ON NEWTON DIAGRAMS OF PLURISUBHARMONIC POLYNOMIALS

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Abstract. Each extreme edge of the Newton diagram of a plurisubharmonic polynomial on $\mathbb{C}^2$ gives rise to a plurisubharmonic polynomial. It is tempting to believe that the union of the extreme edges or the convex hull of said union will do the same. If true, then the latter would provide useful strategies for the bumping of plurisubharmonic polynomials on $\mathbb{C}^2$, but whether they are true has been elusive until now. We construct a plurisubharmonic polynomial P on $\mathbb{C}^2$ with precisely two extreme edges E_1 and E_2, such that neither $E_1 \cup E_2$ nor $\mathrm{Conv}(E_1 \cup E_2)$ yields a plurisubharmonic polynomial.

1. Introduction

It is a well-known fact that it is possible to solve the $\overline{\partial}$-equation with supnorm estimates for sufficiently regular $\overline{\partial}$-closed $(0,1)$-forms on bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with boundary of class $\mathcal{C}^2$. This was shown by H. Grauert and I. Lieb [8] and G.M. Henkin [9] in the case of higher boundary regularity and by N. Øvrelid [11] for boundaries of class $\mathcal{C}^2$.

If, however, $\Omega \subseteq \mathbb{C}^n$ is a bounded weakly pseudoconvex domain with boundary of class $\mathcal{C}^\infty$, it is not necessarily possible to solve the $\overline{\partial}$-equation with supnorm estimates. In fact, N. Sibony [14] has constructed a bounded weakly pseudoconvex domain $D \subseteq \mathbb{C}^3$ with $\mathcal{C}^\infty$-boundary which admits a $\overline{\partial}$-closed $(0,1)$-form $\Phi \in \mathcal{C}^\infty_{0,1}(D) \cap \mathcal{C}^0_{0,1}(\overline{D})$, such that the equation $\overline{\partial}\Psi = \Phi$ has no bounded solution on D.

It hence becomes and interesting question which additional assumptions on a bounded weakly pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$ with smooth boundary guarantee the existence of supnorm estimates for solutions of $\overline{\partial}u = f$, where f is a sufficiently regular $\overline{\partial}$-closed $(0,1)$-form on Ω.
R.M. Range [13] has shown that supnorm (and even Hölder) estimates *do* exist for bounded smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$. Later K. Diederich, B. Fischer and J.E. Fornæss [3] obtained estimates for bounded smoothly bounded convex domains of finite type in $\mathbb{C}^n$.

One of the crucial ingredients in Range's argument is the *local bumping* of the domain at a boundary point. Following [2], one defines a local bumping of a smoothly bounded pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$, $n \geq 2$, at a boundary point $\zeta \in \partial\Omega$ to be a triple $(\partial\Omega, U_\zeta, \rho_\zeta)$, such that:

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- $U_\zeta \subseteq \mathbb{C}^n$ is an open neighborhood or ζ,
- $\rho_\zeta \colon U_\zeta \to \mathbb{R}$ is smooth and plurisubharmonic,
- $\rho_\zeta^{-1}(\{0\})$ is a smooth hypersurface in U_ζ that is pseudoconvex from the side $U_\zeta^- := \{z \colon \rho_\zeta(z) < 0\}$,
- $\rho_\zeta(\zeta) = 0$, but $\rho_\zeta < 0$ on $U_\zeta \cap \left(\overline{\Omega} \setminus \{\zeta\}\right)$.

Given a bounded smoothly bounded pseudoconvex domain D of finite type in $\mathbb{C}^2$, Range proceeds by producing a bumping D_p of D at a boundary point $p \in \mathrm{b}D$, fitting large polydiscs centered in D into D_p and thus obtaining good pointwise estimates for holomorphic functions using the Cauchy estimates. This in turn he uses to construct integral kernels for the $\overline{\partial}$-equation satisfying the necessary estimates. The finite type condition is necessary to ensure that the above-mentioned polydiscs are large enough.

When the dimension is increased, however, it becomes much harder to construct local bumpings of the domain. For the remainder of this section let $\Omega \subseteq \mathbb{C}^n$, $n \geq 2$, be a bounded pseudoconvex domain with real-analytic boundary. In this situation, K. Diederich and J.E. Fornæss have shown in [5] that local bumpings always exist at each boundary point. This, however, is a priori not enough to construct good integral kernels and hence obtain supnorm or Hölder estimates for $\overline{\partial}$, since the order of contact between $\partial\Omega$ and the boundary of the bumped out domain at a boundary point $p \in \partial\Omega$ can be a lot higher than the type of p when $n \geq 3$ (the notion of type we are working with is the *D'Angelo 1-type*).
The goal hence becomes to construct a local bumping of Ω at a boundary point $p \in \partial\Omega$, such that the order of contact between $\partial\Omega$ and the boundary of the bumped out domain at p does not exceed the type in any direction. It should be noted that Ω is of finite type, as was shown by K. Diederich and J.E. Fornæss [4].

So let p be a boundary point of Ω. After a holomorphic change of coordinates one can assume that $p = 0$ and that the domain is given as follows:

$$\Omega \cap V = \{(\zeta, z) \in (\mathbb{C} \times \mathbb{C}^{n-1}) \cap V \colon \; \mathrm{Re}(\zeta) + r(z) + \mathcal{O}(|\,\mathrm{Im}(\zeta)|^2, |z| \cdot |\,\mathrm{Im}(\zeta)|) < 0\},$$

where V is a small open neighborhood of $p = 0$ and r is a real-valued real-analytic function defined on an open neighborhood of $0 \in \mathbb{C}^{n-1}$. Furthermore r can be chosen to be of the form

$$r(z) = \sum_{j=2k}^{\infty} P_j(z),$$

where P_j is a homogeneous polynomial in z and $\overline{z}$ of degree j and $P_{2k} \not\equiv 0$ (i.e. the lowest-degree term of r has degree $2k$, which is less or equal to the type of Ω at $p = 0$) and P_{2k} is plurisubharmonic but not pluriharmonic. In the special case $\Omega \subseteq \mathbb{C}^2$ one can show that it is possible to find such a local description, such that $2k$ is actually equal to the type of the domain at $p = 0$. By absorbing all pluriharmonic terms of P_{2k} into the real part of ζ, one can assume that P_{2k} has no pluriharmonic terms.
When $\Omega \subseteq \mathbb{C}^2$, J.E. Fornæss and N. Sibony [6] have shown that the domain can be bumped to order $2k$, the type of the domain. Further A. Noell [10] showed that if P_{2k} is additionally assumed to not be harmonic along any complex line through $0 \in \mathbb{C}^{n-1}$ this is still the case. But if P_{2k} is allowed to be harmonic along complex lines through 0, things become much more complicated.

Noell proceeded by showing that there exist an $\mathbb{R}$-homogeneous function $\widetilde{P}_{2k}\colon \mathbb{C}^{n-1} \to \mathbb{R}$ of degree $2k$ and a constant $\epsilon > 0$, such that

$$P_{2k}(z) - \widetilde{P}_{2k}(z) \geq \epsilon |z|^{2k} \text{ for all } z \in \mathbb{C}^{n-1},$$

and such that $\widetilde{P}_{2k}$ is smooth and strictly plurisubharmonic on $\mathbb{C}^{n-1} \setminus \{0\}$.

The next step is to look for similar results without assuming P_{2k} to not be harmonic along any complex line through 0. In this case, however, one can not expect to obtain an inequality as strong as the one in Noell's result, since that would lead to a violation of the strong maximum principle for subharmonic functions along a complex line through 0 along which P_{2k} is harmonic (i.e. vanishes, since P_{2k} does not have any pluriharmonic terms). A similar argument also shows that one can not expect to get something *strictly* plurisubharmonic on $\mathbb{C}^{n-1} \setminus \{0\}$.

Assume $n = 3$ for the remainder of this section. In this situation G. Bharali and B. Stensønes [2] have obtained bumping results for the polynomial $P_{2k}\colon \mathbb{C}^2 \to \mathbb{R}$ in two different cases. They prove that that P_{2k} is harmonic along at most finitely many complex lines through 0, which, in one of the two cases, allows them to combine local bumpings in conical neighborhoods of said lines using a gluing argument.
Since P_{2k} can be harmonic along complex lines through 0, however, this does not necessarily lead to a bumping of the domain Ω. Addressing this issue in one of the cases considered in [2], G. Bharali [1] has constructed bumpings *of the domain* Ω under a non-restrictive assumption on the remaining terms, which is satisfied in a motivating example in [2].

This paper deals with the problem of finding a bumping for the domain Ω in the case $n = 3$ and provides a counterexample to a proposed strategy.

2. Motivating Examples

Let Ω be a bounded pseudoconvex domain with real-analytic boundary in $\mathbb{C}^3$ and $p \in \partial\Omega$. As in the introduction, after a holomorphic change of coordinates, one can assume that $p = 0$ and that

$$\Omega \cap V = \{(\zeta, z, w) \in \mathbb{C}^3 \cap V \colon \operatorname{Re}(\zeta) + r(z, w) + \mathcal{O}(|\operatorname{Im}(\zeta)|^2, |(z, w)| \cdot |\operatorname{Im}(\zeta)|) < 0\},$$

where V is a small open neighborhood of $p = 0$ and r is a real-valued real-analytic function defined on an open neighborhood of $0 \in \mathbb{C}^2$. Since this paper is on a counterexample, we limit ourselves to the case where r is a plurisubharmonic polynomial. By absorbing all pluriharmonic terms into the real part of ζ, one can assume that r has no pluriharmonic terms. Write

$$r(z, w) = \sum_{j=2k}^{M} P_j(z, w),$$

where P_j is a homogeneous polynomial in $z, \overline{z}, w, \overline{w}$ of degree j and $P_{2k} \not\equiv 0$ is plurisubharmonic.

If the remainder

$$R(z, w) := r(z, w) - P_{2k}(z, w) = \sum_{j=2k+1}^{M} P_j(z, w)$$

is plurisubharmonic then a bumping with the desired properties exists in many cases. The situation is not usually that simple however, so a different strategy is needed when the remainder R is not assumed to be plurisubharmonic.

Example 2.1. Assume Ω is given as follows locally around 0:

$$\Omega \cap V = \{(\zeta, z, w) \in \mathbb{C}^3 \cap V \colon \operatorname{Re}(\zeta) + P(z,w) < 0\},$$

where

$$\begin{aligned} P(z,w) = &|z|^6|w|^8 - 2\operatorname{Re}(z^3w^4\overline{z^5w^3}) + |z|^4|w|^{12} + |z|^{10}|w|^6 - 2\operatorname{Re}(zw^{10}\overline{z^2w^6}) \\ &+ |z|^{18}|w|^4 + |z|^2|w|^{20} - 2\operatorname{Re}(z^9w^2\overline{z^{17}w}) + |z|^{34}|w|^2 + \|(z,w)\|^{1000}. \end{aligned}$$

Define “wedge-wise” holomorphic coordinate changes $\Phi_1, \Phi_2, \Phi_3 \colon \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\begin{aligned} \Phi_1(z,w) &= \left(z^4, w\right), \\ \Phi_2(z,w) &= \left(z, w^2\right), \\ \Phi_3(z,w) &= \left(z, w^8\right). \end{aligned}$$

We compute:

$$\begin{aligned} \left(P \circ \Phi_1\right)(z,w) &= |z|^8|w|^{20} - 2\operatorname{Re}(z^4w^{10}\overline{z^8w^6}) + |z|^{16}|w|^{12} + (\textit{higher-order terms}) \\ &= \left|z^4w^{10} - z^8w^6\right|^2 + (\textit{higher-order terms}), \\ \left(P \circ \Phi_2\right)(z,w) &= |z|^6|w|^{16} - 2\operatorname{Re}(z^3w^8\overline{z^5w^6}) + |z|^{10}|w|^{12} + (\textit{higher-order terms}) \\ &= \left|z^3w^8 - z^5w^6\right|^2 + (\textit{higher-order terms}), \\ \left(P \circ \Phi_3\right)(z,w) &= |z|^{18}|w|^{32} - 2\operatorname{Re}(z^9w^{16}\overline{z^{17}w^8}) + |z|^{34}|w|^{16} + (\textit{higher-order terms}) \\ &= \left|z^9w^{16} - z^{17}w^8\right|^2 + (\textit{higher-order terms}). \end{aligned}$$

For $j \in \{1,2,3\}$, the lowest-order homogeneous summand of $P \circ \Phi_j$ corresponds to the summand $P^{(j)}$ in the Taylor expansion of P around 0, where

$$\begin{aligned} P^{(1)}(z,w) &= \left|zw^{10} - z^2w^6\right|^2, \\ P^{(2)}(z,w) &= \left|z^3w^4 - z^5w^3\right|^2, \\ P^{(3)}(z,w) &= \left|z^9w^2 - z^{17}w\right|^2. \end{aligned}$$

$P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ are plurisubharmonic. This is not a coincidence: P is plurisubharmonic and Φ_j, $j \in \{1,2,3\}$, is holomorphic, so the lowest order homogeneous summand of $P \circ \Phi_j$ is plurisubharmonic as well, which leads to $P^{(j)}$ being plurisubharmonic. $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ have pairwise no monomial in common, so:

$$P = P^{(1)} + P^{(2)} + P^{(3)} + (\textit{remaining terms}),$$

where the (*remaining terms*) consists of a finite (possibly empty) sum of monomials, each appearing with the same coefficient as the corresponding monomial in the Taylor expansion of P around 0. By direct computation one easily verifies that

$$P(z,w) = P^{(1)}(z,w) + P^{(2)}(z,w) + P^{(3)}(z,w) + \|(z,w)\|^{1000}.$$

So we have written P as a sum of four plurisubharmonic weighted-homogeneous polynomials. It is obvious how to bump P. In a more general setting one could attempt to use the bumping results for weighted-homogeneous plurisubharmonic polynomials in [2] to bump each summand separately.

Example 2.2. Assume Ω is given as follows locally around 0:

$$\Omega \cap V = \{(\zeta, z, w) \in \mathbb{C}^3 \cap V \colon \operatorname{Re}(\zeta) + P(z,w) < 0\},$$

where

$$P(z,w) = |z|^6 - 2\operatorname{Re}(z^3\overline{z^2w^2}) + 2|z|^4|w|^4 - 2\operatorname{Re}(z^2w^2\overline{w^{10}}) + |w|^{20} + \|(z,w)\|^{1000}.$$

Analogously to Example 2.1, one defines singular holomorphic coordinate changes $\Phi_1, \Phi_2 \colon \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\Phi_1(z,w) = \left(z^2, w\right),$$
$$\Phi_2(z,w) = \left(z^4, w\right),$$

and computes:

$$(P \circ \Phi_1)(z,w) = |z|^{12} - 2\operatorname{Re}(z^6\overline{z^4w^2}) + 2|z|^8|w|^4 + (\textit{higher-order terms}),$$
$$(P \circ \Phi_2)(z,w) = 2|z|^{16}|w|^4 - 2\operatorname{Re}(z^8w^2\overline{w^{10}}) + |w|^{20} + (\textit{higher-order terms}).$$

For $j \in \{1,2\}$, the lowest-order homogeneous summand of $P \circ \Phi_j$ corresponds to the summand $P^{(j)}$ in the Taylor expansion of P around 0, where

$$P^{(1)}(z,w) = |z|^6 - 2\operatorname{Re}(z^3\overline{z^2w^2}) + 2|z|^4|w|^4,$$
$$P^{(2)}(z,w) = 2|z|^4|w|^4 - 2\operatorname{Re}(z^2w^2\overline{w^{10}}) + |w|^{20}.$$

Analogously to the previous example, one argues that $P^{(1)}$ and $P^{(2)}$ are plurisubharmonic. But now the polynomials $P^{(1)}$ and $P^{(2)}$ share the summand $2|z|^4|w|^4$, so one can *not* proceed analogously to Example 2.1.
Splitting up the shared summand, however, one can write:

$$P(z,w) = \widetilde{P}^{(1)}(z,w) + \widetilde{P}^{(2)}(z,w) + \|(z,w)\|^{1000},$$

where

$$\begin{aligned}\widetilde{P}^{(1)}(z,w) &= |z|^6 - 2\operatorname{Re}(z^3\overline{z^2w^2}) + |z|^4|w|^4 \\ &= \left|z^3 - z^2w^2\right|^2, \\ \widetilde{P}^{(2)}(z,w) &= |z|^4|w|^4 - 2\operatorname{Re}(z^2w^2\overline{w^{10}}) + |w|^{20} \\ &= \left|z^2w^2 - w^{10}\right|^2.\end{aligned}$$

$\widetilde{P}^{(1)}$ and $\widetilde{P}^{(2)}$ are obviously plurisubharmonic and hence we have once again written P as a sum of plurisubharmonic weighted-homogeneous polynomials, each of which we can attempt to bump individually.

So, in both Example 2.1 and Example 2.2, we used certain singular holomorphic coordinate changes to express P as a sum of weighted-homogeneous plurisubharmonic polynomials. While the algorithmic procedure we applied will not always yield such a decomposition, the existence of said coordinate changes is not a coincidence: in both examples, each coordinate change corresponds to an *extreme edge* (see Def. 3.1 below) of the real-valued plurisubharmonic polynomial P.

3. The Problem

Most of the definitions and lemmas in this section are taken from [7]. From now on, all occurring polynomials are assumed to be polynomials with complex coefficients in two complex variables (z, w) and their conjugates $(\overline{z}, \overline{w})$.

Let P be a real-valued polynomial. We write

$$P = \sum_{(A,B)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}} P_{A,B},$$

where $P_{A,B}$ is homogeneous of degree A in $z, \overline{z}$ and homogeneous of degree B in $w, \overline{w}$. Note that this decomposition is unique and that each $P_{A,B}$ is real-valued.

Definition 3.1. Let P be a real-valued polynomial. We define the *Newton diagram* $N(P)$ of P to be the following subset of $\mathbb{R}^2$:

$$N(P) = \{(A, B) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \colon P_{A,B} \not\equiv 0\}.$$

We make the following definitions:

- A non-empty subset $X \subseteq N(P)$ is called an *extreme set* if there exist $a, b \in \mathbb{R}$ with $a < 0$, such that

$$B = aA + b \text{ for all } (A, B) \in X$$
$$B > aA + b \text{ for all } (A, B) \in N(P) \setminus X.$$

- A point $(A_0, B_0) \in N(P)$ is called an *extreme point* if $\{(A_0, B_0)\}$ is an extreme set.
- A subset $E \subseteq N(P)$ is called an *extreme edge* if E is an extreme set of cardinality at least 2.

Remark 3.2. Similar notions appear elsewhere in the literature. In the study of oscillatory integral operators for example, one defines the *Newton polytope* of a real-valued real-analytic function

$$S(x, y) = \sum_{(p,q)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}} c_{pq} x^p y^q$$

defined in a neighborhood of the origin in $\mathbb{R}^2$ as the convex hull of

$$\{(p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \colon c_{pq} \neq 0\} + {\mathbb{R}_{\geq 0}}^2,$$

i.e., in the language of [12], as the convex hull of the union of all the northeast quadrants in ${\mathbb{R}_{\geq 0}}^2$ with corners at those $(p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ for which $c_{pq} \neq 0$. If S is non-trivial, the Newton polygon is an unbounded set with non-empty interior, in contrast to the Newton diagram as defined in Definition 3.1, which is always a finite set. Furthermore, each element of the set $\{(p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \colon c_{p,q} \neq 0\}$ corresponds to a single monomial of S, whereas the analogous assertion is obviously not true in general for the Newton diagram.
Still, these two notions are clearly analogous in the sense that the 0-faces (resp. compact 1-faces) of the Newton polytope play the same role as the the extreme points (resp. extreme edges) of the Newton diagram.

Notation 3.3. Let P be a real-valued polynomial and let $S \subseteq \mathbb{R}^2$. We define the real-valued polynomial P_S as follows:

$$P_S := \sum_{(A,B) \in N(P) \cap S} P_{A,B}.$$

Note that $P_S \equiv 0$ if and only if $N(P) \cap S = \emptyset$.

Notation 3.4. Let P be a real-valued polynomial. We denote the Complex Hessian Matrix or the Levi Matrix of P as H_P,

$$H_P = \begin{pmatrix} \frac{\partial^2 P}{\partial z \partial \bar{z}} & \frac{\partial^2 P}{\partial w \partial \bar{z}} \\ \frac{\partial^2 P}{\partial z \partial \overline{w}} & \frac{\partial^2 P}{\partial w \partial \overline{w}} \end{pmatrix}.$$

The following two lemmas demonstrate that the concepts introduced in this section are significant when considering plurisubharmonic polynomials:

Lemma 3.5 ([7, Lemma 2 on p. 983])**.** *Let P be a real-valued polynomial. Then the Newton diagram $N(P)$ has finitely many extreme sets.*

Lemma 3.6 ([7, Lemmas 3 and 4 on p. 983])**.** *Let P be a real-valued polynomial and furthermore assume that P is plurisubharmonic. Then, for any extreme set X of $N(P)$, the function P_X is a plurisubharmonic weighted-homogeneous polynomial and there exists a natural singular holomorphic change of coordinates Φ of the form $(z, w) \mapsto (z^k, w^l)$ with $k, l \in \mathbb{Z}_{\geq 1}$,* $\gcd(k, l) = 1$*, such that $P_X \circ \Phi$ constitutes the lowest-order homogeneous terms of $P \circ \Phi$.*

In the setting of Example 2.1, the maps Φ_1, Φ_2 and Φ_3 correspond to extreme edges, say E_1, E_2 and E_3, of $N(P)$ in the sense of Lemma 3.6 (it should be noted, however, that $N(P)$ has other extreme edges as well). Since E_1, E_2 and E_3 are pairwise disjoint, the polynomials P_{E_1}, P_{E_2} and P_{E_3} have pairwise no terms in common, so that

$$P_{E_1 \cup E_2 \cup E_3} = P_{E_1} + P_{E_2} + P_{E_3}$$

is plurisubharmonic and

$$P(z, w) = P_{E_1}(z, w) + P_{E_2}(z, w) + P_{E_3}(z, w) + \|(z, w)\|^{1000}.$$

In the setting of Example 2.2, the maps Φ_1 and Φ_2 correspond to the precisely two extreme edges, say E_1 and E_2, of $N(P)$ in the sense of Lemma 3.6. Here, however, E_1 and E_2 are neighboring extreme edges, so that P_{E_1} and P_{E_2} have terms in common, namely $P_{E_1 \cap E_2}$. But $P_{E_1 \cup E_2}$ is plurisubharmonic and we found a splitting

$$P_{E_1 \cup E_2} = \widetilde{P_{E_1}} + \widetilde{P_{E_2}},$$

where $\widetilde{P_{E_j}}$ is a plurisubharmonic polynomial with $N\left(\widetilde{P_{E_j}}\right) \subseteq N(P_{E_j})$, for $j \in \{1, 2\}$.

In attempting to generalize the bumping strategies outlined in Examples 2.1 and 2.2, it becomes desirable to identify subsets of the Newton diagram of a plurisubharmonic polynomial that will yield a plurisubharmonic function in the sense of Notation 3.3. It is the content of Lemma 3.6 that extreme sets, i.e. extreme points and extreme edges, are examples of such subsets.
Specifically, in view of Examples 2.1 and 2.2 and Remark 3.8 below, one could hope that two "neighboring" extreme edges yield a plurisubharmonic function by taking their union or by taking the convex hull of that union. A precise statement of those questions goes as follows:

Question 3.7. Let P be a real-valued polynomial and furthermore assume that P is plurisubharmonic. Let $\mathcal{E}$ denote the (possibly empty) set of extreme edges of $N(P)$.

- Given extreme edges E_1 and E_2 of $N(P)$ with $E_1 \neq E_2$ but $E_1 \cap E_2 \neq \emptyset$, is $P_{E_1 \cup E_2}$ necessarily plurisubharmonic in some neighborhood of the origin?
- Given extreme edges E_1 and E_2 of $N(P)$ with $E_1 \neq E_2$ but $E_1 \cap E_2 \neq \emptyset$, is $P_{\mathrm{Conv}(E_1 \cup E_2)}$ necessarily plurisubharmonic in some neighborhood of the origin?
- Is $P_{\bigcup_{E \in \mathcal{E}} E}$ necessarily plurisubharmonic in some neighborhood of the origin?
- Is $P_{\mathrm{Conv}(\bigcup_{E \in \mathcal{E}} E)}$ necessarily plurisubharmonic in some neighborhood of the origin?

Here, $\mathrm{Conv}(S)$ denotes the convex hull of a subset S of $\mathbb{R}^2$.

In the following section we will construct a plurisubharmonic polynomial with precisely 2 extreme edges, for which the answer to all of these questions is "no".

Remark 3.8. In the last two questions in Question 3.7 one asks whether certain estimates of the Complex Hessian of P are preserved when deleting certain terms of P without affecting the extreme edges.
A similar question in the situation of Remark 3.2 is, roughly speaking, whether certain estimates of S and its partials are preserved when modifying S without affecting the Newton polytope or the terms corresponding to its 1-faces. In [12], D. H. Phong and E. M. Stein have obtained L^2-estimates for oscillatory integral operators that only depend on the (reduced) Newton polytope of the phase S. More recently, in [15], L. Xiao has classified the existence of certain L^p-estimates for oscillatory integral operators entirely in terms of the (reduced) Newton polytope of the phase S. One important ingredient in Xiao's proof is [15, Theorem 4.1 on p. 267]; it says, roughly speaking, that one can partition a neighborhood of the origin in $\mathbb{R}^2$ into finitely many curved triangular regions, on each of which the phase S (and its partials) can be estimated in terms of a monomial (and its partials).
From our point of view, the crucial point is that these monomials correspond to certain compact faces of the Newton polytope of S and that the curved triangular regions are obtained in a way that is very similar to some of the ideas appearing in [7]. Because of this, one could hope that, in the situation of Question 3.7, a similar analysis of the Levi form of P involving the extreme edges would yield a positive answer to one of the questions in Question 3.7.

4. The Counterexample

In order to simplify the computations in the construction announced in the previous section, we state and prove the following lemma:

Lemma 4.1. *Let* $P = \sum_{\alpha \in \mathcal{A}} c_\alpha \cdot |f_\alpha|^2$, *where*

- $\mathcal{A}$ *is a finite set,*
- $c_\alpha \in \{-1, 1\}$ *for all* $\alpha \in \mathcal{A}$,
- $f_\alpha \colon \mathbb{C}^2 \to \mathbb{C}$ *is a holomorphic polynomial for all* $\alpha \in \mathcal{A}$.

Then in $\mathbb{C}^2$ *we have:*

$$\det H_P = \frac{1}{2} \cdot \sum_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{A}} c_\alpha c_\beta \left| \frac{\partial f_\alpha}{\partial z} \cdot \frac{\partial f_\beta}{\partial w} - \frac{\partial f_\beta}{\partial z} \cdot \frac{\partial f_\alpha}{\partial w} \right|^2 .$$

Proof. We calculate:

$$\begin{aligned}
\det H_P &= \left(\sum_{\alpha\in\mathcal{A}} c_\alpha \frac{\partial f_\alpha}{\partial z}\frac{\partial \overline{f_\alpha}}{\partial \overline{z}}\right)\cdot\left(\sum_{\beta\in\mathcal{A}} c_\beta \frac{\partial f_\beta}{\partial w}\frac{\partial \overline{f_\beta}}{\partial \overline{w}}\right) - \left(\sum_{\alpha\in\mathcal{A}} c_\alpha \frac{\partial f_\alpha}{\partial z}\frac{\partial \overline{f_\alpha}}{\partial \overline{w}}\right)\cdot\left(\sum_{\beta\in\mathcal{A}} c_\beta \frac{\partial f_\beta}{\partial w}\frac{\partial \overline{f_\beta}}{\partial \overline{z}}\right)\\
&= \left(\sum_{\alpha\in\mathcal{A}} c_\alpha \frac{\partial f_\alpha}{\partial z}\overline{\left(\frac{\partial f_\alpha}{\partial z}\right)}\right)\cdot\left(\sum_{\beta\in\mathcal{A}} c_\beta \frac{\partial f_\beta}{\partial w}\overline{\left(\frac{\partial f_\beta}{\partial w}\right)}\right)\\
&\quad - \left(\sum_{\alpha\in\mathcal{A}} c_\alpha \frac{\partial f_\alpha}{\partial z}\overline{\left(\frac{\partial f_\alpha}{\partial w}\right)}\right)\cdot\left(\sum_{\beta\in\mathcal{A}} c_\beta \frac{\partial f_\beta}{\partial w}\overline{\left(\frac{\partial f_\beta}{\partial z}\right)}\right)\\
&= \sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{A}} c_\alpha c_\beta \cdot \frac{\partial f_\alpha}{\partial z}\cdot\frac{\partial f_\beta}{\partial w}\cdot\overline{\left(\frac{\partial f_\alpha}{\partial z}\cdot\frac{\partial f_\beta}{\partial w} - \frac{\partial f_\beta}{\partial z}\cdot\frac{\partial f_\alpha}{\partial w}\right)}\\
&= \frac{1}{2}\cdot\sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{A}} c_\alpha c_\beta \cdot \frac{\partial f_\alpha}{\partial z}\cdot\frac{\partial f_\beta}{\partial w}\cdot\overline{\left(\frac{\partial f_\alpha}{\partial z}\cdot\frac{\partial f_\beta}{\partial w} - \frac{\partial f_\beta}{\partial z}\cdot\frac{\partial f_\alpha}{\partial w}\right)}\\
&\quad + \frac{1}{2}\cdot\sum_{(\beta,\alpha)\in\mathcal{A}\times\mathcal{A}} c_\beta c_\alpha \cdot \frac{\partial f_\beta}{\partial z}\cdot\frac{\partial f_\alpha}{\partial w}\cdot\overline{\left(\frac{\partial f_\beta}{\partial z}\cdot\frac{\partial f_\alpha}{\partial w} - \frac{\partial f_\alpha}{\partial z}\cdot\frac{\partial f_\beta}{\partial w}\right)}\\
&= \frac{1}{2}\cdot\sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{A}} c_\alpha c_\beta \cdot \left(\frac{\partial f_\alpha}{\partial z}\cdot\frac{\partial f_\beta}{\partial w} - \frac{\partial f_\beta}{\partial z}\cdot\frac{\partial f_\alpha}{\partial w}\right)\cdot\overline{\left(\frac{\partial f_\alpha}{\partial z}\cdot\frac{\partial f_\beta}{\partial w} - \frac{\partial f_\beta}{\partial z}\cdot\frac{\partial f_\alpha}{\partial w}\right)}\\
&= \frac{1}{2}\cdot\sum_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{A}} c_\alpha c_\beta \left|\frac{\partial f_\alpha}{\partial z}\cdot\frac{\partial f_\beta}{\partial w} - \frac{\partial f_\beta}{\partial z}\cdot\frac{\partial f_\alpha}{\partial w}\right|^2.
\end{aligned}$$

□

Let $f_1, f_2, f_3, g, h\colon \mathbb{C}^2 \to \mathbb{C}$ be the holomorphic monomials given as follows:

$$f_1(z,w) = z^2w^2 \qquad f_2(z,w) = z^{10}w \qquad f_3(z,w) = zw^{10}$$
$$g(z,w) = z^4w^2 \qquad h(z,w) = z^4w^8$$

We now define a real-valued polynomial P:

$$P := |f_1 + f_2 + f_3|^2 + |g+h|^2.$$

It is obvious that P is plurisubharmonic. Intuitively speaking, the Newton diagram $N(P)$ has precisely two extreme edges and lies entirely in the triangle spanned by $N(|f_1|^2)$, $N(|f_2|^2)$ and $N(|f_3|^2)$, with the exception of $N(|h|^2)$, which is "sticking out" of the triangle without creating an extreme edge. Both extreme edges correspond to sides of said triangle. The monomials were specifically chosen to have these properties (among others). We will treat this formally:

Lemma 4.2. *The Newton diagram of P is the following set:*

$$N(P) = \{(4,4),(12,3),(3,12),(20,2),(11,11),(2,20),(8,4),(8,10),(8,16)\}.$$

Furthermore, $N(P)$ has precisely two extreme edges, namely

$$E_1 = \{(4,4),(3,12),(2,20)\} \text{ and } E_2 = \{(4,4),(12,3),(20,2)\},$$

and the following holds on $\mathbb{C}^2$*:*

$$\begin{aligned} P_{E_1\cup E_2} &= |f_1+f_3|^2+|f_1+f_2|^2-|f_1|^2,\\ P_{\mathrm{Conv}(E_1\cup E_2)} &= P-|h|^2\\ &= |f_1+f_2+f_3|^2+|g+h|^2-|h|^2. \end{aligned}$$

The proof of Lemma 4.2 is a straightforward calculation and will be omitted. It should, however, be remarked that, in light of Lemma 4.1, the monomials occurring in the definition of P were chosen so that $P_{E_1\cup E_2}$ and $P_{\mathrm{Conv}(E_1\cup E_2)}$ take this particular form.

In order to show that (for P) the answer to all the questions in Question 3.7 is “no”, it suffices to show that both $P_{E_1\cup E_2}$ and $P_{\mathrm{Conv}(E_1\cup E_2)}$ are *not* plurisubharmonic in any neighborhood of the origin.
By Lemma 4.1 and Lemma 4.2 we have the following on $\mathbb{C}^2$:

$$\begin{aligned} \det H_{P_{E_1\cup E_2}} = &\left|\frac{\partial(f_1+f_3)}{\partial z}\cdot\frac{\partial(f_1+f_2)}{\partial w}-\frac{\partial(f_1+f_2)}{\partial z}\cdot\frac{\partial(f_1+f_3)}{\partial w}\right|^2\\ &-\left|\frac{\partial(f_1+f_3)}{\partial z}\cdot\frac{\partial f_1}{\partial w}-\frac{\partial f_1}{\partial z}\cdot\frac{\partial(f_1+f_3)}{\partial w}\right|^2\\ &-\left|\frac{\partial(f_1+f_2)}{\partial z}\cdot\frac{\partial f_1}{\partial w}-\frac{\partial f_1}{\partial z}\cdot\frac{\partial(f_1+f_2)}{\partial w}\right|^2\\ \leq &\left|\frac{\partial(f_1+f_3)}{\partial z}\cdot\frac{\partial(f_1+f_2)}{\partial w}-\frac{\partial(f_1+f_2)}{\partial z}\cdot\frac{\partial(f_1+f_3)}{\partial w}\right|^2\\ &-\left|\frac{\partial(f_1+f_3)}{\partial z}\cdot\frac{\partial f_1}{\partial w}-\frac{\partial f_1}{\partial z}\cdot\frac{\partial(f_1+f_3)}{\partial w}\right|^2,\\ \det H_{P_{\mathrm{Conv}(E_1\cup E_2)}} = &\left|\frac{\partial(f_1+f_2+f_3)}{\partial z}\cdot\frac{\partial(g+h)}{\partial w}-\frac{\partial(g+h)}{\partial z}\cdot\frac{\partial(f_1+f_2+f_3)}{\partial w}\right|^2\\ &-\left|\frac{\partial(f_1+f_2+f_3)}{\partial z}\cdot\frac{\partial h}{\partial w}-\frac{\partial h}{\partial z}\cdot\frac{\partial(f_1+f_2+f_3)}{\partial w}\right|^2\\ &-\left|\frac{\partial(g+h)}{\partial z}\cdot\frac{\partial h}{\partial w}-\frac{\partial h}{\partial z}\cdot\frac{\partial(g+h)}{\partial w}\right|^2\\ \leq &\left|\frac{\partial(f_1+f_2+f_3)}{\partial z}\cdot\frac{\partial(g+h)}{\partial w}-\frac{\partial(g+h)}{\partial z}\cdot\frac{\partial(f_1+f_2+f_3)}{\partial w}\right|^2\\ &-\left|\frac{\partial(g+h)}{\partial z}\cdot\frac{\partial h}{\partial w}-\frac{\partial h}{\partial z}\cdot\frac{\partial(g+h)}{\partial w}\right|^2. \end{aligned}$$

So, by plugging in and calculating, we get the following inequalities on $\mathbb{C}^2$:

$$\begin{aligned} \det H_{P_{E_1\cup E_2}}(z,w) \leq &\left|(2zw^2+w^{10})\cdot(2z^2w+z^{10})-(2zw^2+10z^9w)\cdot(2z^2w+10zw^9)\right|^2\\ &-\left|(2zw^2+w^{10})\cdot 2z^2w-2zw^2\cdot(2z^2w+10zw^9)\right|^2\\ = &\left|z^2w^2(99z^8w^8+18z^9+18w^9)\right|^2\\ &-\left|18z^2w^{11}\right|^2, \end{aligned}$$

$$\begin{aligned}\det H_{P_{\mathrm{Conv}(E_1\cup E_2)}}(z,w) \leq\ & |(2zw^2+10z^9w+w^{10})\cdot(2z^4w+8z^4w^7)\\ &-(4z^3w^2+4z^3w^8)\cdot(2z^2w+z^{10}+10zw^9)|^2\\ &-\left|(4z^3w^2+4z^3w^8)\cdot 8z^4w^7-4z^3w^8\cdot(2z^4w+8z^4w^7)\right|^2\\ =\ & \left|-2z^4w^2(16w^{15}-38w^6z^9+19w^9-8z^9-4zw^7+2zw)\right|^2\\ &-\left|24z^7w^9\right|^2.\end{aligned}$$

We define two holomorphic polynomials $Q_1, Q_2\colon \mathbb{C}^2 \to \mathbb{C}$ as follows:

$$\begin{aligned}Q_1(z,w) &= 99z^8w^8+18z^9+18w^9,\\ Q_2(z,w) &= 16w^{15}-38w^6z^9+19w^9-8z^9-4zw^7+2zw,\end{aligned}$$

i.e. we have on $\mathbb{C}^2$:

$$\begin{aligned}\det H_{P_{E_1\cup E_2}}(z,w) \leq\ & \left|z^2w^2Q_1(z,w)\right|^2\\ &-\left|18z^2w^{11}\right|^2,\\ \det H_{P_{\mathrm{Conv}(E_1\cup E_2)}}(z,w) \leq\ & \left|-2z^4w^2Q_2(z,w)\right|^2\\ &-\left|24z^7w^9\right|^2.\end{aligned}$$

Since Q_1 is a non-constant holomorphic polynomial on $\mathbb{C}^2$, its vanishing set $V(Q_1)$ is an equidimensional affine algebraic variety of dimension 1 containing $(0,0)$. For $(z,w)\in V(Q_1)$ we have

$$\det H_{P_{E_1\cup E_2}}(z,w) \leq -\left|18z^2w^{11}\right|^2,$$

so that it suffices to show that $V(Q_1)$ contains points (z,w) with $z\neq 0, w\neq 0$ arbitrarily close to $(0,0)$. But that is clear, since both $Q_1(\cdot,0)$ and $Q_1(0,\cdot)$ are non-constant holomorphic polynomials on $\mathbb{C}$ and as such have finitely many zeroes.

Hence $P_{E_1\cup E_2}$ is not plurisubharmonic in any neighborhood or the origin. By considering Q_2 instead of Q_1, we analogously get that $P_{\mathrm{Conv}(E_1\cup E_2)}$ is not plurisubharmonic in any neighborhood of the origin.

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