NTNU – Trondheim
Norwegian University of
Science and Technology

Nonlinear fractional convection-diffusion equations, with nonlocal and nonlinear fractional diffusion

Jørgen Endal

Master of Science in Physics and Mathematics
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Supervisor: Espen Robstad Jakobsen, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Preface

This study marks the completion of my Master of Science degree in Industrial Mathematics at the Norwegian University of Science and Technology. Thus, fulfilling the requirements of the course TMA4900 - Mathematics.

I would like to thank Professor Espen Robstad Jakobsen for his guidance and help throughout this academic year, and especially this semester. My master's thesis could not have been completed in this manner without his advice and remarks.

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Jørgen Endal

Abstract

We study nonlinear fractional convection-diffusion equations with non-local and nonlinear fractional diffusion. By the idea of Kružkov (1970), entropy sub- and supersolutions are defined in order to prove well-posedness under the assumption that the solutions are elements in $L^\infty(\mathbb{R}^d \times (0,T)) \cap C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$. Based on the work of Alibaud (2007) and Cifani and Jakobsen (2011), a local contraction is obtained for this type of equations for a certain class of Lévy measures. In the end, this leads to an existence proof for initial data in $L^\infty(\mathbb{R}^d)$.

Sammendrag

Vi studerer ikke-lineære fraksjonelle konveksjon-diffusjon-ligninger med ikke-lokal og ikke-lineær fraksjonell diffusjon. Ut i fra Kružkovs (1970) ideer vil sub- og superentropiløsninger bli definerte for å bevise velstilthet av løsninger som er i $L^\infty(\mathbb{R}^d \times (0,T)) \cap C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$. Ved hjelp av arbeidet til Alibaud (2007) og Cifani og Jakobsen (2011), finner vi en lokal kontraksjon for den ovennevnte type ligninger for en spesiell klasse av Lévy-mål. Til slutt leder dette til et eksistensbevis for initialdata i $L^\infty(\mathbb{R}^d)$.

Contents

Chapter 1

Introduction

Nonlocal partial differential equations have, in recent years, received a lot of interest. Mainly due to their applications in physics and finance, but also because of their mathematical properties. Some of these mathematical properties will, of course, be the main topic of this study.

The Cauchy problem

$$\begin{cases} \partial_t u + \operatorname{div}(f(u))(x,t) = \mathcal{L}^\mu[A(u(\cdot,t))](x) & (x,t) \in \mathbb{R}^d \times (0,T) \\ u(x,0) = u_0(x) & x \in \mathbb{R}^d \end{cases} \tag{1.1}$$

will be the core of this project, where the assumptions on u and (f, A, u_0, μ) and the definition of $\mathcal{L}^\mu$ are to be discussed later.

1.1 Mathematical background

Equation (1.1) is an extension of the degenerate convection-diffusion equation

$$\partial_t u + \operatorname{div}(f(u)) = \Delta A(u), \tag{1.2}$$

where $A(u)$ could be nonlinear and possibly degenerate. In (1.1) the Laplacian has been interchanged with a more general fractional diffusion term. Actually, $\mathcal{L}^\mu$ is the generator of a pure jump Lévy process, and reversely, a pure jump Lévy process has a generator like $\mathcal{L}^\mu$.

Depending on the data (f, A, u_0, μ), many processes can be described by the general Cauchy problem given by (1.1). In the following, some of them are highlighted:

When $\mu = 0$, (1.1) is the well-known scalar conservation law

$$\partial_t u + \operatorname{div}(f(u)) = 0. \tag{1.3}$$

When $A(u) = u$, (1.1) is the fractional conservation law

$$\partial_t u + \operatorname{div}(f(u)) = \mathcal{L}^\mu[u]. \tag{1.4}$$

Both of the above mentioned consequences of (1.1) use Kružkov's definition of entropy solutions (and the doubling of variables technique) to establish uniqueness. Thus, the well-posedness of (1.3) and (1.4) are well-known (see e.g. [1, 5]). Recently, these results have been extended, in [6], to cover (1.1) for general, singular Lévy measures and nonlinear, possibly degenerate A.

Equation (1.1) is also interesting because of its potential and actual applications. A large variety of physical and financial problems are modeled by anomalous diffusion equations:

The model described by (1.4) appears in physical models describing detonation in gases, semiconductor growth, dislocation dynamics, hydrodynamics, and molecular biology. Further, a slight change in the fractional diffusion gives rise to an application in radiation hydrodynamics.

It is also known that equations like (1.2) are, for instance, used in models describing porous media flow, reservoir simulation, sedimentation processes, and traffic flow. Thus, equation (1.1) might also be applicable at describing these phenomenons.

For references on all of these applications see [6], and references therein.

1.2 Project outline

The purpose of the present project is to show that there exists a unique entropy solution to (1.1). In the process of finding such a unique entropy solution, a local contraction result is obtained for a certain class of Lévy measures, namely the one given by (A.5). That result gives rise to an existence proof for $u_0 \in L^\infty(\mathbb{R}^d)$ (based on [6, Theorem 5.3]). Theorem 4.1.3 iii) and Theorem 5.3.1 are, thus, the main results of this work, and as far as we know, it has not been proved before.

The rest of the project is organized as follows:

It starts by defining the notion of entropy solutions in the sense of Kružkov. Then entropy sub- and supersolutions are also defined in order to establish the forthcoming results for the positive and the negative part of the solution u to (1.1). Further, an auxiliary result is given - the dual equation of (1.1) - in order to find the local and the global contractions. In the end a wide range of results are established both locally and globally; an L^1 contraction (for the positive part and the absolute value of the solution u), a comparison principle, an L^1 bound, an L^∞ bound, a BV bound, uniqueness and existence.

Do notice that the assumption on the Lévy measures varies throughout this project. Assumption (A.4) is the most general, and all results found for that measure are valid for assumption (A.5) (but not the other way around of course).

1.3 Notation

In this section there is given a short overview of some of the notation used throughout this project.

Let $\omega \in \mathbb{R}$. The *positive part*, the *negative part* and the *signum function* are defined respectively:

$$\omega^+ = \max\{\omega, 0\}$$
$$\omega^- = \max\{-\omega, 0\}$$
$$\mathrm{sign}(\omega) = \begin{cases} 1 & \omega > 0 \\ 0 & \omega = 0 \\ -1 & \omega < 0 \end{cases}.$$

The *support* of a function is the closure of the interval where the function is non-zero. It will be denoted with $\mathrm{supp}\{\omega\}$, or with $\omega \in C_c$ (in the last denotation ω is assumed to be continuous as well).

The symbol C_b will denote continuous functions which are bounded.

The derivative of a function u will be denoted by $\frac{\mathrm{d}u}{\mathrm{d}\nu}$ or u_ν when $u = u(\nu)$, or $\partial_\nu u$ when u is dependent on more than one variable. Moreover, Du and D^2u denotes the gradient and the Hessian matrix of u, respectively, with respect to (w.r.t.) the spatial variable x.

$K \subset\subset \mathbb{R}^d$ denotes a compact subset, K, of $\mathbb{R}^d$.

The space $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ denotes all measurable functions such that $\int_K |u(x)|\mathrm{d}x < \infty$ for all $K \subset\subset \mathbb{R}^d$.

The space $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ denotes all measurable functions such that $\max_{t\in[0,T]} \int_K |u(x,t)|\mathrm{d}x < \infty$ for all $K \subset\subset \mathbb{R}^d$, and, in addition, satisfies that $\int_K |u(x,t) - u(x,s)|\mathrm{d}x \to 0$ when $t \to s$ for all $K \subset\subset \mathbb{R}^d$.

Let $\omega(\sigma)$ be a C^∞-function with the following properties

$$0 \le \omega(\sigma) \le 1 \qquad \mathrm{supp}\{\omega\} \subseteq [-1, 1]$$
$$\omega(\sigma) = \omega(-\sigma) \qquad \int_{-1}^{1} \omega(\sigma)\mathrm{d}\sigma = 1$$

and define $\omega_\varepsilon(\sigma) = \frac{1}{\varepsilon}\omega(\frac{\sigma}{\varepsilon})$ with $\lim_{\varepsilon\to 0} \omega_\varepsilon(\sigma) = \delta_0$ in the sense of distributions (with δ_0 being Dirac's delta at the origin). $\omega(\sigma)$ will be given the name *mollifier* throughout this project.

If f is *Lipschitz continuous* (or simply *Lipschitz*), then there exists a constant, say L_f, such that

$$\|f\|_{\mathrm{Lip}} := \sup_{u \neq v} \left| \frac{f(u) - f(v)}{u - v} \right| \le L_f.$$

Let $\mathbf{1}_{[a,b]}(x)$ denote the *indicator function* on the interval $[a, b]$, i.e.:

$$\mathbf{1}_{[a,b]}(x) = \begin{cases} 1, & x \in [a, b] \\ 0, & x \notin [a, b] \end{cases}.$$

The symbol $*$ is reserved for the *convolution product*, defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)\mathrm{d}y.$$

In some calculations the convolution will be taken over (x, t):

$$(f * g)(x,t) = \iint_{\mathbb{R}\times\mathbb{R}^d} f(x-y, t-s)g(y,s)\mathrm{d}y\mathrm{d}s.$$

$B(x,r) = \{y \in \mathbb{R}^d : |x-y| < r\}$ denotes the *open ball* in $\mathbb{R}^d$ with center x and radius $r > 0$.

The notation $L^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ denotes an L^1 function which has the domain $\mathbb{R}^d$ and the range $\mathbb{R}^{d\times d}$.

Let $\mathbb{N}^+$ denote the natural numbers not including zero. That is, $\mathbb{N}^+ = \{1, 2, 3, 4, \ldots\}$.

Let B denote the Borel σ-algebras on $\mathbb{R}^d \setminus \{0\}$. Then $\mu = \mu(B)$ is the Lévy measure (i.e. it satisfies (A.4) or (A.5)). Further, μ^* is defined by

$$\mu^*(B) = \mu(-B),$$

for all Borel σ-algebras on $\mathbb{R}^d \setminus \{0\}$.

Chapter 2

Entropy formulation

2.1 Introduction

This project studies entropy solutions, $u(x,t)$ in $\Omega_T := \mathbb{R}^d \times (0,T)$ (where $d \in \mathbb{N}^+$ denotes the dimension of the space $\mathbb{R}^d$), of the Cauchy problem given by (1.1), where u is the scalar unknown function, div denotes the divergence w.r.t x, and the nonlocal operator $\mathcal{L}^\mu$ is defined for all $\phi \in C_c^\infty(\mathbb{R}^d)$ by

$$\mathcal{L}^\mu[\phi](x) := \int_{\mathbb{R}\setminus\{0\}} \phi(x+z) - \phi(x) - z \cdot D\phi(x)\mathbf{1}_{|z|\le 1}\mathrm{d}\mu(z),$$

where $D\phi$ denotes the gradient of ϕ w.r.t. x and

$$\mathbf{1}_{|z|\le 1} = \begin{cases} 1, & |z| \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

The data (f, A, u_0, μ) are assumed to satisfy the following assumptions:

$$f = (f_1, f_2, \dots, f_d) \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^d) \text{ is a locally Lipschitz function;} \tag{A.1}$$

$$A \in W^{1,\infty}(\mathbb{R}) \text{ is nondecreasing } (A' \ge 0) \text{ and a locally Lipschitz function;} \tag{A.2}$$

$$u_0 \in L^\infty(\mathbb{R}^d); \text{ and} \tag{A.3}$$

$$\mu \ge 0 \text{ is a Radon measure satisfying } \int_{\mathbb{R}^d\setminus\{0\}} \min\{|z|^2, 1\}\mathrm{d}\mu(z) < \infty. \tag{A.4}$$

In addition an extra assumption on μ is needed later:

$$\mu \ge 0 \text{ is a Radon measure satisfying } \int_{\mathbb{R}^d\setminus\{0\}} \min\{|z|^2, \mathrm{e}^{M|z|}\}\mathrm{d}\mu(z) < \infty, \tag{A.5}$$

for some constant $M > 0$.

Remark 2.1.1. i) There is no loss of generality in assuming that $f(0) = 0$ and $A(0) = 0$ since adding or substracting constants to f and A will not change (1.1).

ii) Throughout this project $u \in L^\infty(\Omega_T)$, which implies that f and A are globally Lipschitz.

iii) Assuming that $\mathrm{d}\mu(z) = \mu \mathrm{d}z$ with $\mu(z)$ being Lebesgue measurable and $\mathrm{d}z$ being the Lebesgue measure avoids technical difficulties with the Radon measure, thus, this will be assumed in the rest of this project.

In the calculations that follow below, a continuous and convex function called the entropy is crucial to obtain many results concerning entropy solutions. For simplicity this function will be defined now, together with its pair; the entropy flux. The standard choice of Kružkov is the following entropy-entropy flux pair:

$$\begin{cases} \eta_k(u) = |u-k| & \forall k \in \mathbb{R} \\ q_f(u,k) = \operatorname{sign}(u-k)(f(u)-f(k)) & \forall k \in \mathbb{R} \end{cases}. \tag{2.1}$$

Later, it will be evident that $q_f(u,k)$ is a consequence of the choice of $\eta_k(u)$. Notice that $\eta_{A(k)}(A(u)) = |A(u) - A(k)|$.

The nonlinear fractional vanishing viscosity equation will also be needed later, and is given by

$$\begin{cases} \partial_t u^\varepsilon + \operatorname{div}(f(u^\varepsilon))(x,t) = \mathcal{L}^\mu[A(u^\varepsilon(\cdot,t))](x) + \varepsilon \Delta u^\varepsilon & (x,t) \in \Omega_T \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & x \in \mathbb{R}^d \end{cases}. \tag{2.2}$$

2.2 Properties of $\mathcal{L}^\mu$

In this section some crucial properties of $\mathcal{L}^\mu$, which are needed later, are shown.

For notational simplicity and to handle each term differently, consider the following splitting

$$\mathcal{L}^\mu[\phi](x) = \mathcal{L}^\mu_r[\phi](x) + \mathcal{L}^{\mu,r}[\phi](x) + b^{\mu,r} \cdot D\phi(x),$$

for $\phi \in C_c^\infty(\Omega_T)$, $r > 0$ and $x \in \mathbb{R}^d$, where

$$\mathcal{L}^\mu_r[\phi](x) := \int_{0<|z|\le r} \phi(x+z) - \phi(x) - z \cdot D\phi \mathbf{1}_{|z|\le 1} \mathrm{d}\mu(z),$$

$$\mathcal{L}^{\mu,r}[\phi](x) := \int_{|z|>r} \phi(x+z) - \phi(x) \mathrm{d}\mu(z),$$

$$b^{\mu,r} := -\int_{|z|>r} z \mathbf{1}_{|z|\le 1} \mathrm{d}\mu(z).$$

Lemma 2.2.1. *i) Let A satisfy* (A.2) *and let η_k be defined by* (2.1)*. Then*

$$\eta_k'(u)\mathcal{L}^\mu_r[A(u(\cdot,t))](x) \le \mathcal{L}^\mu_r[\eta_{A(k)}(A(u(\cdot,t))](x).$$

ii) Let A satisfy (A.2) *and let η_k be defined by* (2.1)*. Further, assume that $\phi \in C_c^\infty(\Omega_T)$. Then*

$$\iint_{\Omega_T} b^{\mu,r} \cdot DA(u)\eta_k'(u)\phi \mathrm{d}x\mathrm{d}t = \iint_{\Omega_T} b^{\mu,r} \cdot D\eta_{A(k)}(A(u))\phi \mathrm{d}x\mathrm{d}t.$$

Proof i) For simplicity, leave t out of the calculation (it has no influence on the spatial operator $\mathcal{L}_r^\mu$). Start by writing up the definition of $\mathcal{L}_r^\mu[A(u)](x)$,

$$\mathcal{L}_r^\mu[A(u)](x) = \int_{0<|z|\le r} A(u(x+z)) - A(u(x)) - z \cdot DA(u(x))\mathbf{1}_{|z|\le 1}\mathrm{d}\mu(z),$$

multiply it by $\eta_k'(u)$ and add and subtract $A(k)$ to get

$$\begin{aligned}
\eta_k'(u)\mathcal{L}_r^\mu[A(u)](x) =& \eta_k'(u)\int_{0<|z|\le r} (A(u(x+z)) - A(k)) - (A(u(x)) - A(k)) \\
& \qquad - z \cdot D(A(u(x)) - A(k))\mathbf{1}_{|z|\le 1}\mathrm{d}\mu(z) \\
=& \int_{0<|z|\le r} \eta_k'(u)[(A(u(x+z)) - A(k)) - (A(u(x)) - A(k))] \\
& \qquad + \eta_k'(u)[-z \cdot D(A(u(x)) - A(k))\mathbf{1}_{|z|\le 1}]\mathrm{d}\mu(z) \\
\le& \int_{0<|z|\le r} \eta_{A(k)}(A(u(x+z))) - \eta_{A(k)}(A(u(x))) \\
& \qquad - z \cdot D\eta_{A(k)}(A(u(x)))\mathbf{1}_{|z|\le 1}]\mathrm{d}\mu(z) \\
=& \mathcal{L}_r^\mu[\eta_{A(k)}(A(u))](x),
\end{aligned}$$

this is due to a Kato type of inequality.

ii) A constant disappears whenever one differentiates and $\eta_k'(u)$ is just a constant w.r.t. x, therefore

$$\begin{aligned}
&\iint_{\Omega_T} b^{\mu,r} \cdot DA(u)\eta_k'(u)\phi \mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T} b^{\mu,r} \cdot D((A(u) - A(k))\eta_k'(u))\phi \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Conclude by (A.2). □

Remark 2.2.2. Notice that because of (A.2), $\eta_k'(u)(A(u) - A(k)) = \eta_{A(k)}(A(u))$ holds for η_k defined by (2.1).

Lemma 2.2.3. *Let $\phi, \psi \in C_c^\infty(\Omega_T)$, then*

$$\begin{aligned}
&\iint_{\Omega_T} \phi(x,t)\mathcal{L}_r^\mu[\psi(\cdot,t)](x)\mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T} \psi(x,t)\mathcal{L}_r^{\mu^*}[\phi(\cdot,t)](x)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Remark 2.2.4. The assumption on ψ can be relaxed, and the result holds for $\psi \in C((0,T); C_b^2(\mathbb{R}^d))$.

Proof Utilize these Taylor expansions

$$u(x+z,t) = u(x,t) + \frac{z}{1!} \cdot Du(x,t) + \frac{1}{1!}\int_0^1 (1-\tau) D^2 u(x+\tau z,t) z \cdot z \mathrm{d}\tau, \quad (2.3)$$

where $D^2uz \cdot z = \operatorname{div}(F)$ with $F = (z_1 Du \cdot z, \dots, z_d Du \cdot z)$, and

$$u(x+z,t) = u(x,t) + \frac{1}{0!}\int_0^1 Du(x+\tau z,t) \cdot z \mathrm{d}\tau.$$

Insert these expansions into the definition of $\mathcal{L}_r^\mu[\psi(\cdot,t)](x)$

$$\begin{aligned} &\iint_{\Omega_T} \Bigg[\int_{0<|z|\le \min\{r,1\}} \int_0^1 (1-\tau) D^2\psi(x+\tau z,t) z \cdot z \mathrm{d}\tau \mathrm{d}\mu(z) \\ &+ \int_{1<|z|\le \max\{r,1\}} \int_0^1 D\psi(x+\tau z,t) \cdot z \mathrm{d}\tau \mathrm{d}\mu(z) \Bigg] \phi(x,t) \mathrm{d}x \mathrm{d}t \\ &=: I_{\min} + I_{\max}. \end{aligned} \quad (2.4)$$

For simplicity, the domains $0 < |z| \le \min\{r,1\}$ and $1 < |z| \le \max\{r,1\}$ are, throughout this proof, called α and β respectively.

First, consider $I_{\min}$.

Thanks to Fubini's theorem (ϕ has compact support), and change of variables (in the fourth equality $(x,z) \mapsto (x+\tau z, -z)$), the integration signs can be interchanged, and integration by parts can be utilized:

$$\begin{aligned} I_{\min} &= \iint_{\Omega_T} \int_\alpha \int_0^1 (1-\tau) D^2\psi(x+\tau z,t) z \cdot z \mathrm{d}\tau \mathrm{d}\mu(z) \phi(x,t) \mathrm{d}x \mathrm{d}t \\ &= \int_0^T \int_0^1 (1-\tau) \int_\alpha \int_{\mathbb{R}^d} D^2\psi(x+\tau z,t) z \cdot z \phi(x,t) \mathrm{d}x \mathrm{d}\mu(z) \mathrm{d}\tau \mathrm{d}t \\ &= -\int_0^T \int_0^1 (1-\tau) \int_\alpha \int_{\mathbb{R}^d} D\psi(x+\tau z,t) \cdot z D\phi(x,t) \cdot z \mathrm{d}x \mathrm{d}\mu(z) \mathrm{d}\tau \mathrm{d}t \\ &= -\int_0^T \int_0^1 (1-\tau) \int_\alpha \int_{\mathbb{R}^d} D\psi(x,t) \cdot z D\phi(x+\tau z,t) \cdot z \mathrm{d}x \mathrm{d}\mu^*(z) \mathrm{d}\tau \mathrm{d}t \\ &= \int_0^T \int_0^1 (1-\tau) \int_\alpha \int_{\mathbb{R}^d} \psi(x,t) D^2\phi(x+\tau z,t) z \cdot z \mathrm{d}x \mathrm{d}\mu^*(z) \mathrm{d}\tau \mathrm{d}t \\ &= \iint_{\Omega_T} \int_\alpha \int_0^1 (1-\tau) D^2\phi(x+\tau z,t) z \cdot z \mathrm{d}\tau \mathrm{d}\mu^*(z) \psi(x,t) \mathrm{d}x \mathrm{d}t. \end{aligned}$$

Second, consider $I_{\max}$.

By the same considerations as above:

$$\begin{aligned} I_{\max} &= \iint_{\Omega_T}\int_\beta\int_0^1 D\psi(x+\tau z,t)\cdot z\mathrm{d}\tau\mathrm{d}\mu(z)\phi(x,t)\mathrm{d}x\mathrm{d}t \\ &= \iint_{\Omega_T}\int_\beta\int_0^1 D\phi(x+\tau z,t)\cdot z\mathrm{d}\tau\mathrm{d}\mu^*(z)\psi(x,t)\mathrm{d}x\mathrm{d}t \end{aligned}$$

Combining the results for $I_{\min}$ and $I_{\max}$ gives the desired equality. □

Lemma 2.2.5. *Let $\phi,\psi\in C_c^\infty(\Omega_T)$, then*

$$\begin{aligned} &\iint_{\Omega_T}\phi(x,t)b^{\mu,r}\cdot D\psi(x,t)\mathrm{d}x\mathrm{d}t \\ &= \iint_{\Omega_T}\psi(x,t)b^{\mu^*,r}\cdot D\phi(x,t)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Remark 2.2.6. The assumption on ψ can be relaxed, and the result holds for $\psi\in C((0,T);C_b^1(\mathbb{R}^d))$.

Proof Rearrange the expression, and do integration by parts (ϕ has compact support):

$$\begin{aligned} \iint_{\Omega_T}\phi(x,t)b^{\mu,r}\cdot D\psi(x,t)\mathrm{d}x\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^d}\phi(x,t)\mathrm{div}(b^{\mu,r}\psi(x,t))\mathrm{d}x\mathrm{d}t \\ &= -\int_0^T\int_{\mathbb{R}^d}\psi(x,t)b^{\mu,r}\cdot D\phi(x,t)\mathrm{d}x\mathrm{d}t \\ &= \int_0^T\int_{\mathbb{R}^d}\psi(x,t)b^{\mu^*,r}\cdot D\phi(x,t)\mathrm{d}x\mathrm{d}t \end{aligned}$$

The proof is complete. □

Lemma 2.2.7. *i) Assume that $r>0$ and remember* (A.4). *Let $\phi\in C_c^\infty(\Omega_T)$, then*

$$\|\mathcal{L}_r^\mu[\phi](x)\|_{L^1(\mathbb{R}^d)}\leq\|\phi(x)\|_{W^{2,1}(\mathbb{R}^d)}\int_{0<|z|\leq r}|z|^2\mathrm{d}\mu(z).$$

ii) Remember (A.4). *Let $\phi\in C_c^\infty(\Omega_T)$, then*

$$\begin{aligned} \|\mathcal{L}^\mu[\phi](x)\|_{L^1(\mathbb{R}^d)} \leq{}& \frac{1}{2}\|D^2\phi(x)\|_{L^1(\mathbb{R}^d,\mathbb{R}^{d\times d})}\int_{0<|z|\leq 1}|z|^2\mathrm{d}\mu(z) \\ &+2\|\phi(x)\|_{L^1(\mathbb{R}^d)}\int_{|z|>1}\mathrm{d}\mu(z). \end{aligned}$$

Remark 2.2.8. The assumption on ϕ can be relaxed, and the results hold for e.g. $\phi \in W^{2,1}(\mathbb{R}^d)$.

Proof i) Do the same splitting as in (2.4) in the proof of Lemma 2.2.3:

$$\begin{aligned}
\|\mathcal{L}_r^\mu[\phi]\|_{L^1(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} \Bigg| & \int_{0<|z|\leq \min\{r,1\}} \int_0^1 (1-\tau) D^2\phi(x+\tau z) z \cdot z \mathrm{d}\tau \mathrm{d}\mu(z) \\
&+ \int_{1<|z|\leq \max\{r,1\}} \int_0^1 D\phi(x+\tau z) \cdot z \mathrm{d}\tau \mathrm{d}\mu(z) \Bigg| \mathrm{d}x \\
&\leq \int_{0<|z|\leq \min\{r,1\}} \int_0^1 |z|^2 (1-\tau) \int_{\mathbb{R}^d} |D^2\phi(x+\tau z))| \mathrm{d}x \mathrm{d}\tau \mathrm{d}\mu(z) \\
&+ \int_{1<|z|\leq \max\{r,1\}} \int_0^1 |z| \int_{\mathbb{R}^d} |D\phi(x+\tau z)| \mathrm{d}x \mathrm{d}\tau \mathrm{d}\mu(z) \\
&\leq \int_{0<|z|\leq \min\{r,1\}} |z|^2 \mathrm{d}\mu(z) \|D^2\phi\|_{L^1(\mathbb{R}^d, \mathbb{R}^{d\times d})} \\
&+ \int_{1<|z|\leq \max\{r,1\}} |z| \mathrm{d}\mu(z) \|D\phi\|_{L^1(\mathbb{R}^d, \mathbb{R}^d)} \\
&\leq \int_{0<|z|\leq \min\{r,1\}} |z|^2 \mathrm{d}\mu(z) \|D^2\phi\|_{L^1(\mathbb{R}^d, \mathbb{R}^{d\times d})} \\
&+ \int_{1<|z|\leq \max\{r,1\}} |z|^2 \mathrm{d}\mu(z) \|D\phi\|_{L^1(\mathbb{R}^d, \mathbb{R}^d)} \\
&\leq \int_{0<|z|\leq r} |z|^2 \mathrm{d}\mu(z) \|\phi\|_{W^{2,1}(\mathbb{R}^d)},
\end{aligned}$$

where $\|\phi\|_{W^{2,1}(\mathbb{R}^d)} \geq \|D\phi\|_{L^1(\mathbb{R}^d)} + \|D^2\phi\|_{L^1(\mathbb{R}^d)}$. □

ii) Write up the definition of $\mathcal{L}^\mu[\phi]$ and integrate over $\mathbb{R}^d$:

$$\begin{aligned}
\|\mathcal{L}^\mu[\phi](x)\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left| \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot D\phi \mathbf{1}_{|z|\leq 1} \mathrm{d}\mu(z) \right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \left| \int_{0<|z|\leq 1} \phi(x+z) - \phi(x) - z \cdot D\phi \mathbf{1}_{|z|\leq 1} \mathrm{d}\mu(z) \right| \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \left| \int_{|z|>1} \phi(x+z) - \phi(x) \mathrm{d}\mu(z) \right| \mathrm{d}x.
\end{aligned}$$

Use Taylor expansion in (2.3) to rewrite the first integral, and simply use the triangle inequality in the second integral (Fubini's theorem is applicable because of the compact support of ϕ):

$$\begin{aligned}\|\mathcal{L}^\mu[\phi](x)\|_{L^1(\mathbb{R}^d)} &\leq \int_{0<|z|\leq 1}\int_0^1 (1-\tau)|z|^2\int_{\mathbb{R}^d}|D^2\phi(x+\tau z)|\mathrm{d}x\mathrm{d}\tau\mathrm{d}\mu(z)\\ &+\int_{|z|>1}\int_{\mathbb{R}^d}|\phi(x+z)|+|\phi(x)|\mathrm{d}x\mathrm{d}\mu(z)\\ &=\frac{1}{2}\|D^2\phi(x)\|_{L^1(\mathbb{R}^d,\mathbb{R}^{d\times d})}\int_{0<|z|\leq 1}|z|^2\mathrm{d}\mu(z)\\ &+2\|\phi(x)\|_{L^1(\mathbb{R}^d)}\int_{|z|>1}\mathrm{d}\mu(z),\end{aligned}$$

where the integrals w.r.t. z are integrable due to (A.4). □

Lemma 2.2.9. *Remember* (A.4). *Let* $\phi \in C_c^\infty(\Omega_T)$*, then*

$$\begin{aligned}\|\mathcal{L}^\mu[\phi](x)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{2}\|D^2\phi(x)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^{d\times d})}\int_{0<|z|\leq 1}|z|^2\mathrm{d}\mu(z)\\ +2\|\phi(x)\|_{L^\infty(\mathbb{R}^d)}\int_{|z|>1}\mathrm{d}\mu(z).\end{aligned}$$

Remark 2.2.10. The assumption on ϕ can be relaxed, and the result holds for e.g. $\phi \in W^{2,1}(\mathbb{R}^d)$.

Proof The proof is almost identical to the proof of Lemma 2.2.7 ii). □

2.3 Entropy formulation

In this section entropy solutions and entropy sub- and supersolutions will be defined, and some of their properties are to be proven.

The following proposition will make use of (2.2) in order to establish an entropy formulation for (1.1).

Definition 2.3.1. A classical solution to (2.2) is a function $u(x,t) \in C^1((0,T);C_b^2(\mathbb{R}^d))\cap C(\mathbb{R}^d\times[0,T))$ which satisfies (2.2) at every point in $\mathbb{R}^d\times(0,T)$.

Proposition 2.3.2. *Let* $\{u^\varepsilon\}_{\varepsilon>0}$ *be a uniformly bounded sequence of classical solutions to* (2.2)*. If* $u^\varepsilon \to u$ *in* $C([0,T];L^1_{loc}(\mathbb{R}^d))$*, then* u *satisfies*

$$\begin{aligned}&\iint_{\Omega_T}\eta_k(u)\partial_t\phi + q_f(u,k)\cdot D\phi\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T}\eta_{A(k)}(A(u))\mathcal{L}_r^{\mu^*}[\phi]\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T}\eta_k'(u)\mathcal{L}^{\mu,r}[A(u)]\phi\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T}\eta_{A(k)}(A(u))b^{\mu^*,r}\cdot D\phi\,\mathrm{d}x\mathrm{d}t \geq 0,\end{aligned}\tag{2.5}$$

for the entropy-entropy flux pair defined in (2.1)*, call them* η_k *and* q_f *respectively, with* η_k *continuous and convex,* $q_f = (q_{f,1}, \dots, q_{f,d})$ *such that* $q'_{f,i} = \eta'_k f'_i$ *(*$i = 1, \dots, d$*), and for all* $r > 0$ *and all nonnegative* $\phi \in C_c^\infty(\Omega_T)$*.*

Remark 2.3.3. If $u^\varepsilon \to u$ in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$, then it is known that there exists a subsequence such that $u^\varepsilon \to u$ almost everywhere (a.e.), and this fact will be used in the proof of the above proposition.

Proof Choose a convex $\eta \in C^\infty(\mathbb{R})$ and a nonnegative test function $\phi \in C_c^\infty(\Omega_T)$. Consider (2.2), multiply it with $\eta'(u^\varepsilon)$ and use the fact that the solution should be interpreted in a distributional way. The calculations are given in the following

$$\begin{aligned}
0 &= \iint_{\Omega_T} (\partial_t u^\varepsilon + \operatorname{div}(f(u^\varepsilon)) - \varepsilon\Delta u^\varepsilon - \mathcal{L}^\mu[A(u^\varepsilon)])\eta'(u^\varepsilon)\phi(x,t)\mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T} -\eta(u^\varepsilon)\partial_t\phi - q(u^\varepsilon)\cdot D\phi - \varepsilon\Delta\eta(u^\varepsilon)\phi + \varepsilon\eta''(u)Du\cdot Du\phi \\
&\qquad - \mathcal{L}^\mu[A(u^\varepsilon)]\eta'(u^\varepsilon)\phi\,\mathrm{d}x\mathrm{d}t \\
&\geq \iint_{\Omega_T} -\eta(u^\varepsilon)\partial_t\phi - q(u^\varepsilon)\cdot D\phi - \varepsilon\eta(u^\varepsilon)\Delta\phi \\
&\qquad - \mathcal{L}^\mu[A(u^\varepsilon)]\eta'(u^\varepsilon)\phi\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $q_i = \eta' f'_i$ is introduced, integration by parts is utilized (keeping in mind the compact support of ϕ), the convexity of η (i.e., $\eta'' \geq 0$) is used and, lastly, Fubini's theorem is valid because ϕ has compact support.

Now, let $C^\infty \ni \eta_\delta = \eta_k * \omega_\delta$ and $C^\infty \ni \eta'_\delta = \eta'_k * \omega_\delta$, where $\eta_k \in C(\mathbb{R})$ is defined by (2.1). (Convexity is assured by $\omega_\delta \geq 0$). Taking the limit as $\delta \to 0$ gives η_k and η'_k since they both are in $L^\infty_{\text{loc}}(\mathbb{R}) \subset L^1_{\text{loc}}(\mathbb{R})$ [2, p. 714]. Note that η'_k exists a.e. since η_k is locally Lipschitz (this is due to Rademacher's theorem). Consider

$$0 \leq \iint_{\Omega_T} \eta_\delta(u^\varepsilon)\partial_t\phi + q_\delta(u^\varepsilon)\cdot D\phi + \varepsilon\eta_\delta(u^\varepsilon)\Delta\phi + \mathcal{L}^\mu[A(u^\varepsilon)]\eta'_\delta(u^\varepsilon)\phi\,\mathrm{d}x\mathrm{d}t,$$

Since $\mathcal{L}^\mu[A(u^\varepsilon)] \in L^1(\mathbb{R}^d)$ by Lemma 2.2.7 ii), Lebesgue's dominated convergence theorem can be utilized when $\delta \to 0$, and

$$0 \leq \iint_{\Omega_T} \eta_k(u^\varepsilon)\partial_t\phi + q_f(u^\varepsilon,k)\cdot D\phi + \varepsilon\eta_k(u^\varepsilon)\Delta\phi + \mathcal{L}^\mu[A(u^\varepsilon)]\eta'_k(u^\varepsilon)\phi\,\mathrm{d}x\mathrm{d}t, \quad (2.6)$$

is obtained.

To continue, the nonlocal term needs some further investigation. Consider the integral

$$L = \iint_{\Omega_T} \mathcal{L}^\mu[A(u^\varepsilon)]\eta'_k(u^\varepsilon)\phi\,\mathrm{d}x\mathrm{d}t,$$

and split it in to three parts

$$
\begin{aligned}
L = &\iint_{\Omega_T} \mathcal{L}^{\mu,r}[A(u^\varepsilon)]\eta_k'(u^\varepsilon)\phi \,\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \mathcal{L}_r^{\mu}[A(u^\varepsilon)]\eta_k'(u^\varepsilon)\phi \,\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} b^{\mu,r} \cdot DA(u^\varepsilon)\eta_k'(u^\varepsilon)\phi \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Use Lemma 2.2.1 to obtain

$$
\begin{aligned}
L \le &\iint_{\Omega_T} \mathcal{L}^{\mu,r}[A(u^\varepsilon)]\eta_k'(u^\varepsilon)\phi \,\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \mathcal{L}_r^{\mu}[\eta_{A(k)}(A(u^\varepsilon))]\phi \,\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} b^{\mu,r} \cdot D\eta_{A(k)}(A(u^\varepsilon))\phi \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

To conclude the preliminary manipulations on $\mathcal{L}^\mu$, move the regularity upon the test function ϕ. By Lemma 2.2.3 and 2.2.5

$$
\begin{aligned}
L \le &\iint_{\Omega_T} \mathcal{L}^{\mu,r}[A(u^\varepsilon)]\eta_k'(u^\varepsilon)\phi \,\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \eta_{A(k)}(A(u^\varepsilon))\mathcal{L}_r^{\mu^*}[\phi] \,\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \eta_{A(k)}(A(u^\varepsilon))b^{\mu^*,r} \cdot D\phi \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{2.7}
$$

Combining (2.6) and (2.7) gives

$$
\begin{aligned}
0 \le &\iint_{\Omega_T} \eta_k(u^\varepsilon)\partial_t\phi + q_k(u^\varepsilon) \cdot D\phi + \varepsilon\eta_k(u^\varepsilon)\Delta\phi \,\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \mathcal{L}^{\mu,r}[A(u^\varepsilon)]\eta_k'(u^\varepsilon)\phi + \eta_{A(k)}(A(u^\varepsilon))\mathcal{L}_r^{\mu^*}[\phi] \\
&\qquad + \eta_{A(k)}(A(u^\varepsilon))b^{\mu^*,r} \cdot D\phi \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

It remains to let $\varepsilon \to 0$.

Since η_k and q_f are locally Lipschitz continuous (by convexity and definition respectively), and thus continuous, the convergence of the local terms is easily proven by Lebesgue's dominated convergence theorem.

To ensure the convergence of the nonlocal terms by Lebesgue's dominated convergence theorem, observe that the integrand is dominated by

$$\begin{aligned}
&L_{\eta_k} L_A 2\|u^\varepsilon\|_{L^\infty(\mathbb{R})}|\phi| \int_{|z|>r} \mathrm{d}\mu(z)\\
&+ L_{\eta_k} L_A(\|u^\varepsilon\|_{L^\infty(\mathbb{R})} + k)|\mathcal{L}_r^{\mu^*}[\phi]|\\
&+ L_{\eta_k} L_A(\|u^\varepsilon\|_{L^\infty(\mathbb{R})} + k)|D\phi| \int_{|z|>r} \mathrm{d}\mu^*(z),
\end{aligned}$$

which is integrable by (A.4), Lemma 2.2.7 i) and since $C_c^\infty(\Omega_T)$ is dense in $W^{2,1}(\Omega_T)$. (Notice that $|u^\varepsilon|$ is uniformly bounded by assumption.) Thus, taking the a.e. limit $u^\varepsilon \to u$ gives

$$\begin{aligned}
0 \leq &\iint_{\Omega_T} \eta_k(u)\partial_t\phi + q_f(u,k)\cdot D\phi \,\mathrm{d}x\mathrm{d}t\\
&+ \iint_{\Omega_T} \mathcal{L}^{\mu,r}[A(u)]\eta_k'(u)\phi + \eta_{A(k)}(A(u))\mathcal{L}_r^{\mu^*}[\phi]\\
&\qquad + \eta_{A(k)}(A(u))b^{\mu^*,r}\cdot D\phi \,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

when $\varepsilon \to 0$. □

Definition 2.3.4. A solution $u(x,t) \in L^\infty(\Omega_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ of (1.1) is an entropy solution if it

i) satisfies (2.5) for the entropy-entropy flux pair defined in (2.1), call them η_k and q_f respectively, with η_k continuous and convex, $q_f = (q_{f,1}, \ldots, q_{f,d})$ such that $q_{f,i}' = \eta_k' f_i'$ $(i = 1, \ldots, d)$, and for all $r > 0$ and all nonnegative $\phi \in C_c^\infty(\Omega_T)$, and

ii) $u(\cdot, 0) = u_0(\cdot)$ for almost every $x \in \mathbb{R}^d$.

Remark 2.3.5. Since $u \in L^\infty(\Omega_T)$ and by (A.1) and (A.2), $\eta_k'(u), \eta_k(u), q_k(u)$, $\eta_{A(k)}(A(u)), A(u) \in L^\infty(\Omega_T)$, it follows that the first and fourth integral in (2.5) are well-defined. Since $\mathcal{L}_r^{\mu^*}[\phi] \in C_c^\infty(\Omega_T)$ for $\phi \in C_c^\infty(\Omega_T)$, the second integral is also well-defined. Finally, $\mathcal{L}^{\mu,r}[A(u)] \in L^\infty(\Omega_T)$ when $A(u) \in L^\infty(\Omega_T)$ and thus the third integral is well-defined.

In order to proceed, one needs to define the entropy-entropy flux pairs used in the definitions of sub- and supersolutions:

$$\begin{cases} \eta_k^\pm(u) = (u-k)^\pm & \forall k \in \mathbb{R} \\ q_f^\pm(u,k) = \pm\mathrm{sign}(u-k)^\pm (f(u) - f(k)) & \forall k \in \mathbb{R} \end{cases}, \tag{2.8}$$

where the pair (η_k^+, q_f^+) is associated with entropy subsolutions, and the pair (η_k^-, q_f^-) is associated with entropy supersolutions. It is more convenient to define entropy sub- and supersolutions in order to obtain a wider range of results. The above entropy sub- and supersolutions also satisfy (2.5), but restrictions, $\eta_k'(u) \geq 0$

and $\eta_k'(u) \leq 0$ for entropy sub- and supersolutions respectively, on the entropy $\eta_k(u)$ are needed.

The following results will motivate the definitions in (2.8).

Definition 2.3.6. A classical subsolution to (2.2) is a function $u(x,t) \in C^1((0,T); C_b^2(\mathbb{R}^d)) \cap C(\mathbb{R}^d \times [0,T))$ which satisfies

$$\begin{cases} \partial_t u^\varepsilon + \operatorname{div}(f(u^\varepsilon))(x,t) \leq \mathcal{L}^\mu[A(u^\varepsilon(\cdot,t))](x) + \varepsilon \Delta u^\varepsilon & (x,t) \in \Omega_T \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & x \in \mathbb{R}^d \end{cases}$$

at every point in $\mathbb{R}^d \times (0,T)$.

Proposition 2.3.7. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a uniformly bounded sequence of classical* sub*solutions to* (2.2)*. If $u^\varepsilon \to u$ in $C([0,T]; L^1_{loc}(\mathbb{R}^d))$, then u satisfies*

$$\begin{aligned} &\iint_{\Omega_T} \eta_k(u)\partial_t \phi + q_f(u,k) \cdot D\phi \,\mathrm{d}x\mathrm{d}t \\ &+ \iint_{\Omega_T} \eta_{A(k)}(A(u))\mathcal{L}_r^{\mu^*}[\phi] \,\mathrm{d}x\mathrm{d}t \\ &+ \iint_{\Omega_T} \eta_k'(u)\mathcal{L}^{\mu,r}[A(u)]\phi \,\mathrm{d}x\mathrm{d}t \\ &+ \iint_{\Omega_T} \eta_{A(k)}(A(u)) b^{\mu^*,r} \cdot D\phi \,\mathrm{d}x\mathrm{d}t \geq 0, \end{aligned}$$

for the entropy-entropy flux pair defined in (2.1)*, call them η_k and q_f respectively, with η_k continuous, convex and* nondecreasing*, $q_f = (q_{f,1}, \dots, q_{f,d})$ such that $q_{f,i}' = \eta_k' f_i'$ $(i = 1, \dots, d)$, and for all $r > 0$ and all nonnegative $\phi \in C_c^\infty(\Omega_T)$.*

Proof The proof is quite similar to the proof of Proposition 2.3.2. The main difference is that subsolutions take the place of solutions, and that η_k is nondecreasing $(\eta_k'(u) \geq 0)$.

Remark 2.3.8. The vanishing viscosity equation for supersolutions yields

$$\begin{cases} \partial_t u^\varepsilon + \operatorname{div}(f(u^\varepsilon))(x,t) \geq \mathcal{L}^\mu[A(u^\varepsilon(\cdot,t))](x) + \varepsilon \Delta u^\varepsilon & (x,t) \in \Omega_T \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & x \in \mathbb{R}^d \end{cases}.$$

Multiplying with $\eta'(u) \leq 0$ gives the same inequality as for subsolutions. That is, both entropy sub- and supersolutions satisfy the entropy inequality in (2.5).

In this project, candidates for entropy sub- and supersolutions are the ones defined in (2.8) since these entropies span the entire space of nondecreasing and nonincreasing Kružkov entropies.

The definitions are summed up in the following:

Definition 2.3.9. I) A solution $u(x,t) \in L^\infty(\Omega_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ of (1.1) is a Kruzkov entropy solution if

i) for all nonnegative $\phi \in C_c^\infty(\Omega_T)$ and $k \in \mathbb{R}$

$$\begin{aligned}&\iint_{\Omega_T} |u-k|\partial_t\phi + \operatorname{sign}(u-k)[f(u)-f(k)]\cdot D\phi \,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} |A(u)-A(k)|\mathcal{L}_r^{\mu^*}[\phi]\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} \operatorname{sign}(u-k)\mathcal{L}^{\mu,r}[A(u)]\phi\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} |A(u)-A(k)|b^{\mu^*,r}\cdot D\phi\,\mathrm{d}x\mathrm{d}t \geq 0;\end{aligned}$$

ii) the initial condition must satisfy $u(\cdot,0)=u_0(\cdot)$ for almost every $x \in \mathbb{R}^d$.

II) A subsolution $u(x,t) \in L^\infty(\Omega_T) \cap C([0,T];L^1_{\text{loc}}(\mathbb{R}^d))$ of (1.1) is an entropy subsolution if

i) for all nonnegative $\phi \in C_c^\infty(\Omega_T)$ and $k \in \mathbb{R}$

$$\begin{aligned}&\iint_{\Omega_T} (u-k)^+\partial_t\phi + \operatorname{sign}(u-k)^+[f(u)-f(k)]\cdot D\phi \,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} (A(u)-A(k))^+\mathcal{L}_r^{\mu^*}[\phi]\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} \operatorname{sign}(u-k)^+\mathcal{L}^{\mu,r}[A(u)]\phi\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} (A(u)-A(k))^+b^{\mu^*,r}\cdot D\phi\,\mathrm{d}x\mathrm{d}t \geq 0;\end{aligned}$$

ii) the initial condition must satisfy $u(\cdot,0)=u_0(\cdot)$ for almost every $x \in \mathbb{R}^d$.

III) A supersolution $u(x,t) \in L^\infty(\Omega_T) \cap C([0,T];L^1_{\text{loc}}(\mathbb{R}^d))$ of (1.1) is an entropy supersolution if

i) for all nonnegative $\phi \in C_c^\infty(\Omega_T)$ and $k \in \mathbb{R}$

$$\begin{aligned}&\iint_{\Omega_T} (u-k)^-\partial_t\phi + \operatorname{sign}(u-k)^-[f(k)-f(u)]\cdot D\phi \,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} (A(u)-A(k))^-\mathcal{L}_r^{\mu^*}[\phi]\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} -\operatorname{sign}(u-k)^-\mathcal{L}^{\mu,r}[A(u)]\phi\,\mathrm{d}x\mathrm{d}t\\ &+\iint_{\Omega_T} (A(u)-A(k))^-b^{\mu^*,r}\cdot D\phi\,\mathrm{d}x\mathrm{d}t \geq 0;\end{aligned}$$

ii) the initial condition must satisfy $u(\cdot,0)=u_0(\cdot)$ for almost every $x \in \mathbb{R}^d$.

Remark 2.3.10. i) Since u is assumed to be in $C([0,T];L^1_{\text{loc}}(\mathbb{R}^d))$ and the initial condition satisfies $u(\cdot,0)=u_0(\cdot)$ for almost every $x\in\mathbb{R}^d$, the initial condition is, in fact, imposed in the strong L^1_{loc}-sense:

$$\lim_{t\to 0}\|u(\cdot,t)-u_0(\cdot)\|_{L^1_{\text{loc}}(\mathbb{R}^d)}=0.$$

ii) Notice that the above considerations also apply when $u\in C([0,T];L^1(\mathbb{R}^d))$. In that case the initial condition is imposed in the strong L^1-sense:

$$\lim_{t\to 0}\|u(\cdot,t)-u_0(\cdot)\|_{L^1(\mathbb{R}^d)}=0.$$

Lemma 2.3.11. *$u(x,t)$ is an entropy solution of* (1.1) *if and only if $u(x,t)$ is both an entropy subsolution and an entropy supersolution of* (1.1)*.*

Proof In the case of Kružkov notice that $|u-k|=(u-k)^+ + (u-k)^-$.

For a more general case define

$$\eta_k^{\pm}(u)=\int_{-\|u\|_{L^\infty}}^{u}(\eta_k'(s))^{\pm}\mathrm{d}s,$$

where η_k^+ and η_k^- are the parts of η_k where the derivate is nonnegative and nonpositive respectively. Further, notice that $(\eta_k'(s))^+ - (\eta_k'(s))^- = \eta_k'(s)$. Combined this gives

$$\begin{aligned}\eta_k^+(u)-\eta_k^-(u)+\eta_k(-\|u\|_{L^\infty}) &= \int_{-\|u\|_{L^\infty}}^{u}(\eta_k'(s))^+ - (\eta_k'(s))^-\mathrm{d}x + \eta_k(-\|u\|_{L^\infty})\\ &= \int_{-\|u\|_{L^\infty}}^{u}\eta_k'(s)\mathrm{d}x + \eta_k(-\|u\|_{L^\infty})\\ &= \eta_k(u).\end{aligned}$$

Finally, use $\eta_k^+(u)+\eta_k(-\|u\|_{L^\infty})$ as an entropy in the definition of entropy subsolutions, and use $-\eta_k^-(u)$ as an entropy in the definition of entropy supersolutions (this is done for $(u-k)^+$ and $(u-k)^-$ in Definition 2.3.9 II) and III)). If one adds the two definitions of entropy sub- and supersolutions together, one gets the definition of entropy solutions with $\eta_k(u)$ as an entropy. To conclude, notice that η_k is invariant under translation.

By defining

$$\eta_k(u)=\int_{-\|u\|_{L^\infty}}^{u}\eta_k'(s)\mathrm{d}x,$$

the opposite implication can be obtained. □

Proposition 2.3.12. *Entropy solutions satisfy the following properties:*

i) If $A\in C^2(\mathbb{R})$ and satisfies (A.2)*, then any classical solution to* (1.1) *is an entropy solution.*

ii) Entropy solutions to (1.1) *are weak solutions in the sense that*

$$\iint_{\Omega_T} u\partial_t\phi + f(u)\cdot D\phi + A(u)\mathcal{L}^{\mu^*}[\phi]\mathrm{d}x\mathrm{d}t = 0,$$

for all $\phi \in C_c^\infty(\Omega_T)$*.*

Proof i) Since classical solutions solve (1.1) point-wise in Ω_T, the calculations in the proof of Proposition 2.3.2 will give the desired result. (The required regularity on A is needed in Lemma 2.2.3.)

ii) A modification of (2.5), by the change of variables $(x, z) \mapsto (x + z, -z)$, is needed in the rest of the proof:

$$\begin{aligned}
&\iint_{\Omega_T} \eta_k'(u)\mathcal{L}^{\mu,r}[A(u)]\phi\mathrm{d}x\mathrm{d}t \\
&\leq \iint_{\Omega_T}\int_{|z|>r} \left[\eta_{A(k)}(A(u(x+z,t))) - \eta_{A(k)}(A(u(x,t)))\right] \\
&\qquad \phi(x,t)\mathrm{d}\mu(z)\mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T}\int_{|z|>r} \eta_{A(k)}(A(u(x,t)))\phi(x+z,t) \\
&\qquad - \eta_{A(k)}(A(u(x,t)))\phi(x,t)\mathrm{d}\mu^*(z)\mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T} \eta_{A(k)}(A(u(x,t)))\mathcal{L}^{\mu^*,r}[\phi(\cdot,t)](x)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.9}$$

First, Let $\eta_k(u) = (u-k)^+$ and insert (2.9) into Definition 2.3.9 II). In addition let $k \leq -\|u\|_{L^\infty}$ which gives

$$\begin{aligned}
&\iint_{\Omega_T} u\partial_t\phi + f(u)\cdot D\phi + A(u)\mathcal{L}^{\mu^*}[\phi]\mathrm{d}x\mathrm{d}t \\
&\geq \iint_{\Omega_T} k\partial_t\phi + f(k)\cdot D\phi + A(k)\mathcal{L}^{\mu^*}[\phi]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Second, let $\eta_k(u) = (u-k)^-$ and insert (2.9) into Definition 2.3.9 III). In addition let $k \geq \|u\|_{L^\infty}$ which gives

$$\begin{aligned}
&\iint_{\Omega_T} u\partial_t\phi + f(u)\cdot D\phi + A(u)\mathcal{L}^{\mu^*}[\phi]\mathrm{d}x\mathrm{d}t \\
&\leq \iint_{\Omega_T} k\partial_t\phi + f(k)\cdot D\phi + A(k)\mathcal{L}^{\mu^*}[\phi]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Further, investigate

$$\iint_{\Omega_T} k\partial_t\phi + f(k)\cdot D\phi + A(k)\mathcal{L}^{\mu^*}[\phi]\mathrm{d}x\mathrm{d}t.$$

The first term is zero by integration by parts (compact support of ϕ), the second term is zero by the Divergence theorem (by the compact support of ϕ) and the last term is zero by doing the same considerations as in Lemma 2.2.3 and (2.9). Combined

$$\iint_{\Omega_T} u\partial_t\phi + f(u)\cdot D\phi + A(u)\mathcal{L}^{\mu^*}[\phi]\mathrm{d}x\mathrm{d}t = 0$$

is obtained for entropy solutions and all nonnegative test functions $\phi \in C_c^\infty(\Omega_T)$.

Passing from all nonnegative to all test functions is done by considering test functions on the form $\phi = \phi_+ - \phi_-$, with $\phi_\pm C_c^\infty(\Omega_T)$. □

Chapter 3

Auxiliary result

3.1 Dual equation

In order to establish uniqueness, the dual equation needs to be given. This section will find it by Kružkov's doubling of variables technique.

Proposition 3.1.1. *Let u and v be entropy sub- and supersolutions, respectively, of* (1.1) *with initial data u_0 and v_0. Assume that* (A.1)-(A.4) *hold. Then*

$$\begin{aligned}\iint_{\Omega_T} &\eta(u(x,t),v(x,t))\partial_t\psi(x,t) \\ &+ q(u(x,t),v(x,t))\cdot D\psi(x,t) \\ &+ \eta(A(u(x,t)),A(v(x,t)))\mathcal{L}^{\mu^*}[\psi(\cdot,t)](x)\mathrm{d}x\mathrm{d}t \geq 0,\end{aligned} \tag{3.1}$$

for all nonnegative $\psi \in C_c^\infty(\Omega_T)$, with entropy $\eta(u,v) = (u-v)^+$, and $q(u,v) =$ $\mathrm{sign}(u-v)^+[f(u)-f(v)]$.

Remark 3.1.2. Even though this proposition holds for (A.1)-(A.4), it also holds for (A.1)-(A.3) and (A.5).

Proof The proof will only pay attention to the nonlocal operators $\mathcal{L}_r^\mu$, $\mathcal{L}^{\mu,r}$ and $b^{\mu,r}$ since the calculations of the local operators are assumed to be well-known.

Let $\phi = \phi(x,t,y,s)$ be a nonnegative test function in (x,t) and (y,s) with compact support for $t \in (0,T)$ and $s \in (0,T)$. Further, utilize Kružkov's doubling of variables technique; first described in [3]. That is, let $k = v(y,s)$, fix (y,s) in

Ω_T, and write up Definition 2.3.9 II):

$$\begin{aligned} &\iint_{\Omega_T} (u-v)^+\partial_t\phi + \mathrm{sign}(u-v)^+[f(u)-f(v)]\cdot D\phi \,\mathrm{d}x\mathrm{d}t \\ &+\iint_{\Omega_T} (A(u)-A(v))^+\mathcal{L}_r^{\mu^*}[\phi(\cdot,t,y,s)](x)\,\mathrm{d}x\mathrm{d}t \\ &+\iint_{\Omega_T} \mathrm{sign}(u-v)^+\mathcal{L}^{\mu,r}[A(u(\cdot,t))](x)\phi\,\mathrm{d}x\mathrm{d}t \\ &+\iint_{\Omega_T} (A(u)-A(v))^+ b^{\mu^*,r}\cdot D\phi\,\mathrm{d}x\mathrm{d}t \geq 0. \end{aligned} \tag{3.2}$$

Further, let $k = u(x,t)$ and write up Definition 2.3.9 III) for a fixed point (x,t):

$$\begin{aligned} &\iint_{\Omega_T} (v-u)^-\partial_t\phi + \mathrm{sign}(v-u)^-[f(u)-f(v)]\cdot D\phi \,\mathrm{d}y\mathrm{d}s \\ &+\iint_{\Omega_T} (A(v)-A(u))^-\mathcal{L}_r^{\mu^*}[\phi(x,t,\cdot,s)](y)\,\mathrm{d}y\mathrm{d}s \\ &+\iint_{\Omega_T} -\mathrm{sign}(v-u)^-\mathcal{L}^{\mu,r}[A(v(\cdot,s))](y)\phi\,\mathrm{d}y\mathrm{d}s \\ &+\iint_{\Omega_T} (A(v)-A(u))^- b^{\mu^*,r}\cdot D\phi\,\mathrm{d}y\mathrm{d}s \geq 0. \end{aligned} \tag{3.3}$$

Notice that $(v-u)^- = (u-v)^+$ and $(A(v)-A(u))^- = (A(u)-A(v))^+$ (the last equality is due to (A.2)). Integrate (3.2) over Ω_T with respect to (y,s) and integrate (3.3) over Ω_T with respect to (x,t), use Fubini's theorem on (3.3) (compact support of ϕ will ensure applicability of the theorem), and add the two equations together (let $\mathrm{d}w$ denote $\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s$):

$$\begin{aligned} &\iiiint_{\Omega_T\times\Omega_T} (u-v)^+(\partial_t+\partial_s)\phi \\ &\qquad + \mathrm{sign}(u-v)^+[f(u)-f(v)]\cdot(D_x+D_y)\phi\,\mathrm{d}w \\ &+\iiiint_{\Omega_T\times\Omega_T} (A(u)-A(v))^+(\mathcal{L}_r^{\mu^*}[\phi](x)+\mathcal{L}_r^{\mu^*}[\phi](y))\mathrm{d}w \\ &+\iiiint_{\Omega_T\times\Omega_T} \mathrm{sign}(u-v)^+(\mathcal{L}^{\mu,r}[A(u)](x)-\mathcal{L}^{\mu,r}[A(v)](y))\phi\mathrm{d}w \\ &+\iiiint_{\Omega_T\times\Omega_T} (A(u)-A(v))^+ b^{\mu^*,r}\cdot(D_x+D_y)\phi\mathrm{d}w \geq 0. \end{aligned} \tag{3.4}$$

Consider the third integral in (3.4); call it I:

$$
\begin{aligned}
I = \iiiint_{\Omega_T\times\Omega_T} &\operatorname{sign}(u-v)^+ \\
&\left(\int_{|z|>r} A(u(x+z,t)) - A(u(x,t))\mathrm{d}\mu(z)\right. \\
&\left. - \int_{|z|>r} A(v(y+z,s)) - A(v(y,s))\mathrm{d}\mu(z)\right)\phi\mathrm{d}w \\
= \iiiint_{\Omega_T\times\Omega_T} &\int_{|z|>r} \operatorname{sign}(u-v)^+ \\
&\Big((A(u(x+z,t)) - A(v(y+z,s))) \\
&\qquad - (A(u(x,t)) - A(v(y,s))\Big)\mathrm{d}\mu(z)\phi\mathrm{d}w \\
\leq \iiiint_{\Omega_T\times\Omega_T} &\int_{|z|>r} (A(u(x+z,t)) - A(v(y+z,s)))^+ \\
&\qquad - (A(u(x,t)) - A(v(y,s))^+\mathrm{d}\mu(z)\phi\mathrm{d}w,
\end{aligned}
$$

where the inequality is due to a Kato type of inequality. Continue with the change of variables $(z,x,t,y,s)\mapsto(-z,x+z,t,y+z,s)$:

$$
\begin{aligned}
I &\leq \iiiint_{\Omega_T\times\Omega_T}\int_{|z|>r} (A(u(x+z,t)) - A(v(y+z,s)))^+ \\
&\qquad - (A(u(x,t)) - A(v(y,s))^+\mathrm{d}\mu(z)\phi(x,t,y,s)\mathrm{d}w \\
&= \iiiint_{\Omega_T\times\Omega_T}\int_{|z|>r} (A(u(x+z,t)) - A(v(y+z,s)))^+\phi(x,t,y,s)\mathrm{d}\mu(z) \\
&\quad - \int_{|z|>r} (A(u(x,t)) - A(v(y,s))^+\phi(x,t,y,s)\mathrm{d}\mu(z)\mathrm{d}w \\
&= \iiiint_{\Omega_T\times\Omega_T}\left(\int_{|z|>r}\phi(x+z,t,y+z,s)\mathrm{d}\mu^*(z)\right. \\
&\quad \left. - \int_{|z|>r}\phi(x,t,y,s)\mathrm{d}\mu^*(z)\right)(A(u(x,t)) - A(v(y,s)))^+\mathrm{d}w \\
&= \iiiint_{\Omega_T\times\Omega_T}(A(u)-A(v))^+\tilde{\mathcal{L}}^{\mu^*,r}[\phi(\cdot,t,\cdot,s)](x,y)\mathrm{d}w,
\end{aligned}
$$

where

$$
\tilde{\mathcal{L}}^{\mu,r}[\phi](x,y) := \int_{|z|>r}\phi(x+z,y+z) - \phi(x,y)\mathrm{d}\mu(z).
$$

Further, insert the above into (3.4)

$$\begin{aligned}
&\iiiint_{\Omega_T\times\Omega_T} (u-v)^+(\partial_t+\partial_s)\phi \\
&\qquad\qquad + \operatorname{sign}(u-v)^+[f(u)-f(v)]\cdot(D_x+D_y)\phi\,\mathrm{d}w \\
&+ \iiiint_{\Omega_T\times\Omega_T} (A(u)-A(v))^+(\mathcal{L}_r^{\mu^*}[\phi](x)+\mathcal{L}_r^{\mu^*}[\phi](y))\mathrm{d}w \\
&+ \iiiint_{\Omega_T\times\Omega_T} (A(u)-A(v))^+\tilde{\mathcal{L}}^{\mu^*,r}[\phi(\cdot,t,\cdot,s)](x,y)\mathrm{d}w \\
&+ \iiiint_{\Omega_T\times\Omega_T} (A(u)-A(v))^+b^{\mu^*,r}\cdot(D_x+D_y)\phi\mathrm{d}w \ge 0,
\end{aligned}$$

and notice that

$$\begin{aligned}
&\tilde{\mathcal{L}}^{\mu^*,r}[\phi(\cdot,t,\cdot,s)](x,y)+b^{\mu^*,r}\cdot(D_x+D_y)\phi \\
&= \int_{|z|>r} \phi(x+z,t,y+z,s)-\phi-z\cdot(D_x+D_y)\phi\mathbf{1}_{|z|\le1}\mathrm{d}\mu^*(z).
\end{aligned} \tag{3.5}$$

To proceed, let $r\to0$.

By Lemma 2.2.7, $\mathcal{L}_r^{\mu^*}[\phi](x)$ and $\mathcal{L}_r^{\mu^*}[\phi](y)$ go to zero as r goes to zero. Since the integrand in (3.5) is integrable by a Taylor expansion, r can be sent to zero without any further considerations. For simplicity define

$$\tilde{\mathcal{L}}^{\mu}[\phi](x,y) := \int_{|z|>0} \phi(x+z,y+z)-\phi(x,y)-z\cdot(D_x+D_y)\phi\mathbf{1}_{|z|\le1}\mathrm{d}\mu(z),$$

which gives the following result after letting r go to zero:

$$\begin{aligned}
&\iiiint_{\Omega_T\times\Omega_T} (u-v)^+(\partial_t+\partial_s)\phi \\
&\qquad\qquad + \operatorname{sign}(u-v)^+[f(u)-f(v)]\cdot(D_x+D_y)\phi\,\mathrm{d}w \\
&+ \iiiint_{\Omega_T\times\Omega_T} (A(u)-A(v))^+\tilde{\mathcal{L}}^{\mu^*}[\phi(\cdot,t,\cdot,s)](x,y)\mathrm{d}w \ge 0.
\end{aligned} \tag{3.6}$$

To conclude, the test function ϕ has to be specified. Let

$$\phi(x,t,y,s) = \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right),$$

for some $\varepsilon_1,\varepsilon_2>0$, some $\psi\in C_c^\infty(\Omega_T)$, and with ω being mollifiers. In addition $\hat{\omega}_{\varepsilon_1}(x)=\omega_{\varepsilon_1}(x_1)\dots\omega_{\varepsilon_1}(x_d)$, that is, there is a mollifier in each spatial dimension.

It is trivial to verify that

$$(\partial_t+\partial_s)\phi = \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)(\partial_t+\partial_s)\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)$$

and that

$$(D_x + D_y)\phi = \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)(D_x + D_y)\psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right).$$

A bit more work has to be done on the nonlocal operator:

$$\begin{aligned}
\tilde{\mathcal{L}}^{\mu^*}[\phi(\cdot,t,\cdot,s)](x,y) = &\int_{|z|>0} \phi(x+z,t,y+z,s) - \phi(x,y) \\
&\qquad - z\cdot(D_x + D_y)\phi\mathbf{1}_{|z|\le 1}\mathrm{d}\mu^*(z) \\
= &\int_{|z|>0} \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\psi\left(\frac{x+y}{2}+z, \frac{t+s}{2}\right) \\
&\qquad - \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\psi \\
&\qquad - z\cdot\hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \\
&\qquad (D_x + D_y)\psi\mathbf{1}_{|z|\le 1}\mathrm{d}\mu^*(z) \\
= &\hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \\
&\int_{|z|>0} \psi\left(\frac{x+y}{2}+z, \frac{t+s}{2}\right) \\
&\qquad - \psi - z\cdot(D_x + D_y)\psi\mathbf{1}_{|z|\le 1}\mathrm{d}\mu^*(z) \\
= &\hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \\
&\int_{|z|>0} \psi\left(\frac{x+y}{2}+z, \frac{t+s}{2}\right) \\
&\qquad - \psi - z\cdot D\psi\mathbf{1}_{|z|\le 1}\mathrm{d}\mu^*(z) \\
= &\hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right),
\end{aligned}$$

where $D\psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) = (D_x + D_y)\psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$.

In the following, the limit when $(\varepsilon_1, \varepsilon_2) \to (0,0)$ reduces inequality (3.6) to

$$\begin{aligned}
\iint_{\Omega_T} &\eta(u(x,t),v(x,t))\partial_t\psi(x,t) \\
&+ q(u(x,t),v(x,t))\cdot D\psi(x,t) \\
&+ \eta(A(u(x,t)),A(v(x,t)))\mathcal{L}^{\mu^*}[\psi(\cdot,t)](x)\mathrm{d}x\mathrm{d}t \ge 0,
\end{aligned}$$

which is the inequality sought. Thus, it remains to take the limit. Again, the convergence of the local terms is well-known, and the focus will be on the convergence of the nonlocal term.

The convergence of the nonlocal term is done by a direct argument. That is, define

$$
\begin{aligned}
M := \Bigg| & \iiiint_{\Omega_T\times\Omega_T} \eta(A(u(x,t)),A(v(y,s))) \\
& \qquad \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\mathcal{L}^{\mu^*}\left[\psi\left(\cdot,\frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right)\mathrm{d}w \\
& - \iint_{\Omega_T} \eta(A(u(x,t)),A(v(x,t)))\mathcal{L}^{\mu^*}[\psi(\cdot,t)](x)\mathrm{d}x\mathrm{d}t\Bigg|,
\end{aligned}
$$

and show that $M \to 0$.

Notice that

$$
\iint_{\Omega_T} \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\mathrm{d}y\mathrm{d}s = 1,
$$

and add and subtract

$$
\begin{aligned}
& \iiiint_{\Omega_T\times\Omega_T} \eta(A(u(x,t)),A(v(x,t))) \\
& \qquad \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\mathcal{L}^{\mu^*}\left[\psi\left(\cdot,\frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right)\mathrm{d}w
\end{aligned}
$$

to obtain

$$
\begin{aligned}
M \leq & \iiiint_{\Omega_T\times\Omega_T} |\eta(A(u(x,t)),A(v(y,s))) - \eta(A(u(x,t)),A(v(x,t)))| \\
& \qquad \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\mathcal{L}^{\mu^*}\left[\psi\left(\cdot,\frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right)\mathrm{d}w \\
& + \iiiint_{\Omega_T\times\Omega_T} \eta(A(u(x,t)),A(v(y,s)))\hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \\
& \qquad \left|\mathcal{L}^{\mu^*}\left[\psi\left(\cdot,\frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right) - \mathcal{L}^{\mu^*}[\psi(\cdot,t)](x)\right|\mathrm{d}w \\
=: & \, M_1 + M_2.
\end{aligned}
$$

Observe that $\mathcal{L}^{\mu}[\psi] \in L^1(\Omega_T)$ (by Lemma 2.2.7 ii)) and that $u, v \in L^\infty(\Omega_T)$ (and, thus, $A(u), A(v) \in L^\infty(\Omega_T)$ by (A.2)). In addition, since η is locally Lipschitz, $|\eta(A(u(x,t)),A(v(y,s))) - \eta(A(u(x,t)),A(v(x,t)))| \leq L_\eta|A(v(x,t)) - A(v(y,s))|$.

The convergence of M_1 is ensured by first considering a change in variables

$x-y=y'$ and $t-s=s'$, and then utilizing the continuity of the L^1-translation

$$
\begin{aligned}
M_1 \le& L_\eta \iiiint_{\Omega_T\times\Omega_T} |A(v(x,t))-A(v(y,s))| \\
&\left|\mathcal{L}^{\mu^*}\left[\psi\left(\cdot,\frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right)\right| \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\mathrm{d}w \\
=& L_\eta \iint_{\Omega_T}\iint_{\Omega_T} |A(v(x,t))-A(v(x+y',t+s'))| \\
&\left|\mathcal{L}^{\mu^*}\left[\psi\left(\cdot,t+\frac{s'}{2}\right)\right]\left(x+\frac{y'}{2}\right)\right| \hat{\omega}_{\varepsilon_1}\left(\frac{y'}{2}\right)\omega_{\varepsilon_2}\left(\frac{s'}{2}\right)\mathrm{d}x\mathrm{d}t\mathrm{d}y'\mathrm{d}s' \\
=& L_\eta \iint_{\Omega_T}\left(\iint_{\Omega_T} |(A(v(x,t))-A(v(x+y',t+s'))| \right.\\
&\left.\left|\mathcal{L}^{\mu^*}\left[\psi\left(\cdot,t+\frac{s'}{2}\right)\right]\left(x+\frac{y'}{2}\right)\right|\mathrm{d}x\mathrm{d}t\right) \hat{\omega}_{\varepsilon_1}\left(\frac{y'}{2}\right)\omega_{\varepsilon_2}\left(\frac{s'}{2}\right)\mathrm{d}y'\mathrm{d}s' \\
\le& L_\eta \sup_{|y'|\le\varepsilon_1,|s'|\le\varepsilon_2} \Big\|(A(v(x,t))-A(v(x+y',t+s'))) \\
&\mathcal{L}^{\mu^*}\left[\psi\left(\cdot,t+\frac{s'}{2}\right)\right]\left(x+\frac{y'}{2}\right)\Big\|_{L^1(\Omega_T)} \overset{(\varepsilon_1,\varepsilon_2)\to(0,0)}{\to} 0.
\end{aligned}
$$

The convergence of M_2 is done in a similar manner, and the proof is complete.
□

Chapter 4

Uniqueness

This chapter is devoted to finding local and global contractions - for the positive part of the solution u - to the Cauchy problem given by (1.1).

4.1 Local contraction

This section focuses on obtaining a local contraction. Later this will be utilized to prove a global contraction, uniqueness, a comparison principle, an L^1 bound, an L^∞ bound, a BV bound and even existence.

The main theorem of this section will be stated below after a lemma with the properties of $\tilde{K}$ and K. $\tilde{K}$ is a kernel to $\mathcal{L}^\mu$ (with $\mathrm{d}\mu(z) = \frac{c_\lambda}{|z|^{d+\alpha}}\mathrm{d}z$), and K is a solution to $\partial_t K + L_A(\mathcal{L}^{\mu^*}[K])^+ \leq 0$.

Lemma 4.1.1. *I) Let $\tilde{K}(x,t) := \mathcal{F}^{-1}(e^{-t|\xi|^\alpha})(x)$ for $t > 0$ and $x \in \mathbb{R}^d$. It has the following properties*

- *i) $\tilde{K}$ is nonnegative;*
- *ii) $\tilde{K}(x,t) = t^{-\frac{1}{\alpha}}\tilde{K}(t^{-\frac{1}{\alpha}}x, 1)$ for all $t > 0$;*
- *iii) $\int_{\mathbb{R}^d} \tilde{K}(x,1)\mathrm{d}x = \mathcal{F}(\tilde{K}(x,1))(0) = 1$;*
- *iv) $\{\tilde{K}(x,t)\}_{t>0}$ is an approximate unit as $t \to 0$ in the sense of distributions;*
- *v) $\tilde{K}(x,t+s) = \tilde{K}(x,t) * \tilde{K}(x,s)$ for all $t > 0$ and all $s > 0$;*
- *vi) $u(x,t) = \tilde{K}(x,t) * u(x,0)$ is a solution to $\partial_t u - \mathcal{L}^\mu[u] = 0$; and*
- *vii) $\tilde{K}(x-y,t) = \tilde{K}(y-x,t)$ for all $t > 0$ and $y \in \mathbb{R}^d$.*

II) Let $\tilde{T} = \max\{T, \frac{T}{L_A}\}$, $0 < \tau < \tilde{T}$ and $0 \leq t \leq \tau$. There exists a $\Phi(x,t)$ such that $K(x,t) := (\Phi * \rho_\delta)(x, \frac{1}{L_A}(\tau - t))$, where ρ_δ is a mollifier in both space and time (to be defined later), and $L_A > 0$ is the Lipschitz constant of A. K has the following properties

- *i) K is nonnegative and bounded;*

ii) $\|K(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq 1$ *for all* $t \in [0,\tau]$*; and*

iii) K *is a solution to* $\partial_t K + L_A(\mathcal{L}^{\mu^*}[K])^+ \leq 0$.

Remark 4.1.2. i) The properties given in I) (and their proofs) are found in e.g. [4, Lemma 3.5.1].

ii) The properties given in II) are proved throughout this section.

Theorem 4.1.3. *i) Let* $\mu = 0$*, and let* u *and* v *be entropy sub- and supersolutions of* (1.1) *with* u_0 *and* v_0*, fulfilling* (A.3)*, as initial data, respectively. Then for all* $t \in (0,T)$*,* $M > 0$ *and* $x_0 \in \mathbb{R}^d$

$$\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+ \mathrm{d}x \leq \int_{B(x_0,\,M+L_f t)} (u_0(x) - v_0(x))^+ \mathrm{d}x,$$

where L_f *is the Lipschitz constant of* f *and* f *satisfies* (A.1)*.*

ii) Let $A(u) = u$*, and let* u *and* v *be entropy sub- and supersolutions of* (1.1) *with* u_0 *and* v_0*, fulfilling* (A.3)*, as initial data, respectively. Then for all* $t \in (0,T)$*,* $M > 0$ *and* $x_0 \in \mathbb{R}^d$

$$\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+ \mathrm{d}x \leq \int_{B(x_0,\,M+L_f t)} (\tilde{K}(\cdot,t) * (u_0(\cdot) - v_0(\cdot))^+)(x) \mathrm{d}x,$$

where L_f *is the Lipschitz constant of* f *and* f *satisfies* (A.1)*, and* $\tilde{K}$ *is the kernel of* $\mathcal{L}^\mu$ *satisfying Lemma 4.1.1 I), with* $\mathrm{d}\mu(z) = \frac{c_\lambda}{|z|^{d+\alpha}}\mathrm{d}z$ *symmetric and fulfilling* (A.4)*.*

iii) Let $A(u)$ *be non-constant, nonlinear and possibly degenerate, and let* u *and* v *be entropy sub- and supersolutions of* (1.1) *with* u_0 *and* v_0*, fulfilling* (A.3)*, as initial data, respectively. Then for all* $t \in (0,T)$*,* $M > 0$ *and* $x_0 \in \mathbb{R}^d$

$$\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+ \mathrm{d}x \leq \int_{B(x_0,\,M+1+L_f t)} (K(-\cdot,0) * (u_0(\cdot) - v_0(\cdot))^+)(x) \mathrm{d}x,$$

where L_f *is the Lipschitz constant of* f *and* f *satisfies* (A.1)*,* $L_A > 0$ *is the Lipschitz constant of* A *and* A *satisfies* (A.2)*, and* K *satisfying Lemma 4.1.1 II), with* $\mathrm{d}\mu(z)$ *fulfilling* (A.5)*.*

Remark 4.1.4. i) By assuming that $A(u)$ is non-constant, L_A will never be zero. If $A(u)$ is a constant, then (1.1) is the standard conservation law ($\mu = 0$).

ii) Notice that the +1-factor in $B(x_0, M+1+L_f t)$ in theorem 4.1.3 iii), may seem somewhat arbitrarily chosen, but the +1-factor depends on the choice of K. However, it is just a constant and will not disrupt the result in any way.

iii) Observe that if $\mathrm{d}\mu(z)$ was assumed to be *symmetric* in theorem 4.1.3 iii), the integrand on the right-hand side would look like $(K(\cdot,0)*(u_0(\cdot)-v_0(\cdot))^+)(x)$.

iv) In the proof of the above theorem, it will be evident that $K(-\cdot,0)$ could be interchanged with $(\Phi*\rho_\delta)(-\cdot,\frac{1}{L_A}t)$.

The rest of this section is devoted to the lemmas needed to prove Theorem 4.1.3 iii), and in the end the proof itself will be given.

With the main result in mind, some manipulation of the dual equation, given in Proposition 3.1.1, is needed to continue.

Lemma 4.1.5. *Let $\psi(x,t)=\Gamma(x,t)\Theta(t)$ be nonnegative and in $C_c^\infty(\Omega_T)$, and let $\eta(u,v)(x,t)=(u(x,t)-v(x,t))^+$, $\Gamma(x,t)\in C_c^\infty(\Omega_T)$ nonnegative and $\Theta\in C_c^\infty((0,T))$ nonnegative.*

i) Then the dual equation obtained in Proposition 3.1.1 is

$$0\le\iint_{\Omega_T}\eta(u,v)(x,t)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t$$
$$+\iint_{\Omega_T}\Theta(t)\eta(u,v)\left[\partial_t\Gamma+L_f|D\Gamma|+L_A\big(\mathcal{L}^{\mu^*}\left[\Gamma(\cdot,t)\right](x)\big)^+\right]\mathrm{d}x\mathrm{d}t.$$

ii) Assume further that $\Gamma\in C([0,T);L^1(\mathbb{R}^d))\cap C^\infty(\Omega_T)$ solves

$$\partial_t\Gamma+L_f|D\Gamma|+L_A\big(\mathcal{L}^{\mu^*}\left[\Gamma(\cdot,t)\right](x)\big)^+\le 0.$$

Then for all $t\in(0,T)$

$$\int_{\mathbb{R}^d}\eta(u,v)(x,t)\Gamma(x,t)\mathrm{d}x\le\int_{\mathbb{R}^d}\eta(u_0,v_0)(x)\Gamma(x,0)\mathrm{d}x$$

is obtained.

Proof i) Write up the dual equation, and insert $\psi(x,t)=\Gamma(x,t)\Theta(t)$ into it

$$0\le\iint_{\Omega_T}\eta(u(x,t),v(x,t))\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t$$
$$+\iint_{\Omega_T}\eta(u,v)\partial_t\Gamma(x,t)\Theta(t)+\Theta(t)q(u(x,t),v(x,t))\cdot D\Gamma(x,t)$$
$$+\eta(A(u(x,t)),A(v(x,t)))\Theta(t)\mathcal{L}^{\mu^*}[\Gamma(\cdot,t)](x)\mathrm{d}x\mathrm{d}t.$$

Notice that Cauchy-Schwarz' inequality gives that $|q(u,v)\cdot D\Gamma|\le|q(u,v)||D\Gamma|$. In addition observe that $q(u,v)\le L_f\eta(u,v)$ and $\eta(A(u),A(v))\le L_A\eta(u,v)$ (the last inequality leads to $\eta(A(u),A(v))\mathcal{L}^{\mu}[\Gamma] \leq L_A\eta(u,v)\big(\mathcal{L}^{\mu}[\Gamma]\big)^{+}\big)$. Combining these observations gives

$$\begin{aligned}
0 \leq& \iint_{\Omega_T} \eta(u,v)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \eta(u,v)\partial_t\Gamma(x,t)\Theta(t) + \Theta(t)q(u(x,t),v(x,t))\cdot D\Gamma(x,t) \\
&\qquad + \eta(A(u(x,t)),A(v(x,t)))\Theta(t)\mathcal{L}^{\mu^*}[\Gamma(\cdot,t)](x)\mathrm{d}x\mathrm{d}t \\
\leq& \iint_{\Omega_T} \eta(u,v)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \Theta(t)\eta(u,v)\partial_t\Gamma(x,t) + \Theta(t)L_f\eta(u,v)|D\Gamma(x,t)| \\
&\qquad + \Theta(t)L_A\eta(u,v)\big(\mathcal{L}^{\mu^*}[\Gamma(\cdot,t)](x)\big)^{+}\mathrm{d}x\mathrm{d}t \\
=& \iint_{\Omega_T} \eta(u,v)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \Theta(t)\eta(u,v)\Big[\partial_t\Gamma(x,t) + L_f|D\Gamma(x,t)| + L_A\big(\mathcal{L}^{\mu^*}[\Gamma(\cdot,t)](x)\big)^{+}\Big]\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

and the proof of this part is complete.

ii) Write up the result of Lemma 4.1.5 i)

$$\begin{aligned}
0 \leq& \iint_{\Omega_T} \eta(u,v)(x,t)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \Theta(t)\eta(u,v)\left[\partial_t\Gamma + L_f|D\Gamma| + L_A\big(\mathcal{L}^{\mu^*}\left[\Gamma(\cdot,t)\right](x)\big)^{+}\right]\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

and use the extra assumption on Γ to get

$$0 \leq \iint_{\Omega_T} \eta(u,v)(x,t)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t.$$

Let $\Theta \in C_c^{\infty}((0,T))$ nonnegative be defined by

$$\Theta(t) = \Theta_{\varepsilon}(t) = \int_{-\infty}^{t} \omega_{\varepsilon}(s-t_1) - \omega_{\varepsilon}(s-t_2)\mathrm{d}s, \tag{4.1}$$

where $0 < t_1 < t_2 < T$. $\Theta_{\varepsilon}(t)$ is (loosely speaking) a smooth approximation of a function which is, when ε is small enough, zero near $t=0$ and $t=T$ and equal to one at the points $t=t_1$ and $t=t_2$. Taking the derivative, $\Theta_{\varepsilon}'(t)$, gives $\omega_{\varepsilon}(t-t_1) - \omega_{\varepsilon}(t-t_2)$. Insert this into the reduced dual equation to get

$$\iint_{\Omega_T} \eta(u,v)(x,t)\Gamma(x,t)\omega_{\varepsilon}(t-t_2)\mathrm{d}x\mathrm{d}t \leq \iint_{\Omega_T} \eta(u,v)(x,t)\Gamma(x,t)\omega_{\varepsilon}(t-t_1)\mathrm{d}x\mathrm{d}t.$$

In order to let $\varepsilon \to 0$, consider the right-hand side (the computations for the left-hand side is quite similar), and do a direct argument:

$$\begin{aligned}
&\left| \iint_{\Omega_T} (u-v)^+(x,t)\Gamma(x,t)\omega_\varepsilon(t-t_1)\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^d} (u-v)^+(x,t_1)\Gamma(x,t_1)\mathrm{d}x \right| \\
&\leq \iint_{\Omega_T} \left|(u-v)^+(x,t)\Gamma(x,t) - (u-v)^+(x,t_1)\Gamma(x,t_1)\right| \omega_\varepsilon(t-t_1)\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^d} \left|(u-v)^+(x,t)\Gamma(x,t) - (u-v)^+(x,t+s')\Gamma(x,t+s')\right| \mathrm{d}x\, \omega_\varepsilon(s')\mathrm{d}s' \\
&\leq \sup_{|s'|\leq\varepsilon} \|(u-v)^+(x,t)\Gamma(x,t) - (u-v)^+(x,t+s')\Gamma(x,t+s')\|_{L^1(\mathbb{R}^d)}
\end{aligned}$$

Since $\eta(u,v) \in L^\infty(\Omega_T)$ and $\Gamma \in C([0,T);L^1(\mathbb{R}^d))$, taking $\varepsilon \to 0$ will, by definition of the space $C([0,T);L^1(\mathbb{R}^d))$, send the whole expression to zero. Rename t_2 and let $t_1 \to 0$:

$$\begin{aligned}
&\|\eta(u,v)(x,t_1)\Gamma(x,t_1) - \eta(u_0,v_0)(x)\Gamma(x,0)\|_{L^1(\mathbb{R}^d)} \\
&\leq \|\eta(u,v)\|_{L^\infty(\mathbb{R}^d)}\|\Gamma(x,t_1) - \Gamma(x,0)\|_{L^1(\mathbb{R}^d)} \\
&\quad + \|(\eta(u_0,v_0)(x) - \eta(u,v)(x,t))\Gamma(x,0)\|_{L^1(\mathbb{R}^d)},
\end{aligned}$$

where the first term goes to zero as $t_1 \to 0$ since $\Gamma \in C([0,T);L^1(\mathbb{R}^d))$. The second term, however, needs more inspection. By Definition 2.3.9 II) it is known that $\|u(\cdot,t) - u_0(\cdot)\|_{L^1_{\mathrm{loc}}(\mathbb{R}^d)} \to 0$ as $t \to 0$. Since that holds, there is a subsequence in t such that $\lim_{t\to 0} u(x,t) = u_0(x)$ a.e. Notice that $|\eta(u_0,v_0)(x) - \eta(u,v)(x,t)|\Gamma(x,0)$ is dominated by $(\|\eta(u_0,v_0)\|_{L^\infty} + \|\eta(u,v)\|_{L^\infty})\Gamma(x,0) \in L^1(\mathbb{R}^d)$. Thus, by Lebesgue's dominated convergence theorem, the second term also goes to zero when $t_1 \to 0$.

The above calculations can thus be summed up in

$$\int_{\mathbb{R}^d} \eta(u,v)(x,t)\Gamma(x,t)\mathrm{d}x \leq \int_{\mathbb{R}^d} \eta(u_0,v_0)(x)\Gamma(x,0)\mathrm{d}x,$$

for all $t \in (0,T)$. □

Remark 4.1.6. i) Let $\mu = 0$ in Lemma 4.1.5 i), then

$$\begin{aligned}
0 \leq &\iint_{\Omega_T} \eta(u,v)(x,t)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \Theta(t)\eta(u,v)\left[\partial_t\Gamma + L_f|D\Gamma|\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

ii) Let $A(u) = u$ in Lemma 4.1.5 i), then

$$\begin{aligned}
0 \leq &\iint_{\Omega_T} \eta(u,v)(x,t)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t \\
&+ \iint_{\Omega_T} \Theta(t)\eta(u,v)\left[\partial_t\Gamma + L_f|D\Gamma| + \mathcal{L}^{\mu^*}\left[\Gamma(\cdot,t)\right](x)\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

In previous proofs of local contractions (see e.g. [4, 1, 5] for the proofs of remark 4.1.6 i) and ii)), the idea has - more or less - been to do as in Lemma 4.1.5 ii). That is, one wants to find a solution to

$$\partial_t \Gamma + L_f |D\Gamma| + L_A \big(\mathcal{L}^{\mu^*} \left[\Gamma(\cdot,t)\right](x)\big)^+ \leq 0. \tag{4.2}$$

In particular, a solution to

$$\begin{cases} \partial_t u = (\mathcal{L}^{\mu}[u(\cdot,t)](x))^+ & (x,t) \in \Omega_{\tilde{T}} \\ u(x,0) = u_0(x) & x \in \mathbb{R}^d \end{cases} \tag{4.3}$$

is needed. Notice that $(\mathcal{L}^{\mu}[u(\cdot,t)](x))^+ = \max\{\mathcal{L}^{\mu}[u(\cdot,t)](x), 0\}$, and that $\tilde{T} = \max\{T, \frac{1}{L_A}T\}$ (where $L_A > 0$ is the Lipschitz constant of A).

Further, Γ is to be chosen as a convolution between two functions that are subsolutions of a certain differential equation as well. In the following, a lemma will explain which two subsolutions one needs to find, but first an auxiliary result.

Lemma 4.1.7. *If $\phi \in C_c(\mathbb{R}^d)$ is nonnegative and $f \in C(\mathbb{R}^d)$, then*

$$\max\{(\phi * f)(x), 0\} \leq (\phi * \max\{f, 0\})(x).$$

Proof First, consider these two trivialities

$$0 \leq \max\{f(y), 0\},$$

and

$$f(y) \leq \max\{f(y), 0\}.$$

Second, multiply both inequalities with $\phi(x-y)$ and integrate over $\mathbb{R}^d$ w.r.t. y:

$$0 \leq \int_{\mathbb{R}^d} \phi(x-y) \max\{f(y), 0\} \mathrm{d}y, \tag{4.4}$$

and

$$\int_{\mathbb{R}^d} \phi(x-y) f(y) \mathrm{d}y \leq \int_{\mathbb{R}^d} \phi(x-y) \max\{f(y), 0\} \mathrm{d}y. \tag{4.5}$$

By (4.4),

$$\begin{aligned} &\left(\int_{\mathbb{R}^d} \phi(x-y) \max\{f(y), 0\} \mathrm{d}y\right)^+ \\ &= \max\left\{\int_{\mathbb{R}^d} \phi(x-y) \max\{f(y), 0\} \mathrm{d}y, 0\right\} \\ &= \int_{\mathbb{R}^d} \phi(x-y) \max\{f(y), 0\} \mathrm{d}y. \end{aligned}$$

Therefore, taking the positive part on both sides of (4.5) completes the proof. □

Lemma 4.1.8. *Assume that* $\phi(x,t) \in C^1(\Omega_T) \cap L^\infty((0,T);L^1(\mathbb{R}^d))$ *is nonnegative and solves*

$$\partial_t \phi(x,t) + L_f|D\phi(x,t)| \le 0, \tag{4.6}$$

and that $\psi(x,t) \in C^1((0,T);C^2(\mathbb{R}^d)) \cap L^\infty(\Omega_T)$ *is nonnegative and solves*

$$\partial_t \psi(x,t) + L_A(\mathcal{L}^{\mu^*}[\psi(\cdot,t)](x))^+ \le 0. \tag{4.7}$$

Then $\Gamma(x,t) = (\psi(\cdot,t) * \phi(\cdot,t))(x)$ *solves*

$$\partial_t \Gamma(x,t) + L_f|D\Gamma(x,t)| + L_A\big(\mathcal{L}^{\mu^*}[\Gamma(\cdot,t)](x)\big)^+ \le 0.$$

Proof The proof is mainly a straight forward computation.

Start by doing the time derivative:

$$\partial_t \Gamma(x,t) = \partial_t(\psi(\cdot,t) * \phi(\cdot,t))(x) = (\partial_t \psi(\cdot,t) * \phi(\cdot,t))(x) + (\psi(\cdot,t) * \partial_t \phi(\cdot,t))(x).$$

Continue with the spatial derivative:

$$D\Gamma(x,t) = D(\psi(\cdot,t) * \phi(\cdot,t))(x) = (\psi(\cdot,t) * D\phi(\cdot,t))(x),$$

and

$$\begin{aligned}
|D\Gamma(x,t) =& |(\psi(\cdot,t) * D\phi(\cdot,t))(x)| \\
=& \left| \int_{\mathbb{R}^d} \psi(x-y,t) D\phi(y,t) \mathrm{d}y \right| \\
\le& \int_{\mathbb{R}^d} \psi(x-y,t)|D\phi(y,t)|\mathrm{d}y \\
=& (\psi(\cdot,t) * |D\phi(\cdot,t)|)(x).
\end{aligned}$$

The Lévy-operator requires a bit more work, but with the help of Fubini's theorem, the result follows

$$\begin{aligned}
\mathcal{L}^\mu[\Gamma(\cdot,t)](x) =& \int_{|z|>0} \Gamma(x+z,t) - \Gamma(x,t) - z \cdot D\Gamma \mathbf{1}_{|z|\le 1} \mathrm{d}\mu(z) \\
=& \int_{|z|>0} \int_{\mathbb{R}^d} \psi(x+z-y,t)\phi(y)\mathrm{d}y - \int_{\mathbb{R}^d} \psi(x-y,t)\phi(y,t)\mathrm{d}y \\
& - z \cdot \int_{\mathbb{R}^d} D\psi(x-y,t)\phi(y,t)\mathrm{d}y \mathbf{1}_{|z|\le 1} \mathrm{d}\mu(z) \\
=& \int_{\mathbb{R}^d} \phi(y,t) \int_{|z|>0} \psi(x+z-y,t) - \psi(x-y,t) \\
& - z \cdot D\psi(x-y,t) \mathbf{1}_{|z|\le 1} \mathrm{d}\mu(z)\mathrm{d}y \\
=& \int_{\mathbb{R}^d} \phi(y,t) \mathcal{L}^\mu[\psi(\cdot,t)](x-y)\mathrm{d}y \\
=& (\phi(\cdot,t) * \mathcal{L}^\mu[\psi(\cdot,t)])(x).
\end{aligned}$$

Inserting the result of the Lévy-operator into the max-operator gives

$$\begin{aligned}(\mathcal{L}^{\mu}[\Gamma(\cdot,t)](x))^{+} &= \max\{\mathcal{L}^{\mu}[\Gamma(\cdot,t)](x),0\} = \max\{(\phi(\cdot,t)*\mathcal{L}^{\mu}[\psi(\cdot,t)])(x),0\}\\ &\le (\phi(\cdot,t)*\max\{\mathcal{L}^{\mu}[\psi(\cdot,t)],0\})(x),\end{aligned}$$

where the inequality is due to Lemma 4.1.7.

Combining the above calculations yields

$$\begin{aligned}&\partial_t\Gamma(x,t)+L_f|D\Gamma(x,t)|+L_A\big(\mathcal{L}^{\mu^*}[\Gamma(\cdot,t)]\big)^{+}(x)\\ &\le(\partial_t\psi(\cdot,t)*\phi(\cdot,t))(x)+(\psi(\cdot,t)*\partial_t\phi(\cdot,t))+L_f(\psi(\cdot,t)*|D\phi(\cdot,t)|)(x)\\ &\quad+L_A(\phi(\cdot,t)*\max\{\mathcal{L}^{\mu^*}[\psi(\cdot,t)],0\})(x)\\ &=(\phi(\cdot,t)*(\partial_t\psi(\cdot,t)+L_A\max\{\mathcal{L}^{\mu^*}[\psi(\cdot,t)],0\}))(x)\\ &\quad+(\psi(\cdot,t)*(\partial_t\phi(\cdot,t)+L_f|D\phi(\cdot,t)|))(x)\\ &\le 0,\end{aligned}$$

which completes the proof. □

The main problem is to find ψ in Lemma 4.1.8, and the search for such a ψ starts with another auxiliary result.

In the following, consider nonnegative $\rho\in C^{\infty}$ with $\text{supp}\{\rho\}\subset B(0,1)\times(0,1)$ and $\iint_{B(0,1)\times(0,1)}\rho(x,t)\mathrm{d}x\mathrm{d}y=1$, and let

$$\rho_\delta(x,t)=\frac{1}{\delta^{d+2}}(\frac{x}{\delta},\frac{t}{\delta^2}). \tag{4.8}$$

Then $\lim_{\delta\to 0}\rho_\delta(x,t)=\delta_0(x,t)$ in the sense of distributions (with $\delta_0(x,t)$ being Dirac's delta at the origin).

Lemma 4.1.9. *If Φ is a viscosity solution to* (4.3)*, and ρ_δ is defined by* (4.8)*, then*

$$\Phi_\delta(x,t):=(\Phi*\rho_\delta)(x,t)=\iint_{\mathbb{R}\times\mathbb{R}^d}\Phi(x-y,t-s)\rho_\delta(y,s)\mathrm{d}y\mathrm{d}s \tag{4.9}$$

solves

$$\partial_t\Phi_\delta(x,t)\ge(\mathcal{L}^{\mu}[\Phi_\delta(\cdot,t)](x))^{+}, \tag{4.10}$$

that is, Φ_δ is a classical supersolution to (4.3).

Proof This proof is divided into two parts. The first part shows that the above lemma holds for classical solutions to (4.3), and the second part gives an outline of the proof for viscosity solutions.

Let $\Phi(y,t)$ be a classical solution to (4.3), that is

$$\partial_t\Phi-\max\{\mathcal{L}^{\mu}[\Phi],0\}=0.$$

Multiply both sides of the equation with $\rho_\delta(x-y,t)$, and integrate over $\mathbb{R}^d$ w.r.t. y. In addition, make use of Lemma 4.1.7 (which holds in this case as well) and Fubini's theorem to obtain

$$\begin{aligned}
0 &= \int_{\mathbb{R}^d} \partial_t \Phi(y,t)\rho_\delta(x-y,t)\mathrm{d}y - \int_{\mathbb{R}^d} \max\{\mathcal{L}^\mu[\Phi],0\}\rho_\delta(x-y,t)\mathrm{d}y \\
&\leq \partial_t(\Phi * \rho_\delta)(x,t) - \max\{(\rho_\delta * \mathcal{L}^\mu[\Phi])(x,t),0\} \\
&= \partial_t(\Phi * \rho_\delta)(x,t) - \max\{\mathcal{L}^\mu[(\Phi * \rho_\delta)(\cdot,t)](x),0\} \\
&= \partial_t \Phi_\delta - (\mathcal{L}^\mu[\Phi_\delta])^+.
\end{aligned}$$

Therefore, Φ_δ fulfills (4.10) when Φ is a classical solution to (4.3).

The next paragraphs are devoted to give an outline of the proof for viscosity solutions: First, it will be shown that a finite linear combination of classical solutions to (4.3) is indeed a classical supersolution to (4.3). Second, it will be shown that the convolution given by (4.9) is nothing more than a linear combination of classical solutions to (4.3). Combining the first and the second step will give a sketch of the proof. Therefore, the proof is concluded by giving references to the handling of viscosity solutions (that are not classical solutions) to (4.3).

1) Let u_1 and u_2 be classical solutions to (4.3), and let $\lambda_1, \lambda_2 \geq 0$ satisfy $\sum_{i=1}^2 \lambda_i = 1$. Define $w(x,t) := \sum_{i=1}^2 \lambda_i u_i(x,t) = \lambda_1 u_1(x,t) + \lambda_2 u_2(x,t)$.

The rest of Step 1) is mainly straight forward calculations.

The time and the spatial derivatives of w are given by $\partial_t w = \lambda_1 \partial_t u_1 + \lambda_2 \partial_t u_2$ and $Dw = \lambda_1 Du_1 + \lambda_2 Du_2$. Further, consider the nonlocal operator:

$$\begin{aligned}
\mathcal{L}^\mu[w(\cdot,t)](x) &= \int_{|z|>0} w(x+z) - w(x) - z \cdot Dw \mathbf{1}_{|z|\leq 1} \mathrm{d}\mu(z) \\
&= \lambda_1 \mathcal{L}^\mu[u_1(\cdot,t)](x) + \lambda_2 \mathcal{L}^\mu[u_2(\cdot,t)](x).
\end{aligned}$$

Use the definition of a convex function, given by e.g. [2, p. 705], on the max-function, and insert the above calculations into (4.3):

$$\begin{aligned}
&\partial_t w - \max\{\mathcal{L}^\mu[w],0\} \\
&= \lambda_1 \partial_t u_1 + \lambda_2 \partial_t u_2 - \max\{\lambda_1 \mathcal{L}^\mu[u_1] + \lambda_2 \mathcal{L}^\mu[u_2],0\} \\
&\geq \lambda_1 \partial_t u_1 + \lambda_2 \partial_t u_2 - \lambda_1 \max\{\mathcal{L}^\mu[u_1],0\} - \lambda_2 \max\{\mathcal{L}^\mu[u_2],0\} \\
&= \lambda_1 \left(\partial_t u_1 - \max\{\mathcal{L}^\mu[u_1],0\}\right) + \lambda_2 \left(\partial_t u_2 - \max\{\mathcal{L}^\mu[u_2],0\}\right) \\
&= 0.
\end{aligned}$$

That is, w is a classical supersolution to (4.3). By induction, $w := \sum_{i=1}^n \lambda_i u_i$ is also a classical supersolution to (4.3) for any finite n, with $\sum_{i=1}^n \lambda_i = 1$.

2) Let Φ be a classical solution to (4.3). Define Φ_δ by (4.9), and notice that $\Phi(x-y,t-s)$ is also a classical solution to (4.3) since the equation is translation invariant.

Consider the following partition of $\mathbb{R}^d$, $Q_\alpha(h) = \alpha + \left[-\frac{h}{2}, \frac{h}{2}\right]^d$ with $\alpha \in h\mathbb{Z}^d$.

Notice that $|Q_\alpha(h)| = 1$ for any choice of α. Further, define

$$\rho_{\delta,h}(\alpha,\beta) := \int_{Q_\beta(h)} \int_{Q_\alpha(h)} \rho_\delta(y,s)\mathrm{d}y\mathrm{d}s$$
$$\Phi_{\delta,h}(x,t) := \sum_{\beta\in h\mathbb{Z}} \sum_{\alpha\in h\mathbb{Z}^d} \Phi(x-\alpha,t-\beta)\rho_{\delta,h}(\alpha,\beta),$$

where $Q_\beta(h)$ is a partition of $\mathbb{R}$. Observe that $\Phi_{\delta,h}$ is an approximation of Φ_δ. By the properties of mollifiers (see e.g. [2, p. 714]), $\Phi_{\delta,h} \to \Phi_\delta$ uniformly when $h \to 0$. Since ρ_δ is nonnegative, $\rho_{\delta,h}$ is also nonnegative. Further, due to the compact support of ρ_δ, $\rho_{\delta,h}(\alpha,\beta) > 0$ for only a finite number of α and β. In addition

$$\begin{aligned}
&\sum_{\beta\in h\mathbb{Z}} \sum_{\alpha\in h\mathbb{Z}^d} \rho_{\delta,h}(\alpha,\beta) \\
&= \sum_{\beta\in h\mathbb{Z}} \sum_{\alpha\in h\mathbb{Z}^d} \int_{Q_\beta(h)} \int_{Q_\alpha(h)} \rho_\delta(y,s)\mathrm{d}y\mathrm{d}s \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \rho_\delta(y,s)\mathrm{d}y\mathrm{d}s \\
&= 1.
\end{aligned}$$

That is, $\{\rho_{\delta,h}(\alpha,\beta)\}_{\alpha,\beta}$ plays the role of $\{\lambda_i\}_i$ in Step 1).

Finally, $\Phi_{\delta,h}$ is a linear combination of classical solutions $\Phi(x-\alpha,t-\beta)$ to (4.3). By Step 1), $\Phi_{\delta,h}$ is in fact a classical supersolution to (4.3). Since it is outside the scope of this paper to discuss the stability of the uniform limit of classical supersolutions, this is as far as one gets with classical solutions. It is, however, known that a uniform limit of viscosity supersolutions are still viscosity supersolutions, and, thus, it is up to Step 3) to conclude the proof.

3) To pass from classical solutions to viscosity solutions (that are not classical solutions) see [8, Theorem 6.4] for a proof of the convolution in space, and see [9, Proof of Theorem 3.1 (a)] to conclude that it holds for a convolution in both space and time. □

Proposition 4.1.10. *Let Φ be a viscosity solution to* (4.3) *with nonnegative initial condition $\Phi_0 \in C_c^\infty(\mathbb{R}^d)$. Then $\Phi_\delta(x,t)$ defined by* (4.9) *is nonnegative and bounded, an element in $C([0,\tilde{T}];L^1(\mathbb{R}^d)) \cap L^\infty([0,\tilde{T}];L^1(\mathbb{R}^d)) \cap C^\infty(\Omega_{\tilde{T}})$ and solves* (4.10).

In addition, Φ_δ satisfies

$$\|\Phi_\delta(x,0) - \Phi_0(x)\|_{L^\infty(\mathbb{R}^d)} \le C\delta, \tag{4.11}$$

where C is some constant independent of $\delta > 0$.

Proof It is known, by [7], that (4.3) satisfies the following properties

I) if $u_0 \in W^{1,\infty}(\mathbb{R}^d)$, then there exists an unique viscosity solution to (4.3) such that $u(x,t)$ is bounded and an element in $C(\mathbb{R}^d \times [0,\tilde{T}))$;

II) let u and v be viscosity sub- and supersolutions to (4.3) and let $u_0(x) \leq v_0(x)$ on $\mathbb{R}^d$, then $u(x,t) \leq v(x,t)$ for $(x,t) \in \Omega_{\tilde{T}}$;

III) if u is a solution to (4.3) and has initial condition $u_0 \in W^{1,\infty}(\mathbb{R}^d)$, then $|u(x,t) - u(y,s)| \leq C(|x-y| + |t-s|^{\frac{1}{2}})$ for $(x,t) \in \Omega_{\tilde{T}}$; and

IV) if u is a classical sub- or supersolution to (4.3), then u is a viscosity sub- or supersolution to (4.3).

These properties will be called Property I)-IV) throughout this proof.

Since $\Phi_0(x) \in C_c^\infty(\mathbb{R}^d)$, $\Phi_0(x) \in W^{1,\infty}(\mathbb{R}^d)$, and then there exists a unique, bounded and continuous solution $\Phi(x,t)$ to (4.3) by Property I). Further, since $0 \leq \Phi_0(x)$, Property II) gives that $0 \leq \Phi(x,t)$.

The next paragraphs are devoted to showing that $\Phi(\cdot,t) \leq C\mathrm{e}^{Kt}\mathrm{e}^{-k|\cdot|} \in L^1(\mathbb{R}^d)$ for all $t \in (0,\tilde{T})$, with $\Phi \geq 0$. This inequality will first be established for functions of the type $w_\pm(x,t) = C\mathrm{e}^{Kt}\mathrm{e}^{\pm kx}$. Notice that if $\Phi(x,t) \leq w_+(x,t)$ and $\Phi(x,t) \leq w_-(x,t)$ for all $(x,t) \in \Omega_{\tilde{T}}$, then $\Phi(x,t) \leq C\mathrm{e}^{Kt}\mathrm{e}^{-k|x|}$ for all $(x,t) \in \Omega_{\tilde{T}}$.

To begin with consider the case when $d = 1$.

Let $w_\pm(x,t) = C\mathrm{e}^{Kt}\mathrm{e}^{\pm kx}$ for some $C > 0$, $k > 0$, $K > 0$ and $x \in \mathbb{R}$. Choose C such that $w_\pm(x,0) \geq \Phi_0$. Further, insert $w_\pm$ into (4.3). It can be easily verified that $\partial_t w_\pm = K w_\pm$, but the nonlocal operator requires some more work:

$$
\begin{aligned}
\mathcal{L}^\mu[w_\pm(\cdot,t)](x) &= \int_{|z|>0} w_\pm(x+z,t) - w_\pm(x,t) - z\partial_x w_\pm(x,t)\mathbf{1}_{|z|\leq 1}\mathrm{d}\mu(z) \\
&= C\mathrm{e}^t \left[\int_{0<|z|\leq 1} \mathrm{e}^{\pm k(x+z)} - \mathrm{e}^{\pm kx} \mp zk\mathrm{e}^{\pm kx}\mathrm{d}\mu(z) \right. \\
&\qquad \left. + \int_{|z|>1} \mathrm{e}^{\pm k(x+z)} - \mathrm{e}^{\pm kx}\mathrm{d}\mu(z) \right] \\
&= w_\pm(x,t) \left[\int_{0<|z|\leq 1} \mathrm{e}^{\pm kz} - 1 \mp kz\mathrm{d}\mu(z) \right. \\
&\qquad \left. + \int_{|z|>1} \mathrm{e}^{\pm kz} - 1\mathrm{d}\mu(z) \right] \\
&\leq w_\pm(x,t) \left[k^2 \frac{\mathrm{e}^k}{2} \int_{0<|z|\leq 1} |z|^2\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{e}^{M|z|}\mathrm{d}\mu(z) \right] \\
&= w_\pm(x,t) C_k,
\end{aligned}
$$

where the inequality is due to Taylor's theorem and the fact that $\mathrm{e}^{\pm kz} - 1$ is dominated by $\mathrm{e}^{M|z|}$ if $k \leq M$. In addition,

$$
C_k := k^2 \frac{\mathrm{e}^k}{2} \int_{0<|z|\leq 1} |z|^2\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{e}^{M|z|}\mathrm{d}\mu(z) = k^2\frac{\mathrm{e}^k}{2}\alpha + \beta > 0.
$$

Notice that both α and β are finite by assumption (A.5).

Inserting the above into (4.3) gives

$$\partial_t w_\pm - (\mathcal{L}^\mu[w_\pm])^+ = \partial_t w_\pm + \min\{-\mathcal{L}^\mu[w_\pm], 0\} \geq w_\pm(K - C_k),$$

that is, K must be chosen such that $K - C_k \geq 0$ in order to make $w_\pm$ supersolutions. To establish this, require that K is such that $K \geq C_k$ when $k \leq M$. With these choices, property II) ensures that $\Phi(x,t) \leq w_\pm(x,t)$, that is, $\Phi(\cdot,t) \leq f(x,t) := C\mathrm{e}^{Kt}\mathrm{e}^{-k|\cdot|}$, with $f \in L^\infty([0,\tilde{T}]; L^1(\mathbb{R}^d))$, and, thus, $\Phi \in L^\infty([0,\tilde{T}]; L^1(\mathbb{R}^d))$.

Further, let

$$\Phi_\delta(x,t) = (\Phi * \rho_\delta)(x,t) := \iint_{\mathbb{R}\times\mathbb{R}} \Phi(x-y, t-s)\rho_\delta(y,s)\mathrm{d}y\mathrm{d}s.$$

Observe that since both Φ and ρ_δ are nonnegative and bounded, so is Φ_δ. Moreover, the derivatives can be put upon the mollifier, ρ_δ, and, thus, $\Phi_\delta \in C^\infty(\Omega_{\tilde{T}})$. Now, utilize Tonelli's theorem and the compact support of ρ_δ to get

$$\begin{aligned}
&\int_{\mathbb{R}} \Phi_\delta(x,t)\mathrm{d}x \\
&= \int_{\mathbb{R}} \iint_{\mathbb{R}\times\mathbb{R}} \Phi(x-y,t-s)\rho_\delta(y,s)\mathrm{d}y\mathrm{d}s\mathrm{d}x \\
&= \iint_{\mathbb{R}\times\mathbb{R}} \rho_\delta(y,s) \int_{\mathbb{R}} \Phi(x-y,t-s)\mathrm{d}x\mathrm{d}y\mathrm{d}s \\
&\leq \iint_{\mathbb{R}\times\mathbb{R}} \rho_\delta(y,s) \int_{\mathbb{R}} \max_{t\in[0,\tilde{T}]} \Phi(x-y,\cdot)\mathrm{d}x\mathrm{d}y\mathrm{d}s \\
&= \max_{t\in[0,\tilde{T}]} \|\Phi\|_{L^1(\mathbb{R})}\|\rho_\delta\|_{L^1(\mathbb{R}\times\mathbb{R})} \\
&< \infty.
\end{aligned}$$

That is, $\Phi_\delta(\cdot,t) \in L^1(\mathbb{R})$ for all $t \in [0,\tilde{T}]$. Furthermore, $\Phi_\delta \in L^\infty([0,\tilde{T}]; L^1(\mathbb{R}^d))$. In addition, $\Phi_\delta \in C([0,\tilde{T}); L^1(\mathbb{R}))$ since both Φ and ρ_δ are continuous in time with L^1 values in space. By Lemma 4.1.9, Φ_δ is a supersolution to (4.3).

It remains to prove property (4.11).

Let C be a constant, that might change throughout the calculations, independently of $\delta > 0$. Start by noticing that

$$\iint_{\mathbb{R}\times\mathbb{R}} \rho_\delta(y,s)\mathrm{d}y\mathrm{d}s = 1,$$

and consider the following

$$\begin{aligned}
&|\Phi_\delta(x,0) - \Phi_0(x)| \\
&= \left| \iint_{\mathbb{R}\times\mathbb{R}} \Phi(x-y, 0-s)\rho_\delta(y,s)\mathrm{d}y\mathrm{d}s - \Phi_0(x) \iint_{\mathbb{R}\times\mathbb{R}} \rho_\delta(y,s)\mathrm{d}y\mathrm{d}s \right| \\
&\leq \iint_{\mathbb{R}\times\mathbb{R}} |\Phi(x-y,0-s) - \Phi_0(x)|\rho_\delta(y,s)\mathrm{d}y\mathrm{d}s \\
&= \iint_{\mathbb{R}\times\mathbb{R}} |\Phi(x-y,0-s) - \Phi_0(x-y)| + |\Phi_0(x-y) - \Phi_0(x)|\rho_\delta(y,s)\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Use Property III) to obtain

$$
\begin{aligned}
&|\Phi_\delta(x,0)-\Phi_0(x)| \\
&\le \iint_{\mathbb{R}\times\mathbb{R}} C(|s|^{\frac{1}{2}}+|y|)\rho_\delta(y,s)\mathrm{d}y\mathrm{d}s,
\end{aligned}
$$

and by the compact support of ρ_δ one gets

$$
\begin{aligned}
&|\Phi_\delta(x,0)-\Phi_0(x)| \\
&\le C\left(\sup_{s\in(0,\delta^2)}\{|s|^{\frac{1}{2}}\}+\sup_{y\in(-\delta,\delta)}\{|y|\}\right)\iint_{\mathbb{R}\times\mathbb{R}}\rho_\delta(y,s)\mathrm{d}y\mathrm{d}s \\
&= C\delta.
\end{aligned}
$$

Taking the supremum over all $x\in\mathbb{R}$ will not change the result.

To conclude this proof for a general $d\in\mathbb{N}^+$, let $w_\pm(x,t)=C\mathrm{e}^{Kt}\xi^{\pm kx}$ for some $C>0$, $k>0$, $K>0$ and $x\in\mathbb{R}^d$. Notice that

$$
\xi^x := \prod_{i=1}^d \mathrm{e}^{x_i} = \mathrm{e}^{\sum_{i=1}^d x_i}.
$$

The further calculations are quite similar to the one dimensional case, except that one ends up with, for $k\le M$

$$
C_k := dk^2\frac{\mathrm{e}^{dk}}{2}\int_{0<|z|\le 1}|z|^2\mathrm{d}\mu(z)+\int_{|z|>1}\mathrm{e}^{M|z|}\mathrm{d}\mu(z)=dk^2\frac{\mathrm{e}^{dk}}{2}\alpha+\beta>0,
$$

where Hölder's inequality has been used to get

$$
\begin{aligned}
\left|\sum_{i=1}^d z_i\right|^2 &\le \left(\sum_{i=1}^d |z_i|^2\right)\left(\sum_{i=1}^d |1|^2\right) \\
&= d\sum_{i=1}^d |z_i|^2 = d\sum_{i=1}^d z_i^2 = d|z|^2,
\end{aligned}
$$

and the term e^{dk} is due to the fact that one gets a factor of e^k in each spatial direction.

In addition, all other calculations hold for a general $d\in\mathbb{N}^+$. □

Remark 4.1.11. i) Notice that $\Omega_{\tilde{T}}=\mathbb{R}\times(0,\tilde{T})$ when $d=1$, and $\Omega_{\tilde{T}}=\mathbb{R}^d\times(0,\tilde{T})$ when $d\in\mathbb{N}^+$.

ii) It is not needed that $\mu(z)$ is symmetric in order to establish lemma 4.1.10.

iii) The CGMY-model, described in e.g. [10, Chapter 5.3.9], fulfills assumption (A.5).

Corollary 4.1.12. *Let* $\tilde{T} = \max\{T, \frac{T}{L_A}\}$*,* $0 < \tau < \tilde{T}$ *and* $0 \leq t \leq \tau$*, and let* $K(x,t) := \Phi_\delta(x, \frac{1}{L_A}(\tau - t))$*, where* Φ_δ *is a viscosity supersolution to (4.3) defined by (4.9), and* $L_A > 0$ *is the Lipschitz constant of A. Then* $K(x,t) \in C([0,\tilde{T}); L^1(\mathbb{R}^d)) \cap L^\infty([0,\tilde{T}]; L^1(\mathbb{R}^d)) \cap C^\infty(\Omega_{\tilde{T}})$ *is nonnegative and bounded, solves*

$$\partial_t K + L_A(\mathcal{L}^{\mu^*}[K])^+ \leq 0,$$

satisfies

$$\|K(x,\tau) - \Phi_0(x)\|_{L^\infty(\mathbb{R}^d)} \leq C\delta,$$

where $\Phi_0 \in C_c^\infty(\mathbb{R}^d)$ *is nonnegative and* C *is a constant independent of* $\delta > 0$*, and satisfies*

$$\|K(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq 1 \tag{4.12}$$

for all $t \in [0,\tau]$*.*

Remark 4.1.13. Observe that $K(x,t)$ is well-defined for all $t \in (0,\tilde{T})$. That is, if $t \in (0,T)$, K is still well-defined.

Proof Since $K := \Phi_\delta$, it inherits all the properties of Φ_δ from Lemma 4.1.10.

Further, from Lemma 4.1.10 it is known that $\Phi_\delta(x,t)$ satisfies

$$\partial_t \Phi_\delta - (\mathcal{L}^{\mu^*}[\Phi_\delta])^+ \geq 0.$$

By the chain rule $K(x,t)$ satisfies

$$\partial_t K + L_A(\mathcal{L}^{\mu^*}[K])^+ \leq 0.$$

It only remains to prove (4.12).

Since $K(\cdot,t) \in L^1(\mathbb{R}^d)$ for all $t \in [0,\tau]$, the L^1 norm of K must be equal to some function in time, call it $F(t)$. As $K \in L^\infty([0,\tau]; L^1(\mathbb{R}^d))$, this function has a maximum. To sum up, one gets

$$\|K(\cdot,t)\|_{L^1(\mathbb{R}^d)} = F(t) \leq \max_{t\in[0,\tau]} F(t) := \tilde{C}.$$

If $\|K(\cdot,t)\|_{L^1(\mathbb{R}^d)} \neq 1$, then consider a new function $\hat{K}(x,t) := \frac{1}{\tilde{C}} K(x,t)$. The L^1 norm of $\hat{K}$ is

$$\|\hat{K}(\cdot,t)\|_{L^1(\mathbb{R}^d)} = \frac{1}{\tilde{C}}\|K(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq 1.$$

The scaling of K with $\tilde{C}$ (which is independent of (x,t)) will not change the other properties of K, and therefore, the scaled K will be used throughout the rest of this project. □

The search for ψ is now done, and it only remains to find ϕ in Lemma 4.1.8 before one can prove Theorem 4.1.3.

Lemma 4.1.14. *Let $0 < \tau < T$, $0 \leq t \leq \tau$, $R > L_f T + 1$, $\delta > 0$ and $x_0 \in \mathbb{R}^d$. Further, let $\gamma_\delta \in C_c^\infty(\Omega_T)$, and define it by*

$$\gamma_\delta(x,t) = (\mathbf{1}_{[0,R]} * \omega_\varepsilon)(\sqrt{\delta^2 + |x - x_0|^2} + L_f t), \tag{4.13}$$

where $\mathbf{1}_{[0,R]}$ is the indicator function on the interval $[0, R]$, L_f being the Lipschitz constant of f, $\sqrt{\delta^2 + |x - x_0|^2}$ is a smooth approximation to $|x - x_0|$, and ω_ε is a mollifier. Then

$$\partial_t \gamma_\delta(x,t) + L_f |D\gamma_\delta(x,t)| \leq 0.$$

Remark 4.1.15. i) Notice that $\mathbf{1}_{[0,R]}(x)$ is nonnegative and nonincreasing for $x \in \mathbb{R}^+$.

ii) Observe that the above lemma also holds for $\gamma := \lim_{\delta \to 0} \gamma_\delta$.

Proof The proof is mainly a straight forward computation. Start by taking the time derivative

$$\begin{aligned} \partial_t \gamma_\delta(x,t) =& \partial_t \left((\mathbf{1}_{[0,R]} * \omega_\varepsilon)(\sqrt{\delta^2 + |x - x_0|^2} + L_f t) \right) \\ =& L_f (\mathbf{1}_{[0,R]} * \omega_\varepsilon)'(\sqrt{\delta^2 + |x - x_0|^2} + L_f t), \end{aligned}$$

and continue with the spatial derivative

$$\begin{aligned} D\gamma_\delta(x,t) =& D \left((\mathbf{1}_{[0,R]} * \omega_\varepsilon)(\sqrt{\delta^2 + |x - x_0|^2} + L_f t) \right) \\ =& \frac{x - x_0}{\sqrt{\delta^2 + |x - x_0|^2}} (\mathbf{1}_{[0,R]} * \omega_\varepsilon)'(\sqrt{\delta^2 + |x - x_0|^2} + L_f t). \end{aligned}$$

Insert the above computations into $\partial_t \gamma_\delta(x,t) + L_f |D\gamma_\delta(x,t)|$ to get

$$\begin{aligned} \partial_t \gamma_\delta + L_f |D\gamma_\delta| =& (\mathbf{1}_{[0,R]} * \omega_\varepsilon)' L_f \left(1 + \frac{|x - x_0|}{\sqrt{\delta^2 + |x - x_0|^2}} \right) \\ \leq & 0, \end{aligned}$$

since $(\mathbf{1}_{[0,R]} * \omega_\varepsilon)' \leq 0$ and $1 + \frac{|x - x_0|}{\sqrt{\delta^2 + |x - x_0|^2}} \geq 0$. □

Finally, one can prove the main theorem of this section:

Proof of Theorem 4.1.3 The proofs of i) and ii) are assumed to be known (see e.g. [4, 1, 5]), and the proof of iii) - which follows - is in some extent based upon these references.

Let $0 < \tau < T$, $R > L_f T + 1$, $\delta > 0$ and $x_0 \in \mathbb{R}^d$. Further, define

$$\gamma_\delta(x,t) = (\mathbf{1}_{[0,R]} * \omega_\varepsilon)(\sqrt{\delta^2 + |x - x_0|^2} + L_f t),$$

with

$$\gamma(x,t) := \lim_{\delta \to 0} \gamma_\delta(x,t) = (\mathbf{1}_{[0,R]} * \omega_\varepsilon)(|x - x_0| + L_f t),$$

and

$$\Gamma(x,t) = \begin{cases} (K(\cdot,t) * \gamma_\delta(\cdot,t))(x) & 0 \le t \le \tau \\ 0 & t > \tau \end{cases}. \tag{4.14}$$

Remember that $\gamma_\delta(x,t) \in C_c^\infty(\Omega_T)$, and since both K and γ_δ are bounded, $\Gamma(x,t) \in C_b^\infty(\Omega_T)$. In addition Γ is nonnegative since both K and γ_δ are nonnegative.

Although Proposition 3.1.1 holds for nonnegative test functions in $C_c^\infty(\Omega_T)$, the following will make sure that one can use nonnegative $\psi(x,t) = \Gamma(x,t)\Theta(t)$, where $\Theta \in C_c^\infty((0,T))$ nonnegative and $\Gamma(x,t)$ is given by (4.14).

By the properties of K, and since $\gamma_\delta \in C_c^\infty(\Omega_T)$, $\psi \in C((0,T); L^1(\mathbb{R}^d)) \cap L^1((0,T); W^{2,1}(\mathbb{R}^d))$. In addition, $\partial_t\psi = \partial_t\Gamma\Theta + \Gamma\partial_t\Theta$ is in $L^1(\Omega_T)$, since $\Gamma, \Theta \in L^\infty(\Omega_T)$ and $\partial_t\Gamma, \partial_t\Theta \in L^1(\Omega_T)$. By [5, p. 159], $C_c^\infty(\Omega_T)$ is dense in

$$E = \{w : w \in C((0,T); L^1(\mathbb{R}^d)) \cap L^1((0,T); W^{2,1}(\mathbb{R}^d)) \text{ and } \partial_t w \in L^1(\Omega_T)\}.$$

Proposition 3.1.1 holds for nonnegative test functions in $C_c^\infty(\Omega_T)$, however, $C_c^\infty(\Omega_T)$ is dense in E, which means that functions in E can be approximated by functions in $C_c^\infty(\Omega_T)$. Notice, first, that Lemma 2.2.7 ii) is true for $\psi \in C_b^\infty(\Omega_T) \cap L^1((0,T); W^{2,1}(\mathbb{R}^d))$ (that is, $\mathcal{L}^\mu[\psi] \in L^1(\Omega_T)$). Second, construct a sequence of functions $\psi^\varepsilon \in C_c^\infty(\Omega_T)$ such that $\lim_{\varepsilon\to 0} \|\psi - \psi^\varepsilon\|_E = 0$, for $\psi \in E$. Since $\mathcal{L}^\mu$ is continuous from $C_b^\infty(\Omega_T) \cap L^1((0,T); W^{2,1}(\mathbb{R}^d))$, endowed with the norm of $L^1((0,T); W^{2,1}(\mathbb{R}^d))$, into $L^1(\Omega_T)$, and since $u, v \in L^\infty(\Omega_T)$, taking the limit as $\varepsilon \to 0$ in (3.1) will make sure that $\psi(x,t) = \Gamma(x,t)\Theta(t)$, can be used as a test function in Proposition 3.1.1.

Write up the result of Lemma 4.1.5 i) (valid for the above choice of a test function)

$$\begin{aligned} 0 \le & \iint_{\Omega_T} \eta(u,v)(x,t)\Gamma(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t \\ & + \iint_{\Omega_T} \Theta(t)\eta(u,v) \left[\partial_t\Gamma + L_f|D\Gamma| + L_A\big(\mathcal{L}^{\mu^*}\big)^+ [\Gamma(\cdot,t)](x)\right] \mathrm{d}x\mathrm{d}t, \end{aligned}$$

by inserting the result of Lemma 4.1.8 (let $\phi = \gamma_\delta$ and $\psi = K$) into Lemma 4.1.5 ii) one ends up with

$$\int_{\mathbb{R}^d} \eta(u,v)(x,\tau)\Gamma(x,\tau)\mathrm{d}x \le \int_{\mathbb{R}^d} \eta(u_0,v_0)(x)\Gamma(x,0)\mathrm{d}x,$$

or

$$\begin{aligned} & \int_{\mathbb{R}^d} \eta(u,v)(x,\tau)(K(\cdot,\tau) * \gamma_\delta(\cdot,\tau))(x)\mathrm{d}x \\ & \le \int_{\mathbb{R}^d} \eta(u_0,v_0)(x)(K(\cdot,0) * \gamma_\delta(\cdot,0))(x)\mathrm{d}x. \end{aligned}$$

Now, take the $\liminf_{\delta\to 0}$ on both sides. Use Fatou's lemma on the left-hand side (the integrand is nonnegative and measurable), and use Lebesgue's dominated convergence theorem on the right-hand side (the integrand is dominated by

$(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty})\|K\|_{L^1} 2(\mathbf{1}_{[0,2R]} * \omega_\varepsilon)(|x - x_0|))$ to get

$$\begin{aligned}&\int_{\mathbb{R}^d} \eta(u,v)(x,\tau)(K(\cdot,\tau) * \gamma(\cdot,\tau))(x)\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d} \eta(u_0,v_0)(x)(K(\cdot,0) * \gamma(\cdot,0))(x)\mathrm{d}x.\end{aligned} \tag{4.15}$$

First, consider the right-hand side of (4.15), and make use of Tonelli's theorem to obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} \eta(u_0,v_0)(x) \int_{\mathbb{R}^d} K(x-y,0)\gamma(y,0)\mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \gamma(y,0) \int_{\mathbb{R}^d} \eta(u_0,v_0)(x) K(x-y,0)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \gamma(y,0) \int_{\mathbb{R}^d} \eta(u_0,v_0)(x) K(-(y-x),0)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \gamma(x,0)(K(-\cdot,0) * \eta(u_0,v_0)(\cdot))(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} (\mathbf{1}_{[0,R]} * \omega_\varepsilon)(|x-x_0|)(K(-\cdot,0) * (u_0(\cdot) - v_0(\cdot))^+)(x)\mathrm{d}x.\end{aligned}$$

Second, continue with the left-hand side of (4.15). By the properties of K, given in Lemma 4.1.12, it is known that $\|K(x,\tau) - \Phi_0(x)\|_{L^\infty(\mathbb{R}^d)} \leq C\delta$. With this knowledge at hand consider

$$\begin{aligned}&|(K(\cdot,\tau) * \gamma(\cdot,\tau))(x) - (\Phi_0(\cdot) * \gamma(\cdot,\tau))(x)| \\ &= \int_{\mathbb{R}^d} |K(y,\tau) - \Phi_0(y)|\gamma(x-y,\tau)\mathrm{d}y \\ &\leq \|K(x,\tau) - \Phi_0(x)\|_{L^\infty(\mathbb{R}^d)}\|\gamma(\cdot,\tau)\|_{L^1(\mathbb{R}^d)} \\ &= C\delta\|\gamma(\cdot,\tau)\|_{L^1(\mathbb{R}^d)} \\ &= \tilde{\delta},\end{aligned}$$

with $\tilde{\delta} := C\|\gamma(\cdot,\tau)\|_{L^1(\mathbb{R}^d)}\delta$. Thus, taking the limit inferior as $\tilde{\delta} \to 0$ on both sides of (4.15) will give (the right-hand side is independent of $\tilde{\delta}$), by Fatou's lemma (the integrand is nonnegative and measurable)

$$\begin{aligned}&\liminf_{\tilde{\delta}\to 0} \int_{\mathbb{R}^d} \eta(u,v)(x,\tau)(K(\cdot,\tau) * \gamma(\cdot,\tau))(x)\mathrm{d}x \\ &\geq \int_{\mathbb{R}^d} \eta(u,v)(x,\tau)(\Phi_0(\cdot) * \gamma(\cdot,\tau))(x)\mathrm{d}x.\end{aligned}$$

Now, let $C_c^\infty(\mathbb{R}^d) \ni \Phi_0(x) := \hat{\omega}_{\tilde{\varepsilon}}(x - x_0)$. Since both $\Phi_0(x)$ and $\gamma(x,\tau)$ have compact support, the convolution, $\Phi_0 * \gamma$, also has compact support. Thus, there exists a compactly supported interval in which $(\Phi_0(\cdot) * \gamma(\cdot,\tau))(x) = 1$. Actually, this interval is $|x-x_0| < R-L_f\tau-\varepsilon-\tilde{\varepsilon}$. When ε and $\tilde{\varepsilon}$ is small enough, the interval reduces to $|x-x_0| < R-L_f\tau-1$. Therefore, $(\Phi_0(\cdot)*\gamma(\cdot,\tau))(x) \geq \mathbf{1}_{|x-x_0|\leq R-L_f\tau-1}$ for all $x \in \mathbb{R}^d$. Taking these observations into account one can find this inequality for the left-hand side of (4.15)

$$\int_{\mathbb{R}^d} \eta(u,v)(x,\tau)(K(\cdot,\tau)*\gamma(\cdot,\tau))(x)\mathrm{d}x \geq \int_{\mathbb{R}^d} \eta(u,v)(x,\tau)\mathbf{1}_{|x-x_0|\leq R-L_f\tau-1}\,\mathrm{d}x.$$

With the above considerations in mind, (4.15) can be written as

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathbf{1}_{|x-x_0|\leq R-L_f\tau-1}(u-v)^+(x,\tau)\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d} (\mathbf{1}_{[0,R]}*\omega_\varepsilon)(|x-x_0|)(K(-\cdot,0)*(u_0(\cdot)-v_0(\cdot))^+)(x)\mathrm{d}x.\end{aligned}$$

Take the limit of both sides as $\varepsilon \to 0$ in the above inequality. Recall that $\mathbf{1} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Further, use Lebesgue's dominated convergence theorem on the right-hand side (the integrand is dominated by $(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty})\|K\|_{L^1}2\mathbf{1}_{[0,2R]}(|x - x_0|))$ to get

$$\begin{aligned}&\int_{B(x_0,R-L_f\tau-1)} (u(x,\tau)-v(x,\tau))^+\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d} \mathbf{1}_{[0,R]}(|x-x_0|)(K(-\cdot,0)*(u_0(\cdot)-v_0(\cdot))^+)(x)\mathrm{d}x.\end{aligned}$$

Now, for any real $M > 0$ let $R = M + 1 + L_f\tau$. This yields

$$\begin{aligned}&\int_{B(x_0,M)} (u(x,\tau)-v(x,\tau))^+\mathrm{d}x\\ &\leq \int_{B(x_0,M+1+L_f\tau)} (K(-\cdot,0)*(u_0(\cdot)-v_0(\cdot))^+)(x)\mathrm{d}x\\ &= \int_{B(x_0,M+1+L_f\tau)} (\Phi_\delta(-\cdot,\frac{1}{L_A}\tau)*(u_0(\cdot)-v_0(\cdot))^+)(x)\mathrm{d}x\end{aligned}$$

Finally, since τ could be any value in $(0,T)$, the above calculations apply for any $t \in (0,T)$. □

4.2 Global contraction

This section focuses on establishing a global contraction for (1.1). There are, at least, two ways of doing this, and with the knowledge at hand they require different assumptions. The first way is a corollary to Theorem 4.1.3 iii), and the second way is to obtain a global contraction directly from the dual equation given in Proposition 3.1.1.

Corollary 4.2.1 (Global contraction)**.** *Let* $u, v \in L^\infty(\Omega_T) \cap C([0,T]; L^1_{loc}(\mathbb{R}^d))$ *be sub- and supersolutions, respectively, of* (1.1) *with initial data* u_0 *and* v_0*. Further, let* (A.1)-(A.3) *and* (A.5) *hold. Assume that* $(u_0 - v_0)^+ \in L^1(\mathbb{R}^d)$*. Then for all* $t \in (0,T)$

$$\int_{\mathbb{R}^d} (u(x,t) - v(x,t))^+ \mathrm{d}x \le \int_{\mathbb{R}^d} (u_0(x) - v_0(x))^+ \mathrm{d}x.$$

Proof Write up the local contraction given in Theorem 4.1.3 iii)

$$\begin{aligned} &\int_{B(x_0,\, M)} (u(x,t) - v(x,t))^+ \mathrm{d}x \\ &\le \int_{B(x_0,\, M+1+L_f t)} (K(-\cdot, 0) * (u_0(\cdot) - v_0(\cdot))^+)(x) \mathrm{d}x, \end{aligned}$$

which holds for all $M > 0$, $x_0 \in \mathbb{R}^d$ and $t \in (0,T)$. Since the integrand on the right-hand side of the inequality above is nonnegative, the following holds

$$\begin{aligned} &\int_{B(x_0,\, M+1+L_f t)} (K(-\cdot, 0) * (u_0(\cdot) - v_0(\cdot))^+)(x) \mathrm{d}x \\ &\le \int_{\mathbb{R}^d} (K(-\cdot, 0) * (u_0(\cdot) - v_0(\cdot))^+)(x) \mathrm{d}x. \end{aligned}$$

By Tonelli's theorem and by Corollary 4.1.12

$$\begin{aligned} &\int_{\mathbb{R}^d} (K(-\cdot, 0) * (u_0(\cdot) - v_0(\cdot))^+)(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(-(x-y), 0)(u_0(y) - v_0(y))^+ \mathrm{d}y \mathrm{d}x \\ &= \int_{\mathbb{R}^d} (u_0(y) - v_0(y))^+ \int_{\mathbb{R}^d} (K(-(x-y), 0) \mathrm{d}x \mathrm{d}y \\ &= \|K(-\cdot, 0)\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} (u_0(x) - v_0(x))^+ \mathrm{d}x \\ &\le \int_{\mathbb{R}^d} (u_0(x) - v_0(x))^+ \mathrm{d}x \end{aligned}$$

Altogether, this yields

$$\int_{\mathbb{R}^d} (u(x,t) - v(x,t))^+ \mathbf{1}_{|x-x_0| \le M} \mathrm{d}x \le \int_{\mathbb{R}^d} (u_0(x) - v_0(x))^+ \mathrm{d}x.$$

To conclude, take the limit inferior on both sides as $M \to \infty$, and use Fatou's lemma on the left-hand side (the integrand is nonnegative and measureable):

$$\begin{aligned}
&\liminf_{M\to\infty} \int_{\mathbb{R}^d} (u(x,t)-v(x,t))^+ \mathbf{1}_{|x-x_0|\le M} \mathrm{d}x \\
&\ge \int_{\mathbb{R}^d} (u(x,t)-v(x,t))^+ \liminf_{M\to\infty} \mathbf{1}_{|x-x_0|\le M} \mathrm{d}x \\
&= \int_{\mathbb{R}^d} (u(x,t)-v(x,t))^+ \mathrm{d}x.
\end{aligned}$$

□

By the dual equation derived in Proposition 3.1.1, a global contraction will be deduced in the following. First, the idea is to choose a test function $\psi(x,t) = \psi_r(x)\Theta(t)$, with $\psi_r(x)$ such that $D\psi_r \to 0$, $\mathcal{L}^{\mu^*}[\psi_r] \to 0$ and $\psi_r \to 1$ as $r \to \infty$. Second, $\Theta(t)$ will be defined and utilized to conclude the proof.

The proof of the following proposition will enlighten the details:

Theorem 4.2.2 (Global contraction)**.** *Let $u, v \in L^\infty(\Omega_T) \cap C([0,T]; L^1(\mathbb{R}^d))$ be sub- and supersolutions, respectively, of* (1.1) *with initial data u_0 and v_0. Further, let* (A.1)-(A.4) *hold. Assume that $(u_0 - v_0)^+ \in L^1(\mathbb{R}^d)$. Then for all $t \in (0,T)$*

$$\int_{\mathbb{R}^d} (u(x,t)-v(x,t))^+ \mathrm{d}x \le \int_{\mathbb{R}^d} (u_0(x)-v_0(x))^+ \mathrm{d}x.$$

Remark 4.2.3. Notice that in order to establish the global contraction directly, one needs to assume that $u(\cdot,t) \in L^1(\mathbb{R}^d)$ for all $t \in [0,T]$.

Proof Start by writing up the dual equation from Proposition 3.1.1:

$$\begin{aligned}
\iint_{\Omega_T} &\eta(u(x,t),v(x,t))\partial_t\psi(x,t) \\
&+ q(u(x,t),v(x,t)) \cdot D\psi(x,t) \\
&+ \eta(A(u(x,t)),A(v(x,t)))\mathcal{L}^{\mu^*}[\psi(\cdot,t)](x)\mathrm{d}x\mathrm{d}t \ge 0,
\end{aligned}$$

Let $\psi(x,t) = \psi_r(x)\Theta(t)$, where

$$C_c^\infty(\mathbb{R}^d) \ni \psi_r(x) = (\hat{\omega}(\cdot) * \mathbf{1}_{|\cdot|<r})(x) = \int_{\mathbb{R}^d} \hat{\omega}(x-y)\mathbf{1}_{|y|<r} \mathrm{d}y$$

with $r > 1$, and $\Theta \in C_c^\infty((0,T))$ (to be specified later).

The first integrand in Proposition 3.1.1 is dominated by the function $\|\Theta'\|_{L^\infty}(|u|+|v|)$ which is integrable since $u, v \in C([0,T]; L^1(\mathbb{R}^d))$, thus one can interchange the limit as $r \to \infty$ and the integral signs by Lebesgue's dominated convergence theorem:

$$\begin{aligned}
&\lim_{r\to\infty} \iint_{\Omega_T} \eta(u(x,t),v(x,t))\partial_t\psi(x,t)\mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T} \eta(u(x,t),v(x,t)) \lim_{r\to\infty} \psi_r(x)\Theta'(t)\mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T} \eta(u(x,t),v(x,t)\Theta'(t)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By construction all derivatives of $\psi_r(x)$ vanish for all $||x|-r|>1$, and they are all bounded uniformly in r. The second integrand in Proposition 3.1.1 is therefore dominated by the function $L_f\|\Theta\|_{L^\infty}(|u|+|v|)\|D\psi_1\|_{L^\infty}$ (independently of r) which is integrable, and Lebegue's dominated convergence theorem can be utilized once more to get

$$\begin{aligned}
&\lim_{r\to\infty} \iint_{\Omega_T} q(u(x,t),v(x,t)) \cdot D\psi(x,t)\mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T} \Theta(t) \lim_{r\to\infty} q(u(x,t),v(x,t)) \cdot D\psi_r(x)\mathbf{1}_{||x|-r|<1}\mathrm{d}x\mathrm{d}t \\
&= 0.
\end{aligned}$$

The third integrand in Proposition 3.1.1 is dominated by the function $L_A(|u|+|v|)\sup_{r>1}\|\mathcal{L}^{\mu^*}[\psi_r]\|_{L^\infty}$ which is integrable, and independent of r by Lemma 2.2.9. Lebegue's dominated convergence theorem gives

$$\begin{aligned}
&\lim_{r\to\infty} \iint_{\Omega_T} \eta(A(u(x,t)),A(v(x,t)))\mathcal{L}^{\mu^*}[\psi(\cdot,t)](x)\mathrm{d}x\mathrm{d}t \\
&= \iint_{\Omega_T} \eta(A(u(x,t)),A(v(x,t)))\Theta(t) \lim_{r\to\infty} \mathcal{L}^{\mu^*}[\psi_r](x)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

it thus remains to find $\lim_{r\to\infty}\mathcal{L}^{\mu^*}[\psi_r]$ in order to conclude the first part of the proof. Fix any $x,z\in\mathbb{R}^d$ and let $r>1+|x|$ such that $\psi_r(x)=1$ and

$$|\psi_r(x+z)-\psi(x)| \le |\mathbf{1}_{|x+z|<r-1}-1| = \mathbf{1}_{|x+z|>r-1} \le \mathbf{1}_{|z|>r-1-|x|},$$

since $|x| + |z| \geq |x + z|$. Now, consider

$$
\begin{aligned}
|\mathcal{L}^{\mu^*}[\psi_r](x)| &= \left| \int_{|z|>0} \psi_r(x+z) - \psi_r(x) - z \cdot D\psi_r \mathbf{1}_{|z|\leq 1} \mathrm{d}\mu^*(z) \right| \\
&\leq \int_{0<|z|\leq 1} \int_0^1 (1-\tau)|D^2\psi_r(x+\tau z)||z|^2 \mathbf{1}_{||x|-r|<1} \mathrm{d}\tau \mathrm{d}\mu^*(z) \\
&\quad + \int_{|z|>1} |\psi_r(x+z) - \psi_r(x)| \mathrm{d}\mu^*(z) \\
&\leq \frac{1}{2}\|D^2\psi_1\|_{L^\infty} \int_{0<|z|\leq 1} |z|^2 \mathbf{1}_{||x|-r|<1} \mathrm{d}\mu^*(z) \\
&\quad + \int_{|z|>1} \mathbf{1}_{|z|>r-1-|x|} \mathrm{d}\mu^*(z),
\end{aligned}
$$

which goes to zero as $r \to \infty$.

To sum up the above, the dual equation is reduced to

$$
\iint_{\Omega_T} (u(x,t) - v(x,t))^+ \Theta'(t) \geq 0.
$$

Let $\Theta \in C_c^\infty((0,T))$ be defined by (4.1), and call it Θ_ε. Taking the derivative gives $\Theta'_\varepsilon(t) = \omega_\varepsilon(t-t_1) - \omega_\varepsilon(t-t_2)$. By inserting this into the reduced dual equation one gets the following inequality

$$
\begin{aligned}
&\iint_{\Omega_T} (u(x,t) - v(x,t))^+ \omega_\varepsilon(t-t_2) \mathrm{d}x\mathrm{d}t \\
&\leq \iint_{\Omega_T} (u(x,t) - v(x,t))^+ \omega_\varepsilon(t-t_1) \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Tonelli's theorem (the integrands are positive) ensures that interchanging the order of integration is possible, and, in addition, $\omega_\varepsilon(x) = \omega_\varepsilon(-x)$. Combined this yields

$$
\begin{aligned}
&\int_{\mathbb{R}} \omega_\varepsilon(t_2 - t) \left(\int_{\mathbb{R}^d} (u(x,t) - v(x,t))^+ \mathrm{d}x \right) \mathrm{d}t \\
&\leq \int_{\mathbb{R}} \omega_\varepsilon(t_1 - t) \left(\int_{\mathbb{R}^d} (u(x,t) - v(x,t))^+ \mathrm{d}x \right) \mathrm{d}t,
\end{aligned}
$$

or

$$
(\omega_\varepsilon(\cdot) * \Phi(\cdot))(t_2) \leq (\omega_\varepsilon(\cdot) * \Phi(\cdot))(t_1), \tag{4.16}
$$

with

$$
\Phi(t) = \int_{\mathbb{R}^d} (u(x,t) - v(x,t))^+ \mathrm{d}x.
$$

Since $u, v \in C([0,T]; L^1(\mathbb{R}^d))$ and $(u)^+ \leq |u|$, Φ must be in $C([0,T])$. Further, it is known that $C([0,T])$ is dense in $L^1([0,T]) \subset L^1_{\text{loc}}([0,T])$, [2, Theorem 7, p. 714] ensures that

$$
\lim_{\varepsilon \to 0} (\omega_\varepsilon * \Phi)(t) = \Phi(t),
$$

for all $t \in (0, T)$. Taking the limit as $\varepsilon \to 0$ in (4.16) gives

$$\int_{\mathbb{R}^d} (u(x,t_2) - v(x,t_2))^+ \mathrm{d}x \leq \int_{\mathbb{R}^d} (u(x,t_1) - v(x,t_1))^+ \mathrm{d}x.$$

To conclude, let $t_1 \to 0$ and rename t_2. Letting $t_1 \to 0$ requires some attention:

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} (u(x,t_1) - v(x,t_1))^+ \mathrm{d}x - \int_{\mathbb{R}^d} (u_0(x) - v_0(x))^+ \mathrm{d}x \right| \\
&\leq \int_{\mathbb{R}^d} |(u(x,t_1) - v(x,t_1))^+ - (u_0(x) - v_0(x))^+| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} |(u(x,t_1) - v(x,t_1)) - (u_0(x) - v_0(x))| \mathrm{d}x,
\end{aligned}$$

which goes to zero by Remark 2.3.10 ii).

This proof is based on a similar proof given in [6, p. 11-12]. □

Chapter 5

Consequences

This chapter gives global and local properties of entropy solutions to (1.1). In addition, uniqueness and existence for $u_0 \in L^\infty(\mathbb{R}^d)$ are proven.

5.1 Global properties of entropy solutions

This section will give some global consequences of Theorem 4.2.2 (and Corollary 4.2.1), and lastly uniqueness of (1.1) will be established.

Notice that the global results that follow will hold even if $\mu = 0$ or $A(u) = u$ in (1.1) (see e.g. [4]).

Corollary 5.1.1. *For Theorem 4.2.2 assume that* $u, v \in C([0,T]; L^1(\mathbb{R}^d))$*, and that* (A.1)-(A.3) *and* (A.4) *holds. For Corollary 4.2.1 assume that* $u, v \in C([0,T]; L^1_{loc}(\mathbb{R}^d))$*, and that* (A.1)-(A.3) *and* (A.5) *holds.*

i) (L^1 contraction). *Let u and v be entropy solutions to* (1.1) *with u_0 and v_0 as initial data fulfilling* (A.3)*. Assume that $u_0 - v_0 \in L^1(\mathbb{R}^d)$, then the following is valid*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)},$$

for all $t \in (0,T)$.

ii) (Comparison principle). *Let u and v be entropy sub- and supersolutions to* (1.1) *with u_0 and v_0 as initial data fulfilling* (A.3)*. Assume that $u_0(x) \leq v_0(x)$ a.e. on $\mathbb{R}^d$. Then*

$$u(x,t) \leq v(x,t),$$

a.e. in Ω_T.

iii) (L^1 bound). *Let u be an entropy solution to* (1.1) *with u_0 as initial data fulfilling* (A.3)*. Then by Corollary 5.1.1 i) the following result is true*

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}$$

for all $t \in (0,T)$.

iv) (L^∞ bound). *Let u be an entropy solution to* (1.1) *with u_0 as initial data fulfilling* (A.3). *Then by Corollary 5.1.1 ii) the following result is true*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$$

for almost every x and for all $t \in (0,T)$.

v) (BV bound). *Let u be an entropy solution to* (1.1) *with $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ as initial data. Then by Corollary 5.1.1 i) the following result is true*

$$|u(\cdot,t)|_{BV(\mathbb{R}^d)} \leq |u_0|_{BV(\mathbb{R}^d)}$$

for all $t \in (0,T)$.

Remark 5.1.2. In Corollary 5.1.1 v),

$$|u(\cdot,t)|_{BV(\mathbb{R}^d)} = \sup_{h \neq 0} \frac{\|u(\cdot+h,t) - u(\cdot,t)\|_{L^1(\mathbb{R}^d)}}{|h|} \tag{5.1}$$

is used as a BV-semi-norm. By [1, Appendix A] it is evident that (5.1) can be derived from the notion of finite total variation.

Proof As long as the global contraction is established in Theorem 4.2.2 (and Corollary 4.2.1), this proof will not change much from the proof given in e.g. [4, Corollary 2.4.2] (except for the proof of v)). Nevertheless, for the reader's convenience the proofs will be given in the following:

i) By Theorem 4.2.2 (or Corollary 4.2.1), it is known that for all $t \in (0,T)$

$$\int_{\mathbb{R}^d} (u(x,t) - v(x,t))^+ \mathrm{d}x \leq \int_{\mathbb{R}^d} (u_0(x) - v_0(x))^+ \mathrm{d}x$$

holds for u, v being entropy sub- and supersolutions respectively. Observe that $(u-v)^+ = (v-u)^-$, and interchange the roles of u and v to see that the above inequality holds for $(u-v)^-$ as well. That is, it holds for u, v being entropy super- and subsolutions respectively. Thus, the following holds:

$$\int_{\mathbb{R}^d} |u(x,t) - v(x,t)| \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \mathrm{d}x,$$

or

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)},$$

for entropy solutions by Lemma 2.3.11.

ii) Write up the result of Theorem 4.2.2 (or Corollary 4.2.1)

$$\int_{\mathbb{R}^d} (u(x,t) - v(x,t))^+ \mathrm{d}x \leq \int_{\mathbb{R}^d} (u_0(x) - v_0(x))^+ \mathrm{d}x$$

$$\Updownarrow$$

$$0 \leq \int_{\mathbb{R}^d} \max\{u(x,t) - v(x,t), 0\} \mathrm{d}x \leq \int_{\mathbb{R}^d} \max\{u_0(x) - v_0(x), 0\} \mathrm{d}x = 0.$$

Because of the additional assumption ($u_0 - v_0 \leq 0$ a.e.), the right-hand sides of the above inequalities are zero. Since the left-hand side is always nonnegative, and since the integral is forced to be less or equal to zero, there is no other choice than $u(x,t) - v(x,t) \leq 0$ a.e.

iii) Since $v \equiv 0$ is an entropy solution to (1.1), conclude by taking $v \equiv 0$ in Corollary 5.1.1 i).

iv) Obviously,

$$-\|u_0(x)\|_{L^\infty(\mathbb{R}^d)} \leq u_0(x) \leq \|u_0(x)\|_{L^\infty(\mathbb{R}^d)}.$$

Let $w(x,t) := \|u_0(x)\|_{L^\infty(\mathbb{R}^d)}$, then since w is a constant, it is an entropy solution to (1.1). Use Corollary 5.1.1 ii) to obtain

$$-\|u_0(x)\|_{L^\infty(\mathbb{R}^d)} = -w(x,t) \leq u(x,t) \leq w(x,t) = \|u_0(x)\|_{L^\infty(\mathbb{R}^d)},$$

i.e.,

$$|u(x,t)| \leq w(x,t) = \|u_0(x)\|_{L^\infty(\mathbb{R}^d)},$$

which completes the proof.

v) Since (1.1) is translation invariant, both $u(x,t)$ and $u(x+h,t)$ are solutions to (1.1) at $(x,t) \in \Omega_T$. By Corollary 5.1.1 i) and by Remark 5.1.2,

$$\begin{aligned} |u(\cdot,t)|_{BV(\mathbb{R}^d)} &= \sup_{h\neq 0} \frac{\|u(\cdot+h,t) - u(\cdot,t)\|_{L^1(\mathbb{R}^d)}}{|h|} \\ &\leq \sup_{h\neq 0} \frac{\|u_0(\cdot+h) - u_0(\cdot)\|_{L^1(\mathbb{R}^d)}}{|h|} \\ &= |u_0|_{BV(\mathbb{R}^d)}. \end{aligned}$$

□

Finally, the uniqueness result follows:

Corollary 5.1.3 (Uniqueness). *If $u_0 \in L^\infty(\mathbb{R}^d)$, then there is at most one entropy solution to the initial value problem* (1.1).

Remark 5.1.4. Notice that the uniqueness result is valid for (A.1)-(A.4), and hence also for (A.1)-(A.3) and (A.5) as well.

Proof If $u_0 = v_0$ a.e. on $\mathbb{R}^d$ in Corollary 5.1.1 i), then $u = v$ in $C([0,T];L^1(\mathbb{R}^d))$ and a.e. in Ω_T. □

5.2 Local properties of entropy solutions

This section will give some local consequences of Theorem 4.1.3 iii), and lastly a local uniqueness result will be given for (1.1).

Observe that the properties given in the corollaries below are established for (A.1)-(A.3) and (A.5) only.

Corollary 5.2.1. *Assume that* (A.1)-(A.3) *and* (A.5) *holds. Let* $M > 0$, $x_0 \in \mathbb{R}^d$ *and* L_f *be the Lipschitz constant of* f.

i) (L^1 contraction). *Let* u *and* v *be entropy solutions to* (1.1) *with* u_0 *and* v_0 *as initial data fulfilling* (A.3). *Then the following is valid*

$$\begin{aligned}&\|u(\cdot,t) - v(\cdot,t)\|_{L^1(B(x_0,M))}\\ &\leq \|K(-\cdot,0) * |u_0(\cdot) - v_0(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))},\end{aligned}$$

for all $t \in (0,T)$.

ii) (Comparison principle). *In addition to the assumptions of Theorem 4.1.3 iii), assume that* $u_0(x) \leq v_0(x)$ *a.e. in* $B(x_0, M + 1 + L_f t)$. *Then*

$$u(x,t) \leq v(x,t),$$

a.e. in $B(x_0,M)$ *and for all* $t \in (0,T)$.

iii) (L^1 bound). *Let* u *be an entropy solution to* (1.1) *with* u_0 *as initial data fulfilling* (A.3). *Then by Corollary 5.2.1 i) the following result is true*

$$\|u(\cdot,t)\|_{L^1(B(x_0,M))} \leq \|K(-\cdot,0) * |u_0|\|_{L^1(B(x_0,M+1+L_f t))}$$

for all $t \in (0,T)$.

iv) (L^∞ bound). *Let* u *be an entropy solution to* (1.1) *with* u_0 *as initial data fulfilling* (A.3). *Then by Corollary 5.2.1 ii) the following result is true*

$$\|u(\cdot,t)\|_{L^\infty(B(x_0,M))} \leq \|u_0\|_{L^\infty(B(x_0,M+1+L_f t))}$$

for almost every x *and for all* $t \in (0,T)$.

v) (BV bound). *Let* u *be an entropy solution to* (1.1) *with* $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *as initial data. Then by Corollary 5.2.1 i) the following result is true*

$$\begin{aligned}&|u(\cdot,t)|_{BV(B(x_0,M))}\\ &\leq \sup_{h\neq 0} \frac{\|K(-\cdot,0) * |u_0(\cdot+h) - u_0(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))}}{|h|}\\ &< \infty\end{aligned}$$

for all $t \in (0,T)$.

Proof The proof is quite similar to the proof of Corollary 5.1.1 and is left to the reader, except for the proof of i) which will be needed later, and the proof of v) which is new.

i) By Theorem 4.1.3 iii), it is known that for all $t \in (0,T)$, $M > 0$ and $x_0 \in \mathbb{R}^d$

$$\begin{aligned}&\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+ \mathrm{d}x \\ &\leq \int_{B(x_0,\,M+1+L_f t)} (K(-\cdot,0) * (u_0(\cdot) - v_0(\cdot))^+)(x)\mathrm{d}x\end{aligned}$$

holds for u, v being entropy sub- and supersolutions respectively. Observe that $(u-v)^+ = (v-u)^-$, and interchange the roles of u and v to see that the above inequality holds for $(u-v)^-$ as well. That is, it holds for u, v being entropy super- and subsolutions respectively. Thus, the following holds:

$$\begin{aligned}&\int_{B(x_0,\,M)} |u(x,t) - v(x,t)|\mathrm{d}x \\ &\leq \int_{B(x_0,\,M+1+L_f t)} (K(-\cdot,0) * |u_0(\cdot) - v_0(\cdot)|)(x)\mathrm{d}x,\end{aligned}$$

or

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(B(x_0,M))} \leq \|K(-\cdot,0) * |u_0(\cdot) - v_0(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))},$$

for entropy solutions by Lemma 2.3.11.

v) Since (1.1) is translation invariant, both $u(x,t)$ and $u(x+h,t)$ are solutions to (1.1) at $(x,t) \in \Omega_T$. By Corollary 5.2.1 i),

$$\begin{aligned}&|u(\cdot,t)|_{BV(B(x_0,M))} \\ &= \sup_{h\neq 0} \frac{\|u(\cdot+h,t) - u(\cdot,t)\|_{L^1(B(x_0,M))}}{|h|} \\ &\leq \sup_{h\neq 0} \frac{\|K(-\cdot,0) * |u_0(\cdot+h) - u_0(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))}}{|h|} \\ &= \sup_{h\neq 0} \int_{B(x_0,M+1+L_f t)} \int_{\mathbb{R}^d} K(-(x-y),0)\frac{|u_0(y+h) - u_0(y)|}{|h|}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Further, utilize Tonelli's theorem and Corollary 4.1.12 to see that

$$\begin{aligned}&\sup_{h\neq 0} \int_{B(x_0,M+1+L_f t)} \int_{\mathbb{R}^d} K(-(x-y),0)\frac{|u_0(y+h) - u_0(y)|}{|h|}\mathrm{d}y\mathrm{d}x \\ &\leq \sup_{h\neq 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(-(x-y),0)\frac{|u_0(y+h) - u_0(y)|}{|h|}\mathrm{d}y\mathrm{d}x \\ &= \sup_{h\neq 0} \int_{\mathbb{R}^d} \frac{|u_0(y+h) - u_0(y)|}{|h|} \int_{\mathbb{R}^d} K(-(x-y),0)\mathrm{d}x\mathrm{d}y \\ &= \|K(\cdot,t)\|_{L^1(\mathbb{R}^d)}|u_0|_{BV(\mathbb{R}^d)} \leq |u_0|_{BV(\mathbb{R}^d)} < \infty,\end{aligned}$$

that is, $|u(\cdot,t)|_{BV(B(x_0,M))}$ is finite for all $t \in (0,T)$. □

Finally, the local uniqueness result follows:

Corollary 5.2.2 (Uniqueness)**.** *Assume that* (A.1)-(A.3) *and* (A.5) *holds. Let* $M > 0$*,* $x_0 \in \mathbb{R}^d$ *and* L_f *be the Lipschitz constant of* f*. If* $u_0 \in L^\infty(B(x_0, M + 1 + L_f t))$*, then there is at most one entropy solution to the initial value problem* (1.1) *in* $B(x_0, M)$*.*

Proof If $u_0 = v_0$ a.e in $B(x_0, M + 1 + L_f t)$ in Corollary 5.2.1 i), then for all $t \in (0, T)$ $u(\cdot, t) = v(\cdot, t)$ in $L^1(B(x_0, M))$ and a.e in $B(x_0, M)$. □

5.3 Existence for $u_0 \in L^\infty(\mathbb{R}^d)$

This section is devoted to showing that the Cauchy problem (1.1) has at least one solution. The local contraction found in Theorem 4.1.3 iii) will be used together with an already established existence proof to ensure existence.

Theorem 5.3.1 (Existence)**.** *Assume that* (A.1)-(A.3) *and* (A.5) *hold, then there exists an entropy solution to the initial value problem* (1.1)*.*

Proof By [6, Theorem 5.3] there exists an entropy solution to the initial value problem (1.1) with initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Notice that [6] uses (A.4) in order to establish this proof, however, all results valid for (A.4) hold for (A.5) as well. Thus, their existence proof holds for (A.5).

Let $u_0 \in L^\infty(\mathbb{R}^d)$, and consider approximations $u_{0,n} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \|u_0 - u_{0,n}\|_{L^1_{\text{loc}}(\mathbb{R}^d)} = 0. \tag{5.2}$$

By Corollary 5.2.1 i), it is known that for all $t \in (0, T)$, $M > 0$ and $x_0 \in \mathbb{R}^d$

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1(B(x_0,M))} \leq \|K(-\cdot, 0) * |u_0(\cdot) - v_0(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))}$$

holds for entropy solutions. Further, take the maximum over t to obtain

$$\begin{aligned}
&\|u - v\|_{C([0,T];L^1(B(x_0,M)))} \\
&= \max_{t\in[0,T]} \|u(\cdot, t) - v(\cdot, t)\|_{L^1(B(x_0,M))} \\
&\leq \max_{t\in[0,T]} \|K(-\cdot, 0) * |u_0(\cdot) - v_0(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))}.
\end{aligned}$$

Now, let u_m, u_n be entropy solutions with initial data $u_{0,m}, u_{0,n}$ respectively. By the same considerations as above

$$\begin{aligned}
&\|u_m - u_n\|_{C([0,T];L^1(B(x_0,M)))} \\
&\leq \max_{t\in[0,T]} \|K(-\cdot, 0) * |u_{0,m}(\cdot) - u_{0,n}(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))} \\
&\leq \max_{t\in[0,T]} \|K(-\cdot, 0) * |u_{0,m}(\cdot) - u_0(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))} \\
&\quad + \max_{t\in[0,T]} \|K(-\cdot, 0) * |u_{0,n}(\cdot) - u_0(\cdot)|\|_{L^1(B(x_0,M+1+L_f t))}.
\end{aligned} \tag{5.3}$$

Consider

$$\begin{aligned}&\|K(-\cdot,0) * |u_{0,m}(\cdot) - u_0(\cdot)|\|_{L^1(B(x_0,M+1+L_ft))}\\&= \int_{B(x_0,M+1+L_ft)}\int_{\mathbb{R}^d} K(-y,0)|u_{0,m}(x-y) - u_0(x-y)|\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Remember that convergence in $L^1_{\text{loc}}(\mathbb{R}^d)$ ensures that there exists a subsequence such that one gets a.e. convergence. Thus, equation (5.3) goes to zero by Lebesgue's dominated convergence theorem and (5.2) when $n, m \to \infty$. Therefore, the sequence of entropy solutions $\{u_n\}$ is Cauchy in $C([0,T]; L^1(B(x_0, M)))$, that is, for any ball in $L^1(\mathbb{R}^d)$.

To continue, one wants to cover a compact subset, $K \subset\subset \mathbb{R}^d$, with balls of the same form as in the above result. In order to ensure a countable finite number of balls, let x_0 range over $\mathbb{Z}^d$. Further, let $M = 2$ to ensure that every ball overlaps, and fix an enumeration of the balls $\{B_i\}_{i\in\mathbb{N}^+}$ (which is an indexed family of sets). Since K is compact, there exists a finite subcover of K, say $\bigcup_{i=1}^I B_i \supseteq K$.

Let u_m, u_n be entropy solutions with initial data $u_{0,m}, u_{0,n}$ respectively, and look at (to save typing, take the maximum over time in the end)

$$\begin{aligned}\|u_m - u_n\|_{L^1(K)} &\le \|u_m - u_n\|_{L^1(\bigcup B_i)}\\&\le \sum_{i=1}^I \|u_m - u_n\|_{L^1(B_i)},\end{aligned}$$

which goes to zero when $m, n \to \infty$ for all $i \in [1, 2, \ldots, I]$ by (5.3). Taking the supremum over time will not change the result, and

$$\|u_m - u_n\|_{C([0,T];L^1(K))} = \max_{t\in[0,T]} \|u_m - u_n\|_{L^1(K)} \overset{m,n\to\infty}{\to} 0$$

for all $K \subset\subset \mathbb{R}^d$.

Therefore, the sequence of entropy solutions $\{u_n\}$ is Cauchy in $C([0,T]; L^1(K))$ for all $K \subset\subset \mathbb{R}^d$, and hence, by the definition of $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$, $\{u_n\}$ is Cauchy in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$. Which means that the limit, u, is in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$. Passing to the limit in a similar manner as in the proof of Proposition 2.3.2 (with $\varepsilon := \frac{1}{n}$) proves that u is an entropy solution of (1.1), and the existence result is obtained. □

Remark 5.3.2. By combining Theorem 5.3.1 and Corollary 5.1.3, there *exists* a *unique* entropy solution to the initial value problem (1.1) for (A.1)-(A.3) and (A.5).

Chapter 6

Concluding remarks

6.1 Further work

As stated in the project outline, Theorem 4.1.3 iii) and Theorem 5.3.1 are the main results of this paper. These theorems need an additional assumption on the Lévy measure, namely that

$$\int_{|z|>1} \mathrm{e}^{M|z|} \mathrm{d}\mu(z) < \infty,$$

as stated in (A.5). An obvious way of improving our work is to establish the mentioned theorems for (A.4), that is, the most general Lévy measure. Theorem 5.3.1 relies on Theorem 4.1.3 iii), and, therefore, the reason for assuming (A.5) is found in the proofs of the lemmas needed to prove Theorem 4.1.3 iii).

Proposition 4.1.10 is the first result which needs (A.5). An explanation of and some remarks about the additional assumption will be given in the following:

Our idea has been to find a solution to (4.2) by writing the solution as a convolution between the solution to (4.6) and (4.7) (this is an extension of Alibaud's proof, stated in [5], of a similar result). Finding a solution to (4.7) was not an easy task, and involved - among other things - the notion of viscosity solutions. The main problem with viscosity solutions to (4.3) was that there were no results ensuring the existence of an L^1 function solving (4.3). This is probably due to the fact that the references on viscosity solutions were, compared to our problem, quite general (see e.g. [7]). However, the reason for considering (A.5) in the first place, was to ensure that a viscosity solution to (4.3) indeed belonged to $L^1(\mathbb{R}^d)$; see the calculations done in the proof of Proposition 4.1.10. Our guess, which - when inserted into (4.3) - eventually solved the problem, was $w_\pm(x,t) = C\mathrm{e}^{Kt}\mathrm{e}^{\pm k\sum_i x_i}$ where C, K, k are constants greater than zero. (Notice that if a viscosity solution is less than or equal to both w_+ and w_-, then it is less than or equal to $C\mathrm{e}^{Kt}\mathrm{e}^{-k|\cdot|} \in L^1(\mathbb{R}^d)$.) There probably exist other functions which only demand (A.4) to be assumed, but these were not discovered in this project.

Another way of avoiding (A.5), is to obtain the local contraction in Theorem

4.1.3 iii) using a more straightforward method. That is, a method resembling the one used in the proof of Theorem 4.2.2 for instance, or other methods not depending on finding a solution to (4.2). These alternative methods have, unfortunately, not been the focus of our project.

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