i\subset Q^{(4)}$ and $|Q^{(4)}\cap Q_i|=|Q_i|=|B_i|\times\lambda_1^{(2-p_0^i)/p_0^i}r_i^2$. This finishes the proof of (6.20).

Finally we come to the proof of (6.22). Recall that $\sum_{j=0}^{\infty}\psi_j(z)=1$ for any $z\in Q^{(4)}\backslash E(\lambda_1)$. This motivates us to define the sets

$$\Lambda:=\left\{i\in\Theta:\operatorname{supp}\psi_i\cap(B^{(4)}\times\{t\})\neq\emptyset\ \text{ and }\ |v|+|\tilde{v}|\neq 0\ \text{ on }\ \operatorname{supp}\psi_i\cap(B^{(4)}\times\{t\})\right\}$$

and $\Xi_1:=\Lambda\cap\Theta_1$ and $\Xi_2:=\Lambda\cap\Theta_2$. Then, we can decompose the left hand side of (6.22) as follows:

$$\begin{aligned}\int_{(B^{(4)}\backslash E_t(\lambda_1))\times\{t\}}(|v|^2-|v-\tilde{v}|^2)dx &= \sum_{i\in\Xi_2}\int_{B^{(4)}\times\{t\}}\psi_i(|v|^2-|v-\tilde{v}|^2)dx\\ &\quad+\sum_{i\in\Xi_1}\int_{B^{(4)}\times\{t\}}\psi_i(|v|^2-|v-\tilde{v}|^2)dx:=III+IV.\end{aligned}$$

To estimate III, we write $|v|^2-|v-\tilde{v}|^2=2v\tilde{v}-|\tilde{v}|^2$. Since $i\in\Theta_2$ and $Q_i\cap Q^{(3)}\neq\emptyset$, there holds $|\Lambda^{(4)}\cap(2\Lambda_i)|\ge cs^{-1}$ for some constant c depending only on n. From Remark 6.6 (6.7) and the definition of $\tilde{v}$, we see that

$$\begin{aligned}\int_{B^{(4)}\times\{t\}}\psi_i|\tilde{v}|^2dx &\le c\sum_{j\in I(i)}|v_{Q_j\cap Q^{(4)}}|^2|B^{(4)}\cap(2B_i)|=c\sum_{j\in I(i)}|v_{Q_j\cap Q^{(4)}}|^2\frac{|Q^{(4)}\cap\widehat{Q}_i|}{|\Lambda^{(4)}\cap(2\Lambda_i)|}\\ &\le cs^{-1}\sum_{j\in I(i)}\frac{|Q^{(4)}\cap\widehat{Q}_i|}{|Q^{(4)}\cap Q_j|}\int_{Q_j\cap Q^{(4)}}|v|^2dz\le cs^{-1}\int_{\widehat{Q}_i\cap Q^{(4)}}|u-u_{Q^{(1)}}|^2dz.\end{aligned}$$

Therefore, we use (6.21) and the estimate from above to conclude that

$$\begin{aligned}III &\le c\sum_{i\in\Xi_2}\int_{B^{(4)}\times\{t\}}v\tilde{v}\psi_i dx+cs^{-1}\sum_{i\in\Xi_2}\int_{\widehat{Q}_i\cap Q^{(4)}}|u-u_{Q^{(1)}}|^2dz\\ &\le c\mu\lambda_1|Q^{(4)}\backslash E(\lambda_1)|+cs^{-1}\int_{Q^{(4)}}|u-u_{Q^{(1)}}|^2dz,\end{aligned}$$

since $\{Q_i\}_{i=1}^{\infty}$ has a finite overlap. We now proceed to find the lower bound of IV. Note that we can rewrite $|v|^2-|v-\tilde{v}|^2=|v_{Q_i\cap Q^{(4)}}|^2+2(v-v_{Q_i\cap Q^{(4)}})\tilde{v}-|\tilde{v}-v_{Q_i\cap Q^{(4)}}|^2$ for any fixed $i\in\mathbb{N}$. Since $i\in\Theta_1$, we infer that $|\Lambda^{(4)}\cap(2\Lambda_i)|=2|\Lambda_i|$.

From Remark 6.6 (6.9), we conclude that

$$\int_{B^{(4)}\times\{t\}} \psi_i|\tilde{v} - v_{Q_i\cap Q^{(4)}}|^2 dx \le \int_{B^{(4)}\times\{t\}} \psi_i\Big|\sum_{j\in I(i)} (v_{Q_j\cap Q^{(4)}} - v_{Q_i\cap Q^{(4)}})\psi_j\Big|^2 dx$$
$$\le c\mu \min\{r_i, \varrho\}^2 \lambda_1^{2/p_0^i} \lambda_1^{2(p_0^i - p_1^i)/p_0^i p_1^i} |B^{(4)} \cap (2B_i)| \le c\mu\lambda_1 |Q^{(4)} \cap (2Q_i)|.$$

This implies

$$\sum_{i\in\Xi_1} \int_{B^{(4)}\times\{t\}} \psi_i|\tilde{v} - v_{Q_i\cap Q^{(4)}}|^2 dx \le c\mu\lambda_1 \sum_{i=1}^{\infty} |Q^{(4)} \cap (2Q_i)| \le c\mu\lambda_1 |Q^{(4)}\backslash E(\lambda_1)|.$$

Next, we use (6.20) to get

$$\sum_{i\in\Xi_1} \int_{B_{\varrho_2}\times\{t\}} \psi_i \left[|v_{Q_i\cap Q^{(4)}}|^2 + 2(v - v_{Q_i\cap Q^{(4)}})\tilde{v}\right] dx$$
$$\ge \sum_{i\in\Xi_1} \int_{B^{(4)}\times\{t\}} \psi_i |v_{Q_i\cap Q^{(4)}}|^2 dx - \sum_{i\in\Xi_1} \varepsilon_1\mu\delta_3(i)|B_i||v_{Q_i\cap Q^{(4)}}|^2 - c\sum_{i\in\Xi_1} \mu\varepsilon_1^{-1}\lambda_1|Q^{(4)} \cap Q_i|$$
$$=: IV_1 - IV_2 - IV_3.$$

We only have to consider the case $i \in \Xi_1$ and $\hat{r}_i < (\varrho_2 - \varrho_1)/12$ where $\delta_3(i) \equiv 1$. Observe that in this case $\widehat{Q_i} \subset Q^{(4)}$ and we infer from Remark 6.6 (6.9) that

$$|v_{Q_i\cap Q^{(4)}}| \le |v_{Q_i\cap Q^{(4)}} - v_{Q_j\cap Q^{(4)}}| + |v_{Q_j\cap Q^{(4)}}| \le c\mu r_i \lambda_1^{1/p_0^i} + |v_{Q_j\cap Q^{(4)}}|$$

where $j \in I(i)$. Let $\Xi_1' := \{i \in \Xi_1 : j \in \Xi_1 \text{ for any } j \in I(i)\}$. We decompose IV_2 as follows:

$$IV_2 = \sum_{i\in\Xi_1'} \varepsilon_1\mu|B_i||v_{Q_i\cap Q^{(4)}}|^2 + \sum_{i\in\Xi_1\backslash\Xi_1'} \varepsilon_1\mu|B_i||v_{Q_i\cap Q^{(4)}}|^2 =: IV_{2,1} + IV_{2,2}.$$

We now obtain the estimate for $IV_{2,1}$ as follows:

$$IV_{2,1} \le \sum_{i\in\Xi_1'} \left(\varepsilon_1\mu|B_i|r_i^2\lambda_1^{2/p_0^i} + \varepsilon_1\mu|B_i||v_{Q_j\cap Q^{(4)}}|^2\right)$$
$$\le \varepsilon_1\mu\lambda_1 \sum_{i\in\Xi_1'} |Q_i| + c\varepsilon_1\mu \sum_{i\in\Xi_1} \int_{B^{(4)}\times\{t\}} \psi_i|v_{Q_i\cap Q^{(4)}}|^2 dx \le c\mu\lambda_1|Q^{(4)}\backslash E(\lambda_1)| + IV_1,$$

provided $\varepsilon_1 = \frac{1}{20c\mu} < 1$. To estimate $IV_{2,2}$, we note that for any $i \in \Xi_1\backslash\Xi_1'$ there exists $j(i) \in I(i)$ such that $j(i) \in \Xi_2$. This implies that $|\Lambda_i| = 2\lambda_1^{(2-p_0^i)/p_0^i} r_i^2 \ge c\lambda_1^{(2-p_0^i)/p_0^i} r_{j(i)}^2 \ge cs$. Recalling that $\widehat{Q_i} \subset Q^{(4)}$ we conclude that

$$IV_{2,2} \le \sum_{i\in\Xi_1\backslash\Xi_1'} \mu|B_i| \fint_{Q_i\cap Q^{(4)}} |v|^2 dz \le \frac{c\mu}{s} \int_{Q^{(4)}} |u - u_{Q^{(1)}}|^2 dz.$$

Finally, it is easily seen that $IV_3 \le c\mu\lambda_1|Q^{(4)}\backslash E(\lambda_1)|$. From this, we get the desired estimate (6.22) immediately. □

6.3. **Lipschitz continuity of $\tilde v$ on $B^{(4)} \times S_1$.** In this Subsection we will prove that $\tilde v$ is Lipschitz continuous with respect to the parabolic metric on the set $B^{(4)} \times S_1$. This property will be essential in the proof of the Caccioppoli inequality, since it ensures that $\tilde v$ is an admissible testing function in the weak formulation of the parabolic system. For simplicity of notation, we let Q_1^4 and Q_2^5 stand for $B^{(4)} \times S_1$ and $B^{(5)} \times S_2$ respectively. First, we will show that for any $z \in Q^{(4)}$ the two parabolic metrics d_P and d_z are equivalent. Since $p(\cdot) \geq 2$ and $\lambda_1 \geq 1$, we have for any fixed $z_1, z_1 \in Q^{(6)}$ that

$$d_P(z_1, z_2) \leq \max\left\{|x_1 - x_2|, \sqrt{\lambda_1^{(p(z)-2)/p(z)}|t_1 - t_2|}\right\} = d_z(z_1, z_2). \tag{6.26}$$

On the other hand, since $\lambda_1^{(p(z)-2)/p(z)} \leq \lambda_1^{(p_2-2)/p_2}$, we get

$$d_z(z_1, z_2) \leq \lambda_1^{(p_2-2)/(2p_2)} d_P(z_1, z_2) \tag{6.27}$$

for any $z_1, z_1 \in Q^{(6)}$. Hence, d_P and d_z are equivalent for any $z \in Q^{(4)}$. In this subsection, the constants will depend on λ, λ_1, γ_2, ϱ_1, ϱ_2 and $\|v\|_{L^1(Q^{(4)})}$. Note that this is not a problem, since we will only use the qualitative result that $\tilde v$ is Lipschitz continuous with respect to the parabolic metric.

Lemma 6.12. *Let $\lambda_1 \geq c_E \tilde\lambda$. Then there exists a constant $K > 0$ such that for any $z_1, z_2 \in B^{(4)} \times S_1$ there holds*

$$|\tilde v(z_1) - \tilde v(z_2)| \leq K d_P(z_1, z_2).$$

Proof. In order to prove this lemma, we use the metric version of the integral characterization of Lipschitz continuous functions by Da Prato [9, Theorem 3.1]. For $z_w = (x_w, t_w) \in \overline{Q_1^4}$ we define

$$I_r(z_w) := \frac{1}{|Q_1^4 \cap Q_r(z_w)|^{1+\frac{1}{n+2}}} \int_{Q_1^4 \cap Q_r(z_w)} \left|\tilde v - \tilde v_{Q_1^4 \cap Q_r(z_w)}\right| dz,$$

where we recall that $Q_r(w) \equiv B_r(x_w) \times (t_w - r^2, t_w + r^2)$. Our aim now is to show that $I_r(z_w)$ is bounded independently of z_w and r. To this aim we distinguish between the following four cases:

$$\begin{cases} 2Q_r(z_w) \subset Q_2^5 \setminus E(\lambda_1), \\ 2Q_r(z_w) \cap E(\lambda_1) \neq \emptyset,\ 2Q_r(z_w) \subset Q_2^5 \text{ and } r < \frac{1}{3}\lambda_1^{(2-p_2)/(2p_2)}(\varrho_2 - \varrho_1), \\ 2Q_r(z_w) \cap E(\lambda_1) \neq \emptyset,\ 2Q_r(z_w) \subset Q_2^5 \text{ and } r \geq \frac{1}{3}\lambda_1^{(2-p_2)/2p_2}(\varrho_2 - \varrho_1), \\ 2Q_r(z_w) \setminus Q_2^5 \neq \emptyset. \end{cases}$$

In the first case, we observe that $|Q_1^4 \cap Q_r(z_w)| \geq c_n r^{n+2}$ and this implies

$$\begin{aligned} I_r(z_w) &\leq \frac{c}{r} \fint_{Q_1^4 \cap Q_r(z_w)} \fint_{Q_1^4 \cap Q_r(z_w)} \left|\tilde v(z) - \tilde v(\tilde z)\right| dz d\tilde z \\ &\leq c \sup_{z \in Q_1^4 \cap Q_r(z_w)} \left[|D\tilde v(z)| + r|\partial_t \tilde v(z)|\right], \end{aligned} \tag{6.28}$$

since $\tilde v$ is smooth on $Q_2^5 \setminus E(\lambda_1)$. Now, we consider $z \in Q_1^4 \cap Q_r(z_w)$. Then, we can find $i \in \Theta$ such that $z \in Q_i$. Since $2Q_r(z_w) \subset Q_2^5 \setminus E(\lambda_1)$

we have $d_P(z, E(\lambda_1)) \geq r$. Letting $\hat{z}_i \in E(\lambda_1)$ be a point such that $d_{z_i}(z_i, \hat{z}_i) = d_{z_i}(z_i, E(\lambda_1)) \leq 2\hat{c}r_i$ holds (where $\hat{c} > 1$ denotes the constant from Lemma 4.2), we can use (6.26) to infer that

$$r \leq d_P(z, \hat{z}_i) \leq d_P(z, z_i) + d_P(z_i, \hat{z}_i) \leq d_{z_i}(z, z_i) + d_{z_i}(z_i, E(\lambda_1)) \leq 3\hat{c}r_i.$$

By the definition of $\tilde{v}$, we therefore find that

$$\begin{aligned}|D\tilde{v}(z)| + r|\partial_t\tilde{v}(z)| &\leq \sum_{j\in I(i)} |D\psi_j||v_{Q_j} - v_{Q_i}| + r\sum_{j\in I(i)} |\partial_t\psi_j||v_{Q_j} - v_{Q_i}| \\ &\leq \left[\frac{c}{r_i} + \frac{cr\lambda_1^{(p_2-2)/p_2}}{r_i^2}\right] \fint_{\widehat{Q}_i\cap Q^{(4)}} |v - v_{\widehat{Q}_i\cap Q^{(4)}}|dz \\ &\leq \frac{c}{r_i}\fint_{\widehat{Q}_i\cap Q^{(4)}} |v - v_{\widehat{Q}_i\cap Q^{(4)}}|dz,\end{aligned}$$

where c now depends also on λ_1. In the case $i \in \Theta_2$, we have $|\widehat{Q}_i \cap Q^{(4)}| \geq c\lambda_1^{(2-p_0^i)/p_0^i} r_i^{n+2} \geq c(n, \gamma_2, \lambda_1, \varrho_1, \varrho_2)$ as well as $r_i \geq c(n, \gamma_2, \lambda_1, \varrho_1, \varrho_2)$, so that

(6.29)
$$\frac{c}{r_i}\fint_{\widehat{Q}_i\cap Q^{(4)}} |v - v_{\widehat{Q}_i\cap Q^{(4)}}|dz \leq \frac{2}{r_i|\widehat{Q}_i \cap Q^{(4)}|}\int_{\widehat{Q}_i\cap Q^{(4)}} |v|dz \leq c\,\|v\|_{L^1(Q^{(4)})} \leq c.$$

Moreover, from (6.8) we conclude that (6.29) also holds in the case $i \in \Theta_1$. Therefore, in any case, we find that $|D\tilde{v}(z)| + r|\partial_t\tilde{v}(z)| \leq c$. Since $z \in Q_1^4 \cap Q_r(z_w)$ was arbitrary, we have thus shown that

$$I_r(z_w) \leq c, \tag{6.30}$$

where c depends on $n, L, \gamma_2, \lambda_1, \varrho_1, \varrho_2, \|v\|_{L^1(Q^{(4)})}$, but is independent of z_w and r.

We now turn our attention to the second case. Since $z_w \in \overline{Q_1^4}$, it is easy to check that $|Q_r(z_w) \cap Q_1^4| \geq c(n)|Q_r(z_w)|$. Therefore we obtain

$$I_r(z_w) \leq \frac{c(n)}{|Q_r(z_w)|^{1+\frac{1}{n+2}}}\int_{Q_r(z_w)\cap Q_1^4} 2|\tilde{v} - v| + |v - v_{Q_r(z_w)\cap Q_1^4}|dz =: c(n)(2I_1 + I_2),$$

with the obvious meaning of I_1 and I_2. To estimate I_2, we apply the arguments in the spirit of the proof of Lemma 6.5. In a similar way, we construct a weight function $\hat{\eta} \in C_0^\infty(B_r(x_w) \cap B^{(4)})$ satisfying $\hat{\eta} \geq 0$, $\int_{\mathbb{R}^n} \hat{\eta}dx = 1$ and $|D\hat{\eta}| \leq c\max\{r^{-(1+n)}, \varrho^{-(1+n)}\}$. We let $v_{\hat{\eta}}(t) := \int_{\mathbb{R}^n\times\{t\}} (v\hat{\eta})\,dx$ and use Poincaré's inequality to conclude that

$$\begin{aligned}I_2 &= \frac{c}{r}\fint_{Q_r(z_w)\cap Q_1^4} |v - v_{Q_r(w)\cap Q_1^4}|dz \\ &\leq \frac{c}{r}\fint_{Q_r(z_w)\cap Q_1^4} |v - v_{\hat{\eta}}|dz + \frac{c}{r}\max_{t_1,t_2\in S_1\cap(t_w-r^2,t_w+r^2)} \left|v_{\hat{\eta}}(t_2) - v_{\hat{\eta}}(t_1)\right| \\ &\leq \frac{c}{r}\min\{r, \varrho\}\fint_{Q_r(z_w)\cap Q_1^4} |Dv|dz + \frac{c}{r}\sup_{t_1,t_2\in S_1\cap(t_w-r^2,t_w+r^2)} \left|v_{\hat{\eta}}(t_2) - v_{\hat{\eta}}(t_1)\right| =: I_2^{(1)} + I_2^{(2)},\end{aligned}$$

with the obvious meaning of $I_2^{(1)}$ and $I_2^{(2)}$. In order to estimate $I_2^{(1)}$, we fix a point $\tilde{z} \in 2Q_r(z_w) \cap E(\lambda_1)$. Then we have

$$\begin{aligned} I_2^{(1)} &\le c \fint_{2Q_r(z_w)} \Big(|Du| + (\varrho_2 - \varrho_1)^{-1}|u - u_{Q^{(1)}}| + 1\Big)^{p(\cdot)/\tilde{q}}\, dz \\ &\le c\mu M_{Q^{(4)}}(\tilde{z})^{1/(\tilde{q}(1-\varepsilon))} \le c\mu\lambda_1^{1/\tilde{q}}. \end{aligned}$$

We now consider the term $I_2^{(2)}$. Since $r < \frac{1}{3}\lambda_1^{(2-p_2)/(2p_2)}(\varrho_2 - \varrho_1)$, we have $S_1 \cap (t_w - r^2, t_w + r^2) \subset S_2$. Recalling that $\zeta \equiv 1$ on S_2, this implies that $v(x,t) = [u(x,t) - u_{Q^{(1)}}]\eta(x)$ whenever $t \in S_1 \cap (t_w - r^2, t_w + r^2)$. Therefore, using the Steklov formulation (3.1) of the parabolic system with $\varphi = \eta\hat{\eta}$, we obtain for $h > 0$ and $t_1, t_2 \in S_1 \cap (t_w - r^2, t_w + r^2)$ that

$$\Big|([u]_h)_{\eta\hat{\eta}}(t_2) - ([u]_h)_{\eta\hat{\eta}}(t_1)\Big| \le \int_{t_1}^{t_2}\int_{B^{(3)}\cap B_r(x_w)} |\langle [A(z,Du)]_h, D(\eta\hat{\eta})\rangle| + |\langle [B(z,Du)]_h, \eta\hat{\eta}\rangle| dxdt.$$

Letting $h \downarrow 0$ and using assumption (2.2) we find that

$$\Big|v_{\hat{\eta}}(t_2) - v_{\hat{\eta}}(t_1)\Big| \le c(1 + \|D(\eta\hat{\eta})\|_{L^\infty}) \int_{Q^{(3)}\cap Q_r(z_w)} (1 + |Du| + |F|)^{p(\cdot)-1}\, dz.$$

To estimate the right hand side of the above inequality, we observe that $|D(\eta\hat{\eta})| \le c\mu r^{-(1+n)}$. Using this information we find that

$$\begin{aligned} \Big|v_{\hat{\eta}}(t_2) - v_{\hat{\eta}}(t_1)\Big| &\le \frac{c\mu}{r^{1+n}} \int_{Q^{(3)}\cap Q_r(z_w)} (1 + |Du| + |F|)^{p_2-1}\, dz \\ &\le \frac{c\mu|Q_r(z_w)|}{r^{1+n}} M_{Q^{(4)}}(\tilde{z})^{\frac{p_2-1}{(1-\varepsilon)p_1}} \le c\mu r\lambda_1^{\frac{p_2-1}{p_1}}, \end{aligned}$$

which ensures that $I_2^{(2)} \le c\mu\lambda_1^{(p_2-1)/p_1}$. We now come to the estimate for I_1. Recalling that $\operatorname{supp} v \subset Q^{(3)}$, then we use (6.29) to obtain

$$\begin{aligned} I_1 \le \frac{c}{r^{n+3}} \int_{Q_r(z_w)\setminus E(\lambda_1)} |\tilde{v} - v| dz &\le \frac{c}{r^{n+3}} \sum_{i\in\Theta: 2Q_i\cap Q_r(z_w)\neq\emptyset} \int_{\widehat{Q}_i\cap Q^{(4)}} |v - v_{\widehat{Q}_i\cap Q^{(4)}}| dz \\ &\le \frac{c}{r^{n+3}} \sum_{i\in\Theta: 2Q_i\cap Q_r(z_w)\neq\emptyset} r_i|\widehat{Q}_i \cap Q^{(4)}|. \end{aligned}$$

We let w_1 and w_2 be two points in $2Q_r(z_w)$ satisfying $w_1 \in 2Q_i \cap Q_r(z_w)$ and $w_2 \in 2Q_r(z_w) \cap E(\lambda_1)$. Then, by (6.27), we have

$$r_i \le \tfrac{1}{\hat{c}} d_{z_i}(z_i, w_2) \le \tfrac{1}{\hat{c}}\left[d_{z_i}(z_i, w_1) + d_{z_i}(w_1, w_2)\right] \le \tfrac{1}{\hat{c}}\Big[2r_i + \lambda_1^{(p_2-2)/(2p_2)} d_P(w_1, w_2)\Big].$$

Since $\hat{c} \ge 4$ this proves that $r_i \le c\lambda_1^{(p_2-2)/(2p_2)} r \le cr$. For $i \in \Theta$ with $2Q_i \cap Q_r(w) \neq \emptyset$ we therefore have that $\widehat{Q}_i \subset Q_{2\hat{c}r}(z_w)$. Keeping in mind that at each point at most $c(n)$ of cylinders $2Q_i$ intersect we can further estimate

$$I_1 \le \frac{c}{r^{n+2}} \sum_{i\in\Theta: 2Q_i\cap Q_r(z_w)\neq\emptyset} |\widehat{Q}_i \cap Q_{2\hat{c}r}(z_w)| \le \frac{c}{r^{n+2}} |Q_{2\hat{c}r}(z_w)| \le c.$$

Inserting the estimates for I_1 and I_2 above we have shown that the estimate (6.30) continues to hold in the second case.

Finally we come to the third and fourth case. We first observe that in both cases we have $|Q_1^4 \cap Q_r(z_w)| \geq c(n, \gamma_2, \lambda_1, \varrho_1, \varrho_2)$. Therefore, we conclude from estimate (6.15) in Lemma 6.8 that

$$I_r(z_w) \leq c \int_{Q_1^4 \cap Q_r(z_w)} |\tilde{v}| dz \leq c \|v\|_{L^1(Q^{(4)})},$$

which proves (6.30) in the third and fourth case. At this point, the Lipschitz continuity follows from (6.30) and the integral characterization of Hölder continuous functions from [9, Theorem 3.1]. This finishes the proof of the Lemma. □

We now have the prerequisites to prove the Caccioppoli inequality stated in Theorem 6.1.

6.4. **Proof of Theorem 6.1.** From Lemma 6.12, we know that $D\tilde{v}(\cdot, \tau) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ for any $\tau \in S_1$. Let $t \in \Lambda^{(2)}$ and $t_1 \in S_1 \backslash \Lambda^{(1)}$ with $t_1 < t$. In the Steklov formulation (3.1) of the parabolic system we choose $\varphi(x, \tau) = \eta(x)\chi_\delta(\tau)[\tilde{v}]_h(x, \tau)$ as a testing function, where $h > 0$, $0 < \delta \ll 1$ and

$$\chi_\delta(\tau) = \begin{cases} 0 & \text{on } (-\infty,\ t_1 + h] \cup [t - h,\ \infty) \\ 1 + \frac{\tau - t_1 - h - \delta}{\delta} & \text{on } [t_1 + h,\ t_1 + h + \delta] \\ 1 & \text{on } [t_1 + h + \delta,\ t - h - \delta] \\ 1 - \frac{\tau - t + h + \delta}{\delta} & \text{on } [t - h - \delta,\ t - h] \end{cases}$$

to infer that

$$\begin{aligned} \int_{B^{(4)} \times \{\tau\}} \partial_\tau [u]_h \cdot \eta \chi_\delta [\tilde{v}]_h + \langle [A(z, Du)]_h, \chi_\delta D\left([\tilde{v}]_h \eta\right) \rangle dx & \\ = - \int_{B^{(4)} \times \{\tau\}} [B(z, Du)]_h \cdot \eta \chi_\delta [\tilde{v}]_h dx & \end{aligned} \tag{6.31}$$

for any $\tau \in S_1$. For the first term on the left hand side we compute

$$\begin{aligned} \partial_\tau [u]_h \chi_\delta \cdot [\tilde{v}]_h &= \partial_\tau [v]_h \chi_\delta \cdot [\tilde{v}]_h \\ &= \tfrac{1}{2} \partial_\tau |[v]_h|^2 \chi_\delta + \partial_\tau [v]_h \cdot [\tilde{v} - v]_h \chi_\delta \\ &= \tfrac{1}{2} \partial_\tau |[v]_h|^2 \chi_\delta + \partial_\tau [\tilde{v}]_h \cdot [\tilde{v} - v]_h \chi_\delta - \partial_\tau [\tilde{v} - v]_h \cdot [\tilde{v} - v]_h \chi_\delta \\ &= \tfrac{1}{2} \partial_\tau (|[v]_h|^2 - |[\tilde{v} - v]_h|^2) \chi_\delta + \partial_\tau [\tilde{v}]_h \cdot [\tilde{v} - v]_h \chi_\delta. \end{aligned}$$

Integrating over $B^{(4)} \times (t_1, t)$ and using the fact that $\int_{t_1}^t [f]_h \cdot g \, d\tau = \int_{t_1}^t f \cdot [g]_{-h} d\tau$ whenever $\operatorname{supp} g \subset (t_1 + h, t - h)$ (cf. [8, Lemma 2.10]) we find that

$$\begin{aligned} \int_{t_1}^t \int_{B^{(4)}} \partial_\tau [u]_h \eta \chi_\delta \cdot [\tilde{v}]_h \, dx d\tau &= \frac{1}{2} \int_{t_1}^t \int_{B^{(4)}} \partial_\tau (|[v]_h|^2 - |[\tilde{v} - v]_h|^2) \chi_\delta \, dx d\tau \\ &\quad + \int_{t_1}^t \int_{B^{(4)}} \left[\partial_\tau [\tilde{v}]_h \chi_\delta\right]_{-h} \cdot (\tilde{v} - v) \, dx d\tau \\ &= \frac{1}{2} \int_{t_1}^t \int_{B^{(4)}} \partial_\tau \Big[(|[v]_h|^2 - |[\tilde{v} - v]_h|^2) \chi_\delta\Big] \, dx d\tau \end{aligned}$$

$$
\begin{aligned}
&- \tfrac{1}{2} \int_{t_1}^{t} \int_{B^{(4)}} (|[v]_h|^2 - |[\tilde v - v]_h|^2) \partial_\tau \chi_\delta \, dxd\tau \\
&+ \int_{t_1}^{t} \int_{B^{(4)} \setminus E_\tau(\lambda_1)} [\partial_\tau [\tilde v]_h \chi_\delta]_{-h} \cdot (\tilde v - v) \, dxd\tau \\
=:\,& E_1(\delta, h) + E_2(\delta, h) + E_3(\delta, h).
\end{aligned}
$$

Since $\chi_\delta(t_1) = 0 = \chi_\delta(t)$, we have $E_1(\delta, h) = 0$. For the term $E_2(\delta, h)$ we use the definition of χ_δ to infer that

(6.32)

$$
\begin{aligned}
E_2(\delta, h) &= \tfrac{1}{2\delta} \int_{t-h-\delta}^{t-h} \int_{B^{(4)}} (|[v]_h|^2 - |[\tilde v - v]_h|^2) \, dxd\tau \\
&\quad - \tfrac{1}{2\delta} \int_{t_1+h}^{t_1+h+\delta} \int_{B^{(4)}} (|[v]_h|^2 - |[\tilde v - v]_h|^2) \, dxd\tau \\
&\to \tfrac{1}{2} \int_{B^{(4)} \times \{t\}} \Big[(|v|^2 - |\tilde v - v|^2) \Big] \, dx - \tfrac{1}{2} \int_{B^{(4)} \times \{t_1\}} \Big[(|v|^2 - |\tilde v - v|^2) \Big] \, dx =: I + II,
\end{aligned}
$$

as $\delta, h \downarrow 0$, for a.e. t, t_1 as above. We now turn our attention to the estimate of $E_3(\delta, h)$. We define $Q_t := B^{(4)} \times (t_1, t)$ and observe that the set $Q_t \setminus E(\lambda_1)$ is open. This implies that $[\partial_\tau [\tilde v]_h \chi_\delta]_{-h} \cdot (\tilde v - v) \to \partial_t \tilde v \chi_\delta \cdot (\tilde v - v)$ pointwise a.e. on $Q_t \setminus E(\lambda_1)$ as $h \downarrow 0$. Furthermore, we will ensure that $|[\partial_\tau [\tilde v]_h \chi_\delta]_{-h} \cdot (\tilde v - v)| \le cH$, where H is defined by

$$
H := \sum_{i \in \Theta} |v - \tilde v| \chi_{Q_i} \sup_{2Q_i \cap Q^{(4)}} |\partial_t \tilde v| + \sum_{i \in \Theta_1} r_i^{-1} |v - \tilde v| \chi_{Q_i}.
$$

To this aim we define $N_h = \{ i \in \mathbb{N} : \ h < \lambda_1^{(2-p_0^i)/p_0^i} r_i^2 \}$ and decompose the term under consideration as follows

$$
\begin{aligned}
\Big| [\partial_\tau [\tilde v]_h \chi_\delta]_{-h} \cdot (\tilde v - v) \Big| \le & \sum_{i \in \Theta \cap N_h : Q_i \cap Q_t \neq \emptyset} \Big| [\partial_\tau [\tilde v]_h \chi_\delta]_{-h} \cdot (\tilde v - v) \Big| \chi_{Q_i} \\
& + \sum_{i \in \Theta \setminus N_h : Q_i \cap Q_t \neq \emptyset} \Big| [\partial_\tau [\tilde v]_h \chi_\delta]_{-h} \cdot (\tilde v - v) \Big| \chi_{Q_i} =: H_1 + H_2,
\end{aligned}
$$

with the obvious meaning of H_1 and H_2. In the case $i \in \Theta \cap N_h$, we find that

$$
\sup_{Q_i : Q_i \cap Q_t \neq \emptyset} \Big| [\partial_\tau [\tilde v]_h \chi_\delta]_{-h} \Big| \le \sup_{2Q_i \cap Q^{(4)}} |\partial_t \tilde v|,
$$

which implies that $H_1 \le H$. Since we are interested in small values of h, we may assume that $h < \frac{1}{3} \lambda_1^{(2-p_2)/p_2} (\varrho_2 - \varrho_1)^2$. Then for any $i \in \Theta \setminus N_h$ we have $i \in \Theta_1$. Using the formula for the time derivative of Steklov averages and Lemma 6.12, we find that for $i \in \Theta \setminus N_h$ and $z \in Q_t \cap Q_i$ there holds

$$
|\partial_\tau [\tilde v]_h(z)| = \frac{|\tilde v(x, t+h) - \tilde v(x, t)|}{h} \le \frac{K}{\sqrt{h}} \le K \lambda_1^{(p_2-2)/(2p_2)} r_i^{-1},
$$

which proves that $H_2 \le H$. It remains to prove that H is an integrable function on $Q^{(4)}$. A slight change in the proof of Lemma 6.7 shows that

$$\sup_{2Q_i \cap Q^{(4)}} |\partial_t \tilde{v}| \le c\mu \lambda_1^{(p_2-1)/p_2} r_i^{-1} \quad \text{when } i \in \Theta_1$$

and

$$\sup_{2Q_i \cap Q^{(4)}} |\partial_t \tilde{v}| \le c\mu s^{-1} \varrho \lambda_1^{1/p_1} \quad \text{when } i \in \Theta_2.$$

We now use estimates (6.16), (6.15) from Lemma 6.8 and Tonelli's theorem to get

$$\begin{aligned}\int_{Q^{(4)}} H\,dz &\le c \sum_{i\in\Theta_1} r_i^{-1} \int_{Q_i \cap Q^{(4)}} |v(z) - \tilde{v}(z)| dz + c \sum_{i\in\Theta_2} \int_{Q_i\cap Q^{(4)}} |v(z) - \tilde{v}(z)| dz \\ &\le c \sum_{i\in\Theta_1} |Q_i \cap Q^{(4)}| + c \left[\|v\|_{L^1(Q^{(4)})} + \|\tilde{v}\|_{L^1(Q^{(4)})} \right] \le c\,|Q^{(4)}| + c\,\|v\|_{L^1(Q^{(4)})} < \infty,\end{aligned}$$

where c depends on $n, L, \gamma_2, \lambda_1, \varrho_1, \varrho_2$ and $\|v\|_{L^1(Q^{(4)})}$, but is independent of h. This ensures that H is integrable on $Q^{(4)}$ and therefore, we are allowed to apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{\delta\downarrow 0} \lim_{h\downarrow 0} E_3(\delta, h) = \int_{Q_t \setminus E(\lambda_1)} \partial_t \tilde{v} \cdot (\tilde{v} - v)\,dz =: III.$$

We now start with the estimate of II. Since $2^{-100}s \le |S_1 \setminus \Lambda^{(1)}| \le 2^{100}s$, we can choose $t_1 \in S_1 \setminus \Lambda^{(1)}$ and $t_1 < t$ such that

$$II \le \frac{1}{|S_1 \setminus \Lambda^{(1)}|} \int_{S_1 \setminus \Lambda^{(1)}} \int_{B^{(4)}} (|v|^2 - |\tilde{v} - v|^2)\,dxdt.$$

From estimate (6.15) of Lemma 6.8 and the fact that $|v| \le c(u - u_{Q^{(1)}})$, we obtain

$$II \le \frac{c}{s} \int_{Q^{(3)}} |v|^2 dz + \frac{c}{s} \int_{Q^{(3)} \setminus E(\lambda_1)} (|v|^2 + |\tilde{v}|^2) dz \le \frac{c}{s} \int_{Q^{(3)}} |u - u_{Q^{(1)}}|^2 dz.$$

To deal with III, we apply estimate (6.17) of Lemma 6.8 to get

$$III \le c\lambda_1 |Q^{(4)} \setminus E(\lambda_1)| + \frac{c}{s} \int_{Q^{(3)}} |u - u_{Q^{(1)}}|^2 dz.$$

Next, we integrate the remaining terms of (6.31) with respect to the time variable over (t_1, t) and subsequently pass to the limit $h \downarrow 0$ and $\delta \downarrow 0$. Finally, we decompose the domain of integration into the sets $Q^{(4)} \setminus E(\lambda_1)$ and $E(\lambda_1)$ to obtain

$$\begin{aligned}\int_{t_1}^{t} \int_{E(\lambda_1)} \langle A(z, Du), D(\tilde{v}\eta) \rangle dxdt + \int_{t_1}^{t} \int_{B^{(4)}} \langle B(z, Du), \tilde{v}\eta \rangle dxdt \\ = \int_{Q_t \cap E(\lambda_1)} \ldots\,dxdt + \int_{Q_t \setminus E(\lambda_1)} \ldots dxdt := IV + V.\end{aligned}$$

We now use the growth condition (2.2) and Remark 6.10 to conclude that

$$\begin{aligned} V &\le c \int_{Q^{(4)}\setminus E(\lambda_1)} (1+|Du|+|F|)^{p(\cdot)-1}(\varrho^{-1}|\tilde{v}|+|D\tilde{v}|)dz \\ &\le c\mu\lambda_1|Q^{(4)}\setminus E(\lambda_1)| + cs^{-1}\int_{Q^{(3)}} |u-u_{Q^{(1)}}|^2 dz. \end{aligned}$$

Joining the preceding estimates we find that

$$I + IV \le c\mu\lambda_1|Q^{(4)}\setminus E(\lambda_1)| + \frac{c}{s}\int_{Q^{(3)}} |u-u_{Q^{(1)}}|^2 dz$$

holds true for a.e. $t \in \Lambda^{(1)}$. On the other hand we infer from estimate (6.22) of Lemma 6.11 that

$$I \ge -c\mu\lambda_1|Q^{(4)}\setminus E(\lambda_1)| - \frac{c\mu}{s}\int_{Q^{(4)}} |u-u_{Q^{(1)}}|^2 dz + \frac{1}{2}\int_{E_t(\lambda_1)\times\{t\}} |v|^2 dx$$

and therefore

$$\frac{1}{2}\int_{E_t(\lambda_1)\times\{t\}} |v|^2 dx \le -IV + c\mu\lambda_1|Q^{(4)}\setminus E(\lambda_1)| + \frac{c\mu}{s}\int_{Q^{(4)}} |u-u_{Q^{(1)}}|^2 dz.$$

We multiply both sides by $\lambda_1^{-1-\varepsilon}$ and integrate over $(c_E\tilde{\lambda},\infty)$ with respect to λ_1. Setting $m_{Q^{(4)}} := \max\{c_E\tilde{\lambda},\ M_{Q^{(4)}}^{\frac{1}{1-\varepsilon}}\}$ and $s_1 := \lambda^{(2-p_0)/p_0}\varrho_1^2$ and multiplying the result by ε, we get the following estimate,

$$\begin{aligned} &\frac{1}{2}\int_{B^{(4)}\times\{t\}} |v|^2 m_{Q^{(4)}}^{-\varepsilon} dx \\ &\quad\le -\int_{B^{(4)}\times(t_1,t_0+s_1)} \Big[\langle A(z,Du),D(v\eta)\rangle + \langle B(z,Du),v\eta\rangle\Big] m_{Q^{(4)}}^{-\varepsilon}\,dz \\ &\qquad + c\varepsilon\mu\int_{c_E\tilde{\lambda}}^{\infty} \lambda_1^{-\varepsilon}\Big|\{z\in Q^{(4)}:\ M_{Q^{(4)}}(z) > \lambda_1^{1-\varepsilon}\}\Big| d\lambda_1 + \frac{c\mu}{s\tilde{\lambda}^{\varepsilon}}\int_{Q^{(4)}} |u-u_{Q^{(1)}}|^2 dz \\ &\quad:= -VI + \varepsilon VII + VIII, \end{aligned}$$

Since $\tilde{\lambda} \ge \lambda$, it follows that $VIII \le c\mu s^{-1}\lambda^{-\varepsilon}\int_{Q^{(4)}} |u-u_{Q^{(1)}}|^2 dz$. Next, we use Fubini's theorem and the boundedness of strong maximal function to infer that

$$\begin{aligned} VII &\le c\mu\int_{Q^{(4)}} M_{Q^{(4)}}\,dz \le c\mu\int_{Q^{(4)}} \left(\left|\frac{u-u_{Q^{(1)}}}{\varrho}\right| + |Du| + |F| + 1\right)^{p(\cdot)(1-\varepsilon)} dz \\ &\le c\mu\int_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho}\right|^{p(\cdot)(1-\varepsilon)} dz + c\mu\lambda^{1-\varepsilon}|Q^{(4)}|. \end{aligned}$$

To estimate a lower bound for VI, we note that $D(v\eta)(x,t) = \eta^2(x)Du(x,t) + v(x,t)D\eta(x)$. Therefore, using the ellipticity and growth conditions (2.2) we obtain

$$
\begin{aligned}
VI &\geq \int_{B^{(4)}\times(t_1,t_0+s_1)} \Big[\langle A(z,Du),\eta^2 Du\rangle - |\langle A(z,Du), vD\eta\rangle| - |\langle B(z,Du), v\eta\rangle|\Big]\, m_{Q^{(4)}}^{-\varepsilon}\, dz \\
&\geq \nu \int_{Q^{(1)}} |Du|^{p(\cdot)} m_{Q^{(4)}}^{-\varepsilon}\, dz - \int_{B^{(4)}\times(t_1,t_0+s_1)} |F|^{p(\cdot)} m_{Q^{(4)}}^{-\varepsilon}\, dz \\
&\quad - \frac{c}{\varrho_2-\varrho_1} \int_{B^{(4)}\times(t_1,t_0+s_1)} (1+|F|+|Du|)^{p(\cdot)-1} |u-u_{Q^{(1)}}| m_{Q^{(4)}}^{-\varepsilon}\, dz \\
&:= IV_1 - IV_2 - IV_3.
\end{aligned}
$$

It is easily seen that $IV_2 \leq \int_{Q^{(4)}} |F|^{p(\cdot)(1-\varepsilon)} dz$. In order to estimate IV_1, we introduce the set

$$
E := \Big\{ z \in Q^{(1)} : |Du(z)|^{p(z)} \geq \varepsilon_1 m_{Q^{(4)}}(z) \Big\}
$$

for some $\varepsilon_1 \in (0,1)$ to be specified later. For the integral on E we have

$$
\int_E |Du|^{p(\cdot)(1-\varepsilon)} dz \leq \varepsilon_1^{-\varepsilon} \int_E |Du|^{p(\cdot)} m_{Q^{(4)}}^{-\varepsilon}\, dz \leq c\varepsilon_1^{-\varepsilon} IV_1.
$$

On the other hand, for $z \in Q^{(1)}\backslash E$, we see that either $|Du|^{p(z)} \leq \varepsilon_1 M_{Q^{(4)}}(z)^{\frac{1}{1-\varepsilon}}$ or $|Du|^{p(z)} \leq \varepsilon_1 c_E \tilde{\lambda}$. This implies that

$$
\begin{aligned}
\int_{Q^{(1)}\backslash E} |Du|^{p(\cdot)(1-\varepsilon)} dz &\leq c\varepsilon_1^{1-\varepsilon} \int_{Q^{(1)}} \Big[M_{Q^{(4)}} + \tilde{\lambda}^{1-\varepsilon} \Big] dz \\
&\leq c\varepsilon_1^{1-\varepsilon} \lambda^{1-\varepsilon} |Q^{(4)}| + c\varepsilon_1^{1-\varepsilon} \int_{Q^{(4)}} \left| \frac{u-u_{Q^{(1)}}}{\varrho} \right|^{p(\cdot)(1-\varepsilon)} dz.
\end{aligned}
$$

Summing the previous two estimates, we find that

$$
\varepsilon_1^{\varepsilon} \int_{Q^{(1)}} |Du|^{p(\cdot)(1-\varepsilon)} dz \leq cIV_1 + c\varepsilon_1 \lambda^{1-\varepsilon} |Q^{(4)}| + c\varepsilon_1 \int_{Q^{(4)}} \left| \frac{u-u_{Q^{(1)}}}{\varrho} \right|^{p(\cdot)(1-\varepsilon)} dz.
$$

Rewriting the last inequality and using assumption (6.1) we obtain

$$
\begin{aligned}
cIV_1 &\geq [\varepsilon_1^{\varepsilon} |Q^{(0)}| - c\varepsilon_1 |Q^{(4)}|]\lambda_1^{1-\varepsilon} - c\int_{Q^{(4)}} \left(\left| \frac{u-u_{Q^{(1)}}}{\varrho} \right| + |F| + 1 \right)^{p(\cdot)(1-\varepsilon)} dz \\
&\geq \frac{\lambda_1^{1-\varepsilon} |Q^{(4)}|}{c} - c\int_{Q^{(4)}} \left(\left| \frac{u-u_{Q^{(1)}}}{\varrho} \right| + |F| + 1 \right)^{p(\cdot)(1-\varepsilon)} dz,
\end{aligned}
$$

where we have chosen ε_1 small enough in the last line. Now we come to the estimate of IV_3. Using the definition of $m_{Q^{(4)}}$ and Young inequality with exponents $p(z)(1-\varepsilon)$ and $\frac{p(z)(1-\varepsilon)}{p(z)(1-\varepsilon)-1}$ and assumption (6.2), we find that

$$
\begin{aligned}
IV_3 &\leq \frac{c}{\varrho_2-\varrho_1} \int_{Q^{(4)}} (1+|F|+|Du|)^{p(\cdot)(1-\varepsilon)-1} |u-u_{Q^{(1)}}| dz \\
&\leq \varepsilon \lambda^{1-\varepsilon} |Q^{(4)}| + c_\varepsilon \mu \int_{Q^{(4)}} \left| \frac{u-u_{Q^{(1)}}}{\varrho_2} \right|^{p(\cdot)(1-\varepsilon)} dz,
\end{aligned}
$$

where c_ε indicates that the constant depends on the structural data and ε. From the estimates above we arrive at

$$\begin{aligned}\lambda^{1-\varepsilon}|Q^{(4)}| &+ \sup_{t\in\Lambda^{(1)}} \int_{B^{(1)}\times\{t\}} |u-u_{Q^{(1)}}|^2 m_{Q^{(4)}}^{-\varepsilon}\, dx \\ &\le c_\varepsilon \mu \int_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{p(\cdot)(1-\varepsilon)} dz + c\mu\lambda^{\frac{p_0-2}{p_0}-\varepsilon} \int_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^2 dz \\ &\quad + c\int_{Q^{(4)}} (1+|F|)^{p(\cdot)(1-\varepsilon)} dz + c\mu\varepsilon\lambda^{1-\varepsilon}|Q^{(4)}|.\end{aligned}$$

This proves (6.4). Moreover, choosing $\varrho_1 = \varrho$ and $\varrho_2 = 2\varrho$ we have $\mu \equiv$ constant. This allows us to choose ε small in dependence on n, N, L, γ_2, to obtain

$$\begin{aligned}\lambda^{1-\varepsilon}|Q^{(4)}| &+ \sup_{t\in\Lambda} \int_{B\times\{t\}} |u-u_Q|^2 m_{Q^+}^{-\varepsilon} dx \\ &\le c\int_{Q^{(4)}} \left|\frac{u-u_Q}{\varrho}\right|^{p(\cdot)(1-\varepsilon)} dz + c\lambda^{\frac{p_0-2}{p_0}-\varepsilon} \int_{Q^{(4)}} \left|\frac{u-u_Q}{\varrho}\right|^2 dz + c\int_{Q^{(4)}} (1+|F|)^{p(\cdot)(1-\varepsilon)} dz.\end{aligned}$$

Recalling assumption (6.2), this finishes the proof of the Caccioppoli inequality. □

7. Estimates for the lower order terms

In the final proof, it will be necessary to treat lower order terms, involving the L^2 and $L^{p_2(1-\varepsilon)}$-norm of u. However, these exponents of integrability could be too large, so that Lemma 5.1 is not applicable. This difficulty comes from the fact that we consider a variable exponent of integrability. Therefore, we need the following improvement of Lemma 5.1.

Proposition 7.1. *Let $M \ge 1$ be fixed. Then there exists $\varrho_0 = \varrho_0(n, L, M) > 0$ such that the following holds: Suppose that u is a very weak solution to the parabolic system (2.5) and satisfies the assumptions of Theorem 2.2. Assume that for some parabolic cylinder $Q_{32\varrho}^{(\lambda)}(z_0) \subset \Omega_T$ with $0 < 32\varrho \le \varrho_0$ the following intrinsic coupling holds:*

$$\lambda^{1-\varepsilon} \le \fint_{Q_\varrho^{(\lambda)}(z_0)} (|Du|+|F|+1)^{p(\cdot)(1-\varepsilon)} dz \quad \text{and} \quad \fint_{Q_{16\varrho}^{(\lambda)}(z_0)} (|Du|+|F|+1)^{p(\cdot)(1-\varepsilon)}\, dz \le \lambda^{1-\varepsilon}.$$

Then, for $\sigma = \max\{2, p_2(1-\varepsilon)\}$ there holds:

$$\fint_{Q_{4\varrho}^{(\lambda)}(z_0)} \left|\frac{u-u_{Q_{4\varrho}^{(\lambda)}(z_0)}}{4\varrho}\right|^\sigma dz \le c\lambda^{\frac{\sigma}{p_0}},$$

where $c = c(n, N, \gamma_2, \nu, L)$.

Proof. In the following we abbreviate $\alpha Q \equiv \alpha B\times\alpha\Lambda := Q_{\alpha\varrho}^{(\lambda)}(z_0)$ for $\alpha \ge 1$. Without loss of generality, we may assume that $p_1 < p_2$. Otherwise, the result follows from Corollary 5.2. To begin with the proof, we define the exponent

$$\tilde{p}_1 := \frac{p_1[2(1-\varepsilon)-\varepsilon\sigma]}{2-\varepsilon p_1}$$

and compute

$$\sigma - \tilde{p}_1 = \frac{2[\sigma - p_1(1-\varepsilon)]}{2-\varepsilon p_1}. \tag{7.1}$$

In the following we want to apply Gagliardo-Nirenberg's inequality from Lemma 3.2 with (σ, q, r, θ) replaced by $(\sigma, \tilde{p}_1, 2(1-\varepsilon), \tilde{p}_1/\sigma)$. This will be allowed, once we can ensure that $\frac{\sigma}{\tilde{p}_1} \le 1 + \frac{2(1-\varepsilon)}{n}$ holds true. To ensure this condition, we have to distinguish two cases, whether $\sigma = 2$, or $\sigma = p_2(1-\varepsilon)$. In the case $\sigma = 2$, we recall that $p_1 \ge 2$, so that

$$\frac{\sigma}{\tilde{p}_1} \le 1 + \frac{\varepsilon}{1-2\varepsilon} \le 1 + \frac{2-2\varepsilon}{n},$$

provided $0 < \varepsilon < \frac{1}{4n}$. In the case $\sigma = p_2(1-\varepsilon)$, we have

$$\frac{\sigma}{\tilde{p}_1} = 1 + \frac{\sigma - \tilde{p}_1}{\tilde{p}_1} \le 1 + \frac{\sigma - \tilde{p}_1}{\tilde{p}_1} \le 1 + \sigma - \tilde{p}_1.$$

Since we may assume that $\varepsilon \le \frac{1}{2}$, we may estimate the difference $\sigma - \tilde{p}_1$ as follows:

$$\sigma - \tilde{p}_1 = \frac{2(1-\varepsilon)(p_2-p_1)}{2-\varepsilon p_1} \le 2(1-\varepsilon)(p_2-p_1) \le 2(1-\varepsilon)\omega(64\varrho) \le 2^7(1-\varepsilon)\omega(\varrho_0), \tag{7.2}$$

where we also used the concavity of ω. Next, we choose ϱ_0 in dependence on n, L small enough to have $\omega(\varrho_0) \le \frac{1}{2^6 n}$. This ensures that $\frac{\sigma}{\tilde{p}_1} \le 1 + \frac{2(1-\varepsilon)}{n}$ is satisfied also in the second case when $\sigma = p_2(1-\varepsilon)$ and therefore, we are allowed to apply Gagliardo-Nirenberg's inequality. Now, we choose radii ϱ_1 and ϱ_2 such that $4\varrho \le \varrho_1 < \varrho_2 \le 16\varrho$ and (using the notation from the proof of Caccioppoli's inequality) we write $Q^{(1)} := Q^{(\lambda)}_{\varrho_1}(z_0)$ and $Q^{(4)} := Q^{(\lambda)}_{\varrho_2}(z_0)$. Applying Lemma 3.2 with (σ, q, r, θ) replaced by $(\sigma, \tilde{p}_1, 2-2\varepsilon, \tilde{p}_1/\sigma)$ slice-wise to $(u - u_{Q^{(1)}})(\cdot, t)$ we obtain

$$\begin{aligned} I_\sigma(\varrho_1) &:= \fint_{Q^{(1)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_1}\right|^\sigma dz \\ &\le c \fint_{\Lambda^{(1)}} \left(\fint_{B^{(1)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_1}\right|^{\tilde{p}_1} + |Du|^{\tilde{p}_1} dx\right)\left[\fint_{B^{(1)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_1}\right|^{2-2\varepsilon} dx\right]^{\frac{\sigma - \tilde{p}_1}{2(1-\varepsilon)}} dt, \end{aligned}$$

where $c = c(n, \gamma_2)$. note that the constant in Lemma 3.2 initially depends on σ. Since the dependence on σ is continuous, it can be replaced by a possibly larger constant depending on γ_2 instead. Next, we use Hölder's inequality to obtain for a.e. $t \in \Lambda^{(1)}$ that

$$\fint_{B^{(1)}\times\{t\}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_1}\right|^{2-2\varepsilon} dx = \fint_{B^{(1)}\times\{t\}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_1}\right|^{2-2\varepsilon} m_{Q^{(4)}}^{-\varepsilon(1-\varepsilon)} m_{Q^{(4)}}^{\varepsilon(1-\varepsilon)} dx$$
$$\leq \left[\fint_{B^{(1)}\times\{t\}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_1}\right|^{2} m_{Q^{(4)}}^{-\varepsilon}\, dx\right]^{1-\varepsilon} \left(\fint_{B^{(1)}\times\{t\}} m_{Q^{(4)}}^{1-\varepsilon}\, dx\right)^{\varepsilon}.$$

To proceed further, we apply the Caccioppoli type inequality from Theorem 6.1 to get

$$\begin{aligned}
J &:= \sup_{t\in\Lambda^{(1)}} \fint_{B^{(1)}\times\{t\}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_1}\right|^{2} m_{Q^{(4)}}^{-\varepsilon}\, dx \\
&\leq c\mu\lambda^{(2-p_0)/p_0} \fint_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{p(\cdot)(1-\varepsilon)} dz + c\mu\lambda^{-\varepsilon} \fint_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{2} dz + c\mu\varepsilon\lambda^{\frac{2}{p_0}-\varepsilon} \\
&\leq c\mu\lambda^{(2-p_0)/p_0} \left(\fint_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma} dz\right)^{\frac{p_2(1-\varepsilon)}{\sigma}} + c\mu\lambda^{-\varepsilon} \left(\fint_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma} dz\right)^{\frac{2}{\sigma}} + c\mu\lambda^{\frac{2}{p_0}-\varepsilon},
\end{aligned}$$

where in the last line we applied Hölder's inequality. Here, we also note that the preceding estimates also imply that $I_\sigma(\varrho_1)$ is finite. We now insert the previous computations above and apply Hölder's inequality with exponents $r = \frac{2(1-\varepsilon)}{\varepsilon(\sigma-\tilde{p}_1)}$ and $r' = \frac{2(1-\varepsilon)}{2(1-\varepsilon)-\varepsilon(\sigma-\tilde{p}_1)}$. In this way, we get

$$\begin{aligned}
I_\sigma(\varrho_1) &\leq cJ^{\frac{\sigma-\tilde{p}_1}{2}} \fint_{\Lambda^{(1)}} \left(\fint_{B^{(1)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_1}\right|^{\tilde{p}_1} + |Du|^{\tilde{p}_1} dx\right)\left(\fint_{B^{(1)}\times\{t\}} m_{Q^{(4)}}^{1-\varepsilon}\, dx\right)^{\frac{\varepsilon(\sigma-\tilde{p}_1)}{2(1-\varepsilon)}} dt \\
&\leq cJ^{\frac{\sigma-\tilde{p}_1}{2}} \left(\fint_{Q^{(1)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho_1}\right|^{\tilde{p}_1 r'} + |Du|^{\tilde{p}_1 r'} dz\right)^{\frac{1}{r'}} \left(\fint_{Q^{(1)}} m_{Q^{(4)}}(z)^{1-\varepsilon} dz\right)^{\frac{\varepsilon(\sigma-\tilde{p}_1)}{2(1-\varepsilon)}}.
\end{aligned}$$

Now, we observe that $\tilde{p}_1 r' = p_1(1-\varepsilon)$ and therefore, we are allowed to apply Corollary 5.2 with the choice $p_1(1-\varepsilon)$ for θ to the first integral on the right-hand side. Together with the hypothesis of the proposition we obtain

$$I_\sigma(\varrho_1) \leq cJ^{\frac{\sigma-\tilde{p}_1}{2}} \lambda^{\frac{1-\varepsilon}{r'}} \left(\fint_{Q^{(1)}} m_{Q^{(4)}}(z)^{1-\varepsilon}\, dz\right)^{\frac{\varepsilon(\sigma-\tilde{p}_1)}{2(1-\varepsilon)}}.$$

Moreover, using the definition of $m_{Q^{(4)}}$ and the boundedness of the strong maximal function we find that

$$\begin{aligned}
\fint_{Q^{(1)}} m_{Q^{(4)}}^{1-\varepsilon}\, dz &\leq \tilde{\lambda}^{1-\varepsilon} + \fint_{Q^{(1)}} M_{Q^{(4)}}\, dz \\
&\leq c\lambda^{1-\varepsilon} + c\fint_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho}\right|^{p(\cdot)(1-\varepsilon)} dz \leq c\lambda^{1-\varepsilon} + c\left(\fint_{Q^{(4)}} \left|\frac{u-u_{Q^{(1)}}}{\varrho}\right|^{\sigma} dz\right)^{\frac{p_2(1-\varepsilon)}{\sigma}}.
\end{aligned}$$

Inserting this estimate and the estimate for J above, we conclude that

$$\begin{aligned}
I_\sigma(\varrho_1) &\le c\mu\lambda^{1-\varepsilon+\frac{(2-\varepsilon p_0)(\sigma-\tilde p_1)}{2p_0}} + c\mu\lambda^{1-\varepsilon-\frac{\varepsilon(\sigma-\tilde p_1)}{2}}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz\right)^{\frac{\sigma-\tilde p_1}{\sigma}}\\
&\quad + c\mu\lambda^{1-\varepsilon+\frac{(2-p_0)(\sigma-\tilde p_1)}{2p_0}}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz\right)^{\frac{(\sigma-\tilde p_1)p_2(1-\varepsilon)}{2\sigma}}\\
&\quad + c\mu\lambda^{\frac{1-\varepsilon}{r'}+\frac{(2-\varepsilon p_0)(\sigma-\tilde p_1)}{2p_0}}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz\right)^{\frac{\varepsilon p_2(\sigma-\tilde p_1)}{2\sigma}}\\
&\quad + c\mu\lambda^{\frac{1-\varepsilon}{r'}+\frac{(2-p_0)(\sigma-\tilde p_1)}{2p_0}}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz\right)^{\frac{p_2(\sigma-\tilde p_1)}{2\sigma}}\\
&\quad + c\mu\lambda^{\frac{1-\varepsilon}{r'}-\frac{\varepsilon(\sigma-\tilde p_1)}{2}}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz\right)^{\frac{(2+\varepsilon p_2)(\sigma-\tilde p_1)}{2\sigma}}\\
&:= I + II + III + IV + V + VI.
\end{aligned}$$

Now we are going to estimate the terms I – VI. From the definition of $\tilde p_1$ and (7.1) we infer

$$1-\varepsilon+\frac{(2-\varepsilon p_0)(\sigma-\tilde p_1)}{2p_0} \le 1-\varepsilon+\frac{2-\varepsilon p_1}{2p_1}\cdot\frac{2\sigma-2p_1(1-\varepsilon)}{2-\varepsilon p_1} = \frac{\sigma}{p_1} \le \frac{\sigma}{p_0}+\omega(64\varrho)$$

so that $I \le c\mu\lambda^{\frac{\sigma}{p_0}}$. We now come to the estimate of II. Since $\sigma > \tilde p_1$, we use Young inequality with exponents $\frac{\sigma}{\sigma-\tilde p_1}$ and $\frac{\sigma}{\tilde p_1}$ to obtain for any $\delta \in (0,1)$ that

$$\begin{aligned}
II &\le \delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\left(1-\varepsilon-\frac{\varepsilon(\sigma-\tilde p_1)}{2}\right)\frac{\sigma}{\tilde p_1}}\\
&\le \delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{\sigma}{p_0}},
\end{aligned}$$

where we have used the following identity:

$$\left(1-\varepsilon-\frac{\varepsilon(\sigma-\tilde p_1)}{2}\right)\frac{\sigma}{\tilde p_1} = \left(1-\varepsilon-\frac{\varepsilon(\sigma-p_1(1-\varepsilon))}{2-\varepsilon p_1}\right)\frac{\sigma(2-\varepsilon p_1)}{p_1(2-2\varepsilon-\varepsilon\sigma)} = \frac{\sigma}{p_1}$$

and hence $\lambda^{\left(1-\varepsilon-\varepsilon\frac{\sigma-\tilde p_1}{2}\right)\frac{\sigma}{\tilde p_1}} = \lambda^{\frac{\sigma}{p_1}} \le \lambda^{\frac{\sigma}{p_0}+\omega(64\varrho)} \le c\lambda^{\frac{\sigma}{p_0}}$. Next, we consider the estimate for III. We use Young's inequality with exponents $\frac{2\sigma}{p_2(1-\varepsilon)(\sigma-\tilde p_1)}$ and $\frac{2\sigma}{2\sigma-p_2(1-\varepsilon)(\sigma-\tilde p_1)}$ to find that

$$\begin{aligned}
III &\le \delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\left(1-\varepsilon+\frac{(2-p_0)(\sigma-\tilde p_1)}{2p_0}\right)\frac{2\sigma}{2\sigma-p_2(1-\varepsilon)(\sigma-\tilde p_1)}}\\
&\le \delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{\sigma}{p_0}},
\end{aligned}$$

since $\lambda \geq 1$ and

$$\begin{aligned}&\left(1-\varepsilon+\frac{(2-p_0)(\sigma-\tilde{p}_1)}{2p_0}\right)\frac{2\sigma}{2\sigma-p_2(1-\varepsilon)(\sigma-\tilde{p}_1)}\\&\quad=\frac{\sigma}{p_0}\cdot\frac{2p_0(1-\varepsilon)+(2-p_0)(\sigma-\tilde{p}_1)}{2\sigma-p_2(1-\varepsilon)(\sigma-\tilde{p}_1)}\\&\quad=\frac{\sigma}{p_0}\cdot\frac{p_0(1-\varepsilon)(2-\varepsilon p_1)+(2-p_0)(\sigma-p_1(1-\varepsilon))}{\sigma(2-\varepsilon p_1)-p_2(1-\varepsilon)(\sigma-p_1(1-\varepsilon))}\\&\quad\leq\frac{\sigma}{p_0}+c(\gamma_2)\,(p_2-p_1)\leq\frac{\sigma}{p_0}+c(\gamma_2)\,\omega(64\varrho).\end{aligned}$$

In order to avoid the complicated computations in the estimates for $IV-VI$, we shall deal with the estimates in the two cases separately. We start with the case $\sigma = p_2(1-\varepsilon)$. From (7.2) we deduce that $\lambda^{\sigma-\tilde{p}_1} \leq \lambda^{c\omega(\varrho)} \leq c$. To estimate IV, we use Young's inequality with exponents r and r' (note that $\frac{\varepsilon p_2(\sigma-\tilde{p}_1)}{2\sigma} = \frac{1}{r}$) to obtain

$$IV \leq c\mu\lambda^{\frac{1-\varepsilon}{r'}}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma}dz\right)^{\frac{\varepsilon p_2(\sigma-\tilde{p}_1)}{2\sigma}} \leq \delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma}dz + c(\delta)\mu\lambda^{1-\varepsilon}.$$

To estimate V, we use that $\lambda^{\frac{1-\varepsilon}{r'}+\frac{(2-\varepsilon p_0)(\sigma-\tilde{p}_1)}{2p_0}} \leq c\lambda^{\frac{1-\varepsilon}{r'}} = \lambda^{1-\varepsilon-\frac{\varepsilon(\sigma-\tilde{p}_1)}{2}} \leq c\lambda^{1-\varepsilon}$ and apply Young's inequality with exponents $\frac{2(1-\varepsilon)}{\sigma-\tilde{p}_1}$ and $\frac{2(1-\varepsilon)}{2(1-\varepsilon)-(\sigma-\tilde{p}_1)} = 1-\frac{\sigma-\tilde{p}_1}{2(1-\varepsilon)-(\sigma-\tilde{p}_1)}$ to deduce

$$\begin{aligned}V&\leq c\mu\lambda^{1-\varepsilon}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma}dz\right)^{\frac{\sigma-\tilde{p}_1}{2(1-\varepsilon)}}\\&\leq\delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma}dz+c(\delta)\mu\lambda^{(1-\varepsilon)\left(1-\frac{\sigma-\tilde{p}_1}{2(1-\varepsilon)-(\sigma-\tilde{p}_1)}\right)}\\&\leq\delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma}dz+c(\delta)\mu\lambda^{1-\varepsilon}.\end{aligned}$$

Next we consider the estimate for VI. Again, we use that $\lambda^{\frac{1-\varepsilon}{r'}-\frac{\varepsilon(\sigma-\tilde{p}_1)}{2}} \leq c\lambda^{1-\varepsilon}$. Applying Young's inequality with exponents $\frac{2\sigma}{(2+\varepsilon p_2)(\sigma-\tilde{p}_1)}$ and $\frac{2\sigma}{2\tilde{p}_1-\varepsilon p_2(\sigma-\tilde{p}_1)}$ we obtain

$$\begin{aligned}VI&\leq c\mu\lambda^{1-\varepsilon}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma}dz\right)^{\frac{(2+\varepsilon p_2)(\sigma-\tilde{p}_1)}{2\sigma}}\\&\leq\delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma}dz+c(\delta)\mu\lambda^{(1-\varepsilon)\frac{2\sigma}{2\tilde{p}_1-\varepsilon p_2(\sigma-\tilde{p}_1)}}\\&\leq\delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^{\sigma}dz+c(\delta)\mu\lambda^{1-\varepsilon}.\end{aligned}$$

Therefore, from the estimates above and the fact that $\lambda^{1-\varepsilon} \le \lambda^{\frac{\sigma}{p_0}}$, we conclude that in the case $\sigma = p_2(1-\varepsilon)$ there holds

$$I_\sigma(\varrho_1) \le \delta \fint_{Q^{(4)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{\sigma}{p_0}}. \tag{7.3}$$

In the following, we will show that (7.3) also holds in the second case when $\sigma = 2$. It remains to consider the terms $IV - VI$. Let $\theta = \frac{\tilde p_1}{\sigma}$. In order to estimate IV, we use Young's inequality with exponents $\frac{4}{\varepsilon p_2(2-\tilde p_1)}$ and $\frac{4}{4-\varepsilon p_2(2-\tilde p_1)}$ to get

$$\begin{aligned}
IV &= c\mu\lambda^{\frac{p_0(1-\varepsilon)+(1-\varepsilon p_0)(2-\tilde p_1)}{p_0}} \left(\fint_{Q^{(4)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz\right)^{\frac{\varepsilon p_2(2-\tilde p_1)}{4}} \\
&\le \delta \fint_{Q^{(4)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{4[p_0(1-\varepsilon)+(1-\varepsilon p_0)(2-\tilde p_1)]}{p_0[4-\varepsilon p_2(2-\tilde p_1)]}} \\
&\le \delta \fint_{Q^{(4)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{\sigma}{p_o}},
\end{aligned}$$

where in the last line we have used the following computation:

$$\begin{aligned}
&\frac{4[p_0(1-\varepsilon)+(1-\varepsilon p_0)(2-\tilde p_1)]}{p_0[4-\varepsilon p_2(2-\tilde p_1)]} \\
&\quad = \frac{2}{p_0}\cdot\frac{p_0(1-\varepsilon)(2-\varepsilon p_1) + 2(1-\varepsilon p_0)(2-p_1(1-\varepsilon))}{2(2-\varepsilon p_1) - \varepsilon p_2(2-p_1(1-\varepsilon))} \\
&\quad = \frac{\sigma}{p_0}\cdot\left[1 + \frac{-(2-p_0(1-\varepsilon))(2-\varepsilon p_1) + (2(1-\varepsilon p_0)+\varepsilon p_2)(2-p_1(1-\varepsilon))}{2(2-\varepsilon p_1) - \varepsilon p_2(2-p_1(1-\varepsilon))}\right] \\
&\quad \le \frac{\sigma}{p_0}\cdot\left[1 + \frac{(2-p_1(1-\varepsilon))\varepsilon(p_2+p_1-2p_0)}{2(2-\varepsilon p_1) - \varepsilon p_2(2-p_1(1-\varepsilon))} + c\omega(\varrho)\right] \le \frac{\sigma}{p_0} + c\omega(\varrho).
\end{aligned}$$

The estimate of the term V is similar. Applying Young's inequality with exponents $\frac{4}{p_2(2-\tilde p_1)}$ and $\frac{4}{4-p_2(2-\tilde p_1)}$ we get

$$\begin{aligned}
V &= c\mu\lambda^{\frac{2p_0(1-\varepsilon)+(2-p_0-\varepsilon p_0)(2-\tilde p_1)}{2p_0}} \left(\fint_{Q^{(4)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz\right)^{\frac{p_2(2-\tilde p_1)}{4}} \\
&\le \delta \fint_{Q^{(4)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{2[2p_0(1-\varepsilon)+(2-p_0-\varepsilon p_0)(2-\tilde p_1)]}{p_0[4-p_2(2-\tilde p_1)]}} \\
&\le \delta \fint_{Q^{(4)}} \left|\frac{u - u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{\sigma}{p_0}},
\end{aligned}$$

where in the last line we have used the following computation:

$$\begin{aligned}
&\frac{2[2p_0(1-\varepsilon)+(2-p_0-\varepsilon p_0)(2-\tilde p_1)]}{p_0[4-p_2(2-\tilde p_1)]} \\
&\quad = \frac{2}{p_0}\cdot\frac{p_0(1-\varepsilon)(2-\varepsilon p_1) + (2-p_0-\varepsilon p_0)(2-p_1(1-\varepsilon))}{2(2-\varepsilon p_1) - p_2(2-p_1(1-\varepsilon))} \\
&\quad = \frac{\sigma}{p_0}\cdot\left[1 + \frac{-(2-p_0(1-\varepsilon))(2-\varepsilon p_1) + (2-p_0-\varepsilon p_0+p_2)(2-p_1(1-\varepsilon))}{2(2-\varepsilon p_1) - p_2(2-p_1(1-\varepsilon))}\right]
\end{aligned}$$

$$\le \frac{\sigma}{p_0}\cdot\left[1+\frac{(2-p_1(1-\varepsilon))(p_2-p_0+\varepsilon(p_1-p_0))}{2(2-\varepsilon p_1)-p_2(2-p_1(1-\varepsilon))}+c\omega(\varrho)\right]\le \frac{\sigma}{p_0}+c\omega(\varrho).$$

We now come to the estimate for VI. Here, we apply Young's inequality with exponents $\frac{4}{(2+\varepsilon p_2)(2-\tilde p_1)}$ and $\frac{4}{4-(2+\varepsilon p_2)(2-\tilde p_1)}$ to infer that

$$\begin{aligned} VI &= c\mu\lambda^{1-\varepsilon-\varepsilon(2-\tilde p_1)}\left(\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz\right)^{\frac{(2+\varepsilon p_2)(2-\tilde p_1)}{4}}\\ &\le \delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{4[1-\varepsilon-\varepsilon(2-\tilde p_1)]}{4-(2+\varepsilon p_2)(2-\tilde p_1)}}\\ &\le \delta\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{\sigma}{p_0}}, \end{aligned}$$

where we have used that

$$\begin{aligned} \frac{4[1-\varepsilon-\varepsilon(2-\tilde p_1)]}{4-(2+\varepsilon p_2)(2-\tilde p_1)} &= \frac{2}{p_0}\cdot\frac{p_0[(1-\varepsilon)(2-\varepsilon p_1)-2\varepsilon(2-p_1(1-\varepsilon))]}{2(2-\varepsilon p_1)-(2+\varepsilon p_2)(2-p_1(1-\varepsilon))}\\ &= \frac{\sigma}{p_0}\cdot\left[1+\frac{(p_0(1-\varepsilon)-2)(2-\varepsilon p_1)+(2+\varepsilon p_2-2p_0\varepsilon)(2-p_1(1-\varepsilon))}{2(2-\varepsilon p_1)-(2+\varepsilon p_2)(2-p_1(1-\varepsilon))}\right]\\ &\le \frac{\sigma}{p_0}\cdot\left[1-\frac{\varepsilon(2-p_1(1-\varepsilon))(2p_0-p_2-p_1)}{2(2-\varepsilon p_1)-(2+\varepsilon p_2)(2-p_1(1-\varepsilon))}+c\omega(\varrho)\right]\le\frac{\sigma}{p_0}+c\omega(\varrho). \end{aligned}$$

This ensures that (7.3) holds true also in the second case when $\sigma = 2$. Therefore, from (7.3) we conclude that

$$\begin{aligned} I_\sigma(\varrho_1) &\le \delta 2^{\sigma-1}\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(4)}}}{\varrho_2}\right|^\sigma dz + \delta 2^{\sigma-1}\left|\frac{u_{Q^{(4)}}-u_{Q^{(1)}}}{\varrho_2}\right|^\sigma + c(\delta)\mu\lambda^{\frac{\sigma}{p_0}}\\ &\le \delta 2^\sigma\fint_{Q^{(4)}}\left|\frac{u-u_{Q^{(4)}}}{\varrho_2}\right|^\sigma dz + c(\delta)\mu\lambda^{\frac{\sigma}{p_0}}\\ &= \delta 2^\sigma I_\sigma(\varrho_1)+c(\delta)\mu\lambda^{\frac{\sigma}{p_0}} \end{aligned}$$

holds true for any $0<\delta<1$. Recalling that $\mu = (\frac{\varrho}{\varrho_2-\varrho_1})^\beta$, where β is a constant depends only on the structural parameters, we choose $\delta = 2^{-(\sigma+1)}$ to infer that

$$I_\sigma(\varrho_1)\le \tfrac12 I_\sigma(\varrho_2)+c\left(\tfrac{\varrho}{\varrho_2-\varrho_1}\right)^\beta\lambda^{\frac{\sigma}{p_0}}$$

for any radii ϱ_1,ϱ_2 such that $4\varrho\le\varrho_1<\varrho_2\le 16\varrho$. At this point, we use Lemma 3.1 with ϕ replaced by I_σ to infer that $I_\sigma(4\varrho)\le c\lambda^{\frac{\sigma}{p_0}}$. This proves the desired estimate. □

8. Reverse-Hölder type inequality

Proposition 8.1. *Let $M\ge 1$ be fixed. Then there exists $\varrho_0=\varrho_0(n,L,M)>0$ and $\varepsilon=\varepsilon(n,\gamma_2)>0$ such that the following holds: Suppose that u is a very weak solution to the parabolic system* (2.1) *and that the hypothesis of Theorem 2.2 are satisfied. Finally, assume that for some parabolic cylinder*

$Q^{(\lambda)}_{32\varrho}(z_0) \subset \Omega_T$ with $0 < 32\varrho \le \varrho_0$ and $\lambda \ge 1$ the following intrinsic coupling holds:

$$\lambda^{1-\varepsilon} \le \fint_{Q^{(\lambda)}_{\varrho}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \quad \text{and} \quad \fint_{Q^{(\lambda)}_{16\varrho}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)}\, dz \le \lambda^{1-\varepsilon}.$$

Then we have the following reverse-Hölder inequality:

$$\fint_{Q^{(\lambda)}_{\varrho}(z_0)} |Du|^{p(\cdot)(1-\varepsilon)} dz \le c \left[\fint_{Q^{(\lambda)}_{2\varrho}(z_0)} |Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}} dz \right]^{\bar q} + c \fint_{Q^{(\lambda)}_{2\varrho}(z_0)} (1 + |F|)^{p(\cdot)(1-\varepsilon)} dz$$

where $\bar q = \bar q(n, \gamma_2) > 1$ and $c = c(n, N, \nu, L, \gamma_2)$.

Proof. In the following we abbreviate $\alpha Q \equiv \alpha B \times \alpha\Lambda := Q^{(\lambda)}_{\alpha\varrho}(z_0)$ for $\alpha \ge 1$. From the Caccioppoli inequality from Theorem 6.1, [i.e. estimate (6.3)], we obtain

$$\begin{aligned}
&\fint_{Q} |Du|^{p(\cdot)(1-\varepsilon)} dz \\
&\quad \le c \fint_{2Q} \left| \frac{u - u_Q}{\varrho} \right|^{p(\cdot)(1-\varepsilon)} dz + c\lambda^{\frac{p_0-2}{p_0} - \varepsilon} \fint_{2Q} \left| \frac{u - u_Q}{\varrho} \right|^2 dz + c \fint_{2Q} (1 + |F|)^{p(\cdot)(1-\varepsilon)} dz \\
&\quad \le c I_{p_2(1-\varepsilon)} + c\lambda^{\frac{p_0-2}{p_0} - \varepsilon} I_2 + c \fint_{2Q} (1 + |F|)^{p(\cdot)(1-\varepsilon)} dz,
\end{aligned}$$

where we have abbreviated

$$I_\sigma := \fint_{2Q} \left| \frac{u - u_{2Q}}{\varrho} \right|^\sigma dz$$

for $\sigma = p_2(1-\varepsilon)$ and $\sigma = 2$. For σ as before, we define $q_1 := \frac{n\sigma}{n+2-2\varepsilon}$, so that $q_1 < \sigma$ and $\frac{\sigma}{q_1} = \frac{n+2-2\varepsilon}{n}$. We now apply Gagliardo-Nirenberg's inequality, i.e. Lemma 3.2 with (σ, q, r, θ) replaced by $(\sigma, q_1, 2 - 2\varepsilon, q_1/\sigma)$ slice-wise to $(u - u_{2Q})(\cdot, t)$. In this way we obtain

$$I_\sigma \le c \fint_{2\Lambda} \left(\fint_{2B} \left| \frac{u - u_{2Q}}{\varrho} \right|^{q_1} + |Du|^{q_1} dx \right) \left[\fint_{2B} \left| \frac{u - u_{2Q}}{\varrho} \right|^{2-2\varepsilon} dx \right]^{\frac{\sigma - q_1}{2-2\varepsilon}} dt.$$

Next, we use Hölder's inequality to obtain

$$\begin{aligned}
\fint_{2B} \left| \frac{u - u_{2Q}}{\varrho} \right|^{2-2\varepsilon} (\cdot, t) dx &= \fint_{2B} \left| \frac{u - u_{2Q}}{\varrho} \right|^{2-2\varepsilon} m_{4Q}^{-\varepsilon(1-\varepsilon)} m_{4Q}^{\varepsilon(1-\varepsilon)}(\cdot, t) dx \\
&\le J^{1-\varepsilon} \left[\fint_{2B} m_{4Q}^{1-\varepsilon}(\cdot, t) dx \right]^\varepsilon
\end{aligned}$$

for a.e. $t \in 2\Lambda$, where we have abbreviated

$$J := \sup_{t \in 2\Lambda} \fint_{2B \times \{t\}} \left| \frac{u - u_{2Q}}{\varrho} \right|^2 m_{4Q}^{-\varepsilon}\, dx.$$

Inserting this above and applying Hölder's inequality with exponents $r = \frac{2-2\varepsilon}{\varepsilon(\sigma-q_1)}$ and $r' = \frac{2-2\varepsilon}{2-2\varepsilon-\varepsilon(\sigma-q_1)}$ we get

$$
\begin{aligned}
I_\sigma &\le cJ^{\frac{\sigma-q_1}{2}} \fint_{2\Lambda}\left(\fint_{2B}\left|\frac{u-u_{2Q}}{\varrho}\right|^{q_1} + |Du|^{q_1}dx\right)\left(\fint_{2B} m_{4Q}^{1-\varepsilon}(\cdot,t)dx\right)^{\frac{\varepsilon(\sigma-q_1)}{2-2\varepsilon}} dt \\
&\le cJ^{\frac{\sigma-q_1}{2}}\left[\fint_{2Q}\left|\frac{u-u_{2Q}}{\varrho}\right|^{q_1r'} + |Du|^{q_1r'}dz\right]^{\frac{1}{r'}}\left[\fint_{2Q} m_{4Q}(z)^{1-\varepsilon}dz\right]^{\frac{\varepsilon(\sigma-q_1)}{2-2\varepsilon}}.
\end{aligned}
$$

Observe that $\sigma - q_1 = \frac{2\sigma(1-\varepsilon)}{n+2-2\varepsilon}$ and

$$
q_2 := q_1r' = \frac{n\sigma}{n+2-2\varepsilon}\cdot\frac{2(1-\varepsilon)}{2(1-\varepsilon)-\varepsilon(\sigma-q_1)} = \frac{\sigma n}{n+2-\varepsilon(\sigma+2)}.
$$

To proceed further, we apply the Caccioppoli type inequality from Theorem 6.1 (more precisely, we use (6.4) with the choice $\varrho_1 = 2\varrho$ and $\varrho_2 = 4\varrho$) and subsequently Proposition 7.1 to get

$$
\begin{aligned}
J &\le c\lambda^{\frac{2-p_0}{p_0}}\fint_{4Q}\left|\frac{u-u_{2Q}}{\varrho}\right|^{p(\cdot)(1-\varepsilon)}dz + c\lambda^{-\varepsilon}\fint_{4Q}\left|\frac{u-u_{2Q}}{\varrho}\right|^2 dz + c_1\varepsilon\lambda^{\frac{2}{p_0}-\varepsilon} \\
&\le c\lambda^{\frac{2-p_0}{p_0}}\lambda^{\frac{p_2(1-\varepsilon)}{p_0}} + c\lambda^{\frac{2}{p_0}-\varepsilon} + c_1\varepsilon\lambda^{\frac{2}{p_0}-\varepsilon} \le c\lambda^{\frac{2}{p_0}-\varepsilon}.
\end{aligned}
$$

Moreover, from Proposition 7.1 we infer that

$$
\begin{aligned}
\fint_{2Q} m_{4Q}^{1-\varepsilon}dz &\le \tilde\lambda^{1-\varepsilon} + \fint_{2Q} M_{4Q}\,dz \le c\lambda^{1-\varepsilon} + c\fint_{4Q}\left|\frac{u-u_{2Q}}{\varrho}\right|^{p(\cdot)(1-\varepsilon)}dz \\
&\le c\lambda^{1-\varepsilon} + \lambda^{\frac{p_2(1-\varepsilon)}{p_0}} \le c\lambda^{1-\varepsilon}.
\end{aligned}
$$

Inserting the last two estimates above yields

$$
I_\sigma \le c\lambda^{\frac{\sigma-q_1}{p_0}}\left[\fint_{2Q}\left|\frac{u-u_{2Q}}{\varrho}\right|^{q_2} + |Du|^{q_2}dz\right]^{\frac{1}{r'}}.
$$

Moreover, choosing ϱ_0 small enough to have $p_2 - p_1 \le \omega(32\varrho_0) \le \min\{(n+1)/(n+\frac{3}{2}), 2/n\}$ and $\varepsilon \le \min\{1/1000n, 1/4\gamma_2\}$, we have $q_2 \le p_1(1-\varepsilon)$. We now apply Lemma 5.1 with $(\theta, \tilde\Omega\times T_1, \tilde\Omega\times T_2)$ replaced by $(q_2, 2Q, 2Q)$ to conclude that

$$
\begin{aligned}
I_\sigma &\le c\lambda^{\frac{\sigma-q_1}{p_0}}\left[\left[\fint_{2Q}|Du|^{q_2}dz\right]^{\frac{1}{r'}} + \left[\lambda^{\frac{2-p_0}{p_0}}\fint_{2Q}(1+|Du|+|F|)^{p(\cdot)-1}dz\right]^{q_1}\right] \\
&\le c\lambda^{\frac{\sigma-q_1}{p_0}}\left[\left[\fint_{2Q}(1+|Du|)^{\frac{p(\cdot)q_2}{p_1}}dz\right]^{\frac{1}{r'}} + \left[\lambda^{\frac{2-p_0}{p_0}}\fint_{2Q}(1+|Du|+|F|)^{\frac{p(\cdot)(p_2-1)}{p_2}}dz\right]^{q_1}\right].
\end{aligned}
$$

Now, we will find a lower bound for the exponents appearing on the right-hand side; note that there are three different exponents: $\frac{p(\cdot)q_2}{p_1}$ with the coice $\sigma = 2$ and $\sigma = p_2(1-\varepsilon)$ and $\frac{p(\cdot)(p_2-1)}{p_2}$. To this aim we define

$$
\bar q_1 := \frac{(n+2-4\varepsilon)(1-\varepsilon)}{n}, \quad \bar q_2 := \frac{p_1[n+2-\varepsilon(p_2(1-\varepsilon)+2)]}{np_2} \quad \text{and} \quad \bar q_3 := \frac{p_2(1-\varepsilon)}{p_2-1}.
$$

An easy computation shows that $\bar{q}_1,\ \bar{q}_2 \geq \frac{n+1}{n} > 1$ and $\bar{q}_3 \geq \frac{\gamma_2-\frac{1}{2}}{\gamma_2-1} > 1$. Therefore, letting

$$\bar{q} := \min\left\{\frac{n+1}{n}, \frac{\gamma_2-\frac{1}{2}}{\gamma_2-1}\right\},$$

we can use Hölder's inequality to get from the last estimate that

$$I_\sigma \leq c\lambda^{\frac{\sigma-q_1}{p_0}}\left[\left[\fint_{2Q}(1+|Du|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\frac{\bar{q}q_1}{p_1(1-\varepsilon)}} + \lambda^{\frac{(2-p_0)q_1}{p_0}}\left[\fint_{2Q}(1+|Du|+|F|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\frac{\bar{q}q_1(p_2-1)}{p_2(1-\varepsilon)}}\right].$$

Again by Hölder's inequality and the hypothesis, we obtain for the first term on the right-hand side that

$$\begin{aligned}
&\lambda^{\frac{\sigma-q_1}{p_0}}\left[\fint_{2Q}(1+|Du|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\frac{\bar{q}q_1}{p_1(1-\varepsilon)}}\\
&\quad\leq \lambda^{\frac{\sigma-q_1}{p_0}}\left[\fint_{2Q}(1+|Du|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\bar{q}}\left[\fint_{2Q}(1+|Du|)^{p(\cdot)(1-\varepsilon)}dz\right]^{\frac{q_1}{p_1(1-\varepsilon)}-1}\\
&\quad\leq \lambda^{\frac{\sigma-q_1}{p_0}}\lambda^{\frac{q_1}{p_1}-(1-\varepsilon)}\left[\fint_{2Q}(1+|Du|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\bar{q}}\\
&\quad\leq \lambda^{\frac{\sigma}{p_0}-1+\varepsilon}\left[\fint_{2Q}(1+|Du|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\bar{q}}
\end{aligned}$$

where we have also used that $\lambda^{p_0-p_1} \leq c$. Similarly, we find for the second integral on the right-hand side that

$$\begin{aligned}
&\lambda^{\frac{\sigma-q_1}{p_0}}\lambda^{\frac{(2-p_0)q_1}{p_0}}\left[\fint_{2Q}(1+|Du|+|F|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\frac{\bar{q}q_1(p_2-1)}{p_2(1-\varepsilon)}}\\
&\quad\leq \lambda^{\frac{\sigma+q_1-p_0q_1}{p_0}}\left[\fint_{2Q}(1+|Du|+|F|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\bar{q}}\left[\fint_{2Q}(1+|Du|+|F|)^{p(\cdot)(1-\varepsilon)}dz\right]^{\frac{q_1(p_2-1)}{p_2(1-\varepsilon)}-1}\\
&\quad\leq \lambda^{\frac{\sigma+q_1-p_0q_1}{p_0}}\lambda^{\frac{q_1(p_2-1)}{p_2}-(1-\varepsilon)}\left[\fint_{2Q}(1+|Du|+|F|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\bar{q}}\\
&\quad\leq \lambda^{\frac{\sigma}{p_0}-1+\varepsilon}\left[\fint_{2Q}(1+|Du|+|F|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\bar{q}}
\end{aligned}$$

Inserting this above, we find that

$$I_\sigma \leq c\lambda^{\frac{\sigma}{p_0}-1+\varepsilon}\left[\fint_{2Q}(1+|Du|+|F|)^{\frac{p(\cdot)(1-\varepsilon)}{\bar{q}}}dz\right]^{\bar{q}}.$$

Now, we note that for $\sigma = p_2(1-\varepsilon)$, we have $\lambda^{\frac{\sigma}{p_0}-1+\varepsilon} = \lambda^{\frac{p_2(1-\varepsilon)}{p_0}-1+\varepsilon} = \lambda^{\frac{(p_2-p_0)(1-\varepsilon)}{p_0}} \leq c$, while for $\sigma = 2$, we have $\lambda^{\frac{p_0-2}{p_0}-\varepsilon}\lambda^{\frac{\sigma}{p_0}-1+\varepsilon} = 1$. Therefore, inserting the estimate for I_σ with $\sigma = 2$ and $\sigma = p_2(1-\varepsilon)$ above, we arrive at

$$\fint_Q |Du|^{p(\cdot)(1-\varepsilon)}dz \le c\left[\fint_{2Q}(1+|Du|+|F|)^{\frac{p(\cdot)(1-\varepsilon)}{\tilde q}}dz\right]^{\tilde q} + c\fint_{2Q}(1+|F|)^{p(\cdot)(1-\varepsilon)}dz$$

$$\le c\left[\fint_{2Q}|Du|^{\frac{p(\cdot)(1-\varepsilon)}{\tilde q}}dz\right]^{\tilde q} + c\fint_{2Q}(1+|F|)^{p(\cdot)(1-\varepsilon)}dz.$$

This proves the reverse Hölder inequality. □

9. Proof of the main theorem

In this section we will prove the higher integrability of very weak solutions stated in Theorem 2.2. The idea now, is to prove estimates for $|Du|^{p(\cdot)(1-\varepsilon)}$ on certain upper level sets. The argument uses a certain stopping time argument which allows one to construct a covering of the upper level sets. This method has its origin in [19, 20]; a slightly simplified version can be found in [7]. Since most of the arguments are standard by now, we will only sketch the proof and refer to [7, §7] for the details.

Let $M \ge 1$ and suppose that (2.6) is satisfied. From now on, we consider a cylinder $Q_r \equiv Q_r(\mathfrak{z}_0)$ such that $Q_{2r} \Subset \Omega_T$ and define

(9.1)

$$\lambda_0 := \left[\fint_{Q_{2r}}(|Du|+|F|+1)^{p(\cdot)(1-\varepsilon)}dz\right]^{\frac{p_M}{2-\varepsilon p_M}} \ge 1, \quad \text{where} \quad p_M := \sup_{Q_{2r}} p(\cdot).$$

For fixed $r \le r_1 < r_2 \le 2r$ we consider the concentric parabolic cylinders

$$Q_r \subseteq Q_{r_1} \subset Q_{r_2} \subseteq Q_{2r}\,.$$

In the following we shall consider parameter λ such that

$$\lambda > B\lambda_0 \qquad \text{where} \qquad B := \Big(\frac{8\chi r}{r_2 - r_1}\Big)^{\frac{(n+2)p_M}{2-\varepsilon p_M}}, \tag{9.2}$$

and for $z_0 \in Q_{r_1}$ we consider radii ϱ satisfying

$$\frac{r_2 - r_1}{4\chi} \le \varrho \le \frac{r_2 - r_1}{2}, \tag{9.3}$$

where $\chi = \chi(n, \gamma_1) \ge 5$ denotes the constant from a version of Vitali's covering theorem [7, Lemma 7.1] for non-uniformly parabolic cylinders. Note that this choice ensures that $Q_\varrho^{(\lambda)}(z_0) \subset Q_{r_2}$. Recalling the definition of λ_0 we get by enlarging the domain of integration from $Q_\varrho^{(\lambda)}(z_0)$ to Q_{2r} the following estimate for λ and ϱ as above:

$$\fint_{Q_\varrho^{(\lambda)}(z_0)}(|Du|+|F|+1)^{p(\cdot)(1-\varepsilon)}dz \le \frac{|Q_{2r}|}{|Q_\varrho^{(\lambda)}(z_0)|}\fint_{Q_{2r}}(|Du|+|F|+1)^{p(\cdot)(1-\varepsilon)}dz$$

$$\le \Big(\frac{2r}{\varrho}\Big)^{n+2}\lambda^{\frac{p_0-2}{p_0}}\lambda_0^{\frac{2}{p_M}-\varepsilon} \le \Big(\frac{2r}{\varrho}\Big)^{n+2}B^{\varepsilon-\frac{2}{p_M}}\lambda^{1-\varepsilon} \le \lambda^{1-\varepsilon}\,.$$

As usual, we denoted $p_0 = p(z_0)$. For λ as in (9.2) we consider the upper level set

$$E(\lambda, r_1) := \{z \in Q_{r_1} : |Du(z)|^{p(z)} > \lambda\}.$$

In the following we show that also a reverse inequality holds true for small radii and for the Lebesgue points $z_0 \in E(\lambda, r_1)$. By Lebesgue's differentiation theorem (see [7, (7.9)]) we infer for any $z_0 \in E(\lambda, r_1)$ that

$$\lim_{\varrho \downarrow 0} \fint_{Q_\varrho^{(\lambda)}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)} dz \geq |Du(z_0)|^{p_0(1-\varepsilon)} > \lambda^{1-\varepsilon}.$$

From the preceding reasoning we conclude that the last inequality yields a radius for which the considered integral takes a value larger than $\lambda^{1-\varepsilon}$, and on the other hand, the integral is smaller than $\lambda^{1-\varepsilon}$ for any radius satisfying (9.3). Therefore, the continuity of the integral yields the existence of a maximal radius ϱ_{z_0} in between, i.e. $0 < \varrho_{z_0} < \frac{r_2 - r_1}{4\chi}$ such that

$$\fint_{Q_{\varrho_{z_0}}^{(\lambda)}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)} dz = \lambda^{1-\varepsilon} \tag{9.4}$$

holds and

$$\fint_{Q_\varrho^{(\lambda)}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)} dz < \lambda^{1-\varepsilon} \qquad \forall\, \varrho \in (\varrho_{z_0}, \tfrac{r_2 - r_1}{2}]. \tag{9.5}$$

At this stage we note that $Q_{4\chi\varrho_{z_0}}^{(\lambda)}(z_0) \subseteq Q_{r_2}$ and therefore by (9.4) and (9.5) for $s = 16\varrho_{z_0}$ we conclude, that the assumptions of Proposition 8.1 are fulfilled. Note here that $16 \leq 4\chi$ and therefore $16\varrho_{z_0} \in (\varrho_{z_0}, \frac{r_2 - r_1}{2}]$. We now impose the following bound on the radius r:

$$r \leq r_0 \equiv r_0(n, N, \nu, L, \gamma_2),$$

where r_0 denotes the radius bound from Proposition 8.1 (i.e. $r_0 \equiv \varrho_0$ where ϱ_0 is from Proposition 8.1). We are now allowed to apply Proposition 8.1, which yields the following Reverse-Hölder inequality:

$$\fint_{Q_{\varrho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)(1-\varepsilon)} dz \leq c \Big(\fint_{Q_{2\varrho_{z_0}}^{(\lambda)}(z_0)} |Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}} dz \Big)^{\bar q} + c \fint_{Q_{2\varrho_{z_0}}^{(\lambda)}(z_0)} (|F| + 1)^{p(\cdot)(1-\varepsilon)} dz, \tag{9.6}$$

where $\bar q = \bar q(n, \gamma_2) > 1$ and $c \equiv c(n, N, \nu, L, \gamma_2)$.

Now, for $\eta \in (0,1)$ to be fixed later we consider the upper level sets $E(\eta\lambda, r_1)$ of $|Du|$ defined above, and those of $|F| + 1$, defined by

$$\Phi(\eta\lambda, r_1) := \{ z \in Q_{r_1} : (|F(z)| + 1)^{p(z)} > \eta\lambda \}.$$

If $\eta\lambda > B\lambda_0$, then for a.e. $z_0 \in E(\eta\lambda, r_1)$ there exists a parabolic cylinder $Q_{\varrho_{z_0}}^{(\lambda)}(z_0)$ on which (9.4), (9.5) and (9.6) hold, and, moreover that $Q_{4\chi\varrho_{z_0}}^{(\lambda)}(z_0) \subseteq Q_{r_2}$. We let

$$p_0 \equiv p(z_0), \qquad p_1 \equiv \inf_{Q_{2\varrho_{z_0}}^{(\lambda)}(z_0)} p(\cdot) \qquad \text{and} \qquad p_2 \equiv \sup_{Q_{2\varrho_{z_0}}^{(\lambda)}(z_0)} p(\cdot).$$

Our next aim is to infer a suitable estimate for the $L^{p(\cdot)(1-\varepsilon)}$-norm of Du on the cylinder $Q_{4\chi\varrho_{z_0}}^{(\lambda)}(z_0)$. Using in turn (9.6), (9.4), Hölder's inequality and

(9.5) we obtain similar as in [7, page 238] that

$$
\begin{aligned}
&\fint_{Q^{(\lambda)}_{\varrho_{z_0}}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)} dz \\
&\qquad \le c\Big(\fint_{Q^{(\lambda)}_{2\varrho_{z_0}}(z_0)} |Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}} dz\Big)^{\bar q} + c \fint_{Q^{(\lambda)}_{2\varrho_{z_0}}(z_0)} (|F| + 1)^{p(\cdot)(1-\varepsilon)} dz \\
&\qquad \le c\,\eta^{1-\varepsilon} \fint_{Q^{(\lambda)}_{\varrho_{z_0}}(z_0)} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)} dz \\
&\qquad\quad + \frac{c}{|Q^{(\lambda)}_{2\varrho_{z_0}}(z_0)|} \int_{Q^{(\lambda)}_{2\varrho_{z_0}}(z_0)\cap E(\eta\lambda, r_2)} \lambda^{\frac{(\bar q-1)(1-\varepsilon)}{\bar q}} |Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}} dz \\
&\qquad\quad + \frac{c}{|Q^{(\lambda)}_{2\varrho_{z_0}}(z_0)|} \int_{Q^{(\lambda)}_{2\varrho_{z_0}}(z_0)\cap \Phi(\eta\lambda, r_2)} (|F| + 1)^{p(\cdot)(1-\varepsilon)} dz\,,
\end{aligned}
$$

where $c = c(n, N, \nu, L, \gamma_2)$. Choosing $\eta \equiv \eta(n, N, \nu, L, \gamma_2) > 0$ small enough – i.e. of the form $\eta^{1-\varepsilon} \equiv 1/(2c)$ – we can re-absorb the first integral appearing on the right-hand side into the left. Moreover, using (9.4) and (9.5) with $s = 4\chi\varrho_{z_0}$ we can bound the left-hand side of the preceding inequality from below by $\fint_{Q^{(\lambda)}_{4\chi\varrho_{z_0}}(z_0)} |Du|^{p(\cdot)(1-\varepsilon)} dz$. Multiplying the resulting inequality by $|Q^{(\lambda)}_{4\chi\varrho_{z_0}}(z_0)|$ we obtain

$$
\begin{aligned}
\int_{Q^{(\lambda)}_{4\chi\varrho_{z_0}}(z_0)} |Du|^{p(\cdot)(1-\varepsilon)} dz &\le c \int_{Q^{(\lambda)}_{2\varrho_{z_0}}(z_0)\cap E(\eta\lambda, r_2)} \lambda^{\frac{(\bar q-1)(1-\varepsilon)}{\bar q}} |Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}} dz \\
&\quad + c \int_{Q^{(\lambda)}_{2\varrho_{z_0}}(z_0)\cap \Phi(\eta\lambda, r_2)} (|F| + 1)^{p(\cdot)(1-\varepsilon)} dz\,,
\end{aligned}
\tag{9.7}
$$

for a constant $c \equiv c(n, N, \nu, L, \gamma_2)$.

Now, from (4.4) we infer that

$$
\lambda \le \left(\frac{\beta_n M}{\varrho_{z_0}^{n+2}}\right)^{p_0}. \tag{9.8}
$$

Moreover, imposing a further bound for the radii $r \le r_0$ of the following form:

$$
r \le r_0 \le \tfrac{1}{64M}\,, \tag{9.9}
$$

we get from (4.5) that

$$
\lambda^{(p_2-p_1)/p_0} \le c(n)\, e^{64(n+3)L}. \tag{9.10}
$$

Thus, so far we have shown that for any $\lambda > B\lambda_0$ the level set $E(\lambda, r_1)$ is covered by a family $\mathcal{F} \equiv \{Q^{(\lambda)}_{4\chi\varrho_{z_0}}(z_0)\}$ of parabolic cylinders with center $z_0 \in E(\lambda, r_1)$ whose radii ϱ_{z_0} are bounded by the radius r_0 from (9.9). Furthermore, on each cylinder of the covering we have (9.7) at our hands. From Vitali's covering theorem, i.e. the version for non-uniformly parabolic cylinders from [7, Lemma 7.1] we infer the existence of a countable subfamily $\{Q^{(\lambda)}_{4\varrho_{z_i}}(z_i)\}_{i=1}^{\infty} \subseteq \mathcal{F}$ of pair-wise disjoint parabolic cylinders, such that the χ-times enlarged cylinders $Q^{(\lambda)}_{4\chi\varrho_{z_i}}(z_i)$ cover the set $E(\lambda, r_1)$, i.e. up to a set of measure zero there holds (note that $Q^{(\lambda)}_{4\chi\varrho_{z_i}}(z_i) \subseteq Q_{r_2}$ by construction)

$$E(\lambda, r_1) \subseteq \bigcup_{i=1}^{\infty} Q^{(\lambda)}_{4\chi\varrho_{z_i}}(z_i) \subseteq Q_{r_2}\,.$$

Recalling that the cylinders $\{Q^{(\lambda)}_{4\varrho_{z_i}}(z_i)\}_{i=1}^{\infty}$ are pair-wise disjoint we infer from (9.7) that

$$\int_{E(\lambda,r_1)} |Du|^{p(\cdot)(1-\varepsilon)} dz \le c \int_{E(\eta\lambda,r_2)} \lambda^{\frac{(\bar q-1)(1-\varepsilon)}{\bar q}} |Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}} dz + c \int_{\Phi(\eta\lambda,r_2)} (|F|+1)^{p(\cdot)(1-\varepsilon)} dz\,,$$

with a constant $c \equiv c(n, N, \nu, L, \gamma_2)$. Moreover, on $E(\eta\lambda, r_1)\setminus E(\lambda, r_1)$ we have $|Du|^{p(\cdot)(1-\varepsilon)} \le \lambda^{1-\varepsilon}$ and therefore we may replace the domain of integration $E(\lambda, r_1)$ on the left-hand side by $E(\eta\lambda, r_1)$. Subsequently, replacing $\eta\lambda$ by λ and recalling that $\eta < 1$ depends only on n, ν, L, γ_2 we obtain for any $\lambda \ge B\lambda_0/\eta =: \lambda_1$ that

$$\int_{E(\lambda,r_1)} |Du|^{p(\cdot)(1-\varepsilon)} dz \le c \int_{E(\lambda,r_2)} \lambda^{\frac{(\bar q-1)(1-\varepsilon)}{\bar q}} |Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}} dz + c \int_{\Phi(\lambda,r_2)} (|F|+1)^{p(\cdot)(1-\varepsilon)} dz\,,$$

for a constant $c = c(n, N, \nu, L, \gamma_2)$. Having arrived at this stage we would like to multiply the preceding inequality by $\lambda^{\varepsilon-1}$ with $\varepsilon \in (0, 1]$ and then integrate with respect to λ over (λ_1, ∞). This, formally, would lead in a standard way to the desired higher-integrability of $|Du|$, where ε has to be chosen small enough in between in order to re-absorb certain terms on the left-hand side. However, there is a difficulty in moving terms to the left-hand side since they may be infinite. This technical problem can be treated, by truncating $|Du|^{p(\cdot)}$ (see [2, § 8.4] for example). The precise argument is as follows: For $k \ge \lambda_1$ we define the truncation operator $T_k : [0, \infty) \to [0, k]$ by

$$T_k(\sigma) := \min\{\sigma, k\} \quad \text{and} \quad E_k(\lambda, r_i) := \{z \in Q_{r_i} : T_k(|Du(z)|^{p(z)}) > \lambda\},\ \ i = 1, 2.$$

Since $E_k(\lambda, r_1) = \emptyset$ in the case $k \le \lambda$ and $E_k(\lambda, r_2) \equiv E(\lambda, r_2)$ in the case $k > \lambda$, we can replace in the last inequality $E(\lambda, r_i)$ by $E_k(\lambda, r_i)$ for $i = 1, 2$. Now, we multiply on both sides by $\lambda^{\varepsilon-1}$ and integrate with respect to λ over (λ_1, ∞). In this way we obtain

$$\begin{aligned}\int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \int_{E_k(\lambda,r_1)} |Du|^{p(\cdot)(1-\varepsilon)} dz\, d\lambda &\le c \int_{\lambda_1}^{\infty} \int_{E_k(\lambda,r_2)} \lambda^{-\frac{1-\varepsilon}{\bar q}} |Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}} dz\, d\lambda \\ &\quad + c \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \int_{\Phi(\lambda,r_2)} (|F|+1)^{p(\cdot)(1-\varepsilon)} dz\, d\lambda\,. \end{aligned} \tag{9.11}$$

To the integral on the left-hand side we apply Fubini's theorem and find that

$$\int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \int_{E_k(\lambda,r_1)} |Du|^{p(\cdot)(1-\varepsilon)} dz\, d\lambda = \int_{E_k(\lambda_1,r_1)} |Du|^{p(\cdot)(1-\varepsilon)} \int_{\lambda_1}^{T_k(|Du(z)|^{p(z)})} \lambda^{\varepsilon-1}\, d\lambda dz$$

$$= \frac{1}{\varepsilon}\int_{E_k(\lambda_1,r_1)}\Big[|Du|^{p(\cdot)(1-\varepsilon)}T_k\big(|Du|^{p(\cdot)}\big)^{\varepsilon} - \lambda_1^{\varepsilon}\,|Du|^{p(\cdot)(1-\varepsilon)}\Big]dz\,.$$

Similarly, we obtain for the first integral on the right-hand side that

$$\int_{\lambda_1}^{\infty}\int_{E_k(\lambda,r_2)}\lambda^{-\frac{1-\varepsilon}{\bar q}}|Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}}dz\,d\lambda \le \frac{\bar q}{\bar q-1}\int_{E_k(\lambda_1,r_2)}|Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}}T_k\big(|Du|^{p(\cdot)}\big)^{\frac{\bar q+\varepsilon-1}{\bar q}}dz\,.$$

and for the second integral on the right-hand side we get

$$\int_{\lambda_1}^{\infty}\lambda^{\varepsilon-1}\int_{\Phi(\lambda,r_2)}(|F|+1)^{p(\cdot)(1-\varepsilon)}dz\,d\lambda \le \frac{1}{\varepsilon}\int_{Q_{2r}}(|F|+1)^{p(\cdot)}dz\,.$$

Joining the preceding estimates with (9.11) and multiplying by ε we arrive at

$$\begin{aligned}\int_{E_k(\lambda_1,r_1)}|Du|^{p(\cdot)(1-\varepsilon)}T_k\big(|Du|^{p(\cdot)}\big)^{\varepsilon}dz &\le \frac{c\varepsilon\bar q}{\bar q-1}\int_{E_k(\lambda_1,r_2)}|Du|^{\frac{p(\cdot)(1-\varepsilon)}{\bar q}}T_k\big(|Du|^{p(\cdot)}\big)^{\frac{\bar q+\varepsilon-1}{\bar q}}dz\\ &\quad+\lambda_1^{\varepsilon}\int_{E_k(\lambda_1,r_1)}|Du|^{p(\cdot)(1-\varepsilon)}dz + c\int_{Q_{2r}}(|F|+1)^{p(\cdot)}dz,\end{aligned}$$

for a constant $c = c(n,N,\nu,L,\gamma_2)$. Since $T_k(|Du|^{p(\cdot)}) \le \lambda_1$ on $Q_{r_1}\setminus E_k(\lambda_1,r_1)$, we may replace the domain of integration $Q_{r_1}\setminus E_k(\lambda_1,r_1)$ on the left-hand side by Q_{r_1}. At this stage we perform the choice of ε. Choosing

$$0<\varepsilon\le\varepsilon_0\equiv\varepsilon_0(n,N,\nu,L,\gamma_2,\sigma) := \frac{\bar q-1}{2c\bar q}\,,$$

recalling the definitions of λ_1, i.e. $\lambda_1^{\varepsilon} = (B\lambda_0/\eta)^{\varepsilon} \le B\lambda_0^{\varepsilon}/\eta$ since $B/\eta\ge1$, $\varepsilon\le1$, and of B from (9.2) and taking into account that $T_k(|Du|^{p(\cdot)}) \le |Du|^{p(\cdot)}$, we arrive at

$$\begin{aligned}\int_{Q_{r_1}}|Du|^{p(\cdot)(1-\varepsilon)}T_k\big(|Du|^{p(\cdot)}\big)^{\varepsilon}dz &\le \tfrac12\int_{Q_{r_2}}|Du|^{p(\cdot)(1-\varepsilon)}T_k\big(|Du|^{p(\cdot)}\big)^{\varepsilon}dz\\ &\quad+\frac{c_*(2r)^{\beta}\lambda_0^{\varepsilon}}{(r_2-r_1)^{\beta}}\int_{Q_{2r}}|Du|^{p(\cdot)(1-\varepsilon)}dz + c\int_{Q_{2r}}(|F|+1)^{p(\cdot)}dz\,,\end{aligned}$$

where $c_* := (4\chi)^{\beta}/\eta$ and $\beta := \frac{(n+2)p_M}{2-\varepsilon p_M}$. Since $r\le r_1<r_2\le 2r$ are arbitrary we are in the position to apply Lemma 3.1 to infer that

$$\int_{Q_r}|Du|^{p(\cdot)(1-\varepsilon)}T_k\big(|Du|^{p(\cdot)}\big)^{\varepsilon}dz \le c(\beta)\Big[2^{\beta}c_*\lambda_0^{\varepsilon}\int_{Q_{2r}}|Du|^{p(\cdot)(1-\varepsilon)}dz + c\int_{Q_{2r}}(|F|+1)^{p(\cdot)}dz\Big].$$

Letting $k\to\infty$ (which is possible by Fatou's lemma) we get

$$\int_{Q_r}|Du|^{p(\cdot)}dz \le c\Big[\lambda_0^{\varepsilon}\int_{Q_{2r}}|Du|^{p(\cdot)(1-\varepsilon)}dz + \int_{Q_{2r}}(|F|+1)^{p(\cdot)}dz\Big],$$

for a constant $c\equiv c(n,N,\nu,L,\gamma_2)$. Note that the dependence on β can be eliminated since $p_M\in[2,\gamma_2]$. Finally, passing to averages and recalling the definition of λ_0, i.e. (9.1), we deduce that

(9.12)

$$⨍_{Q_r}|Du|^{p(\cdot)}dz \le c\Big(⨍_{Q_{2r}}(|Du|+|F|+1)^{p(\cdot)(1-\varepsilon)}dz\Big)^{1+\frac{\varepsilon p_M}{2-\varepsilon p_M}} + c⨍_{Q_{2r}}(|F|+1)^{p(\cdot)}dz\,.$$

At this point, it remains to replace in the preceding estimate $\frac{\varepsilon p_M}{2-\varepsilon p_M}$ by $\frac{\varepsilon p_0}{2-\varepsilon p_0}$, where $p_0 \equiv p(\mathfrak{z}_0)$ denotes the value of $p(\cdot)$ evaluated at the center $\mathfrak{z}_0$ of $Q_{2r} \equiv Q_{2r}(\mathfrak{z}_0)$. Using (2.3) and $\varepsilon \le \frac{1}{2}$ we obtain

$$0 \le \frac{\varepsilon p_M}{2-\varepsilon p_M} - \frac{\varepsilon p_0}{2-\varepsilon p_0} \le \frac{2\varepsilon(p_M - p_0)}{(2-\varepsilon p_M)(2-\varepsilon p_0)} \le 2\varepsilon(p_M - p_0) \le 2\varepsilon\omega(4r) \le \omega(4r).$$

The preceding estimate together with $\varepsilon \le 1$ and (2.6) implies

$$\begin{aligned}\Big(\fint_{Q_{2r}} (|Du| + |F| + 1)^{p(\cdot)(1-\varepsilon)} dz \Big)^{\frac{\varepsilon p_M}{2-\varepsilon p_M} - \frac{\varepsilon p_0}{2-\varepsilon p_0}} &\le \Big(\fint_{Q_{2r}} (|Du| + |F| + 1)^{p(\cdot)} dz \Big)^{\omega(4r)} \\ &\le c(n)\,(2r)^{-(n+2)\omega(4r)} M^{\omega(4r)}.\end{aligned}$$

In order to proceed further we use the logarithmic continuity condition (2.4) twice to infer for the terms involving r and M that

$$(4r)^{-\omega(4r)} \le c(L) \qquad \text{and} \qquad M^{\omega(4r)} \le c(L)\,.$$

The second assertion is obtained as follows :

$$M^{\omega(4r)} = \exp\big[\omega(4r)\log M\big] \le \exp\big[\omega(1/M)\log M\big] \le e^L\,,$$

provided $r \le r_0 \le \frac{1}{4M}$. This restriction on the size of r_0 is already implied by the restriction from (9.9). Joining the preceding estimates we find

$$\Big(\fint_{Q_{2r}} (|Du| + |F| + 1)^{p(\cdot)} dz \Big)^{\frac{\varepsilon p_M}{2-\varepsilon p_M} - \frac{\varepsilon p_0}{2-\varepsilon p_0}} \le c(n, L)\,,$$

which together with (9.12) yields the desired estimate (2.7). This finally completes the proof of Theorem 2.2.

Remark 9.1. Here, we briefly discuss if the result of Theorem 2.2 can be extended to $p(\cdot) > \frac{2n}{n+2}$. We think that the result remains to be true in the sub-quadratic case. However, there is an extra technical difficulty in the proof of the localization argument in Lemma 4.1. In order to treat the subquadratic case it would be necessary to improve the estimate $\mathcal{A} \le c\,\varrho^{-\gamma_2} M^2$ in the direction that no negative power of the radius appears, i.e. one would have to show an estimate of the type $\mathcal{A} \le cM$. More precisely, the proof of this inequality can be reduced to showing that

$$\sup_{\varrho>0} \int_{Q^{(\lambda)}_{16\varrho}(z_0)} \left| \frac{u - u_{Q^{(\lambda)}_{16\varrho}(z_0)}}{\varrho} \right|^{p(\cdot)(1-\varepsilon)} dz \le c\,M. \tag{9.13}$$

Assumed that (9.13) holds true, by carefully inspecting the proof of Theorem 2.2 and [7], one could extend Theorem 2.2 to the case $p(\cdot) > \frac{2n}{n+2}$.

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References

[1] S. Antontsev and V. Zhikov. *Higher integrability for parabolic equations of $p(x,t)$-Laplacian type,* Adv. Differential Equations, **10** (9), (2005), 1053-1080.

[2] P. Baroni and V. Bögelein, *Calderón-Zygmund estimates for parabolic $p(x,t)$-Laplacian systems,* Rev. Mat. Iberoam., to appear.

[3] V. Bögelein and A. Zatorska-Goldstein, *Higher integrability of very weak solutions of systems of p(x)-Laplacean type,* J. Math. Anal. Appl. **336** (1), 480-497, (2007).

[4] V. Bögelein, *Regularity results for weak and very weak solutions of higher order parabolic systems,* Doctoral Thesis in Universität Erlangen-Nürnberg, (2007).

[5] V. Bögelein, *Higher integrability for weak solutions of higher order degenerate parabolic systems,* Ann. Acad. Sci. Fenn., Math. **33** (2), 387-412, (2008).

[6] V. Bögelein, *Very weak solutions of higher-order degenerate parabolic systems,* Adv. Differential Equations **14**, No. 1/2, 121-200, (2009).

[7] V. Bögelein and F. Duzaar, *Higher integrability for parabolic systems with non-standard growth and degenerate diffusions,* Publ. Mat. **55**, No. 1, 201-250, (2011).

[8] V. Bögelein, F. Duzaar and G. Mingione, The regularity of general parabolic systems with degenerate diffusion Memoirs of AMS, **221**, No. 1041, American Math. Society, (2013).

[9] G. Da Prato, *Spazi $\mathcal{L}^{(p,\theta)}(\omega,\delta)$ e loro proprieta,* Ann. Mat. Pure Appl., IV., **69** (4), 383-392, (1965).

[10] E. DiBenedetto, Degenerate parabolic equations, Universitext. New York, NY: Springer-Verlag. xv 387, (1993).

[11] E. DiBenedetto and A. Friedman, *Regularity of solutions of nonlinear degenerate parabolic systems,* J. reine ang. Math. (Crelles J.) **349**: 83-128, (1984).

[12] E. DiBenedetto and A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems,* J. reine ang. Math. (Crelles J.) **357**: 1-22, (1985).

[13] L. Diening, M. Růžička and J. Wolf, *Existence of weak solutions for unsteady motions of generalized newtonian fluids,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **9** (1), 1-46, (2010).

[14] L. Diening, P. Harjulehto, P Hästö, and M. Růžička, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, **2017**, Springer, Heidelberg, 509 pp, (2011).

[15] F. W. Gehring, *The L^p integrability of the partial derivatives of a quasiconformal mapping,* Acta Math., **130**, 265-277, (1973).

[16] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems,* Annals of Mathematics Studies **105**, Princeton University Press, Princeton, NJ, (1983).

[17] P. Harjulehto, P. Hästö, Út V. Lê and M. Nuortio, *Overview of differential equations with non-standard growth,* Nonlinear Analysis, **72**, 4551-4574, (2010).

[18] T. Iwaniec and C. Sbordone, *Weak minima of variational integrals,* J. Reine Angew. Math. **454**, 143-161, 1994.

[19] J. Kinnunen and J. L. Lewis, *Higher integrability for parabolic systems of p-Laplacian type,* Duke Math. J., **102**, 253-271, (2000).

[20] J. Kinnunen and J. L. Lewis, *Very weak solutions of parabolic systems of p-Laplacian type,* Ark. Mat., **40**, 105-132, (2002).

[21] T. Kopaliani, *A note on strong maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces*, Proc. A. Razmadze Math. Inst., **145**, 43-46, (2007).

[22] J. L. Lewis, *On the very weak solutions of certain elliptic systems,* Comm. Part. Diff. Equ., **18**, 1515-1537, (1993).

[23] Q. Li, *Some regularity results for certain weakly quasiregular mappings on the Heisenberg group and elliptic equations,* Master Thesis in Univ. of Bergen, Bergen Open Research Archive, (2011).

[24] N. G. Meyers and A. Elcrat, *Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions,* Duke Math. J., **42**, 121-136, (1975).

[25] V. V. Zhikov, *On some variational problems,* Russian J. Math. Phys. **5** (1) (1997), 105-116 (1998).

[26] V. V. Zhikov and S. E. Pastukhova, *On the Property of Higher Integrability for Parabolic Systems of Variable Order of Nonlinearity,* Mat. Zametki, **87** (2), (2010), 179-200, [Math. Notes, **87**, (2010), 169-188].

Verena Bögelein, Department Mathematik, Universität Erlangen–Nürnberg, Cauerstrasse 11, 91058 Erlangen, Germany

E-mail address: boegelein@math.fau.de

Qifan Li, Institutt for matematiske fag, Norwegian University of Science and Technology, 7491 Trondheim, Norway

E-mail address: qifan.li@math.ntnu.no