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Harmonic Functions On Square Lattices: Uniqueness Sets and Growth Properties

Thesis for the degree of Philosophiae Doctor

Trondheim, October 2013

Norwegian University of Science and Technology
Faculty of Information Technology, Mathematics
and Electrical Engineering
Department of Mathematical Sciences



NTNU – Trondheim
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Preface

This thesis is submitted in partial fulfillment of the requirements for the Degree Philosophiae (PhD) at the Norwegian University of Science and Technology (NTNU) in Trondheim, Norway. The thesis represents a culmination of learning experience that has taken place over a period of almost four years from September 2009 to June 2013. The research work has been financed by the Research Council of Norway grant 185359/ V30 and carried out at the Department of Mathematical Science under the supervision of Associate Professor Eugenia Malinnikova. Most of the results presented in this thesis are from the following papers.

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1 Introduction

1.1 Preliminaries

Discrete harmonic functions on the lattice, also known as pre-harmonic functions, were intensively studied in the last century. The main motivation comes from the classical works where the solution of the Dirichlet problem is constructed using discrete approximation. This method can be used to establish both existence and stability of solution to the Dirichlet problem.

The theory of discrete harmonic functions on the lattices dates back to the 1920s, when fundamental works of H. Phillips and N. Wiener [41] (1923), and R. Courant, K. Friedrichs, and H. Lewy [14] (1928) were published. For the next three decades a number of articles followed, we will mention those of J. Capoulade [9] (1932), I. Petrowski [40] (1941), M. Frocht [22] (1946), H. Heilbronn [28] (1949), and S. Verblunsky [47] (1949-50). At the same time the theory of discrete holomorphic functions was developed in the works of J. Ferrand [20] (1944). In the middle of the last century an important contribution to the theory of discrete harmonic functions was done by R. Duffin [18] (1953).

The study of discrete harmonic function in the first half of the last century was at least two-folded, convergence of the numerical methods and approximation of continuous harmonic functions by discrete ones was developed side by side with the study of the properties of discrete functions. One of the original motivations for the study of discrete harmonic functions is that such functions converge to continuous ones. For example to obtain a solution of the Dirichlet problem one may solve discrete problems in lattice domains and pass to the limit as the mesh size of the lattice goes to zero. We refer the reader to the classical works mentioned above and to the article of I. Petrowsky [40]. Connections to random walks give one more side of the theory. We don't discuss it here and refer the readers to the classical work of K. Ito and H. McKean [29]. It turned out that while many fundamental results of continuous theory of harmonic functions have discrete counterparts (as the maximum principle, Green's function, solution of the Dirichlet problem), there are many aspects of discrete potential theory that

are quite different from the continuous ones. These difference provide various interesting problems on discrete Laplacian and special properties of discrete harmonic functions which were recently addressed by researchers in theoretical and applied analysis, as examples we cite articles of C. Kiselman [31], E. Bendo, A. Carmona, and A. Encinas, [4], J. Chanzy [10], A. Rubinstein, J. Rubenstein, and G. Wolansky [43], P. Vivo, M. Casartelli, L. Dall’Asta, and A. Vezzani [48], and P. Nayar [38]. One of the interesting questions is the study of the zero sets of discrete harmonic functions is that equivalent to the study of their uniqueness sets. The aim of our work is to provide basic examples and first results that connect the size of the zero sets of discrete harmonic functions to its growth properties.

In this introduction we first give a short account of discrete harmonic functions on graphs and then describe the main results of the thesis.

1.2 Laplace operators and harmonic functions on graphs

In this section we discuss basic definitions, the maximum principle for discrete harmonic functions on graphs, and the discrete Dirichlet problem. For more general theory of Laplace operators on graphs we refer the reader to the monograph [12].

1.2.1 Definitions and the maximum principle

A graph $G = (V, E)$ consists of two sets, V being the set of vertices and $E \subset V \times V$ the set of edges. We assume that G has no self loops and is undirected, i.e. $(x, x) \notin E$ for any $x \in V$ and if $(x, y) \in E$ then $(y, x) \in E$. We also assume that G is locally finite. That means that for any $x \in V$ the set $O_x = \{y \in V : (x, y) \in E\}$ is finite; the number of points in O_x will be denoted by $d(x)$. Moreover, we mostly consider graphs with $d = \sup_{x \in G} d(x) < +\infty$.

By a function on G we mean a real-valued function on the set of the vertices. We denote by $U(G)$ the set of all functions on G . By a vector field we mean a real-valued function $u : E \rightarrow \mathbf{R}$ on the set of the edges that satisfy $u(x, y) = -u(y, x)$, We denote by $F(G)$ the set of all vector fields on G .

If f is a function on G , then its gradient, ∇f , is the vector field defined by

$$\nabla f(x, y) = f(y) - f(x), \quad (1.1)$$

for any $(x, y) \in E$. Clearly $\nabla f(y, x) = -\nabla f(x, y)$. The divergence of a vector field u is a function on G defined by

$$\operatorname{div}u(x) = \sum_{y:(y,x) \in E} u(x, y). \quad (1.2)$$

Definition. Let f be a function on G . Then the Laplacian of f is a function on G denoted by Δf and defined by

$$\Delta f(x) = \sum_{y:(y,x) \in E} f(y) - d(x)f(x). \quad (1.3)$$

Clearly, for every $f \in U(G)$ it holds that

$$\Delta f(x) = \operatorname{div}(\nabla f)(x).$$

Definition. A function $f \in U(G)$ is said to be discrete harmonic at a vertex x if it satisfies the relation

$$\Delta f(x) = 0. \quad (1.4)$$

We say that f is discrete harmonic on a set $S \subset V$ if it is discrete harmonic at each point of S .

Definition. A function $f \in U(G)$ is said to be discrete subharmonic at a vertex x if $\Delta f(x) \geq 0$.

Definition. Let D be a subset of the vertices, $D \subset V$. Then the (outer) boundary of D denoted bD is defined by

$$bD = \{y \in V \setminus D, \text{ there exists } x \in D \text{ such that } (x, y) \in E\}.$$

The closure of D is defined by $\overline{D} = D \cup bD$.

Definition. Given a graph $G = (V, E)$, a subset $D \subset V$ is said to be connected if for every two points $x, y \in D$, there exists a finite sequence of points $\{x_0, x_1, \dots, x_s\}$ such that $x_0 = x, x_s = y, x_j \in D, x_j$ and x_{j+1} are neighboring points of G , i.e., $(x_j, x_{j+1}) \in E$.

From now on we assume that G is a connected graph. It means that V is simply a connected set of vertices.

Theorem 1. *If f is a function on G that is discrete harmonic on a finite connected set $D \subset V$ and if*

$$\max_{x \in D} f(x) = \max_{x \in \overline{D}} f(x),$$

then f is constant on \overline{D} .

Proof. Let $M = \max_{x \in \overline{D}} f(x)$ and suppose that x_0 is a point of D for which $f(x_0) = M$. Then we have $f(x_0) \geq f(x)$ for any x that is neighboring to x_0 , furthermore, since f is discrete harmonic at x_0 then we have $f(x) = M$ for all neighbors x of x_0 . Since D is connected, \overline{D} is also connected and $f(x) = M$ for all $x \in \overline{D}$. \square

In particular, any discrete harmonic function on a finite connected graph is a constant (take $D = V$). It also follows from the theorem that if D is a finite connected set, $D \neq V$, and f is a discrete harmonic function on D , then

$$\max_{x \in \overline{D}} f(x) = \max_{x \in bD} f(x). \quad (1.5)$$

The relation displayed in (1.5), which is the maximum principle for discrete harmonic functions is an analog of the classical maximum principle for continuous harmonic functions in a bounded domain. The condition of boundedness is now replaced by the condition that D is finite; it is easy to see that such condition is necessary. There exists a non-zero function on the graph $V = \mathbf{Z} \times \{-1, 0, 1\}$ with standard edges of the square lattice that is discrete harmonic on $D = \mathbf{Z} \times \{0\}$, but equal to zero on the boundary of the latter set. We can define

$$u(-n, 0) = u(n, 0) = \frac{1}{2} \left((2 + \sqrt{3})^n + (2 - \sqrt{3})^n \right), \quad n \geq 0. \quad (1.6)$$

We will discuss discrete harmonic functions on subdomains of the lattice in the next chapter. The example above illustrates that on special unbounded subdomains of the lattice maximum principle fails for exponentially growing functions. We will develop this topic further in Chapter 3.

1.2.2 The Dirichlet problem

Let $G = (V, E)$ be a connected finite graph and let $D \subset V$ be a finite subset of V such that $bD \neq \emptyset$. Then the discrete Dirichlet problem is formulated as

$$\begin{cases} \Delta f(x) = 0 & x \in D \\ f(x) = g(x) & x \in bD, \end{cases}$$

where $g : bD \rightarrow \mathbf{R}$ is a given function.

The discrete Dirichlet problem can be considered as a linear algebra problem, with k unknowns, where k is the number of points in D and k equations from the relation $\Delta f = 0$ in D . Moreover, when $g = 0$ the system is homogeneous and from the maximum principle we know that it only admits trivial solution. Then the existence and uniqueness of the solution to the Dirichlet problem for arbitrary g is a simple consequence of the basic theorems of linear algebra. Another classical approach is also to describe harmonic functions as minimizers of the certain energy integral.

Theorem 2. *Let $g : bD \rightarrow \mathbf{R}$. Then the Dirichlet problem above has a unique solution f . If h is a real function defined on \bar{D} which takes the values $g(x)$ on bD , then*

$$\sum_{(x,x') \in E} (\nabla f(x, x'))^2 \leq \sum_{(x,x') \in E} (\nabla h(x, x'))^2$$

and equality holds only if $f(x) = h(x)$ for all x in D .

Proof. Let f be a function for which $E(f)$ is minimum, subject to the boundary condition, where

$$E(f) = \frac{1}{2} \sum_{(x,x') \in E} (\nabla f(x, x'))^2 = \frac{1}{2} \sum_{(x,x') \in E} (f(x') - f(x))^2$$

is the discrete Dirichlet energy.

For each $x \in D$ the minimizer of the discrete Dirichlet energy satisfies

$$\frac{d}{df(x)} E(f) = - \sum_{x':(x,x') \in E} \nabla f(x, x') = - \sum_{x':(x,x') \in E} (f(x') - f(x)) = 0,$$

which implies that $\Delta f(x) = 0$. This proves that f is discrete harmonic.

To show that the solution f is unique we argue as follows. Let F be another discrete harmonic function which takes the value g on the boundary. Then it follows that $F(x) - f(x) = 0$ on the boundary. By the maximum principle we conclude $F(x) = f(x)$ for all x in D which shows the uniqueness of f . \square

In chapter 4 we consider the following generalization of the discrete Dirichlet problem

$$\begin{cases} \Delta f(x) = 0 & x \in D \\ f(x) = g(x) & x \in \Lambda, \end{cases}$$

for some $\Lambda \subset \overline{D}$. For the problem with $\Lambda \neq bD$ one can't use the energy method and reconstruction becomes unstable.

1.2.3 Green's formula

In this section we introduce the discrete version of Green's formula on graphs, which can be found for example in [12].

Proposition 1. *Suppose $G = (V, E)$ be a finite connected graph and D be any non-empty finite subset of V . Then for any two functions $f, g \in U(G)$ we have*

$$\begin{aligned} \sum_{x \in D} \Delta f(x)g(x) = & \\ & -\frac{1}{2} \sum_{x,y \in D, (x,y) \in E} \nabla f(x,y) \nabla g(x,y) + \sum_{x \in D} \sum_{y \in bD, (x,y) \in E} \nabla f(x,y)g(x), \end{aligned}$$

where Δ and ∇ are the discrete Laplace and gradient operators respectively.

Proof. For any two functions $f, g \in U(G)$, we have

$$\begin{aligned} \sum_{x \in D} \Delta f(x)g(x) &= \sum_{x \in D} g(x) \sum_{y: (x,y) \in E} (f(y) - f(x)) = \\ & \sum_{x \in D, (x,y) \in E} g(x)f(y) - \sum_{x \in D} d(x)g(x)f(x) = \\ & \sum_{x,y \in D, (x,y) \in E} g(x)f(y) + \sum_{x \in D, y \in bD, (x,y) \in E} g(x)f(y) - \sum_{x \in D} d(x)g(x)f(x). \end{aligned}$$

And also we have

$$\begin{aligned} \sum_{x,y \in D, (x,y) \in E} \nabla f(x,y) \nabla g(x,y) &= \\ & \sum_{x,y \in D, (x,y) \in E} (f(y) - f(x))(g(y) - g(x)) = \\ & 2 \sum_{x \in D} d_D(x)f(x)g(x) - 2 \sum_{x,y \in D, (x,y) \in E} f(x)g(y), \end{aligned}$$

where $d_D(x)$ is the number of neighbors of the vertex x in D . Now dividing both sides of the above identity by 2 and adding the last two identities we obtain

$$\begin{aligned} \sum_{x \in D} \Delta f(x)g(x) + \frac{1}{2} \sum_{x,y \in D, (x,y) \in E} \nabla f(x,y)\nabla g(x,y) &= \\ \sum_{x \in D, y \in bD, (x,y) \in E} g(x)f(y) - \sum_{x \in D} f(x)g(x)(d(x) - d_D(x)) &= \\ \sum_{x \in D, y \in bD, (x,y) \in E} g(x)[f(y) - f(x)] &= \sum_{x \in D, y \in bD, (x,y) \in E} g(x)\nabla f(x,y). \end{aligned}$$

Then the proposition follows. \square

Corollary. *Let $G = (V, E)$ be a finite graph, and f, g in $U(G)$ be two arbitrary functions. Then*

$$\sum_{x \in V} f(x)\Delta g(x) = \sum_{x \in V} g(x)\Delta f(x) = -\frac{1}{2} \sum_{(x,y) \in E} \nabla f(x,y)\nabla g(x,y).$$

1.2.4 Green's function

Let $D \subset V$ be a finite subset that defines a connected subgraph with non-empty boundary bD . We define a function $G_D : \bar{D} \times \bar{D} \rightarrow \mathbf{R}$ such that for any $g : D \rightarrow \mathbf{R}$ we have

$$\Delta f = g, \quad \text{where } f(x) = \sum_{y \in D} G_D(x,y)g(y).$$

The function G_D defines the inverse of the Laplace operator on D . For a fixed $y \in D$ we have $\Delta G_D(\cdot, y) = \delta_y$ and $G_D(x, y) = 0$ when $x \in bD$. For example when $D \subset \mathbf{Z}^n$, we may first construct a global Green's function $G_0(x, y)$ that satisfies $\Delta G_0(\cdot, y) = \delta_y$ and then take $G_D(x, y) = G_0(x, y) - H_D(x, y)$, where $H_D(\cdot, y)$ is discrete harmonic in D with boundary values $G_0(x, y)$ for $x \in bD$.

Suppose that ϕ_j 's are the Dirichlet eigenfunctions of the discrete Laplacian on D with eigenvalues λ_j . Then we have

$$G_D(x, y) = \sum_j \lambda_j^{-1} d_x^{1/2} \phi_j(x) \phi_j(y) d_y^{-1/2},$$

where d_a is the degree of the vertex a . A detailed treatment of discrete Green's function can be found in [11] and the references therein.

1.2.5 Harmonic measure and the Poisson kernel

Let $D \subset V$ be a finite subset with $bD \neq \emptyset$. For any $A \subset bD$ we define $\omega(x, A, D)$ as the discrete harmonic function in D with boundary values $\omega(x, A, D) = 1$ when $x \in A$ and $\omega(x, A, D) = 0$ when $x \in bD \setminus A$. Clearly such discrete harmonic function exists uniquely, and $0 \leq \omega(x, A, D) \leq 1$ for any $x \in D$. The function $\omega(\cdot, A, D)$ is called the harmonic measure of A with respect to D . In particular, if we take $A = \{y\}$ for some $y \in bD$, we get a discrete harmonic function $\omega(x, \{y\}, D) = P_D(x, y)$. Then for any discrete harmonic function u in D we have

$$u(x) = \sum_{y \in bD} P_D(x, y)u(y).$$

So $P_D(x, y)$ plays the role of the Poisson kernel for the discrete domain D . Another way to look at the Poisson kernel is by taking the “normal derivative” of the Green function defined above and applying the Green formula. For some simple cases, for example when D is a lattice cube, we give a simple formula for the Poisson kernel and prove some estimates in Chapters 2 and Chapter 5.

1.3 Overview of the main results

1.3.1 Harmonic polynomials and generalizations of the Liouville theorem

In the rest of the text we study discrete harmonic functions on subsets on the lattice \mathbf{Z}^n (or sometimes $(h\mathbf{Z})^n$ when we prefer to fix some domains and change the mesh size of the lattice). In Chapter 2 we present some results from a long dated work of H. Heilbronn [28] and subsequent work of B. Murdoch [37], that seems to be completely forgotten. Recently some of their results were rediscovered in [48, 38].

The main topics of this Chapter are discrete harmonic polynomials (it is easy to see that usual harmonic polynomials in general are not discrete harmonic) and Liouville theorem and its generalizations. We fill in some of the details of the arguments of Heilbronn, suggest new proofs for some of his results and collect some well-known facts about discrete harmonic functions. We also generalize one of the results of H. Heilbronn and show that each

discrete harmonic function on a cube in \mathbf{Z}^n coincides with some discrete harmonic polynomial.

1.3.2 Discrete harmonic functions on product domains

Continuing the study of discrete harmonic functions on subsets of \mathbf{Z}^n , we consider unbounded cylinder domains and prove some discrete versions of the Phragmén-Lindelöf theorem in Chapter 3.

Let Ω be a bounded subdomain of \mathbf{Z}^n and $D = \Omega \times \mathbf{Z}^k$. The following statement holds

Theorem 3. *Let v be a discrete subharmonic function in D such that $v \leq 0$ on ∂D . Let λ_1 be the first eigenvalue of the discrete Dirichlet problem for the Laplacian in Ω and b be the positive solution to the equation*

$$\cosh b = 1 + \frac{1}{2k} \lambda_1.$$

Suppose that

$$v(x, y) \leq o(1) \exp(b\|y\|_1), \quad \text{when } \|y\|_1 \rightarrow \infty. \quad (1.7)$$

Then $v \leq 0$ on D .

We also give some quantitative version of the result when $k = 1$, using estimates of the harmonic measure in truncated cylinders, and discuss eigenvalues of discrete Dirichlet problem.

Another result which would be discussed in Chapter 3 is the following discrete version of the three-line theorem.

Theorem 4. *Let u be a discrete harmonic function in $\llbracket 0, M+1 \rrbracket \times \mathbf{Z}^n$, where $\llbracket 0, M+1 \rrbracket = [0, M+1] \cap \mathbf{Z}$. Suppose that u satisfies (1.7) and*

$$\{u(0, j)\}_{j \in \mathbf{Z}^n} \in l^2(\mathbf{Z}^n), \quad \{u(M+1, j)\}_{j \in \mathbf{Z}^n} \in l^2(\mathbf{Z}^n).$$

Let us further denote

$$m(k) = \|u_x(k, j)\|_{l^2(\mathbf{Z}^n)}^2 + \sum_{l=1}^n \|u_{y_l}(k, j)\|_{l^2(\mathbf{Z}^n)}^2 \quad \text{for } k = 0, 1, \dots, M,$$

where u_x and u_{y_l} are discrete partial derivatives of u . Then $m(k)$ is finite and satisfies

$$m(k) \leq (m(0))^{1-\frac{k}{M}} (m(M))^{\frac{k}{M}}.$$

The result is obtained by treating values of a discrete harmonic function on hyperplanes as the Fourier coefficients of some continuous function.

1.3.3 Determining sets of discrete Laplacian

Further, we study discrete harmonic functions on cubes or squares. The space of the discrete harmonic functions on such set Q is finite dimensional, the dimension, \dim_Q , is equal to the number of boundary points (for example if $Q = \llbracket 1, N - 1 \rrbracket^2$, then its boundary contains $4N - 4$ points). A discrete subset of a cube is called a determining set for the discrete Laplace operator if it contains exactly \dim_Q points and is also a uniqueness set for discrete harmonic functions. The notation of determining sets were introduced in [43].

In Chapter 4 we discuss reconstruction of discrete harmonic function from a determining set. This is a linear algebra problem that may have a very large conditioning number. For example we fix a square and a set Λ as below.

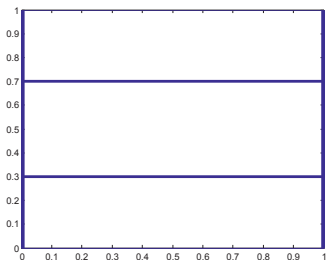


Figure 1.1: Model set Λ_b

Then when we change the mesh size $h = N^{-1}$ of the lattice, the conditioning number growth exponentially, like C^N . We suggest regularization procedure and get conditional stability for such reconstruction. One of the main tools is the logarithmic convexity estimates for the norms of a discrete harmonic function over parallel segments.

1.3.4 Unique continuation for discrete harmonic functions

In Chapter 5 we suggest a simple proof of the discrete version of three sphere inequality for harmonic functions. It is clear that there is no classical unique continuation theorem for discrete harmonic functions, as one can construct functions that vanish on any finite subset of the lattice without being zero

identically. However, there is a unique continuation inequality that involves the mesh size of the lattice.

We define by Q_d the cube $[-d, d]^n \subset \mathbf{R}^n$ and by Q_d^h its discretization, $Q_d^h = Q_d \cap (h\mathbf{Z})^n$. Then we prove the following.

Theorem 5. *Suppose that $r < R < 1$. Then there exist positive constants C, δ, α that depend on r, R with $\alpha, \delta < 1$ such that for any $h = N^{-1}$, $N \in \mathbb{N}$, and any h -discrete harmonic function u in Q_1^h that satisfies $\max_{Q_r^h} |u(x)| \leq \varepsilon$ and $\max_{Q_1^h} |u(x)| \leq M$, then the inequality*

$$\max_{Q_R^h} |u(x)| \leq C(\varepsilon^\alpha M^{1-\alpha} + \delta^{\sqrt{N}} M),$$

holds.

Explicit values for the constants are given for the case $r < R < 2r < 2^{-3n-3}$. We also discuss the nature of additional term $\delta^{\sqrt{N}} M$ that goes to zero as $N \rightarrow \infty$.

2 Discrete harmonic polynomials and a generalization of the Liouville theorem

In this Chapter we present discrete harmonic polynomials and zero sets of discrete harmonic polynomials on square lattice, discrete harmonic interpolation in higher dimensions, the Liouville's theorem and its generalization. Our starting point is the work of Heilbronn [28]. For the first section we will set our graph to be a square lattice with vertices in \mathbf{Z}^n and edges of the form $(x \pm e_j, x)$, where $x = (x_1, x_2, \dots, x_n) \in \mathbf{Z}^n$. We denote the discrete Laplacian on this standard lattice by Δ_d and use Δ for the usual continuous Laplace operator.

2.1 Discrete harmonic polynomials

2.1.1 Definition and examples

In this section we consider polynomials that are discrete harmonic. We start by pointing out that in general polynomials which are harmonic in the usual continuous sense are not necessarily discrete harmonic.

Theorem 6. *For every integer $k \geq 1$, there are exactly*

$$h_n(k) = \binom{n+k-2}{n-1} \frac{2k+n-1}{k} \quad (2.1)$$

linearly independent discrete harmonic polynomials of degree not exceeding k .

Proof. For $k = 1$ the theorem is trivial since the $n+1$ polynomials $1, x_1, \dots, x_n$ are all discrete harmonic. Hence we may assume that $k \geq 2$. An easy (and standard) count shows that there are

$$\binom{n+k}{n}$$

linearly independent polynomials of degree not exceeding k . We will show that every polynomial of degree not exceeding $k - 2$ can be represented in the form

$$g(x) = \Delta_d f(x), \quad \deg f(x) \leq k.$$

To show this we consider the discrete Laplacian $\Delta_d : P_k \rightarrow P_{k-2}$ where P_k is the set consisting of all polynomials of degree not exceeding k .

Note that $\Delta_d f_k = \Delta f_k + r$ where $f_k \in P_k$, and $r \in P_{k-3}$. This can be easily checked on monomials. We remark also that the usual Laplacian $\Delta : P_k \rightarrow P_{k-2}$ is onto since $\dim(\text{Ker} \Delta P_k) = \dim(P_k) - \dim(P_{k-2})$, see for example [3].

We will show by induction on k that if $g \in P_{k-2}$, then there exists $f \in P_k$ such that $\Delta_d f = g$. For $k = 2$ the claim is trivial. Let $g \in P_{k-2}$. Then there exists $h \in P_k$ such that $\Delta h = g$ and that

$$\Delta_d h = g + g_1 \quad \text{where } g_1 \in P_{k-3}. \quad (2.2)$$

By the induction hypothesis we can find $h_1 \in P_{k-1}$ such that $\Delta_d h_1 = g_1$. Then we take $f = h - h_1 \in P_k$ and get $\Delta_d f = g$. Thus we have

$$\dim(\text{Ker} \Delta_d P_k) = \dim(P_k) - \dim(\text{Im} \Delta_d P_k) = \binom{n+k}{n} - \binom{n+k-2}{n}.$$

Computing further the difference above yields

$$\begin{aligned} \binom{n+k}{n} - \binom{n+k-2}{n} &= \frac{(n+k)!}{n!k!} - \frac{(k-2+n)!}{n!(k-2)!} \\ &= \frac{(n+k-2)! [(n+k)(k+n-1) - k(k-1)]}{n(n-1)!k(k-1)(k-2)!} \\ &= \frac{(k+n-2)!}{(n-1)!(k-1)!} \frac{(2k+n-1)}{k} = \binom{n+k-2}{n-1} \frac{2k+n-1}{k}, \end{aligned}$$

which completes the proof of the theorem. \square

Observe that in the proof we used the fact that for any $f_k \in P_k$ we have $\Delta_d f_k = \Delta f_k + r$ for some $r \in P_{k-3}$. This fact further implies the following

Corollary. *If f is a discrete harmonic polynomial of degree k then $f = h + r$ where h is a homogeneous harmonic polynomial of degree k and the degree of r is less than k .*

Example 1. For $n = 2$ the discrete harmonic polynomials up to degree 5 are linear combinations of

$$\begin{aligned} &1; x_1, x_2; x_1^2 - x_2^2, 2x_1x_2; x_1^3 - 3x_1x_2^2, 3x_1^2x_2 - x_2^3; \\ &x_1^4 - 6x_1^2x_2^2 + x_2^4 - (x_1^2 + x_2^2), 4x_1^3x_2 - 4x_1x_2^3; \\ &x_1^5 - 10x_1^3x_2^2 + 5x_1x_2^4 - 10x_1x_2^2, 5x_1^4x_2 - 10x_1^2x_2^3 + x_2^5 - 10x_1^2x_2. \end{aligned}$$

We notice that in this sequence one polynomial of degree k is of the form $\Re(x_1 + ix_2)^k +$ terms of degree not exceeding $k - 2$, while the second one is of the form $\Re(i^{-1}(x_1 + ix_2)^k) +$ terms of degree not exceeding $k - 2$. The above sequence is not uniquely defined and it is not clear if there is any preferable choice. For further examples and discussion we refer the reader to [28, 48].

Example 2. We also give some examples for the case $n = 3$. By Theorem 6 we have

$$\binom{k+1}{2} \frac{2k+2}{k} = (k+1)^2$$

linearly independent discrete harmonic polynomials of degree not exceeding k , i.e., we have $2k + 1$ linearly independent discrete harmonic polynomials of degree k . The following 25 polynomials generate all harmonic discrete polynomials of degree up to 4

$$\begin{aligned} &1; x_1, x_2, x_3; x_1x_2, x_2x_3, x_1x_3, x_1^2 - x_2^2, x_2^2 - x_3^2; x_1x_2x_3, \\ &x_1^3 - 3x_1x_2^2, x_1^3 - 3x_1x_3^2, x_2^3 - 3x_2x_1^2, x_2^3 - 3x_2x_3^2, x_3^3 - 3x_3x_1^2, x_3^3 - 3x_3x_2^2; \\ &x_1x_2(x_1^2 + x_2^2 - 6x_3^2), x_1x_3(x_1^2 + x_3^2 - 6x_2^2), x_2x_3(x_2^2 + x_3^2 - 6x_1^2) \\ &x_1^4 - 6x_1^2x_2^2 + x_2^4 - (x_1^2 + x_2^2), x_1^4 - 6x_1^2x_3^2 + x_3^4 - (x_1^2 + x_3^2) \\ &x_2^4 - 6x_2^2x_3^2 + x_3^4 - (x_2^2 + x_3^2), x_1x_2(x_1^2 - x_2^2), x_1x_3(x_1^2 - x_3^2), x_2x_3(x_2^2 - x_3^2). \end{aligned}$$

We remind that $h_n(k)$ are given by (2.1) for $k \geq 1$ and define $h_n(0) = 1$, $h_n(-1) = 0$. We will use the following result in our later consideration.

Proposition 2. *There exists a sequence of polynomials $\{P_{k,m,j}\}$, where k and m are non-negative integers, $1 \leq j \leq h_n(k) - h_n(k-1)$, such that $\Delta_d P_{k,m,j} = P_{k,m-1,j}$ for $m \geq 1$, $\Delta_d P_{k,0,j} = 0$, the degree of $P_{k,m,j}$ is equal to $2m + k$, and for each l the polynomials $\{P_{k,m,j}, k + 2m \leq l\}$ form a basis of the space of all polynomials of degree less than or equal to l .*

Proof. We will construct those polynomials by induction on $2m + k$. First we define $P_{0,0,1} = 1$ and $P_{1,0,j} = x_j, j = 1, \dots, n$. Now suppose we already

have all polynomials $P_{k,m,j}$ with $k + 2m \leq l$ we will construct ones with $k + 2m = l + 1$.

It follows from Theorem 6 that for each polynomial $P_{k,m,j}$ with $k + 2m = l - 1$ there exists a polynomial that we call $P_{k,m+1,j}$ such that its degree is equal to $l + 1$ and $\Delta_d P_{k,m+1,j} = P_{k,m,j}$. We also choose new polynomials $P_{l+1,0,j}$ as linear independent discrete harmonic polynomials of degree exactly $l + 1$, we have $h_n(l + 1) - h_n(l)$ of them. Further, since $\{P_{k,m,j}, k + 2m = l - 1\}$ are linearly independent polynomials of degree $l - 1$, we conclude that the system $\{P_{k,m+1,j}, k + 2m = l - 1, m \geq 0\} \cup \{P_{l+1,0,i}\}$ is also linearly independent. \square

2.1.2 Zeros of discrete harmonic polynomials

We discuss zeros of discrete harmonic polynomials and interpolation of discrete harmonic functions by harmonic polynomials first for the case of dimension 2.

We consider the following square subdomains of the lattice. Let K_N be the domain whose interior points are

$$|x_1| \leq N, |x_2| \leq N.$$

We will also need the interior boundary of K_N defined by $\partial K_N = K_N \setminus K_{N-1}$. Further, let U_{2N} , be the domain whose interior points are

$$|x_1| + |x_2| \leq 2N.$$

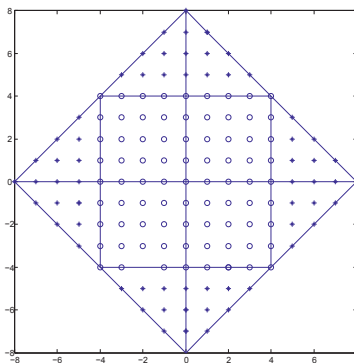


Figure 2.1: Graph of K_4 and U_8

Figure 2.1 shows the domain K_4 and U_8 ; the points of K_4 are denoted by o and points of U_8 are denoted by $o \cup *$.

The following statement is straightforward but it turns to be useful.

Lemma 1. *If p is discrete harmonic function on U_{2N} and $p = 0$ on K_N then $p = 0$ on U_{2N} .*

The next theorem was proved in [28] (see also [48]). We repeat the proof given by Heilbronn and fill in the necessary details.

Theorem 7. *If N is a positive integer, and if f is a discrete harmonic function on $K_N \subset \mathbf{Z} \times \mathbf{Z}$, then there exists a discrete harmonic polynomial p such that $f = p$ on K_N .*

Proof. Let f be discrete harmonic function on the domain K_N . We want to find a discrete harmonic polynomial p such that

$$f(m, n) = p(m, n) \text{ for all } (m, n) \in \partial K_N.$$

Then by the maximum principle it will follow that $f = p$ on K_N . We note that ∂K_N consists of $4(2N + 1) - 4 = 8N$ points. Let $p_0 = 1$ and for each $j = 1, \dots, 4N - 1$ let $p_{j,1}$ and $p_{j,2}$ be two linearly independent discrete harmonic polynomials of degree j ; further let $p_{4N,1}$ be a discrete harmonic polynomials of the form

$$\begin{aligned} p_{4N,1} &= \Re(x_1 + ix_2)^{4N} + \text{polynomial of lower degree} \\ &= x_1^{4N} - \binom{4N}{2} x_1^{4N-2} x_2^2 + \dots + \text{polynomial of lower degree.} \end{aligned}$$

We claim that for any f there exist constants $C_0, C_{1,1}, \dots, C_{4N,1}$ such that

$$f = C_0 p_0 + C_{1,1} p_{1,1} + C_{1,2} p_{1,2} + \dots + C_{4N,1} p_{4N,1} \text{ on } \partial K_N.$$

This system has a solution for any data $\{f(m, n), (m, n) \in \partial K_N\}$ if and only if the only solution to the system

$$0 = C_0 p_0 + C_{1,1} p_{1,1} + C_{1,2} p_{1,2} + \dots + C_{4N,1} p_{4N,1} \text{ on } \partial K_N \quad (2.3)$$

is the trivial one $C_0 = C_{1,1} = C_{1,2} = \dots = C_{4N,1} = 0$. Assume that

$$q = C_0 p_0 + C_{1,1} p_{1,1} + C_{1,2} p_{1,2} + \dots + C_{4N,1} p_{4N,1}$$

and $q(m, n) = 0$, for any $(m, n) \in \partial K_N$. Clearly q is discrete harmonic on K_N . Since $q = 0$ on ∂K_N , it also holds that $q = 0$ on U_{2N} .

Now let us consider q on the horizontal axis $(x_1, 0)$,

$$\begin{aligned} q(x_1, 0) &= C_{4N,1}p_{4N,1}(x_1, 0) + \cdots + C_{1,1}p_{1,1}(x_1, 0) + C_0p_0(x_1, 0) \\ &= C_{4N,1}x_1^{4N} + \text{polynomials of degree less than } 4N. \end{aligned}$$

We have $q(k, 0) = 0$ for $k = -2N, \dots, -1, 0, 1, \dots, 2N$, while the degree of $q(x_1, 0)$ does not exceed $4N$. Hence $q(x_1, 0) = 0$ and in particular $C_{4N,1} = 0$. Then q is a polynomial of degree less than or equal to $4N - 1$. Similarly $q(0, x_2)$ has $4N + 1$ zeros and thus $q(0, x_2) = 0$. We have $q(x_1, x_2) = x_1x_2q_1(x_1, x_2)$ where $\deg q_1 \leq 4N - 3$. We then consider the following polynomials $q_1(x_1, 1)$, $q_1(x_1, -1)$, $q_1(1, x_2)$, $q_1(-1, x_2)$ each of them is of degree at most $\leq 4N - 3$, and each has zeros at the points $-2N + 1, -2N + 2, \dots, -1, 1, \dots, 2N - 1$. Therefore $q_1(x_1, 1) = 0$, $q_1(x_1, -1) = 0$, $q_1(1, x_2) = 0$, $q_1(-1, x_2) = 0$. Hence

$$q_1(x_1, x_2) = (x_1 - 1)(x_1 + 1)(x_2 - 1)(x_2 + 1)q_2(x_1, x_2),$$

where $\deg q_2(x_1, x_2) \leq 4N - 7$. Continuing this process we obtain polynomial $q_s(x_1, x_2)$ of degree less than or equal to $4N + 1 - 4s$ that has zeros at all points $(m, n) \in U_{2N}$ if $|n| \geq s$, $|m| \geq s$. The polynomial $q_s(x_1, s)$ has $4N - 4s + 2$ roots and the same is true for $q_s(x_1, -s)$, $q_s(s, x_2)$, $q_s(-s, x_2)$. Thus,

$$q_s(x_1, x_2) = (x_1 - s)(x_1 + s)(x_2 - s)(x_2 + s)q_{s+1}(x_1, x_2),$$

where $q_{s+1}(x_1, x_2)$ is of degree $\leq 4N - 4s - 3$ and $q_{s+1}(x_1, x_2)$ has zeros at all points $(m, n) \in U_{2N}$ such that $|n| \geq s$, $|m| \geq s$. Take $s = N - 1$, then we have

$$q_{N-1}(x_1, x_2) = (x_1 - N + 1)(x_1 + N - 1)(x_2 - N + 1)(x_2 + N - 1)q_N(x_1, x_2),$$

where $q_N(x_1, x_2)$ is of degree less than or equal to 1 and it has zeros at the points (N, N) , $(N, -N)$, $(-N, -N)$, $(-N, N)$. This implies that $q_N = 0$ and hence $q = 0$. \square

Corollary. *There exists a non-zero harmonic polynomial of degree $4N$ with zero values at each point of $K_N \subset \mathbf{Z} \times \mathbf{Z}$. But there is no non-zero discrete harmonic polynomial of degree strictly less than $4N$ with zero values on K_N .*

Proof. Let $p_0, p_{1,1}, \dots, p_{4N,1}, p_{4N,2}$ be discrete harmonic polynomials defined as above. We want to find constants $C_0, C_{1,1}, \dots, C_{4N,1}, C_{4N,2}$ such that

$$C_0 p_0(a) + C_{1,1} p_{1,1}(a) + \dots + C_{4N,1} p_{4N,1}(a) + C_{4N,2} p_{4N,2}(a) = 0$$

for any point $a = (m, n) \in K_N$.

Since we are dealing with discrete harmonic polynomials where one could apply the maximum principle, it is enough to check the above condition on the boundary. Thus, we have $8N + 1$ unknowns and $8N$ equations. Which implies that the system has a non trivial solution. Therefore there exists a non zero harmonic polynomial of degree $4N$ with zero values on K_N .

The second part follows from the proof of Theorem 7, where we proved that there is no non-zero solution of equation (2.3) and thus no non-zero polynomial of degree $< 4N$ with zero values on K_N . \square

2.1.3 Discrete harmonic interpolation in higher dimensions

We give another proof of Theorem 7 that works in any dimension. We find it essential to present this alternative proof since we have not succeeded in generalizing Heilbroon's proof to higher dimensional settings. Let us define

$$K_N^{(n)} = \{(x_1, \dots, x_n) \in \mathbf{Z}^n : |x_1| \leq N, \dots, |x_n| \leq N\}$$

and more generally

$$K_{M,N}^{(n)} = \{(x_1, \dots, x_n) \in \mathbf{Z}^n : |x_1| \leq M, \dots, |x_{n-1}| \leq M, |x_n| \leq N\}.$$

Having setting this we will prove the following.

Theorem 8. *If f is a discrete harmonic function on $K_N^{(n)}$, then there exists a discrete harmonic polynomial P on \mathbf{Z}^n , $\deg P \leq 6N(n-1) + 1$, such that $f = P$ on $K_N^{(n)}$.*

We first claim that there exists a function g discrete harmonic in $K_{3N,N}^{(n)}$ such that $f = g$ in $K_N^{(n)}$ (for example, we can always extend a discrete harmonic function from a rectangle $[-K, K] \times [-L, L]$ to a large rectangle $[-K, K] \times [-M, M]$ by adding arbitrarily values at the points on $\{\pm K\} \times$

$[\pm L + 1, \pm M]$, and similar construction works in higher dimensions). Further, values of g on $K_N^{(n)}$ are determined by its values on two squares $K_{3N}^{(n-1)} \times \{-N + 1, -N\}$ and we will be done if we show that there exists a discrete harmonic polynomial that coincides with g on the set $K_{3N}^{(n-1)} \times \{-N + 1, -N\}$. By changing variables we may instead consider the set $K_{3N}^{(n-1)} \times \{0, 1\}$. Furthermore we can find two polynomials G_0 and G_1 of $n - 1$ variables such that $g(x, 0) = G_0(x)$ and $g(x, 1) = G_1(x)$ when $x \in \mathbf{Z}^{n-1}$ (this is standard multivariate polynomial interpolation on a grid). The polynomials G_0, G_1 can be chosen of degree less than or equal to $6N(n - 1)$. For the detail we refer the reader to [17, Chapter 4]. So we have reduced the Theorem to the following statement.

Lemma 2. *Let G_0 and G_1 be polynomials of $n - 1$ variables with degree less than or equal to M . Then there exists a discrete harmonic polynomial P on \mathbf{Z}^n such that $P(x, 0) = G_0(x)$, $P(x, 1) = G_1(x)$ and the degree of P is less than or equal to $M + 1$.*

Proof. We first find polynomials of one variable q_j that satisfy

$$\Delta_d q_j(t) = q_j(t + 1) + q_j(t - 1) - 2q_j(t) = t^{j-2}$$

when $j \geq 2$, $q_j(0) = 0$ and $q_j(1) = 0$, we have

$$q_j(t) = c_{j,j}t^j + c_{j,j-1}t^{j-1} + \dots + c_{j,1}t.$$

We also let $q_0 = 1$ and $q_1 = t$. Now we look for $P(x, x_n)$ in the form

$$P(x, x_n) = \sum_{j=0}^{M+1} q_j(x_n)Q_j(x),$$

and the conditions become $\Delta_d P(x, x_n) = 0$, $P(x, 0) = Q_0(x) = G_0(x)$, and $P(x, 1) = Q_0(x) + Q_1(x) = G_1(x)$. We have

$$\begin{aligned} \Delta_d P(x, x_n) &= \sum_{j=0}^{M+1} (\Delta_d q_j(x_n)Q_j(x) + q_j(x_n)\Delta_d Q_j(x)) = \\ &= \sum_{j=2}^{M+1} x_n^{j-2}Q_j(x) + \sum_{j=0}^{M+1} q_j(x_n)\Delta_d Q_j(x) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{M-1} x_n^j \left(Q_{j+2}(x) + \sum_{k=j}^{M+1} c_{k,j} \Delta_d Q_k(x) \right) + \\
&\quad x_n^M \sum_{k=M}^{M+1} c_{k,M} \Delta_d Q_k(x) + x_n^{M+1} c_{M+1,M+1} \Delta_d Q_{M+1}(x).
\end{aligned}$$

Now since Q_0 and Q_1 are given polynomials of degree less than or equal to M , it is sufficient to find sequence of polynomials Q_2, \dots, Q_{M+1} such that the degree of Q_j is less than or equal to $M - j + 1$ and

$$Q_{j+2}(x) + \sum_{k=j}^{M+1} c_{k,j} \Delta_d Q_k = 0, \quad j = 0, \dots, M + 1.$$

Now, by comparing coefficients we treat the equations as a linear system. Each polynomial Q_j gives us unknowns (coefficients) and the total number of unknowns we get is

$$\sum_{k=0}^{M-1} \binom{n-1+k}{n-1}.$$

The number of equations is exactly the same. The right-hand sides for this linear system comes from the given polynomials Q_0 and Q_1 . To show that there is a solution, we have to show that the relation $Q_0 = Q_1 = 0$ gives only trivial solution $Q_2 = \dots = Q_{M+1} = 0$. If there exists a non-trivial solution, we choose polynomial Q_l , $l \geq 2$ that has the highest degree. We thus have

$$Q_l = - \sum_{k=l-2}^{M+1} c_{k,l-2} \Delta_d Q_k,$$

which is not possible, since the degree of the polynomial on the left-hand side is greater than the degree of the one on the right-hand side. \square

The difference between Theorem 8 and Theorem 7 is that in the latter we get exact degree of the polynomial and the solution is unique, while the former gives a non-unique solution but holds in any dimension. There exists also a more constructive way to find the interpolation polynomial using the basis described in Proposition 2. Applications of such discrete harmonic polynomial interpolation can be found in [48].

We remark also that the discrete Laplacian has no rotational invariant structure and discrete harmonic polynomials seem to be not so natural as in the continuous case, where they correspond to eigenfunctions on the Laplace–Beltrami operator on the unit sphere. For discrete harmonic functions on a cube a more natural basis is obtained by taking products of eigenfunctions on the base of the cube and some exponential function of the last variable as it is described in Chapter 3.

2.2 Liouville’s theorem and its generalizations

2.2.1 Liouville’s theorem

We introduce the following notation for the directional forward difference of the function f in \mathbf{Z}^n ;

$$d_j f(x) = f(x + e_j) - f(x) \text{ for } 1 \leq j \leq n,$$

where e_1, e_2, \dots, e_n are the standard coordinate vectors for \mathbf{Z}^n .

The following theorem is a well known discrete analog of the Liouville’s theorem.

Theorem 9. *If f is discrete harmonic everywhere and satisfies the inequality*

$$|f(x)| \leq M \tag{2.4}$$

for all x , where M is a constant, then f is a constant.

Proof. We follow the proof given in [28]. We assume that

$$d_1 f(x) = g(x) \tag{2.5}$$

is not zero everywhere. Since $|g(x)| \leq 2M$, there exists m such that $m = \sup_x |g(x)|$ then

$$|g(x)| \leq m \leq 2M, \tag{2.6}$$

everywhere, and for any $\epsilon > 0$ there exists x such that $g(x) > m - \epsilon$.

We choose an integer l such that $lm > 2M$, and a positive δ such that $l(m - (2n)^l \delta) > 2M$. Then we can find a point $x^{(0)}$ such that

$$g(x^{(0)}) > m - \delta. \tag{2.7}$$

We put $x^{(\lambda)} = x^{(0)} + \lambda e_1$ for $0 < \lambda \leq l$. Then since g is discrete harmonic,

$$2ng(x^{(0)}) = \sum_{j=1}^n [g(x^{(0)} + e_j) + g(x^{(0)} - e_j)] = \\ g(x^{(1)}) + g(x^0 - e_1) + g(x^0 + e_2) + g(x^0 - e_2) + \cdots + g(x^0 + e_n) + g(x^0 - e_n).$$

Therefore

$$2ng(x^{(0)}) \leq (2n - 1)m + g(x^{(1)})$$

and by (2.7), we have $g(x^{(1)}) \geq m - 2n\delta$. Applying the same argument again, we obtain by induction for $0 \leq \lambda \leq l$

$$g(x^{(\lambda)}) \geq m - (2n)^\lambda \delta.$$

Hence

$$2M \geq f(x^{(l)}) - f(x^{(0)}) = \sum_{\lambda=0}^{l-1} g(x^{(\lambda)}) \geq \sum_{\lambda=0}^{l-1} (m - (2n)^\lambda \delta) \\ > l(m - (2n)^l \delta) > 2M$$

which is the desired contradiction. Therefore f is constant. \square

2.2.2 A generalization of Liouville's theorem

It turns out that the Liouville theorem remains true if one assume boundedness from below only. The following statement is well known, most famous proof (in dimension two) is probably the probabilistic one. We avoid random walks in this exposition and give an elementary proof (also well known in folklore).

Theorem 10. *If $f : \mathbf{Z}^n \rightarrow \mathbf{R}$ is discrete harmonic function and $f \geq 0$ then f is a constant.*

Proof. Consider the set \mathcal{A} of all non-negative discrete harmonic functions u on \mathbf{Z}^n that satisfy $u(0) = 1$. Let

$$M^2 = \sup_{u \in \mathcal{A}} \sum_{j=1}^n (u(e_j)^2 + u(-e_j)^2),$$

where $\{e_j\}_{j=1}^n$ are as usual standard basis vectors. Then for any non-negative discrete harmonic function f one has

$$M^2 f(x)^2 \geq \sum_{j=1}^n (f(x+e_j)^2 + f(x-e_j)^2). \quad (2.8)$$

First we claim that there exists $u_0 \in \mathcal{A}$ for which the supremum above is achieved (and thus it is finite). Indeed, let u_i be a sequence of discrete harmonic functions for which

$$\sum_{j=1}^n (u_i(e_j)^2 + u_i(-e_j)^2) \rightarrow M^2.$$

We can find a subsequence u_{i_k} such that $u_{i_k}(x)$ converges for any $x \in \mathbf{Z}^n$. Note that the conditions $u(0) = 1$ and $u \geq 0$ imply that $u(x) \leq C(x)$ for any $x \in \mathbf{Z}^n$. If x and y are two neighboring points, then $u(x) \geq (2n)^{-1}u(y)$. Thus we can take $C(x) = (2n)^{-m}$, where m is the length of the shortest path from 0 to x . Now applying the Heine–Borel theorem and the standard diagonal procedure we find a required subsequence. Let $u_0(x) = \lim_k u_{i_k}(x)$. Then clearly $u_0(0) = 1$, u_0 is discrete harmonic and

$$\sum_{j=1}^n (u_0(e_j)^2 + u_0(-e_j)^2) = M^2.$$

Now let $g_k^+(x) = (2n)^{-1}u_0(x+e_k)$ and $g_k^-(x) = (2n)^{-1}u_0(x-e_k)$, we have $u_0 = \sum_k (g_k^+ + g_k^-)$. Then by triangle inequality and (2.8), we have

$$\begin{aligned} M = \|(u_0(\pm e_j))\|_2 &= \left\| \sum_k (g_k^+(\pm e_j) + g_k^-(\pm e_j)) \right\|_2 \leq \\ &\sum_k (\|g_k^+(\pm e_j)\|_2 + \|g_k^-(\pm e_j)\|_2) \leq M \sum_k (g_k^+(0) + g_k^-(0)) = M. \end{aligned}$$

But the equality in the triangle inequality occurs only when all vectors are proportional. Thus

$$g_k^+(\pm e_j) = c_k^+ u_0(\pm e_j), \quad g_k^-(\pm e_j) = c_k^- u_0(\pm e_j),$$

which also implies $g_k^\pm(0) = c_k^\pm$ since all these functions are discrete harmonic. Then

$$1 = u_0(0) = 2n g_j^+(-e_j) = 2n c_j^+ u_0(-e_j) = (2n)^2 c_j^+ g_j^-(0) = (2n)^2 c_j^+ c_j^-.$$

Finally

$$u_0(0) = \sum_k (g_k^+(0) + g_k^-(0)) = \sum_k c_k^+ + c_k^- \geq \sum_k 2\sqrt{c_k^+ c_k^-} = \sum_k d^{-1} = 1.$$

Here the equality holds only if $c_j^+ = c_j^- = (2n)^{-1}$. Then $u_0(\pm e_j) = 1$ and $M^2 = 2n$.

For any non-negative discrete harmonic function satisfying condition (2.8) with $M^2 = 2n$ implies

$$\sum_{j=1}^n ((f(x + e_j) - f(x))^2 + (f(x - e_j) - f(x))^2) \leq 0,$$

from which it follows that f is a constant. □

2.2.3 Discrete harmonic functions of polynomial growth

In this section we present generalizations of the Liouville theorem for discrete harmonic functions of polynomial growth. The results can be found in [28, 37, 38]. We suggest a different approach based on estimates of the Poisson kernel for a cube. Similar estimates are used in Chapter 5 to prove quantitative unique continuation for discrete harmonic functions.

Let $Q_R = [-R, R]^n \cap \mathbf{Z}^n$. Therefore there exists a function $P_R : Q_R \times \partial Q_R \rightarrow [0, 1]$ such that for any discrete harmonic function u on Q_R we have

$$u(x) = \sum_{y \in \partial Q_R} P_R(x, y)u(y).$$

One can find $P_R(\cdot, y_0)$ as the solution of the Dirichlet problem

$$\begin{cases} \Delta P_R(\cdot, y_0) = 0, \\ P_R(y, y_0) = 0, \quad y \in \partial Q_R \setminus \{y_0\}, \\ P_R(y_0, y_0) = 1. \end{cases}$$

This function appears in different combinatorial and geometric problems. We refer the reader to [12] for modern treatment. Here we use only an elementary identity $\sum_{y \in \partial Q_R} P_R(x, y) = 1$ and the following estimates.

Lemma 3. *There exists c_0, c_1 and c_2 that depend only on n such that for all $R \geq 2$*

(i) $P_R(x, y) \leq c_0 R^{1-n}$, when $x \in Q_{\frac{R}{2}}$,

(ii) $|P_R(e_j, y) - P_R(0, y)| \leq c_1 R^{-n}$ and

(iii) $P_R(0, y_0) \geq c_2 R^{1-n}$, when $y_0 = R e_j$.

Proof. Without loss of generality, we assume that $y = (y_1, \dots, y_{n-1}, R)$. For each $K = (k_1, \dots, k_{n-1}) \in ((0, 2R) \cap \mathbf{Z})^{n-1} = \llbracket 1, 2R-1 \rrbracket^{n-1}$ we define a_K to be the only positive solution of the equation

$$\cosh \frac{a_K}{2} = n - \sum_{j=1}^{n-1} \cos \frac{\pi k_j}{2R}.$$

Then

$$f_K(x) = \sinh(a_K(x_n + R)/2) \prod_{j=1}^{n-1} \sin(\pi k_j(x_j + R)/2R)$$

is discrete harmonic and vanishes on all sides of the cube Q_R in \mathbf{Z}^n except the one where y lies. Then

$$P_R(x, y) = \left(\frac{1}{R}\right)^{n-1} \sum_K \prod_{j=1}^{n-1} \sin\left(\pi k_j \frac{x_j + R}{2R}\right) \sin\left(\pi k_j \frac{y_j + R}{2R}\right) \frac{\sinh\left(a_K \frac{x_n + R}{2}\right)}{\sinh a_K R}.$$

It is easy to check that this function is discrete harmonic in the cube Q_R and satisfies the required boundary conditions. In fact, it vanishes when $x_j = \pm R$ and $j \neq n$ and when $x_n = -R$, to compute the values of $P_R(x, y)$ when $x = (x_1, \dots, x_{n-1}, R)$ we note that for this case

$$P_R(x, y) = R^{1-n} \prod_{j=1}^{n-1} \sum_{k_j=1}^{2R-1} \sin\left(\pi k_j \frac{x_j + R}{2R}\right) \sin\left(\pi k_j \frac{y_j + R}{2R}\right).$$

But discrete sin functions that we consider are orthogonal when $x_j \neq y_j$,

$$\sum_{k=1}^{2R-1} \sin\left(\pi k \frac{x_j + R}{2R}\right) \sin\left(\pi k \frac{y_j + R}{2R}\right) = R \delta(x_j, y_j),$$

where $\delta(x, y) = 0$ when $x \neq y$ and $\delta(x, y) = 1$ when $x = y$.

We first prove the inequality in (i). We have

$$\begin{aligned} R^{n-1}P_R(x, y) &= \\ & \left| \sum_K \prod_{j=1}^{n-1} \sin\left(\pi k_j \frac{x_j + R}{2R}\right) \sin\left(\pi k_j \frac{y_j + R}{2R}\right) \frac{\sinh\left(a_K \frac{x_n + R}{2}\right)}{\sinh a_K R} \right| \\ & \leq \sum_K \left| \frac{\sinh\left(a_K \frac{x_n + R}{2}\right)}{\sinh a_K R} \right| \leq \sum_K \exp(a_K(x_n - R)/2) \leq \sum_K \exp\left(\frac{-a_K R}{4}\right). \end{aligned}$$

We note that $a_K \geq 2$ or

$$\left(\frac{a_K}{2}\right)^2 \geq \cosh \frac{a_K}{2} - 1 = \sum_{j=1}^{n-1} (1 - \cos \frac{\pi k_j}{2R}) \geq \frac{1}{4R^2} \sum_{j=1}^{n-1} k_j^2,$$

we used elementary inequalities $1 + x^2 \geq \cosh x$, for $x \in [0, 1)$ and $1 - \cos x \geq \frac{x^2}{\pi^2}$ for $x \in [0, \pi)$. Thus either $a_K \geq 2$ or $a_K \geq \|K\|R^{-1}$, where

$$\|K\|^2 = \sum_j k_j^2 \geq \frac{1}{n-1} \left(\sum_j k_j \right)^2.$$

We then obtain

$$\begin{aligned} \sum_K \exp(-a_K R/4) &\leq \sum_K (\exp(-\|K\|/4) + \exp(-R/2)) \leq \\ & \sum_K \exp(-\|K\|/4) + \sum_K \exp(-R/2) \leq \\ & \left(\sum_{k=1}^{\infty} \exp\left(-\frac{k}{4\sqrt{n-1}}\right) \right)^{n-1} + (2R)^{n-1} \exp(-R/2) \leq C_n, \end{aligned}$$

where the constant C_n depends only on n but not R , and from which the statement in (i) follows.

To prove (ii), assume first that $j \neq n$. Then we have

$$\begin{aligned} R^{n-1}|P_R(e_j, y) - P_R(0, y)| &\leq \\ & \sum_K \left| \left(\sin\left(\pi k_j \frac{R+1}{2R}\right) - \sin\left(\frac{\pi k_j}{2}\right) \right) \right| \exp\left(-\frac{a_K R}{2}\right) \leq \\ & \pi R^{-1} \sum_K k_j \exp\left(-\frac{a_K R}{2}\right). \end{aligned}$$

The same argument as above shows that the last sum is finite. For $j = n$, we have

$$\begin{aligned} R^{n-1}|P_R(e_n, y) - P_R(0, y)| &\leq \sum_K \frac{\sinh(a_K(R+1)/2) - \sinh(a_K R/2)}{\sinh a_K R} \leq \\ &C \sum_K \sinh \frac{a_K}{4} \exp\left(-\frac{a_K R}{2}\right) \leq C \sum_K a_K \exp\left(-\frac{a_K R}{4}\right). \end{aligned}$$

Now, using the formula for a_K we see that

$$a_K^2 \leq 8(\cosh \frac{a_K}{2} - 1) \leq \frac{\pi^2}{R^2} \sum_j k_j^2 \leq \frac{\pi^2 n}{R^2} \prod_j k_j^2.$$

Finally,

$$|P_R(e_n, y) - P_R(0, y)| \leq C R^{-n} \sum_K \prod_j k_j \exp\left(-\frac{a_K R}{4}\right)$$

and the last sum is finite.

It remains to verify the inequality in (iii). To do this we assume that $j = n$ and let $K_0 = (1, \dots, 1)$. Then $a_{K_0} R \leq \pi\sqrt{n}$ and

$$\begin{aligned} P_R(0, y_0) &= \\ R^{1-n} \sum_K \prod_{j=1}^{n-1} \sin(\pi k_j/2)^2 (2 \cosh a_K R/2)^{-1} &\geq \frac{R^{1-n}}{2 \cosh a_{K_0} R/2} \geq c_2 R^{1-n}. \end{aligned}$$

□

Now we can prove the following.

Lemma 4. *Let $f : \mathbf{Z}^n \rightarrow \mathbf{R}$ be a discrete harmonic function. Suppose there exists a polynomial $W : \mathbf{Z}^n \rightarrow \mathbf{R}$ such that $f \geq -W$. Then $|f| \leq R$ for some polynomial $R : \mathbf{Z}^n \rightarrow \mathbf{R}$.*

Proof. Let $y_1 = y_0 + R e_j$. Applying the Poisson formula to a cube $y_0 + Q_R$, we obtain

$$\begin{aligned} f(y_1) \leq P_R(0, R e_j)^{-1} \left(f(y_0) + \sum_{y \in \partial Q_R \setminus \{y_1 - y_0\}} P_R(0, y) |W(y_0 + y)| \right) \leq \\ c_2^{-1} R^{n-1} (f(y_0) + c_0 \max_{y_0 + Q_R} |W|). \end{aligned}$$

The inequality with $y_0 = 0$ and $y_1 = R_1 e_1$ implies that $|f(R_1 e_1)|$ is bounded by a polynomial in R_1 . Now we repeat the same argument for $y_0 = R_1 e_1$ and $y_1 = R_2 e_2$, iterating n times we conclude that $|f(y)|$ is bounded by a polynomial in $|y|$. \square

For probabilistic proof of Lemma 4 we refer the reader to a recent work [38]. The next Theorem was stated and proved in [28].

Theorem 11. *If f is discrete harmonic function everywhere and satisfies the inequality*

$$f(x) = O\left(1 + (|x_1| + \dots + |x_n|)^N\right)$$

everywhere, where N is an integer, then f is a polynomial of degree not exceeding N .

The proof is done by induction on N . For $N = 0$ it is the Liouville theorem. For $N \geq 1$ we consider $d_1 f(x) = f(x + e_1) - f(x)$ and use inequality (ii) of Lemma 3 to conclude that for $d_1 f$ a similar estimate holds with $N - 1$. Now by the induction hypothesis $d_1 f$ is a polynomial of degree less than or equal to $N - 1$, we have $d_1 f(x_1, \dots, x_n) = P_0(x_2, \dots, x_n) + x_1 P_1(x_2, \dots, x_n) + \dots + x_1^{N-1} P_{N-1}(x_2, \dots, x_n)$. For each k there exists a polynomial $s_k(x_1)$ of degree $k + 1$ such that $d_1 s_k = x_1^k$. Then

$$f(x_1, \dots, x_n) = \sum_{k=0}^{N-1} s_k(x_1) P_k(x_2, \dots, x_n) + g(x_2, \dots, x_n).$$

Further, we have $\Delta_d g(x_2, \dots, x_n) = Q(x_2, \dots, x_n)$ is a polynomial of degree less than $N - 2$, as it was proved in the beginning of this chapter, there exists a discrete harmonic polynomial $S(x_2, \dots, x_n)$ of degree less than or equal to N such that $\Delta_d S = Q$. We have $\Delta_d(g - S) = 0$ and $g - S = O(1 + (|x_2| + \dots + |x_n|)^N)$. We reduced the problem to the same one in $n - 1$ dimension. Clearly in dimension 1 any harmonic function is linear and the statement holds. Then by induction in n it holds in any dimension.

We conclude this chapter by the following one-sided polynomial Liouville theorem.

Theorem 12. *Let $f : \mathbf{Z}^n \rightarrow \mathbf{R}$ be a discrete harmonic function. Suppose there exists a polynomial $W : \mathbf{Z}^n \rightarrow \mathbf{R}$ of degree N such that $f \geq -W$. Then f is a polynomial and $\deg f \leq N$.*

Proof. We already know that f is a polynomial and assume that $\deg f = M > N$. Then $f = H + Q$, where $\Delta H = 0$, H is homogeneous of degree M and $\deg Q < \deg H$. Hence we have $H + (Q + W) \geq 0$ on \mathbf{Z}^n , where $\deg(Q + W) < \deg H$. Since there are no non-constant positive harmonic functions (continuous version of Theorem 10), there exists a set U on the sphere \mathbb{S}^{n-1} such that $H < -c < 0$ on U . Clearly there exists an unbounded sequence of points $x_m \in \mathbf{Z}^n$ such that $x_m/|x_m| \in U$. Then we have

$$c|x_m|^M < -H(x_m) \leq (Q + W)(x_m).$$

This is a contradiction since $Q + W$ is a polynomial of degree strictly less than M and for $|x_m|$ large enough the opposite inequality holds. \square

3 Stability estimates for discrete harmonic functions on product domains

We study the Dirichlet problem for discrete harmonic functions in unbounded product domains on multidimensional lattices. First we prove some versions of the Phragmén–Lindelöf theorem, and use Fourier series to obtain a discrete analog of the three-line theorem for the gradients of harmonic functions in a strip. Then we derive some inequalities for the discrete harmonic measure and also use elementary spectral inequalities to obtain stability estimates for Dirichlet problem in cylinder domains.

3.1 Introduction

We consider functions defined on subsets of the multidimensional lattice $(\delta\mathbf{Z})^n$ in \mathbf{R}^n . The usual $2n + 1$ -point discretization of the Laplace operator is denoted by Δ_n or $\Delta_{n,\delta}$ to emphasize the mesh of the lattice, the accurate definition is given below. Then we study the following Dirichlet problem

$$\begin{aligned}\Delta_n u &= 0, \\ u &= f \text{ on } \partial D^\delta, \\ u &\in H_b(D^\delta),\end{aligned}$$

where $H_b(D^\delta)$ is some class of functions of bounded growth in D^δ , and D^δ is an unbounded connected (on the lattice) subset of $(\delta\mathbf{Z})^n$. Our main question is for which $H_b(D^\delta)$ the problem above has a unique solution. Moreover, when the solution is unique we estimate how the error in the boundary data affects the error of the solution. Such estimates are called conditional stability estimates. We suppose a priori that solution belongs to $H_b(D^\delta)$. Since our problem is linear, stability estimate reduces to a bound of some norm of the solution $u \in H_b(D)$ by some norm of its boundary values f .

First, we prove that if $D = \Omega \times \mathbf{R}^k$, where Ω is a bounded domain in \mathbf{R}^n , and u is a discrete harmonic function in $D^\delta = D \cap (\delta\mathbf{Z})^{n+k}$ that satisfies

$$|u(x, y)| \leq C \exp(c\|y\|_1)$$

for some $c = c(\Omega, k)$, and $u = 0$ on ∂D^δ then $u = 0$ (here and in what follows $\|y\|_1 = |y_1| + \dots + |y_k|$, and $\|y\|_\infty = \max\{|y_1|, \dots, |y_k|\}$ where $y = (y_1, \dots, y_k) \in \mathbf{R}^k$). We refer to this statement as a discrete version of the Phragmén–Lindelöf theorem. It implies the uniqueness in the Dirichlet problem in the class of functions of limited growth. We consider more carefully the case $\Omega = [0, 1]$ and solve the Dirichlet problem using Fourier analysis when the boundary data is in l^2 . We obtain

$$\|u(x, \cdot)\|_{l^2} \leq \|f\|_{l^2}.$$

We also use this technique to show that gradients of discrete harmonic functions satisfy the following three-line inequality that resembles three-line theorem of Hadamard,

$$\|\nabla u(\delta k, \cdot)\|_{l^2(\mathbf{Z}^k)} \leq (\|\nabla u(0, \cdot)\|)^{1-\frac{k}{M}} (\|\nabla u(\delta M, \cdot)\|)^{\frac{k}{M}},$$

where $(M+1)\delta = 1$. Both the Phragmén–Lindelöf theorem and Hadamard’s three line theorem are classical results in complex analysis (for example see [45]). Here we want to underline that the result is precise and it is an analog of the continuous theorem proved in [30], the proof is similar but requires a new algebraic identity.

To obtain conditional stability estimates for Dirichlet problem with partial boundary data (see Theorem 17), we study the discrete harmonic measure in the truncated domains $\Omega \times [-N, N]$. We also use elementary properties of the spectrum of the discrete Dirichlet–Laplacian on Ω and some comparison results that can be found in [5, 16]. In particular, we show that if u grows slower than some exponential function, then the maximum principle holds for the infinite cylinder domain.

The remaining part of this Chapter is organized as follows. In the next section we give necessary definitions and results for discrete harmonic functions, including basic properties of the eigenvalues and eigenfunctions of the discrete Laplace operator with Dirichlet boundary condition. We also prove a simple version of the Phragmén–Lindelöf theorem for product domains. In Section 3.3 we use Fourier analysis to study discrete harmonic functions in a strip, in particular we obtain the logarithmic convexity inequality. Our main stability result for the Dirichlet problem in an infinite cylinder is proved in the last section. It follows from estimates of discrete harmonic measure and a more accurate version of the Phragmén–Lindelöf theorem. This Chapter is based on paper [26].

3.2 Preliminaries

3.2.1 Discrete harmonic functions

Suppose that u is a function defined on a subset of the lattice $(\delta\mathbf{Z})^n$. Then the δ -discrete Laplacian of u is defined by

$$\Delta_\delta u(x) = \Delta_{\delta,n} u(x) = \delta^{-2} \left(\sum_{j=1}^n (u(x + \delta e_j) + u(x - \delta e_j)) - 2nu(x) \right),$$

where e_1, e_2, \dots, e_n is the standard coordinate basis for \mathbf{Z}^n and $-\Delta_\delta$ coincides with the combinatorial Laplacian of the lattice where the conductance associated to each edge equals δ^{-2} . This is the discrete version of the Laplace–Beltrami operator in Riemannian manifolds. We refer the reader to T. Biyikoglu, J. Leydold, P. Stadler [5] for the details. Potential theory on finite networks is an active area of investigation, see for example [4] and the references therein.

Definition. A function u is called δ -discrete harmonic at a point x of the lattice $(\delta\mathbf{Z})^n$ if it is defined at x together with all its neighbors and satisfies the equation

$$\Delta_\delta u(x) = 0.$$

So the value of a discrete harmonic function at a lattice point is the average of its values at the $2n$ neighboring points.

Discrete harmonic functions share many properties of continuous ones. For example results on the maximum principle, solution to the Dirichlet problem, Green’s function, and Liouville’s theorem can be found in the very first articles on the subject, see also Y. Colin de Verdière [13] and C. Kiselman [31] for more recent surveys and more general discrete structures. On the other hand, not all results about continuous harmonic functions are easily generalized to the discrete case. For example zero sets of discrete harmonic functions are difficult to compare to those of continuous ones. For any finite square there exists a discrete harmonic polynomial that vanishes at each lattice point of this square. We study growth properties of discrete harmonic functions in cylinders and strips and provide accurate estimates that show to which extend continuous theorems can be generalized to solutions of the discrete equation that arises in the simplest numerical scheme.

We consider discrete harmonic functions on subsets of $(\delta\mathbf{Z})^n$. A subset D^δ is called a (discrete) domain if it is connected, i.e., for any two points x and y in D^δ there exists a sequence $\{x_0, x_1, \dots, x_s\}$ such that $x_0 = x$, $x_s = y$, $x_j \in D^\delta$, x_j and x_{j+1} are neighboring points of the lattice $(\delta\mathbf{Z})^n$.

A point $x \in (\delta\mathbf{Z})^n \setminus D^\delta$ is called a boundary point of D^δ if at least one of the $2n$ neighbors of x is in D^δ . We denote the set of boundary points of D^δ by ∂D^δ , we also use the notation $\overline{D^\delta} = D^\delta \cup \partial D^\delta$. A domain is called finite if it contains only finite number of points, otherwise it is called infinite.

Definition. A function u defined on $D^\delta \cup \partial D^\delta$ is called δ -discrete subharmonic (superharmonic) in D^δ if $\Delta_\delta u \geq 0$ (≤ 0) in D^δ .

Clearly, a function is harmonic in D^δ if it is both subharmonic and superharmonic. The following Maximum principle holds

Theorem. *If u is δ -discrete subharmonic in a finite domain D , then*

$$\max_{\overline{D}} u = \max_{\partial D} u.$$

Simple examples show that the maximum principle does not hold for infinite domains.

3.2.2 Eigenvalues and eigenfunctions for the discrete Dirichlet-Laplacian

In order to prove a version of the Phragmén–Lindelöf theorem for discrete subharmonic functions in cylindrical domains, we need some basic facts about eigenfunctions and eigenvalues of the discrete Dirichlet problem for the Laplacian on the base of the cylinder.

Let Ω be a bounded domain \mathbf{R}^n , $n \geq 1$, with Lipschitz boundary and let $\Omega^\delta = \Omega \cap (\delta\mathbf{Z})^n$. We always assume that $\delta < \delta_0$ is small enough such that Ω^δ is a discrete connected set. We study δ -discrete harmonic functions that are defined on the product domain $D^\delta(\Omega) = \overline{\Omega^\delta} \times (\delta\mathbf{Z})^k$ and vanish on the boundary. We consider the eigenvalues of the continuous n -dimensional Dirichlet–Laplacian on Ω , $\{\lambda_j(\Omega)\}$ and the eigenvalues of the corresponding discrete operators. It is known (see for example [5] or [16]) that the eigenvalues of the following problem

$$\begin{cases} -\Delta_{\delta,n} f = \lambda f & \text{in } \Omega^\delta \\ f = 0 & \text{on } \partial\Omega^\delta \end{cases}$$

are positive, $0 < \lambda_1^\delta < \lambda_2^\delta \leq \dots \leq \lambda_{K^\delta}^\delta$, the first eigenvalue is simple and the corresponding eigenfunction f_1^δ can be chosen strictly positive in Ω^δ . The last statement is the analog of the classical result on the first eigenfunction of Dirichlet–Laplacian, see [15, §6, ch VI]. For the discrete operator it follows from the Perron–Frobenius theorem on positive matrices, see for example [5, Corollary 2.23]. Clearly K^δ is finite in the discrete case and equals the number of points of Ω^δ .

It is also known that $\lambda_k^\delta(\Omega^\delta) \rightarrow \lambda_k(\Omega)$ as $\delta \rightarrow 0$. We don't discuss the limit arguments in this article, but we indicate which of our estimates survey the limit passage as $\delta \rightarrow 0$.

The eigenvalues $\lambda_k^\delta(\Omega^\delta)$ are given by the following minimax principle, see [5, Corollary 2.6],

$$\lambda_k^\delta(\Omega^\delta) = \min_{w \in W_k} \max_{0 \neq g \in w} \frac{\langle g, L_\Omega^\delta g \rangle}{\langle g, g \rangle},$$

where W_k denotes the set of subspaces of dimension at least k and L_Ω^δ is the δ -discrete Dirichlet-Laplacian of Ω . This readily implies that if $\Omega' \supset \Omega$ then

$$\lambda_k^\delta(\Omega') \leq \lambda_k^\delta(\Omega). \quad (3.1)$$

We denote by N_Ω^δ the counting function, $N_\Omega^\delta(\lambda)$ equals the number of eigenvalues $\lambda_k^\delta(\Omega)$ that are less than or equal to λ . Then (3.1) implies

$$N_{\Omega'}^\delta(\lambda) \geq N_\Omega^\delta(\lambda). \quad (3.2)$$

3.2.3 Eigenvalues for the cube

We need some estimates of the growth of the eigenvalues $\lambda_j^\delta(\Omega)$ to prove a precise version of the Phragmén–Lindelöf theorem in the last section of this Chapter. We obtain them by comparing the eigenvalues to those of a large cube Q containing Ω . The latter ones can be found explicitly. Let $Q_R = (0, R)^n$, where $R \in \mathbb{N}$ and let $M = 1/\delta \in \mathbb{N}$. We consider the following problem

$$\begin{cases} -\Delta_{\delta,n} f = \lambda f & \text{in } Q_R^\delta \\ f = 0 & \text{on } \partial Q_R^\delta. \end{cases}$$

This is an eigenvalue problem for a matrix of the size $(R\delta^{-1}-1)^n \times (R\delta^{-1}-1)^n$. Let $J = \{1, 2, \dots, R\delta^{-1}-1\}$, for any $\bar{k} \in J^n$, $\bar{k} = (k_1, \dots, k_n)$ the function

$$f_{\bar{k}}(x_1, \dots, x_n) = \prod_{j=1}^n \sin \frac{k_j \pi}{R} x_j$$

is an eigenfunction and the corresponding eigenvalue is

$$\lambda_k^\delta = 2\delta^{-2} \left(n - \sum_{j=1}^n \cos \frac{k_j \pi \delta}{R} \right).$$

Using the elementary inequality $1 - \cos x \geq 2\pi^{-2}x^2$, when $x \in (0, \pi)$ we obtain

$$\lambda_k^\delta \geq 4R^{-2} \sum_{j=1}^n k_j^2.$$

We use the following elementary inequality for the counting function for the cube

$$N_{Q_R}^\delta(\lambda) \leq C_n(R)(\lambda^{n/2} + 1), \quad (3.3)$$

where the constant does not depend on δ . This inequality is an illustration of the Weyl's asymptotic for the counting function for eigenvalues of Dirichlet–Laplacian.

3.2.4 Phragmén–Lindelöf theorems in cylindrical domains

Let Ω be a bounded subdomain of \mathbf{R}^n and $D^\delta = \Omega^\delta \times (\delta\mathbf{Z})^k$. Clearly, $\Delta_{\delta, n+k}u(x, y) = \Delta_{\delta, n}u(x, y) + \Delta_{\delta, k}u(x, y)$, where the first Laplacian is taking with respect to x -variables and the second with respect to y -variables. Let f_1^δ be the first eigenfunction of the Dirichlet–Laplacian in Ω^δ defined above. As we noted f_1^δ is strictly positive on Ω^δ and we have the following positive harmonic function in D^δ

$$u^\delta(x, y) = f_1^\delta(x) \cosh b^\delta y_1 \cosh b^\delta y_2 \dots \cosh b^\delta y_k,$$

where

$$\cosh \delta b^\delta = 1 + \frac{1}{2k} \delta^2 \lambda_1^\delta. \quad (3.4)$$

In the discrete setting the function f_1^δ is strictly positive; this makes the proof of our first theorem of Phragmén–Lindelöf type more simple than the proof of a similar result for continuous functions, see for example [8, 36, 50].

Theorem 13. *Let v be a δ -discrete subharmonic function in D^δ such that $v \leq 0$ on $\partial\Omega^\delta \times (\delta\mathbf{Z})^k$. Let λ_1^δ be the first eigenvalue of the δ -discrete Dirichlet*

problem for the Laplacian in Ω and b^δ be the positive solution to the equation $\cosh \delta b^\delta = 1 + \frac{1}{2k} \delta^2 \lambda_1^\delta$. Suppose that

$$v(x, y) \leq o(1) \exp(b^\delta \|y\|_1), \quad \text{when } \|y\|_1 \rightarrow \infty.$$

Then $v \leq 0$ on D^δ .

Proof. We want to compare $v(x, y)$ to a multiple of $u^\delta(x, y)$ on $\overline{\Omega}^\delta \times [-N, N]^k$. On the part of the boundary $\partial\Omega^\delta \times (\delta\mathbf{Z})^k$ we have $v \leq 0$ and $u^\delta = 0$ because $f_1^\delta = 0$ on $\partial\Omega^\delta$. On the other part of the boundary, $\|y\|_1 \geq N$ and

$$v(x, y) \leq C_N \exp(b^\delta \|y\|_1) \leq \frac{2^k C_N}{\min_{\Omega^\delta} f_1^\delta} u^\delta(x, y),$$

where $C_N \rightarrow 0$ as $N \rightarrow \infty$.

The maximum principle for subharmonic functions implies that

$$v(x, y) \leq \frac{2^k C_N}{\min_{\Omega^\delta} f_1^\delta} u^\delta(x, y), \quad \text{where } x \in \Omega^\delta, \ y \in (\delta\mathbf{Z})^k, \ \|y\|_\infty \leq N.$$

Now if we fix (x, y) and let N grow to infinity, we obtain $v(x, y) \leq 0$. \square

The theorem holds for subharmonic functions with all estimates from above only. If we have a discrete harmonic function h and apply the above statement to h and $-h$ we obtain the uniqueness for the Dirichlet problem in D^δ in the class of functions

$$H_b(D^\delta) = \{u : D^\delta \rightarrow \mathbf{R} : |u(x, y)| = o(\exp(b^\delta \|y\|_1)), \ \|y\|_1 \rightarrow \infty\}.$$

Corollary. Let u and v be δ -discrete harmonic functions on D^δ such that $u, v \in H_b(D^\delta)$. If $u = v$ on $\partial(\Omega^\delta) \times (\delta\mathbf{Z})^k$ then $u = v$ on D^δ .

Proof. Let $g = u - v$. Then g is δ -discrete harmonic in D^δ and $g = 0$ on $\partial(\Omega^\delta) \times (\delta\mathbf{Z})^k$. Moreover $|g(x, y)| \leq |u(x, y)| + |v(x, y)|$ and therefore

$$|g(x, y)| \leq C_N \exp(b^\delta \|y\|_1), \quad \text{when } \|y\|_1 \geq N,$$

where $C_N \rightarrow 0$ as $N \rightarrow \infty$. Then $g \leq 0$ on D^δ by Theorem 13. In the same way we obtain $-g \leq 0$ and thus $u = v$. \square

We note that $b^\delta \rightarrow \sqrt{\lambda_1(\Omega)/k}$ when $\delta \rightarrow 0$, however Theorem 13 does not survive a limit argument as $\delta \rightarrow 0$. In the last section we provide an estimate for δ -discrete harmonic functions in truncated cylinders that allow us to prove a more accurate version of the Phragmén–Lindelöf theorem.

3.3 Discrete harmonic functions on strips

In this section we study quantitative uniqueness for discrete harmonic functions and their gradients on strips $S = (0, 1) \times \mathbf{R}^n$. We remark that eigenvalues of Dirichlet–Laplacian on $[0, 1]^\delta$ are $\lambda_l^\delta = 2\delta^{-2}(1 - \cos 2\pi l\delta)$. In particular the Phragmén–Lindelöf theorem proved in the last section implies the uniqueness in the Dirichlet problem for discrete harmonic functions that satisfy

$$|u(x, y)| = o(\exp(b^\delta \|y\|_1)), \quad \|y\|_1 \rightarrow \infty \quad (3.5)$$

where

$$\cosh \delta b^\delta = \frac{n+1}{n} - \frac{1}{n} \cos 2\pi\delta. \quad (3.6)$$

3.3.1 Tempered harmonic functions in a strip

Now we consider *tempered* harmonic functions in the strip and use the Fourier representation to solve the Dirichlet problem.

Definition. Let u be a δ -discrete function on S^δ . Then u is said to be tempered if

$$\sum_{k=0}^{1/\delta} \sum_{j \in \mathbf{Z}^n} |u(\delta k, \delta j)|^2 < \infty.$$

Theorem 14. *Let u be a δ -discrete harmonic function in S^δ such that (3.5) holds, and $\delta^{-1} = L$ for some positive integer L . Suppose that*

$$\sum_{j \in \mathbf{Z}^n} |u(0, \delta j)|^2 < \infty \quad \text{and} \quad \sum_{j \in \mathbf{Z}^n} |u(1, \delta j)|^2 < \infty.$$

Then $\{u(\delta k, \delta j)\}_{j \in \mathbf{Z}^n} \in l^2(\mathbf{Z}^n)$ for each $k = 1, 2, \dots, L-1$, and

$$\sum_{j \in \mathbf{Z}^n} |u(\delta k, \delta j)|^2 \leq \sum_{j \in \mathbf{Z}^n} |u(0, \delta j)|^2 + \sum_{j \in \mathbf{Z}^n} |u(1, \delta j)|^2.$$

Proof. Let

$$\varphi_0(t) = \sum_{j \in \mathbf{Z}^n} u(0, \delta j) e^{2\pi i j \cdot t}, \quad \text{and} \quad \varphi_L(t) = \sum_{j \in \mathbf{Z}^n} u(1, \delta j) e^{2\pi i j \cdot t}$$

for $t \in [0, 1]^n$. Then $\varphi_0, \varphi_L \in L^2([0, 1]^n)$.

For each $t \in [0, 1]^n$ we define q such that $q(t) \geq 1$ and

$$q(t) + q(t)^{-1} = 2(n+1) - 2 \sum_{l=1}^n \cos 2\pi t_l.$$

More precisely $q(t) = \lambda(t) + \sqrt{\lambda^2(t) - 1}$ and then $q(t)^{-1} = \lambda(t) - \sqrt{\lambda^2(t) - 1}$, where

$$\lambda(t) = n + 1 - \sum_{l=1}^n \cos 2\pi t_l.$$

Now for $k = 1, \dots, L-1$ we consider

$$\varphi_k(t) = \frac{q(t)^k - q(t)^{-k}}{q(t)^L - q(t)^{-L}} \varphi_L(t) + \frac{q(t)^{L-k} - q(t)^{L-k}}{q(t)^L - q(t)^{-L}} \varphi_0(t).$$

Since $q \geq 1$, we have

$$q(t)^k - q(t)^{-k} \leq q(t)^L - q(t)^{-L}, \quad \text{and} \quad q(t)^{L-k} - q(t)^{-L+k} \leq q(t)^L - q(t)^{-L}.$$

Then $\varphi_k \in L^2([0, 1]^n)$ and $\|\varphi_k\|_2 \leq \|\varphi_0\|_2 + \|\varphi_L\|_2$. Thus

$$\varphi_k(t) = \sum_{j \in \mathbf{Z}^n} v(k, j) e^{2\pi i j \cdot t},$$

where $\{v(k, j)\}_{j \in \mathbf{Z}^n} \in l^2(\mathbf{Z}^n)$. Remark that

$$q(t) = \frac{1 + q^2(t)}{2\lambda(t)} \quad \text{and therefore} \quad q^k(t) = \frac{q^{k-1}(t) + q^{k+1}(t)}{2\lambda(t)}.$$

Then

$$\varphi_k(t) = \frac{\varphi_{k-1}(t) + \varphi_{k+1}(t)}{2\lambda(t)}$$

and

$$\varphi_k = \frac{1}{2(n+1)} \left[\varphi_{k+1} + \varphi_{k-1} + \varphi_k \left(\sum_{l=1}^n e^{2\pi i t_l} + e^{-2\pi i t_l} \right) \right].$$

Hence the Fourier coefficients $v(k, j)$ satisfy

$$v(k, j) = \frac{v(k-1, j) + v(k+1, j) + \sum_{l=1}^n (v(k, j - e_l) + v(k, j + e_l))}{2(n+1)}.$$

It means that v is a discrete harmonic function on $[1, L - 1] \times \mathbf{Z}^n$. We have that $v(0, j) = u(0, \delta j)$ and $v(L, j) = u(1, \delta j)$. Note also that

$$|v(k, J)|^2 \leq \sum_{j \in \mathbf{Z}^n} |v(k, j)|^2 = \|\varphi_k\|_{L^2([0,1]^n)}^2 \leq \left(\|\varphi_0\|_{L^2([0,1]^n)} + \|\varphi_L\|_{L^2([0,1]^n)} \right)^2.$$

Thus $v(k, J)$ is bounded, in particular $|v(k, y)| = o(\exp(b^\delta \|y\|_1))$ when $\|y\|_1 \rightarrow \infty$. Finally, by Corollary in Section 3.2.4 we have $v(k, j) = u(\delta k, \delta j)$ and $\{u(\delta k, \delta j)\}_{j \in \mathbf{Z}^n} \in l^2(\mathbf{Z}^n)$ with the required estimate. \square

Remark. We have also proved that if u is a δ -discrete harmonic function on S^δ that is square-summable along the hyperplanes $\{\delta k\} \times (\delta \mathbf{Z})^n$, then there exist two functions $a_1, a_2 \in L^2([0, 1]^n)$ such that

$$u(\delta k, \delta j) = \int_{[0,1]^n} (a_1(t)q(t)^k + a_2(t)q(t)^{-k}) e^{-2\pi j \cdot t} dt, \quad (3.7)$$

where q is defined by $q(t) \geq 1$,

$$q(t) + q^{-1}(t) = 2(n + 1) - 2 \sum_{l=1}^n \cos 2\pi t_l.$$

Reviewing the computations in the proof of the lemma, we see that

$$a_1(t) = \frac{\varphi_L(t) - q(t)^{-L}\varphi_0(t)}{q(t)^L - q(t)^{-L}}, \quad a_2(t) = \frac{q(t)^L\varphi_0(t) - \varphi_L(t)}{q(t)^L - q(t)^{-L}}.$$

Thus the theorem provides a constructive procedure for solution of the Dirichlet problem for tempered harmonic function in a strip as well as a stability estimate for this procedure.

3.3.2 Three line theorem for discrete harmonic functions

In this subsection we prove a three line theorem for the gradients of discrete harmonic functions, the corresponding continuous result and its connections to the interpolation theory can be found in [30].

Definition. Let u be a δ -discrete function on a subdomain of the lattice $(\delta\mathbf{Z})^{n+1}$. Its discrete partial derivatives are defined by

$$u_x(x, y) = \delta^{-1} (u(x + \delta, y) - u(x, y)) \quad \text{and}$$

$$u_{y_l}(x, y) = \delta^{-1} (u(x, y + \delta e_l) - u(x, y)).$$

For the case of the strip $S = [0, 1] \times \mathbf{R}^n$ all partial derivatives in y -variables are defined on the same domain, while u_x is defined on $[0, 1 - \delta] \times \mathbf{R}^n$.

Definition. The discrete gradient of a discrete function u on a subdomain of the lattice $(\delta\mathbf{Z})^{n+1}$ is defined as

$$\nabla u(x, y) = (u_x(x, y), u_{y_1}(x, y), u_{y_2}(x, y), \dots, u_{y_n}(x, y)).$$

Theorem 15. Let u be a δ -discrete harmonic function in $[0, 1] \times \mathbf{R}^n$, $\delta^{-1} = M + 1$ for some positive integer M . Suppose that u satisfies (3.5) and

$$\{u(0, \delta j)\}_{j \in \mathbf{Z}^n} \in l^2(\mathbf{Z}^n), \quad \{u(1, \delta j)\}_{j \in \mathbf{Z}^n} \in l^2(\mathbf{Z}^n).$$

Let further

$$m(k) = \delta^2 \|u_x(\delta k, \delta j)\|_{l^2(\mathbf{Z}^n)}^2 + \delta^2 \sum_{l=1}^n \|u_{y_l}(\delta k, \delta j)\|_{l^2(\mathbf{Z}^n)}^2 \quad \text{for } k = 0, 1, \dots, M.$$

Then

$$m(k) \leq (m(0))^{1 - \frac{k}{M}} (m(M))^{\frac{k}{M}}.$$

Proof. Using formula (3.7) and the definition of the partial derivatives, we get

$$u_x(\delta k, \delta j) = \delta^{-1} \int_{[0, 1]^n} (a_1(t)q(t)^k(q(t) - 1) + a_2(t)q(t)^{-k}(q(t)^{-1} - 1)) e^{-2\pi j \cdot t} dt,$$

and

$$\|u_x(\delta k, \delta j)\|_{l^2(\mathbf{Z}^n)}^2 = \delta^{-2} \|a_1(t)q(t)^k(q(t) - 1) + a_2(t)q(t)^{-k}(q(t)^{-1} - 1)\|_{L^2([0, 1]^n)}^2.$$

Further,

$$u_{y_l}(\delta k, \delta j) = \delta^{-1} \int_{[0, 1]^n} (a_1(t)q(t)^k + a_2(t)q(t)^{-k}) e^{-2\pi j \cdot t} (e^{-2\pi t_l} - 1) dt,$$

$$\|u_{y_l}(\delta k, \delta j)\|_{L^2(\mathbf{Z}^n)}^2 = \delta^{-2} \|(a_1(t)q(t)^k + a_2(t)q(t)^{-k})(e^{-2\pi t_l} - 1)\|_{L^2([0,1]^n)}^2.$$

Then, adding up the identities above, we get

$$\begin{aligned} m(k) &= \delta^2 \|u_x(\delta k, \delta j)\|_{L^2(\mathbf{Z}^n)}^2 + \delta^2 \sum_{l=1}^n \|u_{y_l}(\delta k, \delta j)\|_{L^2(\mathbf{Z}^n)}^2 = \\ & \quad \|a_1(t)q(t)^k(q(t)-1) + a_2(t)q(t)^{-k}(q(t)^{-1}-1)\|_{L^2([0,1]^n)}^2 + \\ & \quad \sum_{l=1}^n \|a_1(t)q(t)^k(e^{-2\pi i t_l} - 1) + a_2(t)q(t)^{-k}(e^{-2\pi i t_l} - 1)\|_{L^2([0,1]^n)}^2. \end{aligned} \quad (3.8)$$

We note that $q(t)$ is real and by the definition $q(t) + q(t)^{-1} = 2(n+1) - 2\sum_{l=1}^n \cos 2\pi t_l$, therefore

$$(q(t)-1)(q(t)^{-1}-1) = 2 \sum_{l=1}^n \cos 2\pi t_l - 2n = - \sum_{l=1}^n (e^{-2\pi i t_l} - 1)(e^{2\pi i t_l} - 1). \quad (3.9)$$

Finally,

$$\begin{aligned} \delta^2 m(k) &= \\ & \quad \|a_1(t)q(t)^k(q(t)-1)\|_{L^2([0,1]^n)}^2 + \|a_2(t)q(t)^{-k}(q(t)^{-1}-1)\|_{L^2([0,1]^n)}^2 + \\ & \quad \sum_{l=1}^n \|a_1(t)q(t)^k(e^{-2\pi i t_l} - 1)\|_{L^2([0,1]^n)}^2 + \\ & \quad \|a_2(t)q(t)^{-k}(e^{-2\pi i t_l} - 1)\|_{L^2([0,1]^n)}^2. \end{aligned} \quad (3.10)$$

Each term in the right hand side of the last formula can be written in the form $s(k) = \|b(t)q(t)^{\pm k}\|_2^2$ for some $b \in L^2([0,1]^n)$ and $q(t)^{\pm k} \in L^\infty([0,1]^n)$. By Hölder's inequality, we have

$$\begin{aligned} s(k) &= \|b(t)q^k(t)\|_{L^2([0,1]^n)}^2 \leq \\ & \quad \left(\int_{[0,1]^n} |b(t)|^2 dt \right)^{1-\frac{k}{M}} \left(\int_{[0,1]^n} |b(t)|^2 q^2(t) dt \right)^{\frac{k}{M}} \leq (s(0))^{1-\frac{k}{M}} (s(M))^{\frac{k}{M}}. \end{aligned}$$

Applying the same computation for each term and using the lemma below we conclude the proof of the theorem. \square

Lemma 5. *If each function $m_i : [0, 1, \dots, M] \rightarrow \mathbf{R}_+$ satisfies the inequality*

$$m(k) \leq [m(0)]^{1-\frac{k}{M}} [m(M)]^{\frac{k}{M}},$$

then the sum $m(k) = \sum_i m_i(k)$ satisfies the same inequality.

Proof. It suffices to prove the statement when $m(k) = m_1(k) + m_2(k)$ is the sum of two functions. Let $\alpha = k/M$ then we have

$$\begin{aligned} m(k) &= m_1(k) + m_2(k) \leq m_1(0)^{1-\alpha} m_1(M)^\alpha + m_2(0)^{1-\alpha} m_2(M)^\alpha = \\ &= m(0)^{1-\alpha} m(M)^\alpha \left[\left(\frac{m_1(0)}{m(0)} \right)^{1-\alpha} \left(\frac{m_1(M)}{m(M)} \right)^\alpha + \left(\frac{m_2(0)}{m(0)} \right)^{1-\alpha} \left(\frac{m_2(M)}{m(M)} \right)^\alpha \right]. \end{aligned}$$

And the lemma follows from the elementary inequality

$$x^{1-\alpha} y^\alpha + (1-x)^{1-\alpha} (1-y)^\alpha \leq 1$$

when $x, y \in [0, 1]$ and $\alpha \in [0, 1]$. □

Remark. The proof of Theorem 15 above is similar to that of the continuous three-line theorem, see [30]. In the continuous case the passage from (3.8) to (3.10) is trivial, in discrete case we fortunately have the identity (3.9).

For continuous harmonic functions similar three balls or three spheres theorem can be obtain, see for example [33, 35]. There are no trivial generalizations of those results as a harmonic function can vanish on any finite square without being identically zero.

3.4 Harmonic measure and stability estimates

In this section we study δ -discrete harmonic functions that are defined on the cylinder $D^\delta(\Omega) = \Omega^\delta \times (\delta\mathbf{Z})$. Discrete harmonic measure on truncated cylinder is estimated first, then we apply these estimates to give a more precise version of the Phragmén–Lindelöf theorem and prove some stability results.

3.4.1 Discrete harmonic measure

Let now $\mathcal{H}_0(D^\delta)$ denote the space of δ -discrete harmonic functions on $D^\delta(\Omega)$ that vanish on the boundary. Such function is uniquely determined by its values on two layers $\Omega^\delta \times \{a\}$ and $\Omega^\delta \times \{b\}$ (where it may attain arbitrary values) and the dimension of $\mathcal{H}_0(D^\delta)$ equals $2K^\delta$, where K^δ is the number of points in Ω^δ .

We note that for a function $u(x) = u(x', x_{n+1})$ on $D^\delta(\Omega)$ we have

$$\begin{aligned} \Delta_{\delta, n+1} u(x', x_{n+1}) &= \\ \Delta_{\delta, n} u(x', x_{n+1}) + \delta^{-2} (u(x', x_{n+1} + \delta) + u(x', x_{n+1} - \delta) - 2u(x', x_{n+1})). \end{aligned}$$

Let $\{f_k^\delta\}_{k=1}^{K^\delta}$ be the sequence of eigenfunctions of the Dirichlet–Laplacian in Ω^δ , discussed in 3.2.2. Then it is easy to check that the following functions form a basis for $\mathcal{H}_0(D^\delta)$.

$u_k^\delta(x) = f_k^\delta(x') \cosh(a_k^\delta x_{n+1})$, $v_k^\delta(x) = f_k^\delta(x') \sinh(a_k^\delta x_{n+1})$, $k = 1, 2, \dots, K^\delta$, where a_k^δ is the positive solution of

$$\cosh \delta a_k^\delta = 1 + \frac{1}{2} \delta^2 \lambda_k^\delta.$$

Now we calculate the discrete harmonic measure of the bases of a truncated cylinder. Let g_N^δ be the δ -discrete harmonic function on $D_N^\delta(\Omega) = \bar{\Omega}^\delta \times ([-N, N] \cap (\delta\mathbf{Z}))$ defined by its boundary values

$$\begin{cases} g_N^\delta(x', \pm N) &= 1 & x' \in \Omega^\delta \\ g_N^\delta(x', x_{n+1}) &= 0 & x' \in \partial\Omega^\delta, -N \leq x_{n+1} \leq N. \end{cases}$$

Lemma 6. *The harmonic measure $g_N^\delta(x) = g_N^\delta(x', x_{n+1})$ is given by*

$$g_N^\delta(x', x_{n+1}) = \sum_{k=1}^{K^\delta} d_k^\delta f_k^\delta(x') \frac{\cosh(a_k^\delta x_{n+1})}{\cosh a_k^\delta N},$$

where $d_k^\delta = \sum_{x' \in \Omega^\delta} f_k^\delta(x')$.

Proof. Clearly g_N^δ is an even function with respect to x_{n+1} and therefore it can be written as

$$g_N^\delta(x', x_{n+1}) = \sum_{k=1}^{K^\delta} C_k f_k^\delta(x') \cosh(a_k^\delta x_{n+1}), \quad (3.11)$$

where the coefficients C_k satisfy the linear system of equations

$$1 = \sum_{k=1}^{K^\delta} C_k f_k^\delta(x') \cosh(a_k^\delta N),$$

for each $x' \in \Omega^\delta$. Since functions $\{f_k^\delta\}_{k=1}^{K^\delta}$ form an orthonormal basis, we obtain

$$C_k \cosh a_k^\delta N = \sum_{x'} f_k^\delta(x') = d_k^\delta. \quad (3.12)$$

Substituting (3.12) in (3.11) we get the required formula. \square

We conclude this subsection by one auxiliary inequality. We note that the values of the function $g_N^\delta(x', x_{n+1})$ on the middle hyperplane $\{x_{n+1} = 0\}$ are given by

$$g_N^\delta(x', 0) = \sum_{k=1}^{K^\delta} d_k^\delta f_k^\delta(x') \frac{1}{\cosh a_k^\delta N}.$$

Then a linear combination of the values of u on $\Omega^\delta \times \{0\}$ admits the following estimate:

$$\begin{aligned} \sum_{x'} w(x') g_N^\delta(x', 0) &= \sum_{x'} \sum_{k=1}^{K^\delta} d_k^\delta w(x') f_k^\delta(x') \frac{1}{\cosh a_k^\delta N} \leq \\ &= \sum_{k=1}^{K^\delta} \frac{|d_k^\delta|}{\cosh a_k^\delta N} \left(\sum_{x'} |w(x')|^2 \right)^{1/2}, \end{aligned} \quad (3.13)$$

we applied the Cauchy–Schwarz inequality and used that eigenfunctions f_k^δ are normalized by $\sum_{x'} |f_k^\delta(x')|^2 = 1$.

3.4.2 Phragmén–Lindelöf theorem for δ -discrete subharmonic functions

Now we prove a version of the Phragmén–Lindelöf theorem for δ -discrete subharmonic functions in truncated cylinder $D_N^\delta(\Omega)$. We want to show that if a subharmonic function is positive inside the cylinder, say at some points on the section $\Omega^\delta \times \{0\}$, then it grows at least exponentially. Moreover, we can give estimates on the truncated cylinders and not only asymptotic result as in Theorem 13.

We use the following notation $u^+ = \max\{0, u\}$.

Theorem 16. *Suppose u is a δ -discrete subharmonic function on $D_N^\delta(\Omega)$ such that $u(x', x_{n+1}) = 0$ when $x' \in \partial\Omega^\delta$ and u satisfies the following positivity condition on $\Omega \times \{0\}$*

$$\sum_{x' \in \Omega^\delta} u^+(x', 0)^2 = A^2 K^\delta > 0.$$

Then

$$\max_{\Omega^\delta \times [-N, N]} u(x', x_{n+1}) \geq \frac{A}{2} \left(\sum_k \exp(-a_k^\delta N) \right)^{-1}, \quad (3.14)$$

where $a_k^\delta = \delta^{-1} \cosh^{-1}(1 + \frac{1}{2} \delta^2 \lambda_k^\delta)$. In particular, there exists a constant C_Ω that depends only on Ω such that

$$\max_{\Omega^\delta \times [-N, N]} u(x', x_{n+1}) \geq C_\Omega A \exp(a_1^\delta N), \quad (3.15)$$

for any $N \in \mathbb{N}$ and any $\delta < \delta_0$.

The inequality (3.14) is more precise than (3.15). We write the constant explicitly and, as soon as λ_k^δ are known, the right hand side of (3.14) can be estimated. Clearly, the right hand side of (3.14) is of order $\exp(a_1^\delta N)$ when $N \rightarrow \infty$. This is expressed accurately in inequality (3.15). The constant C_Ω is not explicit, but it depends neither on N nor on δ , so we can also fix N and let δ go to zero to get estimates of continuous functions that can be approximated by discrete subharmonic ones.

Proof. Let $M_N = \max_{|x_{n+1}|=N} u(x', x_{n+1})$. Then by the maximum principle,

$$u(x', x_{n+1}) \leq M_N g_N^\delta(x', x_{n+1}) \quad \text{on} \quad \Omega^\delta \times [-N, N],$$

where g_N^δ is the harmonic measure from Lemma 6, clearly $g_N^\delta \geq 0$. Taking the linear combination over $x' \in \Omega^\delta$ with non-negative coefficients $w(x') = u^+(x', 0)$ and using (3.13), we obtain

$$\begin{aligned} \sum_{x'} u^+(x', 0)^2 &= \sum_{x'} u^+(x', 0) u(x', 0) \leq \\ &M_N \sum_{k=1}^{K^\delta} \frac{|d_k^\delta|}{\cosh a_k^\delta N} \left(\sum_{x'} |u^+(x', 0)|^2 \right)^{1/2}. \end{aligned}$$

Then we have

$$M_N \geq \left(\sum_{x'} u^+(x', 0)^2 \right)^{1/2} \left(\sum_{k=1}^{K^\delta} \frac{|d_k^\delta|}{\cosh a_k^\delta N} \right)^{-1} = A(K^\delta)^{1/2} \left(\sum_{k=1}^{K^\delta} \frac{|d_k^\delta|}{\cosh a_k^\delta N} \right)^{-1}.$$

Applying the Cauchy–Schwarz inequality, we get

$$|d_k^\delta| = \left| \sum_{x'} f_k^\delta(x') \right| \leq \left(\sum_{x'} (f_k^\delta(x'))^2 \right)^{\frac{1}{2}} \left(\sum_{x'} 1 \right)^{\frac{1}{2}} \leq (K^\delta)^{\frac{1}{2}}.$$

Now, we combine the last two inequalities and obtain

$$M_N \geq A \left(\sum_{k=1}^{K^\delta} \frac{1}{\cosh a_k^\delta N} \right)^{-1}.$$

Then (3.14) follows from the following inequality

$$\sum_{k=1}^{K^\delta} \frac{1}{\cosh a_k^\delta N} \leq 2 \sum_{k=1}^{K^\delta} \exp(-a_k^\delta N).$$

To prove (3.15) we may assume that δ is small (otherwise we have an upper bound for K^δ). We partition the eigenvalues λ_k^δ into two groups. We choose a positive number c and define $I_1 = \{k : \lambda_k^\delta < c\delta^{-2}\}$ and $I_2 = \{k : \lambda_k^\delta \geq c\delta^{-2}\}$. Let also $c_0 = \cosh^{-1}(1+c)$, then

$$\sum_{k \in I_2} \exp(-a_k^\delta N) \leq \sum_{k \in I_2} \exp(-\delta^{-1} c_0 N) \leq K^\delta \exp(-\delta^{-1} c_0 N) \leq C_0 \exp(-a_1^\delta N),$$

when δ is small enough, since $K^\delta \leq C\delta^{-n}$ and $a_1^\delta \rightarrow (\lambda_1(\Omega))^{1/2}$ as $\delta \rightarrow 0$.

For the second part of the sum we have $\delta\sqrt{\lambda_k^\delta} < c$. We consider the function $\alpha : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ defined by

$$\cosh \alpha(s) = 1 + \frac{1}{2}s^2.$$

Then $a_k^\delta = \delta^{-1} \alpha(\delta \sqrt{\lambda_k^\delta})$ and a simple calculation gives

$$\alpha'(s) = \frac{2}{\sqrt{4+s^2}}.$$

Denoting the minimum of the derivative of α on $[0, c]$ by d , we obtain

$$a_k^\delta \geq a_1^\delta + d((\lambda_k^\delta)^{1/2} - (\lambda_1^\delta)^{1/2}).$$

Now we partition I_1 further into $J_l = \{k : l \leq (\lambda_k^\delta)^{1/2} - (\lambda_1^\delta)^{1/2} < l+1\}$, $l = 0, 1, \dots$ and let $|J_l|$ denote the cardinality of J_l . We consider any cube Q such that $\Omega \subset Q$ and apply inequalities (3.2) and (3.3) to obtain

$$|J_l| \leq N_\Omega^\delta \left(\left((\lambda_1^\delta)^{\frac{1}{2}} + l + 1 \right)^2 \right) \leq N_Q^\delta \left(\left((\lambda_1^\delta)^{\frac{1}{2}} + l + 1 \right)^2 \right) \leq C_\Omega (l+1)^n,$$

for each $l = 0, 1, \dots$. Finally, we obtain

$$\begin{aligned} \sum_{k \in I_1} \exp(-a_k^\delta) &\leq \sum_{l=0}^{\infty} \sum_{k \in J_l} \exp(-a_k^\delta) \leq \sum_{l=0}^{\infty} \exp(-(a_1^\delta + ld)N) |J_l| \leq \\ &C_\Omega \exp(-a_1^\delta N) \sum_{l=0}^{\infty} (l+1)^n \exp(-ldN). \end{aligned}$$

The last sum is finite and can be bounded by a constant independent of $N \in \mathbb{N}$ and δ . This concludes the proof of the theorem. \square

One of the differences between the continuous and discrete cases lies in the formulas connecting eigenvalues λ and corresponding numbers a . For the continuous case one has $a(\lambda) = \sqrt{\lambda}$ while for the discrete case the formula becomes

$$a^\delta(\lambda) = \delta^{-1} \cosh^{-1} \left(1 + \frac{1}{2} \delta^2 \lambda \right).$$

This function resembles $\sqrt{\lambda}$ on the interval $[0, c\delta^{-2}]$ but grows as $\log \lambda$ when $\lambda \rightarrow \infty$. To deal with the discrete case we have partitioned the set of eigenvalues into two parts.

3.4.3 Stability estimates for solution of the Dirichlet problem

A standard argument shows that estimates of the harmonic measure imply conditional stability estimates for harmonic function. We apply it for truncated cylinders and prove the following.

Theorem 17. *Let h be a δ -discrete harmonic function in the truncated cylinder $D_N^\delta(\Omega)$ with boundary values f on $\partial\Omega^\delta \times [-N, N]$ and such that $|h(x', \pm N)| \leq M_N$. Then*

$$\max_{x'} |h(x', 0)| \leq \max |f| + C_\Omega(M_N + \max |f|) \exp(-a_1^\delta N). \quad (3.16)$$

In particular, if h is harmonic in $D^\delta(\Omega)$, $|h(x', x_{n+1})| = o(\exp(a_1^\delta |x_{n+1}|))$ when $|x_{n+1}| \rightarrow \infty$ and h is bounded on the boundary $\partial\Omega \times (\delta\mathbf{Z})$, then h is bounded by the same constant in $D^\delta(\Omega)$.

Proof. Let v be the δ -discrete harmonic function in the truncated cylinder $D_N^\delta(\Omega) = (\Omega \times (-N, N))^\delta$ that solves the following Dirichlet problem.

$$\Delta_{n+1, \delta} v = 0, \quad v(x', \pm N) = 0, \quad x' \in \Omega^\delta, \quad \text{and } v(x', x_{n+1}) = f(x', x_{n+1}), \quad x' \in \partial\Omega^\delta.$$

By the maximum principle for the bounded domain $D_N^\delta(\Omega)$ we have $|v(x', x_{n+1})| \leq \max |f|$. Then $u = h - v$ is δ -discrete harmonic function on $D_N^\delta(\Omega)$ that vanishes on the part $\partial\Omega^\delta \times [-N, N]$ of the boundary and satisfies

$$\max_{\Omega^\delta \times [-N, N]} |u(x', x_{n+1})| \leq \max |f| + M_N.$$

We compare it to a multiple of the harmonic measure g_N^δ and use the estimate

$$|g_N^\delta(x', 0)| \leq C_\Omega \exp(-a_1^\delta N)$$

that follows from the proof of Theorem 16. Then we obtain

$$|u(x', 0)| \leq C_\Omega(M_N + \max |f|) \exp(-a_1^\delta N).$$

This implies (3.16). The second statement of the theorem follows from (3.16). \square

4 Stability and regularization for determining sets of discrete Laplacian

We study the determining sets for discrete harmonic functions on the square lattices. The stability and regularization of the reconstruction of harmonic functions from its values on a part of a domain is discussed. For some specific configurations we use the logarithmic convexity estimates to obtain error bounds and propose an optimal choice of the mesh size of discretization.

4.1 Introduction

4.1.1 Background

In this Chapter the question of reconstruction of a harmonic function by its values on a given set is discussed. We study the following model problem

$$\begin{cases} \Delta u = 0 \text{ in } [0, 1] \times [0, 1], \\ u|_{\Lambda} = g, \end{cases} \quad (4.1)$$

where $\Lambda \subset [0, 1] \times [0, 1]$ and g is a given function. A simple example is given by $\Lambda = \partial([0, 1] \times [0, 1])$, for which (4.1) becomes the classical Dirichlet problem. Another limit case is the Cauchy problem

$$\begin{cases} \Delta u = 0 \text{ in } [0, 1] \times [0, 1], \\ u(0, y) = g_1, \quad u(1, y) = g_2, \\ u(x, 0) = f_0, \quad u_y(x, 0) = f_1, \end{cases}$$

which is known to be ill-posed. Different stability estimates and regularization techniques are known for the Cauchy problem, see [2, 6] and the references therein. The discretization of the Cauchy problem and conditional stability was discussed in [21, 42].

We are interested in the situation when only part of the boundary is accessible but some measurements can be done inside the domain, so our

typical example is given by

$$\Lambda_b = (\{0, 1\} \times [0, 1]) \cup ([0, 1] \times \{1/2 - b, 1/2 + b\}), \quad 0 < b \leq 1/2.$$

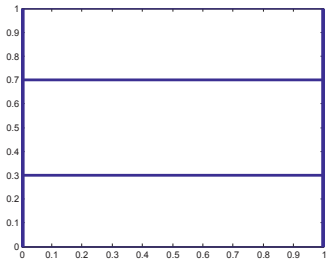


Figure 4.1: The graph of Λ_b

Clearly, a harmonic function is uniquely determined by its values on Λ_b . We discuss a reconstruction method, the stability of this reconstruction, and possible regularization procedures. It is well-known that initial problem is not well-posed when Λ contains a part inside the domain $[0, 1] \times [0, 1]$, not all functions g are admissible and a small variation of the data g may result in a large perturbation of the solution u . Regularization is a standard tool for such problems, instead of solving the initial problem (4.1) exactly, we look for the solution u_0 of the following extremal problem

$$\begin{cases} \Delta u_0 = 0 \text{ in } [0, 1] \times [0, 1], \\ \|u_0|_{\Lambda} - g\| \leq \epsilon, \\ \|u_0\| \rightarrow \min. \end{cases} \quad (4.2)$$

This allows us to treat noisy data g ; ϵ is typically the data error. We are interested in the approximation error $\|u - u_0\|$ and we obtain estimates on the discrete level, using for simplicity the standard five-point difference approximation for the Laplace operator. We specify the norms later.

The starting point of our discussion is the notion of the determining sets for the discrete Laplacian by A. Rubinstein, J. Rubinstein and G. Wolansky, [43]. We measure also the stability of determining sets and suggest a regularization method for reconstruction. For model sets Λ_b we use the technique of logarithmic convexity estimates to obtain approximation bounds following the ideas of R. Falk and P. Monk, [21] and of H. Reinhardt, H. Han and

D. Háo, [42]. However, we don't assume zero initial values on vertical sides of the boundary but elaborate the data on these sides into the problem (4.2).

The notion of determining sets was suggested in [43] in connection to the phase reconstruction problem. The instability of the reconstruction was also addressed and the authors suggested using overdetermined reconstruction and least square methods. Our approach is different, the estimates are based on the simple geometry of the model set, however the regularization procedure can be meaningful for more general configurations of determining sets. This Chapter is based on [25].

4.1.2 Determining sets of discrete Laplacian

We consider discrete harmonic functions on two-dimensional lattice $(h\mathbf{Z})^2$, where $h = 1/N$ is the mesh size and N is a positive integer. Let

$$G_h = \{(hi, hj) \mid 0 \leq i \leq N, \quad 0 \leq j \leq N\} \quad (4.3)$$

be the discretization of the square $[0, 1] \times [0, 1]$. There are three types of points in G_h : the set $I_h = \{(hi, hj) \mid 1 \leq i \leq N-1, \quad 1 \leq j \leq N-1\}$ of interior points that consists of all points that have four neighbors in G_h , the set B_h of boundary points that consists of $4(N-1)$ points that have only three neighbors in G_h , and the four vertices of the square that have no interior point as a neighbor.

A function $u : G_h \rightarrow \mathbf{R}$ is said to be discrete harmonic if

$$\Delta_h u(x, y) = 0 \quad \text{for all } (x, y) \in I_h, \quad \text{where} \quad (4.4)$$

$$\Delta_h u(x, y) = \frac{u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x, y)}{h^2}.$$

Since the four vertices of the square $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$ play no role in equation (4.4), we eliminate them from G_h and denote the new set by \tilde{G}_h .

It is well known that a discrete harmonic function is uniquely determined by prescribing its values on the boundary B_h of G_h , where it can attain arbitrary values. The space of discrete harmonic functions on G_h has dimension

$4(N - 1)$. We consider subsets D of \tilde{G}_h consisting of $4(N - 1)$ points. Following [43], we call such a set D a determining set if prescribing the values of a discrete harmonic function on D determines the function uniquely. In other words, a set D of $4(N - 1)$ points in \tilde{G}_h is called a determining set, or a D -set for short, if $\Delta_h u = 0$ on I_h , $u = 0$ on D implies $u = 0$ everywhere on \tilde{G}_h .

The problem of deciding whether a given set is a D -set is equivalent to checking whether the appropriate system of $(N - 1)^2$ homogeneous linear equations has a unique solution. It could also be reduced to the problem of non-singularity of a $4(N - 1) \times 4(N - 1)$ matrix, see [43] for details. We go back to this in the next section when we calculate the stability constants of determining sets.

The first example of a D -set is the boundary set B_h . One can check that replacing one point on the boundary B_h by some point in I_h provides a new D -set. Another simple example of a D -set is the discrete version of our model set Λ_b defined as

$$\Lambda_{b,h} = \Lambda_b \cap \tilde{G}_h,$$

where $1/2 \pm b \in h\mathbf{Z}$. Our special attention to this model set is connected to the origin of determining sets and problems of phase reconstruction, we refer the reader to [43] for details.

The following example of a D -set that is fairly evenly distributed in the domain \tilde{G} is also studied in [43]. Let (hi_0, hj_0) be a point in \tilde{G}_h , we call it a center. Consider cells around this center of the form

$$S_k = \{(hi, hj) : \max(|i - i_0 + j - j_0|, |i - i_0 - j + j_0|) = k\}.$$

Suppose that $C \subset \tilde{G}_h$ contains all four points of S_1 and for $k = 2, \dots, N$ it contains exactly four points of S_k one on each edge, but not the four vertices, i.e.,

$$C \cap S_k \neq \{(hi_0 + hk, hj_0), (hi_0, hj_0 + hk), (hi_0 - hk, hj_0), (hi_0, hj_0 - hk)\}.$$

Then C is a D -set, we call it a centered D -set with center at (hi_0, hj_0) .

Our main aim is to estimate stability of the reconstruction of a discrete harmonic function from its values on a D -set and describe a regularization method for this reconstruction for some specific D -sets.

The remaining part of this Chapter is organized in the following way. In the next section we define stability constants of determining sets and

calculate those constants for model sets. In Section 4.3 we prove logarithmic convexity for discrete harmonic functions adjusted to our model sets and obtain conditional stability. Regularization is discussed in the last Section. We work first with a combination of L^2 and L^∞ norms and use maximum principle to incorporate boundary values on horizontal sides. In the last section we also show that our main result holds for L^2 -norms, some estimates for eigenvalues of the discrete Laplacian on the square are needed.

4.2 Stability Constants of D -sets

4.2.1 Definitions and elementary estimates

We denote the supremum norm of a discrete function u in \tilde{G}_h by

$$\|u\|_{\infty, \tilde{G}_h} = \max_{(x,y) \in \tilde{G}_h} |u(x,y)|.$$

We also consider the standard L^2 -norm of a discrete function in \tilde{G}_h

$$\|u\|_{2, \tilde{G}_h} = \left(\frac{1}{(N+1)^2} \sum_{(x,y) \in \tilde{G}_h} |u(x,y)|^2 \right)^{\frac{1}{2}}.$$

Definition. The (uniform) stability constant of a D -set K is denoted by $s_\infty(K)$ and is defined as

$$s_\infty(K) = \max \left\{ \|u\|_{\infty, \tilde{G}_h}, \quad \Delta_h u = 0 \text{ in } I_h, \quad |u| \leq 1 \text{ on } K \right\}. \quad (4.5)$$

Similarly, the L^2 -stability constant of a D -set K is defined by

$$s_2(K) = \max \left\{ \|u\|_{2, \tilde{G}_h}, \quad \Delta_h u = 0 \text{ in } I_h, \quad |u| \leq 1 \text{ on } K \right\}. \quad (4.6)$$

We remark that by the maximum principle

- (i) $s_2(B_h) = (N+1)^{-1} \sqrt{(N+1)^2 - 4}$ and $s_\infty(B_h) = 1$;
- (ii) $s_\infty(K) \geq s_2(K)$ for any D -set K and $s_\infty(K) \geq s_\infty(B_h)$, $s_2(K) \geq s_2(B_h)$.

The motivation of the definition above is the following straightforward (and naive) estimate. If $\Delta_h u = 0$, $|u - g| < \epsilon$ on K and if u^* is the exact solution of the discrete problem $\Delta_h u^* = 0$ on I_h , and $u^* = g$ on K , then

$$\|u - u^*\|_{\infty, \tilde{G}_h} \leq s_\infty(K)\epsilon, \quad \|u - u^*\|_{2, \tilde{G}_h} \leq s_2(K)\epsilon.$$

Let L_h be the matrix of the discrete Laplace operator on the square \tilde{G}_h , we normalize it such that each row contains one 4, four -1 and zeros. The kernel of L_h is $4(N - 1)$ dimensional. Let $F_1, F_2, \dots, F_{4(N-1)}$ be a basis for the null space of the Laplacian matrix L_h and let F_h be the matrix whose columns are the basis vectors, the rows of F_h correspond to points in \tilde{G}_h . For computational purposes we may either define F_h as a basis for $\text{Null}(L_h)$, since the matrix L_h has a very simple representation in terms of Kronecker tensor products, or we can write down the standard basis explicitly, using discrete harmonic functions of the form

$$u_k(x, y) = \sin \pi k x \cosh a_k y, \quad v_k(x, y) = \sin \pi k x \sinh a_k y,$$

$$u_k^*(x, y) = \sin \pi k y \cosh a_k x, \quad v_k^*(x, y) = \sin \pi k y \sinh a_k x, \quad 1 \leq k \leq N - 1,$$

where a_k is the only positive solution of $\cos \pi h k + \cosh a_k h = 2$.

Let K be a subset of \tilde{G} consisting of $4(N - 1)$ points. The rows of the matrix L_h correspond to the equations of the discrete Laplacian and the columns correspond to the points in \tilde{G}_h . We rearrange the columns of L_h so that the last columns correspond to the points of K and call the new matrix $L_{h,K}$. Then

$$L_{h,K} = [A B],$$

where A is an $(N - 1)^2 \times (N - 1)^2$ matrix and B is an $(N - 1)^2 \times 4(N - 1)$ matrix. We rearrange also the rows of the matrix F_h so that the last $4(N - 1)$ rows correspond to the points in K . The matrix obtained is called $F_{h,K}$. Then the matrix $F_{h,K}$ can be written as two matrices E which is $(N - 1)^2 \times 4(N - 1)$ matrix that corresponds to all the rows in $\tilde{G}_h \setminus K$ and $4(N - 1) \times 4(N - 1)$ matrix R that corresponds to all the rows in the D -set K . That is

$$F_{h,K} = \begin{bmatrix} E \\ R \end{bmatrix}. \quad (4.7)$$

The following statements are equivalent where the details about them can be consulted in [43].

- (i) K is a D -set ;
- (ii) The matrix A is invertible ;
- (iii) The matrix R is invertible.

Definition. The infinity norm of the matrix $M = \{m_{ij}\}$ is defined by

$$\|M\|_\infty = \max_i \sum_j |m_{ij}| = \max\{\|M\mathbf{v}\|_\infty : \|\mathbf{v}\|_\infty \leq 1\}.$$

Similarly, using L^2 -norms of the vectors $\|\mathbf{w}\|_2 = (L^{-1} \sum_j |w_j|^2)^{1/2}$, where L is the number of components of the vector \mathbf{w} , we define

$$\|M\|_2 = \max\{\|M\mathbf{v}\|_2 : \|\mathbf{v}\|_\infty \leq 1\}.$$

We remark that the last norm is not the usual L^2 -operator norm, it is L^∞ to L^2 norm. The reason for the appearance of those norms lies in our choice of the error norm, we assume that the data error (measurement error) is uniformly bounded.

Lemma 7. *The stability constants for a determining set K equal*

$$s_\infty(K) = \|A^{-1}B\|_\infty = \|ER^{-1}\|_\infty \quad \text{and} \quad s_2(K) = \left\| \begin{bmatrix} I \\ A^{-1}B \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} I \\ ER^{-1} \end{bmatrix} \right\|_2.$$

The lemma follows readily from the definition. The first formula gives a (rather pessimistic) bound for the stability constant.

Corollary. *If $K \subset \tilde{G}_h$ is a D -set, then $s_\infty(K) \leq 4(N-1)^2 8^{(N-1)^2}$.*

Proof. We note that $\|A^{-1}B\|_\infty \leq \|A^{-1}\|_\infty \|B\|_\infty$ and $\|B\|_\infty \leq 8$. To estimate $\|A^{-1}\|_\infty$ we note that entries of A are integers and $|\det(A)| \geq 1$ since $\det(A) \neq 0$. We estimate the cofactor A_{ij} of A . Remind that the matrix contains one 4 in each row, four -1 , and zeros for the remaining entries. Then by induction $|A_{ij}| \leq 4 \cdot 8^{q-2}$, where q is the size of the matrix A . Then $\|A^{-1}\|_\infty \leq 4q \cdot 8^{q-2}$. \square

We don't know how precise the estimate above is and what the geometry of the worst configuration of a D -set could be.

4.2.2 Stability constants for model D -sets

In this section we study model D -sets like $\Lambda_{b,h}$ and estimate their stability constants. We show that the stability constants grow exponentially with N . Thus the approximation estimates we get by solving exact linear systems with noisy data are relatively poor. A regularization method is discussed in the next section.

We start by looking at the extremal case when b is as small as possible, we assume that $N = 2n + 1$ is odd and $b = 1/2N$. Let $\Lambda_{*,h} = \Lambda_{b,h}$. Clearly, this set is obtained from the boundary B_h by replacing all the points on the sides $(x, 0)$ and $(x, 1)$ by interior points (x, hn) and $(x, h(n+1))$ respectively. It is clear that $\Lambda_{*,h}$ is a D -set. We want to estimate $s_\infty(\Lambda_{*,h})$ and $s_2(\Lambda_{*,h})$.

Definition. Let the sequence $\{a_k\}$ be defined by $a_0 = 1$, $a_1 = 7$ and

$$a_{k+1} = 6a_k - a_{k-1}, \quad k \geq 1. \quad (4.8)$$

Then the k^{th} term of the sequence is given by the formula

$$a_k = c_1 \lambda_1^k + c_2 \lambda_2^k,$$

where λ_1 and λ_2 are the roots of the equation

$$\lambda^2 - 6\lambda + 1 = 0.$$

Solving for c_1 and c_2 and using the values of a_0 and a_1 we get

$$a_k = \left(\frac{1 + \sqrt{2}}{2} \right) \left(3 + 2\sqrt{2} \right)^k + \left(\frac{1 - \sqrt{2}}{2} \right) \left(3 - 2\sqrt{2} \right)^k.$$

Proposition 3. *The uniform stability constant for the set $\Lambda_{*,h}$ equals*

$$s_\infty(\Lambda_{*,h}) = a_n, \quad \text{where } h^{-1} = N = 2n + 1.$$

Proof. Let u be a discrete harmonic function on \tilde{G}_h such that $|u| \leq 1$ on $\Lambda_{*,h}$. Then we have

$$|u(hi, hn)| \leq 1, \quad |u(hi, h(n+1))| \leq 1 \text{ and also } |u(0, hj)| \leq 1, \quad |u(1, hj)| \leq 1,$$

for all $1 \leq i, j \leq N - 1$. From $\Delta_h u = 0$ we also know that

$$\begin{aligned} u(x, h(n+2)) - u(x, h(n+1)) = \\ 3u(x, h(n+1)) - u(x-h, h(n+1)) - u(x+h, h(n+1)) - u(x, hn). \end{aligned}$$

Therefore for any $x = hi$, $1 \leq i \leq N - 1$ we have

$$|u(x, h(n+2)) - u(x, h(n+1))| \leq 6. \quad (4.9)$$

Now we want to show that for any $x = hi$, $1 \leq i \leq N - 1$, and $k = 1, \dots, n$

$$|u(x, h(n+k+1)) - u(x, h(n+k))| \leq a_k - a_{k-1} \text{ by induction.} \quad (4.10)$$

If $k = 1$, then we get (4.9). Assume that (4.10) holds for any $k < K$. Then also

$$|u(x, h(n+k+1))| \leq a_k, \quad \text{for } k < K. \quad (4.11)$$

Therefore

$$\begin{aligned} & |u(x, h(n+K+1)) - u(x, h(n+K))| \leq \\ & \quad |u(x, h(n+K)) - u(x, h(n+K-1))| + \\ & \quad 2|u(x, h(n+K))| + |u(x-h, h(n+K))| + |u(x+h, h(n+K))| \\ & \leq a_{K-1} - a_{K-2} + 4a_{K-1} = 5a_{K-1} - a_{K-2} = a_K - a_{K-1}. \end{aligned}$$

On the other hand, an example shows that if a discrete harmonic function has alternative ± 1 values at the points $(0, hj)$, $(1, hj)$, (hi, hn) , and $(hi, h(n+1))$, then

$$\begin{aligned} |u(hi, hn)| &= |u(hi, h(n+1))| = 1, \quad \text{for } 0 \leq i \leq N, \\ |u(hi, h(n+2))| &= a_1, \quad \text{for } 1 \leq i \leq N-1, \\ |u(hi, h(n+3))| &= a_2, \quad \text{for } 2 \leq i \leq N-2 \\ &\vdots \\ |u(hi, h(2n+1))| &= a_n \quad \text{for } i = n, n+1. \end{aligned}$$

Therefore combining all the above inequalities we obtain $s_\infty(\Lambda_{*,h}) = a_n$. \square

Corollary. *The L^2 -stability constant for $\Lambda_{*,h}$ satisfies the inequalities*

$$a_n(N+1)^{-1} \leq s_2(\Lambda_{*,h}) \leq a_n.$$

The first inequality follows from the example described above while the second one is obtained by comparing L^2 and L^∞ norms. The corollary implies that $s_2(\Lambda_{*,h})$ grows exponentially in N .

Our first model set $\Lambda_{*,h}$ is an example of a set for which we have a simple formula for the uniform stability constant. Measurements of the function on this set correspond to measurements of u and its partial derivative u_y on the middle horizontal line of the square. For this case we should choose the norm for the derivative more carefully. We refer the reader to [21, 42] for estimates of the solutions of discrete Cauchy problem. We do similar but more elementary estimates for model sets $\Lambda_{b,h}$ with $0 < b < \frac{1}{2}$ in the next sections. First we show that the stability constants still grow exponentially.

Along with the sequence $\{a_k\}$ defined by (4.8) we consider another sequence $\{c_k\}$ defined by $c_0 = 1$, $c_1 = 5$ and

$$c_{k+1} = 6c_k - c_{k-1}.$$

Proposition 4. *Let $\Lambda_{b,h} \subset \tilde{G}_h$, where $h = 1/N$ and $1/2 \pm b \in h\mathbf{Z}$. We define $l = N(1/2 - b) \in \mathbf{Z}_+$. Then the following estimates hold.*

$$c_l \leq s_\infty(\Lambda_{b,h}) \leq a_l, \quad c_l(N+1)^{-1} \leq s_2(\Lambda_{b,h}) \leq a_l.$$

Before we sketch the proof, which is similar to that of Proposition 3, we note that

$$c_k = \left(\frac{2 + \sqrt{2}}{4} \right) (3 + 2\sqrt{2})^k + \left(\frac{2 - \sqrt{2}}{4} \right) (3 - 2\sqrt{2})^k. \quad (4.12)$$

Thus we obtain

$$(3 + 2\sqrt{2})^{N(1/2-b)} \approx s_\infty(\Lambda_{b,h}) \geq s_2(\Lambda_{b,h}) \geq cN^{-1}(3 + 2\sqrt{2})^{N(1/2-b)}.$$

Proof. Let u be a discrete harmonic function on \tilde{G} such that $|u| \leq 1$ on $\Lambda_{b,h}$ and let also $1/2 + b = s/N$. Then by the maximum principle we have

$$|u(x, h(s-1))| \leq 1, \quad |u(x, hs)| \leq 1,$$

when $x = ih$, $0 \leq i \leq N$. As above we show by induction that $|u(x, h(s+j))| \leq a_j$ it gives the estimate $s_\infty(\Lambda_{b,h}) \leq a_l$, where $l = N - s = N(1/2 - b)$. To estimate $s_\infty(\Lambda_{b,h})$ from below we consider once again a discrete harmonic function u that takes values ± 1 on $\Lambda_{b,h}$ with alternating signs. We have

$$|u(x, hs)| = 1 = c_0, \quad |u(x, h(s-1))| \leq 1, \quad x = hi, \quad 0 \leq i \leq N.$$

We claim that $u(x, h(s+j)) - u(x, h(s+j-1))$ has the same sign as $u(x, hs)$ when $j \geq 0$, $x = hi$, $j \leq i \leq N - j$ and

$$|u(x, h(s+j)) - u(x, h(s+j-1))| \geq c_j - c_{j-1}, \quad |u(x, h(s+j))| \geq c_j.$$

For $j = 1$ we get

$$u(x, h(s+1)) - u(x, hs) = 3u(x, hs) - u(x+h, hs) - u(x-h, hs) - u(x, h(s-1)) = 5u(x, hs) - u(x, h(s-1)).$$

Then $u(x, h(s+1)) - u(x, hs)$ has the same sign as $u(x, hs)$ and $|u(x, h(s+1)) - u(x, hs)| \geq 4$. Clearly, $|u(x, h(s+1))| \geq 5$. Then, by induction we obtain

$$u(x, h(s+j)) - u(x, h(s+j-1)) = (u(x, h(s+j-1)) - u(x, h(s+j-2))) + 2u(x, h(s+j-1)) + (-u(x+h, h(s+j-1)) + (-u(x-h, h(s+j-1))),$$

all four summands have the same sign and

$$|u(x, h(s+j)) - u(x, h(s+j-1))| \geq c_j - c_{j-1} + 4c_j = c_{j+1} - c_j,$$

from which the estimate for $s_2(\Lambda_{b,h})$ follows. \square

Another series of examples of D -sets with exponentially growing stability constants is given by centered D -sets. Let us also note that while boundary values of a harmonic functions are arbitrary, its values on a subset inside the domain are restrictions of real analytic functions and it is not surprising that the problem of reconstruction is very unstable.

The exponential growth of the stability constants implies that using the naive approach with data error of order ϵ we get approximation error for the discrete problem of order ϵA^N . If we want any estimate for the approximation that goes to zero when ϵ goes to zero, we should choose $N \leq C|\log \epsilon|$. However, the discretization error (that should be added to ϵ) is of order h^2 (see the discussion in the Section 4.4.1). Thus our final approximation estimate blows up.

To improve this scheme we first prove some stability estimates under a priori bounds and then use a regularization method. Instead of solving the linear system exactly we consider approximate solution that minimizes some norm. It gives good estimates with a priori boundedness of solution (conditional stability). For general surveys of regularization techniques in discrete problems we refer to [27, 39].

4.3 Conditional stability

4.3.1 Statements

In this section we consider conditional stability estimates for the set $\Lambda_{b,h}$, where $N = h^{-1}$ and $1/2 \pm b \in h\mathbf{Z}$. Let w be an h -discrete harmonic function. We want to estimate the L^2 -norm of w over the square, assuming that w is small on $\Lambda_{b,h}$ and bounded by some constant uniformly. After a simple reduction we suppose that w vanishes on the vertical part of the boundary of the square and prove the logarithmic convexity estimates for the norms of the function w over horizontal segments. The estimates are similar to those obtained in [21], see also [42]. We use here a slightly different approach separating odd and even parts of the function. Further, we get better estimates for norms over proper subdomains.

Let V_h be the points on the vertical sides of the square \tilde{G}_h and $L_{b,h}$ be the horizontal part of the set $\Lambda_{b,h}$ such that $\Lambda_{b,h} = V_h \cup L_{b,h}$ is a disjoint union.

Theorem 18. *Suppose that N is odd and $b \in (0, 1/2)$ is such that $N/2 + Nb \in \mathbf{Z}$. Let w be a discrete harmonic function on \tilde{G}_h that satisfies $|w| \leq B$ and $\sup_{V_h} |w| \leq \epsilon$. Let also*

$$2\delta^2 = \|w\|_{2,L_{b,h}}^2 = \left(\frac{1}{N} \sum_{j=0}^N |w(hj, 1/2 - b)|^2 \right) + \left(\frac{1}{N} \sum_{j=0}^N |w(hj, 1/2 + b)|^2 \right).$$

Then

$$\|w\|_{2,\tilde{G}_h}^2 \leq C \left(\frac{(B + \epsilon)^2 + (\delta + \epsilon)^2}{2 \log((B + \epsilon)/(\delta + \epsilon)) - A} + (B + \epsilon)^2 h + \epsilon^2 \right), \quad (4.13)$$

where C and A are some constants that depend on b but not on h .

Clearly the right-hand side of (4.13) goes to zero as $\epsilon + \delta \rightarrow 0$ and $h \rightarrow 0$. We obtain also the following interior estimate.

Theorem 19. *Let w satisfy the assumptions of the theorem and let*

$$P_{h,t} = \{(hj, hk) : j, k \in \mathbf{Z}, hj \in [0, 1], hk \in [t, 1 - t]\},$$

where $0 < t < 1/2 - b$, $t \in h\mathbf{Z}$. Then

$$\|w\|_{2,P_{h,t}} \leq C(B + \epsilon)^{1-\alpha}(\delta + \epsilon)^\alpha, \quad (4.14)$$

where C depends on b and $\alpha = t(1/2 - b)^{-1}$.

In order to prove (4.13) and (4.14) we first reduce the inequality to the case when $\epsilon = 0$. We express w as the sum of two functions w_1 and w_2 , where w_1 is the solution of the discrete Dirichlet Problem

$$\begin{cases} \Delta_h w_1 = 0 \\ w_1|_{V_h} = w \\ w_1|_{H_h} = 0. \end{cases}$$

Here H_h denotes the set of all points on the horizontal sides of the square \tilde{G}_h .

By the maximum principle $\|w_1\|_{\infty, \tilde{G}_h} \leq \epsilon$. Therefore we have

$$\|w_2\|_{\infty, \tilde{G}_h} = \|w - w_1\|_{\infty, \tilde{G}_h} \leq \|w\|_{\infty, \tilde{G}_h} + \|w_1\|_{\infty, \tilde{G}_h} \leq B + \epsilon.$$

Moreover, $w_2(x, y) = 0$ on V_h and

$$\begin{aligned} \|w_2\|_{2, L_{b,h}} &= \\ \|w - w_1\|_{2, L_{b,h}} &\leq \|w\|_{2, L_{b,h}} + \|w_1\|_{2, L_{b,h}} \leq (2\delta^2)^{\frac{1}{2}} + (2\epsilon^2)^{\frac{1}{2}} = \sqrt{2}(\delta + \epsilon). \end{aligned}$$

Then we obtain

$$\|w_2\|_{2, L_{b,h}}^2 \leq 2(\delta + \epsilon)^2.$$

We prove (4.13) for the case $\epsilon = 0$ in the next subsection. It gives the following estimate for w_2

$$\|w_2\|_{2, \tilde{G}_h}^2 \leq C \left(\frac{(B + \epsilon)^2 + (\delta + \epsilon)^2}{2 \log((B + \epsilon)/(\delta + \epsilon)) - A} + (B + \epsilon)^2 h \right).$$

Then, since $\|w\|_{2, \tilde{G}_h}^2 \leq \|w_2\|_{2, \tilde{G}_h}^2 + \|w_1\|_{2, \tilde{G}_h}^2 \leq \|w_2\|_{2, \tilde{G}_h}^2 + \epsilon$, (4.13) follows. Similarly, it suffices to prove (4.14) for the case $\epsilon = 0$.

4.3.2 Logarithmic convexity

From now on till the end of this section we assume that $|w| \leq B$, $w = 0$ on V_h and

$$2\delta^2 = \|w\|_{2, L_{b,h}}^2 = \left(\frac{1}{N} \sum_j |w(hj, 1/2 - b)|^2 \right) + \left(\frac{1}{N} \sum_j |w(hj, 1/2 + b)|^2 \right).$$

We divide w into two parts as

$$w_e(x, y) = (w(x, y) + w(x, 1 - y))/2, \text{ and } w_o(x, y) = (w(x, y) - w(x, 1 - y))/2.$$

Let $N = 2n + 1$, for $k = 1, 2, \dots, n + 1$ we define the vectors

$$\mathbf{w}_e^k = \{ w_e(h, h(n+k)), w_e(2h, h(n+k)), \dots, w_e((N-1)h, h(n+k)) \} \text{ and}$$

$$\mathbf{w}_o^k = \{ w_o(h, h(n+k)), w_o(2h, h(n+k)), \dots, w_o((N-1)h, h(n+k)) \}.$$

Our aim is to estimate the L^2 - norm of $\mathbf{w}^k = \mathbf{w}_o^k + \mathbf{w}_e^k$. We use the notation

$$(m_k^e)^2 = \|\mathbf{w}_e^k\|_2^2 = \frac{1}{N} \sum_{i=1}^{N-1} |w_e(hi, h(n+k))|^2$$

and similarly $(m_k^o)^2 = \|\mathbf{w}_o^k\|_2^2$. Since w_e and w_o are discrete harmonic functions, we have

$$\mathbf{w}_e^{k+1} = 2\mathbf{w}_e^k - \mathbf{w}_e^{k-1} + L\mathbf{w}_e^k, \quad \mathbf{w}_o^{k+1} = 2\mathbf{w}_o^k - \mathbf{w}_o^{k-1} + L\mathbf{w}_o^k,$$

where L is $(N-1) \times (N-1)$ symmetric, tridiagonal matrix given by

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 & 0 \end{pmatrix}.$$

Using the standard basis $\{\mathbf{v}_j\}$ of normalized eigenvectors of L (see above for our choice of normalization), $\mathbf{v}_j = \{\sqrt{2} \sin(\pi j l / N)\}_{l=1,2,\dots,N-1}$, and by the symmetry properties of w_e and w_o , we obtain

$$\mathbf{w}_e^k = \sum_{j=1}^{N-1} c_j^e \cosh a_j h k \mathbf{v}_j, \quad \mathbf{w}_o^k = \sum_{j=1}^{N-1} c_j^o \sinh a_j h k \mathbf{v}_j,$$

where $\cosh(a_j h) = 2 - \cos(\pi h j)$. We have $a_1 \leq a_2 \leq \dots \leq a_{N-1}$ and $a_1 = a_1(h) \geq a$, where a is an absolute constant. Clearly, $\lim_{h \rightarrow 0} a_1(h) = \pi$.

We remind that a (discrete) function $F : \{1, 2, \dots, L\} \rightarrow \mathbf{R}_+$ is called logarithmically convex if

$$F(j)^2 \leq F(j-1)F(j+1)$$

for $j = 2, \dots, L-1$. It is a standard exercise to check that this inequality implies

$$F(j_1)^{j_2-j_0} \leq F(j_0)^{j_2-j_1} F(j_2)^{j_1-j_0}, \quad 1 \leq j_0 \leq j_1 \leq j_2 \leq L.$$

Setting $b = (2l-1)/2N$ we will first consider the ‘‘even part’’ w_e . We have $m_l^e \leq \delta$ and $m_{n+1}^e \leq B$. Further, we also have

$$(m_k^e)^2 = \sum_j (c_j^e \cosh a_j h k)^2.$$

Note that m_k^e is obviously an increasing function in k . Furthermore, $(m_k^e)^2$ is a logarithmically convex function of k since it is a sum of positive logarithmically convex functions. Thus for $l \leq k \leq n+1$, it holds that

$$(m_k^e)^{n+1-l} \leq (m_l^e)^{(n+1-k)} B^{(k-l)}.$$

We rewrite the above inequality as

$$(m_{j+l}^e)^2 \leq (m_l^e)^2 q^{j/(n+1-l)}, \quad j = 0, \dots, n+1-l,$$

where $q = \left(\frac{B}{m_l^e}\right)^2$. Then we obtain

$$\begin{aligned} \|w_e\|_{2, \tilde{G}_h}^2 &= \frac{2N}{(N+1)^2} \sum_{k=1}^{n+1} (m_k^e)^2 = \frac{2N}{(N+1)^2} \left[\sum_{k=1}^{l-1} (m_k^e)^2 + \sum_{k=l}^{n+1} (m_k^e)^2 \right] \leq \\ &\quad \frac{2}{N} \left[(l-1) (m_l^e)^2 + \sum_{j=0}^{n-l} (m_{j+l}^e)^2 + (m_{n+1}^e)^2 \right]. \end{aligned}$$

The middle sum can be estimated by the following integral (a similar estimate can be found in [42]),

$$\begin{aligned} \sum_{j=0}^{n-l} (m_{j+l}^e)^2 &\leq \\ (m_l^e)^2 \sum_{j=0}^{n-l} q^{j/(n+1-l)} &\leq (n+1-l) (m_l^e)^2 \int_0^1 q^t dt = (n+1-l) (m_l^e)^2 \frac{q-1}{\log q}. \end{aligned}$$

Therefore

$$\|w_e\|_{2, \tilde{G}_h}^2 \leq \frac{2}{N} \left[(l-1)\delta^2 + (n+1-l) \frac{B^2}{2 \log(B/\delta)} + B^2 \right].$$

To finish the proof we should repeat the calculations for w_o . We have that

$$(m_k^o)^2 = \sum_j (c_j^o \sinh a_j h k)^2$$

is increasing in k but not logarithmically convex. However,

$$(m_k^o)^2 + \frac{1}{2} \sum_j (c_j^o)^2$$

is logarithmically convex. Note also that

$$\delta^2 \geq (m_l^o)^2 \geq \sinh a_1 h l \sum_j (c_j^o)^2 \geq \sinh ab \sum_j (c_j^o)^2,$$

from which we get $\frac{1}{2} \sum_j (c_j^o)^2 \leq C_b \delta^2$, where C_b depends only on b . As above we write

$$\begin{aligned} \|w_o\|_{2, \tilde{G}_h}^2 &= \frac{2N}{(N+1)^2} \sum_{k=1}^{n+1} (m_k^o)^2 = \frac{2N}{(N+1)^2} \left[\sum_{k=1}^{l-1} (m_k^o)^2 + \sum_{k=l}^{n+1} (m_k^o)^2 \right] \leq \\ &\frac{2}{N} \left[(l-1) (m_l^o)^2 + \sum_{j=0}^{n-l} (m_{j+l}^o)^2 + (m_{n+1}^o)^2 \right]. \end{aligned}$$

Then the middle sum can be estimated by the logarithmic convexity

$$\sum_{j=0}^{n-l} (m_{j+l}^o)^2 \leq \sum_{j=0}^{n-l} \left((m_{j+l}^o)^2 + C_b \delta^2 \right) \leq [(m_l^o)^2 + C_b \delta^2] \sum_{j=0}^{n-l} \tilde{q}^{\frac{j}{n+1-l}},$$

where $\tilde{q} = \frac{B^2 + C_b \delta^2}{\delta^2 + C_b \delta^2}$. Then arguing as before we obtain,

$$\sum_{j=0}^{n-l} (m_{j+l}^o)^2 \leq (n+1-l) \frac{B^2 + C_b \delta^2}{2 \log(B/\delta) - \log(1 + C_b)}.$$

Therefore we get

$$\|w_o\|_{2,\tilde{G}_h}^2 \leq \frac{4}{N} \left[(l-1)\delta^2 + (n+1-l) \frac{B^2 + C_b\delta^2}{2\log(B/\delta) - \log(1+C_b)} + B^2 \right].$$

Finally, combining inequalities for even and odd parts and taking C and A large enough, we obtain

$$\|w\|_{2,\tilde{G}_h}^2 \leq C \left(\frac{B^2 + \delta^2}{2\log(B/\delta) - A} + B^2 h \right),$$

from which (4.13) readily follows.

4.3.3 Interior estimate

Now we prove (4.14). We use the notations and techniques of the previous computations. To estimate the norm over a subrectangle $[0, 1] \times [t, 1-t]$ we let $1-t = (n+l+s)h$, $s < n+1-l = p$ and consider

$$\begin{aligned} \frac{2}{l+s} \sum_{k=1}^{l+s} (m_k^\epsilon)^2 &\leq \frac{2}{l+s} (m_l^\epsilon)^2 \left(l + \sum_{j=0}^s q^{j/p} \right) \leq 2 (m_l^\epsilon)^2 q^{s/p} \\ &= 2 (m_l^\epsilon)^2 \left(\frac{B^2}{(m_l^\epsilon)^2} \right)^{s/p} = 2\delta^{2\alpha} B^{2(1-\alpha)}, \end{aligned}$$

where $\alpha = 1 - s/p = t(1/2 - b)^{-1}$. Thus

$$\|w_e\|_{2,P_{h,t}}^2 \leq 4B^{2-2\alpha}\delta^{2\alpha}.$$

Similarly, for the odd part we have

$$\begin{aligned} \|w_o\|_{2,P_{h,t}}^2 &= \frac{2}{l+s} \sum_{k=1}^{l+s} (m_k^o)^2 = \frac{2}{l+s} \left[\sum_{k=1}^l (m_k^o)^2 + \sum_{j=0}^s (m_j^o)^2 \right] \\ &\leq \frac{2}{l+s} \left[l(m_l^o)^2 + \sum_{j=0}^s ((m_l^o)^2 + C_b\delta^2) \tilde{q}^{j/p} \right] \leq \\ &\frac{2}{l+s} [l\delta^2 + (\delta^2 + C_b\delta^2) s] \tilde{q}^{s/p} = \frac{2}{l+s} \delta^2 (l+s + sC_b) \left(\frac{B^2 + C_b\delta^2}{\delta^2 + C_b\delta^2} \right)^{s/p} \leq \\ &\frac{2}{l+s} \delta^2 (l+s + sC_b) \left(\frac{2B^2}{\delta^2} \right)^{s/p} \leq C'_b B^{2\frac{s}{p}} \delta^{2(1-s/p)} = C'_b B^{2-2\alpha} \delta^{2\alpha}, \end{aligned}$$

where C'_b now depends on b . Finally, we have $\|w\|_{2,P_{h,t}}^2 \leq C''_b B^{2-2\alpha} \delta^{2\alpha}$ and (4.14) follows.

4.4 Regularization

4.4.1 Discretization error

We assume that U is a (continuous) harmonic function on $[0, 1] \times [0, 1]$ which is (at least) α -Lipschitz up to the boundary for some $0 < \alpha < 1$, i.e.

$$|U(x_1, y_1) - U(x_2, y_2)| \leq L \left((x_1 - x_2)^2 + (y_1 - y_2)^2 \right)^{\alpha/2},$$

and bounded by some constant M . For any $h = 1/N$ we define the discretization u^h of U as the solution of the discrete Dirichlet problem on \tilde{G}_h which coincide with U at the boundary B_h . This is a simple example of discretization of the Dirichlet problem that goes back to classical works of Philips and Wiener (1923) and Courant, Friedrichs and Lewy (1928). The discrete Laplace equation appears also in the finite element method for continuous Laplacian and various results on the approximation rate (depending on the smoothness of the solution) are available. The standard estimate (see for example [44]) says that the difference between U and u^h is of order h^2 provided that U has bounded fourth order derivatives. We don't assume such smoothness but only α -Lipschitz condition on the boundary. One of the early results on the uniform convergence rate for this case is due Walsh and Young, [49] and it gives error of order $h^{2\alpha/(\alpha+3)}$ when $\alpha \in (0, 1)$. We assume that we have some error function $e(h)$ such that

$$\|U - u^h\|_{\infty} \leq e(h).$$

So if we for example know that $U \in C^4$ on the closed square and a bound for the derivatives is known, then $e(h) = Ch^2$. If U is assumed to satisfy only α -Lipschitz condition with a bound for the Lipschitz constant; then we take $e(h) = Ch^\beta$ for some $\beta > 0$.

4.4.2 Convergence rate

Let U be a harmonic function on $[0, 1] \times [0, 1]$. We assume that the values of U are available on the set Λ_b with error ϵ . To reconstruct the function U we

first choose $h = h(\epsilon)$, that may also depend on the smoothness of U . Then we measure the data g^h on $\Lambda_{b,h}$, and solve the following finite-dimensional extremal problem

$$\begin{cases} \Delta_h v^h = 0, \\ \|v^h - g^h\|_{\infty, V_h} \leq \epsilon, \\ \|v^h - g^h\|_{2, L_{b,h}} \leq \epsilon + e(h), \\ \|v^h\|_{\infty, \tilde{G}_h} \rightarrow \min, \end{cases} \quad (4.15)$$

where V_h is the vertical part of $\Lambda_{b,h}$ and $L_{b,h}$ is its horizontal part.

Theorem 20. *Let U be a bounded harmonic function, $|U| \leq M$ on $[0, 1] \times [0, 1]$, α -Lipschitz up to the boundary for some $0 < \alpha < 1$, and let g^h be a function on $\Lambda_{b,h}$ such that*

$$\|g^h - U\|_{\infty, \Lambda_{b,h}} \leq \epsilon.$$

We define v^h to be the solution of (4.15). Let $\epsilon_0 = \min\{\epsilon, e(h)\}$ and suppose that ϵ_0/M is small enough. Then the estimate

$$\|U - v^h\|_{2, \tilde{G}_h} \leq CM (\log(M/\epsilon_0) - A)^{-1/2} \quad (4.16)$$

holds where U is restricted to the lattice $(h\mathbf{Z})^2 \cap [0, 1]^2$ and C and A depend on b .

Furthermore, for $t \in (0, 1/2 - b)$ we have

$$\|U - v^h\|_{2, P_{h,t}} \leq CM^{1-\alpha} \epsilon_0^\alpha, \quad (4.17)$$

where C depends on b and $\alpha = t(1/2 - b)^{-1}$.

Proof. First, we note that u^h satisfies the first three equations of (4.15) and by the maximum principle $\|u^h\|_{\infty} \leq M$; then $\|v^h\|_{\infty} \leq M$.

We know also that $\|u^h - U\|_{\infty} \leq e(h)$ and we want to estimate $\|u^h - v^h\|_{2, \tilde{G}_h}$. Let $w = u^h - v^h$, we have $\|w\|_{\infty} \leq 2M$,

$$\|w\|_{\infty, V_h} \leq \|u^h - g^h\|_{\infty, V_h} + \|g^h - v^h\|_{\infty, V_h} \leq 2\epsilon, \quad \text{and}$$

$$\|w\|_{2, L_{b,h}} \leq \|u^h - U\|_{2, L_{b,h}} + \|U - g^h\|_{2, L_{b,h}} + \|g^h - v^h\|_{2, L_{b,h}} \leq 2(\epsilon(h) + \epsilon).$$

By Theorem 18 we obtain

$$\|u^h - v^h\|_{2, \tilde{G}_h}^2 \leq C \left(\frac{M^2 + e(h)^2 + \epsilon^2}{2 \log((M + \epsilon)/(\epsilon(h) + \epsilon)) - A} + M^2 h + \epsilon^2 \right).$$

For ϵ_0/M small enough we get (4.16). Similarly, applying Theorem 19, we get (4.17). \square

The convergence errors suggest that we choose h such that $e(h)$ is comparable to ϵ , for our regularity assumptions it means that $\log 1/h \asymp \log 1/\epsilon$. Then we obtain $\|U - v^h\|_2 \leq C_{b,M}(|\log \epsilon|)^{-1/2}$ for ϵ small enough and $\|U - v^h\|_{2,P_{h,t}} \leq C_{b,M}\epsilon^\alpha$. The last estimate and standard elliptic type arguments imply a power-type estimate for supremum norm in proper subdomains.

4.4.3 L^2 -norms

We choose to work (partly) with the supremum norms in (4.15) since they describe well the measurement error and give a clear argument for our regularization procedure. However, in practical computations L^2 -norms have certain advantages and the aim of this subsection is to show that our main results (with some minor changes) can be obtained when dealing with L^2 -norms consistently. We consider now the following problem.

$$\begin{cases} \Delta_h v^h = 0, \\ \|v^h - g^h\|_{2,\Lambda_{b,h}} \leq \epsilon + e(h), \\ \|v^h\|_{2,H_h} \rightarrow \min, \end{cases} \quad (4.18)$$

where H_h is the set of points on the horizontal sides of the square. Then we state the following version of Theorem 20.

Theorem 21. *Let U be a bounded harmonic function, $|U| \leq M$ on $[0, 1] \times [0, 1]$, α -Lipschitz up to the boundary for some $0 < \alpha < 1$, and let g^h be a function on $\Lambda_{h,b}$ such that*

$$\|g^h - U\|_{2,\Lambda_{b,h}} \leq \epsilon.$$

We define v^h to be the solution of (4.18). Let $\epsilon_0 = \min\{\epsilon, e(h)\}$ and suppose that $\epsilon_0 \log 1/h M^{-1}$ is small enough. Then the estimate

$$\|U - v^h\|_{2,\tilde{G}_h} \leq CM (\log(M/\epsilon_0 \log 1/h) - A)^{-1/2} \quad (4.19)$$

holds where U is restricted to the lattice $(h\mathbf{Z})^2 \cap [0, 1]^2$ and C and A depend on b .

Moreover, for $t \in (0, 1/2 - b)$ we also have

$$\|U - v^h\|_{2,P_{h,t}} \leq CM^{1-\alpha}(\epsilon_0 \log 1/h)^\alpha, \quad (4.20)$$

where C depends on b and $\alpha = t(1/2 - b)^{-1}$.

The proof is similar to that of the proof of Theorem 20, but instead of the maximum principle which allowed us to estimate the function w_1 in Section 4.3.1 we apply the following estimate.

Lemma 8. *Suppose that u is a discrete harmonic function on \tilde{G}_h and $u|_{H_h} = 0$. Then $\|u\|_{2,L_{b,h}} \leq c_0 \log 1/h \|u\|_{2,V_h}$, where c_0 is an absolute constant.*

If we compare to the case of supremum norms, we loose $\log 1/h$ in the norm (as a multiplicative factor), but since h is chosen in such a way that $\log 1/h \asymp \log 1/\epsilon$, we still have $\epsilon_1 = \epsilon \log 1/\epsilon$ going to 0 as ϵ goes to zero, so we don't loose much in the final estimate.

Proof. We may assume that $u(0, hj) = 0$ for $j = 0, 1, \dots, N$ and that $u(1, hj) = \mathbf{u}$ is an $(N - 1)$ -vector with

$$\|\mathbf{u}\|_2^2 = N^{-1} \sum_{j=1}^{N-1} |u(1, hj)|^2 = d^2.$$

As in Section 4.3.2, we consider the matrix L and its eigenvectors $\mathbf{v}_j = \{\sqrt{2} \sin(\pi j l / N)\}_{l=1}^{N-1}$. We have $\mathbf{u} = \sum_j c_j \mathbf{v}_j$ and $\sum_j c_j^2 = d^2$. Then

$$u(hk, hl) = \sum_{j=1}^{N-1} c_j \frac{\sinh a_j hk}{\sinh a_j} \sqrt{2} \sin \pi j hl,$$

where a_j is the only positive solution of $\cosh a_j h = 2 - \cos \pi h j$.

We define the function $\alpha : [0, \pi] \rightarrow [0, \infty]$ by

$$\cosh \alpha(t) = 2 - \cos t.$$

It is well defined since $2 - \cos t \geq 1$, $\alpha(0) = 0$ and

$$\alpha'(t) = \frac{\sin(t)}{\sinh \alpha(t)} = \frac{2 \sin(\frac{t}{2}) \cos(\frac{t}{2})}{\sqrt{(2 - \cos(t))^2 - 1}} = \frac{\cos(\frac{t}{2})}{\sqrt{1 + \sin^2(\frac{t}{2})}} > 0.$$

for all $t \in [0, \pi)$ and $\alpha'(t) \geq \frac{1}{\sqrt{3}}$, $t \in [0, \frac{\pi}{2}]$. Therefore

$$\alpha(t) \geq \frac{1}{\sqrt{3}} t, \quad 0 \leq t \leq \frac{\pi}{2}. \quad (4.21)$$

We note also that from the Taylor expansion of hyperbolic cosine and cosine functions we get $\alpha(t) = t + o(t^3)$. Thus we have

$$\lim_{t \rightarrow 0} \frac{\alpha(t)}{t} = 1. \quad (4.22)$$

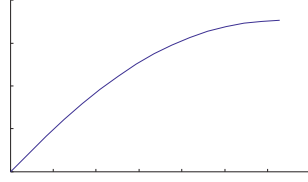


Figure 4.2: Graph of $\alpha(t)$

We have $a_j = h^{-1}\alpha(\pi jh)$, then $1/\sqrt{3} \leq a_1 \leq a_2 \leq \dots \leq a_{N-1}$. We estimate the L^2 -norm of u over an arbitrary horizontal segment in $[0, 1] \times [0, 1]$, applying the Cauchy–Schwarz inequality

$$\begin{aligned} N^{-1} \sum_{k=1}^{N-1} |u(hk, hl)|^2 &= 2N^{-1} \sum_{k=1}^{N-1} \left(\sum_{j=1}^{N-1} c_j \frac{\sinh a_j hk}{\sinh a_j} \sin \pi jhl \right)^2 \leq \\ &2N^{-1} \sum_{k=1}^{N-1} \left(\sum_{j=1}^{N-1} c_j^2 \sin^2 \pi jhl \right) \left(\sum_{j=1}^{N-1} \frac{\sinh^2 a_j hk}{\sinh^2 a_j} \right) \leq \\ &2N^{-1} \sum_{j=1}^{N-1} c_j^2 \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \frac{\sinh^2 a_j hk}{\sinh^2 a_j} \leq 2d^2 N^{-1} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \frac{\sinh^2 a_j hk}{\sinh^2 a_j}. \end{aligned}$$

Now an elementary estimate gives for $a > a_0$

$$\sum_{k=1}^{N-1} \frac{\sinh^2 ahk}{\sinh^2 a} \leq c(ah)^{-1} e^{-ha}.$$

Thus we obtain

$$N^{-1} \sum_{k=1}^{N-1} |u(hk, hl)|^2 \leq 2cd^2 \sum_{j=1}^{N-1} \frac{e^{-ha_j}}{a_j}.$$

Finally, $a_j = h^{-1}\alpha(\pi j h) \geq \frac{1}{\sqrt{3}}\pi j$ when $j \leq N/4$ and $a_j \geq \frac{\pi N}{4\sqrt{3}}$ when $j \geq N/4$. This gives

$$N^{-1} \sum_{k=1}^{N-1} |u(hk, hl)|^2 \leq c_1 d^2 \left(\sum_{j=1}^{N/4} \frac{e^{-\pi h j / \sqrt{3}}}{j} + c_2 \right) \leq c_3 d^2 \log N,$$

where c_1 , c_2 and c_3 are some absolute constants. □

4.4.4 Concluding remarks

We discussed determining sets and reconstruction of harmonic function in two-dimensional square. The same methods can be applied for higher dimensional cubes. We refer the reader to [43] for generalization of determining sets for higher dimensions. For our work, in particular, the regularization for reconstruction from model sets can be repeated in higher dimensions. We may also consider a slightly more general operators than the Laplacian. Similar to that of in [21], we can repeat the argument for the solution of the uniformly elliptic equations of the form $\operatorname{div}(a(x)\nabla U(x, y)) = 0$, where $(x, y) \in \mathbf{R}^2$ and a does not depend on the second coordinate.

We remark also that in computing stability constants in Section 4.2.2 one could calculate interior stability constants for proper subrectangles $P_{h,t}$ defined by

$$s_2(K, P_{h,t}) = \max \left\{ \|u\|_{2, P_{h,t}}, \quad \Delta_h u = 0 \text{ on } I_h, \quad |u| \leq 1 \text{ on } K \right\}.$$

Repeating the arguments of Proposition 4, one would get $s_2(\Lambda_{b,h}, P_{h,t}) \geq c_{l-s} N^{-1}$, where $s = tN$. Thus interior stability constant still grows exponentially in N and the advantage of the regularization becomes so immediate.

5 Quantitative unique continuation for discrete harmonic functions

We suggest an elementary quantitative unique continuation argument for harmonic functions that can be generalized to the discrete case of harmonic functions on the lattice. The analog of the three balls theorem for discrete harmonic functions that we obtain contains an additional term that depends on the mesh size of the lattice and goes to zero when the mesh size goes to zero.

5.1 Introduction

5.1.1 Motivation

In this Chapter we give a quantitative version of the following simple observation: a discrete harmonic functions on the lattice \mathbf{Z}^n may vanish identically on a large square without being zero everywhere but there is a version of three balls theorem for discrete harmonic functions on large scales.

Quantitative unique continuation is an important tool in the study of solutions of elliptic and parabolic problems. It has many applications, including stability estimates for the Cauchy problem, see [2]. The simplest quantitative unique continuation statement is a three balls theorem. For classical harmonic functions it follows from logarithmic convexity of the L^2 -norms, that in turn is obtained using the rotational symmetry and ellipticity of the Laplace operator and can be proved by expansions in eigenfunctions of the Laplace–Beltrami operator on the sphere [33]. However, the logarithmic convexity can be generalized to general elliptic equations and it was successfully used in unique continuation problems [1, 23].

The situation with unique continuation changes drastically when one considers discrete models for elliptic equations. It is easy to construct a discrete harmonic function (even a discrete harmonic polynomial, see Chapter 2) on a lattice \mathbf{Z}^n that vanishes on a large cube of the lattice without being zero identically. On the other hand, we understand that some version of unique

continuation should hold at least when we fix the domains and let the mesh size of the lattice go to zero. An obstacle for an elementary estimate similar to logarithmic convexity for continuous harmonic functions is that discrete Laplacian is not rotationally symmetric.

Recently quantitative uniqueness (from Cauchy data) for discrete models of elliptic PDEs were obtained by Carleman type inequalities we refer readers to [7, 19] for motivation and interesting results; we mention also an earlier work [32] that contains discrete Carleman estimates. It is known that the mesh size of the discretization appears in propagation of smallness inequalities obtained by this method. The Carleman inequalities are the most common tool for quantitative unique continuation due to their flexibility, they can be adjusted to very general setting. However, even for the simplest case of the five-point discrete Laplacian on \mathbf{Z}^2 the quantitative uniqueness by Carleman estimates is very technical. We suggest another approach that is based on the analyticity of the Poisson kernel and thus can be applied only in the case of good equations but is simple and gives direct constructive estimates for discrete harmonic functions on n -dimensional lattices. It is based on [24].

We consider the standard lattice $(h\mathbf{Z})^n$ in n -dimensional space \mathbf{R}^n , we always assume that $N = h^{-1}$ is a positive integer. A function $u : (h\mathbf{Z})^n \rightarrow \mathbf{R}$ is called h -discrete harmonic at a point $x \in (h\mathbf{Z})^n$ if

$$2nu(x) = \sum_{j=1}^n u(x + he_j) + u(x - he_j),$$

where $\{e_j\}$ is the standard orthonormal basis for \mathbf{R}^n . Different logarithmic convexity estimates for discrete harmonic functions can be found in [21, 26, 25, 42], in all these cases the norms are taken over parallel segments or parallel lines.

In this Chapter we obtain an analog of the three sphere theorem for discrete harmonic functions. We define by Q_d the cube $[-d, d]^n \subset \mathbf{R}^n$ and by Q_d^h its discretization, $Q_d^h = Q_d \cap (h\mathbf{Z})^n$. More generally for any set $E \subset \mathbf{R}^n$ we denote $E^h = E \cap (h\mathbf{Z})^n$. Then our main result is the following.

Theorem 22. *Suppose that $r < R < 1$. Then there exist positive constants C, N_0, δ, α that depend on r, R with $\alpha, \delta < 1$ such that for any $h = N^{-1}$, $N \in \mathbb{N}$, $N > N_0$ and any h -discrete harmonic function u in Q_1^h that satisfies*

$\max_{Q_r^h} |u(x)| \leq \varepsilon$ and $\max_{Q_1^h} |u(x)| \leq M$ the following inequality holds:

$$\max_{Q_R^h} |u(x)| \leq C(\varepsilon^\alpha M^{1-\alpha} + \delta^{\sqrt{N}} M).$$

The remaining part of the Chapter is organized in the following way. First, we give a new proof of the three balls theorem for continuous harmonic functions that can be adjusted to the discrete case. Then we continue to prove the main result of the Chapter. For the case $R < 2r < 2^{-3n-3}$ we suggest simple formulas for α and δ , for other cases one has to iterate the estimate in a standard way.

5.1.2 Continuous case

The proposition below is three balls theorem for harmonic functions. It is well known that the standard approach is to obtain logarithmic convexity for L^2 -norms and then use elliptic estimates to obtain L^∞ estimates, see [33] for details. We give another elementary proof that will be extended to discrete situation in the next section. We work in n -dimensional Euclidean space \mathbf{R}^n and fix n , so our constants may depend on the dimension.

Proposition 5. *Let $0 < r < R < 1/4$. There exist constants $C > 0$ and $\alpha \in (0, 1)$ such that for any harmonic function u in the unit ball with*

$$\max_{|x| \leq r} |u(x)| = \varepsilon, \quad \max_{|x| \leq 1} |u(x)| = M,$$

the inequality

$$\max_{|x|=R} |u(x)| \leq C\varepsilon^\alpha M^{1-\alpha} \tag{5.1}$$

holds.

Proof. We have

$$u(x) = \int_{S^{n-1}} P(x, y) u(y) d\sigma(y),$$

where $P(x, y) = \gamma_n(1 - |x|^2)(|x - y|)^{-n}$ is the standard Poisson kernel for the unit ball. We fix a point x_0 such that $|x_0| \leq R$. The idea of the proof is to approximate $P(x_0, y)$ by a linear combination of the form $\sum_{k=1}^m c_k P(x_k, y)$ with $|x_k| \leq r$. We will need two estimates, one for the error $r_m(x_0, y)$ of the approximation and another for the sum of the absolute values of the coefficients of the approximating linear combination.

We choose points x_k on the segment $[0, rR^{-1}x_0]$, $x_k = t_k r R^{-1} x_0$, $t_k \in (0, 1)$, and consider the standard Lagrange interpolation of the function $f(t) = P(trR^{-1}x_0, y)$, then

$$c_k = \prod_{j \neq k} \frac{r^{-1}R - t_j}{t_k - t_j}.$$

Considering the polynomial $H_m(t) = (t - t_1) \dots (t - t_m)$, we have

$$|c_k| \leq (R/r)^m |H'_m(t_k)|^{-1}.$$

Now we choose t_1, \dots, t_m to be the Chebyshev nodes, $t_k = \cos(\pi \frac{2k-1}{2m})$. Then $H_m(t) = 2^{1-m} T_m(t)$, where T_m is the Chebyshev polynomial of the first kind. We have $H'_m(t) = m 2^{1-m} U_{m-1}(t)$, where U_{m-1} is the Chebyshev polynomial of the second kind, see for example [17, Chapter 2]. Therefore

$$U_{m-1}(t_k) = U_{m-1}\left(\cos\left(\pi \frac{2k-1}{2m}\right)\right) = \frac{\sin\left(\pi \frac{2k-1}{2}\right)}{\sin\left(\pi \frac{2k-1}{2m}\right)} = \frac{(-1)^{k-1}}{\sin\left(\pi \frac{2k-1}{2m}\right)}.$$

Then $|H'_m(t)| \geq m 2^{1-m}$ and $|c_k| \leq m^{-1} (2R/r)^m$.

In order to estimate the error of the approximation, we use an analytic extension of the function $f(t) = P(trR^{-1}x_0, y)$ to the disk of radius $1/2r$ centered at the origin on the complex plane, see for example [17, Chapter 4] for the residue method in the interpolation error estimate. We have

$$f(z) = \gamma_m \frac{1 - r^2 R^{-2} |x_0|^2 z^2}{\left(\sum_j (rR^{-1}x_{0,j}z - y_j)^2\right)^{n/2}},$$

this extension is bounded by a constant A_n . We consider the function

$$\Omega(z) = \frac{f(z)H_m(R/r)}{(z - R/r)H_m(z)}$$

which is meromorphic in $\{|z| < 1/2r\}$ and has simple poles at points R/r and t_1, \dots, t_m . Then by the residue theorem, we get (see also [17, Theorem 4.3.3]),

$$\begin{aligned} |r_m(x_0, y)| &= |P(x_0, y) - \sum_{k=1}^m c_k P(x_k, y)| = |f(R/r) - \sum_{k=1}^m c_k f(t_k)| = \\ &= \left| \frac{1}{2\pi i} \int_{|z|=1/2r} \frac{f(z)H_m(R/r)}{(z - R/r)H_m(z)} dz \right| \leq \frac{A_n}{1 - R} \left(\frac{2R}{1 - 2r} \right)^m. \end{aligned}$$

Thus we have the following two estimates.

$$\sum_{k=1}^m |c_k| \leq \left(\frac{2R}{r}\right)^m = B^m, \quad |r_m(x_0, y)| \leq aq^m,$$

for some B, a, q such that $q < 1$. Then we have

$$\begin{aligned} |u(x_0)| &= \left| \int_{S^{n-1}} P(x_0, y) u(y) d\sigma(y) \right| \leq \\ &\sum_{k=1}^m |c_k| \left| \int_{S^{n-1}} P(x_k, y) u(y) d\sigma(y) \right| + \left| \int_{S^{n-1}} r_m(x_0, y) u(y) d\sigma(y) \right| \leq \\ &\sum_{k=1}^m |c_k| |u(x_k)| + c_n \max_{|y|=1} |r_m(x_0, y)| |u(y)| \leq B^m \varepsilon + a_1 q^m M. \end{aligned} \quad (5.2)$$

To minimize the sum we choose $m = [(\log M - \log \varepsilon) (\log B - \log q)^{-1}] + 1$, where $[t]$ is the largest integer less than or equal to t , and obtain the required inequality (5.1) with $C = B + a_1 = 2Rr^{-1} + A_n c_n (1 - R)^{-1}$ and $\alpha = 1 - \log B (\log B - \log q)^{-1}$. \square

The Chebyshev nodes is the standard choice in the interpolation problems, when we want to have a good control over the coefficients, more generally the Fekete points of a given compact set $K \subset \mathbf{R}$ can be chosen. They appear also in the quantitative propagation of smallness from the sets of positive capacity, see [34].

5.2 Discrete case

5.2.1 Poisson kernel and its holomorphic extension

We start by the following discrete version of the Poisson integral representation

$$u(x) = \sum_{y \in \partial Q_1^h} u(y) P_h(x, y), \quad (5.3)$$

where for each $y \in \partial Q_1^h$, the function $P_h(x, y)$ is h -discrete harmonic in the variable x in Q_1^h , and satisfies the boundary conditions $P_h(y, y) = 1$ and $P_h(z, y) = 0$ for any $z \in \partial Q_1^h \setminus \{y\}$.

We will write down an analytic expression for $P_h(x, y)$. Note that since we consider discrete function with finitely many values, its analytic extension is not unique. Without loss of generality, we assume that $y = (y_1, \dots, y_{n-1}, 1)$. For each $K = (k_1, \dots, k_{n-1}) \in ((0, 2N) \cap \mathbf{Z})^{n-1} = \llbracket 1, 2N - 1 \rrbracket^{n-1}$ we define a_K^h to be the only positive solution of the equation

$$\cosh \frac{ha_K^h}{2} = n - \sum_{j=1}^{n-1} \cos \frac{\pi k_j h}{2}.$$

Then

$$f_K^h(x) = \sinh(a_K^h(x_n + 1)/2) \prod_{j=1}^{n-1} \sin(\pi k_j(x_j + 1)/2)$$

is h -discrete harmonic and vanishes on all sides of the cube except the one where y lies. It is easy to check that

$$P_h(x, y) = \left(\frac{1}{N}\right)^{n-1} \sum_K \prod_{j=1}^{n-1} \sin\left(\pi k_j \frac{x_j + 1}{2}\right) \sin\left(\pi k_j \frac{y_j + 1}{2}\right) \frac{\sinh\left(a_K^h \frac{x_n + 1}{2}\right)}{\sinh a_K^h},$$

where summation is taken over $K \in \llbracket 1, 2N - 1 \rrbracket^{n-1}$. Similar computations were done in Section 2.2.3.

Proposition 6. *For any $y \in \partial Q_1^h$ and $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in [-1/2, 1/2]^{n-1}$, $j = 1, \dots, n$, the function $f(t) = P_h(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n, y)$ has a holomorphic extension to the domain $\Omega = \{z : -1/2 \leq \Re z \leq 1/2, -1/16 \leq \Im z \leq 1/16\} \subset \mathbf{C}$ that satisfies $|f(z)| \leq CN^{1-n}$ for any $z \in \Omega$.*

Proof. The holomorphic extension is given by the formula above. We need to prove the estimate. We repeat some of the estimates in Chapter 2 subsection 2.2.3. First, we note that either $ha_K^h \geq 2$ or

$$\left(\frac{ha_K^h}{2}\right)^2 \geq \cosh \frac{ha_K^h}{2} - 1 = \sum_{j=1}^{n-1} (1 - \cos \frac{\pi k_j h}{2}) \geq \frac{1}{4} \sum_{j=1}^{n-1} k_j^2 h^2.$$

Thus either $a_K^h \geq 2N$ or $a_K^h \geq \|K\|$, where $\|K\|^2 = \sum_j k_j^2 \geq n^{-1} \left(\sum_j k_j\right)^2$.

We consider two cases $j = n$ and $j \neq n$. First, if $j = n$, then for $|\Re z| \leq 1/2$, we have

$$\begin{aligned} |f(z)| &\leq CN^{1-n} \sum_K \exp(-a_K^h/4) \leq \\ &CN^{1-n} \sum_K \exp(-\|K\|/4) + CN^{1-n}(2N)^n \exp(-N/2) \leq \\ &CN^{1-n} \left(\sum_{k=1}^{\infty} \exp\left(-\frac{k}{4\sqrt{n}}\right) \right)^{n-1} + CN^{1-n}(2N)^n \exp(-N/2) \leq C_n N^{1-n}. \end{aligned}$$

Otherwise, if $j \neq n$, and for $|\Im z| \leq 1/16$ we have that

$$\begin{aligned} |f(z)| &\leq CN^{1-n} \sum_K \exp(\pi k_j/32 - a_K^h/4) \leq \\ &CN^{1-n} \sum_K \exp(-a_K^h/32) + CN^{1-n}(2N)^n \exp(-N/4 + \pi N/16) \leq C_n N^{1-n}. \end{aligned}$$

□

5.2.2 Proof of the main result

We also need a discrete version of Chebyshev's nodes.

Lemma 9. *Suppose that $M > m^2$. Then there exists a polynomial $H_{m,M}(t) = (t - s_1)\dots(t - s_m)$, where $s_j \in M^{-1}\mathbf{Z} \cap [-1, 1]$ such that $|H'_{m,M}(s_j)| \geq m2^{1-m}$ for any $j = 1, \dots, m$.*

Proof. Let $t_k = \cos((2k-1)\pi/2m)$ be the classical Chebyshev nodes. An elementary estimate shows that $|t_j - t_k| \geq m^{-2}$ when $j \neq k$. We choose $s_j \in M^{-1}\mathbf{Z}$ such that $|s_j - 2t_j + 1| \leq (2M)^{-1}$. Then

$$|s_j - s_k| \geq 2|t_j - t_k| - M^{-1} \geq |t_j - t_k|.$$

We thus have

$$|H'_{m,M}(s_j)| = \prod_{k \neq j} |s_j - s_k| \geq \prod_{k \neq j} |t_j - t_k| \geq m2^{1-m}.$$

□

Combining the statements above and repeating the argument from the previous section, we obtain the following.

Lemma 10. *Suppose that $r < R < 2r < 2^{-3n-3}$. Then there exist constants A, B, q that depend on r, R with $q < 1$ such that for any h -discrete harmonic function u in $[-1, 1]^n$ and any $m < \sqrt{rh^{-1}}$ we have*

$$\max_{Q_R^h} |u(x)| \leq A(B^m \max_{Q_r^h} |u(x)| + q^m \max_{Q_1^h} |u(x)|).$$

Proof. We may assume that $r, R \in h\mathbf{Z}$. We consider the following chain of rectangles $R_0 = [-r, r]^n, R_1 = [-R, R] \times [-r, r]^{n-1}, \dots, R_n = [-R, R]^n$. We want to prolongate the estimate from R_j to R_{j+1} . Let $x = (x_1, \dots, x_j, \dots, x_n) \in \partial R_j \setminus R_{j-1}$, then $r < |x_j| \leq R$. For each $y \in \partial Q_1^h$ we consider the Poisson kernel $P_h(x, y)$ as a function of x_j . More precisely, we fix $y \in \partial Q_1^h$ and define

$$f(t) = P_h(x_1, \dots, tx_j|x_j|^{-1}r, \dots, x_n, y).$$

Further, by Proposition 6, f can be extended to a holomorphic function in the domain $D = \{z \in \mathbf{C} : |\Re z| \leq (2r)^{-1}, |\Im z| \leq (16r)^{-1}\}$, where it satisfies $|f(z)| \leq CN^{1-n}$.

We let $M = Nr$ and choose s_1, \dots, s_m as in Lemma 9. Applying the Lagrange interpolation with nodes s_j we approximate $f(R/r)$ by $\sum_k c_k f(s_k)$, where

$$c_k = \prod_{j \neq k} \frac{r^{-1}R - s_j}{s_k - s_j}.$$

By Lemma 9 we get $|c_k| \leq (r^{-1}R + 1)^m 2^{m-1} m^{-1}$. Then

$$\sum_k |c_k| \leq \left(\frac{2(r+R)}{r} \right)^m.$$

The error of the approximation is

$$\begin{aligned} |r_j(x, y)| &= |P(x, y) - \sum_k c_k P(x_k, y)| = \\ &= \left| \frac{1}{2\pi} \int_{\partial D} \frac{f(z) H_{m,M}(R/r)}{(z - R/r) H_{m,M}(z)} dz \right| \leq CN^{1-n} (16(R+r))^m. \end{aligned}$$

We obtain $\max_{R_j} |u(x)| \leq A_1(B_1^m \max_{R_{j-1}} |u(x)| + q_1^m \max_{Q_1^h} |u(x)|)$, where $q_1 = 16(R+r)$ and $B_1 = 2 + 2Rr^{-1}$. Iterating this estimate $n-1$ times we

obtain a desired estimate with $B = B_1^n$, $q = q_1 B_1^{n-1}$ and $A = A_1^n B_1 / (B_1 - 1)$. We have to check that $q < 1$ and indeed

$$q = 16(R+r)2^{n-1}(R+r)^{n-1}r^{1-n} = 2^{n+3}(R+r)^n r^{1-n} < 2^{3n+3}r < 1.$$

□

Finally, we prove Theorem 22 for the case $R < 2r < 2^{-3n-3}$. Take $m_0 = [(\log M - \log \epsilon)(\log B - \log q)^{-1}] + 1$. If $m_0 < \sqrt{rh^{-1}}$ then applying Lemma 10 with $m = m_0$, we obtain

$$\max_{Q_R^h} |u(x)| \leq C \varepsilon^\alpha M^{1-\alpha}.$$

If $m_0 \geq \sqrt{rh^{-1}}$, then we apply the Lemma with $m = [\sqrt{rh^{-1}}]$ and get

$$\max_{Q_R^h} |u(x)| \leq A_2 q^m M \leq C \delta^{\sqrt{N}} M,$$

where $\delta = q^{\sqrt{r}} < 1$.

A standard argument with a chain of squares and iteration of the estimate gives the following.

Corollary. *Let Ω be a connected domain in \mathbf{R}^n , O be an open subset of Ω , and $K \subset \Omega$ be a compact set. Then there exists C, α and $\delta < 1$ and N_0 large enough such that for any $N \in \mathbf{Z}, N > N_0$, $h = N^{-1}$ and any h -harmonic function u on Ω^h we have*

$$\max_{K^h} |u| \leq C \left(\left(\frac{\max_{O^h} |u|}{\max_{\Omega^h} |u|} \right)^\alpha + \delta^{\sqrt{N}} \right) \max_{\Omega^h} |u|.$$

5.2.3 Concluding remarks

It is easy to see that a discrete version of three balls (or three cubes) theorem should have an error term that depend on the mesh-size of the lattice. It could be also reformulated in the following way. Given $r < R < 1$ there exist C, α and a function $d(N)$ such that $d(N) \rightarrow 0$ as $N \rightarrow \infty$ and any discrete harmonic function u on $[-N, N] \cap \mathbf{Z}^n$ satisfies the inequality

$$\max_{|x| \leq NR} |u| \leq C \left(\max_{|x| \leq rN} |u|^\alpha \max_{|x| \leq N} |u|^{1-\alpha} + d(N) \max_{|x| \leq N} |u| \right).$$

We have proved that one can take $d(N) = \delta^{\sqrt{N}}$. It is clear that a zero function on a cube can be extended non-trivially to a harmonic function on \mathbf{Z}^n . For example from a square $[-M, M]^2$ one may extend the function to a strip $[-M, M] \times \mathbf{Z}$ with arbitrary values at the points $(\pm M, y)$ with $|y| > M$ on the sides of the strip, then lay-wise the function is uniquely extended to a discrete harmonic function on \mathbf{Z}^2 . The same argument works in higher dimensions.

To give an estimate for $d(N)$ from below, we construct a particular function with zero values on $\llbracket -rN, rN \rrbracket^n$ and non-zero extension to $\llbracket -N, N \rrbracket^n$. For simplicity we do the calculations for $n = 2$ and the case when $r < 1/2 < R$, the example can be extended to higher dimensions and general situation easily. Let u be zero on $\llbracket -N, N \rrbracket \times \llbracket -N, -1 \rrbracket$ and let the first non-zero row on $\llbracket -N, N \rrbracket \times \{0\}$ be $(-1, 0, \dots, 0)$. In all the consequent rows the boundary values are taken to be zeros, so the next row is $(0, 1, 0, \dots, 0)$. We consider each row as an $2N - 1$ -vector of values of u at points $\{-N+1, \dots, 0, \dots, N-1\} \times \{k\}$. Then we have a sequence of $2N - 1$ vectors $w_0 = 0$, $w_1 = (1, 0, \dots, 0)$ and $w_{k+1} = (A+2I)w_k - w_{k-1}$, $k \geq 1$, where A is the tridiagonal $(2N-1) \times (2N-1)$ matrix with values 2 on the main diagonal and -1 on diagonals right above and below the main one. To estimate w_k we repeat the standard computation of eigenvalues and eigenvectors of A .

Clearly, for every $j = 1, \dots, 2N-1$, the vector $v_j = \{\sin(j s \pi / 2N)\}_{1 \leq s \leq 2N-1}$ is an eigenvector for A with the corresponding eigenvalue $\lambda_j = 2 - 2 \cos(j \pi / 2N)$. Moreover $\{v_j\}_{j=1}^{2N-1}$ form an orthogonal basis for \mathbf{R}^{2N-1} with $\|v_j\|_2 = \sqrt{N}$. Then we have

$$w_1 = \sum_j \kappa_j v_j, \quad \kappa_j = \frac{1}{\sqrt{N}} \sin(j \pi / 2N), \quad \text{and} \quad w_k = \sum_j \kappa_j \frac{q_j^k - q_j^{-k}}{q_j - q_j^{-1}} v_j,$$

where q_j is one of the two positive real roots of the equation $q^2 - (\lambda_j + 2)q + 1 = 0$; $1/q_j$ is the other root, we choose $q_j > 1$. We have $0 < \lambda_1 \leq \dots \leq \lambda_{2N-1} < 4$, then $1 < q_1 < q_2 < \dots < q_{2N-1} < 2 + \sqrt{3}$. Finally,

$$|w_k^{(s)}| = \left| \sum_{j=1}^{2N-1} \frac{1}{\sqrt{N}} \sin(j \pi / 2N) \sin(j \pi s / 2N) \frac{q_j^k - q_j^{-k}}{q_j - q_j^{-1}} \right| \leq (2 + \sqrt{3})^k \sqrt{N}.$$

Now when $r < 1/2 < R$ we have $u(x) = 0$ on K_{rN} and $\max_{x \in K_{RN}} |u(x)| \geq 1$. Hence

$$d(N) \geq (\max_{x \in K_N} |u(x)|)^{-1} \geq \frac{1}{\sqrt{N}} (2 + \sqrt{3})^{-N}.$$

Thus the error term $d(N)$ cannot go to zero faster than q^N for $q < 2 - \sqrt{3}$. It would be interesting to find the correct asymptotic behavior of $d(N)$.

Our interest to quantitative unique continuation comes partly from the question of conditional stability of the discrete Cauchy problem. A general scheme of applying three balls inequalities for such stability estimates is described in [2]. It would be interesting to use it for the discrete case as well.

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APPENDIX

In this appendix we present the MATLAB code for the stability constants $s_\infty(\Lambda_{*,h})$ for our model set $\Lambda_{*,h}$. The computations are based on the formula in Lemma 7 of Chapter 4. We compute also an L^2 -stability constant $\tilde{s}_2(\Lambda_{*,h})$ of $\Lambda_{*,h}$, where

$$\tilde{s}_2(K) = \max \left\{ \|u\|_{2,\tilde{G}_h}, \Delta_h u = 0 \text{ in } I_h \text{ and } \|u\|_{2,K} \leq 1 \right\}.$$

Note that it is different from the stability constant defined in Chapter 4.2.1. First we calculate the Laplacian matrix L_h using the Kronecker tensor product. The rows of the matrix L_h correspond to the equations of the discrete Laplacian and the columns correspond to the points in \tilde{G}_h . This matrix L_h has very simple representation in terms of the Kronecker tensor product. The Kronecker tensor product of two $A \in \mathbf{R}^{m \times n}$, $B \in \mathbf{R}^{p \times q}$ matrices is defined as the matrix

$$\text{kron}(A, B) = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix} \in \mathbf{R}^{mp \times nq}.$$

Then we rearrange the columns of L_h such that the first N^2 columns correspond to the points in $\tilde{G}_h \setminus \Lambda_{*,h}$ and the last $4N$ columns correspond to the points of the set $\Lambda_{*,h}$. The obtained matrix is denoted by

$$L_{h,K} = [A B],$$

where A is $N^2 \times N^2$ matrix and B is an $N^2 \times 4N$. By Lemma 7 the stability constants $\tilde{s}_2(\Lambda_{*,h})$ and $s_\infty(\Lambda_{*,h})$, for the model set $\Lambda_{*,h}$ are given by

$$s_\infty(\Lambda_{*,h}) = \|A^{-1}B\|_\infty \quad \tilde{s}_2(\Lambda_{*,h}) = \left\| \begin{bmatrix} I \\ A^{-1}B \end{bmatrix} \right\|_2.$$

The standard stability constant $s_\infty(\Lambda_{*,h})$ can also be computed using the formula

$$a_k = \left(\frac{1 + \sqrt{2}}{2} \right) (3 + 2\sqrt{2})^k + \left(\frac{1 - \sqrt{2}}{2} \right) (3 - 2\sqrt{2})^k,$$

discussed in Chapter 4, Section 4.2.2.

```

function [S]=STABILITYCONSTANT(N)
T1,N = -spdiags(ones(N, 1), 0, N, N + 2) + 2*spdiags(ones(N, 1), 1, N, N + 2)
        -spdiags(ones(N, 1), 2, N, N + 2);
    J = spdiags(ones(N, 1), 1, N, N + 2);
T2,N×N = kron(T1,N, J) + kron(J, T1,N);
L = T2,N×N(:, any(T2,N×N));
Lh = full (L);
    A = Lh (:, 1 : N);
    B = Lh (:, N + 1);
    index=N + 2;
    for row 1 : N/2 - 1
        A = [A, Lh(:, index : index + N - 1)];
        index = index + N;
        B = [B, Lh(:, index : index + 1)];
        index = index + 2;
    end
    B = [B, Lh(:, index : index + 2 * (N + 2) - 3)];
    index = index + 2 * (N + 2) - 2;
    for row = 1 : N/2 - 1
        B = [B, Lh(:, index : index + 1)];
        index = index + 2;
        A = [A, Lh(:, index : index + N - 1)];
        index = index + N;
    end
    B = [B, Lh(:, index)];
    index = index + 1;
    A = [A, Lh(:, index : end)];
    LhK = [A B];
    s∞(Λ*,h) = norm (inv(A) * B, inf); the uniform stability constant for the
set Λ*,h.
    s̃2(Λ*,h) = norm ( [ I
                      A-1B ], 2 ); the L2- stability constant for the set
Λ*,h.
    The table below shows some values of the Stability constants s∞(Λ*,h) and s̃2(Λ*,h)
for N = 2, 4, 6, ..., 40. We refer Chapter 4, Lemma 7 for the detail.

```

N	$s_\infty(\Lambda_{*,h})$	$\tilde{s}_2(\Lambda_{*,h})$
2	7	6.1644
4	41	36.9210
6	329	220.1458
8	1.3930e+03	1.3046e+03
10	8.1190e+03	7.6966e+03
12	4.7321e+04	4.5267e+04
14	2.7581e+05	2.6569e+05
16	1.6075e+06	1.5571e+06
18	9.3693e+06	9.1165e+06
20	5.4608e+07	5.3332e+07
22	3.1828e+08	3.1181e+08
24	1.8551e+09	1.8222e+09
26	1.0812e+10	1.0645e+10
28	6.3018e+10	6.2167e+10
30	3.6730e+11	3.6297e+11
32	2.1408e+12	2.1188e+12
34	1.2477e+13	1.2366e+13
36	7.2723e+13	7.2166e+13
38	4.2387e+14	4.2108e+14
40	2.4708e+15	2.4566e+15

Table 5.1: Stability Constants for Model set $\Lambda_{*,h}$

From the table above we see that $s_\infty(\Lambda_{*,h}) \leq \tilde{s}_2(\Lambda_{*,h})$ and the ratio of the standard stability constants $s_\infty(\Lambda_{*,h})$ to the L^2 -stability constant $\tilde{s}_2(\Lambda_{*,h})$ becomes close to 1 as N increases.

We have the following remarks for the MATHLAB code above.

1. For $N > 40$, the matrix A is close to singular or badly scaled. Therefore the results may be inaccurate.

2. The matrix

$$T_{1,N} = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & \dots 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & \dots 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & \dots 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & \dots 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & 1 \end{pmatrix}.$$

is $N \times (N + 2)$ symmetric tridiagonal matrix.

3. The matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

is $N \times (N + 2)$ symmetric tridiagonal matrix.

4. $T_{2,N \times N}$ is $N^2 \times (N + 2)^2$ matrix with four zero columns obtained from the Kronecker products of the matrices $T_{1,N}$ and J . The MATLAB code $T_{2,N \times N}(:, any(T_{2,N \times N}))$ removes all the zero columns of $T_{2,N \times N}$ to get the discrete Laplacian matrix L_h .
5. L_{hK} is an $N^2 \times (N^2 + 4N)$ discrete Laplacian matrix obtained from L_h after rearranging all the columns of the matrix L_h such that the first N^2 columns correspond to all the points in $\widetilde{G}_h \setminus \Lambda_{*,h}$ and the remaining $N^2 + 4N$ columns correspond to all the points in the set $\Lambda_{*,h}$.