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The 2-Kronecker Quiver and Systems of Linear Differential Equations

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Problem Description

Let Q be a quiver with two vertices and two arrows going in the same direction, known as the 2-Kronecker quiver. Let $V = (V_1, V_2, A, B)$ be a representation of the quiver, where V_1 and V_2 are vector spaces over a field k, and A and B are linear transformations from V_1 into V_2. It can be shown that there are only three classes of indecomposable representations over this quiver. This thesis considers two problems:

1. The main problem of this thesis is the problem of classifying all the indecomposable representations of the 2-Kronecker quiver over an algebraically closed field.

2. The other problem mentioned in this thesis is the problem of solving a system of linear differential equations, $Ax = B\dot{x}$, where A and B are $m \times n$-matrices.

Abstract

We present a way of classifying all the indecomposable representations of the 2-Kronecker quiver over an algebraically closed field. We do this by constructing classes of irregular indecomposable representations by the coxeter functor, and by constructing a class of regular indecomposable representations by mathematical induction using projective resolutions and the Ext^1(A,B) functor.

When the indecomposable representations have been classified, we use the decomposition of any representation into a finite direct sum of indecomposable representations to evaluate some systems of linear differential equations on the form $Ax = Bx'$, where A and B are $m \times n$-matrices.

Sammendrag

I denne oppgaven presenteres en måte å klassifisere de ikke-dekomponerbare representasjonene av 2-Kronecker *koggeret* (Engelsk:quiver) over en algebraisk lukket kropp. To klasser av ikke-dekomponerbare representasjoner blir bestemt ved å benytte Coxeterfunktoren på noen få, velkjente, ikke-dekomponerbare representasjoner. Den siste klassen blir bestemt ved hjelp av Ext^1-funktoren og betraktninger om den projektive oppløsningen av ikke-dekomponerbare representasjoner.

I tillegg til å klassifisere de ikke-dekomponerbare representasjonene som beskrevet, blir denne klassifiseringen brukt til å studere lineære likningssystemer av formen $Ax = Bx'$, hvor A og B er $m \times n$-matriser.

Contents

Acknowledgements

I would like to express my deepest gratitude to those who have helped me complete this thesis. First, I would like to thank my supervisor, Sverre Olaf Smalø. He has shown great patience in his supervision of this thesis, and without his tireless efforts to help me understand the subject matter, this thesis simply could not have been completed. I would also like to thank the staff at the mathematics department at NTNU for their help and guidance along the way. Finally, I would like to thank my family and friends for their support, both emotionally and financially, throughout my time at the university.

Introduction

The objective of this thesis is to classify all the indecomposable representations of the 2-Kronecker quiver over an algebraically closed field. We also consider how this classification might prove useful for solving systems of differential equations.

The 2-Kronecker quiver is a quiver with two vertices, and two arrows both going in the same direction.

Given a representation V of a quiver over a field, k. V is indecomposable if it cannot be written as a direct sum of representations, $V = V_1 \oplus V_2$. The focus of this thesis is to classify all indecomposable representations for the given quiver, that is, to find a finite number of classes of representations such that every indecomposable representation of the quiver is an element in one of these classes.

The classification of the indecomposable representations over the 2-Kronecker quiver has been known for a long time. In [2], the indecomposable representations have been classified by the use of representation theory for hereditary algebras, and by tools such as the Auslander-Reiten quiver. However, the aim of this thesis is to achieve the classification in a simpler way, by a more *ad hoc* theoretical approach.

In chapter 1, we establish some basic definitions and properties of modules and sequences of modules, and finally study the baer sum, which is a useful tool in the evaluation of exact sequences.

Chapter 2 is considered to be the main body of this thesis, and contains the construction of the three different classes of indecomposable representations, and a proof that all indecomposable representations are contained in one of these classes.

In chapter 3, we consider one possible application for the classification obtained in chapter 2.

For the proofs provided in this thesis, the symbol $\square$ is used to indicate completion of the proof.

Chapter 1

Preliminary Results

In this chapter, we will give some basic definitions, and use these to derive some useful properties of modules and sequences of modules. Although the reader is assumed to be familiar with most of the concepts contained in this chapter, the chapter provides a general introduction to some of this theory to make the remainder of the thesis more accessible to any reader. For more about the fundamental background, and for some of the definitions omitted, see [1], [2], [3], and [5].

1.1 Free, projective, and injective modules

Definition.
For a ring, R, a *left R-module*, M is an additive abelian group, and a mapping $(r, m) \mapsto rm$ of $R \times M$ into M such that the following holds:

(i)	$r(m_1 + m_2) = rm_1 + rm_2,$
(ii)	$(r_1 + r_2)m = r_1m + r_2m,$
(iii)	$(r_1r_2)m = r_1(r_2m),$
(iv)	$1m = m, \text{if } 1 \in R$

where $r, r_1, r_2 \in R, m, m_1, m_2 \in M$.

Throughout this chapter, unless otherwise stated, whenever we refer to "modules", we really mean left R-modules for a ring R.

Definition.
Let R be a ring. Let A and B be left R-modules. Then a mapping $f : A \to B$ is called an R-homomorphism if

$$\begin{array}{ll} (i) & f(x+y) = f(x) + f(y) \\ (ii) & f(rx) = rf(x) \end{array}$$

for all $x, y \in A$, and $r \in R$.

Definition.
Let R be a ring with unity. An R-module, F, is *a free R-module* if it admits a basis. That is, if there exists a set $X = \{x_j\}_{j \in J} \subseteq F$ such that X is linearly independent, and for each $f \in F$,

$$f = \sum_{j \in J} c_j x_j, c_j \in R,$$

and only finitely many $c_j \neq 0$.

Proposition 1.1. *Let R be a ring with unity, and let M be a left R-module, then there exists a free left R-module F, and a surjective R-homomorphism $\phi : F \to M$.*

Proof: One may construct a free left R-module, F, and an R-homomorphism $\phi : F \to M$ as follows:
Let $F = \{h : M \to R \,|\, |M \setminus h^{-1}(0)| < \infty\}$. (All functions h such that there is a finite number of elements not mapped to zero.)
Let $f_1, f_2 \in F$. Define an addition on F as follows: $(f_1 + f_2)(m) = f_1(m) +_R f_2(m)$. Here, $+_R$ is the addition operator in R. By construction, F contains additive inverses and a zero element.
For $r \in R$, $(r \cdot f)(m) = r \cdot f(m) \in R$. Thus, F is an additive abelian group, and it is easy to confirm that it satisfies the remaining conditions of an R-module.
To show that F is a free R-module, it is enough to show that it is an R-module that admits a basis.
To see that F has a basis, consider for each $m \in M$ the kronecker function

$$\delta_m(x) = \begin{cases} 1 & \text{if } x = m, \\ 0 & \text{if } x \neq m. \end{cases}$$

Now for any $f \in F$,

$$f(x) = \sum_{m \in M} f(m)\delta_m(x),$$

so the set $\{\delta_m | m \in M\}$ spans F. Also, this set is linearly independent, since for each $x \in M$, we have:

$$\sum_{m \in M} r_m \delta_m(x) = r_x, \text{ when } r_i \in R, \qquad \sum_{m \in M} r_m \delta_m = 0 \Rightarrow r_m = 0 \ \forall m \in M.$$

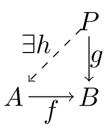
Thus, F admits a basis, and F is a free left R-module. Define the R-homomorphism

$$\phi : F \to M, \phi(f) = \sum_{m \in M} f(m) \cdot m.$$

$$\begin{aligned} \phi(f_1 + f_2) &= \phi(f_1) + \phi(f_2) \\ \phi(r \cdot f_1) &= r \cdot \phi(f_1) \end{aligned}$$

As $\phi(\delta_m) = m$, this is surjective. □

Definition. Let R be a ring. Let A and B be left R-modules. A left R-module, P, is a *projective module* if for every surjective R-homomorphism $f : A \to B$, and every R-homomorphism $g : P \to B$, there exists an R-homomorphism $h : P \to A$ such that $fh = g$.



Proposition 1.2. *A free left $R-$module is projective.*

Proof: Let F be a free left R-module. Then there exists a set $\mathcal{B}$ s.t. $\mathcal{B}$ is a basis for F. Consider any left R-modules A and B and any surjective R-homomorphism $f : A \to B$ and any R-homomorphism $g : F \to B$. As f is surjective, one may choose a mapping $h' : \mathcal{B} \to A$ such that

$$h'(x') \in \{a \in A | f(a) = g(x')\}, \forall x' \in \mathcal{B},$$

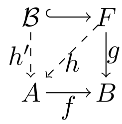
by the axiom of choice. As any element $x \in F$ is uniquely determined by

$$x = \sum_{x'_i \in \mathcal{B}} r_i x'_i, r_i \in R,$$

this gives rise to an R-homomorphism

$$h : F \to A$$
$$h(x) = \sum_{x'_i \in \mathcal{B}} r_i h'(x'_i)$$

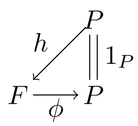
which is uniquely determined by the choice of h', and we have that $gh(x) = f(x)$. Thus, F is a projective module.



□

Proposition 1.3. *A projective module is a summand of a free module.*

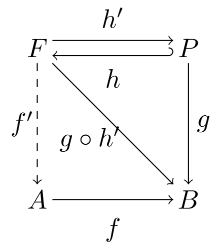
Proof: By Proposition 1.1, for any projective module P, there exists a free module F and a surjective R-homomorphism $\phi : F \to P$, such that the following diagram commutes:



Now, $\phi \circ h = 1_P \Rightarrow P \simeq \text{Im } h$. $F = \text{Im } h \oplus \ker \phi$. Hence, P is a summand of a free module. □

Proposition 1.4. *Let P be a direct summand of a free R-module F. Then P is a projective module.*

Proof: Let P be a direct summand of a free R-module, F. Let $h : P \hookrightarrow F$ be an inclusion, and $h' : F \to P$ be an R-homomorphism such that $h' \circ h = 1_P$. Let A and B be R-modules, such that there exists an R-homomorphism $g : P \to B$, and a surjective R-homomorphism $f : A \to B$. As F is projective by Proposition 1.2, and as $f \circ h' : F \to B$ defines an R-homomorphism from F to B, there exists an R-homomorphism $f' : F \to A$, such that $f \circ f' = g \circ h'$. Then, by composition by h on the right, one obtains the relation $f \circ f' \circ h = g$, hence one have obtained an R-homomorphism $f' \circ h : P \to A$, and thus, P is projective.



□

Proposition 1.5. *Let $P_1, P_2, ..., P_i$ be projective modules. Then $P = \bigoplus_{j=1}^{i} P_j$ is projective.*

Proof: Let $P_1, P_2, ..., P_i$ be projective modules. By proposition 1.3, projective modules are summands of free modules. Let $F_1, F_2, ..., F_i$ be free modules such that P_j is a summand of F_j for all $j \in \{1, ..., i\}$. Now we have that

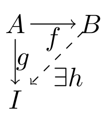
$$P = \bigoplus_{j=1}^{i} P_j$$

is a direct summand of the module

$$F = \bigoplus_{j=1}^{i} F_j$$

F is a free module by the definition of a free module. Thus, a direct sum of projective modules is a summand of a free module. By proposition 1.4, a summand of a free module is projective. □

Definition. Let R be a ring. Let A and B be left R-modules. A left R-module, I is an *injective module* if for every injective R-homomorphism $f : A \to B$, and every R-homomorphism $g : A \to I$, there exists an R-homomorphism $h : B \to I$ such that $hf = g$.



1.2 Exact sequences

Definition. An *exact sequence* of R-modules, is a sequence of R-modules, $\{A_i\}_{i\in\mathbb{Z}}$, and morphisms $f_i : A_i \to A_{i+1}$, such that Im f_i=ker f_{i+1}, $\forall i \in \mathbb{Z}$,

$$\cdots \to A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \to \cdots$$

Definition. The *length* of an R-module M, is defined as ∞ or the number n of submodules in the longest chain of submodules of M such that:

$$N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = M,$$

where N_i are submodules of M for $i \in \{1, 2, ..., n\}$.

Remark. By the Jordan-Hölder Theorem, see [2, Theorem 1.2, p. 3], the length of an R-module of finite length, n, is independent of the choice of submodules.

Remark. For a vector space, V, $\ell(V) = \dim(V)$.

Proposition 1.6. *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \to 0$$

be an exact sequence, where f is a monomorphism, and h is surjective. If A, B, C, and D are R-modules of finite length, then $\ell(A) + \ell(C) = \ell(B) + \ell(D)$.

Proof: Construct the short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{p} B/\text{Im } f = E \to 0,$$

where p is the projection in the obvious way. That this sequence is "short", means that f is injective and p is surjective. By a corollary of the Jordan-Hölder theorem, see[2, Corollary 1.3, p. 4], one obtains

$$\ell(A) + \ell(E) = \ell(B). \tag{1.1}$$

Also, there is the short exact sequence

$$0 \to E \xrightarrow{g'} C \xrightarrow{h} D \to 0,$$

where $g' : E \to C$, by $g'(b + \text{Im} f) = g(b)$, and hence

$$\ell(E) + \ell(D) = \ell(C). \tag{1.2}$$

Combining equation 1.1 and equation 1.2, one obtains equation 1.3.

$$\ell(A) + \ell(E) - \ell(D) - \ell(E) = \ell(B) - \ell(C) \Rightarrow \ell(A) + \ell(C) = \ell(B) + \ell(D) \quad (1.3)$$

□

Definition. An R-module M is called *noetherian* if for every ascending sequence of R-submodules of M,

$$M_1 \subset M_2 \subset M_3 \subset \cdots$$

there exists a positive integer k such that $M_k = M_{k+1} = M_{k+2} \cdots$.

Definition. An R-module M is called *artinian* if for every descending sequence of R-submodules of M,

$$M_1 \supset M_2 \supset M_3 \supset \cdots$$

there exists a positive integer k such that $M_k = M_{k+1} = M_{k+2} \cdots$.

Remark. An R-module of finite length is both artinian and noetherian.

1.3 Extension modules

Definition.
An *extension* of a module B by A, is a short exact sequence

$$0 \to B \xrightarrow{f} C \xrightarrow{g} A \to 0$$

Definition. A short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is *split* if there exists a map $j : C \to B$ with $gj = 1_C$.

Definition. Two extensions $\alpha : 0 \to B \xrightarrow{f} C \xrightarrow{g} A \to 0$ and $\beta : 0 \to B \xrightarrow{f'} C' \xrightarrow{g'} A \to 0$ are *equivalent* if there exists a map $\phi : C \to C'$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\alpha & : & 0 & \longrightarrow & B & \xrightarrow{f} & C & \xrightarrow{g} & A & \longrightarrow & 0 \\
 & & & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\
\beta & : & 0 & \longrightarrow & B & \xrightarrow{f'} & C' & \xrightarrow{g'} & A & \longrightarrow & 0
\end{array}$$

Moreover, by the Five Lemma, see [5, p. 90], if the two extensions α and β are equivalent, then ϕ is an isomorphism.

Example. *$R = \mathbb{Z}$. Take the extensions as follows:*

$$\alpha : 0 \to \mathbb{Z}_3 \xrightarrow{\lambda_3} \mathbb{Z}_9 \xrightarrow{p} \mathbb{Z}_3 \to 0$$
$$\beta : 0 \to \mathbb{Z}_3 \xrightarrow{\lambda_6} \mathbb{Z}_9 \xrightarrow{p} \mathbb{Z}_3 \to 0$$

where p is the projection onto $\mathbb{Z}_3$ by isomorphism with the quotient ring $\mathbb{Z}_9/\mathbb{Z}_3$, and

$$\lambda_i : \mathbb{Z}_i \to \mathbb{Z}_j, \lambda_i(x) = ix.$$

To show that α and β are not equivalent, it is enough to show that no $\phi : \mathbb{Z}_9 \to \mathbb{Z}_9$ can make both of the following diagrams commute:

$$\text{Left side: } \begin{array}{ccc} \mathbb{Z}_3 & \xrightarrow{\lambda_3} & \mathbb{Z}_9 \\ \| & & \downarrow \phi \\ \mathbb{Z}_3 & \xrightarrow{\lambda_6} & \mathbb{Z}_9 \end{array} \qquad \text{Right side: } \begin{array}{ccc} \mathbb{Z}_9 & \xrightarrow{p} & \mathbb{Z}_3 \\ \downarrow \phi & & \| \\ \mathbb{Z}_9 & \xrightarrow{p} & \mathbb{Z}_3 \end{array}$$

In order to make the left side commute, $\phi(0) = 0, \phi(3) = 6, \phi(6) = 3$. Hence, $\phi = \lambda_2$ or $\phi = \lambda_5$. But considering the right side, $p\phi(1) = \bar{2} \neq \bar{1} = p(1)$. Hence, the right side diagram does neither commute for $\phi = \lambda_2$ nor $\phi = \lambda_5$.

Definition.
Let X be a left R-module. Then a *projective presentation of X* is an exact sequence

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0$$

where P_1 and P_0 are projective modules.

1.4 Categories and functors

Definition. A *category*, $\mathcal{C}$ consists of

1. a class of *objects*, obj $\mathcal{C}$,

2. a class of *morphisms* between objects, hom $\mathcal{C}$,
3. and a *composition of morphims.* This is a binary operation such that when $a, b, c \in$ obj $\mathcal{C}$, hom $(a, b)\times$ hom $(b, c) \mapsto$ hom (a, c). Let $f \in$ hom (a, b), $g \in$ hom (b, c), then the composition is denoted by $gf \in$ hom (a, c).

such that

a) the composition of morphisms is associative,
$f \in$ hom (a, b), $g \in$ hom (b, c), $h \in$ hom (c, d), $(hg)f = h(gf)$.

b) for every object $x \in$ obj $\mathcal{C}$, there exists an identity morphism, $1_x : x \to x$, such that for any morphism $f : a \to b$, $1_b f = f 1_a$.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor*, F from $\mathcal{C}$ to $\mathcal{D}$ is a mapping that

1. associates to each object $a \in \mathcal{C}$ an object $F(a) \in \mathcal{D}$.
2. associates to each morphism $f : a \to b \in \mathcal{C}$ a morphism
$F(f) : F(a) \to F(b) \in \mathcal{D}$ such that
   a) identity morphims are preserved, and
   b) compositions are well behaved.

   That is, $F(1_a) = 1_{F(a)}$, and $F(gf) = F(g)F(f)$ for all objects $a, b, c \in \mathcal{C}$ and all morphims $f, g \in \mathcal{C}$, such that: $f : a \to b$ and $g : b \to c$.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant functor*, F from $\mathcal{C}$ to $\mathcal{D}$ is a mapping that

1. associates to each object $a \in \mathcal{C}$ an object $F(a) \in \mathcal{D}$.
2. associates to each morphism $f : a \to b \in \mathcal{C}$ a morphism
$F(f) : F(b) \to F(a) \in \mathcal{D}$ such that
   a) identity morphims are preserved, and
   b) compositions are well behaved.

   That is, $F(1_a) = 1_{F(a)}$, and $F(gf) = F(f)F(g)$ for all objects $a, b, c \in \mathcal{C}$ and all morphims $f, g \in \mathcal{C}$, such that: $f : a \to b$, and $g : b \to c$.

Example. *We may define a category whose objects are sets, x, where the morphisms from the set x_1 to the set x_2 are taken to be all mappings of sets from x_1 to x_2. This category is called the* category of sets, *and is denoted by Set.*

Example. *We may define a category whose objects are all abelian groups, g, where the morphisms from the abelian groups g_1 to the abelian group g_2 are all group homomorphisms from g_1 to g_2. This category is called the* category of abelian groups*, and is denoted by Ab.*

1.5 The functors Hom$(-, x)$ and Ext$^1(-, x)$

Definition. To each $x \in \text{Set}$, the contravariant Hom-functor, $\text{Hom}(-, x) : \mathcal{C} \to \text{Set}$ is a functor given by mapping:

1. an object $a \in \mathcal{C}$ to the set of morphisms mapping a to x, $\text{Hom}(a, x)$, and
2. each morphism $f : a \to b$ to the function

$$\begin{aligned} \text{Hom }(f, x) : \text{Hom }(b, x) &\to \text{Hom }(a, x) \\ g &\mapsto gf, \qquad\qquad \forall g \in \text{Hom}(b, x) \end{aligned}$$

The extension functor, $\text{Ext}^n(-, x)$, is a useful functor for canonically constructing exact sequences from short sequences, and repairing exactness lost when using the Hom-functor. Also, there are some useful results when it comes to deciding wheter or not a module is either projective or injective by use of $\text{Ext}^1(-, x)$. The derivation of the different characteristics and propositions concerning the $\text{Ext}^1(-, x)$ functor is beyond the scope of this paper, but a useful result will be cited to prove the main theorem of chapter 2.

Proposition 1.7. *If $Ext^1_R(C, A) = \{0\}$, then every extension of A by C splits.*

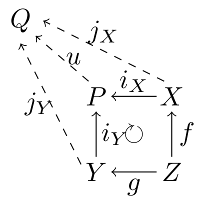
Proof: The proof is omitted in this thesis, but a complete proof of this proposition may be found in [5, p. 421].

Obviously, two exact sequences being equivalent by the definition of equivalence in section 1.3, is an equivalence relation, and it can be seen that $\text{Ext}^1(A, B)$ is really a group of residual classes of exact sequences up to this equivalence relation.

Definition. Let A and B be R-modules.
$\text{Ext}^1_R(A, B) = \left\{0 \to B \xrightarrow{f} C \xrightarrow{g} A \to 0\right\} \big/ \sim$

1.6 Pushout and Pullback diagrams

Definition. For two morphisms $f : Z \to X$, $g : Z \to Y$, the *pushout* of f and g consists of an object, P, and two morphisms $i_X : X \to P$ and $i_Y : Y \to P$, s.t. $i_X \circ f = i_Y \circ g$.



Also, the pushout must be *universal* with respect to this diagram. That is, for any other Q, j_X, j_Y such that $j_X \circ f = j_Y \circ g$, there must be a unique morphism $u : P \to Q$ such that $j_Y = u \circ i_Y$ and $j_X = u \circ i_X$.

To prove that the pushout exists, it is enough to show that at least one module and pair of morphisms exists satisfying these requirements. For all morphisms f and g, we may construct an object, P, and morphisms i_X and i_Y such that:
$P = \text{Coker}\begin{pmatrix} f \\ -g \end{pmatrix}$. We define the image of $\begin{pmatrix} f \\ -g \end{pmatrix}$ by:

$$I = \{(f(z), -g(z))|z \in Z\}.$$

The morphism i_X is the compositions of the inclusion l_X with the projection p, given by the definition of the cokernel, such that $i_X = pl_X$,

$$l_X : X \hookrightarrow X \times Y, \qquad l_X(z) = (z, 0),$$
$$p : X \times Y \to P, \qquad p(x, y) = (x, y) + I.$$

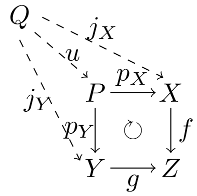
Thus, we see that $i_X(z) = (z, 0) + I$.

The morphism i_Y is the composition of the inclusion l_Y with the projection p, such that $i_Y = pl_Y$,

$$l_Y : Y \hookrightarrow X \times Y, \qquad l_Y(z) = (0, z).$$

From this, we see that $i_Y(z) = (0, z) + I$.
This construction is a pushout, and thus, the pushout exists.

Definition. For two morphisms $f : X \to Z$, $g : Y \to Z$, the *pullback* of f and g consists of an object, P, and two morphisms $p_X : P \to X$ and $p_Y : P \to Y$, s.t. the following diagram commutes, that is, $f \circ p_X = g \circ p_Y$:



Also, p_X and p_Y must be universal with respect to this property.

To prove that the pullback exists, it is enough to show that at least one object and two morphisms exists satisfying these requirements.
For all morphisms f and g, we may construct an object, P, and morphisms p_X and p_Y such that: $P = \ker \begin{pmatrix} f \\ -g \end{pmatrix}$. The morphism p_X is the composition of the inclusion m with the projection q_X, such that $p_X = q_X m$,

$$m : P \hookrightarrow X \times Y, \qquad m \text{ is inclusion of the kernel of } \begin{pmatrix} f \\ -g \end{pmatrix},$$

$$q_X : X \times Y \to X, \qquad q_X(x, y) = x.$$

The morphism p_Y is the composition of the inclusion m with the projection q_Y, such that $p_Y = q_Y m$,

$$q_Y : X \times Y \to Y, \qquad q_Y(x, y) = y.$$

This construction is a pullback, and thus, the pullback exists.

1.7 Baer sum

The Baer sum of two extensions can be used as an operation in order to make $\mathrm{Ext}^1_R(A, B)$ an abelian group. It works by applying pushout and pullback to operate on a pair of extensions,

$$\alpha : 0 \to B \xrightarrow{f} C \xrightarrow{g} A \to 0$$

$$\beta : 0 \to B \xrightarrow{f'} C' \xrightarrow{g'} A \to 0$$

as follows:

Pullback:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B \oplus B & \xrightarrow{i} & H & \xrightarrow{k} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} & & \\
0 & \longrightarrow & B \oplus B & \xrightarrow[\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)]{} & C \oplus C' & \xrightarrow[\left(\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right)]{} & A \oplus A & \longrightarrow & 0
\end{array}
$$

By applying "pullback along $A \oplus A$" one obtains the pullback, H, j, k. As the identity and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ are both injective, j is injective. By commutativity of the right square of the exact sequences, we get:

$$
\begin{aligned}
H &= \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}^{-1} \{(a,a) | a \in A\} \\
&= \{(c, c') \in C \oplus C' | g(c) = g'(c')\}
\end{aligned}
$$

For the diagram to commute, it needs to satisfy

$$
\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) k(c, c') = (g(c), g'(c')), (c, c') \in H
$$

Thus, we solve the different commutative diagrams and obtain:

$$
\begin{aligned}
k((c, c')) &= g(c) \\
j((c, c')) &= (c, c') \\
i((b, b')) &= (f(b), f'(b'))
\end{aligned}
$$

Pushout:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B \oplus B & \xrightarrow{i} & H & \xrightarrow{k} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (1\ 1)} & & \downarrow{\scriptstyle m} & & \| & & \\
0 & \longrightarrow & B & \xrightarrow[l]{} & K & \xrightarrow[n]{} & A & \longrightarrow & 0
\end{array}
$$

By applying "pushout along $B \oplus B$", one obtains K, m, l which is a pushout of i and $(1\ 1)$. As both $(1\ 1)$ and the identity are surjective, m is surjective.

$$
K \simeq \operatorname{Coker} \begin{pmatrix} i \\ -1 \ \ -1 \end{pmatrix} \simeq H/I,
$$

where

$$I = \{(f(b), -f'(b))|b \in B\}, I \subseteq H \subseteq C \oplus C'.$$

Baer sum:

$$\alpha + \beta : 0 \to B \xrightarrow{l} K \xrightarrow{n} A \to 0$$

Where the functions l and n are given as follows:
$l : B \to K$ is the composition $B \xrightarrow[b \mapsto (f(b),0)]{} H \xrightarrow[(h,h') \mapsto (h,h')+I]{} K$
$n : K \to A$ is uniquely induced by $k : H \to A$, as $k(I) = 0$.

$$l : B \to K, l(b) = (f(b), 0) + I = (0, -f'(b)) + I$$
$$n : K \to A, n((k, k') + I) = g(k).$$

Example. *Consider the extensions*

$$\alpha : 0 \to \mathbb{Z}_3 \xrightarrow{\lambda_3} \mathbb{Z}_9 \xrightarrow{p} \mathbb{Z}_3 \to 0$$
$$\beta : 0 \to \mathbb{Z}_3 \xrightarrow{\lambda_6} \mathbb{Z}_9 \xrightarrow{p} \mathbb{Z}_3 \to 0$$

used in the example of section 1.3.
To compute the Baer sum, $\gamma = \alpha + \beta$, we first construct a submodule H of $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ such that
$H = \{(z, z') \in \mathbb{Z}_9 \oplus \mathbb{Z}_9|\ p(z) = p(z')\}$*. Explicitly, this submodule is given by:*

$$\begin{aligned} H = \{&(0,0),(0,3),(0,6),(1,1),(1,4),(1,7),(2,2),(2,5),(2,8),\\ &(3,0),(3,3),(3,6),(4,1),(4,4),(4,7),(5,2),(5,5),(5,8),\\ &(6,0),(6,3),(6,6),(7,1),(7,4),(7,7),(8,2),(8,5),(8,8)\} \simeq \mathbb{Z}_9 \oplus \mathbb{Z}_3. \end{aligned}$$

Secondly, construct the submodule $I = \{(f(z), -f'(z)|z \in \mathbb{Z}_3\}$
$I = \{(0,0),(3,3),(6,6)\} \simeq \mathbb{Z}_3$*. Thus,*

$$K \simeq H/I = \{\overline{(0,0)}, \overline{(0,3)}, \overline{(0,6)}, \overline{(1,1)}, \overline{(1,4)}, \overline{(1,7)}, \overline{(2,2)}, \overline{(2,5)}, \overline{(2,8)}\}.$$

Here $\overline{(z, z')}$ denotes the coset $(z, z') + I$.

By the isomorphism

$$\phi : K \to \mathbb{Z}_3 \oplus \mathbb{Z}_3$$
$$\begin{aligned} &\phi((0,0)) = (0,0), && \phi((0,3)) = (0,1), && \phi((0,6)) = (0,2),\\ &\phi((1,1)) = (1,0), && \phi((1,4)) = (1,1), && \phi((1,7)) = (1,2),\\ &\phi((2,2)) = (2,0), && \phi((2,5)) = (2,1), && \phi((2,8)) = (2,2), \end{aligned}$$

we have that $K \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

$$l : \mathbb{Z}_3 \to \mathbb{Z}_3 \oplus \mathbb{Z}_3, l(z) = (0, z).$$
$$n : \mathbb{Z}_3 \oplus \mathbb{Z}_3 \to \mathbb{Z}_3, n(z, z') = z.$$

We have the Baer sum:

$$\alpha + \beta = \gamma : 0 \to \mathbb{Z}_3 \xrightarrow{(0\ 1)} \mathbb{Z}_3 \oplus \mathbb{Z}_3 \xrightarrow{\binom{1}{0}} \mathbb{Z}_3 \to 0.$$

Remark. α, β and γ are all the possible elements of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_3, \mathbb{Z}_3)$, and these form an abelian group isomorphic to $\mathbb{Z}_3$!

Chapter 2

Kronecker Quiver

In this chapter, we study the finite oriented graph known as the 2-Kronecker Quiver, and try to classify all indecomposable representations of this quiver over an algebraically closed field. Some of the more well-known propositions of this chapter are not proven, but rather referred to from the sources where the reader is provided with complete proofs.

2.1 Motivation

The 2-Kronecker Quiver, here denoted by Q, is an oriented graph with two vertices, 1 and 2, and two arrows, α and β, both going in the same direction, as shown in Figure 2.1.

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

Figure 2.1: The 2-Kronecker Quiver, Q.

By assigning to each vertex a finite dimensional vector space over a field, k, a k-vector space, and to each arrow a linear transformation between the vector spaces, we obtain a *representation of* Q over a field k. This representation may be written as a four-tuple $(k^m, k^n, l_\alpha, l_\beta)$, where k^m and k^n are the vector spaces assigned to vertices 1 and 2 respectively, and l_α and l_β are linear maps assigned to arrows α and β respectively. A map between two such representations, $(k^m, k^n, l_\alpha, l_\beta)$ and $(k^{m'}, k^{n'}, l'_\alpha, l'_\beta)$ is a pair of linear maps,

$$f_1 : k^m \to k^{m'}$$
$$f_2 : k^n \to k^{n'}$$

such that:

$$l'_\alpha f_1 = f_2 l_\alpha$$
$$l'_\beta f_1 = f_2 l_\beta.$$

This way we get the category of finite dimensional representations of the quiver, Q, where the objects are the representations and the morphisms are the maps between representations.
Let Λ be the path algebra defined in equation 2.1.

$$\Lambda = kQ \simeq \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & 0 & c \end{pmatrix} \middle| a, b, c, d \in k \right\} \tag{2.1}$$

Proposition 2.1. *The category of representations of Q over k, rep(Q, k) is equivalent as a category to the category of Λ-modules of finite k-dimension, mod Λ.*

Proof: For a proof of this proposition, see [2, p. 57].

Each finite dimensional module is both artinian and noetherian, so by the Krull-Remak-Schmidt Theorem, see [7, Theorem 3.3, p.7], every Λ-module may be written as a direct sum of indecomposable Λ-modules. Thus, in order to prove something for a general Λ-module, it may be enough to show that it holds for the indecomposable Λ-modules.

2.2 Constructing indecomposable Λ-modules

Let $J = (V_1, V_2, l_\alpha, l_\beta)$ be interpreted as a Λ-module corresponding to the representation of Q over k, where:

$$\begin{aligned} V_1 &= \text{vector space assigned to vertex 1} \\ V_2 &= \text{vector space assigned to vertex 2} \\ l_\alpha &= \text{linear transformation assigned to arrow } \alpha \\ l_\beta &= \text{linear transformation assigned to arrow } \beta. \end{aligned}$$

For the remainder of this chapter, the vector spaces assigned to the vertices will be denoted by k^n where $n \in \mathbb{N}$, unless otherwise stated. As we have an equivalence of categories between mod Λ and rep(Q, k), we use the same notation to describe the representations and the Λ-modules corresponding to them throughout the text, however, whether the notation is referring to a module or a representation is explicitly stated in every case where it might be unclear.

2.2.1 Coxeter functors

To study indecomposable objects of a category of representations of a quiver without oriented cycles, we may use a powerful tool named the coxeter functor, introduced by Bernstein, Gel'fand and Ponomarev, see [8]. The general idea, is that the coxeter functor is a functor constructed in such a way that indecomposable objects are either mapped to other indecomposable objects, or to a zero object. Explicitly, for the case we are concerned with, we have the coxeter functors

$$\begin{aligned} C^+ &: \text{mod } \Lambda \to \text{mod } \Lambda \\ C^- &: \text{mod } \Lambda \to \text{mod } \Lambda \end{aligned}$$

which maps indecomposable Λ-modules to indecomposable Λ-modules or the module $0 = (0, 0, 0, 0)$.

Constructing the coxeter functor

Starting out more general, we have a quiver, Q', given by equation 2.2.

$$Q' = \{Q_0, Q_1, h : Q_1 \to Q_0, t : Q_1 \to Q_0\} \tag{2.2}$$

Q_0 : Is a finite set, called the *vertices* in the quiver.

Q_1 : Is a finite set, called the *arrows* of the quiver.

$h(q)$: Is the *head* of the arrow $q \in Q_1$.

$t(q)$: Is the *tail* of the arrow $q \in Q_1$.

Head and *tail* of an arrow is defined such that for an arrow

$$q : i \to j, h(q) = j, t(q) = i.$$

Definition. A quiver is said to contain *oriented cycles* if there exists a finite composition of arrows $q = q_1 q_2 ... q_n$ such that $h(q) = t(q)$.

Definition. A *representation of* Q' over a field k, is for each vertex $i \in Q_0$ a k-vector space, $V(i)$, and for each arrow $q : i \to j \in Q_1$ a linear transformation $l_q : V(i) \to V(j)$.

Remark. Let Q' be a quiver with n vertices and m arrows. Then a representation V will be denoted by $V = (V(1), ..., V(n), l_1, ..., l_m)$.

Definition. A *map between two representations* V and V' over the same quiver, Q', is defined as a set of linear maps

$$\{f_i : V(i) \to V(i)'\}_{i \in Q_0}$$

such that if l_q and l'_q are linear transformations assigned to the same arrow $q : i \to j$ in the different representations, we have:

$$l'_q f_i = f_j l_q$$

This is the same as the linear maps $\{f_i\}_{i \in Q_0}$ making the diagram

$$\begin{array}{ccc} V(i) & \xrightarrow{f_i} & V(i)' \\ \downarrow{\scriptstyle l_q} & & \downarrow{\scriptstyle l'_q} \\ V(j) & \xrightarrow{f_j} & V(j)' \end{array}$$

commute for any $i, j \in Q_0$, and $q : i \to j \in Q_1$.

Definition. A *sink*, is a vertex $i \in Q_0$ such that $t(q) \neq i, \forall q \in Q_1$.

Definition. The partial coxeter functor of a sink i, C_i^+, is defined by:

$$C_i^+ : rep(Q', k) \to rep(Q_i', k)$$

where Q_i' is the quiver obtained when *reversing the direction* of every arrow $q \in Q'$ such that $h(q) = i$. That is, replacing each arrow $q : j \to i \in Q'$ by the arrow $q' : i \to j \in Q_i'$.

a) Let $V(j)$ be the vector space assigned to vertex $j \in Q_0$.

$$C_i^+(V(j)) = \begin{cases} V(j), & j \neq i \\ V(i)' = \ker g, & j = i. \end{cases}$$

where

$$g : \bigoplus_{\substack{q \in Q_1 \\ h(q)=i}} V(t(q)) \longrightarrow V(i).$$

Let $v_{t(q)} \in V(t(q))$, then g is defined by

$$g\left((v_{t(q)})_{\substack{q \in Q_1 \\ h(q)=i}} \right) = \sum_{\substack{q \in Q_1 \\ h(q)=i}} l_q(v_{t(q)}).$$

b) Let $l_q : V(j) \to V(k)$, be the linear transformation assigned to the arrow $q : j \to k \in Q_1$.

$$C_i^+(l_q) = \begin{cases} l_q, & k \neq i, \\ l_{q'} : C_i^+(V(i)) \xrightarrow{\text{incl.}} \bigoplus_{\substack{q \in Q_1 \\ h(q)=i}} V(t(q)) \xrightarrow{\text{proj.}} V(t(q)), & k = i. \end{cases}$$

Assuming the quiver we are considering does not have any oriented cycles, we may describe how C^+ acts on a representation of the quiver, V, by the following algorithm:

1. Locate a sink, i.
2. Apply C_i^+ to the representation.
3. Locate a previously unchanged sink, $j \neq i$, in the quiver Q_i'.

4. Apply C_j^+ to the representation $C_i^+(V)$.

5. Repeat the procedure until every vertex in the quiver have been a sink exactly once.

In other words, by changing the numbering of our vertices to fit to the order in which they appear as a sink for our given choice of order of operations, we may write:

$$C^+(V) = C_n^+ C_{n-1}^+ ... C_2^+ C_1^+(V)$$

Definition. A *source*, is a vertex $i \in Q_0$ such that $h(q) \neq i, \forall q \in Q_1$.

Definition. The partial coxeter functor of a source i, C_i^-, is defined by:

$$C_i^- : rep(Q', k) \to rep(Q_i'', k)$$

where Q_i'' is the quiver obtained when reversing the direction of every arrow $q \in Q'$ such that $t(q) = i$.

a) Let $V(j)$ be the vector space assigned to vertex $j \in Q_0$.

$$C_i^-(V(j)) = \begin{cases} V(j), & j \neq i \\ V(i)' = \text{Coker } g, & j = i. \end{cases}$$

where

$$g : V(i) \to \bigoplus_{\substack{q \in Q_1 \\ t(q)=i}} V(h(q))$$

Let $v_i \in V(i)$, then g is defined by:

$$g(v_i) = \bigoplus_{\substack{q \in Q_1 \\ t(q)=i}} l_q(v_i)$$

b) Let $l_q : V(j) \to V(k)$, be the linear transformation assigned to the arrow $q : j \to k \in Q_1$.

$$C_i^-(l_q) = \begin{cases} l_q, & j \neq i, \\ l_{q'} : V(k) \xrightarrow{\text{incl.}} \bigoplus_{\substack{q \in Q_1 \\ h(q)=i}} V(h(q)) \xrightarrow{\text{proj.}} C_i^-(V(i)), & j = i. \end{cases}$$

Assuming the quiver we are considering does not contain any oriented cycles, we may describe how C^- acts upon a representation of a quiver, V, by the following algorithm:

1. Locate a source, i.
2. Apply C_i^- to the representation.
3. Locate a previously unchanged source in Q_i'', $j \neq i$.
4. Apply C_j^- to the representation $C_i^-(V)$.
5. Repeat the procedure until every vertex has appeared as a source exactly once.

In other words, by changing the numbering of our vertices to fit to the order in which they appear as a source, we may write:

$$C^-(V) = C_n^- C_{n-1}^- ... C_2^- C_1^-(V)$$

Remark. For any indecomposable representation $V \ncong S_i$. $C_i^- C_i^+(V) = V$. S_i is the representation with a one dimensional vector space assigned to the vertex i, zero spaces assigned to every vertex $j \neq i$, and zero maps assigned to every arrow $q \in Q_1$.

Example. *Consider the quiver without cycles given by:*

$$1 \xrightarrow{\ \alpha\ } 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3$$

Construct a representation of this quiver over a field k by assigning to each vertex $i \in \{1, 2, 3\}$ a vector space $V(i)$ over k, and to each arrow $\chi \in \{\alpha, \beta, \gamma\}$ a linear map l_χ. Denote such a representation by

$$V = (V(1), V(2), V(3), l_\alpha, l_\beta, l_\gamma).$$

Now consider the representation $V_1 = \left(k, k^2, k^3, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$.

$$k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} k^2 \underset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}{\rightrightarrows}} k^3$$

Applying the coxeter functor C^+ to this representation will give us $C^+(V_1) = C_1^+C_2^+C_3^+(V_1)$. Starting out, there is only one sink to consider: $V(3)$. Computing, we get:

$$C_3^+(V(3)) = ker\ g, g = (l_\beta, l_\gamma) : k^2 \oplus k^2 \to k^3,$$

$$C_3^+(V(3)) = ker\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ a \\ -a \\ 0 \end{pmatrix} \middle| a \in k \right\} \simeq k.$$

The new representation, V_1', obtained is:

$$k \xrightarrow{\binom{1}{0}} k^2 \underset{\binom{0}{1}}{\overset{\binom{1}{0}}{\longleftleftarrows}} k$$

This quiver has a new sink, the vertex 2, and hence, we need to compute $C_2^+(V')$ in the same way:

$$C_2^+(V(2)) = ker\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a \\ -a \\ 0 \end{pmatrix} \middle| a \in k \right\} \simeq k.$$

The new representation, V_1'', obtained is:

$$k \xleftarrow{\ 1\ } k \underset{0}{\overset{1}{\rightrightarrows}} k$$

Now, there is only one vertex left to consider, 1, and this is actually turned into a sink from the last use of the partial coxeter functor, so we apply $C_1^+(V'')$, and finally obtain $C^+(V)$:

$$C_1^+(V(1)) = ker\ 1 = 0.$$

$$0 \xrightarrow{\ 0\ } k \underset{0}{\overset{1}{\rightrightarrows}} k$$

The quiver is now the same quiver we started with, and we see that

$$C^+\left(\left(k, k^2, k^3, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = (0, k, k, 0, 1, 0).$$

In a similar way, we may apply C^- to the representation $V_2 = C^+(V_1)$ by considering the sources of the quiver. First, we consider the source at vertex 1:

$$C_1^-(V(1)) = Coker\ 0 = k/Im\ 0 \simeq k$$

The new representation, V_2', obtained is:

$$k \xleftarrow{\quad 0 \quad} k \underset{0}{\overset{1}{\rightrightarrows}} k$$

Now the source is located at vertex 2:

$$C_2^-(V(2)) = Coker \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = k \oplus k \oplus k / Im \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \simeq k^2$$

The new representation, V_2'', obtained is:

$$k \xrightarrow{\quad 1 \quad} k^2 \underset{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}{\leftleftarrows}} k$$

And finally, the last source is now at vertex 3:

$$C_3^-(V(3)) = Coker \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = k \oplus k \oplus k \oplus k / Im \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \simeq k^3$$

The new representation obtained is:

$$k \xrightarrow{\quad 1 \quad} k^2 \underset{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}{\rightrightarrows}} k^3$$

And hence, we see that: $C^-(V_2) = V_1$, and $C^-C^+(V_1) = V_1$.

2.2.2 Coxeter functor and indecomposables of Q.

Let $V = (k^m, k^n, A, B)$ be a representation of the quiver Q, as defined in section 2.1. In order to obtain all the indecomposable representations of Q over a field k, or "the indecomposables of Q", one possible idea is to start out by finding different types of indecomposables and obtain indecomposables of a similar form by applying the coxeter functor to these. The indecomposable representations are either *irregular*, or *regular*, and the irregular indecomposables may all be derived from a finite number of indecomposable representations simply by applying the coxeter functors C^+ or C^-.[8, Theorem 1.3, p. 25].

Proposition 2.2. *Let $V = (k^m, k^n, A, B)$ be a representation of the quiver Q over the field k, such that $m > n$. Now $C^+(V)$ is indecomposable if and only if V is indecomposable.*

Proof: Let Λ be the path algebra defined in equation 2.1. A Λ-module, M, of finite length, is indecomposable if and only if the endomorphism ring $\mathrm{End}_\Lambda(M)$ is local, see [2, Theorem 2.2, p. 33]. So in order to prove that the coxeter functor preserves indecomposable modules, it is enough to observe the endomorphism rings of M and $C^+(M)$. Consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_i^+(V(i)) & \longrightarrow & \bigoplus_{\substack{q\in Q_1\\ h(q)=i}} V(t(q) & \longrightarrow & V(i) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle g_i} & & \Big\| & & \Big\downarrow{\scriptstyle f_i} & & \\
0 & \longrightarrow & C_i^+(V(i)) & \longrightarrow & \bigoplus_{\substack{q\in Q_1\\ h(q)=i}} V(t(q) & \longrightarrow & V(i) & \longrightarrow & 0
\end{array}$$

As the diagram commutes, any homomorphism $f_i : V(i) \to V(i)$ induces a homomorphism $g_i : C_i^+(V(i)) \to C_i^+(V(i))$, and the converse is also true.
$\Rightarrow$) Assume that M is indecomposable, and $M \not\simeq S_i$. We have that $\mathrm{End}(M) \simeq \mathrm{End}(C^+(M))$, and as $\mathrm{End}(M)$ is local, $\mathrm{End}(C^+(M))$ is local, and thus, $C^+(M)$ is an indecomposable module.
$\Leftarrow$) Assume that $C^+(M)$ is indecomposable, and $C_i^+(M) \not\simeq S_i$ for the quiver Q_i. We have that $\mathrm{End}(C^+(M)) \simeq \mathrm{End}(M)$, and as $\mathrm{End}(C^+(M))$ is local, so is $\mathrm{End}(M)$, and hence, M is an indecomposable module.
A Λ-module M is indecomposable if and only if $C^+(M)$ is indecomposable. The same holds for representations of the quiver Q over k, by proposition 2.1. □

Proposition 2.3. *Let $V = (k^m, k^n, A, B)$ be a representation of the quiver Q over the field k, such that $m < n$. Now $C^-(V)$ is indecomposable if and only if V is indecomposable.*

Proof: The proof is similar to the proof of proposition 2.2.

General requirements for indecomposable modules

Before we start considering the use of the coxeter functor on an indecomposable representation, we first need to derive some basic properties for the indecomposable representations, or by equivalence, for the indecomposable Λ-modules.

Proposition 2.4. *Let $J = (k^m, k^n, A, B)$ be an indecomposable Λ-module, with $m > 0$. Then Im A + Im B $= k^n$*

Proof: Assume Im A + Im $B = k^{n'}$. Let $n'' = n - n'$. One might write

$$J = (k^m, k^n, A, B) = (k^m, k^{n'}, \bar{A}, \bar{B}) \oplus (0, k^{n''}, 0_{n''}, 0_{n''})),$$

where $0_{n''}$ is the zero matrix of dimensions $1 \times n''$, and $\bar{A}$ and $\bar{B}$ are A and B when removing the rows simultaneously taking every element to zero for both A and B. J is decomposable for any $n'' \neq 0$. □

Proposition 2.5. *Let $J = (k^m, k^n, A, B)$ be an indecomposable Λ-module, with $n > 0$. Then ker $A \cap$ ker $B = 0$*

Proof: Let K=ker$A \cap$ ker$B = k^{m'}$. Then

$$J = (k^{m-m'}, k^n, \bar{A}, \bar{B}) \oplus (k^{m'}, 0, 0^{m'}, 0^{m'}),$$

where $0^{m'}$ is the zero matrix of dimensions $m' \times 1$, and $\bar{A}$ and $\bar{B}$ are A and B restricted to the vector space $k^m \setminus K$. Thus J is decomposable for any $m' \neq 0$. □

Coxeter functor of a representation of Q.

Let $V = (k^m, k^n, A, B)$ be an indecomposable representation of the quiver Q over k.

$$k^m \underset{B}{\overset{A}{\rightrightarrows}} k^n$$

Obviously, for the quiver, Q, there is only one sink, the vertex 2. Hence, we may let the coxeter functor C^+ act on V by:

$$C^+(V) = C_1^+ C_2^+(V)$$

Starting out:

$$C_2^+(k^n) = \text{ker } (A\ B)$$

And we get the new representation, V', such that:

$$V' : \quad k^m \underset{B'}{\overset{A'}{\leftleftarrows}} \text{ker } (A\ B)$$

Here, dim (ker $(A\ B)$) is given by constructing the short exact sequence:

$$0 \to \ker\,(A\ B) \hookrightarrow k^m \oplus k^m \xrightarrow{(A\ B)} k^n \to 0$$

where $(A\ B)$ is surjective by proposition 2.4. Now as the length of a vector space is equal to its dimension, we get that dim (ker $(A\ B)) = m + m - n =$ $2m - n$, and hence, ker $(A\ B) \simeq k^{2m-n}$.

$$C^+(V) = C_1^+(V')$$

$$C_1^+(k^n) \simeq ker(A'\ B'),$$

And we get the representation $C^+(V)$ such that:

$$C^+(V): \quad ker(A'\ B') \underset{B''}{\overset{A''}{\rightrightarrows}} k^{2m-n}$$

By constructing an exact sequence in the similar way as before, we get

$$0 \to \ker\,(A'\ B') \hookrightarrow k^{2m-n} \oplus k^{2m-n} \xrightarrow{(A'\ B')} k^n \to 0$$

dim (ker $(A'\ B')) = 2m - n + 2m - n - m = 3m - 2n$, ker $(A'\ B') \simeq k^{3m-2n}$.

$$C^+(k^m, k^n, A, B) = (k^{3m-2n}, k^{2m-n}, A'', B'').$$

By this relation, we may define the *coxeter matrix*, Φ, which is a matrix describing what happens to the dimensions of the vector spaces in an indecomposable Λ-module when applying the coxeter functor C^+ on it.

2.2.3 Coxeter Matrix

Let the dimensions of an indecomposable Λ-module

$$J = (k^m, k^n, A, B)$$

be given by the dimension vector

$$d_J = \left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right).$$

When we use the coxeter functor C^+ to move from one indecomposable Λ-module, J_1, to another indecomposable Λ-module, $J_2 = C^+(J_1)$, the dimension vectors d_{J_1} and d_{J_2} are related by the *coxeter matrix*, Φ:

$$\Phi = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix},$$

Applying the coxeter functor C^+ on a module q times, will change the dimension vectors by Φ^q, given by:

$$\Phi^q = q \cdot \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is given by the binomial expression of $(A + I)^q$ for quadratic matrices $A = \left(\begin{smallmatrix} 2 & -2 \\ 2 & -2 \end{smallmatrix}\right)$ and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, as $\left(\begin{smallmatrix} 2 & -2 \\ 2 & -2 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

The inverse coxeter functor, C^- may be studied in a way similar to how we approached C^+, and it gives an inverse coxeter matrix, Φ^{-1}, such that:

$$\Phi^{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix},$$

Applying the inverse coxeter functor C^- to a module, V, q times, will change the dimension vectors by Φ^{-q}, given by:

$$\Phi^{-q} = q \cdot \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have that

$$\Phi\Phi^{-1} = \Phi^{-1}\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.2.4 Indecomposables in the class N_1

To find indecomposable Λ-modules, it is a good idea to start with the simple modules $S_1 = (k, 0, 0, 0)$ and $S_2 = (0, k, 0, 0)$, which are obviously indecomposable. By constructing injective Λ-modules of the simple Λ-modules S_1 and S_2, we obtain the injective Λ-modules

$$I_1 \simeq S_1 = (k, 0, 0, 0),$$

and

$$I_2 = (k^2, k, \begin{pmatrix}1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\end{pmatrix}).$$

respectively. Applying the coxeter functor C^+ to I_1 gives us:

$$C^+(I_1) = C_1 = \left(k^3, k^2, \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$

Applying the coxeter matrix C^+ on I_2, gives us the module:

$$C_2 = \left(k^4, k^3, \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right)$$

Now,

$$C_1 = C^+(I_1)$$
$$C_2 = C^+(I_2),$$

and both C_1 and C_2 are indecomposable Λ-modules, as I_1 and I_2 are indecomposable Λ-modules. Expanding on this idea, by applying C^+ consecutively on the modules attained this way, we may construct an entire class of indecomposable Λ-modules on the form of equation 2.3,

$$N_1 = \left\{(k^{n+1}, k^n, (i_n \ 0), (0 \ i_n)) \middle| \ n \in \mathbb{N}\right\}. \tag{2.3}$$

Where i_n for the rest of this thesis denotes the $n \times n$-identity matrix.

2.2.5 Indecomposables in the class N_2

Construct the projective Λ-modules

$$P_1 = \left(k, k^2, \begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\end{pmatrix}\right),$$

and

$$P_2 \simeq S_2 = (0, k, 0, 0)$$

over S_1 and S_2 respectively. We obtain another class of indecomposable Λ-modules on the form given in equation 2.4, by applying C^- to P_1 and P_2 in a similar manner to the construction we performed in section 2.2.4.

$$N_2 = \left\{\left(k^n, k^{n+1}, \begin{pmatrix}i_n\\ 0\end{pmatrix}, \begin{pmatrix}0\\ i_n\end{pmatrix}\right) \middle| \ n \in \mathbb{N}\right\}. \tag{2.4}$$

2.2.6 Indecomposables in the class N_3

Jordan canonical form

The last class of indecomposable Λ-modules we construct, may be simplified by restricting our study to deal with the cases where k is an *algebraically closed* field, that is, all the irreducible polynomials in the polynomial ring $k[x]$ are of degree 1. When this is the case, any square matrix A, being a linear transformation from k^n to k^n, may be written on *Jordan canonical form* as a direct product of matrices on the form of equation 2.5, called *Jordan blocks*. For further explaination of the Jordan canonical form of a matrix, see for instance [1, p. 423].

$$JB_{n_i}^{\lambda} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \tag{2.5}$$

$JB_{n_i}^{\lambda}$ is an $n_i \times n_i$-matrix, where λ is an eigenvalue of the matrix A. Let the *algebraic multiplicity*, m_a^{λ} of a fixed eigenvalue λ of A be given by:

$$m_a^{\lambda} = \sum_{\substack{n_i \in \mathbb{N}, \\ \exists JB_{n_i}^{\lambda}}} n_i.$$

Described with words, m_a^{λ} is the dimension of a direct sum of all Jordan blocks in the Jordan canonical form of A containing the eigenvalue λ. The number of different such Jordan blocks, m_g^{λ}, is called the *geometric multiplicity* of the eigenvalue λ. The dimension of the direct sum of all Jordan blocks in the Jordan canonical form of a matrix, A, is given by

$$\sum_{\lambda} m_a^{\lambda} = n,$$

where the sum is taken over every different eigenvalue of A.

Indecomposables in the class N_3

Using the well behaved structure of the Jordan blocks, we may construct a third class of indecomposable Λ-modules by considering the Λ-module

$$S_{\lambda}^{*} = (k, k, 1, \lambda), \lambda \in k \cup \{\infty\},$$

where $\lambda = \infty$ corresponds to the module $(k, k, 0, 1)$.
We see that:

$$\dim(\mathrm{Ext}^1_k(S^*_\lambda, S^*_\gamma)) \begin{cases} 1 & \text{if } \lambda = \gamma \\ 0 & \text{if } \lambda \neq \gamma \end{cases},$$

We obtain a class of indecomposable modules by taking $S^*_\lambda, \lambda \in k$ extended with itself. These are the Λ-modules in the class N_3', given by equation 2.6.

$$N_3' = \left\{ \left(k^n, k^n, i_n, JB^\lambda_n\right) \middle|\ n \in \mathbb{N}, \lambda \in k \right\} \tag{2.6}$$

Also, S^*_∞ extended with itself gives us the Λ-modules in the class N_3'', given by equation 2.7.

$$N_3'' = \left\{ \left(k^n, k^n, JB^0_n, i_n\right) \middle|\ n \in \mathbb{N} \right\} \tag{2.7}$$

The modules in the class N_3'' may be denoted as modules of the form of equation 2.6 with $\lambda = \infty$. Thus, the classes N_3' and N_3'' may be written as a single class of Λ-modules, N_3, given by equation 2.8.

$$N_3 = \left\{ \left(k^n, k^n, i_n, JB^\lambda_n\right) \middle|\ n \in \mathbb{N}, \lambda \in k \cup \{\infty\} \right\} \tag{2.8}$$

2.3 Main Theorem

The three classes N_1, N_2, and N_3 does in fact cover all the indecomposable Λ-modules, as will be shown in the next section. Due to this fact, we arrive at the main theorem of this thesis:

Theorem 2.6. *Let k be an algebraically closed field. Let M be a Λ-module. Then:*

$$\begin{aligned} M \simeq\ & \left(k^{m_1}, k^{m_1+1}, \left(\begin{smallmatrix} i_{m_1} \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ i_{m_1} \end{smallmatrix}\right)\right) \oplus \cdots \oplus \left(k^{m_r}, k^{m_r+1}, \left(\begin{smallmatrix} i_{m_r} \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ i_{m_r} \end{smallmatrix}\right)\right) \\ & \oplus (k^{n_1+1}, k^{n_1}, (i_{n_1}\ 0), (0\ i_{n_1})) \oplus \cdots \oplus (k^{n_s+1}, k^{n_s}, (i_{n_s}\ 0), (0\ i_{n_s})) \\ & \oplus (k^{l_1}, k^{l_1}, i_{l_1}, JB^{\lambda_1}_{l_1}) \oplus \cdots \oplus (k^{l_t}, k^{l_t}, i_{l_t}, JB^{\lambda_t}_{l_t}), r, s, t \in \mathbb{N}, \\ & \lambda_t \in k \cup \{\infty\} \end{aligned}$$

Where i_n is the identity matrix of dimension $n \times n$, and $JB^{\lambda_i}_{l_i}$ is the Jordan block of dimension $l_i \times l_i$ with eigenvalue λ_i.

2.4 Proof of Main Theorem

If N_1, N_2 and N_3 are all indecomposable Λ-modules, Theorem 2.6 follows immediately from the Krull-Remak-Schmidt theorem, see [7, Theorem 3.3, p.7]. Hence, we need only show that the classes N_1, N_2, and N_3 contain all indecomposable Λ-modules. Considering an arbitrary indecomposable Λ-module

$$J = (k^m, k^n, A, B)$$

the cases we need to study may be divided into three separate cases:

1. The case where $m < n$.
2. The case where $m > n$.
3. The case where $m = n$.

These three will turn out to correspond to the three different classes of indecomposable modules discovered in the previous section.

2.4.1 Indecomposables in the classes N_2 and N_1

In the following proofs, the specifics of the linear transformations in the representations are not always shown, as the proofs generally rely on dimension arguments from using the coxeter functor.

Proposition 2.7. *Let $J = (k^m, k^n, A, B)$ be an indecomposable Λ-module. If $m < n$, then $n = m + 1$.*

Proof: Assuming the module J is not isomorphic to P_1 or P_2, in which case it would belong to the class N_2, we may use the coxeter functor C^+ on the module, to obtain a Λ-module on the form $(k^{3m-2n}, k^{2m-n}, A'', B'')$. By continued application of the coxeter functor, the dimension of the vector spaces will be given by Φ^q, as defined in section 2.2.3. This yields modules

$$(k^{(2q+1)m-2qn}, k^{2qm-(2q-1)n}, A_q, B_q), q \in \mathbb{N}.$$

As $m < n$, this means that after using the coxeter functor C^+ a finite number of times, the modules generated will contain vector spaces of negative dimension, imaginary vector spaces so to say. The coxeter functor will only transform an indecomposable module into such "imaginary modules" if it is used on one of the projective modules, P_1 or P_2. So at the last point before the dimensions turn negative, the module must have been reduced to one of the projective modules. Thus, we arrive at equation 2.9 if the projective module obtained is P_1 and equation 2.10 if the projective module obtained is P_2:

$$\begin{aligned}
&\Phi^q \left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) && (2.9)\\
&(2q+1)m - 2qn = 0,\\
&2qm - (2q-1)n = 1\\
&\Rightarrow \underline{n = m+1}\\
&\Phi^q \left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) && (2.10)\\
&(2q+1)m - 2qn = 1,\\
&2qm - (2q-1)n = 2\\
&\Rightarrow \underline{n = m+1}
\end{aligned}$$

□

Proposition 2.8. *Let $J = (k^m, k^n, A, B)$ be an indecomposable Λ-module. If $m > n$, then $m = n + 1$*

Proof: The proof is similar to the proof of Proposition 2.7, but instead of using the coxeter functor C^+, we may look at what happens when applying C^- to an indecomposable Λ-module, J. Assume J is not equal to I_1 or I_2, in which case it would belong to the class N_1 of. C^-, will yield negative dimensions after being applied some finite number of times, as the dimensions are given by Φ^{-q}, and $m > n$. Thus, as applying C^- to an indecomposable Λ-module only yields imaginary modules when being applied to one of the injective Λ-modules, I_1 or I_2. We arrive at equations 2.11 and 2.12, by the same reasoning as for proposition 2.7.

$$\begin{aligned}
&\Phi^{-q} \left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) && (2.11)\\
&-(2q-1)m + 2qn = 1,\\
&-2qm + 2(q+1)n = 0\\
&\Rightarrow \underline{m = n+1}\\
&\Phi^{-q} \left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) && (2.12)\\
&-(2q-1)m + 2qn = 2,\\
&-2qm + (2q+1)n = 1\\
&\Rightarrow \underline{m = n+1}
\end{aligned}$$

□

2.4.2 Indecomposables in the class N_3

By now, the modules in the classes N_1 and N_2 have been shown to be the only indecomposable Λ-modules where $n \neq m$ only by relying on the theory of the coxeter functor. However, to obtain the proof of existence and uniqueness of the last class of indecomposable modules, we need to apply the $\mathrm{Ext}^1_\Lambda(-, x)$ functor described in section 1.5. We start by looking at the well-behaved structure of the 2-Kronecker quiver.

Proposition 2.9. *For any indecomposable Λ-module,*

$$J = (k^m, k^n, A, B), n < 2m, m \neq 0,$$

there exists a projective presentation

$$0 \to p_2^{(m,n)} \to p_1^{(m,n)} \to J \to 0$$

where $p_2^{(m,n)}$ and $p_1^{(m,n)}$ are given by:

$$p_2^{(m,n)} = (P_2)^{2m-n},$$
$$p_1^{(m,n)} = (P_1)^m.$$

Proof: Let $J = (k^m, k^n, A, B)$ be an indecomposable left Λ-module. Now construct

$$p_1^{m,n} = (k^m, k^{2m}, A' = \left(\begin{smallmatrix} i_m \\ 0 \end{smallmatrix}\right), B' = \left(\begin{smallmatrix} 0 \\ i_m \end{smallmatrix}\right)),$$
$$p_2^{m,n} = (0, k^{2m-n}, 0, 0),$$

where A' and B' are inclusions into the first m copies of k and the last m copies of k respectively. $p_2^{m,n}$ and $p_1^{m,n}$ are direct sums of a finite number of copies of projective modules P_2 and P_1 respectively, and thus, $p_2^{m,n}$ and $p_1^{m,n}$ are also projective, by proposition 1.5. This gives the projective presentation

$$0 \to p_2^{m,n} \to p_1^{m,n} \to M \to 0$$

more precisely given by:

$$
0 \longrightarrow
\begin{array}{ccccc}
0 & \xrightarrow{0} & k^m & \xrightarrow{1} & k^m \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
k^{2m-n} & \xrightarrow{i} & k^{2m} & \xrightarrow{(A\ B)} & k^n
\end{array}
\longrightarrow 0
$$

where i is inclusion into $\ker{(A\ B)} \subseteq k^{2m}$. □

Proposition 2.10. *Let $M = (k^m, k^m, A, B)$ be a left Λ-module. Then M belongs to indecomposable class N_3 or it is decomposable by Theorem 2.6.*

Proof: We may prove this by mathematical induction:
<u>$m = 1$:</u>
For the case where $m = 1$, consider the case of the module K defined by equation 2.13,

$$K = K_{(x,y)} = (k, k, x, y). \tag{2.13}$$

In the case where $(x, y) = (0, 0)$, the module would decompose as

$$K_{(0,0)} = (k, k, 0, 0) = (k, 0, 0, 0) \oplus (0, k, 0, 0),$$

which satisfies Theorem 2.6. Now, assume $(x, y) \neq (0, 0)$, as the module would otherwise be decomposable. To create a projective resolution of this module, consider the projective modules P_1 and P_2, and the exact sequence:

$$0 \to P_2 \to P_1 \to K_{(x,y)} \to 0$$

given by:

$$
0 \longrightarrow
\begin{array}{ccccc}
0 & \xrightarrow{0} & k & \xrightarrow{1} & k \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
k & \xrightarrow{\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)} & k^2 & \xrightarrow{(x\ y)} & k
\end{array}
\longrightarrow 0
$$

Now, define $K' = K_{(x',y')}$. To calculate the $\mathrm{Ext}^1_\Lambda(K, K')$, consider the exact sequence

$$0 \to \mathrm{H}(K, K') \to \mathrm{H}(P_1, K') \to \mathrm{H}(P_2, K') \to \mathrm{E}(K, K') \to 0$$

where

$$\mathrm{H}(A, B) = \mathrm{Hom}_\Lambda(A, B),$$
$$\mathrm{E}(A, B) = \mathrm{Ext}^1_\Lambda(A, B).$$

The homomorphisms including the projective modules, $\mathrm{H}(P_1, K')$ and $\mathrm{H}(P_2, K')$, are completely determined by the first and second vector space respectively, hence, the length of their homomorphism group are both equal to 1 (they are of k-dimension 1).
Also $E(P_1, K') = E(P_2, K') = 0$.
By Proposition 1.6,

$$\ell(\mathrm{E}(K, K')) = \ell(\mathrm{H}(K, K')).$$

H (K, K') is the set of homomorphisms such that the following diagram commutes:

$$\begin{array}{ccc} k & \xrightarrow{\quad 1 \quad} & k \\ x \Big\downarrow\Big\downarrow y & & x' \Big\downarrow\Big\downarrow y' \\ k & \xrightarrow{\quad \gamma \quad} & k \end{array}$$

this is only possible for $(x, y) = \gamma(x', y'), \gamma \in k$. If $K \simeq K'$, $\mathrm{H}(K, K') \simeq k$, if not $\mathrm{H}(K, K') = 0$, and so $\ell(\mathrm{E}(K, K')) = 1$ or 0 respectively. Thus, the module may be written as $K_{(1,\lambda)} \simeq K_{(x,y)}$ by

$$\lambda = \begin{cases} x^{-1}y & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

Hence, for $m = 1$, the module belongs to the indecomposable class N_3.
<u>$m = n$:</u>
Now assume that the statement holds for $m = n - 1$. Let

$$M_n = (k^n, k^n, A, B)$$

be an indecomposable matrix. If both A and B are of full rank, this module is isomorphic to

$$M_n^* = (k^n, k^n, I, A^{-1}B),$$

and we may change the bases of V_1 and V_2 such that $A^{-1}B$ is on Jordan canonical form[1, Theorem 5.4, p. 423]. Thus the module belongs to N_3.
If A is not of full rank, then $k \subseteq \ker A$, such that we have the exact sequence:

$$0 \to S_\infty^* \to M_n \to L \to 0$$

explicitly given by:

$$
\begin{array}{ccccccccc}
 & & k & \xrightarrow{i} & k^n & \xrightarrow{p} & k^{n-1} & & \\
0 & \longrightarrow & 0 \Big\downarrow\Big\downarrow B|_k & & \Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow & \longrightarrow & 0 \\
 & & k & \xrightarrow{i} & k^n & \xrightarrow{p} & k^{n-1} & &
\end{array}
$$

where i is the inclusion, p is the projection onto the quotient spaces of smaller dimension, and $B|_k \simeq 1_k$.
By the induction hypothesis, the module

$$L = (k^{n-1}, k^{n-1}, A', B')$$

is either indecomposable or on the form of Theorem 2.6. Hence,

$$
\begin{aligned}
L \simeq\ & \left(k^{m_1}, k^{m_1+1}, \left(\begin{smallmatrix} i_{m_1} \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ i_{m_1} \end{smallmatrix}\right)\right) \oplus \cdots \oplus \left(k^{m_r}, k^{m_r+1}, \left(\begin{smallmatrix} i_{m_r} \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ i_{m_r} \end{smallmatrix}\right)\right) \\
& \oplus (k^{n_1+1}, k^{n_1}, (i_{n_1}\ 0), (0\ i_{n_1})) \oplus \cdots \oplus (k^{n_s+1}, k^{n_s}, (i_{n_s}\ 0), (0\ i_{n_s})) \\
& \oplus (k^{l_1}, k^{l_1}, i_{l_1}, JB_{l_1}^{\lambda_1}) \oplus \cdots \oplus (k^{l_t}, k^{l_t}, i_{l_t}, JB_{l_t}^{\lambda_t}), r, s, t \in \mathbb{N}, \\
& \lambda_t \in k \cup \{\infty\}
\end{aligned}
$$

.
Here

$$\sum_{i=1}^{r} m_i = \sum_{j=1}^{s} n_j,$$

as the vector spaces have the same dimension.
As

$$\mathrm{E}((k^{m_1}, k^{m_1+1}, \left(\begin{smallmatrix} i_{m_1} \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ i_{m_1} \end{smallmatrix}\right)), (k, k, 0, 1)) = 0,$$

and

$$\mathrm{E}((k^{l_i}, k^{l_i}, I_{l_i}, JB_{l_i}^{\lambda_i}), (k, k, 0, 1)) = 0, \forall \lambda_i \neq \infty,$$

any such summands on the form of N_2, or N_3 with eigenvalues $\lambda_i \neq \infty$ would make the initial short sequence split, by Proposition 1.7, and hence,
$M_n \simeq L \oplus (k, k, 0, 1)$, which is a direct sum on the form of theorem 2.6.

By this point it has been shown that the module M_n is either on the form of Theorem 2.6, or it is indecomposable, and

$$L = (k^{s_1}, k^{s_1}, JB_{s_1}^{0}, i_{s_1}) \oplus \cdots \oplus (k^{s_t}, k^{s_t}, JB_{s_t}^{0}, i_{s_t}).$$

Assume that the module M_n is indecomposable. Thus, we may write the exact sequence explicitly by:

$$
\begin{array}{ccccccccccc}
 & k & \xrightarrow{\;\;i\;\;} & k^n & \xrightarrow{\;\;p\;\;} & k^{s_1} & & & k^{s_t} & & \\
0 \longrightarrow & 0 \Big\downarrow\!\Big\downarrow 1 & & A \Big\downarrow\!\Big\downarrow B & & JB^0_{s_1} \Big\downarrow\!\Big\downarrow i_{s_1} & \bigoplus \;\cdots\; \bigoplus & & JB^0_{s_t} \Big\downarrow\!\Big\downarrow i_{s_t} & \longrightarrow 0 & \\
 & k & \xrightarrow{\;\;i\;\;} & k^n & \xrightarrow{\;\;p\;\;} & k^{s_1} & & & k^{s_t} & &
\end{array}
$$

As each i_{s_u} is of full rank for all $u \in \{1, ..., t\}$, and 1 is of full rank, this implies that $\ker(\oplus_{u=1}^{t} s_{i_u}) = 0$, $\ker 1 = 0$, hence $\ker B = 0$. Thus, as B is of full rank, we may apply a change of basis in k^n, such that

$$M_n \simeq (k^n, k^n, B^{-1}A, I),$$

where $B^{-1}A$ may be conjugated to be on the Jordan canonical form by a simultaneous change of basis of V_1 and V_2. This Jordan canonical form can not contain more than one Jordan block, as M_n is assumed to be indecomposable, and hence, the Jordan canonical form of $B^{-1}A$ must be the single Jordan block with eigenvalue $\lambda = 0$, as the matrix $B^{-1}A$ would otherwise be of full rank, and it is assumed that $\ker A \neq 0$. Thus, the indecomposable matrix M is on the form of N_3. A similar approach will yield the remaining modules of the class N_3 by assuming that B is not of full rank. $\square$

Chapter 3

Systems of linear differential equations

In this chapter, we use the decomposition of representations of the 2-Kronecker quiver, Q over an algebraically closed field, k, by theorem 2.6 to try to find a way to make it easier to solve systems of linear differential equations.

3.1 Systems of differential equations

A system of homogenous linear differential equations are systems on the form of equation 3.1.

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} \tag{3.1}$$

For a system on this form, obtaining the solutions of the system is pretty straightforward, see for instance [4, p. 164-168]. However, the initial system may not be as well behaved. Consider the system in equation 3.2.

$$\mathbf{A}\mathbf{x} = \mathbf{B}\dot{\mathbf{x}} \tag{3.2}$$

A and **B** are of the same dimension, $m \times n$, but otherwise, general matrices without any initial restrictions, thus, this system seems too general to be easily solved. Nevertheless, there are are some ways of manipulating **A** and **B** which will make us able to put the matrices on some very specific forms.

Proposition 3.1. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$-matrices. Let $\boldsymbol{P}$ be an invertible $m \times m$-matrix. Then solving the system*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\dot{\boldsymbol{x}} \tag{3.3}$$

is equivalent to solving the system

$$\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{P}\boldsymbol{B}\dot{\boldsymbol{x}}. \tag{3.4}$$

Proof: If equation 3.3 has a solution, the same solution will still solve the system when applying the same invertible matrix on both sides of the equation. If equation 3.4 has a solution, as $\mathbf{P}$ is invertible, we may apply the same invertible matrix, $\mathbf{P}^{-1}$, to both sides of the equation while maintaining the same solution, hence:

$$\mathbf{P}^{-1}\mathbf{P}\mathbf{A}\mathbf{x} = \mathbf{P}^{-1}\mathbf{P}\mathbf{B}\dot{\mathbf{x}} \Rightarrow \mathbf{A}\mathbf{x} = \mathbf{B}\dot{\mathbf{x}}$$

and thus, equation 3.3 has a solution. □

Proposition 3.2. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$-matrices. Let $\boldsymbol{Q}$ be an invertible $n \times n$-matrix. Then solving the system*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\dot{\boldsymbol{x}} \tag{3.5}$$

is equivalent to solving the system

$$\boldsymbol{A}\boldsymbol{Q}\boldsymbol{y} = \boldsymbol{B}\boldsymbol{Q}\dot{\boldsymbol{y}}. \tag{3.6}$$

Proof: For any matrix $\mathbf{M}$, we have that differentiating a vector, $\mathbf{x}$, and then applying the matrix is the same as applying the matrix and then differentiating, $\mathbf{M}\dot{\mathbf{x}} = (\dot{\mathbf{M}\mathbf{x}})$. Let $\mathbf{Q}$ be an invertible $n \times n$ matrix, and let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$-matrices. The following systems are thus equivalent:

$$\mathbf{A}\mathbf{x} = \mathbf{B}\dot{\mathbf{x}}, \tag{3.7}$$

$$\mathbf{A}\mathbf{Q}\mathbf{Q}^{-1}\mathbf{x} = \mathbf{B}\mathbf{Q}\mathbf{Q}^{-1}\dot{\mathbf{x}},$$

$$\mathbf{A}\mathbf{Q}\mathbf{Q}^{-1}\mathbf{x} = \mathbf{B}\mathbf{Q}(\dot{\mathbf{Q}^{-1}\mathbf{x}}),$$

$$\mathbf{A}\mathbf{Q}\mathbf{y} = \mathbf{B}\mathbf{Q}\dot{\mathbf{y}}. \tag{3.8}$$

The last system is obtained by the substitution $\mathbf{x} = \mathbf{Q}\mathbf{y}$, and hence, we have solutions to the system of equation 3.6 if and only if we have solutions to the system of equation 3.5. The matrix describing the relationship between $\mathbf{x}$ and $\mathbf{y}$ is an invertible matrix, which is an isomorphism, thus the solutions for $\mathbf{x}$ are isomorphic to the solutions for $\mathbf{y}$. □

Proposition 3.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be linear transformations from k^n to k^m. Then the solutions to the system*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\dot{\boldsymbol{x}}$$

is independent of the choice of basis for k^n and k^m.

Proof: This follows immediately from proposition 3.1 and proposition 3.2, as a change of basis in k^l corresponds to multiplication by an invertible $l \times l$-matrix. □

Remark. Let $\mathbf{A}$ and $\mathbf{B}$ be the matrices in a system on the form of equation 3.2. As the solutions of a system is independent of the choice of basis, we may change our bases to obtain the matrices $\mathbf{A}'$ and $\mathbf{B}'$ on very specific forms, and solve the system given by these matrices instead. This is an interesting approach in theory, although actually finding the changes of bases required may be difficult in practice.

3.2 Matrix decomposition

Definition. Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$-matrices describing linear transformations from a vector space k^n to a vector space k^m. The matrices are *simultaneously decomposable* if there exists a change of bases in k^n and k^m such that we may write $\mathbf{A} \simeq \mathbf{A}_1 \oplus \mathbf{A}_2$ and $\mathbf{B} \simeq \mathbf{B}_1 \oplus \mathbf{B}_2$, where $\mathbf{A}_i$ and $\mathbf{B}_i$ are both $m_i \times n_i$-matrices for $i \in \{1, 2\}$.

Remark. A direct sum of matrices, $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ is a block diagonal matrix such that:

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{A}_1 & 0 \\ \hline 0 & \mathbf{A}_2 \end{array}\right)$$

Remark. Let $\mathbf{A}$ be a linear transformation between vector spaces V_1 and V_2. For the remainder of this chapter, we will use the terminology that a pair of matrices $(\boldsymbol{A}, \boldsymbol{B})$ *contains a summand* $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ to describe that $\mathbf{A}$ and $\mathbf{B}$ are simultaneously decomposable in such a way that $\mathbf{A}_1$ is a summand of $\mathbf{A}$, and $\mathbf{B}_1$ is a summand of $\mathbf{B}$.

Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$-matrices, that is, linear maps from a vector space k^n to a vector space k^m. Now, the 4-tuple (k^n, k^m, A, B) corresponds to a representation of the quiver Q over a field k, and hence, this representation is on the form given by theorem 2.6. As this is the case, we see that the pair of matrices $(\mathbf{A}, \mathbf{B})$ contains a finite number of summands, each summand being an element in one of three classes D_1, D_2 or D_3:

$$\begin{aligned} D_1 &= \{(i_n\ 0), (0\ i_n)\}, n \in \mathbb{N} \\ D_2 &= \{\left(\begin{smallmatrix} i_m \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ i_m \end{smallmatrix}\right)\}, m \in \mathbb{N} \\ D_3 &= \{i_l, JB_l^\lambda\}, l \in \mathbb{N}, \lambda \in k \cup \{\infty\}, \end{aligned}$$

where $\lambda = \infty$ corresponds to the pair JB_l^0, i_l.
In other words, there exists a way to simultaneously decompose $\mathbf{A}$ and $\mathbf{B}$, such that instead of solving the system

$$\mathbf{Ax} = \mathbf{B\dot{x}},$$

we can choose to solve a finite number of systems on the forms:

$$\begin{aligned} (i_n\ 0)\mathbf{x}_n &= (0\ i_n)\dot{\mathbf{x}}_n, n \in \mathbb{N}, \\ \left(\begin{smallmatrix} i_m \\ 0 \end{smallmatrix}\right)\mathbf{x}_m &= \left(\begin{smallmatrix} 0 \\ i_m \end{smallmatrix}\right)\dot{\mathbf{x}}_m, m \in \mathbb{N}, \\ i_l\mathbf{x}_l &= JB_l^\lambda\dot{\mathbf{x}}_l, l \in \mathbb{N}. \end{aligned}$$

Here, $\mathbf{x}_i$ denotes the vector of unknowns of dimension $i \times 1$, $\forall i \in \mathbb{N}$. We will consider what happens to the partial systems containing each of these classes of summands in order to get information about the general system.

3.3 Preliminary considerations

Let $(\mathbf{A}, \mathbf{B})$ be a pair of $m \times n$-matrices.

1. If $n > m$, the pair of matrices must contain direct summands in the class D_1.

2. If $m > n$, the pair of matrices must contain direct summands in the class D_2.

Thus, if we show that one of these summands yields systems that are unsolveable or have infinitely many solutions, the same will be the case for any pair of matrices containing such a summand.
For the case $m = n$, there are three different alternatives:

a) The pair of matrices may be written as a direct sum of an equal number of summands in the classes D_1 and D_2.

b) The pair of matrices may be written as a direct sum of summands in the class D_3 alone.

c) The pair of matrices may be written as a direct sum of an equal number of summands in the classes D_1 and D_2, *and* summands in the class D_3.

As either one of these may be the case, the fact that the matrices are square matrices does not provide specific information about the summands of the pair of matrices.

3.4 Direct summand in the class D_1

The direct summands in the first class, D_1, corresponding to modules on the form of class N_1, yields systems of infinitely many solutions, as information about one of the unknows are lost due to the matrices having rank less than the dimension of the codomain. We can see this by solving the system for a summand in the class D_1 explicitly:

$$\mathbf{Ax} = \mathbf{B\dot{x}} \Rightarrow \begin{pmatrix} i_n & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & i_n \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_{n+1} \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x'_2 \\ x'_3 \\ \vdots \\ x'_{n+1} \end{pmatrix}$$

$$\Rightarrow \text{System dependent upon unknown } x_{n+1}$$

$$\Rightarrow \text{Infinitely many solutions to the system. No information about } x_{n+1}.$$

Thus, if the initial pair of matrices contains direct summands corresponding to the class N_1, we can not have a uniquely determined solution to any initial value problem on this form.

3.5 Direct summand in the class D_2

Direct summands in the second class, D_2 yields only trivial solutions, as

$$\mathbf{Ax} = \mathbf{B\dot{x}} \Rightarrow \begin{pmatrix} i_n \\ 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ i_n \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1' \\ \vdots \\ x_{n-1}' \\ x_n' \end{pmatrix} \Rightarrow \mathbf{x} = \vec{0}$$

Hence, the summands in this class yield only trivial solutions to their part of the differential equation.

3.6 Direct summand in the class D_3

The third class of direct summands, summands in the class D_3, gives us completely determined systems for any eigenvalue, $\lambda \in k \setminus \{0\}$.

$$i_n\mathbf{x} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \dot{\mathbf{x}} \Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1' + x_2' \\ \vdots \\ \lambda x_{n-1}' + x_n' \\ \lambda x_n' \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} C_1 t^{n-1} e^{\lambda t} + C_2 t^{n-2} e^{\lambda t} + \cdots + C_n e^{\lambda t} \\ C_2 t^{n-2} e^{\lambda t} + C_3 t^{n-3} e^{\lambda t} + \cdots + C_n e^{\lambda t} \\ \vdots \\ C_{n-1} t e^{\lambda t} + C_n e^{\lambda t} \\ C_n e^{\lambda t} \end{pmatrix}$$

Here, the coefficients C_i for $i \in \{1, ..., n\}$ corresponds to solutions of initial value problems.
For the eigenvalue $\lambda = 0$, the system "shifts" the position of the derivatives, such as for summands in the class D_1, except for the last coordinate, which instead of an unknown becomes zero, and the solutions become the trivial solution.

$$i_n\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \dot{\mathbf{x}} \Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_2' \\ \vdots \\ x_n' \\ 0 \end{pmatrix} \Rightarrow \mathbf{x} = \vec{0}.$$

For the eigenvalue $\lambda = \infty$, all the information available is that every value is dependent upon an unknown, x_1, where $x_1^{(n)} = 0$.

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \mathbf{x} = i_n \dot{\mathbf{x}} \Rightarrow \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ 0 \end{pmatrix} = \begin{pmatrix} x'_1 \\ \vdots \\ x'_{n-1} \\ x'_n \end{pmatrix}$$

$$\Rightarrow x_n \text{ is a polynomial of degree at most } n-1.$$

$$\mathbf{x} = \begin{pmatrix} a_1 \\ a_1 x + a_2 \\ \vdots \\ a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \end{pmatrix}$$

However, even as we see that the summands in the class D_3 of the system is completely solveable in theory, this approach assumes that we are able to find the decomposition of the initial pair of matrices such that we find all summands in the class D_3. This is the same as assuming that we are able to put an arbitrary matrix, $\mathbf{M}$, on Jordan canonical form.

3.6.1 Rational canonical form

In order to obtain a Jordan canonical form of a matrix, $\mathbf{M}$, we must be able to find it's eigenvalues, which may be very hard in principle. When dealing with pairs of matrices, $(\mathbf{A}, \mathbf{B})$, containing summands in the class D_3, we may use this information to construct another kind of simultaneous decomposition of $\mathbf{A}$ and $\mathbf{B}$. As the pair $(\mathbf{A}$,$\mathbf{B})$ contains a summand in the class D_3, we know that there exists directs summands $\mathbf{A}'$ in $\mathbf{A}$ and $\mathbf{B}'$ in $\mathbf{B}$, such that $\mathbf{A}'$ and $\mathbf{B}'$ are two square matrices. We also know that at least one of $\mathbf{A}'$ and $\mathbf{B}'$ is of full rank.

Assuming $\mathbf{B}'$ is of full rank

Assume $\mathbf{B}'$ is of full rank. Then solving the system

$$\mathbf{A}'\mathbf{x} = \mathbf{B}'\dot{\mathbf{x}}$$

is the same as solving the system

$$\mathbf{A}''\mathbf{x} = \mathbf{B}''\dot{\mathbf{x}},$$

where $\mathbf{A}''$ and $\mathbf{B}''$ are matrices such that $\mathbf{B}''$ is on the form of an identity matrix, and matrix $\mathbf{A}''$ is the rational canonical form of $\mathbf{A}'$.[1] By this construction, we avoid the problem of obtaining a matrix on the form of a Jordan block. As the rational canonical form is obtained through conjugation by a pair of matrices $\mathbf{M}$, $\mathbf{M}^{-1}$, the change of basis required to put $\mathbf{A}'$ in the rational canonical form will not distort the identity matrix, as the identity matrix is preserved under conjugation. Assume that the minimal polynomial $m(x)$ of $\mathbf{A}'$, is equal to the characteristic polynomial $f(x)$ of $\mathbf{A}'$. When this is the case, the rational canonical form is in the form of the companion matrix of $f(x)$. Thus, we get the system of linear differential equations as follows:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \mathbf{x} = i_n \dot{\mathbf{x}} \Rightarrow \begin{pmatrix} -a_0 x_n \\ x_1 - a_1 x_n \\ \vdots \\ x_{n-1} - a_{n-1} x_n \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}$$

This gives the recursive relations:

$$x_{n-1} = a_{n-1} x_n + x_n'$$

$$\Rightarrow x_{n-2} = a_{n-2} x_n + x_{n-1}' = a_{n-2} x_n + a_{n-1} x_n' + x_n''$$

$$\mathbf{x} = \begin{pmatrix} x_n^{(n-1)} + \sum_{i=1}^{n-1} a_{n-i} x_n^{(n-1-i)} \\ \vdots \\ a_{n-1} x_n + x_n' \\ x_n \end{pmatrix}$$

Considering x_1 in this way, we get:

$$x_1' = -a_0 x_n = \frac{d}{dt} \left(x_n^{(n-1)} + \sum_{i=1}^{n-1} a_{n-i} x_n^{(n-1-i)} \right)$$

Which shows that if $a_0 = 0$, x_n is constant, and this corresponds to the solutions of the system with summands in the class D_3 where $\lambda = \infty$. If $a_0 \neq 0$, 0 is not a root in the characteristic polynomial of $\mathbf{A}'$, and thus, $\mathbf{A}'$ is of full rank, as the eigenvalue of the Jordan block corresponding to $\mathbf{A}'$ is non-zero. If this is the case, these solutions are included in the solutions obtained in the next part.

[1] The definition of the rational canonical form of a matrix is omitted in this thesis, but the reader is referred to [1] for the definition used in this section.

Assuming $\mathbf{A}'$ is of full rank

Assume instead that $\mathbf{A}'$ is of full rank. Now a change of basis in the vector spaces gives us a matrix $\mathbf{A}''$ which is an identity matrix, and a matrix $\mathbf{B}''$ which is the rational canonical form of $\mathbf{B}'$. By assuming that the minimal polynomial of $\mathbf{B}'$ is equal to the characteristic polynomial of $\mathbf{B}'$, we get the system:

$$i_n\mathbf{x} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \dot{\mathbf{x}} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -a_0 x_n' \\ x_1' - a_1 x_n' \\ \vdots \\ x_{n-1}' - a_{n-1} x_n' \end{pmatrix}$$

$$\Rightarrow x_j = -\sum_{i=0}^{j-1} a_i x_n^{(j-i)}$$

$$\Rightarrow x_n = -a_{n-1}x_n' - \sum_{i=0}^{n-2} a_i x_n^{(n-i-2)}$$

If $a_0 = 0$, the characteristic polynomial has a root $\lambda = 0$, and as it is possible to put $\mathbf{B}'$ on the form of a Jordan block, the characteristic polynomial of $\mathbf{B}'$ has exactly one root. This implies that $a_i = 0$ for all $i \in \{1, ..., n-1\}$, and thus, this yields only trivial solutions to the system, corresponding to summands in D_3 with eigenvalue $\lambda = 0$. If $a_0 \neq 0$, the recursive formula given, yields solutions of the form:

$$\Rightarrow x_j = \sum_{i=0}^{j} C_i t^{(n-i)} e^{\gamma t}$$

But as these solutions are isomorphic to the solutions given by solving the system with $\mathbf{B}''$ as a Jordan block, where $\lambda \in k \setminus \{0\}$, we deduce that finding the unknown, γ, corresponds to finding the eigenvalue of the Jordan block. Finding the solutions of the rational canonical form is equivalent to finding the eigenvalue of the Jordan block, and we have not managed to reduce the problem further.

Remark. Here, we assumed that the minimal polynomial was equal to the characteristic polynomial of the matrix in order to keep our computations simple. However, if the characteristic polynomial is not equal to the minimal polynomial, we obtain a block diagonal matrix which have blocks of the same form as the one we just computed. In this case, we would by the same reasoning deduce that solving the system corresponds to finding the eigenvalue of the Jordan block.

3.7 Final remarks

We have seen that the representations over the 2-Kronecker quiver is intimately related to the problem of decomposing a general system of linear differential equations. However, this approach is only feasible in principle, as obtaining a simultaneous decomposition of a pair of matrices is a highly non-trivial task in itself. Even if we were able to find the number of summands on the given form, and their dimensions, solving the system we are left with is equivalent to finding the eigenvalues of arbitrary quadratic matrices, which in turn is equivalent to finding all roots of the characteristic polynomial of an arbitrary matrix. So in conclusion, the connection discovered between solving systems of differential equations and the representations of the 2-Kronecker quiver, though interesting and somewhat surprising, does not help us solve the systems of linear equations in general, as finding the roots of a general polynomial of degree higher than or equal to 5 is not possible, by the Abel-Ruffini theorem, see [3, p. 470].

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