AN IMPROVEMENT OF THE KOLMOGOROV–RIESZ COMPACTNESS THEOREM

HARALD HANCHE-OLSEN, HELGE HOLDEN, AND EUGENIA MALINNIKOVA

Abstract. The purpose of this short note is to provide a new and very short proof of a result by Sudakov [10], offering an important improvement of the classical result by Kolmogorov–Riesz on compact subsets of Lebesgue spaces.

Introduction

The classical compactness theorem of Kolmogorov–Riesz reads as follows [5]: A subset $\mathcal{F}$ of $L^p(\mathbb{R}^n)$, with $1 \le p < \infty$, is totally bounded if, and only if,

(a) $\mathcal{F}$ is bounded,

(b) for every $\varepsilon > 0$ there is some R so that, for every $f \in \mathcal{F}$,

$$\int_{|x|>R} |f(x)|^p \, dx < \varepsilon^p,$$

(c) for every $\varepsilon > 0$ there is some $\rho > 0$ so that, for every $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$ with $|y| < \rho$,

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \, dx < \varepsilon^p.$$

The purpose of the current paper is to show that the boundedness condition (a) is redundant.

This was discovered by Sudakov [10] in 1957, but the paper appears undeservedly to have been lost in obscurity. We want to revive the result and present a novel and very short proof of the redundancy of (a).

The Kolmogorov–Riesz compactness theorem was discovered by Kolmogorov [7] in 1931. He stated the result for a subset of $L^p(\mathbb{R}^n)$, with $1 < p < \infty$, and the functions in the subset all supported in a common compact set (thus essentially replacing $\mathbb{R}^n$ by a bounded subset of $\mathbb{R}^n$). Tamarkin [11] extended the result to the case of unbounded support by adding the assumption (b), and Tulajkov [12] extended the result to include $p = 1$. At the same time M. Riesz [9] proved a similar result. See [5, 6] for a historical account of this result, various generalizations, and a proof.

The fact that condition (a) is not needed was only discovered in 1957 by Sudakov [10]. The late discovery of this fact is probably due to a mistake by Tamarkin [11], who presented an erroneous "example" in which (b) and (c) are claimed to be true, but (a) is false.[1] Sudakov [10] states that Tamarkin's mistake was discovered by Natanson, but gives no reference. The result by Sudakov has recently been revisited

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[1]His example was as follows. Consider the family $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}} \subset L^p(\mathbb{R})$, where $f_n(x) = (f(x)+n)\mathbf{1}_{(0,1)}(x)$ for any $f \in L^p(\mathbb{R})$. Clearly, $\mathcal{F}$ satisfies (b), but neither condition (a) nor (c), and $\mathcal{F}$ is not totally bounded.

in the context of metric measure spaces in [4]. See also [2, 8]. These proofs all rely intrinsically on the approach by Sudakov, but are applied to more general spaces.

The Kolmogorov–Riesz compactness theorem is really a classical textbook result, and it is always stated as giving necessary and sufficient conditions for a subset of a Lebesgue space to be compact. The fact that one condition is not needed should be more widely known, and this is our reason for publishing this result.

The improved Kolmogorov–Riesz–Sudakov compactness result

Thanks to Sudakov's discovery, the original Kolmogorov–Riesz theorem admits the following improvement:

Theorem 1 (Kolmogorov–Riesz–Sudakov)**.** *Let $1 \le p < \infty$. A subset $\mathcal{F}$ of $L^p(\mathbb{R}^n)$ is totally bounded if, and only if,*

(i) *for every $\varepsilon > 0$ there is some R so that, for every $f \in \mathcal{F}$,*

$$\int_{|x|>R} |f(x)|^p \, dx < \varepsilon^p,$$

(ii) *for every $\varepsilon > 0$ there is some $\rho > 0$ so that, for every $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$ with $|y| < \rho$,*

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \, dx < \varepsilon^p.$$

Remark. Observe that in the case where $\mathcal{F}$ is a subset of $L^p(\Omega)$, where Ω is a bounded subset of $\mathbb{R}^n$, only the condition of "L^p equicontinuity", that is, condition (ii), is necessary and sufficient for $\mathcal{F}$ to be totally bounded. However, this condition must be interpreted with care, by identifying $L^p(\Omega)$ with a subspace of $L^p(\mathbb{R}^n)$. Thus the behavior of functions in $\mathcal{F}$ at the boundary of Ω will influence whether (ii) holds or not. This can be illustrated by the failure of Tamarkin's example; see the footnote in the introduction.

Before embarking on the proof, we establish some notation. Throughout, $B_r(x)$ denotes the open ball of radius r centered at $x \in \mathbb{R}^n$. We sometimes write B_r instead of $B_r(0)$. We write $\mathbf{1}_A$ for the characteristic function of a set $A \subseteq \mathbb{R}^n$. The translation operator T_y is defined by $T_y f(x) = f(x-y)$. When $\Omega \subseteq \mathbb{R}^n$, we identify $L^p(\Omega)$ with the set of functions in $L^p(\mathbb{R}^n)$ vanishing outside Ω. We write X_1 for the closed unit ball of any normed space X.

In light of the classical Kolmogorov–Riesz theorem, see, e.g., [5], the following is all that is required to prove Theorem 1:

Proof that (b) *and* (c) *imply* (a). Assume that conditions (b) and (c) are satisfied. Due to condition (b) we only need to bound the norm uniformly on some sufficiently large ball. The idea is that by (c), small translation are uniformly close to the identity in the $L^p(\mathbb{R}^n)$ norm. By restricting to a ball, and repeating the small translation, we can get an estimate of the norm on a ball by the norm on a translated ball that is contained in the domain of integration in (b), which gives the uniform bound we want.

More precisely, fix $\varepsilon = 1$ and let $R > 0$ and $\rho > 0$ be the corresponding quantities given by (b) and (c). For any $f \in \mathcal{F}$, using the triangle inequality and a translation, we infer

$$\begin{aligned} \|f\mathbf{1}_{B_R(z)}\|_p &\le \|(T_y f - f)\mathbf{1}_{B_R(z)}\|_p + \|f\mathbf{1}_{B_R(z-y)}\|_p \\ &\le \|(T_y f - f)\|_p + \|f\mathbf{1}_{B_R(z-y)}\|_p \\ &\le 1 + \|f\mathbf{1}_{B_R(z-y)}\|_p. \end{aligned}$$

Here $y \in \mathbb{R}^n$ is any nonzero vector with $|y| < \rho$. By induction, we find that

$$\|f\mathbf{1}_{B_R(0)}\|_p \le N + \|f\mathbf{1}_{B_R(-Ny)}\|_p.$$

Choosing N so that $N|y| > 2R$, we see that $B_R(-Ny) \cap B_R(0) = \emptyset$, and

$$\|f\|_p = \|f\mathbf{1}_{B_R(0)}\|_p + \|f\mathbf{1}_{\mathbb{R}^n \setminus B_R(0)}\|_p \le N + 2,$$

uniformly in f.

□

Sudakov states the theorem with the translate $T_y f$ in (ii) replaced by the *Steklov mean*

$$S_h f(x) = |B_h|^{-1} \int_{B_h} f(x+y)\,dy = |B_h|^{-1} f * \mathbf{1}_{B_h}(x)$$

for sufficiently small h, where $|B_h|$ denotes the volume of B_h. Clearly, the revised condition follows from (ii), but the converse is far from obvious. We show that Sudakov's condition can also be used instead of (ii) to estimate the L^p-norm:

Theorem 2 (Kolmogorov–Riesz–Sudakov)**.** *Theorem 1 holds with condition (ii) replaced by*:

(ii') *For every $\varepsilon > 0$ there is some $\rho > 0$ so that, for every $f \in \mathcal{F}$ and h with $0 < h < \rho$,*

$$\int_{\mathbb{R}^n} |f(x) - S_h f(x)|^p\,dx < \varepsilon^p.$$

We will need a lemma.

Lemma 3. *Assume that p and q are conjugate exponents with $1 \le p < \infty$, and that $\phi \in L^q(\mathbb{R}^n)$ has compact support. If $p = 1$, assume further that ϕ is continuous. Let $K \subset \mathbb{R}^n$ be compact. Then the map $\Phi\colon L^p(K) \to L^p(\mathbb{R}^n)$ defined by $\Phi f = \phi * f$ is compact.*

Proof. First note that $y \mapsto T_y\phi$ is a continuous map $\mathbb{R}^n \to L^q(\mathbb{R}^n)$ (see, e.g., [3, Prop. 20.1]). It immediately follows that the set of functions $\{\, \phi * f \mid f \in L^p(\mathbb{R}^n)_1 \,\}$ is equicontinuous, since

$$\begin{aligned} |\phi * f(x-y) - \phi * f(x)| &= |(T_y\phi - \phi) * f(x)| \\ &\le \|T_y\phi - \phi\|_q \cdot \|f\|_p \le \|T_y\phi - \phi\|_q \end{aligned}$$

for any $f \in L^p(\mathbb{R}^n)_1$ (the continuity of ϕ when $p = 1$ is required in order to make $\|T_y\phi - \phi\|_q$ small even in that case). A similar estimate shows that this set of functions is uniformly bounded. Since all functions $\phi * f$ with $f \in L^p(K)$ are supported by the compact set $K + \operatorname{supp}\phi$, we can now employ the Arzelà–Ascoli theorem to conclude that $\{\, \phi * f \mid f \in L^p(K)_1 \,\}$ is totally bounded in the uniform norm. Again, because of the shared compact support, this implies compactness in $L^p(\mathbb{R}^n)$. □

Proof of Theorem 2. In the proof of Theorem 1, we employed repeated translations to move the support of $T_{-Ny}f$ outside a large ball. Here we use instead repeated applications of the convolution operator (the Steklov mean) S_h for a similar purpose, getting some weighted average of f. We cannot move the whole weight to the complement of some fixed ball as before, however. Instead, we notice that the total weight is one, but some fixed part of it is moved to this complement.

To make this presice, we start by fixing R as given by (i) and ρ as given by (ii'), both with $\varepsilon = 1$. Let $0 < h < \rho$, and put $\phi = |B_h|^{-1}\mathbf{1}_{B_h}$. Select a natural number N so that $Nh > 2R$, and put

$$\psi = \phi^{*N} = \underbrace{\phi * \cdots * \phi}_{N \text{ times}}.$$

Writing the N-fold convolution as an $(N-1)$-fold integral over $z_1, \dots, z_{N-1} \in \mathbb{R}^n$ and setting $z_N = x - z_1 - \dots - z_{N-1}$, we can write this as

$$\psi(x) = \int \cdots \int_{z_1+\cdots+z_N=x} \phi(z_1)\cdots\phi(z_N)\,dz_1\cdots dz_{N-1},$$

from which it follows that $\psi(x) > 0$ when $|x| < Nh$, and $\psi(x) = 0$ otherwise. Note also that $\int_{\mathbb{R}^n} \psi\,dx = 1$.

Now fix some $f \in \mathcal{F}$, and define

$$A(y) = \|f\mathbf{1}_{B_R(y)}\|_p = \Bigl(\int_{B_R} |f(x+y)|^p\,dx\Bigr)^{1/p}.$$

Our task is to find a bound for $A(0)$, independent of f. Together with (i), this will establish a uniform bound on $\|f\|_p$ for $f \in \mathcal{F}$.

The function A is continuous, since we can also write $A(y) = \|(T_y f)\mathbf{1}_{B_R(0)}\|_p$. Further, condition (i) implies that $A(y) < 1$ for $|y| \geq 2R$, so A is certainly bounded. Let $M = \sup_{y\in\mathbb{R}^n} A(y)$.

To estimate $A(y)$, we break it up as follows:

$$A(y) \leq \|(f*\psi)\mathbf{1}_{B_R(y)}\|_p + \|(f*\psi - f)\mathbf{1}_{B_R(y)}\|_p. \tag{1}$$

For the first term, the continuous Minkowski inequality (see, e.g., [3, Prop. 4.3 (p. 227)]) yields

$$\begin{aligned}\|(f*\psi)\mathbf{1}_{B_R(y)}\|_p &= \Bigl(\int_{B_R(y)}\Bigl|\int_{\mathbb{R}^n} f(x-u)\psi(u)\,du\Bigr|^p dx\Bigr)^{1/p} \\ &\leq \int_{\mathbb{R}^n}\Bigl(\int_{B_R(y)} |f(x-u)^p|\,dx\Bigr)^{1/p}\psi(u)\,du \\ &= A*\psi(y).\end{aligned}$$

As for the second term of (1), first note that condition (*ii'*) with $\varepsilon = 1$ can be written $\|f*\phi - f\|_p < 1$. Furthermore, $\|g*\phi\|_p \leq \|g\|_p$ for any $g \in L^p(\mathbb{R}^n)$ (as seen, e.g., by another application of the continuous Minkowski inequality). Thus we find $\|f*\phi^{*(k+1)} - f*\phi^{*k}\|_p \leq \|(f*\phi - f)*\phi^{*k}\|_p \leq \|f*\phi - f\|_p < 1$, so we have

$$\|f*\phi^{*k} - f\|_p \leq k \qquad (f \in \mathcal{F}). \tag{2}$$

In particular, $\|f*\psi - f\|_p \leq N$, and so (1) reduces to

$$A \leq A*\psi + N.$$

However,

$$\begin{aligned}A*\psi(y) &= \int_{\mathbb{R}^n} A(u)\psi(y-u)\,du \\ &\leq M\int_{B_{2R}} \psi(y-u)\,du + \int_{\mathbb{R}\setminus B_{2R}} \psi(y-u)\,du \\ &\leq M\gamma + 1,\end{aligned}$$

where

$$\gamma = \max_{y\in\mathbb{R}^n} \int_{B_{2R}} \psi(y-u)\,du < 1.$$

Indeed, note that the above integral is a continuous function of y, with compact support, so it achieves its maximum. But the integral is always less than 1, because the integrand is strictly positive in a ball of radius $Nh > 2R$.

To summarize, we have $M = \sup_y A(y) \le M\gamma + 1 + N$, and therefore $M \le (1+N)/(1-\gamma)$. Since this estimate is independent of f, we have now proved that $\mathcal{F}$ is bounded in $L^p(\mathbb{R}^n)$.

To finish the proof, let $\varepsilon > 0$, once more pick $R > 0$ and $\rho > 0$ according to conditions (i) and (ii'), and let $\phi = |B_h|^{-1}\mathbf{1}_{B_h}$, where $0 < h < \rho$. Define the linear map $\Phi_R \colon \mathcal{F} \to L^p(\mathbb{R}^n)$ by

$$\Phi_R f = (f\mathbf{1}_{B_R}) * \phi * \phi$$

(we may replace $\phi * \phi$ by ϕ, if $p \ne 1$). It is compact, by Lemma 3. Therefore, since $\mathcal{F}$ is bounded, $\Phi_R\mathcal{F}$ is totally bounded. Now, for any $f \in \mathcal{F}$,

$$\|f - \Phi_R f\|_p \le \|f - f * \phi * \phi\|_p + \|(f - f\mathbf{1}_{B_R}) * \phi * \phi\|_p < 2\varepsilon + \varepsilon = 3\varepsilon.$$

Here the first norm estimate comes from (2), while the second one is due to (i) and the general fact that $\|g * \phi\|_p \le \|g\|_p$.

Thus any member of $\mathcal{F}$ is within a distance 3ε of some member of the totally bounded set $\Phi_R\mathcal{F}$, and so $\mathcal{F}$ itself is totally bounded. $\square$

Review of the original proof of Sudakov

For the benefit of the reader we review Sudakov's original argument, which is interesting for two reasons. First of all it is quite different from other proofs of this theorem, and, furthermore, it uses only conditions (i) and (ii') without involving the uniform boundedness. We start by stating and proving two general results.

Theorem 4 (Mazur, see [1, p. 466]). *Let G be a bounded subset of a Banach space X. Assume that (U_k) is a sequence of compact operators on X converging to the identity operator in the strong operator topology, i.e., $\|U_k x - x\| \to 0$ for all $x \in X$. Then G is totally bounded if, and only if, $\|U_k x - x\| \to 0$ uniformly for $x \in G$.*

Proof. First, assume that $\|U_k x - x\| \to 0$ uniformly for $x \in G$. Then for any $\varepsilon > 0$, there is some k so that $\operatorname{dist}(x, U_k G) < \varepsilon$ for all $x \in G$. The image $U_k G$ is totally bounded, because G is bounded and U_k is compact. The total boundedness of G follows.

Conversely, assume G is totally bounded. Apply the Banach–Steinhaus theorem to get a uniform bound $\|U_k\| \le M$ for all k. If $\varepsilon > 0$, there is an ε-net $F \subseteq G$: A finite set so that every point in G is within a distance ε from some member of F. If k is large enough, $\|U_k y - y\| \le \varepsilon$ for all $y \in F$. For any $x \in G$, then, there is some $y \in F$ with $\|y - x\| < \varepsilon$, and so

$$\|U_k x - x\| \le \|U_k(x - y)\| + \|U_k y - y\| + \|y - x\| < M\varepsilon + \varepsilon + \varepsilon = (M+2)\varepsilon.$$

Since M is fixed and ε is arbitrary, $\|U_k x - x\| \to 0$ uniformly for $x \in G$. $\square$

Lemma 5 (Sudakov [10]). *Assume that X is a Banach space, and $G \subseteq X$. Assume also that U is a compact operator on X so that 1 is not an eigenvalue of U, and $\|Ux - x\| \le M < \infty$ for all $x \in G$. Then G is bounded.*

Proof. Since U is compact and 1 is not an eigenvalue, $1 \notin \sigma(U)$, and so $U - I$ is invertible. So for any $x \in G$, $\|x\| \le \|(U-I)^{-1}\| \cdot \|Ux - x\| \le \|(U-I)^{-1}\|M$. $\square$

A different proof of Theorem 2. We prove only that (i) and (ii') imply total boundedness. For the other direction, refer to the earlier proof (see page 3).

For any $\varepsilon > 1$, choose R according to condition (i), and define a continuous cutoff function v_R:

$$v_R(x) = \begin{cases} 1 & |x| < R+1, \\ R+2-|x| & R+1 \le |x| \le R+2, \\ 0 & |x| > R+2. \end{cases}$$

Thus $\|f - fv_R\|_p < \varepsilon$ for any $f \in \mathcal{F}$. If we can show that $\mathcal{F}v_R$ is totally bounded for every $R > 0$, it immediately follows that $\mathcal{F}$ is totally bounded.

We now observe that condition (*ii'*) is still satisfied if $\mathcal{F}$ is replaced by $\mathcal{F}v_R$. To see this, note that

$$\begin{aligned}\|fv_R - S_h(fv_R)\|_p &\le \|(f - S_hf)v_R\|_p + \|(S_hf)v_R - S_h(fv_R)\|_p \\ &\le \|f - S_hf\|_p + \|(S_hf)v_R - S_h(fv_R)\|_p.\end{aligned}$$

Next,

$$S_hf(x)v_R(x) - S_h(fv_R)(x) = |B_h(x)|^{-1}\int_{B_h(x)} f(y)\big(v_R(x) - v_R(y)\big)\,dy.$$

Note that $|v_R(x) - v_R(y)| \le |x - y| < h$ whenever $y \in B_h(x)$, and furthermore $v_R(x) - v_R(y) = 0$ if in addition $|x| \le R$, provided we ensure that $h < 1$. Under this assumption, then,

$$|S_hf(x)v_R(x) - S_h(fv_R)| \le hS_h|f\mathbf{1}_{\mathbb{R}\setminus B_R}|(x),$$

and therefore

$$\|(S_hf)v_R - S_h(fv_R)\|_p \le h\|S_h|f\mathbf{1}_{\mathbb{R}\setminus B_R}|\|_p \le h\|f\mathbf{1}_{\mathbb{R}\setminus B_R}\|_p < h.$$

And so we get

$$\|fv_R - S_h(fv_R)\|_p \le \|f - (S_hf)\|_p + h,$$

and it follows that $\mathcal{F}v_R$ does indeed satisfy (*ii'*). Thus we can replace $\mathcal{F}$ with $\mathcal{F}v_R$ in the remainder of the proof.

From now on, we assume without loss of generality that $\operatorname{supp} f \subseteq K$ for all $f \in \mathcal{F}$, where $K \subset \mathbb{R}$ is compact. Let $\phi_k = |B_{1/k}|^{-1}\mathbf{1}_{B_{1/k}}$. Then $f * \phi_k = S_{1/k}f \to f$ in the L^p norm, uniformly for $f \in \mathcal{F}$; and the same is true for $f * \phi_k * \phi_k$.

Define the operator $\Phi_k \colon L^p(K) \to L^p(K)$ by $\Phi_kf = (f * \phi_k * \phi_k)\mathbf{1}_K$. Lemma 3 ensures that Φ_k is compact.

We claim that 1 is not an eigenvalue of Φ_k. Assuming this, we can use Lemma 5 to conclude that $\mathcal{F}$ is bounded, and then Mazur's theorem (Theorem 4) implies that $\mathcal{F}$ is totally bounded, thus finishing the proof.

To prove the claim, assume the contrary, and let a nonzero $f \in L^p(K)$ satisfy $f = (f * \psi)\mathbf{1}_K$, where $\psi = \phi_k * \phi_k$. Without loss of generality, we may assume that $f(x) > 0$ for some x. Note that $f * \psi$ is continuous, and so f has a maximum value $c > 0$. Let $C \subseteq K$ be the compact set $\{\, x \in \mathbb{R}^n \mid f = c \,\}$, and consider any point x on the boundary of C. Then we have

$$c = f(x) = \int_{\mathbb{R}^n} f(x - y)\psi(y)\,dy.$$

Since $f \le c$, and $f(x - y) < c$ for y in some open set in which $\psi(y) > 0$, we get

$$\int_{\mathbb{R}^n} f(x - y)\psi(y)\,dy < c\int_{\mathbb{R}^n} \psi(y)\,dy = c,$$

and so we arrive at the contradiction $c < c$. This completes the proof. □

References

[1] S. Banach. *Theory of Linear Operations.* North-Holland, 1987.
Published originally as *Théorie des opérations linéaires* (In French) in 1932.

[2] Yu. Brudnyi. Compactness criteria for spaces of measurable functions. *St. Petersburg Math. J.* 26 (2015) 49–68.

[3] E. DiBenedetto. *Real Analysis.* Birkhäuser, Boston, 2002.

[4] P. Górka and H. Rafeiro. Light side of compactness in Lebesgue spaces: Sudakov theorem. *Ann. Acad. Sci. Fenn.* 42 (2017) 135–139.

[5] H. Hanche-Olsen and H. Holden. The Kolmogorov–Riesz compactness theorem. *Expo. Math.* **28** (2010) 385–394.

[6] H. Hanche-Olsen and H. Holden. Addendum to "The Kolmogorov–Riesz compactness theorem." *Expo. Math.* **34** (2016) 243–245.

[7] A. N. Kolmogorov. Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel. (In German) *Nachr. Ges. Wiss. Göttingen* **9** (1931), 60–63.
English translation: On the compactness of sets of functions in the case of convergence in the mean, in V. M. Tikhomirov (ed.), *Selected Works of A. N. Kolmogorov, Vol. I*, Kluwer, Dordrecht, 1991, pp. 147–150.

[8] H. Rafeiro. Kolmogorov compactness criterion in variable exponent Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* 150 (2009) 105–113.

[9] M. Riesz. Sur les ensembles compacts de fonctions sommables. (In French) *Acta Szeged Sect. Math.* **6** (1933), 136–142.
Also in L. Gårding, L. Hörmander (eds.), *Marcel Riesz—Collected Papers*, Springer, Berlin, 1988, pp. 458–464.

[10] V. N. Sudakov. Criteria of compactness in function spaces. (In Russian) *Upsekhi Math. Nauk.* **12** (1957) 221–224.

[11] J. D. Tamarkin. On the compactness of the space L_p. *Bull. Amer. Math. Soc.* **32** (1932), 79–84.

[12] A. Tulajkov. Zur Kompaktheit im Raum L_p für $p = 1$. (In German) *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl. I* 1933, nr. 39, 167–170.

(Hanche-Olsen)
Department of Mathematical Sciences, NTNU Norwegian University of Science and Technology, NO–7491 Trondheim, Norway
Email address: harald.hanche.olsen@ntnu.no
URL: https://www.ntnu.edu/employees/harald.hanche-olsen

(Holden)
Department of Mathematical Sciences, NTNU Norwegian University of Science and Technology, NO–7491 Trondheim, Norway
Email address: helge.holden@ntnu.no
URL: https://www.ntnu.edu/employees/helge.holden

(Malinnikova)
Department of Mathematical Sciences, NTNU Norwegian University of Science and Technology, NO–7491 Trondheim, Norway
Email address: eugenia.malinnikova@ntnu.no
URL: https://www.ntnu.edu/employees/eugenia.malinnikova