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An Introduction to Distributions and Sobolev Spaces, and a Study of the Fractional Partial Differential Operator $1 + (-\Delta)^{s/2}$

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Master of Science in Mathematics
Submission date: December 2013
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December 18, 2013

Abstract

In this thesis we derive the basic theory for distributions, fractional and classical Sobolev spaces and the direct method of variations, and apply this theory to discuss solutions of $(1+(-\Delta)^{\frac{s}{2}})u = f$ for $s \in (0,2)$ on both bounded open sets and all of $\mathbb{R}^n$. We use both the direct methods of variations and Lax-Milgram to give sufficient conditions on f for existence of sobolev and distributional solutions. We also discuss the spectrum of the operator $1+(-\Delta)^{\frac{s}{2}}$ and give a characterisation of all eigenvalues and eigenfunctions on bounded open sets.

Sammendrag

I denne oppgaven utleder vi den grunnleggende teorien for distribusjoner, fraksjonelle og klassiske Sobolevrom og variasjonskalkulus, og anvender denne teorien til diskutere løsninger av ligningen $(1+(-\Delta)^{\frac{s}{2}})u = f$ for $s \in (0,2)$ på begrensede, åpne mengder, samt hele $\mathbb{R}^n$. Vi bruker både variasjonskalkulus og Lax-Milgram til å gi tilstrekkelige krav på f for eksistensen av Sobolev– og distribusjonsløsninger. Vi diskuterer også spektrumet til operatoren $1 + (-\Delta)^{\frac{s}{2}}$ og gir en karakterisering av alle egenverdier og egenfunksjoner på begrensede, åpne mengder.

Preface

This thesis is written throughout the year 2013 in order to complete the degree Master of Mathematics at the Norwegian University of Science and Technology.

The topic of the thesis was chosen more or less as a "let's try something new" adventure. Before starting this endeavour I have had several courses in algebra, topology and real analysis, but with regards to the themes of the thesis, I had only a course in Fourier analysis, where distributions were introduced at a basic level, and an abstract course in functional analysis (where there never was any mention of differential equations). Sobolev spaces were known only by the most basic definition, and the modern field of partial differential equations was truly *terra incognita*. Stepping into the office of my supervisor, Førsteamanuensis/Associate Professor Mats Ehrnström, and telling him that I wanted to write a thesis about Fourier analysis, Distributions and Sobolev spaces, with particular stress on the latter, in the area partial differential equations, I had little idea where I was going. Fortunately, he did, and early in the process of writing the thesis it became clear that I will go on to do a PhD, and then the goal of the thesis became largely a preparation: to learn relevant theory and how to apply it, and how to convey results to the reader. The work started vigorously in January 2013, and with the exception of an exceptionally long summer break, went on throughout the year.

I owe a great deal of gratitude to my supervisor Mats Ehrnström for his thorough guidance in the writing of this thesis, and I would also like to give special thanks to Anastasiia Sergeevna Tkalich, who gave me the power and motivation to continue my work when it came to a halt, and to Sunniva Bøe, who supported me most of the way.

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Introduction

In many problems of both theoretical and applied mathematics it is not sufficient to consider only classical solutions of differential equations in order to find solutions. This led to the introduction of distributions which took its modern form during the first half of the 20th century [18]. Distributions were first introduced by Sergey Lvovich Sobolev in the 1930's [19], who simply called them functionals. The concept was further developed by Laurent Schwartz during the 1940's where he gave a more complete theory of distributions, and also introduced the name *distribution* [18]. Sobolev introduced distributions as a tool for finding solutions to partial differential equations [19] and since their introduction, distributions has been used with great success in the theory of differential equations. Another important tool is Sobolev spaces, which, like distributions, have become a large field during the 20th century, and is connected to distributions: a Sobolev space is a subspace of an L^p space such that the distributional derivatives of the elements, up to some order, is contained in the L^p space [20]. Combining both integrability and differentiability criteria, these spaces have very suitable compactness properties and are natural homes for weak (and strong) solutions of partial differential equations. These concepts were first introduced with more classical types of differential equations in mind, but has been adapted for fractional differential equations. In fact, the history of fractional derivatives is more ancient than that of distributions and Sobolev spaces, and it started from a note by Leibniz [14] where he discusses the meaning of derivatives of order one half, which led to the development of a theory of derivatives of arbitrary order. However, for centuries fractional derivatives was of a purely theoretical interest, but in the latter part of the 20th century it was discovered that many physical problems are better described by fractional differential equations (for some examples of applications there are several textbooks devoted to it, for instance [16]), and it is an area of active research both purely theoretically and with regards to applications.

In this thesis we provide an almost self-contained treatment of the basic theory of tempered distributions, the Fourier transform and Sobolev spaces, both fractional and integer order, and use this theory to define fractional derivatives and discuss the fractional differential operator $1 + (-\Delta)^{\frac{s}{2}}$ for $s \in (0, 2)$. In Chapter 1 we first derive the main properties of the Schwartz space and the Fourier transform. In particular we prove the invariance of the Schwartz space under the Fourier transform, as well as how to extend the Fourier transform from the Schwartz space to $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, with special emphasis on $L^2(\mathbb{R}^n)$, on which we prove that the Fourier transform is a unitary operator. The important tool of mollification which we use extensively in the first chapters is also introduced. With this foundation, we introduce tempered distributions and give sufficient criteria for a function to be a distribution, and show that there exists distributions which are not functions. We define distributional differentiation and the Fourier transform on tempered distributions, proving that tempered distributions are invariant under it. The space $\mathscr{D}'(\Omega)$ is also introduced, but we restrict ourselves to noting that it is a superspace of tempered distributions, and give sufficient criteria for a function to be an element of it. All results are stated with complete proofs, except for the fact that smooth, compactly supported functions are dense in the space of distributions, which we do not need in this thesis but state because of its fundamental importance in the theory of distributions. Many of the proofs are done independently by the author, while the rest are based on [9] and [10].

Distributional differentiation as defined in Chapter 1 provides the tools to give the definition of classical Sobolev spaces $W_p^k(\mathbb{R}^n)$ for $1 \leq p < \infty$ and all non-negative integers k in Chapter 2. Some basic properties of these spaces are proved, before the attention is turned to fractional Sobolev spaces. By using the unitarity of the Fourier transform on $L^2(\mathbb{R}^n)$, we prove that the Fourier transform maps weighted L^2 spaces with weight $(1 + |x|^2)^{k/2}$ unitarily onto $W_2^k(\mathbb{R}^n)$. The spaces $H^s(\mathbb{R}^n)$ then occur by interpolating between these spaces by continuously varying the exponent of the weights. An alternate definition of fractional Sobolev spaces of positive order, denoted by $W_2^s(\mathbb{R}^n)$, is given by means of singular integrals, inspired by Hölder continuity, and it is proved that $H^s = W_2^s$. Lastly we give a proof of the Sobolev embedding theorem, that is, the embedding of Sobolev spaces into spaces of bounded, continuous functions, for $W_2^s(\mathbb{R}^n)$. A counter-example for the critical exponent $s = l + n/2$, where l is the order of the space embedded into, is given at the end of the chapter.

In Chapter 3 we define Sobolev spaces, both fractional and classical, on arbitrary open sets as the set of all distributions on the set which is the restriction of some element in $W_p^s(\mathbb{R}^n)$ or $H^s(\mathbb{R}^n)$ to the set, with the norm being defined as the infimum of the $W_p^k(\mathbb{R}^n)$ or $H^s(\mathbb{R}^n)$ norm of all such elements. We prove that these spaces inherit the basic properties of Sobolev spaces on $\mathbb{R}^n$, before we turn to Sobolev spaces on $\mathbb{R}^n_+$, on which we prove an extension theorem to Sobolev spaces on $\mathbb{R}^n$. This proof is done by using partitions of unity, and extending smooth functions and using their density in the Sobolev spaces. Then, in Chapter 4, we turn to Sobolev spaces on bounded, open sets, and by partitions of unity, many situations may be reduced to the case on $\mathbb{R}^n_+$, and so it is straightforward to extend the extension theorem to Sobolev spaces on bounded open sets with sufficiently smooth boundary. All results on Sobolev spaces up until this point are, in large part, based on [10], with alterations done and additional details and remarks given by the author where deemed helpful for the reader. The most important results in this chapter is the compactness and embedding result, the most important of which being Rellich-Kondrachov and an extension of it to fractional spaces of order $0 < s < 1$. The proof of Rellich-Kondrachov is taken from [7]. The chapter is completed by introducing traces and the spaces $W_{p,0}^k(\Omega)$ and $H_0^s(\Omega)$, that is the Sobolev spaces where all the elements have zero trace on the boundary, and proving the Poincaré inequality.

The last background theory we develop is the direct methods of variations. In Chapter 5 we introduce the idea behind the method with an example and some discussions of the challenges, and we prove a very general theorem giving sufficient conditions for a functional depending on the spatial coordinate, a function $u \in W_p^1(\Omega)$ and its gradient to attain its infimum on $W_p^1(\Omega)$ for any open set $\Omega \subseteq \mathbb{R}^n$ (cf. Theorem 5.3). This theorem and its proof is taken from [24], with minor alterations. The other results are formulated and proved independently by the author, with some inspiration for the general exposition taken from [7]. We finish the chapter by applying the direct methods of variations to a classical problem.

Lastly, in Chapter 6, we apply the theory of the first five chapters to fractional derivatives and operators. Using a result about the Fourier transform of the derivative of a function, we give a definition of fractional differentiation by means of the Fourier transform; a definition which, from our results in Chapter 1, holds for all elements in the space of tempered distributions. Then we introduce the operator $1 + (-\Delta)^{\frac{s}{2}}$, $s \in (0, 2)$, and show its intimate relationship with the space H^s. The rest of Chapter 6 is divided into two main parts: the first part is devoted to the existence of solutions to the equation $(1 + (-\Delta)^{\frac{s}{2}})u = f$ on open sets as well as all of $\mathbb{R}^n$, and the second part to the spectral theory for this operator.

For $f \in H^{-s/2}(\Omega)$, we prove that there exists a solution $u \in H_0^{s/2}(\Omega)$ to the equation $(1 + (-\Delta)^{\frac{s}{2}})u = f$ in Ω, with $u = 0$ outside Ω for all bounded, open sets Ω. This is proved in two ways: by using the direct methods of variations, and by using the Lax-Milgram theorem. On $\mathbb{R}^n$, we prove existence of solutions with even weaker assumptions on f; if $f \in H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, then there exists a solution $u \in H^{r+s}(\mathbb{R}^n)$. This is proved directly, by constructing a solution, and in addition we find a fundamental solution to $(1+(-\Delta)^{\frac{s}{2}})u = f$. We do not claim that these results are new (indeed, we even comment on a more powerful result that has been proved, although by different methods than those we employ), but all theorems are formulated and proved independently by the author.

Finally we turn to the eigenvalue problem. We first give a simple proof of the fact that there are no non-trivial eigenvalues of $1 + (-\Delta)^{\frac{s}{2}}$ on $\mathbb{R}^n$. Therefore we turn our attention to open, bounded sets and prove that in this case there exists a countably infinite number of distinct, positive eigenvalues that form a sequences that diverges to infinity. Furthermore, the eigenfunctions form a orthogonal basis for both $L^2(\Omega)$ and $H_0^{s/2}(\Omega)$. The main method of the proof of existence is the direct method of variations. The proof of the existence of an eigenvalue on bounded domains is done independently by the author, while the rest of the section owes a lot to the article [23], which proves all our results for a general class of fractional, elliptic operators that are intimately related to $1 + (-\Delta)^{\frac{s}{2}}$.

1 Tempered Distributions and their Fourier Transforms

Perhaps the most obvious attribute of a function satisfying a differential equation is differentiability. In solving the kinds of differential equations we will consider in this paper, we also rely heavily on the Fourier transform as introduced below. Unfortunately, the set of Fourier transformable functions seems at first sight very small, and the set of functions that are also differentiable even smaller. This restriction is unfortunate not only in a theoretical sense, but also in a practical sense: it may be physically feasible to consider solutions that are not differentiable in the classical way. This problem has been solved by introducing the concepts of distributions, which is a generalisation of functions, and weak and distributional differentiation. We collect the basic theory in this chapter.

1.1 The Schwartz Space

Notation: Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_0^n = \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) : \alpha_i \in \mathbb{N}_0, 1 \leq i \leq n\}$, the set of all multi-indices of length n. For $\alpha \in \mathbb{N}_0^n$, we define the norm $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. For $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, we write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}}$.

Notation We will by *domain* mean an open but not necessarily bounded subset of $\mathbb{R}^n$.

Definition 1.1. *The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is the set of all complex-valued functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \mathrm{D}^\beta \varphi(x)| < \infty \tag{1.1}$$

holds for all $\alpha, \beta \in \mathbb{N}_0^n$.

Definition 1.2. *We define a family of seminorms on $\mathscr{S}(\mathbb{R}^n)$:*

$$P_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\alpha \mathrm{D}^\beta \varphi(x)|. \tag{1.2}$$

From these semi-norms we can define a topological structure on $\mathscr{S}$. We say that a sequence φ_n in $\mathscr{S}$ converges to φ if

$$P_{\alpha,\beta}(\varphi_n - \varphi) \to 0 \tag{1.3}$$

for all α, β as above. This is denoted by $\varphi_n \underset{\mathscr{S}}{\longrightarrow} \varphi$.

Remark: The limit φ is unique, since (1.3) implies uniform convergence on $\mathbb{R}^n$.

Lemma 1.3. *The space $\mathscr{S}$ is invariant under differentiation and multiplication by polynomials.*

Proof. This is immediate from Definition 1.1. □

Definition 1.4. *For a an open set $\Omega \subset \mathbb{R}^n$, $1 \leq p < \infty$, $L^p(\Omega)$ denotes the Banach space normed by*

$$\|f\|_{p,\Omega} = \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p}. \tag{1.4}$$

If it is clear which space we are talking about, we will drop the Ω in the subscript and just write $\| \cdot \|_p$. The elements of $L^p(\Omega)$ are equivalence classes of measurable functions $f : \Omega \to \mathbb{C}$ for which the norm above is finite, and f is equivalent to g if $f = g$ a.e. in the Lebesgue measure. We call measurable, complex-valued functions f on Ω such that $\|f\|_p < \infty$ for p-integrable. For $p = \infty$, $L^p(\Omega)$ is the space of all essentially bounded, measurable, complex-valued functions on Ω, with the ess supp *norm. $L^p_{loc}(\Omega)$ is the space of all locally p-integrable functions on Ω, i.e. $f \in L^p_{loc}(\Omega)$ if $f \in L^p(K)$ for all $K \Subset \Omega$, that is, for all K with compact support in Ω.*

Remark: We will generally identify an equivalence class $[f] \in L^p(\Omega)$ with a representative function and refer to the elements of $L^p(\Omega)$ as functions.

Lemma 1.5. *The mapping*

$$\varphi \mapsto [\varphi], \quad \varphi \in \mathscr{S}(\mathbb{R}^n) \tag{1.5}$$

defines a continuous embedding $\mathscr{S}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$.

Proof. The case $p = \infty$ is immediate from Definition 1.1. We first prove that $\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. For any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have $\varphi(x) = (1 + |x|^{n+1})^{-1/p}(1 + |x|^{n+1})^{1/p}\varphi(x)$. Evaluating $\|\varphi\|_p$ yields

$$\begin{aligned} &\int_{\mathbb{R}^n} |(1 + |x|^{n+1})^{-1/p}(1 + |x|^{n+1})^{1/p}\varphi(x)|^p \, \mathrm{d}x \\ &\leq \sup_{x \in \mathbb{R}^n} |(1 + |x|^{n+1})\varphi(x)^p| \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+1}} \, \mathrm{d}x < \infty. \end{aligned} \tag{1.6}$$

The last inequality follows from the definition 1.1 and the fact that $(1 + | \cdot |^{n+1})^{-1} \in L^1(\mathbb{R}^n)$.

Next we prove the continuity of the embedding. Given $\varepsilon > 0$, $\varphi_n \underset{\mathscr{S}}{\longrightarrow} 0$ implies there exists an $N \in \mathbb{N}$ such that for all $k \geq N$ and $x \in \mathbb{R}^n$, $|(1 + |x|^{n+1})^{1/p}\varphi_k(x)| < \varepsilon$. Thus for all $k \geq N$,

$$\int_{\mathbb{R}^n} |\varphi_k(x)|^p \, \mathrm{d}x \leq \varepsilon^p \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^{n+1})} \, \mathrm{d}x. \tag{1.7}$$

This proves the result. ∎

Definition 1.6. *For an open set $\Omega \subseteq \mathbb{R}^n$ we define*

$$\mathscr{D}(\Omega) = \{\varphi \in C^\infty : \operatorname{supp} \varphi \Subset \Omega\}. \tag{1.8}$$

Let

$$\omega(x) = \begin{cases} ce^{\frac{-1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{1.9}$$

where c is chosen such that $\int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x = 1$. Set $\omega_h(x) = h^{-n}\omega(x/h)$ for $h > 0$. Then ω_h satisfies

$$\operatorname{supp} w_h = \{x : x \in \mathbb{R}^n, |x| \leq h\}, \qquad \int_{\mathbb{R}^n} \omega_h(x)\,\mathrm{d}x = 1. \tag{1.10}$$

It is easy to check that ω is infinitely differentiable.

Definition 1.7. *For $f \in L^p_{loc}(\mathbb{R}^n)$, $1 \leq p \leq \infty$ we define the mollification of f*

$$f_h(x) = (f * \omega_h)(x) = \int_{\mathbb{R}^n} \omega_h(x-y) f(y)\,\mathrm{d}y. \tag{1.11}$$

Theorem 1.8. *For any open set $\Omega \subset \mathbb{R}^n$, $\mathscr{D}(\Omega)$ is dense in $L^p(\Omega)$, $1 \leq p < \infty$. Thus $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.*

Proof. Clearly, $\mathscr{D}(\Omega) \subset L^p(\Omega)$, $1 \leq p < \infty$, for any open set $\Omega \subset \mathbb{R}^n$. Note that every function $f \in L^p(\Omega)$ can be approximated by step functions, $g = \sum_i a_i \chi_{A_i}$, where the A_i's are connected and compact sets (for instance, it is sufficient to only consider cubes). We proceed by mollification of characteristic functions of such sets. Let

$$(\chi_A)_h(x) = (\omega_h * \chi_A)(x) = \int_{\mathbb{R}^n} \omega_h(x-y)\chi_A(y)\,\mathrm{d}y \tag{1.12}$$

be the mollification of χ_A. We note that $(\chi_A)_h$ is infinitely differentiable (see Proposition 1.9 below) and equations (1.10) and (1.12) imply

$$\operatorname{supp}(\chi_A)_h = \{x : \inf_{a \in A} |x-a| \leq h\}. \tag{1.13}$$

If we choose h such that $\inf_{a \in A} |x-a| \leq h$ implies $x \in \Omega$, which is always possible since A is compact in Ω, then $(\chi_A)_h$ has compact support in Ω. In other words, $(\chi_A)_h \in \mathscr{D}(\Omega)$ for h sufficiently small. Furthermore, equations (1.10), (1.11) and (1.12) implies

$$(\chi_A)_h(x) = \chi_A(x) \quad \text{if} \quad \operatorname{dist}(x, \partial A) > h,\ x \in \mathbb{R}^n. \tag{1.14}$$

Let

$$S_h = \{x \in R^n : \operatorname{dist}(x, \partial A) \leq h\}. \tag{1.15}$$

Then equation (1.14) implies

$$\left(\int_{\mathbb{R}^n} |(\chi_A)_h(x) - \chi_A(x)|^p\,\mathrm{d}x\right)^{1/p} = \left(\int_{S_h} |(\chi_A)_h(x) - \chi_A(x)|^p\,\mathrm{d}x\right)^{1/p}. \tag{1.16}$$

Since both $(\chi_A)_h$ and χ_A are positive functions bounded by 1, we have $|(\chi_A)_h(x) - \chi_A(x)| \leq 1$ for all $x \in \mathbb{R}^n$. We then get

$$\left(\int_{\mathbb{R}^n} |(\chi_A)_h(x) - \chi_A(x)|^p\,\mathrm{d}x\right)^{1/p} \leq \left(\mathcal{L}^n(S_h)\right)^{1/p}, \tag{1.17}$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure. Clearly, $\mathcal{L}^n(S_h) \to 0$ as $h \to 0$, and so $(\chi_A)_h \to \chi_A$ in $L^p(\Omega)$. Thus we can approximate step functions, and therefore all p-integrable functions, by elements of $\mathscr{D}(\Omega)$. For the last part, we note that $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$, and so the result follows. □

We prove some properties of mollification, with focus on what will be needed later.

Proposition 1.9. *Let $f \in L^p_{loc}(\mathbb{R}^n)$, $1 \le p \le \infty$. Then the following holds:*

(i) f_h is infinitely differentiable.

(ii) If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$ then $\|f_h\|_p \le \|f\|_p$.

(iii) If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ then $f_h \to f$ in $L^p(\mathbb{R}^n)$.

Proof. It is not yet clear that the integral in (1.11) always exists for $1 < p < \infty$ (the cases $p = 1$, $p = \infty$ follows immediately from (1.10)). Let $1 < p < \infty$. Then

$$\begin{aligned} |f_h(x)| &\le \int_{\mathbb{R}^n} \omega_h(x-y)^{1/p'} \omega_h(x-y)^{1/p} |f(y)| \,\mathrm{d}y \\ &\le \left(\int_{\mathbb{R}^n} \omega_h(x-y) \,\mathrm{d}y \right)^{1/p'} \left(\int_{\mathbb{R}^n} \omega_h(x-y) |f(y)|^p \,\mathrm{d}y \right)^{1/p} \end{aligned} \tag{1.18}$$

where $1/p' + 1/p = 1$ and we used Hölder's inequality. By definition, the first integral in the second line of (1.18) is 1. Since $f \in L^p_{loc}(\mathbb{R}^n)$ and $\operatorname{supp} w_h$ is compact in $\mathbb{R}^n$, the second integral converges.

(i)

$$\frac{f_h(x + ae_i) - f_h(x)}{a} = \int_\Omega \frac{1}{a}(\omega_h(x + ae_i - y) - \omega_h(x-y)) f(y) \,\mathrm{d}y \tag{1.19}$$

where Ω is some open, bounded set (cf. (1.10)). Since

$$\frac{1}{a}(\omega_h(x + ae_i - y) - \omega_h(x-y)) \to \frac{\partial \omega_h}{\partial x_i}(x-y) \tag{1.20}$$

uniformly on Ω, one can for a small enough find an integrable, dominating function and use Lebesgue's dominated convergence theorem to get

$$\frac{\partial f_h}{\partial x_i}(x) = \int_{\mathbb{R}^n} \frac{\partial \omega_h}{\partial x_i}(x-y) f(y) \,\mathrm{d}y. \tag{1.21}$$

By similar arguments,

$$\mathrm{D}^\alpha f_h(x) = \int_{\mathbb{R}^n} \mathrm{D}^\alpha \omega_h(x-y) f(y) \,\mathrm{d}y, \quad \text{for any } \alpha \in \mathbb{N}_0^n \tag{1.22}$$

(ii) Equation (1.18) implies, for $1 \le p < \infty$,

$$\begin{aligned}\int_{\mathbb{R}^n} |f_h(x)|^p \,\mathrm{d}x &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \omega_h(x-y)|f(y)|^p \,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} |f(y)|^p \int_{\mathbb{R}^n} \omega_h(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} |f(y)|^p \,\mathrm{d}y.\end{aligned} \tag{1.23}$$

If $p = \infty$, then

$$|f_h(x)| \le \|f\|_\infty \int_{\mathbb{R}^n} \omega_h(x-y)\,\mathrm{d}y = \|f\|_\infty, \quad x \in \mathbb{R}^n. \tag{1.24}$$

This proves (ii).

(iii) Since $\int_{\mathbb{R}^n} \omega_h(x)\,\mathrm{d}x = 1$, we may write

$$\begin{aligned}f_h(x) - f(x) &= \int_{\mathbb{R}^n} \omega_h(x-y)(f(y)-f(x))\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \omega_h(x-y)^{1/p'}\omega_h(x-y)^{1/p}(f(y)-f(x))\,\mathrm{d}y\end{aligned} \tag{1.25}$$

where p' is such that $1/p + 1/p' = 1$. Here we use the convention that if $p = 1$, then $p' = \infty$ and $1/p' = 0$. Then, by Hölder's inequality

$$|f_h(x) - f(x)| \le \left(\int_{\mathbb{R}^n} \omega_h(x-y)\,\mathrm{d}y\right)^{1/p'} \left(\int_{\mathbb{R}^n} \omega_h(x-y)|f(y)-f(x)|^p\,\mathrm{d}y\right)^{1/p}. \tag{1.26}$$

By definition, the first integral on the right-hand side of (1.26) is 1. Hence

$$\begin{aligned}\int_{\mathbb{R}^n} |f_h(x) - f(x)|^p \,\mathrm{d}x &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \omega_h(x-y)|f(y)-f(x)|^p \,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{|z|\le h} \omega_h(z) \int_{\mathbb{R}^n} |f(y+z)-f(y)|^p \,\mathrm{d}y\,\mathrm{d}z \\ &\le \left(\sup_{|z|\le h} \int_{\mathbb{R}^n} |f(y+z)-f(y)|^p\,\mathrm{d}y\right) \int_{|z|\le h} \omega_h(z)\,\mathrm{d}z \\ &= \sup_{|z|\le h} \int_{\mathbb{R}^n} |f(y+z)-f(y)|^p\,\mathrm{d}y.\end{aligned} \tag{1.27}$$

Now it remains to prove that the final equation in (1.27) converges to zero as h goes to zero. Given $\varepsilon > 0$, we know from Theorem 1.8 that there exists a function $g \in \mathscr{D}(\mathbb{R}^n)$ such that $\|f - g\|_p < \varepsilon/3$. Since g has compact support and is uniformly continuous in $\mathbb{R}^n$, there is a $\delta > 0$ such that

$$\|g(\cdot + z) - g(\cdot)\|_p < \varepsilon/3, \quad |z| \le \delta. \tag{1.28}$$

Using the triangle inequality we get

$$\begin{aligned}\|f(\cdot+z)-f(\cdot)\|_p &= \|f(\cdot+z)-g(\cdot+z)+g(\cdot+z)-g(\cdot)+g(\cdot)-f(\cdot)\|_p \\ &\le \|f(\cdot+z)-g(\cdot+z)\|_p+\|g(\cdot+z)-g(\cdot)\|_p+\|f-g\|_p \\ &< \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon, \quad |z|\le\delta\end{aligned} \tag{1.29}$$

This implies

$$\sup_{|z|\le h}\int_{\mathbb{R}^n}|f(y+z)-f(y)|^p\,\mathrm{d}y\to 0, \quad \text{as} \quad h\to 0. \tag{1.30}$$

□

1.2 The Fourier Transform

Definition 1.10. *Let $f\in L^1(\mathbb{R}^n)$. Then*

$$\mathscr{F}f(\xi)=\widehat{f}(\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(x)e^{-ix\cdot\xi}\,\mathrm{d}x \tag{1.31}$$

is called the Fourier Transform of f. We also define

$$\check{\mathscr{F}}f(\xi)=\widehat{f}(-\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(x)e^{ix\cdot\xi}\,\mathrm{d}x. \tag{1.32}$$

Here $x\cdot\xi$ in the exponential is the regular scalar product in $\mathbb{R}^n$.

Remark: Lemma 1.5 ensures that the Fourier transform is defined for all $\varphi\in\mathscr{S}(\mathbb{R}^n)$.
Remark: The convention adopted here (in terms of placement of the factor 2π) is chosen because we want $\mathscr{F}$ to be a unitary operator on $L^2(\mathbb{R}^n)$ (cf. Theorem 1.16) and because of perceived ease of bookkeeping in the following results.

Proposition 1.11. *Let $f,g\in L^1(\mathbb{R}^n)$. Then the following holds:*

(i) $\mathscr{F}[f(x-a)](\xi)=e^{-ia\xi}\mathscr{F}f(\xi)$

(ii) $\mathscr{F}[e^{iax}f(x)](\xi)=\mathscr{F}f(\xi-a)$

(iii) $\int_{\mathbb{R}^n}f(x)\widehat{g}(x)\,\mathrm{d}x=\int_{\mathbb{R}^n}\widehat{f}(x)g(x)\,\mathrm{d}x$

Proof. (i) and (ii) can easily be proved by direct computation.
(iii) It is clear from Definition 1.10 that $\widehat{f}$ is bounded by $\|f\|_1$, so the integrals are defined. The proof then follows by a direct application of Fubini's theorem. □

Remark: Proposition 1.11 (iii) is often called the "change of hats" formula, for obvious reasons.

Proposition 1.12. *(i) If $f\in L^1(\mathbb{R}^n)$ is such that $x^\alpha f\in L^1(\mathbb{R}^n)$ for some $\alpha\in\mathbb{N}_0^n$, then $\mathrm{D}^\alpha\mathscr{F}f(\xi)$ exists, and*

$$\mathrm{D}^\alpha\mathscr{F}f(\xi)=\mathscr{F}[(-i)^{|\alpha|}x^\alpha f(x)](\xi). \tag{1.33}$$

(ii) If $f \in C^k(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and if all the derivatives $\mathrm{D}^\alpha f$, $|\alpha| \le k$, are in $L^1(\mathbb{R}^n)$, then

$$\mathscr{F}[\mathrm{D}^\alpha f](\xi) = i^{|\alpha|}\xi^\alpha \, \mathscr{F} f(\xi). \tag{1.34}$$

Proof. (i) The function $\xi \mapsto f(x)e^{-ix\cdot\xi}$ is infinitely differentiable and by assumption $\mathrm{D}^\alpha_\xi f(x)e^{-ix\cdot\xi}$ is integrable with respect to x. Therefore we may differentiate under the integral sign and the result follows.

(ii) All derivative up to order k are by assumption sufficiently smooth and integrable, so we may use integration by parts and the result follows. □

Corollary 1.13. *If $\varphi \in \mathscr{S}(\mathbb{R}^n)$, then*

(i) $\mathscr{F}\varphi$ is infinitely differentiable.

(ii) $\mathscr{F}[\mathrm{D}^\alpha\varphi]$ exists for all $\alpha \in \mathbb{N}_0^n$.

Proof. This follows from Lemmas 1.3 and 1.5 and Proposition 1.12. □

Theorem 1.14. *The space $\mathscr{S}$ is invariant under the Fourier transform $\mathscr{F}$, and, in fact, the Fourier transform is a linear 1-to-1 mapping $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ that maps $\mathscr{S}$ onto itself and is continuous in the topology on $\mathscr{S}$. The inverse mapping is $\mathscr{F}^{-1} = \check{\mathscr{F}}$.*

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. The linearity of $\mathscr{F}$ follows from Definition 1.10. We start by proving that $\widehat{\varphi} \in \mathscr{S}(\mathbb{R}^n)$, i.e. $\sup_{\xi\in\mathbb{R}^n} |\xi^\alpha \mathrm{D}^\beta \widehat{\varphi}(\xi)| < \infty$ for all $\alpha, \beta \in \mathbb{N}_0^n$. From Proposition 1.11 (iii), we have

$$|\xi^\alpha \mathrm{D}^\beta \widehat{\varphi}(\xi)| = (2\pi)^{-\frac{n}{2}} |\xi^\alpha \int_{\mathbb{R}^n} (-i)^{|\beta|} x^\beta e^{-ix\cdot\xi} \varphi(x)\,\mathrm{d}x|. \tag{1.35}$$

For $|\xi| \le 1$, it follows from Lemmas 1.3 and 1.5 that (1.35) is finite. For $|\xi| > 1$, we note that the $(-i)^{|\beta|}x^\beta\varphi(x)$ is infinitely differentiable and all derivatives are integrable, so we may use integration by parts.

$$\begin{aligned} &|\frac{\xi^\alpha}{(-i)^{|\alpha|}\xi^\alpha} \int_{\mathbb{R}^n} \mathrm{D}^\alpha_x((-i)^{|\beta|} x^\beta \varphi(x)) e^{-ix\cdot\xi}\,\mathrm{d}x| \\ &\le \int_{\mathbb{R}^n} |D^\alpha((-i)^{|\beta|}x^\beta\varphi(x))|\,\mathrm{d}x < \infty, \quad |\xi| > 1. \end{aligned} \tag{1.36}$$

The last inequality follows from Lemmas 1.3 and 1.5. This proves that $\widehat{\varphi} \in \mathscr{S}(\mathbb{R}^n)$, since the last expression in (1.36) does not depend on ξ. Since $\check{\mathscr{F}}\varphi(\xi) = \mathscr{F}\varphi(-\xi)$, it follows that $\check{\mathscr{F}}\varphi \in \mathscr{S}(\mathbb{R}^n)$ as well. Next we prove that $\check{\mathscr{F}}\mathscr{F}\varphi = \mathscr{F}\check{\mathscr{F}}\varphi = \varphi$.

For $\varepsilon > 0$, $x \in \mathbb{R}^n$, we define the Gaussian function $g_\varepsilon(x) = e^{-\frac{\varepsilon^2|x|^2}{2}}$. Its Fourier transform is

$$\widehat{g_\varepsilon}(\xi) = \varepsilon^{-n} e^{-\frac{|y|^2}{2\varepsilon^2}}. \tag{1.37}$$

We may apply Proposition 1.11 (ii) and (iii) to φ and $e^{ix\cdot\xi}g_\varepsilon(x)$ and use the transformation $y = \varepsilon z$ to get

$$\int_{\mathbb{R}^n} \widehat{\varphi}(\xi) g_\varepsilon(\xi) e^{ix\cdot\xi}\,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi(\xi + \varepsilon z) e^{-\frac{|z|^2}{2}}\,\mathrm{d}z \tag{1.38}$$

Lebesgue's dominated convergence theorem is applicable on both sides of (1.38) with respect to $\varepsilon \downarrow 0$. Thus the left-hand side of (1.38) converges to $(2\pi)^{\frac{n}{2}} \check{\mathscr{F}} \mathscr{F} \varphi(x)$, while the right-hand side converges to $(2\pi)^{\frac{n}{2}} \varphi(x)$.

Next we prove the continuity of $\mathscr{F}$. Let $\varphi_n \underset{\mathscr{S}}{\longrightarrow} 0$. From Proposition 1.12 and its corollary we have $|\xi^\alpha \mathrm{D}^\beta \widehat{\varphi}_n(\xi)| = |\mathscr{F}[(x^\beta \mathrm{D}^\alpha \varphi_n(x))](\xi)|$. By assumption $(x^\alpha \mathrm{D}^\beta \varphi_n(x)) \to 0$ pointwise and from Definition 1.10 we have that $|\widehat{\psi}(\xi)| \leq \|\psi\|_1$ for $\psi \in \mathscr{S}(\mathbb{R}^n)$. This proves the result. Similar arguments hold for $\check{\mathscr{F}}$.

Lastly, we prove that the mapping is onto. For every $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have shown that $\psi = \check{\mathscr{F}} \varphi \in \mathscr{S}(\mathbb{R}^n)$. Thus $\varphi = \mathscr{F} \psi$. □

Proposition 1.15. *If $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$, then*

(i) $\mathscr{F}(\varphi * \psi) = \mathscr{F} \varphi \mathscr{F} \psi$.

(ii) $\mathscr{F}(\varphi\psi) = \mathscr{F} \varphi * \mathscr{F} \psi$.

Proof. One may check that $\mathscr{S}(\mathbb{R}^n)$ is closed under convolution. Then (i) can be computed by Fubini's theorem:

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi * \psi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \left(\int_{\mathbb{R}^n} \varphi(x-y)\psi(y)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \psi(y) \left(\int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x-y)\,\mathrm{d}x \right) \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \psi(y) e^{-iy\xi} \mathscr{F} \varphi(\xi)\,\mathrm{d}y = \mathscr{F} \psi \mathscr{F} \varphi. \end{aligned} \tag{1.39}$$

(ii) Clearly (i) is true for $\mathscr{F}^{-1}$. Using (i) and Theorem 1.14;

$$\mathscr{F}^{-1}(\mathscr{F} \varphi * \mathscr{F} \psi) = \mathscr{F}^{-1} \mathscr{F} \varphi \mathscr{F}^{-1} \mathscr{F} \psi = \varphi\psi. \tag{1.40}$$

Taking the Fourier transform on both sides of (1.40) gives (ii). □

In light of Theorem 1.8, it is possible to extend the Fourier transform to $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Then, by Theorem 1.8, there is a sequence $\{f_n\}_n \subset \mathscr{S}(\mathbb{R}^n)$ such that $f_n \to f$ in $L^p(\mathbb{R}^n)$. For each f_n, $\mathscr{F} f_n$ is defined and $\{\mathscr{F} f_n\}_n$ is clearly a Cauchy sequence in $L^p(\mathbb{R}^n)$. Since $L^p(\mathbb{R}^n)$ is a Banach space, the sequence converges, and we define

$$\mathscr{F} f = \lim_{n\to\infty} \mathscr{F} f_n. \tag{1.41}$$

This limit is independent of how we choose $\{f_n\}_n$. In the particular case of $p = 2$, we have an interesting and highly useful result.

Theorem 1.16. *$\mathscr{F}$ and $\mathscr{F}^{-1}$ are unitary operators on $L^2(\mathbb{R}^n)$ and $\mathscr{F}^* = \mathscr{F}^{-1}$, meaning that $\mathscr{F}\,\mathscr{F}^{-1} = \mathscr{F}^{-1}\,\mathscr{F} = \mathrm{id}$ in $L^2(\mathbb{R}^n)$.*

Proof. Let $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$. Then $\mathscr{F}\,\varphi, \mathscr{F}\,\psi \in L^2(\mathbb{R}^n)$ and we have

$$\begin{aligned}\langle \mathscr{F}\,\varphi, \mathscr{F}\,\psi\rangle_2 &= \int_{\mathbb{R}^n} \mathscr{F}\,\varphi(\xi)\overline{\mathscr{F}\,\psi(\xi)}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \mathscr{F}\,\varphi(\xi)\,\mathscr{F}^{-1}\,\overline{\psi}(\xi)\,\mathrm{d}\xi \\ &= \langle \varphi, \psi\rangle_2.\end{aligned} \tag{1.42}$$

The last equality follows from the change of hats formula, 1.11 (iii). As equation (1.42) shows, the Fourier transform on $L^2(\mathbb{R}^n)$ preserves the inner product, so that, by the discussion above, $\mathscr{F}$ and $\mathscr{F}^{-1}$ are unitary operators. □

We give an example as a small demonstration of the power of the Fourier transform.
Example: Consider the PDE

$$\partial_t u(x,t) = k\Delta_x u(x,t) + u(x,y), \quad t \geq 0,\, x \in \mathbb{R}^n, \quad \text{with} \quad u(x,0) = f(x) \in \mathscr{S}(\mathbb{R}^n). \tag{1.43}$$

Applying the Fourier transform with respect to x on both sides and using Proposition 1.12 we get

$$\partial_t \widehat{u}(\xi,t) = -k|\xi|^2\widehat{u}(\xi,t) + \widehat{u}(\xi,t) = (-k|\xi|^2+1)\widehat{u}(\xi,t). \tag{1.44}$$

This is an ODE with solution

$$\widehat{u}(\xi,t) = e^{(-k|\xi|^2+1)t}h(\xi) \tag{1.45}$$

for some function h. Using our initial conditions, we find $h(\xi) = \widehat{f}(\xi)$. Thus, using proposition 1.15, we find

$$u(x,t) = e^t(f * \mathscr{F}^{-1}(e^{-k|\xi|^2 t})(x). \tag{1.46}$$

1.3 Tempered Distributions

Definition 1.17. *A tempered distribution is a continuous linear functional on $\mathscr{S}(\mathbb{R}^n)$. In other words,*

$$T : \mathscr{S} \to \mathbb{C}$$

is a distribution if the following holds:

$$\begin{aligned}&T(a\varphi + b\psi) = aT(\varphi) + bT(\psi), \\ &\varphi_n \underset{\mathscr{S}}{\longrightarrow} \varphi \Rightarrow T(\varphi_n) \to T(\varphi),\end{aligned}$$

for all $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$ and all constants $a, b \in \mathbb{C}$. We denote the space of tempered distributions by $\mathscr{S}'(\mathbb{R}^n)$, also called the continuous dual of $\mathscr{S}(\mathbb{R}^n)$.

For a function f such that $f\varphi \in L^1(\mathbb{R}^n)$ for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we can define T_f by

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,\mathrm{d}x. \tag{1.47}$$

T_f is clearly linear and if it is continuous as well, it is a tempered distribution. Distributions that can be expressed as an integral are called *regular distributions.*

Definition 1.18. *Let f be a measurable function. If there exists a constant $C > 0$ and an $N \in \mathbb{N}$ such that*

$$|f(x)| \leq C(1 + |x|)^N \tag{1.48}$$

for all $x \in \mathbb{R}^n$, then f is said to be slowly increasing.

Proposition 1.19. *Every slowly increasing function f defines regular distribution*

Proof. We have $f(x)\varphi(x) = (f(x)/(1+|x|)^N)((1+|x|)^N\varphi(x)) \leq C(1+|x|)^N\varphi(x)$ for some constant C, which is integrable by Lemmas 1.3 and 1.5. Furthermore, if $\{\varphi_n\}_n \subset \mathscr{S}(\mathbb{R}^n)$ and $\varphi_n \underset{\mathscr{S}}{\longrightarrow} 0$, then, by Definition 1.2, $C(1+|x|)^N\varphi_n(x) \underset{\mathscr{S}}{\longrightarrow} 0$ as well. This, together with Lemma 1.5 implies $\int_{\mathbb{R}^n} f(x)\varphi_n(x)\,\mathrm{d}x \to 0$ as $n \to \infty$. This proves that f is continuous on $\mathscr{S}(\mathbb{R}^n)$, and hence a tempered distribution. □

Remark: Slowly increasing functions are also called tempered functions. This explains the name tempered distributions.

Proposition 1.20. *Every function $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, defines a regular distribution.*

Proof. Let $\{\varphi_n\}_n \subset \mathscr{S}(\mathbb{R}^n)$ and $\varphi_n(x) \underset{\mathscr{S}}{\longrightarrow} 0$. Continuity follows trivially from Hölder's inequality, since

$$\|f\varphi_n\|_1 \leq \|f\|_p \|\varphi_n\|_q \tag{1.49}$$

where q is such that $1/p + 1/q = 1$. By Lemma 1.5 the right hand side of (1.49) goes to 0 as $n \to \infty$. □

There is one immediate concern now. Is the Schwartz space large enough to distinguish two functions that defines regular distributions? Indeed even more is true, and we prove a more general result below.

Remark: Recall that we in general identify an equivalence class of measurable functions with a representative function. If two functions are equal a.e. then they obviously define the same distribution.

Lemma 1.21. *Let Ω be an open set in $\mathbb{R}^n$ and $f \in L^1_{loc}(\Omega)$. If*

$$\int_\Omega f(x)\varphi(x)\,\mathrm{d}x = 0 \tag{1.50}$$

for all $\varphi \in \mathscr{D}(\Omega)$, then $f = 0$ a.e. in Ω.

Proof. Let $f \in L^1_{loc}(\Omega)$ be as above, $\overline{K_1} \subsetneq K_2 \Subset \Omega$ and $f_2 = f\chi_{K_2}$. Then f_2 is integrable. Let ω_h be as in (1.9). We then have

$$(f_2)_h(x) = \int_{\mathbb{R}^n} f_2(y)\omega_h(x-y)\,\mathrm{d}y = \int_\Omega f(y)\omega_h(x-y)\,\mathrm{d}y \tag{1.51}$$

for $x \in K_1$ and h small enough, say $0 < h \leq h_0$. By assumption, the right-hand side of (1.51) is zero. Thus $(f_2)_h(x) = 0$ for $x \in K_1$, $0 < h \leq h_0$. By Proposition 1.9 (iii), $(f_2)_h \to f_2$ in $L^1(\mathbb{R}^n)$ as $h \to 0$. This implies $f_2(x) = 0$ for $x \in K_1$, but $f_2 = f$ in K_1, and since K_1 was arbitrary, this proves $f = 0$ a.e. in Ω. □

If f is a tempered distribution and, say, continuously differentiable and bounded, then

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x)\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} f(x)\frac{\partial \varphi}{\partial x_i}(x)\,\mathrm{d}x. \tag{1.52}$$

By lemma 1.3 and Proposition 1.19, $\frac{\partial f}{\partial x_i}$ defines a continuous and linear operation on $\mathscr{S}(\mathbb{R}^n)$. In other words, it is a distribution. We have proved that $T_{\partial_{x_i} f}(\varphi) = -T_f(\partial_{x_i}\varphi)$, but the right-hand side of 1.52 makes sense for all tempered distributions f, regardless of whether or not they are differentiable. In fact, since the test functions are infinitely differentiable, we can iterate the procedure above to get "derivatives" of all orders. This motivates the following definition.

Definition 1.22. *For a tempered distribution T, we define $\mathrm{D}^\alpha T$ by*

$$\mathrm{D}^\alpha T(\varphi) = (-1)^{|\alpha|}T(\mathrm{D}^\alpha \varphi), \quad \varphi \in \mathscr{S}(\mathbb{R}^n). \tag{1.53}$$

By Definition 1.2 and Lemma 1.3, differentiation in this sense is a (linear) operator $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. It follows from the discussion above that if f is sufficiently smooth and its derivatives are regular distributions, then $\mathrm{D}^\alpha T_f = T_{\mathrm{D}^\alpha f}$.

Until now we have only considered regular distributions, but there are non-regular distributions. As an example, consider the heaviside function $u = \chi_{[0,\infty)}$ (whether the interval is open or half-open is irrelevant) on $\mathbb{R}$. By lemma 1.20 it is a tempered distribution. It is not differentiable as a function, but consider its distributional derivative

$$\mathrm{D}T_u(\varphi) = -T_u(\varphi') = -\int_{\mathbb{R}} u(x)\varphi'(x)\,\mathrm{d}x = -\int_0^\infty \varphi'(x)\,\mathrm{d}x = \varphi(0), \quad \varphi \in \mathscr{S}(\mathbb{R}^n). \tag{1.54}$$

There cannot exist a function f such that $\int_{\mathbb{R}^n} f(x)\varphi(x)\,\mathrm{d}x = \varphi(0)$ for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$, for f would have to be zero a.e., but then the integral would be zero as well. In other words $\mathrm{D}T_u$ is not a regular distribution. Non-regular distributions are called *singular distributions.*

The Fourier Transform on $\mathscr{S}'(\mathbb{R}^n)$

Suppose $f \in L^1(\mathbb{R}^n)$. Then, by lemma 1.11 (iii)

$$\int_{\mathbb{R}^n} \widehat{f}(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\widehat{\varphi}(x)\,\mathrm{d}x, \quad \varphi \in \mathscr{S}(\mathbb{R}^n). \tag{1.55}$$

Theorem 1.14 ensures that the operation on a test function φ defined by the right-hand side of (1.55) is continuous and linear on $\mathscr{S}(\mathbb{R}^n)$ for a tempered distribution f, even if $\widehat{f}$ is not defined. This motivates the following definition.

Definition 1.23. *Let $T \in \mathscr{S}(\mathbb{R}^n)$. Then $\mathscr{F}\,T = \widehat{T}$ and $\mathscr{F}^{-1}\,T = \check{T}$ are given by*

$$\mathscr{F}\,T(\varphi) = T(\mathscr{F}\,\varphi) \quad \textit{and} \quad \mathscr{F}^{-1}\,T(\varphi) = T(\mathscr{F}^{-1}\,\varphi), \quad \varphi \in \mathscr{S}(\mathbb{R}^n). \tag{1.56}$$

Definition 1.24. *If T is a tempered distribution and the function g is smooth and tempered (that is, slowly increasing cf. Definition 1.18), we define the distribution gT by*

$$gT(\varphi) = T(g\varphi) \tag{1.57}$$

Definition 1.25. *Let $\{T_n\}_n$ be a sequence in $\mathscr{S}'(\mathbb{R}^n)$. We say that T_n converges to T if*

$$T_n(\varphi) \to T(\varphi) \tag{1.58}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

Remark: This is a weak form of convergence, but the most natural one for our purpose. Clearly, both differentiation and multiplication with smooth functions (as defined above) are continuous in this sense.

Theorem 1.26. *(i) Both $\mathscr{F}$ and $\mathscr{F}^{-1}$ map $\mathscr{S}'(\mathbb{R}^n)$ 1-to-1 and onto itself. Furthermore, they are continuous in the sense of tempered distributions and $\mathscr{F}^{-1}$ is the inverse of $\mathscr{F}$.*

(ii) Let $T \in \mathscr{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Then

$$\mathscr{F}(\mathrm{D}^\alpha T) = i^{|\alpha|}x^\alpha(\mathscr{F}\,T) \quad \textit{and} \quad \mathscr{F}(x^\alpha T) = i^{|\alpha|}\mathrm{D}^\alpha(\mathscr{F}\,T). \tag{1.59}$$

Proof. (i) Linearity follows from construction. Let $T_n \to T$ in $\mathscr{S}'(\mathbb{R}^n)$. Then

$$\mathscr{F}\,T_n(\varphi) = T_n(\mathscr{F}\,\varphi) \to T(\mathscr{F}\,\varphi) = \mathscr{F}\,T(\varphi). \tag{1.60}$$

This proves continuity. That $\mathscr{F}$ is 1-to-1 follows from Theorem 1.14. Similar arguments holds for $\mathscr{F}^{-1}$. That the mappings are onto can be proved in exactly the same way as for $\mathscr{S}$ in Theorem 1.14. Let $T \in \mathscr{S}'(\mathbb{R}^n)$. Then

$$\mathscr{F}\,\mathscr{F}^{-1}\,T(\varphi) = T(\mathscr{F}\,\mathscr{F}^{-1}\,\varphi) = T(\varphi) = \mathscr{F}^{-1}\,\mathscr{F}\,T(\varphi), \quad \varphi \in \mathscr{S}(\mathbb{R}^n). \tag{1.61}$$

This proves that $\mathscr{F}^{-1}$ is the inverse of $\mathscr{F}$ on $\mathscr{S}'(\mathbb{R}^n)$.

(ii) Let $T \in \mathscr{S}'(\mathbb{R}^n)$. Then, by Definitions 1.22 and 1.23 and Proposition 1.12 and its corollary,

$$\begin{aligned} \mathscr{F}(\mathrm{D}^\alpha T)(\varphi) &= (\mathrm{D}^\alpha T)(\mathscr{F}\varphi) = (-1)^{|\alpha|} T(\mathrm{D}^\alpha \mathscr{F}\varphi) \\ &= i^{|\alpha|} T(\mathscr{F}[x^\alpha \varphi]) = i^{|\alpha|}(\mathscr{F}T)(x^\alpha \varphi) \\ &= i^{|\alpha|}(x^\alpha \mathscr{F}T)(\varphi), \quad \varphi \in \mathscr{S}(\mathbb{R}^n). \end{aligned} \tag{1.62}$$

This proves the first formula. The second one can be proved in the same straightforward manner. □

1.4 The Space $\mathscr{D}'(\Omega)$

We will later need distributions on arbitrary domains $\Omega \subset \mathbb{R}^n$, but $\mathscr{S}(R^n)$ has no natural equivalent on $\Omega \subsetneq \mathbb{R}^n$. We therefore need a more general type of distributions. Differentiability being the most desired property, it is natural to consider $\mathscr{D}(\Omega)$ as a starting point. However, to make sense of continuous functionals on $\mathscr{D}(\Omega)$, we need a sense of convergence in $\mathscr{D}(\Omega)$.

Definition 1.27. *A sequence $\{\varphi_n\}_n \subset \mathscr{D}(\Omega)$ is said to converge to φ in $\mathscr{D}(\Omega)$ if there exists a set $K \Subset \Omega$ such that*

$$\operatorname{supp} \varphi_n \subset K, \quad \textit{for all } n \in \mathbb{N}, \tag{1.63}$$

and

$$\sup_{x \in \Omega} \mathrm{D}^\alpha(\varphi_n - \varphi) \to 0, \quad \textit{for all } \alpha \in \mathbb{N}_0^n. \tag{1.64}$$

This is denoted by $\varphi_n \underset{\mathscr{D}}{\longrightarrow} \varphi$.

Similarly to the definition of $\mathscr{S}'(\mathbb{R}^n)$ as the continuous dual of $\mathscr{S}(\mathbb{R}^n)$, we have the following definition:

Definition 1.28. *Let $\Omega \subset \mathbb{R}^n$. The space $\mathscr{D}'(\Omega)$ is the space of all functionals $T : \mathscr{D}(\Omega) \to \mathbb{C}$ such that*

$$\begin{aligned} &T(a\varphi + b\psi) = aT(\varphi) + bT(\psi), \\ &\varphi_n \underset{\mathscr{D}}{\longrightarrow} \varphi \Rightarrow T(\varphi_n) \to T(\varphi), \end{aligned}$$

for all $\varphi, \psi \in \mathscr{D}(\Omega)$ and all constants $a, b \in \mathbb{C}$.

Remark: Since $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ and convergence in $\mathscr{D}(\mathbb{R}^n)$ implies convergence in $\mathscr{S}(\mathbb{R}^n)$, it follows that $\mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$.

We will not need a more precise characterisation of the elements of $\mathscr{D}'(\Omega)$ beyond the remark above and the following proposition.

Proposition 1.29. *Let $\Omega \subset \mathbb{R}^n$. If $f \in L^p_{loc}(\Omega)$, $1 \le p \le \infty$, then f defines a regular distribution.*

Proof. Linearity is obvious. Let $\{\varphi_n\}_n \subset \mathscr{D}(\Omega)$ be such that $\varphi_n \underset{\mathscr{D}}{\to} 0$. Then every φ_n has support contained in the same set, say $K \Subset \Omega$, and we get

$$\int_\Omega |f(x)\varphi_n(x)|\,\mathrm{d}x = \int_K |f(x)\varphi_n(x)|\,\mathrm{d}x \le \|f\|_{p,K}\|\varphi_n\|_{p',K} \to 0 \tag{1.65}$$

where we used Hölder's inequality. □

An interesting question at this point, is how far have we moved away from functions in introducing distributions. An answer is contained in the next theorem.

Theorem 1.30. *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $T \in \mathscr{D}'(\Omega)$ there exists a sequence $\{\varphi_n\}_n \subset \mathscr{D}(\Omega)$ such that $\varphi_n \to T$ in $\mathscr{D}'(\Omega)$.*

Proof. See for instance [11], Theorem 4.1.5. □

Remark: Since $\mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$, it follows that the above result also holds for $\mathscr{S}'(\mathbb{R}^n)$.

Support of a Distribution

The support of a distribution is defined much like the support of a measurable function (essential support, as it is also called, to distinguish it from the usual support of a continuous function, which clearly does not make sense for general measurable functions). Clearly, if T is a tempered distribution and

$$T(\varphi) = 0, \quad \text{for all} \quad \varphi \in \mathscr{D}(\Omega) \tag{1.66}$$

for some open set $\Omega \subset \mathbb{R}^n$, then we want $\Omega \cap \operatorname{supp} T = \emptyset$. Let

$$B_\delta(x) = \{y \in \mathbb{R}^n : |x-y| < \delta\}, \quad x \in \mathbb{R}^n. \tag{1.67}$$

Definition 1.31. *Let T be a distribution. Then we define the support of T:*

$$\operatorname{supp} T = \{x \in \mathbb{R}^n : T|_{B_\delta(x)} \neq 0 \quad \textit{for all} \quad \delta > 0\} \tag{1.68}$$

For a locally integrable function f, the support of f and the support of the distribution it defines coincides. That is,

$$\operatorname{supp} f = \operatorname{supp} T_f. \tag{1.69}$$

2 Sobolev Spaces on $\mathbb{R}^n$

2.1 The Spaces $W_p^k(\mathbb{R}^n)$

Here we will interpret $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ as a tempered distribution according to Proposition 1.20. In particular, we may take derivatives of all orders of $f \in L^p(\mathbb{R}^n)$.

Definition 2.1. *Let $k \in \mathbb{N}_0$ and $1 \leq p < \infty$. We define*

$$W_p^k(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \mathrm{D}^\alpha f \in L^p(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}_0^n, |\alpha| \leq k\}. \tag{2.1}$$

The spaces $W_p^k(\mathbb{R}^n)$ are called classical Sobolev spaces.

Remark: As mentioned $\mathrm{D}^\alpha f \in L^p(\mathbb{R}^n)$ must be interpreted in the sense of distributions. That is, there exists a $g \in L^p(\mathbb{R}^n)$ such that $g = \mathrm{D}^\alpha f$ as distributions. This means

$$\int_{\mathbb{R}^n} g(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) D^\alpha \varphi(x)\,\mathrm{d}x \tag{2.2}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$. While $\mathrm{D}^\alpha f$ always exists as a distribution, it is not necessarily in $L^p(\mathbb{R}^n)$, nor even a regular distribution, as shown in Section 1.2. Thus a g as above does not always exist. Of course, if f is sufficiently smooth, then g is just the ordinary derivative of f.

Remark: If the distributional derivative of a regular distribution is itself a distribution, it is often called a weak derivative. Thus one will often see Sobolev spaces described as the space of weakly differentiable functions belonging to some L^p space in the literature. Clearly, distributional differentiation is even weaker than weak differentiation, but coincides with it when the weak derivative exists.

Theorem 2.2. *The space $W_p^k(\mathbb{R}^n)$ furnished with the norm*

$$\|f\|_{W_p^k(\mathbb{R}^n)} = (\sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha f\|_p^p)^{1/p} \tag{2.3}$$

is a Banach space.

Proof. First we need to check that $\|f\|_{W_p^k(\mathbb{R}^n)}$ really is a norm. Clearly

$$\|\lambda f\|_{W_p^k(\mathbb{R}^n)} = |\lambda| \|f\|_{W_p^k(\mathbb{R}^n)}, \tag{2.4}$$

and

$$\|f\|_{W_p^k(\mathbb{R}^n)} = 0 \Longleftrightarrow f = 0 \quad \text{a.e.} \tag{2.5}$$

If $f, g \in W_p^k(\mathbb{R}^n)$, then

$$\begin{aligned} \|f + g\|_{W_p^k(\mathbb{R}^n)} &= \left(\sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha f + \mathrm{D}^\alpha g\|_p^p \right)^{1/p} \\ &\leq \left(\sum_{|\alpha| \leq k} (\|\mathrm{D}^\alpha f\|_p + \|\mathrm{D}^\alpha g\|_p)^p \right)^{1/p} \\ &\leq \left(\sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha f\|_p^p \right)^{1/p} + \left(\sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha g\|_p^p \right)^{1/p} \qquad (2.6) \\ &= \|f\|_{W_p^k(\mathbb{R}^n)} + \|g\|_{W_p^k(\mathbb{R}^n)}, \end{aligned}$$

where we used Minkowski's inequality going from the second to the third line. Next we show completeness. Let f_j be a Cauchy sequence in $W_p^k(\mathbb{R}^n)$. This implies $\mathrm{D}^\alpha f_j$ for $|\alpha| \le k$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$, which is a Banach space. Thus $\mathrm{D}^\alpha f_j \to f^\alpha$ for some $f^\alpha \in L^p(\mathbb{R}^n)$. Let $f^0 = f$. Using Hölder's inequality, we have

$$\|(\mathrm{D}^\alpha f_j - f^\alpha)\varphi\|_1 \le \|\mathrm{D}^\alpha f_j - f^\alpha\|_p \|\varphi\|_{p'} \to 0, \quad \varphi \in \mathscr{S}(\mathbb{R}^n), \tag{2.7}$$

and

$$\|(f_j - f)\mathrm{D}^\alpha \varphi\|_1 \le \|f_j - f\|_p \|\mathrm{D}^\alpha \varphi\|_{p'} \to 0, \quad \varphi \in \mathscr{S}(\mathbb{R}^n), \tag{2.8}$$

where p' is such that $1/p + 1/p' = 1$. Putting this together we get

$$\int_{\mathbb{R}^n} f^\alpha(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\mathrm{D}^\alpha \varphi\,\mathrm{d}x. \tag{2.9}$$

Thus $\mathrm{D}^\alpha f = f^\alpha$ for $|\alpha| \le k$ and $f_j \to f$ in $W_p^k(\mathbb{R}^n)$. □

Theorem 2.3. *Let $1 \le p < \infty$ and $k \in \mathbb{N}_0$. Then $\mathscr{S}(\mathbb{R}^n) \subset W_p^k(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{D}(\mathbb{R}^n)$, and therefore $\mathscr{S}(\mathbb{R}^n)$, is dense in $W_p^k(\mathbb{R}^n)$.*

Proof. The inclusions follows immediately from Definition 2.1. We show that $\mathscr{D}(\mathbb{R}^n)$ is dense in $W_p^k(\mathbb{R}^n)$. Let $f \in W_p^k(\mathbb{R}^n)$ and let f_h be its mollification. Then

$$\begin{aligned}(\mathrm{D}^\alpha f_h)(x) &= \int_{\mathbb{R}^n} \mathrm{D}_x^\alpha \omega_h(x-y) f(y)\,\mathrm{d}y \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \left(\mathrm{D}_y^\alpha \omega_h(x-y)\right) f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \omega_h(x-y)\mathrm{D}^\alpha f(y)\,\mathrm{d}y = (\mathrm{D}^\alpha f)_h(x).\end{aligned} \tag{2.10}$$

From Proposition 1.9 (ii) we know that

$$\|\mathrm{D}^\alpha f - \mathrm{D}^\alpha f_h\|_p \to 0 \quad \text{as} \quad h \to 0, \quad |\alpha| \le k. \tag{2.11}$$

Thus f_h is a smooth function belonging to $W_p^k(\mathbb{R}^n)$ and $f_h \to f$ in $W_p^k(\mathbb{R}^n)$. Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ be such that $\varphi(x) = 1$ for $|x| < 1$. Then $\varphi f_h \in \mathscr{D}(\mathbb{R}^n) \cap W_p^k(\mathbb{R}^n)$ and $\varphi(2^{-j}\cdot) f_h \to f_h$ in $W_p^k(\mathbb{R}^n)$ as $j \to \infty$. This proves the result. □

2.2 Fractional Sobolev Spaces on $\mathbb{R}^n$

Due to the fact that $L^2(\mathbb{R}^n)$ is a Hilbert space, we can define an inner product on $W_2^k(\mathbb{R}^n)$ as well,

$$\langle f, g\rangle_{W_2^k(\mathbb{R}^n)} = \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} \mathrm{D}^\alpha f(x) \overline{\mathrm{D}^\alpha g(x)}\,\mathrm{d}x. \tag{2.12}$$

By Theorem 2.2, $W_2^k(\mathbb{R}^n)$ with this inner product is a Hilbert space.

Proposition 1.12 relates the integrability of $x^\alpha f$ to the differentiability of $\mathscr{F} f$, and Theorem 1.16 established that the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$. We investigate the consequences of these results with regards to Sobolev spaces. With this in mind, we make the following definition:

Definition 2.4. *Let*

$$w_s(x) = (1 + |x|^2)^{s/2}, \quad s \in \mathbb{N},\, x \in \mathbb{R}^n. \tag{2.13}$$

We define the weighted L^2 *space*

$$L^2(\mathbb{R}^n, w_s) = \{f \text{ measurable} : w_s f \in L^2(\mathbb{R}^n)\}. \tag{2.14}$$

Remark: $L^2(\mathbb{R}^n, w_s)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{R}^n,w_s)} = \int_{\mathbb{R}^n} w_s(x) f(x) \overline{w_s(x) g(x)}\, \mathrm{d}x = \langle w_s f, w_s g\rangle_2. \tag{2.15}$$

Other weights than w_s are of course possible, but w_s serves a special purpose.

Theorem 2.5. *The Fourier transform* $\mathscr{F}$ *and its inverse* $\mathscr{F}^{-1}$ *generate unitary maps from* $W_2^k(\mathbb{R}^n)$ *onto* $L^2(\mathbb{R}^n, w_k)$*, and vice versa.*

Proof. Let $f \in W_2^k(\mathbb{R}^n)$. Using Theorem 1.16 and Proposition 1.11 we get

$$\begin{aligned} \|f\|^2_{W_2^k(\mathbb{R}^n)} &= \sum_{|\alpha|\le k} \|\mathrm{D}^\alpha f\|_2^2 \\ &= \sum_{|\alpha|\le k} \|\mathscr{F}\, \mathrm{D}^\alpha f\|_2^2 \\ &= \sum_{|\alpha|\le k} \|i^{|\alpha|} \xi^\alpha \mathscr{F} f\|_2^2 \\ &= \int_{\mathbb{R}^n} \left(\sum_{|\alpha|\le k} |\xi^\alpha|^2\right) |\mathscr{F} f(\xi)|^2\, \mathrm{d}\xi. \end{aligned} \tag{2.16}$$

For all $k \in \mathbb{N}$, there exists constants c_k,C_k such that $c_k w_k(x) \le \sum_{|\alpha|\le k} |x^\alpha| \le C_k w_k(x)$ for all $x \in \mathbb{R}^n$, thus the last line in (2.16) represents an equivalent norm to $\|\widehat{f}\|_{L^2(\mathbb{R}^n,w_k)}$. We will use the symbol $\sim$ to denote equivalence between norms. This proves that $\mathscr{F}$ is a isometric map from $W_2^k(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n, w_k)$. Given $g \in L^2(\mathbb{R}^n, w_k)$, then, by Proposition 1.12 adapted to $\mathscr{F}^{-1}$ and Theorem 1.16,

$$\mathrm{D}^\alpha \mathscr{F}^{-1} g = i^{|\alpha|} \mathscr{F}^{-1}(x^\alpha g) \in L^2(\mathbb{R}^n) \tag{2.17}$$

for $|\alpha| \le k$. This proves that the mapping is onto, and thus unitary since it is isometric. The proofs for $\mathscr{F}^{-1}$ and the mapping(s) in the opposite direction are similar. □

Theorem 2.5 tells us that $W_2^k(\mathbb{R}^n) = \mathscr{F} L^2(\mathbb{R}^n, w_k)$, and this can be taken as a definition of $W_2^k(\mathbb{R}^n)$. However, Definition 2.4 and $\mathscr{F} L^2(\mathbb{R}^n, w_s)$ makes sense not only for $s \in \mathbb{N}$, but for all $s \in \mathbb{R}$ and this gives rise to fractional Sobolev spaces.

Definition 2.6. *Let* $s \in \mathbb{R}$ *and* w_s *be as in* (2.13)*. We define*

$$H^s(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n) : \mathscr{F} f \in L^2(\mathbb{R}^n, w_s)\}. \tag{2.18}$$

For $s \geq 0$, $\mathscr{F} f \in L^2(\mathbb{R}^n, w_s)$ implies $\mathscr{F} f \in L^2(\mathbb{R}^n)$, which by Theorem 1.16 implies $f \in L^2(\mathbb{R}^n)$, so all elements of $H^s(\mathbb{R}^n)$ are functions. However, for $s < 0$, $\mathscr{F} f \in L^2(\mathbb{R}^n, w_s)$ is weak criteria, which allows for a greater variety of elements; even non-regular distributions. Indeed, for s small enough we have $\delta \in H^s(\mathbb{R}^n)$. Let us prove this. It is simply a matter of writing out the definition to see that $\mathscr{F} \delta = 1$, the function with constant value 1. Then we require

$$(1 + |x|^2)^{s/2} 1 \in L^2(\mathbb{R}^n). \tag{2.19}$$

We know this is the case when $s < -n/2$.

Proposition 2.7. *Let $s \in \mathbb{R}$. Then the following holds:*

(i) $H^s(\mathbb{R}^n)$ furnished with the inner product

$$\langle f, g \rangle_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} w_s(x) \mathscr{F} f(x) \overline{w_s(x) \mathscr{F} g(x)} \, \mathrm{d}x \tag{2.20}$$

is a Hilbert space.

(ii) $\mathscr{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, and $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Proof. (i) If $f \in L^2(\mathbb{R}^n)$ then $\mathscr{F}^{-1} w_{-s} f$ is in $H^s(\mathbb{R}^n)$. Thus the mapping $f \mapsto w_s \mathscr{F} f$ maps $H^s(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$ and it is by definition an isometric map. It follows that $H^s(\mathbb{R}^n)$ is a Hilbert space with the inner product defined above.

(ii) The inclusions are immediate from Definition 2.6. Let $f \in H^s(\mathbb{R}^n)$. Then $w_s \mathscr{F} f \in L^2(\mathbb{R}^n)$, and by Theorem 1.8 there exists a sequence $\{\psi_n\}_n \subset \mathscr{S}(\mathbb{R}^n)$ such that

$$\psi_n \to w_s \mathscr{F} f \quad \text{in} \quad L^2(\mathbb{R}^n). \tag{2.21}$$

By Lemma 1.3 and Theorem 1.14, $\varphi_n = \mathscr{F}^{-1}(w_{-s}\psi) \in \mathscr{S}(\mathbb{R}^n)$, and it follows from (2.21) that $\varphi_n \to f$ in $H^s(\mathbb{R}^n)$. □

Remark: Since $\mathscr{D}(\mathbb{R}^n)$ can be continuously embedded into $\mathscr{S}(\mathbb{R}^n)$, it follows that $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

By Theorem 2.5 and Definition 2.6 we have $W_2^k(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$. Furthermore, it follows from Definitions 2.1 and 2.6, respectively, that

$$\begin{aligned} W_p^{k_1}(\mathbb{R}^n) \subset W_p^{k_2}(\mathbb{R}^n) \quad & k_1, k_2 \in \mathbb{N}_0, \quad k_2 < k_1, \\ H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n) \quad & -\infty < s_2 < s_1 < \infty. \end{aligned} \tag{2.22}$$

This means that the spaces $H^s(\mathbb{R}^n)$ with the continuous parameter s fills in the gaps between the spaces $W_2^k(\mathbb{R}^n)$ with the discrete parameter k (it should be noted that there are other ways to interpolate between the spaces $W_2^k(\mathbb{R}^n)$; the spaces $H^s(\mathbb{R}^n)$ is just one way).

Our current definition of fractional Sobolev spaces is elegant, but perhaps a bit mysterious. It seems natural that one should be able define fractional Sobolev spaces, at least for $s > 0$, without any reference to the Fourier transform. Our original definition of Sobolev spaces was in terms of the existence and integrability of (distributional) derivatives of functions, so we need some fractional extension of differentiation.

Definition 2.8. *Let $BC^k(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, be the space of complex-valued, k-times differentiable functions such that*

$$\|f\|_{BC^k(\mathbb{R}^n)} = \sum_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |\mathrm{D}^\alpha f(x)| < \infty. \tag{2.23}$$

Here B stands for bounded.

With this definition in mind, a reasonable extension to $s = k + \sigma$, $0 < \sigma < 1$, would be the space normed by

$$\|f\|_{BC^s(\mathbb{R}^n)} = \|f\|_{BC^k(\mathbb{R}^n)} + \sum_{|\alpha|=k} \sup_{0\neq h\in\mathbb{R}^n} \sup_{x\in\mathbb{R}^n} \frac{|\mathrm{D}^\alpha f(x+h) - \mathrm{D}^\alpha f(x)|}{|h|^\sigma}. \tag{2.24}$$

An appropriate definition for the norm $\|\cdot\|_{W_2^s(\mathbb{R}^n)}$, $s = k+\sigma$, $k \in \mathbb{N}_0$ and $0 < \sigma < 1$ is then given by

$$\begin{aligned} \|f\|_{W_2^s(\mathbb{R}^n)} &= \left(\|f\|^2_{W_2^k(\mathbb{R}^n)} + \sum_{|\alpha|=k} \iint_{\mathbb{R}^{2n}} \frac{|\mathrm{D}^\alpha f(x+h) - \mathrm{D}^\alpha f(x)|^2}{|h|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}h \right)^{1/2} \qquad (2.25) \\ &= \left(\|f\|^2_{W_2^k(\mathbb{R}^n)} + \sum_{|\alpha|=k} \int_{\mathbb{R}^n} \frac{\|\mathrm{D}^\alpha f(\cdot+h) - \mathrm{D}^\alpha f(\cdot)\|_2^2}{|h|^{n+2\sigma}} \,\mathrm{d}h \right)^{1/2} \end{aligned}$$

where the factor $|h|^n$ is added for convergence purposes. This gives us another definition of fractional Sobolev spaces.

Definition 2.9. *Let $s = k + \sigma$, $k \in \mathbb{N}_0$ and $0 < \sigma < 0$. We define*

$$W_2^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{W_2^s(\mathbb{R}^n)} < \infty\}. \tag{2.26}$$

It follows from (2.25) that if $s = k+\sigma$, $k \in \mathbb{N}_0$, $0 < \sigma < 0$, then $W_2^s(\mathbb{R}^n) \subset W_2^k(\mathbb{R}^n)$. One can define fractional Sobolev spaces for any $1 \leq p < \infty$ in the same way as above, and these spaces are called Slobodeckij spaces. However, we are only interested in the case $p = 2$, so we will restrict our focus to this. Of course, any result we prove for W_2^s where we do not use any properties of L^2 will hold for every other Slobodeckij space.

Theorem 2.10. *Let $s = k + \sigma$, $k \in \mathbb{N}_0$ and $0 < \sigma < 1$. Then $\mathscr{D}(\mathbb{R}^n)$, and therefore $\mathscr{S}(\mathbb{R}^n)$ is dense in $W_2^s(\mathbb{R}^n)$, and*

$$H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n). \tag{2.27}$$

Proof. First we show that $\mathscr{S}(\mathbb{R}^n) \subset W_2^s(\mathbb{R}^n)$. From Theorem 2.3 we know $\mathscr{S}(\mathbb{R}^n) \subset W_2^k(\mathbb{R}^n)$. For $\delta > 0$, $\int_{|h|>\delta} |h|^{-(n+\sigma)}\,\mathrm{d}h < \infty$, so we need only consider h close to zero. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then, by Definition 1.1

$$|\varphi(x+h) - \varphi(x)| \leq c_t |h| (1 + |x|^t)^{-1} \tag{2.28}$$

for all $t > 0$ and some constant $c_t > 0$ dependent on t when h is close to zero. Let $\delta > 0$. Since $\mathscr{S}(\mathbb{R}^n)$ is invariant under differentiation, we may as well consider φ instead of $\mathrm{D}^\alpha\varphi$. Then

$$\int_{|h|<\delta} \frac{\|\varphi(x+h)-\varphi(x)\|_2^2}{|h|^{n+2\sigma}}\,\mathrm{d}h \leq \int_{|h|<\delta} \frac{c_t^2\|(1+|x|^t)^{-1}\|_2^2}{|h|^{n-2(1-\sigma)}}\,\mathrm{d}h \tag{2.29}$$

which is finite if we take $t > n+1$ since (as already noted) $(1+|\cdot|^{n+1})^{-1} \in L^2(\mathbb{R}^n)$ and $\sigma < 1$. This proves $\mathscr{S}(\mathbb{R}^n) \subset W_2^s(\mathbb{R}^n)$.

Let $f \in W_2^s(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$. We wish to show that $f\varphi \in W_2^s(\mathbb{R}^n)$, in order to prove that it is sufficient to consider f with compact support, so that we may use mollification to find an approximating function in $\mathscr{D}(\mathbb{R}^n)$. We use the classic "add and subtract" trick of calculus:

$$(\varphi f)(x+h)-(\varphi f)(x) = f(x)(\varphi(x+h)-\varphi(x)) + \varphi(x+h)(f(x+h)-f(x)). \tag{2.30}$$

Furthermore we have the simple inequality

$$|\varphi(x+h)-\varphi(x)| \leq |h|\sum_{l=1}^{n}\sup_{x\in\mathbb{R}^n}|\frac{\partial\varphi}{\partial x_l}(x)| \tag{2.31}$$

for h small (again, it is sufficient to only consider the behaviour for small h). It then follows from (2.25) that $f\varphi \in W_2^s(\mathbb{R}^n)$.

Now let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ be such that $\varphi(x) = 1$ if $|x| \leq 1$ and $|\varphi(x)| \leq 1$ outside the unit ball. Let $\varphi_j(x) = \varphi(2^{-j})$, $j \in \mathbb{N}$. Then $(\varphi_j f)(x) \to f(x)$ in $\mathbb{R}^n$, and $|\varphi_j f| \leq |f|$ almost everywhere for all $j \in \mathbb{N}$. Furthermore, Leibniz rule holds for distributional derivatives and $\mathrm{D}^\alpha\varphi_j$ is uniformly bounded for every $\alpha \in \mathbb{N}_0^n$, thus $|\mathrm{D}^\alpha(\varphi_j f)| \leq C_\alpha|\mathrm{D}^\alpha f|$ almost everywhere for all $|\alpha| \leq k$ and for some constant C_α depending on α. Lebesgue's dominated convergence then implies that $\varphi_j f \to f$ in $W_2^s(\mathbb{R}^n)$. This shows that every function in $W_2^s(\mathbb{R}^n)$ can be approximated by functions in $W_2^s(\mathbb{R}^n)$ with compact support, so it is sufficient to approximate such functions to prove the density of $\mathscr{D}(\mathbb{R}^n)$, and thus also $\mathscr{S}(\mathbb{R}^n)$. Let $f \in W_2^s(\mathbb{R}^n)$ have compact support and

$$f_t(x) = \int_{\mathbb{R}^n} \omega(y)f(x-ty)\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \quad 0 < t \leq 1 \tag{2.32}$$

be its mollification. From the proof of Theorem 2.3 we know that $f_t \to f$ in $W_2^k(\mathbb{R}^n)$. To prove convergence in $W_2^s(\mathbb{R}^n)$, we consider

$$\int_{\mathbb{R}^n} \frac{\|\mathrm{D}^\alpha f(\cdot+h)-\mathrm{D}^\alpha f(\cdot)+\mathrm{D}^\alpha f_t(\cdot)-\mathrm{D}^\alpha f_t(\cdot+h)\|_2^2}{|h|^{n+2\sigma}}\,\mathrm{d}h \tag{2.33}$$

Again, only h close to zero may cause problems. From Proposition 1.9 (ii) we know that $\|f_t\|_2 \leq \|f\|_2$. By equation (2.10), this implies

$$\int_{|h|\leq\delta} \frac{\|\mathrm{D}^\alpha f_t(\cdot+h)-\mathrm{D}^\alpha f_t(\cdot)\|_2^2}{|h|^{n+2\sigma}}\,\mathrm{d}h \leq \int_{|h|\leq\delta} \frac{\|\mathrm{D}^\alpha f(\cdot+h)-\mathrm{D}^\alpha f(\cdot)\|_2^2}{|h|^{n+2\sigma}}\,\mathrm{d}h \tag{2.34}$$

for any $\delta > 0$. By assumption, the $W_2^s(\mathbb{R}^n)$ norm of f is finite, and so we can for any $\varepsilon > 0$ find a $\delta > 0$ such that the right-hand side of (2.34) is less than ε. It follows that $f_t \to f$ in $W_2^s(\mathbb{R}^n)$. Thus $\mathscr{D}(\mathbb{R}^n)$, and therefore also $\mathscr{S}(\mathbb{R}^n)$, is dense in $W_2^s(\mathbb{R}^n)$. It is then sufficient to prove

$$\int_{\mathbb{R}^n} w_s(\xi)^2 |\mathscr{F} f(\xi)|^2 \, \mathrm{d}\xi \sim \|f\|_{W_2^s(\mathbb{R}^n)}, \quad f \in \mathscr{S}(\mathbb{R}^n), \tag{2.35}$$

in order to prove (2.27). To show this, we first concentrate on the second term in (2.25). Using Theorem 1.16 we get

$$\int_{\mathbb{R}^n} \frac{\|\mathrm{D}^\alpha f(\cdot + h) - \mathrm{D}^\alpha f(\cdot)\|_2^2}{|h|^{n+2\sigma}} \, \mathrm{d}h = \int_{\mathbb{R}^n} \frac{\|\mathscr{F}(\mathrm{D}^\alpha f(\cdot + h) - \mathrm{D}^\alpha f(\cdot))\|_2^2}{|h|^{n+2\sigma}} \, \mathrm{d}h. \tag{2.36}$$

Using Proposition 1.11 (i) and Proposition 1.12 and its corollary, we arrive at

$$\mathscr{F}(\mathrm{D}^\alpha f(\cdot + h) - \mathrm{D}^\alpha f(\cdot))(\xi) = (e^{i\xi h} - 1) i^{|\alpha|} \xi^\alpha \mathscr{F} f(\xi). \tag{2.37}$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\|\mathrm{D}^\alpha f(\cdot + h) - \mathrm{D}^\alpha f(\cdot)\|_2^2}{|h|^{n+2\sigma}} \, \mathrm{d}h &= \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\mathscr{F} f(\xi)|^2 \int_{\mathbb{R}^n} \frac{|e^{i\xi h} - 1|^2}{|h|^{n+2\sigma}} \, \mathrm{d}h \, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\xi|^{2\sigma} |\mathscr{F} f(\xi)|^2 \int_{\mathbb{R}^n} \frac{|e^{i \frac{\xi}{|\xi|} \tilde{h}} - 1|^2}{|\tilde{h}|^{n+2\sigma}} \, \mathrm{d}\tilde{h} \, \mathrm{d}\xi \end{aligned} \tag{2.38}$$

where we used the coordinate transformation $h = \tilde{h}/|\xi|$ going from the first line to the second. Observe that the integral over $\tilde{h}$ is independent of ξ and finite. Hence

$$\int_{\mathbb{R}^n} \frac{\|\mathrm{D}^\alpha f(\cdot + h) - \mathrm{D}^\alpha f(\cdot)\|_2^2}{|h|^{n+2\sigma}} \, \mathrm{d}h = c \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\xi|^{2\sigma} |\mathscr{F} f(\xi)|^2 \, \mathrm{d}\xi, \tag{2.39}$$

where the constant c is independent of f. Combining the equation above with the calculations done in the proof of Theorem 2.5, we get

$$\begin{aligned} \|f\|_{W_2^s(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \left(\sum_{|\alpha| \le k} |\xi^\alpha|^2 \right) |\mathscr{F} f(\xi)|^2 \, \mathrm{d}\xi + c \sum_{|\alpha| = k} \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\xi|^{2\sigma} |\mathscr{F} f(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^n} \left(\left(\sum_{|\alpha| \le k} |\xi^\alpha|^2 \right) + c \sum_{|\alpha| = k} |\xi^\alpha|^2 |\xi|^{2\sigma} \right) |\mathscr{F} f(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}. \end{aligned} \tag{2.40}$$

As noted in the proof of Theorem 2.5, $\sum_{|\alpha| \le k} |\xi^\alpha| \sim w_k$ and $|\xi^\alpha| |\xi|^\sigma \sim |\xi|^{k+\sigma} \sim w_{k+\sigma}$, and so we get

$$\|f\|_{W_2^s(\mathbb{R}^n)} \sim \left(\int_{\mathbb{R}^n} w_s(\xi)^2 |\mathscr{F} f(\xi)|^2 \mathrm{d}\xi \right)^{1/2} \tag{2.41}$$

where $s = k + \sigma$. This proves (2.27). □

Remark: We have proved that $H^s = W_2^s$ for $s \geq 0$, but we will still distinguish these spaces: when working with the norm defined by the singular integral, we will use the notation W_2^s, and when working with the Fourier transform definition we will use H^s. For many of the theoretical theorems regarding the properties of these spaces, the $W_2^s(\mathbb{R}^n)$ norm will be more convenient to use, but in chapter 6 we will return to the Fourier definition and use the notation $H^s(\mathbb{R}^n)$. In particular we have to use H^s for $s < 0$, as W_2^s is not defined in this case.

2.3 Sobolev Embedding

Theorem 2.11. *Let $BC^l(\mathbb{R}^n)$, $l \in \mathbb{N}_0$, be as in Definition 2.8 and $s > l + \frac{n}{2}$. Then the embedding*

$$\mathrm{id} : W_2^s(\mathbb{R}^n) \hookrightarrow BC^l(\mathbb{R}^n) \tag{2.42}$$

exists in the sense that for each equivalence class $[f] \in W_2^s(\mathbb{R}^n)$ there exists a representative function $f \in BC^l(\mathbb{R}^n)$.

Proof. From Theorem 2.10 we know that $\mathscr{S}(\mathbb{R}^n)$ is dense in $W_2^s(\mathbb{R}^n)$. Considering the fact that both $BC^l(\mathbb{R}^n)$ and $W_2^s(\mathbb{R}^n)$ are Banach spaces, it is sufficient to prove that there exists a number $c > 0$ such that

$$\sum_{|\alpha| \leq l} \sup_{x \in \mathbb{R}^n} |\mathrm{D}^\alpha \varphi(x)| \leq c \|\varphi\|_{W_2^s(\mathbb{R}^n)}, \quad \varphi \in \mathscr{S}(\mathbb{R}^n). \tag{2.43}$$

Using Theorem 1.14, Proposition 1.12, our previous considerations regarding $|x^\alpha| \sim w_{|\alpha|}(x)$ and Hölder's inequality we get

$$\begin{aligned} |\mathrm{D}^\alpha \varphi(x)| &= |\mathrm{D}^\alpha (\mathscr{F}^{-1} \mathscr{F} \varphi)(x)| = |\mathscr{F}^{-1}(\xi^\alpha \mathscr{F} \varphi(\xi))(x)| \\ &= c \Big| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha (\mathscr{F} \varphi)(\xi) \, \mathrm{d}\xi \Big| \\ &\leq c' \int_{\mathbb{R}^n} w_s(\xi) |\mathscr{F} \varphi(\xi)| w_{l-s}(\xi) \, \mathrm{d}\xi \\ &\leq c' \left(\int_{\mathbb{R}^n} w_s(x)^2 |\mathscr{F} \varphi(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} w_{s-l}^{-2}(\xi) \, \mathrm{d}\xi \right)^{1/2}. \end{aligned} \tag{2.44}$$

The last integral converges since $s - l > \frac{n}{2}$ and by Theorem 2.10 the first integral in the last line of (2.44) represents an equivalent norm in $W_2^s(\mathbb{R}^n)$. □

The lower bound on s cannot be improved. That is, the theorem does not hold in general for $s = l + \frac{n}{2}$. To see this, consider the sequence of functions

$$f_j(x) = \begin{cases} 1, & |x| \leq \frac{1}{j^2} \\ -1 - \frac{\log|x|}{\log j}, & \frac{1}{j^2} \leq |x| \leq \frac{1}{j} \\ 0, & |x| \geq \frac{1}{j} \end{cases} \tag{2.45}$$

on $\mathbb{R}^2$. It is easy to check that these functions are continuous, and that

$$\sup_{x\in\mathbb{R}^2} |f_j(x)| = 1, \quad j \in \mathbb{N}. \tag{2.46}$$

Thus $|f_j(x)| \leq \chi_{B(0,1)}(x)$ for every $x \in \mathbb{R}^2$, where $B(0,1)$ is the ball centred at 0 with radius 1. Lebesgue's dominated convergence theorem then gives $\|f_j\|_2 \to 0$ since $f_j \to 0$ pointwise.

Consider now the distributional derivative $\frac{\partial f_j}{\partial x_1}(x)$, $i = 1, 2$. Dividing the area of integration and using partial integration in each part, we get

$$\begin{aligned} &-\int_{|x|\leq 1/j^2} f(x)\frac{\partial \varphi}{\partial x_i}(x)\,\mathrm{d}x - \int_{1/j^2\leq|x|\leq 1/j} f(x)\frac{\partial \varphi}{\partial x_i}(x)\,\mathrm{d}x \\ &= \int_{1/j^2\leq|x|\leq 1/j} \frac{2}{\log j}\frac{x_i}{|x|^2}\varphi(x)\,\mathrm{d}x. \end{aligned} \tag{2.47}$$

In other words, $\frac{\partial f_j}{\partial x_i}(x) = \frac{2}{\log j}\frac{x_i}{|x|^2}$ for $1/j^2 \leq |x| \leq 1/j$ and zero elsewhere. This implies $f_j \in W_2^1(\mathbb{R}^2)$. Furthermore

$$|\frac{\partial f_j}{\partial x_i}(x)| \leq \frac{1}{|x|}\chi_{B(0,1)}(x), x \in \mathbb{R}^2. \tag{2.48}$$

The right-hand side is integrable, so Lebesgue's dominated convergence theorem implies $\|\frac{\partial f_j}{\partial x_i}\|_2 \to 0$, $i = 1, 2$, as $j \to \infty$, since $\frac{\partial f_j}{\partial x_i}(x) \to 0$ pointwise. This, combined with the calculations above, implies

$$\|f_j\|_{W_2^1(\mathbb{R}^n)} \to 0 \quad \text{as } j \to \infty. \tag{2.49}$$

Equations (2.46) and (2.49) implies that $W_2^1(\mathbb{R}^n)$ cannot be continuously embedded in $BC^0(\mathbb{R}^2)$.

3 Sobolev Spaces on $\mathbb{R}^n_+$

Sobolev spaces on $\mathbb{R}^n$ was defined in terms on tempered distributions, but for general sets $\Omega \subsetneq \mathbb{R}^n$ there does not exist something like Schwartz functions. There are two natural ways to proceed.

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary, open set and $W_p^s(\mathbb{R}^n)$ be either as in Definition 2.1 with $1 \leq p < \infty$ and $s \in \mathbb{N}_0$, or as in Definition 2.9 with $p = 2$ and $s > 0$. We define*

$$W_p^s(\Omega) = \{f \in L^p(\Omega) : \text{there exists } g \in W_p^s(\mathbb{R}^n) \text{ with } g|_\Omega = f\} \tag{3.1}$$

with norm

$$\|f\|_{W_p^s(\Omega)} = \inf\{\|g\|_{W_p^s(\mathbb{R}^n)} : g \in W_p^s(\mathbb{R}^n),\, g|_\Omega = f\}. \tag{3.2}$$

For $s < 0$, we define

$$H^s(\Omega) = \{f \in \mathscr{D}'(\Omega) : \text{there exists } g \in H^s(\mathbb{R}^n) \text{ with } g|_\Omega = f\}. \tag{3.3}$$

Here $f \in L^p(\Omega)$ must be understood in the sense of distributions, and $g|_\Omega = f$ means $g(\varphi) = f(\varphi)$ for all $\varphi \in \mathscr{D}(\Omega)$.

Remark: It is also common to define $W_p^k(\Omega)$ to be the space of all $f \in L^p(\Omega)$ such that $\mathrm{D}^\alpha f \in L^p(\Omega)$, $|\alpha| \leq k$, where $\mathrm{D}^\alpha f$ is the distributional derivative, with norm corresponding to (2.3). If Ω is smooth enough, this will give the same result, as we will see.

We wish to show that $W_p^s(\Omega)$ inherits some properties from $W_p^s(\mathbb{R}^n)$ and for this we need a basic result from functional analysis.

Proposition 3.2. *Let E be a Banach (Hilbert) space and $M \subset E$ a closed subset. Then the quotient space*

$$E/M, \tag{3.4}$$

where $x \sim y$ if $x - y \in M$, with the quotient norm

$$\|[x]\|_{E/M} = \inf_{y \in [x]} \|y\|_E = \inf_{m \in M} \|x - m\|_E \tag{3.5}$$

is a Banach (Hilbert) space.

Proof. Since M is closed, it is clear that the norm defined above is actually a norm. As is usual in quotient spaces, we write $\pi(x)$ for $[x]$, the mapping of x to its equivalence class. Let $\{\pi(x_n)\}_n$ be a Cauchy sequence in E/M. Since $\{\pi(x_n)\}_n$ is Cauchy, it is sufficient to prove that a subsequence converges. Choose n_1 such that $\|\pi(x_m) - \pi(x_n)\|_{E/M} < 1/2$ for all $n, m \geq n_1$ and choose n_2 such that $\|\pi(x_m) - \pi(x_n)\|_{E/M} < 1/2^2$ for all $n, m \geq n_2$. Proceeding like this, we get a sequence, which we again denote $\{\pi(x_n)\}_n$, such that

$$\|\pi(x_{n+1}) - \pi(x_n)\|_{E/M} < \frac{1}{2^n}, \quad \text{for all} \quad n \geq 1. \tag{3.6}$$

By the definition of the norm of E/M, this implies that there exists a sequence $\{m_n\}_n \subset M$ such that

$$\|x_{n+1} - x_n - m_n\|_E < \frac{1}{2^n}. \tag{3.7}$$

Writing $m_n = y_{n+1} - y_n$ with $y_1 = 0$ and $y_n \in M$, we get that $\{x_n - y_n\}_n$ is a Cauchy sequence in E, and hence has a limit $x \in E$. This implies $\pi(x_n) \to \pi(x)$ in E/M. □

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then $W_p^s(\Omega)$ is Banach space, and a Hilbert space if $p = 2$, and*

$$\mathscr{D}(\Omega) \subset W_p^s(\Omega) \subset L^p(\Omega) \subset \mathscr{D}'(\Omega). \tag{3.8}$$

Furthermore, the restrictions $\mathscr{D}(\mathbb{R}^n)|_\Omega$ and $\mathscr{S}(\mathbb{R}^n)|_\Omega$ are dense in $W_p^s(\Omega)$.

Proof. Let $\Omega^c = \mathbb{R}^n \setminus \Omega$ and

$$\widetilde{W}_p^s(\Omega^c) = \{g \in W_p^s(\mathbb{R}^n) : \operatorname{supp} g \subset \Omega^c\}. \tag{3.9}$$

Since Ω^c is closed, it follows from Definition 1.31 and 1.27 that if $\{g_n\}_n \subset \widetilde{W}_p^s(\Omega^c)$ and $g_n \to g$ in $W_p^s(\mathbb{R}^n)$, then $g \in \widetilde{W}_p^s(\Omega^c)$. Now consider the space

$$W_p^s(\mathbb{R}^n)/\widetilde{W}_p^s(\Omega^c). \tag{3.10}$$

By proposition 3.2 it is a Banach space, and if $p = 2$ it is a Hilbert space. If $f, g \in W_p^s(\mathbb{R}^n)$ are such that $f(x) = g(x)$ for all $x \in \Omega$, then $\text{supp}(f - g) \subset \Omega^c$, hence $[f] = [g]$ in $W_p^s(\mathbb{R}^n)/\widetilde{W}_p^s(\Omega^c)$. Thus we see that

$$W_p^s(\Omega) \approx W_p^s(\mathbb{R}^n)/\widetilde{W}_p^s(\Omega^c), \tag{3.11}$$

meaning that the spaces are isomorphic, and hence $W_p^s(\Omega)$ is a Banach space (and a Hilbert space if $p = 2$). The inclusions are immediate from Definition 3.1, and the density of $\mathscr{D}(\mathbb{R}^n)|_\Omega$ and $\mathscr{S}(\mathbb{R}^n)|_\Omega$ follows from Theorem 2.3 and Theorem 2.10. □

We now restrict our attention to $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. As will become apparent later, many problems on open sets $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary can be reduced to the case $\mathbb{R}^n_+$. We begin by describing a tool that will be useful in the sequel.

3.1 Partitions of Unity

Let $\Omega \subset \mathbb{R}^n$ be a compact set (that is, bounded and closed) and set

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < \varepsilon\}, \quad \varepsilon > 0. \tag{3.12}$$

We cover Ω by finitely many open balls B_i with radius $r_i > 0$, $i = 1, ..., I$. For $\delta > 0$ we let B_i^δ be the ball concentric with B_i with radius δr_i. Since Ω is closed and the finite set of balls B_i is an open covering of Ω, there exists an $\varepsilon' > 0$ such that

$$\Omega_{\varepsilon'} \subset \bigcup_{i=1}^{I} B_i. \tag{3.13}$$

Hence it is possible to choose $\varepsilon > 0$ and $0 < \delta < 1$ such that

$$\Omega_\varepsilon \subset \bigcup_{i=1}^{I} B_i^\delta. \tag{3.14}$$

Using Proposition 1.9 one can find functions

$$\psi_i \in \mathscr{D}(B_i) \quad \text{with} \quad \psi_i(x) = 1, \quad x \in B_i^\delta, \quad i = 1, ..., I. \tag{3.15}$$

For instance, set $\gamma = (1 + \delta)/2$ and consider $(\chi_{B_i^\gamma})_h$ with $h < r_i - \gamma r_i = \gamma r_i - \delta r_i$. By the same argument there exists functions

$$\psi \in \mathscr{D}(\Omega_\varepsilon) \quad \text{with} \quad \psi(x) = 1, \quad x \in \Omega. \tag{3.16}$$

We define

$$\varphi(x) = \sum_{i=1}^{I} \psi_i(x) \in \mathscr{D}(\mathbb{R}^n), \tag{3.17}$$

which has the property that $\varphi(x) \geq 1$ for $x \in \Omega_\varepsilon$. Thus we can define

$$\varphi_i(x) = \frac{\psi_i(x)\psi(x)}{\varphi(x)} \in \mathscr{D}(B_i \cap \Omega_\varepsilon), \quad i = 1, ..., I \tag{3.18}$$

which has the property that

$$\sum_{i=1}^{I} \varphi_i(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \Omega_\varepsilon, \end{cases} \tag{3.19}$$

We call $\{\varphi_i\}_{i=1}^I$ a *partition of unity* of Ω subordinate to the cover $\{B_i\}_{i=1}^I$.

3.2 Extensions

It is often easier to work in $\mathbb{R}^n$ rather than some subset of $\mathbb{R}^n$ (for instance with regards to approximation, as $\mathscr{D}(\Omega)$ is in general not dense in $W_p^k(\Omega)$), and we therefore wish to extend functions from $W_p^s(\mathbb{R}_+^n)$ to $W_p^s(\mathbb{R}^n)$. For a function $f \in W_p^l(\mathbb{R}_+^n)$, distributional differentiation as we have defined it is not a pointwise operation and does not require continuity. One could therefore hope that extending f by zero outside $\mathbb{R}_+^n$ would do the trick. Certainly, $f \in L^p(\mathbb{R}^n)$ in that case, but $\mathrm{D}^\alpha f$ may no longer be a regular distribution, since we require

$$\int_{\mathbb{R}^n} \mathrm{D}^\alpha f(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\mathrm{D}^\alpha\varphi(x)\,\mathrm{d}x \tag{3.20}$$

for a much larger class of functions φ. Indeed, consider $\chi_{[a,b]} \in \mathscr{D}'([a,b])$ for $a < b$. A simple calculation yields $\partial_x \chi_{[a,b]}(\varphi) = 0$ for all $\varphi \in \mathscr{D}([a,b])$, since $\varphi(a) = \varphi(b) = 0$. Thus $\chi_{[a,b]} \in W_1^1([a,b])$. Regarding $\chi_{[a,b]}$ as an element of $\mathscr{D}'(\mathbb{R})$ or $\mathscr{S}'(\mathbb{R})$, we know that differentiating it yields δ-distributions, which are not regular distributions, i.e. $\chi_{[a,b]} \notin W_1^1(\mathbb{R})$. Thus the existence of a general way to extend functions is not obvious. We will rely on the density of smooth functions in $W_p^s(\mathbb{R}_+^n)$.

Theorem 3.4. *For any $L \in \mathbb{N}$ there exists a linear and bounded extension operator* ext^L *defined on $W_p^s(\mathbb{R}_+^n)$, $1 \leq p < \infty$, $s = 0, ..., L$ for $p \neq 2$ and $0 \leq s \leq L$ for $p = 2$, such that*

$$\mathrm{ext}^L : \begin{cases} BC^l(\mathbb{R}_+^n) \hookrightarrow BC^l(\mathbb{R}^n), & l = 0, ..., L, \\ W_p^l(\mathbb{R}_+^n) \hookrightarrow W_p^l(\mathbb{R}^n), & l = 0, ..., L,\ 1 \leq p < \infty, \\ W_2^s(\mathbb{R}_+^n) \hookrightarrow W_2^s(\mathbb{R}^n), & 0 < s < L, \end{cases} \tag{3.21}$$

with

$$\left(\mathrm{ext}^L f\right)|_{\mathbb{R}_+^n} = f \textit{ for any } f \in BC^l \cup W_p^l(\mathbb{R}_+^n) \cup W_2^s(\mathbb{R}_+^n). \tag{3.22}$$

Proof. We define

$$\mathbb{Z}^n = \{m = (m_1, ..., m_n) \in \mathbb{R}^n : m_j \in \mathbb{Z}, \quad j = 1, ..., n\}. \tag{3.23}$$

Let $\{B_m\}_{m\in\mathbb{Z}^n}$ be a set of balls of equal radius (that is, congruent) centred at m with a suitable radius $r > 1$ such that $\mathbb{R}^n \subset \bigcup_{m\in\mathbb{Z}^n} B_m$, and let $\{\varphi_m : m \in \mathbb{Z}^n\}$ be a partition of unity subordinate to $\{B_m\}_{m\in\mathbb{Z}^n}$. Since the balls are congruent, we may assume $\varphi_m(x) = \varphi(x - m)$, $m \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$. Partition of unity was defined only for compact sets with a finite covering, but, clearly, all points and indeed all bounded sets have non-empty intersection with only finitely many balls. Thus only finitely many $\varphi_m(x) \neq 0$ for any $x \in \mathbb{R}^n$. We have

$$\varphi_m \in \mathscr{D}(B_m), \quad 0 \leq \varphi \leq 1, \quad \sum_{m\in\mathbb{Z}^n} \varphi_m(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{3.24}$$

Due to the regular spacing of B_m, and since $\varphi_m(x) = \varphi(x - m)$ and φ has bounded derivatives, it follows that

$$0 < c \leq \inf_{x\in\mathbb{R}^n} \sup_{m\in\mathbb{Z}^n} \mathrm{D}^\alpha \varphi_m(x) \leq m_\alpha \tag{3.25}$$

for all $\alpha \in \mathbb{N}_0^n$ for some constants c and m_α, where m_α depends on α. Using Leibniz rule we deduce

$$\|f\|_{BC^l(\mathbb{R}^n_+)} \sim \sup_{m\in\mathbb{Z}^n} \|\varphi_m f\|_{BC^l(\mathbb{R}^n_+)}, \quad f \in BC^l(\mathbb{R}^n_+). \tag{3.26}$$

Furthermore, it follows from Definition 1.22) that Leibniz rule also holds for distributional differentiation. Using this and the fact that the derivatives of φ is bounded, we may also derive

$$\|f\|_{W^l_p(\mathbb{R}^n)} \sim \left(\sum_{m\in\mathbb{Z}^n} \|\varphi_m f\|^p_{W^l_p(\mathbb{R}^n)} \right)^{1/p}, \quad f \in W^l_p(\mathbb{R}^n). \tag{3.27}$$

The corresponding relation for $p = 2$, $0 < s = k + \sigma < L$ is not immediately clear. We prove that it is given by

$$\|f\|_{W^s_2(\mathbb{R}^n)} \sim \left(\sum_{m\in\mathbb{Z}^n} \left(\|\varphi_m f\|^2_{W^k_2(\mathbb{R}^n)} + \sum_{|\alpha|=k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\mathrm{D}^\alpha(\varphi_m f)(x) - \mathrm{D}^\alpha(\varphi_m f)(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x \,\mathrm{d}y \right) \right)^{1/2}. \tag{3.28}$$

The first term in the parenthesis on the right-hand side is already established. For the second term, recall that φ_m is supported in an open ball B_m. Thus the integrals can be taken over $|x-m| \leq c$ for some constant $c > 0$ and $|x-y| \leq 1$. The problem can then be reduced to whether

$$\int_{|x-m|\leq c} \int_{|x-y|\leq 1} \frac{|(\varphi_m g)(x) - (\varphi_m g)(y)|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y$$
$$\leq c' \int_{|x-m|\leq x} \int_{|x-y|\leq 1} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y + c' \int_{|x-m|\leq c} |g(x)|^2 \, \mathrm{d}x \tag{3.29}$$

for some $c' > 0$. This follows from the "add and subtract" trick

$$(\varphi_m g)(x) - (\varphi_m g)(y) = \varphi_m(y)(g(x) - g(y)) + g(x)(\varphi_m(x) - \varphi_m(y)) \tag{3.30}$$

and $|\varphi_m(x) - \varphi_m(y)|^2 \leq c|x-y|^2$.

The next step is to decompose the lattice $\mathbb{Z}^n$,

$$\mathbb{Z}^n = \bigcup_{j=0}^{J} \mathbb{Z}^n_j, \tag{3.31}$$

where

$$\mathbb{Z}^n_0 = \{x \in \mathbb{R}^n : x = Mm, \, m \in \mathbb{Z}^n\} \quad \text{and} \quad \mathbb{Z}^n_j = m^{(j)} + \mathbb{Z}^n_0, \quad j = 1, ..., J \tag{3.32}$$

with $M \in \mathbb{N}$, $m^{(j)} \in \mathbb{Z}^n$ and $J \in \mathbb{N}$ suitably chosen such that the intersection of the closure of two balls B_m belonging to the same sub-lattice is empty. This implies, by the discussion above,

$$\|f\|_{BC^l(\mathbb{R}^n_+)} \sim \sum_{j=0}^{J} \sup_{m\in\mathbb{Z}^n_j} \|\varphi_m f\|_{BC^l(\mathbb{R}^n_+)} \tag{3.33}$$

and

$$\|f\|_{W^l_p(\mathbb{R}^n)} \sim \sum_{j=0}^{J} \| \sum_{m\in\mathbb{Z}^n_j} \varphi_m f\|_{W^l_p(\mathbb{R}^n)}. \tag{3.34}$$

Due to the construction of $\mathbb{Z}^n_j$, this localizes the extension problem and implies

$$\|f\|_{W^l_p(\mathbb{R}^n_+)} \sim \left(\sum_{m\in\mathbb{Z}^n} \|\varphi_m f\|^p_{W^l_p(\mathbb{R}^n_+)} \right)^{1/p}. \tag{3.35}$$

Thus it is sufficient to extend functions f on $\mathbb{R}^n_+$ with

$$\operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| < 1, x_n \geq 0\}. \tag{3.36}$$

Let $\lambda_1 < ... < \lambda_{L+1} < -1$ and define

$$(\mathrm{ext}^L f)(x) = \begin{cases} f(x), & x_n \geq 0, \\ \sum_{k=1}^{L+1} a_k f(x', \lambda_k x_n), & x_n < 0, \end{cases} \tag{3.37}$$

where $x = (x', x_n) \in \mathbb{R}^n$. If $x_n < 0$ then $\lambda_k x_n > 0$, hence $\text{ext}^L f$ is well defined. Furthermore, $\text{ext}^L f$ has compact support in $\mathbb{R}^n$, and $\text{supp}\,\text{ext}^L f$ depends on $\text{supp}\, f$ and the choice of coefficients λ_i's and a_i's. Now we need to find an appropriate choice of coefficients a_k, $k = 1, ..., L+1$. First we consider $f \in BC^l(\mathbb{R}^n_+)$. We wish to find a choice of a_k's such that $\text{ext}^L f \in BC^l(\mathbb{R}^n)$. The only place $\text{ext}^L f$ may fail to be l times differentiable is in the x_n direction at $x_n = 0$. By definition,

$$\lim_{x_n \to 0^+} \frac{\partial^r}{\partial x_n^r}(\text{ext}^L f)(x) = \frac{\partial^r f}{\partial x_n^r}(x', 0), \quad r = 0, ..., L, \tag{3.38}$$

and

$$\begin{aligned}\lim_{x_n \to 0^-} \frac{\partial^r}{\partial x_n^r}(\text{ext}^L f)(x) &= \lim_{x_n \to 0^-} \sum_{k=1}^{L+1} a_k \lambda_k^r \frac{\partial^r f}{\partial(\lambda_k x_n)^r}(x', \lambda_k x_n) \\ &= \frac{\partial^r f}{\partial x_n^r}(x', 0) \sum_{k=1}^{L+1} a_k \lambda_k^r, \quad r = 0, ..., L. \end{aligned}\tag{3.39}$$

Thus the a_k's need to be chosen such that

$$\sum_{k=1}^{L+1} a_k \lambda_k^r = 1, \quad r = 0, ..., L, \tag{3.40}$$

which can always be done since Vandermonde's matrix

$$\begin{pmatrix} 1 & \lambda_1 & ... & \lambda_1^L \\ 1 & \lambda_2 & ... & \lambda_2^L \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_{L+1} & \dots & \lambda_{L+1}^L \end{pmatrix} \tag{3.41}$$

has non-zero determinant when all the λ_i's are distinct. That is, the rows are linearly independent. Since $\text{ext}^L f = f$ for $x \in \mathbb{R}^n_+$ and $\text{ext}^L f(x)$ depends on f (times a constant) at finitely many points in $\mathbb{R}^n_+$ for $x \in \mathbb{R}^n_-$, we deduce

$$\sum_{|\alpha| \le l} \sup_{x \in \mathbb{R}^n} |\mathrm{D}^\alpha (\text{ext}^L f)(x)| \le c \sum_{|\alpha| \le l} \sup_{x \in \mathbb{R}^n_+} |\mathrm{D}^\alpha f(x)| \tag{3.42}$$

for some constant $c > 0$ and all $f \in BC^l(\mathbb{R}^n_+)$ with (3.36). This proves the theorem for $BC^l(\mathbb{R}^n_+)$.

Furthermore, for a smooth function f the value of $\text{ext}^L f$ and its derivatives at each point $x \in \mathbb{R}^n_-$ is bounded by some constant independent of f times the value of f (and its derivatives) at finitely many points in $\mathbb{R}^n_+$. Thus

$$\|ext^L f\|_{p, \mathbb{R}^n_-} \le c \|f\|_{p, \mathbb{R}^n_+} \tag{3.43}$$

and similarly for its derivatives. Therefore

$$\| \operatorname{ext}^L f\|_{W_p^l(\mathbb{R}^n)} \le c \left(\sum_{|\alpha|\le l} \|D^\alpha f\|_{p,\mathbb{R}^n_+}^p \right)^{1/p} \le c\|f\|_{W_p^l(\mathbb{R}^n_+)} \tag{3.44}$$

for some $c > 0$. The last inequality in (3.44) follows by the definition of the norm in $W_p^l(\mathbb{R}^n_+)$. From Proposition 3.3 we know that $\mathscr{D}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ and $\mathscr{S}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ are dense in $W_p^l(\mathbb{R}^n_+)$ and so the result for $W_p^l(\mathbb{R}^n_+)$ follows by taking limits.

Now let $p = 2$ and $s = k + \sigma < L$, $k \in \mathbb{N}_0$. By Proposition 3.3, $\mathscr{D}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ and $\mathscr{S}(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ are dense in $W_2^s(\mathbb{R}^n_+)$ as well, and

$$\left(\sum_{|\alpha|\le k} \|\mathrm{D}^\alpha f\|_{2,\mathbb{R}^n_+}^2 + \sum_{|\alpha|=k} \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \le \|f\|_{W_2^s(\mathbb{R}^n_+)} \tag{3.45}$$

for smooth functions, so by the preceding steps, all that remains to prove is that

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\mathrm{D}^\alpha(\operatorname{ext}^L f)(x) - \mathrm{D}^\alpha(\operatorname{ext}^L f)(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y \\ &\qquad \le c \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{3.46}$$

for compactly supported, smooth functions. Again we write $x = (x', x_n)$ and $y = (y', y_n)$. The area of integration on the left-hand side of (3.46) can be decomposed into

$$\{(x,y) \in \mathbb{R}^{2n} : x_n y_n \ge 0\} \quad \text{and} \quad \{(x,y) \in \mathbb{R}^{2n} : x_n y_n < 0\}. \tag{3.47}$$

For $x_n \ge 0$ and $y_n \ge 0$, the integrand on the left-hand side equals the integrand on the right-hand side in (3.46), and if $x_n < 0$ and $y_n < 0$, then the integrands differ only by a constant. Hence the integral over $\{(x,y) \in \mathbb{R}^{2n} : x_n y_n \ge 0\}$ can be estimated from above by the right-hand side. For $x_n > 0$ and $y_n < 0$ (the case $x_n < 0$, $y_n > 0$ is similar) we have to prove

$$\begin{aligned} &\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_-} \frac{|\mathrm{D}^\alpha f(x', x_n) - \sum_{k=1}^{L+1} a_k \lambda_k^r \mathrm{D}^\alpha f(y', \lambda_k y_n)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y \\ &\qquad \le c \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.48}$$

where $r = |\alpha|$. Since (3.40) holds, the numerator of the integrand in the left-hand side of (3.48) can be written as

$$|\sum_{k=1}^{L+1} a_k \lambda_k^r \left(\mathrm{D}^\alpha f(x', x_n) - \mathrm{D}^\alpha f(y', \lambda_k y_n) \right)|^2. \tag{3.49}$$

Using the inequality

$$\left(\sum_{j=1}^{n} x_j\right)^2 \leq n \sum_{j=1}^{n} x_j^2, \tag{3.50}$$

the problem can be reduced to

$$\int_{R^n_+} \int_{\mathbb{R}^n_-} \frac{|g(x', x_n) - g(y', \lambda y_n)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x \,\mathrm{d}y \leq c \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x \,\mathrm{d}y \tag{3.51}$$

with $\lambda < -1$. If $\lambda y_n \leq x_n$, then $x_n - \lambda y_n \leq x_n - y_n$, and if $\lambda y_n > x_n$, then $|x_n - \lambda y_n| \leq |\lambda||y_n| \leq |\lambda||x_n - y_n$.

Replacing $|x-y|^2$ on the left-hand side of (3.51) by

$$|x' - y'| + \lambda^{-2}(x_n - \lambda y_n)^2 \leq |x-y|^2 \tag{3.52}$$

one obtains an estimate proving (3.51). This proves the theorem for $W_2^s(\mathbb{R}^n_+)$. □

From the proof of Theorem 3.4 we can extract the following result:

Proposition 3.5. *(i) Let $1 \leq p < \infty$ and $k \in \mathbb{N}_0$. Then*

$$\left(\sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha f\|^p_{p,\mathbb{R}^n_+}\right)^{1/p} \sim \|f\|_{W^k_p(\mathbb{R}^n_+)} \tag{3.53}$$

is an equivalent norm on $W^k_p(\mathbb{R}^n_+)$.

(ii) Let $s = k + \sigma$, $k \in \mathbb{N}_0$ and $0 < \sigma < 1$. Then

$$\left(\sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha f\|^2_{2,\Omega} + \sum_{|\alpha|=k} \iint_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}} \,\mathrm{d}x \,\mathrm{d}y\right)^{1/2} \sim \|f\|_{W^s_2(\mathbb{R}^n_+)} \tag{3.54}$$

is an equivalent norm on $W^s_2(\mathbb{R}^n_+)$.

Proof. (i) Since $\left(\operatorname{ext}^L f\right)|_{\mathbb{R}^n_+} = f$ for $f \in W^k_p(\mathbb{R}^n_+)$, it follows from Definition 3.1 that

$$\|f\|_{W^k_p(\mathbb{R}^n_+)} \leq \|\operatorname{ext}^L f\|_{W^k_p(\mathbb{R}^n)}. \tag{3.55}$$

It then follows from (3.44) that

$$\left(\sum_{|\alpha|\leq l} \|D^\alpha f\|^p_{p,\mathbb{R}^n_+}\right)^{1/p} \leq \|f\|_{W^k_p(\mathbb{R}^n_+)} \leq c \left(\sum_{|\alpha|\leq l} \|D^\alpha f\|^p_{p,\mathbb{R}^n_+}\right)^{1/p}. \tag{3.56}$$

(ii) The inequality in (3.55) holds also for $W^s_2(\mathbb{R}^n_+)$, $s = k+\sigma$, and the result then follows from (3.46). □

4 Sobolev Spaces on Domains

We now return to $W_p^s(\Omega)$ for open sets $\Omega \subset \mathbb{R}^n$. We wish to extend Theorem 3.4 and Proposition 3.5 to open sets $\Omega \subset \mathbb{R}^n$.

Definition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that the boundary $\partial\Omega$ is C^k if for each point $x' \in \partial\Omega$ there exists an $r > 0$ and a real valued function $h \in BC^k(\mathbb{R}^{n-1})$ such that*

$$\Omega \cap B(x', r) = \{x \in B(x', r) : x_n > h(x_1, ..., x_{n-1})\} \tag{4.1}$$

where $B(x', r)$ denotes the open ball centred at x' with radius r. If $\partial\Omega$ is C^k for all $k \in \mathbb{N}$, then we say that $\partial\Omega$ is C^∞.

If Ω is an open and bounded set such that $\partial\Omega$ is C^k, then one can for each point x' on $\partial\Omega$ "straighten out" the boundary near x' by C^k diffeomorphisms. Let $B(x', r)$ and h be as above. Then the function defined by

$$\psi(x) = (x_1, ..., x_{n-1}, x_n - h(x_1, ..., x_{n-1})), \quad x \in B(x', r) \tag{4.2}$$

is a C^k diffeomorphism with inverse

$$\psi^{-1}(y) = (y_1, ..., y_{n-1}, y_n + h(y_1, ..., y_{n-1})) \tag{4.3}$$

such that

$$\psi(B(x', r) \cap \Omega) \subset \mathbb{R}^n_+ \quad \text{and} \quad \psi(B(x', r) \cap \partial\Omega) \subset \mathbb{R}^{n-1} \times \{0\}. \tag{4.4}$$

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set such that $\partial\Omega$ is C^k. Then the following holds:*

(i) For any $L = 1, ..., k$ there exists a linear and bounded extension operator ext^L_Ω defined on $W_p^s(\Omega)$, $1 \le p < \infty$, $s = 0, ..., L$ for $p \neq 2$ and $0 \le s \le L$ for $p = 2$, such that

$$\text{ext}^L : \begin{cases} BC^l(\Omega) \hookrightarrow BC^l(\mathbb{R}^n), & l = 0, ..., L, \\ W_p^l(\Omega) \hookrightarrow W_p^l(\mathbb{R}^n), & l = 0, ..., L,\ 1 \le p < \infty, \\ W_2^s(\Omega) \hookrightarrow W_2^s(\mathbb{R}^n), & 0 < s < L, \end{cases} \tag{4.5}$$

with

$$\left(\text{ext}^L f\right)|_\Omega = f \text{ for all } f \in BC^l(\Omega) \cup W_p^l(\Omega) \cup W_2^s(\Omega). \tag{4.6}$$

(ii) For $1 \le p < \infty$ and $l = 0, 1, ..., k$,

$$\left(\sum_{|\alpha| \le l} \|\mathrm{D}^\alpha f\|^p_{p,\Omega} \right)^{1/p} \sim \|f\|_{W_p^l(\Omega)} \tag{4.7}$$

is an equivalent norm on $W_p^l(\Omega)$.

(iii) For $s = l + \sigma$, $l = 0, 1, ..., k-1$ and $0 < \sigma < 1$,

$$\left(\sum_{|\alpha| \leq l} \|\mathrm{D}^\alpha f\|_{2,\Omega}^2 + \sum_{|\alpha|=l} \iint_{\Omega \times \Omega} \frac{|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha f(y)|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2} \sim \|f\|_{W_2^s(\Omega)} \quad (4.8)$$

is an equivalent norm on $W_2^s(\Omega)$.

Proof of (i). (i) It follows from Proposition 3.3 that it is sufficient to extend only smooth functions f from Ω to $\mathbb{R}^n$ and therefore extending BC^k functions will certainly suffice. Choose balls $B(x', r)$ according to Definition 4.1 for points $x' \in \partial\Omega$ such that $\partial\Omega$ is covered. Since $\partial\Omega$ is closed and bounded, it is compact, thus it is sufficient with only finitely many balls, say $\{B_j\}_{j=1}^J$, to cover it. Now let Ω_0 be an open set such that

$$\overline{\Omega_0} \subset \Omega \quad \text{and} \quad \Omega \subset \Omega_0 \cup \left(\bigcup_{j=1}^J B_j \right). \quad (4.9)$$

This gives a finite open covering of Ω. Let $\{\varphi_j\}_{j=0}^J$ be a partition of unity subordinate to this cover according to Section 3.1. Then f may be decomposed as

$$f(x) = \varphi_0(x) f(x) + \sum_{j=1}^J \varphi_j(x) f(x), \quad x \in \Omega. \quad (4.10)$$

Then

$$\operatorname{supp} \varphi_j f \subset B_j \cap \overline{\Omega}, \quad j = 1, ..., J. \quad (4.11)$$

According to the discussion before Theorem 4.2, there exists C^k diffeomorphisms ψ_j with (4.4) for each B_j, $j = 1, ..., J$. Define

$$g_j(y) = (\varphi_j f) \circ (\psi_j)^{-1}(y), \quad j = 1, ..., J. \quad (4.12)$$

Since g_j is a composition of k-times differentiable functions, g_j is k-times differentiable. Furthermore, by the properties of ψ_j and $\varphi_j f$, g_j satisfies the following:

$$\operatorname{supp} g_j = \psi_j(\operatorname{supp} \varphi_j f) \subset \overline{\mathbb{R}^n_+}. \quad (4.13)$$

This is exactly the same situation as in Theorem 3.4, and the procedure used there yields functions $\operatorname{ext}^L g_j$ with

$$\operatorname{supp} \operatorname{ext}^L g_j \subset \psi_j(B_j). \quad (4.14)$$

Recall that in the proof of Theorem 3.4 we extended functions $f \in BC^l(\mathbb{R}^n_+)$ directly, but for $W_p^l(\mathbb{R}^n_+)$ and $W_2^s(\mathbb{R}^n_+)$ we extended smooth functions and relied on their density in these spaces. However, there we wanted operators ext^L for all $L \in \mathbb{N}$. If we restrict ourself to a given $k \in \mathbb{N}$, and the spaces $W_p^l(\mathbb{R}^n_+)$ and $W_2^s(\mathbb{R}^n_+)$ with $\mathbb{N} \ni l \leq k$ and $s < k$, we see that in the calculations done and arguments used in the proof of Theorem 3.4, we only need f to be k-times differentiable.

Mapping back to our original coordinates again gives functions

$$h_j(x) = (\mathrm{ext}^L g_j) \circ \psi_j(x), \quad \operatorname{supp} h_j \subset B_j, \quad h_j|_\Omega = \varphi_j f. \tag{4.15}$$

Putting $h_j = 0$ outside B_j, we find our extension operator:

$$\mathrm{ext}_\Omega^L f = \varphi_0 f + \sum_{j=1}^{J} h_j. \tag{4.16}$$

(ii) Since everything is reduced to the $\mathbb{R}^n_+$ case, this can be proved by a transformation of (3.53).

(iii) This can be proved by a transformation of (3.54), but requires some extra work compared to (ii). See for instance [10].

□

As previously noted, an alternative definition for Sobolev spaces on domains is

$$\widetilde{W}_p^k(\Omega) = \{f \in L^p(\Omega) : \mathrm{D}^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n, |\alpha| \le k\} \tag{4.17}$$

normed by

$$\|f\|_{\widetilde{W}_p^k(\Omega)} = \left(\sum_{|\alpha| \le k} \|\mathrm{D}^\alpha f\|_{p,\Omega}^p \right)^{1/p}. \tag{4.18}$$

However, under certain conditions Ω, this coincides with Definition 3.1.

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set such that $\partial\Omega$ is of class C^L. Then*

$$W_p^k(\Omega) = \widetilde{W}_p^k(\Omega), \text{ for all } k \le L. \tag{4.19}$$

Proof. By definition, $f \in W_p^k(\Omega)$ implies there exists a $g \in W_p^k(\mathbb{R}^n)$ such that $g|_\Omega = f$. In this case we clearly have $g|_\Omega \in \widetilde{W}_p^k(\Omega)$. Thus, in general

$$W_p^k(\Omega) \subset \widetilde{W}_p^k(\Omega). \tag{4.20}$$

The extension theorem above (Theorem 4.2) also holds for $\widetilde{W}_p^k$, so in the case when $\partial\Omega$ is C^L we get

$$\widetilde{W}_p^k(\Omega) \ni f = \mathrm{ext}_\Omega^L f|_\Omega, \quad \mathrm{ext}_\Omega^L f \in W_p^k(\mathbb{R}^n) \tag{4.21}$$

which implies $W_p^k(\Omega) \supset \widetilde{W}_p^k(\Omega)$ for all $k \le L$. Thus $W_p^k(\Omega) = \widetilde{W}_p^k(\Omega)$ for all $k \le L$. □

4.1 Embeddings

In this section we collect some result on the embedding of Sobolev spaces. Some will be stated without proof, but with references to a full proof.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set such that $\partial\Omega$ is C^1, and let $1 \leq p < n$. Then $W_p^1(\mathbb{R}^n)$ is compactly imbedded in $L^q(\Omega)$, written*

$$W_p^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \tag{4.22}$$

for all $1 \leq q < p^*$, where $p^* = \frac{np}{n-p}$.

To prove the theorem, we will need an inequality called the Gagliardo–Nirenberg–Sobolev inequality, which we state as its own theorem.

Theorem 4.5. *If $1 \leq p < n$, then*

$$\|g\|_{p^*} \leq c\|\nabla g\|_p, \tag{4.23}$$

where $c = \frac{p(n-1)}{n-p}$, for all $g \in C^1(\mathbb{R}^n)$ with compact support.

Proof. Since g has compact support, we have

$$g(x) = \int_{-\infty}^{x_i} \partial_{x_i} g(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)\, \mathrm{d}y_i \tag{4.24}$$

for $i = 1, ..., n$ and any $x \in \mathbb{R}^n$. This implies

$$|g(x)| \leq \int_{\mathbb{R}} |\nabla g(x_1, ..., y_i, ..., x_n)|\, \mathrm{d}y_i, \ i = 1, ..., n, \tag{4.25}$$

which in turn implies

$$|g(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |\nabla g(x_1, ..., y_i, ..., x_n)|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}}, \ i = 1, ..., n. \tag{4.26}$$

Integrating this inequality with respect to x_1 yields

$$\begin{aligned} \int_{\mathbb{R}} |g(x)|^{\frac{n}{n-1}}\, \mathrm{d}x_1 &\leq \int_{\mathbb{R}} \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |\nabla g|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\ &= \left(\int_{\mathbb{R}} |\nabla g|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} |\nabla g|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\ &\leq \left(\int_{\mathbb{R}} |\nabla g|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla g|\, \mathrm{d}x_1\, \mathrm{d}y_i \right)^{\frac{1}{n-1}}, \end{aligned} \tag{4.27}$$

where the last inequality follows from the generalized Hölder inequality. Integrating (4.27) with respect to x_2 and using the generalized Hölder inequality and then proceeding likewise for $x_3, ...x_n$, one can derive

$$\int_{\mathbb{R}^n} |g(x)|^{\frac{n}{n-1}} \, \mathrm{d}x \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |\nabla g| \, \mathrm{d}x_1 ... \, \mathrm{d}y_i ... \, \mathrm{d}x_n \right)^{\frac{1}{n-1}}$$
$$= \left(\int_{\mathbb{R}^n} |\nabla g(x)| \, \mathrm{d}x \right)^{\frac{n}{n-1}}. \tag{4.28}$$

Now apply the inequality (4.28) to $|g|^\gamma$, for some $\gamma > 0$:

$$\left(\int_{\mathbb{R}^n} |g(x)|^{\frac{\gamma n}{n-1}} \, \mathrm{d}x \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla |g(x)|^\gamma | \, \mathrm{d}x = \gamma \int_{\mathbb{R}^n} |g(x)|^{\gamma-1} |\nabla g(x)| \, \mathrm{d}x$$
$$\leq \gamma \left(\int_{\mathbb{R}^n} |g(x)|^{\frac{(\gamma-1)p}{p-1}} \, \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla g(x)|^p \, \mathrm{d}x \right)^{1/p}, \tag{4.29}$$

where the last inequality follows from Hölder's inequality. If we choose $\gamma = \frac{p(n-1)}{n-p}$, then

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} = \frac{np}{n-p} = p^* \tag{4.30}$$

in which case (4.29) becomes

$$\left(\int_{\mathbb{R}^n} |g(x)|^{p^*} \, \mathrm{d}x \right)^{1/p^*} \leq \gamma \left(\int_{\mathbb{R}^n} |\nabla g(x)|^p \, \mathrm{d}x \right)^{1/p}. \tag{4.31}$$

Since γ depends only on n and p, this proves the result. □

Before we proceed with the proof of Theorem 4.4, let us clarify what is stated in the theorem. A compact linear embedding means that there exists a constant $c > 0$ such that $\|f\|_{q,\Omega} \leq c\|f\|_{W_p^s(\Omega)}$ for all $f \in W_p^s(\Omega)$ and that every bounded sequence in $W_p^s(\Omega)$ has a subsequence that converges in $L^q(\Omega)$.

Proof of Theorem 4.4. We prove first that $W_p^1(\Omega) \subset L^q(\Omega)$. In general, for a set Ω such that $\mathcal{L}^n(\Omega) < \infty$, i.e. a bounded set, we have

$$\|g\|_{q,\Omega}^q = \|1 \cdot |g|^q\|_{1,\Omega} \leq \|1\|_{p/(p-q),\Omega} \||g|^q\|_{p/q,V} = \mathcal{L}^n(\Omega)^{(p-q)/p} \|g\|_{p,\Omega}^q \tag{4.32}$$

for $1 \leq q < p \leq \infty$. In other words, for $\Omega \subset \mathbb{R}^n$ with finite measure, the spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, are nested. Thus, in our case, $\|f\|_{q,\Omega} \leq C\|f\|_{p^*,\Omega}$ for $1 \leq q \leq p^*$ and some constant $C > 0$. Therefore it suffices to prove that $f \in W_p^1(\Omega)$ implies $f \in L^{p^*}(\Omega)$. Since $\partial\Omega$ is C^1, Theorem 4.2 implies that there exists an extension $\mathrm{ext}_\Omega^1 f = \bar{f} \in W_p^1(\mathbb{R}^n)$ with compact support and satisfying

$$\|\bar{f}\|_{W_p^1(\mathbb{R}^n)} \leq C\|f\|_{W_p^1(\Omega)}, \quad \bar{f}|_\Omega = f \tag{4.33}$$

for some $C > 0$. By Theorem 2.3 there exists a sequence $\{f_n\}_n \subset \mathscr{D}(\mathbb{R}^n)$ such that

$$f_n \to \bar{f} \quad \text{in } W_p^1(\mathbb{R}^n), \tag{4.34}$$

and according to Theorem 4.5, we have

$$\|f_n - f_m\|_{p^*} \leq C\|\nabla f_n - \nabla f_m\|_p \tag{4.35}$$

for all $m, n \geq 1$. Furthermore, (4.34) implies that $\{\nabla f_n\}_n$ is convergent in $L^p(\mathbb{R}^n)$, which then implies, by the inequality above, that

$$f_n \to \bar{f} \quad \text{in } L^{p^*}(\mathbb{R}^n). \tag{4.36}$$

Now, combining (4.34) and (4.36) with the inequality from Theorem 4.5 for f_n gives the bound

$$\|\bar{f}\|_{p^*} \leq C\|\nabla \bar{f}\|_p. \tag{4.37}$$

By this inequality and the definition of $\bar{f}$, we have

$$\|f\|_{p^*,\Omega} \leq \|\bar{f}\|_{p^*} \leq C\|\nabla \bar{f}\|_p \leq C\left(\|\bar{f}\|_p + \|\nabla \bar{f}\|_p\right) = C\|\bar{f}\|_{W_p^1(\mathbb{R}^n)} \leq C'\|f\|_{W_p^1(\Omega)} \tag{4.38}$$

for some $C' > 0$ independent of f. This proves that $W_p^1(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq p^*$.

Now consider a bounded sequence $\{f_n\}_n \subset W_p^1(\Omega)$. In view of Theorem 4.2, we may assume $\Omega = \mathbb{R}^n$ and that each f_n has compact support in some bounded open set $V \subset \mathbb{R}^n$ and we have

$$\sup_n \|f_n\|_{W_p^1(V)} < \infty. \tag{4.39}$$

According to the Arzela-Ascoli theorem, a sufficient condition for a sequence of functions defined on a closed and bounded set to have a convergent subsequence is that it is uniformly bounded and equicontinuous. However, Arzela-Ascoli only applies to continuous functions. Therefore, we first consider the sequence of mollified functions $\{(f_n)_h\}_n$. Proposition 1.9 and equation (4.39) implies that the sequence is uniformly bounded, and furthermore we have

$$|\nabla(f_n)_h| \leq \int_{\mathbb{R}^n} \omega_h(x-y)|f_n(y)| \,\mathrm{d}y \leq \|\omega_h\|_\infty \|f_n\|_{1,V} \leq C \tag{4.40}$$

for some constant $C > 0$ independent of n. This implies that the sequence is equicontinuous as well. According to (1.10) and (1.11) we may assume that each $(f_n)_h$ has compact support in V, by taking h small enough. Thus Arzela-Ascoli applies and hence there exists a subsequence $\{(f_{n_j})_h\}_j \subset \{(f_n)_h\}_n$ which converges uniformly on V. In particular

$$\limsup_{j,k\to\infty} \|(f_{n_j})_h - (f_{n_k})_h\|_{q,V} = 0. \tag{4.41}$$

Now, if f_n is smooth, then

$$\begin{aligned}(f_n)_h(x) - f_n(x) &= \int_{\mathbb{R}^n} \omega(y)((f_n(x-hy) - f_n(x))\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \omega(y) \int_0^1 \frac{d}{dt} f_n(x-hty)\,\mathrm{d}t\,\mathrm{d}y \\ &= -h \int_{\mathbb{R}^n} \omega(y) \int_0^1 \nabla f_n(x-hty))\cdot y\,\mathrm{d}t\,\mathrm{d}y.\end{aligned} \tag{4.42}$$

Thus, by (1.10) we get

$$\begin{aligned}\int_V |(f_n)_h(x) - f_n(x)|\,\mathrm{d}x &\leq h \int_{|y|\leq 1} \omega(y) \int_0^1 \int_V |\nabla f_n(x-hty)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\leq h \int_V |\nabla f_n(z)|\,\mathrm{d}z.\end{aligned} \tag{4.43}$$

Since, by Proposition 3.3, $\mathscr{D}(\mathbb{R}^n)|_V$ is dense in $W_p^1(V)$, the estimate above holds for any $f_n \in W_p^1(V)$. Furthermore, since V is bounded, we have for any $g \in L^1(V)$ that

$$\|g\|_{1,V} = \|1\cdot|g|\|_{1,V} \leq \|1\|_{p/(p-1),V}\|g\|_{p,V} = \mathcal{L}^n(V)^{(p-1)/p}\|g\|_{p,V} \tag{4.44}$$

for $p \geq 1$. Hence

$$\|(f_n)_h - f_n\|_{1,V} \leq h\|\nabla f_n\|_{1,V} \leq hC\|\nabla f_n\|_{p,V} \tag{4.45}$$

for some C depending only on the (Lebesgue) measure of V. Equation (4.39) implies that there exists a constant $M > 0$ such that $\|\nabla f_n\|_{p,V} \leq M$ for all $n \in \mathbb{N}$. Together with (4.45), this implies

$$(f_n)_h \to f_n \quad \text{in } L^1(V), \text{ uniformly in } n. \tag{4.46}$$

However, we want uniform convergence in $L^q(V)$. Since $q < p^*$, we may use the interpolation inequality to obtain

$$\|(f_n)_h - f_n\|_{q,V} \leq \|(f_n)_h - f_n\|_{1,V}^t \|(f_n)_h - f_n\|_{p^*,V}^{1-t}, \tag{4.47}$$

where $1/q = t + (1-t)/p^*$ and $0 < t < 1$. Then (4.39), Theorem 4.5 and (4.46) together implies

$$(f_n)_h \to f_n \quad \text{in } L^q(V), \text{ uniformly in } n. \tag{4.48}$$

Thus we can for each $l \in \mathbb{N}$ find an $h > 0$ such that

$$\|(f_n)_h - f_n\|_{q,V} \leq \frac{1}{2^{l+1}}, \quad n = 1, 2, ... \tag{4.49}$$

which by the triangle inequality and (4.41) implies

$$\limsup_{j,k\to\infty} \|f_{n_j} - f_{n_k}\|_{q,V} \leq \frac{1}{2^l}. \tag{4.50}$$

Hence we can choose n_l such that

$$\|f_{n_j} - f_{n_k}\| \leq \frac{1}{2^l}, \quad j,k \geq l, \quad l = 1,2,... \tag{4.51}$$

This gives a subsequence $\{f_{n_l}\}_l \subset \{f_n\}_n$ that converges in $L^q(V)$. □

Remark: It follows from Definition 3.1 that $W_p^k(\Omega) \subset W_p^1(\Omega)$ if $k > 1$ and $W_2^s(\Omega) \subset W_2^1(\Omega)$ if $s > 1$. Furthermore, it is clear from the definition of the norms in $W_p^k(\Omega)$ and $W_2^s(\Omega)$ that a bounded sequence in $W_p^k(\Omega)$ or $W_2^s(\Omega)$, $k, s > 1$ is bounded in $W_p^1(\Omega)$ or $W_2^1(\Omega)$, respectively. Thus Theorem 4.4 holds also for $k, s > 1$. In fact, when $k > 1$, the bound p^* can be improved (that is, enlarged), but for our purposes this is not needed.

The inequality in (4.32) gives the following embedding result on bounded sets $\Omega \subset \mathbb{R}^n$.

Proposition 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $1 \leq q < p < \infty$ and $k \in \mathbb{N}_0$. Then*

$$W_p^k(\Omega) \hookrightarrow W_q^k(\Omega). \tag{4.52}$$

Proof. Let $f \in W_p^k(\Omega)$. Then, by (4.32), we have

$$\|f\|_{W_q^k(\Omega)} \leq \left(\sum_{\alpha \leq k} \mathcal{L}^n(\Omega)^{\frac{p-q}{q}} \|\mathrm{D}^\alpha f\|_p^p \right)^{1/p} = \mathcal{L}^n(\Omega)^{\frac{p-q}{pq}} \|f\|_{W_p^k(\Omega)}. \tag{4.53}$$

This proves the result. □

As already noted, we have $W_p^{k+l}(\Omega) \subset W_p^k(\Omega)$ for $k, l \in \mathbb{N}_0$ and $W_2^{s+\varepsilon}(\Omega) \subset W_2^s(\Omega)$ for $\varepsilon > 0$, but if Ω is sufficiently smooth, more is true.

Proposition 4.7. *Let $\Omega \subset \mathbb{R}^n$ be bounded set such that $\partial\Omega$ is C^∞. Let $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, $s \in \mathbb{R}_+$ and $0 < \varepsilon$. Then*

$$W_p^{k+l}(\Omega) \hookrightarrow\hookrightarrow W_p^k(\Omega) \tag{4.54}$$

and

$$W_2^{s+\varepsilon}(\Omega) \hookrightarrow\hookrightarrow W_2^s(\Omega) \tag{4.55}$$

Proof. For a proof of this proposition and more general results, see [4], [5] and [6]. □

The main limitation of Theorem 4.4 for us is that it does not (in the form we stated it) hold for $0 < s < 1$ for $W_2^s(\Omega)$ (and in this case we only need $\partial\Omega$ to be of class C^1). However, Proposition 4.7 gives us compact embedding also in this case, since $W_2^0(\Omega) = L^2(\Omega)$, albeit for a lower exponent: only $L^2(\Omega)$ and not $L^{p^*}(\Omega)$, but this will be sufficient for our purposes.

4.2 Traces

Later, we wish to look for solutions of partial differential equations on domains Ω in the spaces $W_p^k(\Omega)$. In order to do this, we need to assign boundary values to Sobolev functions. This may seem somewhat problematic, as elements of $W_p^k(\Omega)$ are equivalence classes of measurable functions, i.e. allowed to vary on a set of zero measure, and indeed, if $\Omega \subset \mathbb{R}^n$, then $\mathcal{L}^n(\partial\Omega) = 0$. Fortunately, the assumption that the functions are weakly differentiable gives, in some cases, sufficient regularity to make sense of pointwise values as the following theorem shows.

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set such that $\partial\Omega$ is of class C^1. Then there exists a bounded linear operator*

$$\mathrm{tr}_{\partial\Omega} : W_p^1(\Omega) \to L^p(\partial\Omega) \tag{4.56}$$

such that if $f \in W_p^1(\Omega) \cap C(\bar{\Omega})$, then $\mathrm{tr}_{\partial\Omega} f = f|_{\partial\Omega}$ on $\partial\Omega$.

Proof. First assume $\Omega = \mathbb{R}^n_+$ and $f \in BC^1(\mathbb{R}^n_+)$. We may apply the decomposition argument used in the proof of Theorem 3.4 and assume

$$\operatorname{supp} f \subset \{x \in \overline{\mathbb{R}^n_+} : |x| < 1\}. \tag{4.57}$$

If f is real-valued, we can for a fixed $x' \in \mathbb{R}^{n-1}$, $|x'| < 1$ choose a $\tau = \tau(x') \in [0, 1]$ such that

$$\int_0^1 f(x', x_n)\,\mathrm{d}x_n = f(x', \tau). \tag{4.58}$$

We then obtain

$$\begin{aligned} |f(x', 0)|^p &= |f(x', \tau) - \int_0^\tau \frac{\partial f}{\partial x_n}(x', x_n)\,\mathrm{d}x_n|^p \\ &\le c\left(|f(x', \tau)|^p + |\int_0^\tau \frac{\partial f}{\partial x_n}(x', x_n)\,\mathrm{d}x_n|^p\right) \\ &\le c\left(\left(\int_0^1 |f(x', x_n)|\,\mathrm{d}x_n\right)^p + \left(\int_0^1 |\frac{\partial f}{\partial x_n}(x', x_n)|\,\mathrm{d}x_n\right)^p\right) \end{aligned} \tag{4.59}$$

where $c > 0$ can be chosen independently of f. Since $p \ge 1$ and the measure of the interval of integration is 1, we may apply Jensen's inequality and obtain

$$|f(x', 0)|^p \le c \int_0^1 \left(|f(x', x_n)|^p + |\frac{\partial f}{\partial x_n}(x', x_n)|^p\right)\mathrm{d}x_n. \tag{4.60}$$

This inequality can be extended to complex-valued functions $f \in BC^1(\mathbb{R}^n_+)$ satisfying (4.57). Since $\partial\Omega$ is of class C^1, we may use the equivalent norm $\left(\sum_{|\alpha|\le 1} \|\mathrm{D}^\alpha f\|_{p,\Omega}^p\right)^{1/p}$ on $W_p^1(\Omega)$ given in Theorem 4.2. Integration over $x' \in \mathbb{R}^{n-1}$ on both sides of (4.60) then yields

$$\int_{\mathbb{R}^{n-1}} |f(x', 0)|^p\,\mathrm{d}x' \le c \int_{\mathbb{R}^n} \left(|f(x', x_n)|^p + |\frac{\partial f}{\partial x_n}(x', x_n)|^p\right)\mathrm{d}x \tag{4.61}$$

which proves the result for $\Omega = \mathbb{R}^n_+$. For a general open, bounded set Ω with $\partial\Omega$ of class C^1 we may apply the same procedure as used in the proof of Theorem 4.2 to reduce it to the case $\Omega = \mathbb{R}^n_+$. We repeat the procedure here.

For points $x' \in \partial\Omega$, choose open balls $B(x', r)$ according to Definition 4.1 such that $\partial\Omega$ is covered. Since $\partial\Omega$ is compact, a finite number of balls will suffice, say $\{B_j\}_{j=1}^J$. Now let Ω_0 be an open set such that

$$\overline{\Omega_0} \subset \Omega \quad \text{and} \quad \Omega \subset \Omega_0 \cup \left(\bigcup_{j=1}^{J} B_j \right). \tag{4.62}$$

This gives a finite open covering of Ω, and we can, according to Section 3.1, find a partition of unity subordinate to this cover, say $\{\varphi_j\}_{j=0}^J$. Then f may be decomposed as

$$f(x) = \varphi_0(x) f(x) + \sum_{j=1}^{J} \varphi_j(x) f(x), \quad x \in \Omega. \tag{4.63}$$

Then

$$\operatorname{supp} \varphi_j f \subset B_j \cap \overline{\Omega} \tag{4.64}$$

and there exists C^1 diffeomorphisms ψ_j with (4.4) (cf. the discussion prior to Theorem 4.2) for each B_j, $j = 1, ..., J$. Define

$$g_j(y) = (\varphi_j f) \circ (\psi_j)^{-1}(y), \; j = 1, ..., J. \tag{4.65}$$

Then $g_j \in BC^1(\mathbb{R}^n_+)$, and we may apply the same arguments as above to find

$$\operatorname{tr}_{\partial\mathbb{R}^n_+} g_j \in L^p(\partial\mathbb{R}^n_+) \tag{4.66}$$

with (4.61). We then define

$$h_j(x) = (\operatorname{tr}_{\partial\mathbb{R}^n_+} g_j) \circ \psi_j(x) \tag{4.67}$$

and note that $\operatorname{supp} h_j \subset \partial\Omega \cap B_j$. We then find our trace operator:

$$\operatorname{tr}_{\partial\Omega} f = \sum_{j=1}^{J} h_j. \tag{4.68}$$

□

Remark: Theorem 4.8 also holds for $W^k_p(\Omega)$, $k > 1$ and $W^s_2(\Omega)$, $s > 1$, since these are subspaces of $W^1_p(\Omega)$ and $W^1_2(\Omega)$ respectively.

Definition 4.9. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set such that $\partial\Omega$ is of class C^1. Then we denote the space of all functions $f \in W^k_p(\Omega)$, $k \geq 1$, with $\operatorname{tr}_{\partial\Omega} f = 0$ by $W^k_{p,0}(\Omega)$.*

There is an alternative way of defining $W^k_{p,0}(\Omega)$, that has the advantage of being valid regardless of the smoothness of $\partial\Omega$ and the exponent k: it even holds for $W^s_2(\Omega)$ for $0 < s < 1$, and indeed for $H^s(\Omega)$ for $s < 0$.

Definition 4.10 (Alternative definition of $W^k_{p,0}(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then we define $W^s_{p,0}(\Omega)$, $1 \le p < \infty$ with $s \in \mathbb{N}_0$ for $p \neq 2$, $s \in \mathbb{R}$ for $p = 2$, as the closure of $\mathscr{D}(\Omega)$ in the $W^s_p(\Omega)$ norm.*

When $\partial\Omega$ is of class C^1 and $s \ge 1$, then these two definitions coincide. This is the content of the next theorem.

Theorem 4.11. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set such that $\partial\Omega$ is of class C^1 and let $f \in W^1_p(\Omega)$. Then $\mathrm{tr}_{\partial\Omega} f = 0$ if and only if f there is a sequence $\{f_n\}_n \subset \mathscr{D}(\Omega)$ such that*

$$f_n \to f \quad \text{in } W^1_p(\Omega). \tag{4.69}$$

Proof. See Theorem 2 in chapter 5.5 in [7]. □

We may note that for $W^s_{p,0}(\Omega)$, extension to $W^s_p(\mathbb{R}^n)$ is trivial. Any function in $\mathscr{D}(\Omega)$ may simply be extended by zero outside Ω regardless of the properties of $\partial\Omega$, and so the result follows by taking limits. Since Theorem 4.4 depended on being able to extend the functions to $\mathbb{R}^n$, we may now revisit the theorem for $W^s_{p,0}$, giving a version that we rely on in Chapter 5 and 6.

Theorem 4.12. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, and let $1 \le p < n$. Then the following holds:*

(i)

$$W^1_{p,0}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \tag{4.70}$$

for all $1 \le q < p^*$, where $p^* = \frac{np}{n-p}$.

(ii)

$$W^s_{2,0}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \tag{4.71}$$

for all $0 < s < 1$.

We will only sketch the proof as the main details of (i) are already done in the proof of Theorem 4.4 and (ii) follows from Proposition 4.7.

Proof sketch. (i) In Theorem 4.4 we extended $W^1_p(\Omega)$–functions to $W^1_p(\mathbb{R}^n)$ by applying the extension theorem, Theorem 4.2, to prove the inclusion $W^1_p(\Omega) \hookrightarrow L^{p^*}(\Omega)$. Similarly for the compactness, we extended elements of $W^1_p(\Omega)$ and mollified them to apply Arzela-Ascoli. These extensions require that $\partial\Omega$ is of class C^1. However, $\mathscr{D}(\Omega)$ is by Definition 4.10 dense in $W^1_{p,0}(\Omega)$, and extending these functions to $\mathscr{D}(\mathbb{R}^n)$ is done simply by setting them to be zero outside Ω. All arguments done in the proof of Theorem 4.4 can then be carried out as before, without considering the regularity of the boundary $\partial\Omega$.

(ii) It is possible to derive this from Proposition 4.7, noting again that we do not need to put any requirements on $\partial\Omega$ to extend functions in $W^s_{2,0}(\Omega)$ to $W^s_2(\Omega)$. □

We may now also state and prove the useful Poincaré inequality.

Theorem 4.13. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded such that $\partial\Omega$ is C^1, and let $1 \le p < n$. If $f \in W^1_{p,0}(\Omega)$, then there exists a constant C dependent only on p, q, n and Ω such that*

$$\|f\|_{q,\Omega} \le C\|\nabla f\|_{p,\Omega} \tag{4.72}$$

for all q such that $1 \le q \le p^*$. In particular

$$\|f\|_{p,\Omega} \le C\|\nabla f\|_{p,\Omega}. \tag{4.73}$$

Proof. Theorem 4.11 implies that there exists a sequence $\{f_n\}_n \subset \mathscr{D}(\Omega)$ such that $f_n \to f$ in $W^1_p(\Omega)$. Extending f_n by zero outside Ω, we may apply Theorem 4.5 and in the limit obtain

$$\|f\|_{p^*,\Omega} \le C\|\nabla f\|_{p,\Omega}. \tag{4.74}$$

Since Ω is bounded, equation (4.32) implies

$$\|f\|_{q,\Omega} \le C\|\nabla f\|_{p,\Omega} \tag{4.75}$$

for $1 \le q \le p^*$. □

5 Variational Methods

The calculus of variations deals with finding maxima or minima of functionals on function spaces. Many problems in analysis, in particular differential equations, can be recast as functional equations $\mathrm{D}E(u) = 0$, where one looks for the solution u among a suitable class of admissible functions belonging to some Banach space V. Thus the problem of finding a solution to a PDE can be restated as finding a minimum of a functional, since the derivative will be zero at the minimum.

The validity of this approach is perhaps better illustrated by an example than by abstract explainations.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Consider the functional

$$I(v) = \int_\Omega |\nabla v(x)|^2 \,\mathrm{d}x \tag{5.1}$$

where $v : \Omega \to \mathbb{R}$. We postpone the question of smoothness and regularity conditions on v. Suppose u is a minimiser of I over a suitable set, say

$$I(u) = \min\{I(v) : v = f \quad \text{on } \partial\Omega\}. \tag{5.2}$$

Now, for a function $\varphi \in \mathscr{D}(\Omega)$, we consider the function

$$\alpha(t) = \int_\Omega |\nabla(u + t\varphi)(x)|^2 \,\mathrm{d}x. \tag{5.3}$$

Expanding this expression, we obtain

$$\alpha(t) = \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x + 2t \int_\Omega \nabla u(x) \cdot \nabla\varphi(x) \,\mathrm{d}x + t^2 \int_\Omega |\nabla\varphi(x)|^2 \,\mathrm{d}x. \tag{5.4}$$

Clearly, α is differentiable with respect to t, and by (5.2), α has an interior minimum at $t = 0$. In other words, $\alpha'(0) = 0$. That is,

$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x) \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathscr{D}(\Omega). \tag{5.5}$$

Assuming u is sufficiently smooth (that is, at least C^2), we may use integration by parts to obtain (since φ vanishes on $\partial\Omega$)

$$\int_\Omega (\Delta u(x)) \, \varphi(x) \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathscr{D}(\Omega). \tag{5.6}$$

Lemma 1.21 implies that if (5.6) holds, then $\Delta u = 0$ in Ω. Then, since $u = f$ on $\partial\Omega$ by assumption, we see that u solves the classical Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned} \tag{5.7}$$

We need to consider in which space we should look for such a minimiser, and there are many things to take into consideration. First and foremost, the space should contain a solution, and the larger the space, the more likely it is to contain a solution. But we also need to consider how weak solutions we will allow; are we satisfied with distributional solutions, or do we want classical solutions? Since the minimiser will inevitably have to be found by minimising sequences, the compactness and convergence properties of the space will be of utmost importance. Considering the theory we have developed so far, it is no surprise that the Sobolev spaces are ideal spaces for this purpose, containing both classical solutions and regular distributions, as well as having desirable compactness properties (cf. Chapter 4).

Let us formalise the procedure in the example above. Let $\Omega \subset \mathbb{R}^n$ be an open set. We consider functions

$$L : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \tag{5.8}$$

and functionals of the form

$$I(u) = \int_\Omega L(x, u(x), \nabla u(x)) \, \mathrm{d}x, \tag{5.9}$$

for some set of admissible functions $u : \bar{\Omega} \to \mathbb{R}$. That a minimiser of the functional considered in the example above solves a partial differential equation is no coincidence or a property of our particular choice of L, as the next proposition shows. First, let us fix some notation.

We write

$$L = L(x, z, p) = L(x_1, ..., x_n, z, p_1, ..., p_n), \quad x \in \Omega, \ z \in \mathbb{R}, \ p \in \mathbb{R}^n \tag{5.10}$$

and we set

$$\begin{aligned} \mathrm{D}_x L &= (L_{x_1}, ..., L_{x_n}) \\ \mathrm{D}_z L &= L_z \\ \mathrm{D}_p L &= (L_{p_1}, ..., L_{p_n}). \end{aligned} \tag{5.11}$$

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded such that $\partial\Omega$ is of class C^1. If L and I are as above, with L being C^2 in addition, and u is a minimiser of I among the set of smooth functions satisfying $u = g$ on $\partial\Omega$, then u solves the boundary value problem given in* (5.15).

Proof. Choose any function $\varphi \in \mathscr{D}(\Omega)$ and consider the function

$$\alpha(t) = I(u + t\varphi), \quad t \in \mathbb{R}. \tag{5.12}$$

By assumption, $\alpha(\cdot)$ has a minimum at $t = 0$, since $u + t\varphi = u = g$ on $\partial\Omega$. Therefore $\alpha'(0) = 0$. Furthermore, we have

$$\alpha'(t) = \int_\Omega \left[\sum_{i=1}^n L_{p_i}(x, u + t\varphi, \nabla u + t\nabla\varphi) \frac{\partial \varphi}{\partial x_i} + L_z(x, u + t\varphi, \nabla u + t\nabla\varphi)\varphi \right] \mathrm{d}x, \tag{5.13}$$

which implies, since L is smooth and φ has compact support,

$$\begin{aligned} 0 = \alpha'(0) &= \int_\Omega \sum_{i=1}^n L_{p_i}(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + L_z(x, u, \nabla u)\varphi \,\mathrm{d}x \\ &= \int_\Omega \left(-\sum_{i=1}^n \left(L_{p_i}(x, u, \nabla u)\right)_{x_i} + L_z(x, u, \nabla u) \right) \varphi \,\mathrm{d}x. \end{aligned} \tag{5.14}$$

Since this holds for all $\varphi \in \mathscr{D}(\Omega)$, Lemma 1.21 implies that u solves the boundary value problem

$$\begin{aligned} -\sum_{i=1}^n \left(L_{p_i}(x, u, \nabla u)\right)_{x_i} + L_z(x, u, \nabla u) &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{5.15}$$

□

5.1 Existence of Minimisers

We have so far established that a minimiser of (5.9) solves a partial differential equation, but the existence of such a minimiser, and how to find it, is not clear in the general case. There are some conditions which the space of admissible functions and the functional must satisfy, and these conditions are what we will now investigate.

For the functional to attain a global minimum, it is clear that being bounded below is a necessary condition, however, this is not in general sufficient for it to attain its infimum. Consider for example the function e^x, $x \in \mathbb{R}$. It is certainly bounded below, but it does not attain its infimum. The problem in this case is that the infimum of e^x is its limit at infinity. A reasonable way to avoid such a situation is to require that $I(u)$ grows as $|u| \to \infty$.

More precisely, for $1 < q < \infty$ fixed, we will assume that there exist a constant $\alpha > 0$ and a function $\gamma \in L^1(\Omega)$ such that

$$L(x, z, p) \geq \alpha |p|^q - \gamma(x) \quad \text{for almost every } p, z, x. \tag{5.16}$$

This implies

$$I(u) \geq \alpha \|\nabla u\|_{q,\Omega}^q - \|\gamma\|_{1,\Omega} \tag{5.17}$$

for some constant $\alpha > 0$, which further implies

$$I(u) \to \infty \quad \text{when } \|\nabla u\|_{q,\Omega}^q \to \infty. \tag{5.18}$$

Condition (5.17) is called called a *coercivity* condition on I.

This requirement further encourages us to look for $u \in W_q^1(\Omega)$ such that $u - g \in W_{q,0}^1(\Omega)$. To simplify notation, we denote this space as

$$A = \{u \in W_q^1(\Omega) : u - g \in W_{q,0}^1(\Omega)\}. \tag{5.19}$$

However, condition (5.17) is not enough. That is, it is not sufficient for I to attain its infimum. Let

$$m = \inf_{u \in A} I(u) \tag{5.20}$$

and let $\{u_n\}_n \subset A$ be a *minimising sequence*, that is

$$I(u_n) \to m \quad \text{as } n \to \infty. \tag{5.21}$$

We wish to show that $\{u_n\}_n$ or a subsequence thereof converges to a minimiser. That is, we want $u_n \to u$ as $n \to \infty$ such that $I(u) = m$. The convergence of $\{I(u_n)\}_n \subset \mathbb{R}$ implies that

$$\sup_{n \in \mathbb{N}} |I(u_n)| < \infty, \tag{5.22}$$

which by (5.17) implies

$$\sup_{n \in \mathbb{N}} \|\nabla u_n\|_{q,\Omega} < \infty. \tag{5.23}$$

Let $v \in A$. Then the traces of u_n and v are equal, and thus $u_n - v \in W_{q,0}^1(\Omega)$. We may then apply the Poincaré inequality, Theorem 4.13, to this difference and obtain

$$\|u_n\|_{q,\Omega} \leq \|u_n - v\|_{q,\Omega} + \|v\|_{q,\Omega} \leq C\|\nabla u_n - \nabla v\|_{p,\Omega} + C'. \tag{5.24}$$

This, together with (5.23) implies that

$$\sup_n \|u_n\|_{W_q^1(\Omega)} < \infty. \tag{5.25}$$

However, in an infinite dimensional space such as $W_q^1(\Omega)$, boundedness does not imply compactness. On the other hand, $L^q(\Omega)$ is a reflexive space for $1 < q < \infty$, which we are assuming. It therefore follows from the Banach-Alaoglu theorem that there exists a subsequence $\{u_{n_j}\}_j \subset \{u_n\}_n$ and a function $u \in W^1_q(\Omega)$ such that

$$\begin{aligned} u_{n_j} &\rightharpoonup u \text{ weakly in } L^q(\Omega) \\ \nabla u_{n_j} &\rightharpoonup \nabla u \text{ weakly in } L^q(\Omega). \end{aligned} \tag{5.26}$$

We have gained convergence by going to the weak topology, but we have lost something in the process as well, for in most (interesting) cases, I will not be continuous with respect to weak convergence. Thus we cannot infer that

$$I(u) = \lim_{j\to\infty} I(u_{n_j}), \tag{5.27}$$

i.e. we cannot conclude that u is a minimiser. We therefore need to impose another criterion on I.

Proposition 5.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set. If for any sequence $v_n, v \in W^1_q(\Omega)$, $1 \le q < \infty$, for all $n \in N$ such that*

$$\begin{aligned} v_n &\to v \textit{ in } L^1(\Omega) \\ \nabla v_n &\rightharpoonup \nabla v \textit{ weakly in } L^1(\Omega), \end{aligned} \tag{5.28}$$

the functional I defined by

$$I(u) = \int_\Omega L(x, u, \nabla u)\,\mathrm{d}x \tag{5.29}$$

satisfies (5.17) *and is lower semi-continuous with respect to the convergence in* (5.28), *then the following holds:*

(i) If $\partial\Omega$ is of class C^1, then I attains its infimum on $W^1_q(\Omega)$.

(ii) I attains its infimum on $W^1_{q,0}(\Omega)$.

Proof. Let $\{u_n\}_n$ be a minimising sequence for I in $W^1_q(\Omega)$. We will prove that any such minimising sequence has a subsequence that converges in the sense of (5.28). By the discussion above,

$$\sup_n \|u_n\|_{W^1_q(\Omega)} < \infty. \tag{5.30}$$

If $\partial\Omega$ is of class C^1 there exists, by Theorem 4.4, a subsequence (denoted by $\{u_n\}_n$) such that $u_n \to u$ in $L^1(\Omega)$. Since $\{\nabla u_n\}_n$ is bounded in $L^q(\Omega)$, it follows from the discussion prior to Proposition 5.2 that there exists a subsequence $\{u_n\}_n$ such that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^q(\Omega)$.

If $\{u_n\}_n \subset L^q(\Omega)$ and $u_n f \to uf$ in $L^1(\Omega)$ for all $f \in L^{q'}(\Omega)$, where $1/q + 1/q' = 1$, then $u_n f \to uf$ in $L^1(\Omega)$ for all $f \in L^\infty(\Omega)$ since $L^\infty(\Omega) \subseteq L^{q'}(\Omega)$. In other words, weak convergence in $L^q(\Omega)$ implies weak convergence in $L^1(\Omega)$. It follows that for any minimising sequence $\{u_n\}_n \subset W^1_q(\Omega)$ of I, there exists a $u \in W^1_q(\Omega)$ and a subsequence (again denoted by $\{u_n\}_n$) such that

$$\begin{aligned} u_n &\to u \text{ in } L^1(\Omega) \\ \nabla u_n &\rightharpoonup \nabla u \text{ in } L^1(\Omega). \end{aligned} \tag{5.31}$$

Thus we have by lower semi-continuity of I with respect to this convergence that

$$I(u) \leq \liminf_{n\to\infty} I(u_n). \tag{5.32}$$

If $\{u_n\}_n$ is a minimising sequence for I in $W^1_{q,0}(\Omega)$, then we may apply Theorem 4.12 to get strong convergence of (a subsequence) $\{u_n\}_n$ in $L^1(\Omega)$ without any conditions on $\partial\Omega$. The weak convergence of (a subsequence of) $\{\nabla u_n\}_n$ depends only on the boundedness of the domain and not on the boundary, as we saw above, and so we get, by lower semi-continuity,

$$I(u) \leq \liminf_{n\to\infty} I(u_n) \tag{5.33}$$

also in this case. □

Checking if the functional is lower semi-continuous in the sense of Proposition 5.2 directly is in general difficult, and would entail showing that the function attains its infimum directly. Therefore we want conditions for I that are easier to check and that implies lower semi-continuity. Such conditions are given in the theorem below.

Theorem 5.3. *Assume $L : \Omega \times \mathbb{R} \times \mathbb{R}^n$ is a Caratheodory function that satisfies the following:*

(i) $L(x, z, \cdot)$ is convex in p for almost every x, z.

(ii) $L(x, z, p) \geq \gamma(x)$ for almost every x, z, p and $\gamma \in L^1(\Omega)$.

Then, if $u_k, u \in W^1_{q,loc}(\Omega)$ and $u_k \to u$ in $L^1(\Omega')$ and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^1(\Omega')$ for every $\Omega' \Subset \Omega$, it follows that

$$I(u) \leq \liminf_{k\to\infty} I(u_k). \tag{5.34}$$

Proof. We may assume that $\{I(u_k)\}_k$ is finite and convergent, and we may also assume $L \geq 0$, since we may otherwise replace L by $L - \gamma$. Let $\Omega' \Subset \Omega$. Since $L^1(\Omega')$ is convex and $\nabla u_k \rightharpoonup u$ weakly in $L^1(\Omega')$, there exists for any $k_0 \in \mathbb{N}$ a sequence $\{P_l\}_{l\geq k_0}$ of convex linear combinations of ∇u_k's,

$$P_l = \sum_{k=k_0}^{l} \alpha^l_k \nabla u_k, \quad 0 \leq \alpha^l_k \leq 1, \quad \sum_{k=k_0}^{l} \alpha^l_k = 1, \; l \geq k_0, \tag{5.35}$$

such that $P_l \to \nabla u$ strongly in $L^1(\Omega')$. See for instance Theorem 3.13 in [17] for a proof of this result in its most general form. By strong convergence in $L^1(\Omega')$ we also have, by passing to a subsequence if necessary, pointwise convergence almost everywhere in Ω'. Therefore, by Fatou's lemma

$$\int_{\Omega'} L(x, u, \nabla u)\,\mathrm{d}x \leq \liminf_{l\to\infty} \int_{\Omega'} L(x, u, P_l)\,\mathrm{d}x. \tag{5.36}$$

Now, since L is convex in its third argument by assumption, the following holds for any $k_0 \in \mathbb{N}$, any $l \geq k_0$ and almost every $x \in \Omega'$:

$$L(x, u, P_l) = L\left(x, u, \sum_{k=k_0}^{l} \alpha_k^l \nabla u_k\right) \leq \sum_{k=k_0}^{l} \alpha_k^l L(x, u, \nabla u_k). \tag{5.37}$$

From (5.36) we then obtain

$$\int_{\Omega'} L(x, u, \nabla u)\,\mathrm{d}x \leq \sup_{k\geq k_0} \int_{\Omega'} L(x, u, \nabla u_k)\,\mathrm{d}x. \tag{5.38}$$

Since this holds for every $k_0 \in \mathbb{N}$, this implies

$$\int_{\Omega'} L(x, u, \nabla u)\,\mathrm{d}x \leq \limsup_{k\to\infty} \int_{\Omega'} L(x, u, \nabla u_k)\,\mathrm{d}x. \tag{5.39}$$

In order to complete the proof, we need a relation between $L(x, u, \nabla u_k)$ and $L(x, u_k, \nabla u_k)$. More precisely, we will prove that there exists a subsequence $\{u_k\}_k$ such that

$$L(x, u_k, \nabla u_k) - L(x, u, \nabla u_k) \to 0 \tag{5.40}$$

in measure, locally in Ω. Assume to the contrary that there exists a set $\Omega' \Subset \Omega$ and $\varepsilon > 0$ such that for

$$\Omega_k = \{x \in \Omega' : |L(x, u_k, \nabla u_k) - L(x, u, \nabla u_k)| \geq \varepsilon\} \tag{5.41}$$

we have

$$\liminf_{k\to\infty} \mathcal{L}^n(\Omega_k) \geq 2\varepsilon. \tag{5.42}$$

By weak convergence, $\{\nabla u_k\}_k$ is uniformly bounded in $L^1(\Omega')$. Hence

$$\mathcal{L}^n\{x \in \Omega' : |\nabla u_k(x)| \geq l\} \leq \frac{1}{l}\int_{\Omega} |\nabla u_k|\,\mathrm{d}x \leq \frac{C}{l} \leq \varepsilon, \tag{5.43}$$

if l is large enough, say $l \geq l_\varepsilon$ for some l_ε depending on ε. Setting $\tilde{\Omega}_k = \{x \in \Omega_k : |\nabla u_k| \leq l_\varepsilon\}$ we have, by (5.42)

$$\liminf_{k\to\infty} \mathcal{L}^n(\tilde{\Omega}_k) \geq \varepsilon. \tag{5.44}$$

Thus, setting $\Omega^K = \bigcup_{k\geq K} \tilde{\Omega}_k$, we have

$$\liminf_{k\to\infty} \mathcal{L}^n(\Omega^K) \geq \varepsilon, \tag{5.45}$$

for all $K \in \mathbb{N}$. Furthermore, the inclusions $\Omega^{K+1} \subset \Omega^K \subset \Omega'$ hold for all K and so for $\Omega^\infty = \bigcap_{K\in\mathbb{N}} \Omega^K$ we have $\mathcal{L}^n(\Omega^\infty) \geq \varepsilon$. Since $\{u_k\}_k$ converges to u in $L^1(\Omega')$, there exists a subsequence that converges to u pointwise almost everywhere. Neglecting a set of measure zero, we may therefore assume that $L(x, z, p)$ is continuous in (z, p) (it is by assumption continuous almost everywhere) and that $u_k(x) \to u(x)$ as $k \to \infty$ for every point $x \in \Omega^\infty$.

Now let $x \in \Omega^\infty$. Since we are neglecting a set of measure zero, $\nabla u_k(x)$ is unambiguously defined and as noted above, bounded uniformly in k. Thus there exists a subsequence (relabelled) and an element $p \in \mathbb{R}^n$ such that $\nabla u_k(x) \to p$ as $k \to \infty$. Then, by continuity

$$L(x, u(x), \nabla u_k(x)) \to L(x, u(x), p). \tag{5.46}$$

But, since $u_k(x) \to u(x)$, we also have

$$L(x, u_k(x), \nabla u_k(x)) \to L(x, u(x), p). \tag{5.47}$$

Since this holds for every $x \in \Omega^\infty$, except for in a set of measure zero, this contradicts our assumptions on Ω_k. Thus we have convergence in measure, and for any $\varepsilon > 0$ and any $k_0 \in \mathbb{N}$ there exists an $k \geq k_0$ and a set $\Omega^k_\varepsilon \subset \Omega$ such that

$$|L(x, u_k(x), \nabla u_k(x)) - L(x, u(x), \nabla u_k(x))| < \varepsilon \tag{5.48}$$

for all $x \in \Omega^k_\varepsilon$ and $\mathcal{L}^n(\Omega \backslash \Omega^k_\varepsilon) < \varepsilon$. Hence we can choose a subsequence (still labelled with k) such that, upon replacing ε by $\varepsilon_k = 2^{-k}$, there is a set $\Omega^k_{\varepsilon_k} \subset \Omega'$ with $\mathcal{L}^n(\Omega' \setminus \Omega^k_{\varepsilon_k}) < 2^{-k}$ such that (5.48) holds for $x \in \Omega^k_{\varepsilon_k}$ with $\varepsilon = \varepsilon_k$. It follows that for any $\varepsilon > 0$, if we choose $k_0 = k_0(\varepsilon) > |\log_2 \varepsilon|$ and set $\Omega_\varepsilon = \bigcup_{k \geq k_0} \Omega^k_{\varepsilon_k}$, then $\mathcal{L}^n(\Omega' \setminus \Omega_\varepsilon) < \varepsilon$ and (5.48) holds uniformly for all $x \in \Omega_\varepsilon$ and all $k \geq k_0(\varepsilon)$. Furthermore, for $\varepsilon < \delta$, we have $\Omega_\varepsilon \supset \Omega_\delta$.

Next we cover Ω with disjoint bounded sets $\Omega^m \Subset \Omega$, $m \in \mathbb{N}$. Given $\varepsilon > 0$, choose a sequence $\{\varepsilon^m\}_m$ with $\varepsilon^m > 0$ for all $m \in \mathbb{N}$, such that

$$\sum_{m\in\mathbb{N}} \mathcal{L}^n(\Omega^m)\varepsilon^m < \varepsilon. \tag{5.49}$$

Passing to a subsequence, if necessary, we can for each Ω^m and ε^m choose k^m_0 and $\Omega^m_\varepsilon \subset \Omega^m$ such that $\mathcal{L}^n(\Omega^m_\varepsilon) < \varepsilon^k$ and

$$|L(x, u_k(x), \nabla u_k(x)) - L(x, u(x), \nabla u_k(x)| < \varepsilon^m \tag{5.50}$$

uniformly for $x \in \Omega^k_\varepsilon$, $k \geq k^m_0$. If $\varepsilon < \delta$, we may assume $\Omega^k_\varepsilon \supset \Omega^k_\delta$ for all $k \in \mathbb{N}$. Then, defining, for any $K \in \mathbb{N}$, $\Omega^{(K)} = \bigcup_{k=1}^K \Omega^k$ and $\Omega^{(K)}_\varepsilon = \bigcup_{k=1}^K \Omega^k_\varepsilon$, we have

$$\begin{aligned}
\int_{\Omega^{(K)}_\varepsilon} L(x, u, \nabla u)\,\mathrm{d}x &\leq \limsup_{k\to\infty} \int_{\Omega^{(K)}_\varepsilon} L(x, u, \nabla u_k)\,\mathrm{d}x \\
&\leq \limsup_{k\to\infty} \int_{\Omega^{(K)}_\varepsilon} L(x, u_k, \nabla u_k)\,\mathrm{d}x + \varepsilon \\
&\leq \limsup_{k\to\infty} I(u_k) + \varepsilon = \liminf_{k\to\infty} I(u_k) + \varepsilon.
\end{aligned} \tag{5.51}$$

The way constructed the sets $\Omega_\varepsilon^{(K)}$ implies that $\mathcal{L}^n\left(\Omega^{(K)} \setminus \Omega_\varepsilon^{(K)}\right) \to 0$ as $\varepsilon \downarrow 0$, and $\mathcal{L}^n\left(\Omega \setminus \Omega^{(K)}\right) \to 0$ as $K \to \infty$. Thus the result follows from the Monotone Convergence Theorem, letting $\varepsilon \downarrow 0$ and then $K \to \infty$. □

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $L : \Omega \times \mathbb{R} \times \mathbb{R}^n$ be as in Theorem 5.3, with condition (ii) replaced by $L(x, z, p) \geq \alpha|p|^q + \gamma(x)$ for almost every x, z, p, some constant $\alpha > 0$ and some $1 < q < \infty$, $\gamma \in L^1(\Omega)$. Then, assuming that the set of admissible functions is non-empty, there exists a $u \in W^1_{q,0}(\Omega)$ such that*

$$I(u) = \inf_{v \in W^1_{q,0}(\Omega)} I(v). \tag{5.52}$$

Furthermore, if $\partial\Omega$ is of class C^1, then there exists a $u \in W^1_q(\Omega)$ such that

$$I(u) = \inf_{v \in W^1_q(\Omega)} I(v). \tag{5.53}$$

Proof. The condition $L(x, z, p) \geq |p|^q + \gamma(x)$ for almost every x, z, p and some $1 < q < \infty$, $\gamma \in L^1(\Omega)$ is stronger than condition (ii) in Theorem 5.3, and so Theorem 5.3 proves that I is lower semi-continuous with respect to the convergence (5.28). The result then follows from Proposition 5.2. □

Remark: Even though Theorem 5.3 proves lower semi-continuity with respect to convergence in $L^1(\Omega)$ of sequences $\{u_k\}_n \subset W^1_1(\Omega)$ under certain conditions, Theorem 5.4 does not in general hold for $W^1_1(\Omega)$. While Theorem 4.4 gives strong convergence of u_k in $L^1(\Omega)$, we have no results that give weak convergence of ∇u_k in $L^1(\Omega)$. It is possible to prove that ∇u_k converges in some sense, but its limit is in general not in $L^1(\Omega)$, but in BMO, the space of functions with *bounded mean oscillation*.

5.2 Solving the Dirichlet Problem by Variational Methods

Let us consider the Dirichlet problem again. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, and consider the differential equation

$$\Delta u = f \quad \text{in } \Omega \tag{5.54}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{5.55}$$

where $f \in L^2(\Omega)$. We consider the functional

$$I(u) = \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x - \int_\Omega f(x)u(x) \, \mathrm{d}x. \tag{5.56}$$

Considering the boundary condition and that we need $|\nabla u\|^2$ to be integrable over Ω, it is natural to work in the space $W^1_{2,0}(\Omega)$. Then $fu \in L^1(\Omega$, since $f, u \in L^2(\Omega)$. Thus the functional satisfies condition (5.17) with $q = 2$. We wish to show that the functional (5.56) is convex in ∇u.

Lemma 5.5. *The functional I as defined in* (5.56) *is convex. That is*

$$I(tu + (1-t)v) \le tI(u) + (1-t)I(v) \tag{5.57}$$

for all $u, v \in W^1_{2,0}(\Omega)$ and all $t \in [0,1]$.

Proof.

$$\begin{aligned} I(tu + (1-t)v) &= \int_\Omega |t\nabla u(x) + (1-t)\nabla v(x)|^2 \,\mathrm{d}x - \int_\Omega f(x)\,(tu(x) + (1-t)v(x))\,\mathrm{d}x \\ &\le \int_\Omega t|\nabla u(x)|^2 + (1-t)|\nabla v(x)|^2 \,\mathrm{d}x - \int_\Omega tf(x)u(x) + (1-t)f(x)v(x)\,\mathrm{d}x \qquad (5.58) \\ &= tI(u) + (1-t)I(v), \end{aligned}$$

where we used the fact that $w \mapsto |w|^2$ is convex in going from the first to the second line. □

The functional (5.56) thus satisfies all the criteria of Theorem 5.4 and hence there exists a minimiser $u \in W^1_{2,0}(\Omega)$. Recalling the calculations done in (5.3)-(5.5), we have by approximation that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x - \int_\Omega f(x)v(x) = 0 \quad \text{for all } v \in W^1_{2,0}(\Omega). \tag{5.59}$$

Thus u is a weak solution of the problem 5.54.

6 Fractional Operators

Now we turn our attention to fractional operators, the main example being the fractional Laplacian: $(-\Delta)^{\frac{s}{2}}$ for $s \in (0,2)$. Our first goal is to find a usable definition for this. If $u \in \mathscr{S}(\mathbb{R}^n)$ (or $\mathscr{S}'(\mathbb{R}^n)$) we have, by Proposition 1.12 and its corollary, the following equality:

$$\mathscr{F}((-\Delta)u)(\xi) = |\xi|^2\,\mathscr{F}\,u(\xi). \tag{6.1}$$

With that in mind we define $(-\Delta)^{\frac{s}{2}}$ as follows:

Definition 6.1. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. Then*

$$(-\Delta)^{\frac{s}{2}}u(x) = \mathscr{F}^{-1}(|\xi|^s\,\mathscr{F}\,u(\xi))(x), \quad x \in \mathbb{R}^n. \tag{6.2}$$

Recalling Definition 2.6 and the fact that the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$, it is natural to extend this definition to the space $H^s(\mathbb{R}^n)$.

Proposition 6.2. *Let $s \in (0,2)$ and let $u \in H^s(\mathbb{R}^n)$. Then the operator $1 + (-\Delta)^{\frac{s}{2}}$ defines a continuous mapping $H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and we have following relation:*

$$\|u\|_{H^s(\mathbb{R}^n)} \sim \|(1 + (-\Delta)^{\frac{s}{2}})u\|_2. \tag{6.3}$$

Proof. Using Theorem 1.16 and Definition 6.1, we have

$$\|(1+(-\Delta)^{\frac{s}{2}})u\|_2 = \|\mathscr{F}(1+(-\Delta)^{\frac{s}{2}})u\|_2 = \|(1+|\xi|^s)\,\mathscr{F}\,u\|_2 \sim \|u\|_{H^s(\mathbb{R}^n)}. \tag{6.4}$$

This implies that the mapping is continuous. □

An alternative definition of the fractional Laplace is

$$\begin{aligned}(-\Delta)^{\frac{s}{2}}u(x) &= C(n,s)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{n+s}}\,\mathrm{d}y \\ &= -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+s}}\,\mathrm{d}y, \end{aligned} \tag{6.5}$$

where $C(n,s)$ is given by

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-cos(x_1)}{|x|^{n+s}}\,\mathrm{d}x\right)^{-1}. \tag{6.6}$$

This is an equivalent definition, but we will not prove it here since we will not need it because for the properties and results we are interested in, we claim that the Fourier definition is easier to work with. For a proof of the equivalence of these definitions, we refer to [15].

In the sequel, we will work with the operator $1+(-\Delta)^{\frac{s}{2}}$. The fractional Laplace operator $(-\Delta)^{\frac{s}{2}}$ is more studied in the literature than our operator, but the methods and results are almost the same for these operators on domains. The main difference is that $(-\Delta)^{\frac{s}{2}}u = (-\Delta)^{\frac{s}{2}}(u+c)$ for any constant c (this is a direct consequence of the Riemann-Lebesgue lemma, stating that the Fourier transform of any constant is zero), and so one must consider homogeneous Sobolev spaces, and we wish to avoid this.

6.1 The Operator $1+(-\Delta)^{\frac{s}{2}}$ on Domains

First we want to investigate the Dirichlet problem for the operator $1+(-\Delta)^{\frac{s}{2}}$ on a bounded, open set $\Omega \subset \mathbb{R}^n$. However, from Definition 6.1, it is clear the operator $1+(-\Delta)^{\frac{s}{2}}$ is non-local, and the values of $(1+(-\Delta)^{\frac{s}{2}})u$ in Ω will depend not only on its values at $\partial\Omega$ like with Δ, but also on its values in $\mathbb{R}^n\setminus\Omega$. We will therefore consider the following problem:

$$\begin{aligned}(1+(-\Delta)^{\frac{s}{2}})u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n\setminus\Omega.\end{aligned} \tag{6.7}$$

Remark: It is possible to work with the traditional Dirichlet problem for $1+(-\Delta)^{\frac{s}{2}}$ without any further assumptions on u outside $\bar{\Omega}$, but this is beyond the scope of this paper. The problem and operator will, in general, not be the same as in (6.7).

Remark: Having noted the non-local nature of the fractional Laplace operator, it is worth mentioning that there exist a relation between the fractional Laplace and an extension problem [2]. That is, the value of the fractional Laplace of a function in $\mathbb{R}^n$ can be related to a local problem in $\mathbb{R}^n \times \mathbb{R}_+$, and so one can get around its non-local nature. For the results we wish to prove, we do not need this, but it is very useful for proving many properties and inequalities for the operator.

A more general problem than (6.7) is considered in [22], although for $(-\Delta)^{\frac{s}{2}}$, with f also depending on the function u in addition to the spatial variable. They prove existence of solutions in homogeneous Sobolev spaces in this case, under certain conditions on $f(x,u)$, using the Mountain Pass Lemma. In our simpler case, one does not need such heavy machinery. We want to find solutions to (6.7) using the theory developed in previous chapters, and considering the condition $u = 0$ in $\mathbb{R}^n \setminus \Omega$ it is natural to consider $u \in H_0^r(\Omega)$ for some $r \in \mathbb{R}$. To make u globally defined, we can trivially consider $H_0^r(\Omega) \subset H^r(\mathbb{R}^n)$ by extending functions by zero outside Ω. Our main theorem is the following:

Theorem 6.3. *For any bounded, open set $\Omega \subset \mathbb{R}^n$, and for any $f \in H^{-s/2}(\Omega)$, the problem* (6.7) *admits at least one solution $u \in H_0^{s/2}(\Omega)$.*

We will prove this theorem in two different ways: by means of the direct method of variations, and by the Lax-Milgram Theorem. The first proof relies on two lemmas.

The operator $1 + (-\Delta)^{\frac{s}{2}}$ is defined as $\mathscr{F}^{-1}\left((1+|\xi|^s)\,\mathscr{F}\,\cdot\right)$, and we define the square root of the operator, $(1+(-\Delta)^s)^{\frac{1}{2}}$, as $\mathscr{F}^{-1}\left((1+|\xi|^s)^{\frac{1}{2}}\,\mathscr{F}\,\cdot\right)$.

Lemma 6.4. *Let $f \in H^{-s/2}(\Omega)$. Then a minimiser of the functional I on $H_0^{s/2}(\Omega) \subset H^{s/2}(\mathbb{R}^n)$ defined by*

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} u\right)^2 \mathrm{d}x - \int_\Omega f u \,\mathrm{d}x, \tag{6.8}$$

solves (6.7) *in the sense of distributions.*

Proof. We define the function

$$\alpha(t) = \frac{1}{2}\int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} (u+t\varphi)\right)^2 \mathrm{d}x - \int_\Omega f(u+t\varphi)\,\mathrm{d}x \tag{6.9}$$

where $\varphi \in \mathscr{D}(\Omega)$. Expanding (6.9) and evaluating the derivative with respect to t at $t=0$ yields

$$\alpha'(0) = \int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} u\right)(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}\varphi\,\mathrm{d}x - \int_\Omega f\varphi\,\mathrm{d}x = 0. \tag{6.10}$$

By Theorem 1.16 and Definition 6.1, this equals

$$\begin{aligned}
\alpha'(0) &= \int_{\mathbb{R}^n} \left(\mathscr{F}\left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u\right)\right) \overline{\mathscr{F}\left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}\varphi\right)} \mathrm{d}\xi - \int_\Omega f\varphi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left((1+|\xi|^s)^{\frac{1}{2}} \mathscr{F} u\right) (1+|\xi|^s)^{\frac{1}{2}} \overline{\mathscr{F}\varphi} \mathrm{d}\xi - \int_\Omega f\varphi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} ((1+|\xi|^s) \mathscr{F} u) \overline{\mathscr{F}\varphi} \mathrm{d}\xi - \int_\Omega f\varphi \,\mathrm{d}x \qquad (6.11) \\
&= \int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})u\right) \varphi \,\mathrm{d}x - \int_\Omega f\varphi \,\mathrm{d}x \\
&= \int_{\Omega} \left((1+(-\Delta)^{\frac{s}{2}})u\right) \varphi \,\mathrm{d}x - \int_\Omega f\varphi \,\mathrm{d}x = 0.
\end{aligned}$$

The second to last equality follows from $\varphi \in \mathscr{D}(\Omega)$. Since $\varphi \in \mathscr{D}(\Omega)$ is arbitrary, Lemma 1.21 implies that

$$(1+(-\Delta)^{\frac{s}{2}})u = f \qquad (6.12)$$

in Ω, in the sense of distributions. □

Remark: The non-local nature of $(-\Delta)^{\frac{s}{2}}$ and the calculations done in (6.11) show the reason to take the integral in (6.8) over $\mathbb{R}^n$ instead of Ω.

Remark: $1+(-\Delta)^{\frac{s}{2}}$ maps $H_0^{s/2}$ to $H^{-s/2}$. An element of $H^{-s/2}$ is in general not a function, but a distribution, so from Lemma 6.4 we cannot guarantee any more than the existence of solutions in the distributional sense. On the other hand, it is clear that if the minimiser u and f satisfy some stronger regularity condition, say $u \in H_0^s(\Omega)$ and $f \in L^2(\Omega)$, then $(1+(-\Delta)^{\frac{s}{2}})u$ will be a function.

Lemma 6.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open and $f \in H^{-s/2}(\Omega)$. Then the functional*

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u\right)^2 \mathrm{d}x - \int_\Omega fu \,\mathrm{d}x \qquad (6.13)$$

attains its minimum on $H_0^{s/2}(\Omega) \subset H^{s/2}(\mathbb{R}^n)$.

Proof. We first note that since $u = 0$ in $\mathbb{R}^n \setminus \Omega$, we may write

$$I(u) = \int_{\mathbb{R}^n} \frac{1}{2}\left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u\right)^2 - fu \,\mathrm{d}x. \qquad (6.14)$$

We wish to apply the results from Chapter 5, and our first step is to prove lower semi-continuity. Using the notation of Theorem 5.3, we have $L = L(x, u, (-\Delta)^{\frac{s}{4}}u) = \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u\right)^2 - fu$. Clearly,

$$|(1+|\xi|^s)^{\frac{1}{2}} \mathscr{F} u| \sim |(1+|\xi|^{\frac{s}{2}}) \mathscr{F} u| \qquad (6.15)$$

which implies

$$\|(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u\|_2 \sim \|(1+(-\Delta)^{\frac{s}{4}})u\|_2. \tag{6.16}$$

Proposition 6.2 then implies

$$\frac{1}{2}\int_{\mathbb{R}^n}\left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u\right)^2 \,\mathrm{d}x \geq \alpha\|(-\Delta)^{\frac{s}{4}}u\|_2^2 \tag{6.17}$$

for some constant $\alpha > 0$. Furthermore, since $f \in H^{-s/2}(\Omega)$, we have $fu \in L^1(\Omega)$, and also $fu \in L^1(\mathbb{R}^n)$ since $u = 0$ in $\mathbb{R}^n \setminus \Omega$, and combining this with the inequality (6.17) we get

$$I(u) \geq \alpha\|(-\Delta)^{\frac{s}{4}}u\|_2^2 - \|fu\|_1. \tag{6.18}$$

Convexity of L in the third factor is straightforward to check and thus I satisfies the criteria of Theorem 5.3. The only property of the sequences $\{u_n\}_n$ and $\{\nabla u_n\}_n$ used in the proof of Theorem 5.3 is strong local L^1 convergence and weak local L^1 convergence, respectively, and it then follows from that theorem that I is lower semi-continuous with respect to the convergence

$$\begin{aligned} u_n &\to u \text{ in } L^1(\Omega') \text{ for all } \Omega' \Subset \mathbb{R}^n \\ (-\Delta)^{\frac{s}{4}}u_n &\rightharpoonup (-\Delta)^{\frac{s}{4}}u \text{ weakly in } L^1(\Omega') \text{ for all } \Omega' \Subset \mathbb{R}^n. \end{aligned} \tag{6.19}$$

The next step is to prove that a minimising sequence $\{u_n\}_n \subset H_0^{s/2}(\Omega)$ of I converges to a $u \in H_0^{s/2}(\Omega)$ in this way.

Let $\{u_n\}_n \subset H_0^{s/2}(\Omega)$ be a minimising sequence for the functional I. This implies $\sup_n \|u_n\|_{H^{s/2}(\mathbb{R}^n)} < \infty$. Since Ω is bounded, Theorem 4.12 implies that there exists a subsequence (denoted $\{u_n\}_n$) such that $u_n \to u$ strongly in $L^2(\Omega)$ for some u, which by boundedness of domain implies strong convergence in $L^1(\Omega)$. Since $u_n = 0$ in $\mathbb{R}^n \setminus \Omega$ for every $n \in \mathbb{N}$, we may set $u = 0$ in $\mathbb{R}^n \setminus \Omega$ and we get

$$u_n \to u \text{ in } L^1(\Omega') \text{ for all } \Omega' \Subset \mathbb{R}^n. \tag{6.20}$$

Since $\sup_n \|u_n\|_{H^{s/2}(\mathbb{R}^n)} < \infty$, it follows from Proposition 6.2 and equation (6.16) that $\sup_n \|(1+(-\Delta)^{\frac{s}{4}})u_n\|_2 < \infty$, and thus

$$\sup_n \|(-\Delta)^{\frac{s}{4}}u_n\|_{2,\Omega'} < \infty \text{ for all } \Omega' \Subset \mathbb{R}^n. \tag{6.21}$$

Banach-Alaoglu and boundedness of domain then gives

$$(-\Delta)^{\frac{s}{4}}u_n \rightharpoonup (-\Delta)^{\frac{s}{4}}u \text{ weakly in } L^1(\Omega') \text{ for all } \Omega' \Subset \mathbb{R}^n. \tag{6.22}$$

It then follows that

$$I(u) \leq \liminf_{n\to\infty} I(u_n) \tag{6.23}$$

and $I(u) = \inf_{v\in H_0^{s/2}(\Omega)} I(v)$. □

Proof I of Theorem 6.3. This follows immediately from Lemmas 6.4 and 6.5 $\square$

We will also prove Theorem 6.3 by an application of Lax-Milgram. For clarity we state the Lax-Milgram Theorem here, without proof.

Theorem 6.6. *Let H be a Hilbert space. If*

$$B : H \times H \to \mathbb{R} \tag{6.24}$$

is a bilinear, coercive and continuous mapping, that is, there exist constants $\alpha, \beta > 0$ such that

$$|B(u,v)| \leq \alpha \|u\|_H \|v\|_H \quad \textit{for all } u, v \in H \tag{6.25}$$

and

$$\beta \|u\|_H^2 \leq B(u,u), \quad \textit{for all } u \in H, \tag{6.26}$$

and $f : H \to \mathbb{R}$ is a bounded linear functional on H, then there exists a unique element $u \in H$ such that

$$B(u,v) = f(v) \tag{6.27}$$

for all $v \in H$.

Proof. See for instance Theorem 1, Section 6.2.1 in [7]. $\square$

The formulation of Theorem 6.6 is quite abstract, but the result is highly useful, as we will provide a small demonstration of below.

Proof II of Theorem 6.3. We wish to apply Theorem 6.6. Proposition 3.3 states that $H^{s/2}(\Omega)$ is a Hilbert space. Recalling Lemma 6.4, we define a mapping $B : H_0^{s/2}(\Omega) \times H_0^{s/2}(\Omega) \to \mathbb{R}$ by

$$B(u,v) = \int_{\mathbb{R}^n} \left((1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} u \right) (1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v \, \mathrm{d}x, \; u, v \in H_0^{s/2}(\Omega). \tag{6.28}$$

The bilinearity of B follows immediately from the linearity of $(1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}$ and the linearity of integration. Furthermore, by Hölder's inequality

$$\begin{aligned} |B(u,v)| &\leq \int_{\mathbb{R}^n} | \left((1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} u \right) (1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v | \, \mathrm{d}x \\ &\leq \|(1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} u\|_2 \|(1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v\|_2. \end{aligned} \tag{6.29}$$

From (6.15), (6.16), Proposition 6.2 and (6.29), we can then find a constant α such that

$$|B(u,v)| \leq \alpha \|(1 + (-\Delta)^{\frac{s}{4}}) u\|_2 \|(1 + (-\Delta)^{\frac{s}{4}}) v\|_2 = \alpha \|u\|_{H^{s/2}(\mathbb{R}^n)} \|v\|_{H^{s/2}(\mathbb{R}^n)}. \tag{6.30}$$

Since $u, v \in H_0^{s/2}(\Omega)$ were arbitrary, this proves that B is continuous.

Using Proposition 6.2 again we have

$$\|u\|_{H_0^{s/2}(\Omega)}^2 = \|u\|_{H^{s/2}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |(1 + (-\Delta)^{\frac{s}{4}}) u|^2 \, \mathrm{d}x \tag{6.31}$$

Coercivity then follows from equations (6.15), (6.16).

Next we define a bounded, linear functional on $H_0^{s/2}(\Omega)$ by, abusing notation,

$$f(v) = \int_\Omega fv \, \mathrm{d}x. \tag{6.32}$$

Linearity is immediate from the definition, and boundedness follow from the duality of $H^{s/2}$ and $H^{-s/2}$. Thus, by Theorem 6.6, there exists a unique $u \in H_0^{s/2}(\Omega)$ such that

$$B(u,v) = f(v) \quad \text{for all } v \in H_0^{s/2}(\Omega). \tag{6.33}$$

Since $\mathscr{D}(\Omega) \subset H_0^{s/2}(\Omega)$, the result then follows from the calculations done in Lemma 6.4. $\square$

Remark: Proof I only gives existence of solutions and not uniqueness, since it is not clear if there are other minimisers. However, Proof II gives uniqueness as well.

6.2 The Operator $1 + (-\Delta)^{\frac{s}{2}}$ on $\mathbb{R}^n$

Now we turn our attention to the problem

$$(1 + (-\Delta)^{\frac{s}{2}})u = f \quad \text{in } \mathbb{R}^n. \tag{6.34}$$

Again we will work with $u \in H^r(\mathbb{R}^n)$.

Theorem 6.7. *For any $f \in H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, there exists $u \in H^{r+s}(\mathbb{R}^n)$ solving* (6.34).

In this case we may even give a constructive proof.

Proof. We claim that a solution is given by

$$u = \mathscr{F}^{-1}\left(\frac{\mathscr{F} f}{1 + |\xi|^s}\right). \tag{6.35}$$

First we prove that u defined by (6.35) belongs to $H^{r+s}(\mathbb{R}^n)$. By Definition 2.6, $\mathscr{F} f \in L^2(\mathbb{R}^n, w_r)$, and thus $\frac{\mathscr{F} f}{1+|\xi|^s} \in L^2(\mathbb{R}^n, w_{r+s})$. This implies, again by Definition 2.6, that

$$\mathscr{F}^{-1}\left(\frac{\mathscr{F} f}{1 + |\xi|^s}\right) \in H^{r+s}(\mathbb{R}^n). \tag{6.36}$$

Applying the operator $1 + (-\Delta)^{\frac{s}{2}}$ on u yields

$$\begin{aligned}
(1 + (-\Delta)^{\frac{s}{2}})\, \mathscr{F}^{-1}\left(\frac{\mathscr{F} f}{1 + |\xi|^s}\right) &= \mathscr{F}^{-1}\left((1 + |\xi|^s)\, \mathscr{F}\left(\mathscr{F}^{-1}\left(\frac{\mathscr{F} f}{1 + |\xi|^s}\right)\right)\right) \\
&= \mathscr{F}^{-1}\left((1 + |\xi|^s)\frac{\mathscr{F} f}{1 + |\xi|^s}\right) \\
&= \mathscr{F}^{-1}(\mathscr{F} f) = f.
\end{aligned} \tag{6.37}$$

$\square$

From this proof we can also derive a fundamental solution of 6.34, but first we will define what we mean by this.

Definition 6.8. *Let L be a constant coefficient linear partial differential operator,*

$$L = \sum_{|\alpha| \leq m} c_\alpha \mathrm{D}^\alpha, \tag{6.38}$$

where $m \in N$ and the c_α's are scalars. Then we say that $E \in \mathscr{D}'(\mathbb{R}^n)$ is a fundamental solution of L if

$$LE = \delta. \tag{6.39}$$

One may make a similar definition for fractional differential operators.

Proposition 6.9. *The distribution*

$$\mathscr{F}^{-1}\left(\frac{1}{1+|\xi|^s}\right) \in \mathscr{S}'(\mathbb{R}^n) \tag{6.40}$$

is a fundamental solution to (6.34).

Proof. We first prove that (6.40) is a tempered distribution. Since $(1 + |\xi|^s)^{-1}$ is bounded and measurable, it is a tempered distribution by Proposition 1.19. Therefore $\mathscr{F}^{-1}\left((1+|\xi|^s)^{-1}\right)$ is defined in the sense of distributions, and belongs to $\mathscr{S}'(\mathbb{R}^n)$.

Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}(1+(-\Delta)^{\frac{s}{2}})\,\mathscr{F}^{-1}\left(\frac{1}{1+|\xi|^s}\right)(\varphi) &= \mathscr{F}^{-1}(1)(\varphi) \\ &= 1(\mathscr{F}^{-1}\,\varphi) \\ &= \int_{\mathbb{R}^n} \mathscr{F}^{-1}\,\varphi(\xi)\,\mathrm{d}\xi = \varphi(0).\end{aligned} \tag{6.41}$$

Thus

$$(1+(-\Delta)^{\frac{s}{2}})\,\mathscr{F}^{-1}\left(\frac{1}{1+|\xi|^s}\right) = \delta. \tag{6.42}$$

□

Remark: The main feature of a fundamental solution F of a differential operator L is that for every test-function φ, $u = F * \varphi$ solves $Lu = \varphi$. Thus we could already guess from (6.35), recalling Proposition 1.15, that (6.40) is a fundamental solution.

6.3 Spectral Theory for the Operator $1 + (-\Delta)^{\frac{s}{2}}$

In this section we will look for eigenvalues and eigenfunctions of the operator $1+(-\Delta)^{\frac{s}{2}}$. On $\mathbb{R}^n$, the problem reads as follows

$$(1+(-\Delta)^{\frac{s}{2}})u = \lambda u \text{ in } \mathbb{R}^n,\ \lambda \in \mathbb{C}. \tag{6.43}$$

However, this problem is trivial, as shown by the following result:

Theorem 6.10. *The eigenvalue problem* (6.43) *assumes no non-trivial solutions.*

Proof. Assume (6.43) holds for $u \neq 0$. Applying the Fourier transform to both sides of (6.43) yields

$$(1 + |\xi|^s)\, \mathscr{F} u(\xi) = \lambda\, \mathscr{F} u(\xi) \text{ for almost every } \xi \in \mathbb{R}^n. \tag{6.44}$$

Thus $1 + |\xi|^s = \lambda$ for almost every $\xi \in \mathbb{R}^n$, which is impossible unless $s = 0$. □

Therefore, we will consider the eigenvalue problem for this operator on open and bounded sets, and we will consider the weak formulation of it. That is, for an open and bounded set $\Omega \subset \mathbb{R}^n$, we will search for $\lambda \in \mathbb{R}$ and solutions $u \in H_0^{s/2}(\Omega) \subset H^{s/2}(\mathbb{R}^n)$ (again we consider it as a subset by extending functions in $H_0^{s/2}$ by zero outside Ω) of

$$\int_{\mathbb{R}^n} (1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} u(x)(1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v(x)\, \mathrm{d}x = \lambda \int_{\Omega} u(x)v(x)\, \mathrm{d}x, \tag{6.45}$$

for all $v \in H_0^{s/2}(\Omega)$. We wish to find all eigenvalues and eigenfunctions in this case, and discuss some of their properties. First we may note that we need only consider $\lambda \in \mathbb{R}$, since the whole spectrum of $1 + (-\Delta)^{\frac{s}{2}}$ is real.

Proposition 6.11. *The spectrum of* $1 + (-\Delta)^{\frac{s}{2}} : H_0^s(\Omega) \to L^2(\Omega)$ *is real, and contains only the eigenvalues of the operator. That is, the continuous and residual spectrum are empty.*

Proof. For any open set $\Omega \subset \mathbb{R}^n$, the operator is defined on $H_0^s(\Omega) \subset L^2(\Omega)$, which is a dense subset by Theorem 1.8 since $\mathscr{D}(\Omega) \subset H_0^s(\Omega)$, and we have, by Theorem 1.16, Proposition 1.11 (iii) and Definition 6.1, that

$$\begin{aligned}
\langle (1 + (-\Delta)^{\frac{s}{2}})u, v\rangle_{L^2(\Omega)} &= \int_{\mathbb{R}^n} ((1 + |\xi|^s)\, \mathscr{F} u)\, \mathscr{F}^{-1}\, \overline{v}\, \mathrm{d}\xi \\
&\int_{\mathbb{R}^n} (\mathscr{F} u)\, (1 + |\xi|^s)\, \mathscr{F}^{-1}\, \overline{v}\, \mathrm{d}\xi \\
&\int_{\mathbb{R}^n} u\overline{(1 + (-\Delta)^{\frac{s}{2}})v}\, \mathrm{d}x = \langle u, (1 + (-\Delta)^{\frac{s}{2}})v\rangle_{L^2(\Omega)}.
\end{aligned} \tag{6.46}$$

This proves that $1+(-\Delta)^{\frac{s}{2}}$ is symmetric. Now let λ be an eigenvalue of an eigenfunction u. Then

$$\begin{aligned}
(\lambda - \overline{\lambda})\|u\|_{2,\Omega}^2 &= \{\lambda u, u\}_{L^2(\Omega)} - \{u, \lambda u\}_{L^2(\Omega)} \\
&= \langle (1 + (-\Delta)^{\frac{s}{2}})u, u\rangle_{L^2(\Omega)} - \langle u, (1 + (-\Delta)^{\frac{s}{2}})u\rangle_{L^2(\Omega)} = 0.
\end{aligned} \tag{6.47}$$

This proves the first part of the statement.

If the inverse of $1 - \lambda + (-\Delta)^{\frac{s}{2}}$ exists, it is given by $\mathscr{F}^{-1}\left(\frac{\mathscr{F}\,\cdot}{1-\lambda+|\xi|^s}\right)$, which is defined on a dense subset of $L^2(\Omega)$ and is bounded ($\lambda = 1$ corresponds to the trivial eigenvalue 0 of $(-\Delta)^{\frac{s}{2}}$), so the continuous and residual spectrums are empty. □

Remark: From the calculations done in the proof above we may also derive that

$$(1+(-\Delta)^{\frac{s}{2}})u, u\rangle_{L^2(\Omega)} = \|(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u\|_2^2 \geq 0 \tag{6.48}$$

which implies that $1+(-\Delta)^{\frac{s}{2}}$ is a positive operator.

Proposition 6.11 implies that if we characterise all the eigenvalues of $1+(-\Delta)^{\frac{s}{2}}$, we will have characterised the whole spectrum, and this will be our goal.

The next theorem provides the basis of our discourse.

Theorem 6.12. *The functional defined by*

$$I(u) = \int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u\right)^2 \, \mathrm{d}x \tag{6.49}$$

on the space

$$H = \{u \in H_0^{s/2}(\Omega) \subset H^{s/2}(\mathbb{R}^n) : \|u\|_{2,\Omega} = 1\} \tag{6.50}$$

attains its minimum on this space for an $e_1 \in H$ *such that*

$$\int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}e_1\right)(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}v \, \mathrm{d}x = \lambda_1 \int_\Omega e_1 v \, \mathrm{d}x \tag{6.51}$$

for all $v \in H_0^{s/2}(\Omega)$*, where* $\lambda_1 = I(e_1) > 0$.

Proof. Step 1: First we prove that I attains its infimum on the space H. Note that our functional is the same as the functional in Lemmas 6.4 and 6.5 defined by (6.8) with the term $\int_\Omega fu \, \mathrm{d}x$ removed. The proof given in Lemma 6.5 for the lower semi-continuity of (6.8) on $H_0^{s/2}(\Omega)$ did not rely on the last term, and thus result is true also for the functional (6.49), and for any subspace of $H_0^{s/2}(\Omega)$.

Let $\{u_n\}_n$ be a minimising sequence for I in H. By coercivity of I on H, the sequence is bounded. From the proof of Lemma 6.5 we know that a subsequence converges weakly to some $e_1 \in H_0^{s/2}(\Omega)$, and furthermore, by the remark above,

$$I(e_1) \leq \liminf_{n\to\infty} I(u_n). \tag{6.52}$$

What remains to prove is that the limit e_1 is in H, that is, that H is weakly closed. According to Theorem 4.12, the embedding $H_0^{s/2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. This implies that $u_n \to e_1$ strongly in $L^2(\Omega)$ as $n \to \infty$, which implies $\|u_n\|_{2,\Omega} \to \|e_1\|_{2,\Omega}$. By assumption $\|u_n\|_{2,\Omega} = 1$ for all $n \in \mathbb{N}$, and thus $\|e_1\|_{2,\Omega} = 1$ and $e_1 \in H$.

Step 2: We show that the e_1 found in step 1 satisfies (6.51). Let $\varepsilon \in (-1,1)$, $v \in H_0^{s/2}(\Omega)$, $c_\varepsilon = \|e_1 + \varepsilon v\|_{2,\Omega}$ and $e_{1,\varepsilon} = (e_1 + \varepsilon v)/c_\varepsilon$. Observe that $e_{1,\varepsilon} \in H$, and note that $I(\cdot)^{1/2}$ defines a norm equivalent to $\|\cdot\|_{H_0^{s/2}(\Omega)}$ on $H_0^{s/2}(\Omega)$ (cf. Proposition 6.2). For the rest of the chapter, we will take this as our norm on $H_0^{s/2}(\Omega)$, and we define the following inner product, which induces the norm:

$$\langle u, v\rangle_{H_0^{s/2}(\Omega)} = \int_{\mathbb{R}^n} (1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}u(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}}v \, \mathrm{d}x. \tag{6.53}$$

We have the following relations:

$$c_\varepsilon^2 = \int_\Omega |e_1 + \varepsilon v|^2 \,\mathrm{d}x = \|e_1\|_{2,\Omega}^2 + 2\varepsilon \int_\Omega e_1(x)v(x)\,\mathrm{d}x + o(\varepsilon) \tag{6.54}$$

and

$$\begin{aligned} \|e_1 + \varepsilon v\|^2_{H_0^{s/2}(\Omega)} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{s}{2}} |\mathscr{F}(e_1+\varepsilon v)|^2 \,\mathrm{d}\xi \\ &= \|e_1\|_{H_0^{s/2}(\Omega)} + 2\varepsilon \langle e_1, v\rangle_{H_0^{s/2}(\Omega)} + o(\varepsilon). \end{aligned} \tag{6.55}$$

Considering our remarks about the norm and inner product above, (6.54) and (6.55) implies that

$$\begin{aligned} I(e_{1,\varepsilon}) &= \frac{\|e_1\|^2_{H_0^{s/2}(\Omega)} + 2\varepsilon\langle e_1, v\rangle_{H_0^{s/2}(\Omega)} + o(\varepsilon)}{1 + 2\varepsilon \int_\Omega e_1(x)v(x)\,\mathrm{d}x + o(\varepsilon)} \\ &= \frac{\Big(I(e_1) + 2\varepsilon\langle e_1, v\rangle_{H_0^{s/2}(\Omega)} + o(\varepsilon)\Big)\left(1 - 2\varepsilon\int_\Omega e_1(x)v(x)\,\mathrm{d}x + o(\varepsilon)\right)}{1 - 4\varepsilon^2 \left(\int_\Omega e_1(x)v(x)\,\mathrm{d}x\right)^2 + o(\varepsilon)} \qquad (6.56) \\ &= I(e_1) + 2\varepsilon\left(\langle e_1, v\rangle_{H_0^{s/2}(\Omega)} - I(e_1)\int_\Omega e_1(x)v(x)\,\mathrm{d}x\right) + o(\varepsilon). \qquad (6.57) \end{aligned}$$

If $\langle e_1, v\rangle_{H_0^{s/2}(\Omega)} - 2I(e_1)\int_\Omega e_1(x)v(x)\,\mathrm{d}x$ is non-zero, it is possible to choose $\varepsilon \in (-1,1)$ such that $I(e_{1,\varepsilon}) < I(e_1)$, since for small ε, the aforementioned term will by definition dominate the $o(\varepsilon)$ term. This contradicts the minimality of e_1, and we thus have

$$\langle e_1, v\rangle_{H_0^{s/2}(\Omega)} - I(e_1)\int_\Omega e_1(x)v(x)\,\mathrm{d}x = 0, \tag{6.58}$$

which by (6.53) is the same as

$$\int_{\mathbb{R}^n} (1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_1 (1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v \,\mathrm{d}x = I(e_1)\int_\Omega e_1(x)v(x)\,\mathrm{d}x. \tag{6.59}$$

Since $v \in H_0^{s/2}(\Omega)$ was chosen arbitrarily, this proves the result. □

The procedure used to prove Theorem 6.12 cannot be extended to unbounded sets $\Omega \subset \mathbb{R}^n$. To get convergence of (a subsequence of) a minimising sequence $\{u_n\}_n$ in $H_0^{s/2}(\Omega)$, we used the Theorems 4.12 and Banach-Alaoglu, both of which hold only on bounded domains. We show in Theorem 6.10 that there are no non-trivial eigenvalues on $\mathbb{R}^n$.

We will now investigate some of the properties of e_1 and λ_1.

Proposition 6.13. *Either $e_1 \geq 0$ or $e_1 \leq 0$ a.e. in $\mathbb{R}^n$.*

Proof. By definition, $e_1 = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, so we only need to consider its values in Ω. First we note that $|e_1| \in H$, and

$$||e_1(x)| - |e_1(y)|| \leq |e_1(x) - e_1(y)| \tag{6.60}$$

and if $e_1(x) > 0$ and $e_1(y) < 0$, the inequality is strict. We claim that this implies $I(|e_1|) < I(e_1)$ if $\mathcal{L}^n(\{x \in \Omega : e_1(x) < 0\})$ and $\mathcal{L}^n(\{x \in \Omega : e_1(x) > 0\})$ are both non-zero. To see this, note that if the inequality (6.60) is strict for a set of positive measure, it follows from the definition of the $W_2^{s/2}(\mathbb{R}^n)$ norm (cf. (2.25)) that

$$\||e_1|\|^2_{W_2^{s/2}(\mathbb{R}^n)} < \|e_1\|^2_{W_2^{s/2}(\mathbb{R}^n)}. \tag{6.61}$$

Furthermore, in the proof of Theorem 2.10, we showed that

$$\|e_1\|^2_{W_2^{s/2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + c|\xi|^s) \, |\mathscr{F} f(\xi)|^2 \, \mathrm{d}\xi \tag{6.62}$$

for $s \in (0, 2)$ (cf. (2.40)). Recall that, by Definition 6.1, that

$$I(e_1) = \int_{\mathbb{R}^n} \left(\mathscr{F}^{-1} \left((1 + |\xi|^s)^{\frac{1}{2}} \mathscr{F} e_1 \right) \right)^2 \mathrm{d}\xi. \tag{6.63}$$

Thus $\| \cdot \|^2_{W_2^{s/2}(\mathbb{R}^n)}$ is nothing but $I(\cdot)$ with a constant c instead of 1 in front of the ξ term. Thus, by (6.61), $I(|e_1|) < I(e_1)$. This contradicts the minimality of e_1, and therefore either $\mathcal{L}^n(\{x \in \Omega : e_1(x) < 0\}) = 0$ or $\mathcal{L}^n(\{x \in \Omega : e_1(x) > 0\}) = 0$. □

Proposition 6.14. *If $u \in H_0^{s/2}(\Omega)$ solves*

$$\int_{\mathbb{R}^n} \left((1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} u(x) \right) (1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v(x) \, \mathrm{d}x = \lambda_1 \int_{\Omega} u(x) v(x) \, \mathrm{d}x \tag{6.64}$$

for every $v \in H_0^{s/2}(\Omega)$, then $u = \gamma e_1$, for some $\gamma \in \mathbb{R}$.

Proof. Suppose $g_1 \in H_0^{s/2}$ is a non-zero eigenfunction corresponding to λ_1, with $g_1 \neq e_1$. We claim that any eigenfunction $e \in H_0^{s/2}(\Omega)$ corresponding to λ_1 with $\|e\|_{2,\Omega} = 1$ is a minimiser of (6.49) on H. By assumption, e satisfies

$$\int_{\mathbb{R}^n} \left((1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e \right) (1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v \, \mathrm{d}x = I(u) \int_{\Omega} e(x) v(x) \, \mathrm{d}x \tag{6.65}$$

for every $v \in H_0^{s/2}(\Omega)$. Choosing $v = e$, we get $I(e) = \lambda_1 = I(e_1)$, since $\|e\|_{2,\Omega} = 1$. Therefore Proposition 6.13 holds for any eigenfunction in H corresponding to λ_1, and, by normalization, for any eigenfunction in $H_0^{s/2}(\Omega)$ corresponding to λ_1. Thus either $g_1 \geq 0$ a.e. or $g_1 \leq 0$ a.e. in $\mathbb{R}^n$. Define

$$\tilde{g}_1 = \frac{g_1}{\|g_1\|_{2,\Omega}} \tag{6.66}$$

and set $f = e_1 - \tilde{g}_1$. Suppose $f \neq 0$. Then f is also an eigenfunction corresponding to λ_1, and $f \geq 0$ a.e or $f \leq 0$ a.e. Then either $e_1 \geq \tilde{g}_1$ a.e. or $e_1 \leq \tilde{g}_1$ a.e. Since e_1 and $\tilde{g}_1$ both have constant sign, this implies

$$e_1^2 \geq \tilde{g}_1^2 \text{ or } e_1 \leq \tilde{g}_1^2 \text{ a.e. in } \Omega. \tag{6.67}$$

On the other hand,

$$\int_\Omega e_1(x)^2 - \tilde{g}_1(x)^2 \, \mathrm{d}x = \|e_1\|_{2,\Omega}^2 - \|\tilde{g}_1\|_{2,\Omega}^2 = 1 - 1 = 0 \tag{6.68}$$

contradiction our assumption on f. This proves the result. □

Proposition 6.15. *If λ and $\tilde{\lambda}$ are two distinct eigenvalues of problem* (6.45)*, with eigenfunctions e and $\tilde{e} \in H_0^{s/2}(\Omega)$, respectively, then*

$$\langle e, \tilde{e} \rangle_{H_0^{s/2}(\Omega)} = 0 = \int_\Omega e(x)\tilde{e}(x) \, \mathrm{d}x. \tag{6.69}$$

Proof. We may assume $e \neq 0$ and $\tilde{e} \neq 0$. We may also assume both e and $\tilde{e}$ are normalised such that $e, \tilde{e} \in H$. Testing e against $\tilde{e}$ and vice-versa in (6.45), we find

$$\begin{aligned} \int_{\mathbb{R}^n} \Big((1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e(x) \Big) (1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} \tilde{e}(x) \, \mathrm{d}x &= \lambda \int_\Omega e(x)\tilde{e}(x) \, \mathrm{d}x \\ &= \tilde{\lambda} \int_\Omega e(x)\tilde{e}(x) \, \mathrm{d}x. \end{aligned} \tag{6.70}$$

Hence

$$(\lambda - \tilde{\lambda}) \int_\Omega e(x)\tilde{e}(x) \, \mathrm{d}x = 0. \tag{6.71}$$

By assumption, $\lambda \neq \tilde{\lambda}$, which implies

$$\int_\Omega e(x)\tilde{e}(x) \, \mathrm{d}x = 0 \tag{6.72}$$

Inserting this into (6.70), we also get

$$\int_{\mathbb{R}^n} \Big((1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e(x) \Big) (1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} \tilde{e}(x) \, \mathrm{d}x = 0 = \langle e, \tilde{e} \rangle_{H_0^{s/2}(\Omega)}. \tag{6.73}$$

This proves the result. □

Now we come to the main theorem on eigenvalues of $1 + (-\Delta)^{\frac{s}{2}}$, which gives a characterisation of all eigenvalues and eigenfunctions of this operator, and thereby of the whole spectrum (cf. Proposition 6.11). The main ideas of the proof come from [23], with some alterations due to working with slightly different operators in different spaces, and [23] defines the fractional operators considered by means of singular integrals, like in Equation (6.5), instead of using the Fourier transform. It should be noted that [23] works with general elliptic fractional differential operators that satisfy certain criteria with the fractional Laplace operator being the prime example, and their result almost directly holds for our operator. While some arguments used in the preceding results and also in the following theorem are particular to our case, these results can be generalised further than noted above, and they hold for any positive elliptic and self-adjoint operator. See for instance [3] (indeed, we have not proved that our operator is self-adjoint, only symmetric, but in fact many of these results hold for symmetric operators, and self-adjointedness of our operator follows from arguments presented in the cited paper).

Theorem 6.16. *(i) The set of eigenvalues of problem* (6.45) *consists of a sequence* $\{\lambda_k\}_k$ *with*

$$0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_k \leq \lambda_{k+1} \leq ... \tag{6.74}$$

Furthermore, for any $k \in \mathbb{N}$ the eigenvalues can be characterised as follows:

$$\lambda_{k+1} = \inf_{u \in H_{k+1}} I(u), \tag{6.75}$$

where I is as defined in (6.49) *and*

$$H_{k+1} = \{u \in H : \langle u, e_i \rangle_{H_0^{s/2}(\Omega)} = 0, \text{ for all } i = 1, ..., k\} \tag{6.76}$$

with H as in (6.50) *and $\langle \cdot, \cdot \rangle_{H_0^{s/2}(\Omega)}$ as in* (6.53).

(ii) For any $k \in \mathbb{N}$, there exists a function $e_{k+1} \in H_{k+1}$ which attains the infimum in (6.75) *and is an eigenfunction corresponding to λ_{k+1}.*

(iii) The sequence $\{\lambda_k\}_k$ satisfies

$$\lambda_k \to \infty \text{ as } k \to \infty. \tag{6.77}$$

(iv) The sequence $\{e_k\}_k$ of eigenfunctions corresponding to λ_k is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^{s/2}(\Omega)$.

Proof. The proofs of (i), (ii) and (iii) are inter-connected, so in order to avoid repeating lengthy arguments, we will prove them together.

Let λ_k be defined as in (6.75). Since

$$H_{k+1} \subset H_k \subset H_0^{s/2}(\Omega), \tag{6.78}$$

we have

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq \lambda_{k+1} \leq ... \tag{6.79}$$

We prove that $\lambda_1 \neq \lambda_2$. Assume to the contrary that $\lambda_1 = \lambda_2$. Then e_2 is an eigenfunction corresponding to λ_1 and by Proposition 6.14 $e_2 = \gamma e_1$, for $\gamma \neq 0$. On the other hand, $e_2 \in H_2$, which by definition implies

$$0 = \langle e_2, e_1 \rangle_{H_0^{s/2}(\Omega)} = \gamma \langle e_1, e_1 \rangle_{H_0^{s/2}(\Omega)}. \tag{6.80}$$

This is saying that $e_1 = 0$ a.e., which contradicts our previous results. Thus $\lambda_1 \neq \lambda_2$, and (6.74) is proved. Next we prove that the elements of this sequence are eigenvalues of the problem (6.45).

Noting that $H_{k+1} \subset H$ is weakly closed for any $k \in \mathbb{N}$, Theorem 6.12 applies also on this space, and so there exists a function $e_{k+1} \in H_{k+1}$ which attains the infimum in (6.75). Furthermore, Theorem 6.12 applied to H_{k+1} yields

$$\int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_{k+1}\right)(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v \,\mathrm{d}x = \lambda_{k+1} \int_\Omega e_{k+1}(x)v(x)\,\mathrm{d}x, \tag{6.81}$$

for all $v \in H_{k+1}$. In order to prove that λ_{k+1} is an eigenvalue with eigenfunction e_{k+1}, we need to prove that (6.81) holds for all $v \in H_0^{s/2}(\Omega)$. We prove this by induction; that is, we assume it holds for $1, ..., k$ and prove it then also holds for $k+1$. Theorem 6.12 proves that λ_1 is a eigenvalue with eigenfunction e_1, and this is the basis for our induction hypothesis.

By (6.50) and (6.76), we see that

$$H = \operatorname{span}\{e_1, ..., e_k\} \oplus H_{k+1}. \tag{6.82}$$

Thus any $v \in H$ can be decomposed as

$$v = v_1 + v_2, \quad v_1 = \sum_{i=1}^{k} c_1 e_1, \; c_i \in \mathbb{R}, \text{ and } v_2 \in H_{k+1}. \tag{6.83}$$

Inserting $v_2 = v - v_1$ into (6.81), we find

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_{k+1}\right)(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v \,\mathrm{d}x - \lambda_{k+1} \int_\Omega e_{k+1}(x)v(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_{k+1}\right)(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v_1 \,\mathrm{d}x - \lambda_{k+1} \int_\Omega e_{k+1}(x)v_1(x)\,\mathrm{d}x \qquad (6.84)\\
&= \sum_{i=1}^{k} c_i \left(\int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_{k+1}\right)(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_i \,\mathrm{d}x - \lambda_{k+1} \int_\Omega e_{k+1}(x)e_i(x)\,\mathrm{d}x \right).
\end{aligned}$$

By the induction hypothesis, (6.81) holds for e_i tested against e_{k+1} for $i = 1, ..., k$. Recalling (6.53) and (6.76), we find

$$0 = \int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_i\right)(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_{k+1} \,\mathrm{d}x = \lambda_{k+1} \int_\Omega e_i(x)e_{k+1}(x)\,\mathrm{d}x. \tag{6.85}$$

This and (6.84) implies

$$\int_{\mathbb{R}^n} \left((1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_{k+1}\right)(1+(-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} v \,\mathrm{d}x = \lambda_{k+1} \int_\Omega e_{k+1}(x)v(x)\,\mathrm{d}x \tag{6.86}$$

for all $v \in H$. Since every $u \in H_0^{s/2}(\Omega)$ except $u = 0$ can be made an element of H by normalisation, (6.85) holds for all $v \in H_0^{s/2}(\Omega)$. This proves (ii).

In order to complete the proof of (i), we need (iii). Suppose to the contrary that $\lambda_k \to c$ for some $c \in \mathbb{R}_+$. By (ii) we have

$$\|e_k\|^2_{H_0^{s/2}(\Omega)} \sim I(e_k) = \lambda_k, \tag{6.87}$$

which implies $\sup_k \|e_k\|_{H_0^{s/2}(\Omega)} \leq c$. Thus, using Proposition 4.7, we can find a subsequence such that

$$e_{k_i} \to e \text{ for some } e \in L^2(\Omega). \tag{6.88}$$

In particular, $\{e_{k_i}\}_i$ is a Cauchy sequence in $L^2(\Omega)$. On the other hand, the sequence $\{e_{k_i}\}_i$ is by definition orthogonal in $H_0^{s/2}(\Omega)$. Inserting e_{k_i} and e_{k_j}, $i \neq j$, into 6.81 we find that, as they are eigenfunctions,

$$0 = \int_{\mathbb{R}^n} \left((1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_{k_i}\right) (1 + (-\Delta)^{\frac{s}{2}})^{\frac{1}{2}} e_{k_j} \,\mathrm{d}x = \lambda_{k+1} \int_{\Omega} e_{k_i}(x) e_{k_j}(x) \,\mathrm{d}x. \tag{6.89}$$

Thus the sequence is orthogonal in $L^2(\Omega)$ as well, and

$$\|e_{k_i} - e_{k_j}\|^2_{2,\Omega} = \|e_{k_i}\|^2_{2,\Omega} + \|e_{k_j}\|^2_{2,\Omega} = 2. \tag{6.90}$$

This contradicts $\{e_{k_i}\}_i$ being a Cauchy sequence in $L^2(\Omega)$, and therefore our assumption on the limit of λ_k is false. This proves (iii).

To finish the proof of (i), we need to show that eigenvalues we have found are all the eigenvalues. Assume to the contrary that there is an eigenvalue $\lambda \notin \{\lambda_k\}_k$. Let $e \in H$ be a corresponding eigenfunction. Evaluating e against itself in (6.45), we find

$$I(e) = \lambda. \tag{6.91}$$

Since e_1 is a minimiser of I in H, we deduce

$$\lambda = I(e) \geq I(e_1) = \lambda_1. \tag{6.92}$$

Thus, by (iii), there exists a $k \in \mathbb{N}$ such that

$$\lambda_k < \lambda < \lambda_{k+1}. \tag{6.93}$$

We claim that

$$e \notin H_{k+1}. \tag{6.94}$$

Assume to the contrary that it is. By minimality, $\lambda_{k+1} = I(e_{k+1}) \leq I(e) = \lambda$, which contradicts (6.93). From (6.94) we deduce that there exists some $i \in \{1, 2, ..., k\}$ such that $\langle e, e_i \rangle_{H_0^{s/2}(\Omega)} \neq 0$. However, this contradicts Proposition 6.15. This proves (i)

(iv) Orthogonality in both $H_0^{s/2}(\Omega)$ and $L^2(\Omega)$ follows from Proposition 6.15, and the functions are by definition normalised in $L^2(\Omega)$. We first prove that $\{e_k\}_k$ is a basis for $H_0^{s/2}(\Omega)$. Assume to the contrary there exists a non-zero $u \in H_0^{s/2}(\Omega)$ such that

$$\langle u, e_k \rangle_{H_0^{s/2}(\Omega)} = 0 \text{ for all } k \in \mathbb{N}. \tag{6.95}$$

We may assume $\|u\|_{2,\Omega} = 1$, otherwise we could normalise. Since $u \neq e_1$ we know, by Theorem, 6.12 that

$$I(e_1) < I(u). \tag{6.96}$$

On the other hand, $I(u) < \infty$, but (6.77) and (6.75) implies $I(e_k) \to \infty$ as $k \to \infty$. Thus there exists a $k \in \mathbb{N}$ such that

$$I(u) < \lambda_{k+1} = I(e_k) = \inf_{v \in H_{k+1}} I(v). \tag{6.97}$$

This implies $u \notin H_{k+1}$, and so there exists a $j \in 1, ..., k$ such that $\langle u, e_j \rangle_{H_0^{s/2}(\Omega)} \neq 0$, contradicting (6.95). Thus

$$\langle v, e_k \rangle_{H_0^{s/2}(\Omega)} = 0 \text{ for all } k \in \mathbb{N} \text{ implies } v = 0 \text{ a.e.} \tag{6.98}$$

We normalise the eigenfunctions in $H_0^{s/2}(\Omega)$, $\tilde{e}_i = e_i / \|e_i\|_{H_0^{s/2}(\Omega)}$. Given $f \in H_0^{s/2}(\Omega)$, we define

$$f_j = \sum_{i=1}^{j} \langle f, \tilde{e}_i \rangle_{H_0^{s/2}(\Omega)} \tilde{e}_i, \; j \in \mathbb{N}. \tag{6.99}$$

We wish to prove that $f_j \to f$ as $j \to \infty$. Define $v_j = f - f_j$. Then, recalling that $\{\tilde{e}_i\}_i$ is an orthonormal system in $H_0^{s/2}(\Omega)$, we calculate

$$\begin{aligned} 0 \leq \langle v_j, v_j \rangle_{H_0^{s/2}(\Omega)} &= \|f\|^2_{H_0^{s/2}(\Omega)} + \langle f_j, f_j \rangle_{H_0^{s/2}(\Omega)} - 2\langle f, f_j \rangle_{H_0^{s/2}(\Omega)} \\ &= \|f\|^2_{H_0^{s/2}(\Omega)} + \langle f_j, f_j \rangle_{H_0^{s/2}(\Omega)} - 2\sum_{i=1}^{j} \langle f, \tilde{e}_i \rangle^2_{H_0^{s/2}(\Omega)} \\ &= \|f\|^2_{H_0^{s/2}(\Omega)} - \sum_{i=1}^{j} \langle f, \tilde{e}_i \rangle^2_{H_0^{s/2}(\Omega)}. \end{aligned} \tag{6.100}$$

Thus

$$\sum_{i=1}^{j} \langle f, \tilde{e}_i \rangle^2_{H_0^{s/2}(\Omega)} \leq \|f\|^2_{H_0^{s/2}(\Omega)} \text{ for every } j \in \mathbb{N}, \tag{6.101}$$

from which we deduce

$$\sum_{i=1}^{\infty} \langle f, \tilde{e}_i \rangle^2_{H_0^{s/2}(\Omega)} < \infty. \tag{6.102}$$

So

$$\tau_j = \sum_{i=1}^{j} \langle f, \tilde{e}_i \rangle^2_{H_0^{s/2}(\Omega)} \tag{6.103}$$

is a Cauchy sequence in $\mathbb{R}$. Furthermore, using the orthonormality of $\{\tilde{e}_i\}_i$ we find that for $k > j$,

$$\langle v_k - v_j, v_k - v_j \rangle_{H_0^{s/2}(\Omega)} = \sum_{i=j+1}^{k} \langle f, \tilde{e}_i \rangle^2_{H_0^{s/2}(\Omega)} = \tau_k - \tau_j. \tag{6.104}$$

Thus $\{v_j\}_j$ is a Cauchy sequence in $H_0^{s/2}(\Omega)$, and by Proposition 2.7 there exists a $v \in H_0^{s/2}(\Omega)$ such that $v_j \to v$ as $j \to \infty$. From the definition of v, it is straightforward to compute that $\langle v_j, e_i \rangle_{H_0^{s/2}(\Omega)} = 0$ when $j > i$. From (6.98) we then deduce that $v = 0$. Then, since $f_j = f - v_j$, we get

$$f_j \to f - v = f \text{ in } H_0^{s/2}(\Omega) \text{ as } j \to \infty. \tag{6.105}$$

This shows that $\{e_k\}_k$ is an orthogonal basis of $H_0^{s/2}(\Omega)$. Lastly, we prove that it is a basis of $L^2(\Omega)$. Given $g \in L^2(\Omega)$ and $\varepsilon > 0$, Theorem 1.8 states that there exists a function $\varphi \in \mathscr{D}(\Omega)$ such that

$$\|g - \varphi\|_{2,\Omega} < \frac{\varepsilon}{2}. \tag{6.106}$$

Furthermore, since $\mathscr{D}(\Omega) \subset H_0^{s/2}(\Omega)$, there exists, by what we just proved, a function $\tilde{g} \in \operatorname{span}\{e_k : k \in \mathbb{N}\}$ such that

$$\|\varphi - \tilde{g}\|_{\Omega,2} \leq \|\varphi - \tilde{g}\|_{H_0^{s/2}(\Omega)} < \frac{\varepsilon}{2}. \tag{6.107}$$

Using the triangle inequality, we find

$$\|g - \tilde{g}\|_{2,\Omega} \leq \|g - \varphi\|_{2,\Omega} + \|\varphi - \tilde{g}\|_{2,\Omega} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{6.108}$$

Since $g \in L^2(\Omega)$ and $\varepsilon > 0$ were arbitrary, this proves the result. □

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