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**On the structure of certain  
 $C^*$ -algebras arising  
from groups**

Thesis for the degree of Philosophiae Doctor

Trondheim, May 2013

Norwegian University of Science and Technology  
Faculty of Information Technology, Mathematics  
and Electrical Engineering  
Department of Mathematical Sciences



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## Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of Philosophiae Doctor (PhD) at the Norwegian University of Science and Technology (NTNU).

First of all, I would like to thank my advisor Magnus Landstad for all his help and guidance and for introducing me to the international community and arranging my research stay abroad. I would also like to thank my co-advisor Toke Meier Carlsen and the rest of the operator algebra group at NTNU for contributing to a good working environment and learning atmosphere.

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## Outline of thesis

The dissertation consists of a collection of five papers and an introductory section that explains the connection between these papers and puts them into a context. The following five papers are included:

### Paper I

The full group  $C^*$ -algebra of the modular group is primitive. Joint work with Erik Bédos. Published in *Proc. Amer. Math. Soc.*, 140(4):1403-1411, 2012.

### Paper II

Primitivity of some full group  $C^*$ -algebras. Joint work with Erik Bédos. Published in *Banach J. Math. Anal.*, 5(2):44-58, 2011.

### Paper III

Primeness and primitivity conditions for twisted group  $C^*$ -algebras. Accepted for publication in *Math. Scand.*, 2012.

### Paper IV

$C^*$ -algebras generated by projective representations of free nilpotent groups. Submitted for publication, 2013.

### Paper V

Cuntz-Li algebras from  $a$ -adic numbers. Joint work with Steve Kaliszewski and John Quigg. Submitted for publication, 2012.



# Introduction





## 1 $C^*$ -algebras

This section is an attempt to very briefly motivate the study of  $C^*$ -algebras and explain some aspects of the theory related to the thesis, without going too much into detail on the historical background.

The theory of  $C^*$ -algebras can be developed in two different ways, either as certain algebras of bounded operators on Hilbert spaces or as special cases of Banach algebras.

### 1.1 Concrete approach

The motivation for studying operator algebras originally comes from quantum mechanics, and almost every survey on the topic starts with the Heisenberg commutation relation for a free particle,

$$PQ - QP = -iI, \tag{1}$$

where  $P$  and  $Q$  are self-adjoint operators on a Hilbert space  $\mathcal{H}$  representing momentum and position, respectively. It turns out that (1) has nontrivial solutions only if  $\mathcal{H}$  is infinite-dimensional and at least one of  $P$  or  $Q$  is unbounded. However, a theorem by Stone describes a bijective correspondence via “exponentiation” between possibly unbounded self-adjoint operators on  $\mathcal{H}$  and one-parameter unitary subgroups of  $B(\mathcal{H})$ . As a consequence, the Weyl form of (1) is introduced, that is, for every real number  $t$  one defines the bounded unitary operators  $U(t) = e^{itP}$  and  $V(t) = e^{itQ}$  on  $\mathcal{H}$  and observes that

$$U(s)V(t) = e^{ist}V(t)U(s). \tag{2}$$

In this way,  $U$  and  $V$  become unitary representations of  $\mathbb{R}$  on  $\mathcal{H}$ . Moreover, for  $(s, t)$  in  $\mathbb{R}^2$ , set  $W(s, t) = U(s)V(t)$ , and define  $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{T}$  by

$$\sigma((s, t), (s', t')) = e^{its'}. \tag{3}$$

It is then evident that  $\sigma$  is a multiplier (or 2-cocycle) of  $\mathbb{R}^2$  and that  $W$  is a  $\sigma$ -projective unitary representation of  $\mathbb{R}^2$  on  $\mathcal{H}$ .

Equivalently, the pair  $(U, V)$  determines a unitary representation of the real Heisenberg group. Indeed, set  $\widetilde{W}(r, s, t) = e^{ir}U(s)V(t) = e^{ir}W(s, t)$ , and then

$$\widetilde{W}(r, s, t)\widetilde{W}(r', s', t') = e^{i(r+r')}e^{its'}W(s+s', t+t') = \widetilde{W}(r+r'+ts', s+s', t+t').$$

Schrödinger’s solution of (2) are the bounded operators  $U$  and  $V$  on  $L^2(\mathbb{R})$  given by

$$U(s)\psi(t) = \psi(t - s) \quad \text{and} \quad V(s)\psi(t) = e^{-ist}\psi(t), \tag{4}$$

and every pair of irreducible unitary representations of  $\mathbb{R}$  satisfying (2) is unitarily equivalent to this pair. Also, let  $\sigma$  be defined by (3). Then every irreducible  $\sigma$ -projective representation of  $\mathbb{R}^2$  is unitarily equivalent with the one coming from (4).

This uniqueness result is generalized by Mackey to hold for all locally compact second countable abelian groups  $G$  and is called the “Stone-von Neumann theorem”. That is, there is, up to unitary equivalence, only one pair of irreducible unitary representations  $U$  of  $G$  and  $V$  of  $\widehat{G}$ , such that

$$U(a)V(b) = \langle b, a \rangle V(b)U(a)$$

for all  $a \in G$  and  $b \in \widehat{G}$  (see also Example 3.1 below).

In general, states of a quantum mechanical system may be considered as elements  $\psi$  of a Hilbert space  $\mathcal{H}$  and observables as self-adjoint operators  $T$  on  $\mathcal{H}$ , such that the result of a measurement of  $T$  is given by the expected value  $\langle T\psi, \psi \rangle$ . The dynamical evolution of a system is determined by a self-adjoint operator  $H$  through  $T(t) = e^{itH}Te^{-itH}$  or  $\psi(t) = e^{-itH}\psi$  for  $t$  in  $\mathbb{R}$ , so that the expected value at  $t$  is

$$\langle e^{itH}Te^{-itH}\psi, \psi \rangle = \langle Te^{-itH}\psi, e^{-itH}\psi \rangle. \quad (5)$$

Moreover, there is a need for a study of families of operators, for example for the consideration of spectral decomposition of a single operator. The “rings of operators” that Murray and von Neumann considered in the first place are the weakly closed and self-adjoint subalgebras of  $B(\mathcal{H})$  containing the identity operator. This class of operator algebras is now called von Neumann algebras. In this framework, the quantum observables are identified with the self-adjoint elements of such operator algebras.

## 1.2 Abstract approach

One can argue that it is sufficient to consider uniformly closed and self-adjoint subalgebras of  $B(\mathcal{H})$ , and therefore the notion  $C^*$  appears, for closed  $*$ -subalgebra.

Gelfand and Naimark (and Segal) then discover that  $C^*$ -algebras may also be studied abstractly, without any reference to operators on Hilbert spaces. That is, a  $C^*$ -algebra  $A$  can be defined axiomatically as a Banach algebra together with an involution  $A \rightarrow A$ ,  $x \mapsto x^*$  such that

$$\|x^*x\| = \|x\|^2.$$

Then, for every such (abstract)  $C^*$ -algebra  $A$  there exists a Hilbert space  $\mathcal{H}$  and an injective  $*$ -homomorphism  $\pi : A \rightarrow B(\mathcal{H})$ . That is,  $A \cong \pi(A) \subset B(\mathcal{H})$ , as every  $*$ -homomorphism is norm-decreasing and thus continuous.

The algebraic structure in a  $C^*$ -algebra is strong. In fact,  $\|x\|^2$  coincides with the spectral radius of  $x^*x$  so that there is only one norm on a  $*$ -algebra making it a  $C^*$ -algebra.

Furthermore, Gelfand and Naimark show that for every commutative  $C^*$ -algebra  $A$  there is a locally compact Hausdorff space  $X$  such that  $A \cong C_0(X)$ , the set of complex-valued continuous functions on  $X$  vanishing at infinity with pointwise operations and sup-norm. Moreover, two commutative  $C^*$ -algebras are isomorphic if and only if their associated topological spaces are homeomorphic. There is a (contravariant) category equivalence between the category of unital commutative  $C^*$ -algebras with  $*$ -homomorphism and the category of compact Hausdorff spaces with continuous maps. There is also a version of this result relating nonunital commutative  $C^*$ -algebras with locally compact noncompact Hausdorff spaces. Thus, topological properties of  $X$  can be translated into algebraic properties of  $C_0(X)$ , and vice versa, and the theory of  $C^*$ -algebras is often referred to as noncommutative topology in the modern language.

For example, let  $X$  be a compact Hausdorff space. If  $f$  is a projection in  $C(X)$ , that is,  $\overline{f(x)} = f(x) = f(x)^2$  for all  $x \in X$ , then  $f$  can only take the values 0 and 1. Hence,  $X$  is connected if and only if  $C(X)$  is projectionless.

Open and closed sets of  $X$  correspond to ideals and quotients of  $C_0(X)$ , respectively. Clearly,  $C_0(X)$  is simple only if  $X = \{*\}$ . In the noncommutative case, on the other hand, the theory is much more intriguing, and highly nontrivial  $C^*$ -algebras can still be simple.

## 2 Projective unitary representations

The importance of (projective) unitary representations in the theory of  $C^*$ -algebras should now be obvious from the previous section. In particular, the way  $W$  and  $\widetilde{W}$  are obtained above indicate a connection between projective representations of a group and ordinary representations of an extension of that group. In addition, since two states of a quantum mechanical system are equivalent if they are scalar multiples of each other, states are really elements of a projective Hilbert space  $P\mathcal{H} = \mathcal{H}/\mathbb{C}1$ .

The original approach concerns representations of the group  $\mathbb{R}$  and then generalizations to locally compact second countable abelian groups. However, we will delay the discussion of locally compact groups until Section 2.3, and first focus on (arbitrary) discrete groups.

All of the five included papers deal with unitary representations of groups. In Paper III and IV, we study  $C^*$ -algebras associated with projective unitary representations of discrete groups in detail, and the consideration of locally compact groups is needed for Paper V.

## 2.1 Twisted group $C^*$ -algebras

Let  $G$  be a discrete group and  $\mathcal{H}$  a nontrivial Hilbert space. The automorphism group of  $P\mathcal{H}$  is the projective unitary group  $PU(\mathcal{H})$ , that is, the quotient of  $U(\mathcal{H})$  by its center, i.e.

$$PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}1_{\mathcal{H}}.$$

A projective unitary representation of  $G$  is a homomorphism  $G \rightarrow PU(\mathcal{H})$ . Every lift of a projective representation to a map  $U : G \rightarrow U(\mathcal{H})$  satisfies

$$U(a)U(b) = \sigma(a, b)U(ab) \quad (6)$$

for all  $a, b \in G$  and some function  $\sigma : G \times G \rightarrow \mathbb{T}$ . From the associativity of  $U$  and by requiring that  $U(e) = 1_{\mathcal{H}}$ , the identities

$$\begin{aligned} \sigma(a, b)\sigma(ab, c) &= \sigma(a, bc)\sigma(b, c) \\ \sigma(a, e) &= \sigma(e, b) = 1 \end{aligned} \quad (7)$$

must hold for all elements  $a, b, c \in G$ . Motivated by these observations, any function  $\sigma : G \times G \rightarrow \mathbb{T}$  satisfying (7) and is called a *multiplier of  $G$* , and any map  $U : G \rightarrow U(\mathcal{H})$  satisfying (6) is called a  *$\sigma$ -projective unitary representation of  $G$  on  $\mathcal{H}$* .

The lift of a homomorphism  $G \rightarrow PU(\mathcal{H})$  up to  $U$  is not unique, but any other lift is of the form  $\beta U$  for some function  $\beta : G \rightarrow \mathbb{T}$ . Consequently, one says that two multipliers  $\sigma$  and  $\tau$  are *similar* and writes  $\sigma \sim \tau$  if

$$\tau(a, b) = \beta(a)\beta(b)\overline{\beta(ab)}\sigma(a, b) \quad (8)$$

for all  $a, b \in G$  and some  $\beta : G \rightarrow \mathbb{T}$ . The set of similarity classes of multipliers of  $G$  is an abelian group under pointwise multiplication. This group was originally called the *Schur multiplier* of  $G$ , but it coincides with the second cohomology group  $H^2(G, \mathbb{T})$  consisting of *2-cocycles on  $G$  with values in  $\mathbb{T}$* .

Let  $\sigma$  be a multiplier of  $G$ . The Banach  $*$ -algebra  $\ell^1(G, \sigma)$  is defined as the Banach space  $\ell^1(G)$  together with twisted convolution and involution given by

$$\begin{aligned} (f * g)(a) &= \sum_{b \in G} f(b)\sigma(b, b^{-1}a)g(b^{-1}a) \\ f^*(a) &= \overline{\sigma(a, a^{-1})f(a^{-1})} \end{aligned}$$

for elements  $f, g$  in  $\ell^1(G)$ .

The full twisted group  $C^*$ -algebra  $C^*(G, \sigma)$  is the universal enveloping algebra of  $\ell^1(G, \sigma)$ , that is, the completion of  $\ell^1(G, \sigma)$  with respect to the norm  $\|\cdot\|_{\max}$  given by

$$\|f\|_{\max} = \sup\{\|\pi(f)\| : \pi \text{ is representation of } \ell^1(G, \sigma)\}.$$

Let  $i_\sigma$  denote the canonical injection of  $G$  into  $C^*(G, \sigma)$ . Then  $C^*(G, \sigma)$  satisfies the following universal property. Every  $\sigma$ -projective unitary representation of  $G$  on some Hilbert space  $\mathcal{H}$  (or in some  $C^*$ -algebra  $A$ ) factors uniquely through  $i_\sigma$ .

The *left regular  $\sigma$ -projective unitary representation*  $\lambda_\sigma$  of  $G$  on  $B(\ell^2(G))$  is given by

$$(\lambda_\sigma(a)\xi)(b) = \sigma(a, a^{-1}b)\xi(a^{-1}b).$$

The integrated form of  $\lambda_\sigma$  on  $\ell^1(G, \sigma)$  is defined by

$$\lambda_\sigma(f) = \sum_{a \in G} f(a)\lambda_\sigma(a).$$

The reduced twisted group  $C^*$ -algebra and the twisted group von Neumann algebra of  $(G, \sigma)$ ,  $C_r^*(G, \sigma)$  and  $W^*(G, \sigma)$  are, respectively, the  $C^*$ -algebra and the von Neumann algebra generated by  $\lambda_\sigma(\ell^1(G, \sigma))$ , or equivalently by  $\lambda_\sigma(G)$ .

If  $\tau$  is similar with  $\sigma$ , then in all three cases, the operator algebras associated with  $(G, \tau)$  and  $(G, \sigma)$  are isomorphic.

Moreover, there is a natural one-to-one correspondence between the representations of  $C^*(G, \sigma)$  and the  $\sigma$ -projective unitary representations of  $G$ . In particular,  $\lambda_\sigma$  extends to a  $*$ -homomorphism of  $C^*(G, \sigma)$  onto  $C_r^*(G, \sigma)$ . If  $G$  is amenable, then  $\lambda_\sigma$  is faithful. Also, if  $\sigma = 1$ , faithfulness of  $\lambda_\sigma$  implies that  $G$  is amenable, but it is not known whether this holds for nontrivial  $\sigma$ . The reason why the argument does not carry over to the twisted case is that there is in general no trivial 1-dimensional representation of  $C^*(G, \sigma)$ .

## 2.2 Cohomology of groups

Let  $G$  be a discrete group. Denote the group of all multipliers of  $G$  by  $Z^2(G, \mathbb{T})$  and the group of all trivial multipliers of  $G$  by  $B^2(G, \mathbb{T})$ , so that

$$H^2(G, \mathbb{T}) = Z^2(G, \mathbb{T})/B^2(G, \mathbb{T}),$$

where  $\mathbb{T}$  is regarded as a  $\mathbb{Z}G$ -module for which  $G$  acts trivially. To analyze the projective representations of  $G$  and its associated twisted group  $C^*$ -algebras, it is useful to compute its cohomology group  $H^2(G, \mathbb{T})$ , and there are several approaches. In particular, it is interesting to study how the structure of  $C^*(G, \sigma)$  varies with  $\sigma$  and to determine precisely when  $C^*(G, \sigma)$  and  $C^*(G, \tau)$  are isomorphic also when  $\tau \not\sim \sigma$ .

For the results mentioned below, we refer to the books by Brown [3] and Weibel [25].

**Central extensions** Recall that two extensions  $E_1$  and  $E_2$  of  $\mathbb{T}$  by  $G$  are equivalent if there exists a homomorphism  $f : E_1 \rightarrow E_2$  such that the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & E_1 & \longrightarrow & G \longrightarrow e \\ & & \downarrow = & & \downarrow f & & \downarrow = \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & E_2 & \longrightarrow & G \longrightarrow e \end{array}$$

commutes. Then  $f$  must be an isomorphism by the “five-lemma”. Denote by  $\text{Ext}(G, \mathbb{T})$  the set of equivalence classes of (algebraic) central extensions of  $\mathbb{T}$  by  $G$ . Then there is a bijection

$$\text{Ext}(G, \mathbb{T}) \cong H^2(G, \mathbb{T}).$$

In particular, if  $\sigma$  is a multiplier of  $G$ , then the corresponding extension is given as the group  $G^\sigma$  defined by the product

$$(z, a)(w, b) = (zw\sigma(a, b), ab) \tag{9}$$

on  $\mathbb{T} \times G$ . The trivial element in  $\text{Ext}(G, \mathbb{T})$  corresponds to the direct product  $\mathbb{T} \times G$  and is the only split extension. Since every extension of a free group splits,  $H^2(G, \mathbb{T})$  is trivial for all free groups  $G$ . Moreover, if  $G$  is abelian, then every abelian central extension of  $\mathbb{T}$  by  $G$  is trivial in  $\text{Ext}(G, \mathbb{T})$ . Hence, a multiplier of an abelian group is trivial if and only if it is symmetric.

**Homology of groups** The universal coefficient theorem gives an isomorphism

$$H^2(G, \mathbb{T}) \cong \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{T}),$$

that is, a “duality” between homology and cohomology of groups. Here is a few examples from Brown’s book:

- Let  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  be the cyclic group of order  $n$ . Then  $H_2(\mathbb{Z}_n, \mathbb{Z}) = \{0\}$ .
- Let  $F(S)$  be a free group on a set  $S$ . Then  $H_2(F(S), \mathbb{Z}) = \{0\}$ .
- More generally, let  $G = \langle S \mid R \rangle$ , that is,  $G = F/N$  where  $F = F(S)$  is the free group on the set  $S$  and  $N$  is the normal subgroup of  $F$  generated by the relations  $R$ . Then Hopf’s formula gives that

$$H_2(G, \mathbb{Z}) = (N \cap [F, F])/[F, N].$$

One can apply a Mayer-Vietoris sequence to compute the homology of a free product of groups and obtain that

$$H_2(G_1 * G_2, \mathbb{Z}) \cong H_2(G_1, \mathbb{Z}) \oplus H_2(G_2, \mathbb{Z}).$$

By dualizing, we get that

$$H^2(G_1 * G_2, \mathbb{T}) \cong H^2(G_1, \mathbb{T}) \oplus H^2(G_2, \mathbb{T}),$$

and an explicit description of the multipliers is given in [15, Section 5] and [19, Section 4]. The facts stated above illustrates that there are several ways to see that  $H^2(\mathbb{F}_n, \mathbb{T})$  is trivial for all  $n \geq 1$ . Moreover, the Künneth formula gives that

$$H_2(G_1 \times G_2, \mathbb{Z}) \cong H_2(G_1, \mathbb{Z}) \oplus H_2(G_2, \mathbb{Z}) \oplus (H_1(G_1, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(G_2, \mathbb{Z})),$$

where  $H_1(G, \mathbb{Z})$  is the abelianization of  $G$ , and thus,

$$H^2(G_1 \times G_2, \mathbb{T}) \cong H^2(G_1, \mathbb{T}) \oplus H^2(G_2, \mathbb{T}) \oplus \text{Hom}(G_1, \text{Hom}(G_2, \mathbb{T})).$$

By applying Mackey's theorem [14, Theorem 9.4], one can compute the multipliers up to similarity as explained in [19, Section 3].

**Group extensions and semidirect products** If one has a short exact sequence of discrete groups,

$$e \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow e,$$

one may try to apply a Lyndon-Hochschild-Serre spectral sequence on this to compute the group homology or cohomology. Of course, in general there might be complicated to compute the boundary maps.

The drawback with applying purely homological techniques is that one does not get an explicit description of the multipliers up to similarity. To study twisted group  $C^*$ -algebras, we want a concrete family of multipliers in  $Z^2(G, \mathbb{T})$ , that represents the similarity classes in  $H^2(G, \mathbb{T})$ .

In some cases, for example,  $G = \mathbb{Z}^n$  or  $\mathbb{Z}_n$ , the group  $Z^2(G, \mathbb{T})$  is known as well. Apart from this, one of the most important techniques used for explicit computations is given by Mackey in [14, Theorem 9.6] for semidirect products. That is, this applies if the short exact sequence above splits, but it is easier to handle the calculations if  $H$  is abelian.

In Paper IV [21, Section 2], we apply this technique to give an explicit description of a family of representatives of  $H^2(G, \mathbb{T})$  when  $G$  is a free nilpotent group of class 2.

### 2.3 Locally compact groups

We now give a brief explanation of projective unitary representations of locally compact groups and twisted group  $C^*$ -algebras associated with these.

First, if  $G$  is a topological group that is  $T_0$  (i.e. points are topologically distinguishable), then it is also completely regular and Hausdorff, so  $T_0$  is therefore often part of the definition of a topological group (see [10, p. 83]). In particular, when we consider a locally compact group, we will always mean a locally compact  $T_0$  group.

Every locally compact group  $G$  will be equipped with a left Haar measure  $\mu$  and the other spaces in question with the obvious Borel measures. Then a multiplier  $\sigma$  of  $G$  is a measurable function  $G \times G \rightarrow \mathbb{T}$  such that (7) hold, and a  $\sigma$ -projective unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$  is a measurable function  $G \rightarrow \mathcal{U}(\mathcal{H})$  satisfying (6). As above, we say that two multipliers  $\sigma$  and  $\tau$  are similar if there is a measurable function  $G \rightarrow \mathbb{T}$  such that (8) holds. The topological structure of  $H^2(G, \mathbb{T}) = Z^2(G, \mathbb{T})/B^2(G, \mathbb{T})$  is handled by Moore in [16].

The Banach  $*$ -algebra  $L^1(G, \sigma)$  is defined as the Banach space  $L^1(G)$  together with twisted convolution and involution given by

$$(f * g)(a) = \int_G f(b)\sigma(b, b^{-1}a)g(b^{-1}a) d\mu(b),$$

$$f^*(a) = \Delta(a)^{-1}\overline{\sigma(a, a^{-1})f(a^{-1})},$$

for elements  $f, g$  in  $L^1(G)$  and the modular function  $\Delta$  of  $G$ .

Similarly as for the discrete case, one defines the left regular  $\sigma$ -projective unitary representation  $\lambda_\sigma$  of  $G$  on  $B(L^2(G))$ , as well as the full and reduced twisted group  $C^*$ -algebras and the group von Neumann algebra. Also,  $\lambda_\sigma$  is faithful on  $C^*(G, \sigma)$  whenever  $G$  is amenable. However,  $C^*(G, \sigma)$  and  $C_r^*(G, \sigma)$  are unital only if  $G$  is discrete and the canonical map  $i_\sigma$  is in general a map from  $G$  into the multiplier algebra of  $C^*(G, \sigma)$ . The twisted group  $C^*$ -algebras are separable precisely when  $G$  is second countable.

To avoid topological issues, we now assume that  $G$  is a second countable locally compact group (or an arbitrary discrete group, see [26, Section D.3]). For a multiplier  $\sigma$  of  $G$ , the algebraic central extension  $G^\sigma$  of  $\mathbb{T}$  by  $G$  defined by (9) has a unique second countable locally compact topology such that the Borel structures of  $G^\sigma$  and  $\mathbb{T} \times G$  coincide and  $\mu_{G^\sigma} = \mu_{\mathbb{T}} \times \mu_G$  is a left Haar measure on  $G^\sigma$ , by [13]. If  $\text{Ext}(G, \mathbb{T})$  denotes the set of locally compact second countable central extensions of  $\mathbb{T}$  by  $G$ , then  $\sigma \mapsto G^\sigma$  gives a bijection  $H^2(G, \mathbb{T}) \rightarrow \text{Ext}(G, \mathbb{T})$ , by [16].

There is a 1-1-correspondence between  $\sigma$ -projective unitary representations of  $G$  and unitary representations of  $G^\sigma$  satisfying  $(z, e) \mapsto z1_{\mathcal{H}}$ , and  $C^*(G, \sigma)$  is a



quotient of  $C^*(G^\sigma)$ .

**Compact groups and the twisted Peter-Weyl theorem** Let  $G$  be a compact group and  $\sigma$  a multiplier of  $G$ . If  $U$  and  $U'$  are two  $\sigma$ -projective unitary representations of  $G$  on  $\mathcal{H}$  and  $\mathcal{H}'$ , respectively, we can in the usual way form the  $\sigma$ -projective unitary representation  $U \oplus U'$  on  $\mathcal{H} \oplus \mathcal{H}'$ .

Let  $(\widehat{G}, \sigma)$  denote the set of all equivalence classes of irreducible  $\sigma$ -projective representations of  $G$ . We reserve the symbol  $d_U$  for the dimension of the representation space for  $[U] \in (\widehat{G}, \sigma)$ . Then the following hold:

- Each irreducible  $\sigma$ -projective unitary representation of a compact group  $G$  is finite-dimensional. The left regular  $\sigma$ -projective unitary representation  $\lambda$  of  $G$  is unitarily equivalent to the direct sum of irreducible ones, that is,

$$\lambda \simeq \bigoplus_{[U] \in (\widehat{G}, \sigma)} \bigoplus_{j=1}^{d_U} U.$$

- The twisted group  $C^*$ -algebra decomposes into a direct sum of matrix algebras, that is,

$$C^*(G, \sigma) \cong \bigoplus_{[U] \in (\widehat{G}, \sigma)} M_{d_U}(\mathbb{C}),$$

and for  $f \in L^1(G, \sigma)$ , the isomorphism is given by

$$f \mapsto (U(f))_{[U] \in (\widehat{G}, \sigma)}.$$

- For every nontrivial  $a \in G$ , there exists an irreducible  $\sigma$ -projective unitary representation  $U$  of  $G$  such that  $U(a) \neq I$ .

In particular,  $C^*(G, \sigma)$  is residually finite-dimensional, that is, has a separating family of finite-dimensional representations.

### 3 Dynamical systems and crossed products

In this section we consider (twisted) group actions on  $C^*$ -algebras. Again, this is motivated by topological dynamics in the commutative case. To see this, let  $X$  be a compact metric space and  $\varphi$  a homeomorphism  $X \rightarrow X$ . Then  $\varphi$  induces an action of  $\mathbb{Z}$  on  $C(X)$  by  $n \cdot f(x) = f(\varphi^{-n}(x))$ .

More generally, let  $G$  be a locally compact group acting on a locally compact Hausdorff space  $X$ , i.e.  $(X, G)$  is a transformation group. Define the induced

action  $\alpha$  of  $G$  on  $C_0(X)$  by  $\alpha_g(f)(x) = f(g^{-1} \cdot x)$ . Then  $(C_0(X), G, \alpha)$  is a so-called  $C^*$ -dynamical system.

Dynamical systems are dealt with in the enclosed Paper I and II (the unital twisted case) and Paper V (the separable case).

### 3.1 $C^*$ -dynamical systems

In general, a  $C^*$ -dynamical system is a triple  $(A, G, \alpha)$  consisting of a  $C^*$ -algebra  $A$ , a locally compact group  $G$ , and a continuous homomorphism  $\alpha : G \rightarrow \text{Aut } A$  (i.e.  $g \mapsto \alpha_g(a)$  is continuous for all  $a \in A$ ). There are two cases that are of particular interest:

- $A$  is separable and  $G$  is second countable,
- $A$  is unital and  $G$  is discrete.

Let  $(A, G, \alpha)$  be a  $C^*$ -dynamical system and let  $C_c(G, A)$  be the set of continuous functions  $G \rightarrow A$  with compact support. For  $f, g \in C_c(G, A)$ , define  $f * g$ ,  $f^*$ , and  $\|f\|_1$  by

$$\begin{aligned} f * g(a) &= \int_G f(b) \alpha_b(g(b^{-1}a)) d\mu(b), \\ f^*(a) &= \Delta(a^{-1}) \alpha_a(f(a^{-1})^*), \\ \|f\|_1 &= \int_G \|f(a)\| d\mu(a). \end{aligned}$$

Banach space valued integration and the Bochner integral are treated in Williams' book [26, Section B.1]. The completion of  $C_c(G, A)$  with respect to  $\|\cdot\|_1$  is a Banach  $*$ -algebra denoted by  $L^1(A, G, \alpha)$ . A covariant representation of  $(A, G, \alpha)$  is a pair  $(\pi, U)$  consisting of a representation  $\pi$  of  $A$  on a Hilbert space  $\mathcal{H}$  and a unitary representation  $U$  of  $G$  on  $\mathcal{H}$  satisfying

$$\pi \circ \alpha_a = \text{Ad}(U(a))\pi$$

for all  $a \in G$ . There is 1-1 correspondence between covariant representations of  $(A, G, \alpha)$  and representations of  $L^1(A, G, \alpha)$ . In particular, a covariant representation  $(\pi, U)$  of  $(A, G, \alpha)$  induces a representation  $\pi \times U$  of  $L^1(A, G, \alpha)$  given by

$$(\pi \times U)(f) = \int_G \pi(f(a))U(a) d\mu(a).$$

For  $f \in C_c(G, A)$ , define

$$\begin{aligned} \|f\|_{\max} &= \sup\{\|\pi(f)\| : \pi \text{ is representation of } L^1(A, G, \alpha)\} \\ &= \sup\{\|\pi \times U(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\}. \end{aligned}$$

The completion of  $C_c(G, A)$  with respect to  $\|\cdot\|_{\max}$  is the crossed product of  $A$  by  $G$  and is denoted by  $A \rtimes_{\alpha} G$ .

An interesting example of a transformation group  $C^*$ -algebra is when  $G$  is a locally compact group and  $H$  is a subgroup of  $G$  acting on  $G$  by left translation. A special case of this example is when  $H = G$ .

**Example 3.1** (The Stone-von Neumann theorem, part II). Let  $G$  be a locally compact group. Then

$$C_0(G) \rtimes_{\text{lt}} G \cong \mathcal{K}(L^2(G)),$$

where  $\mathcal{K}(L^2(G))$  is the compact operators on  $L^2(G)$ , which is a simple  $C^*$ -algebra.

### 3.2 Coactions and duality theory

If  $G$  is a locally compact abelian group, then  $C^*(G) \cong C_r^*(G) \cong C_0(\widehat{G})$  via the Fourier transform. Moreover, if  $(A, G, \alpha)$  is a  $C^*$ -dynamical system and  $G$  is abelian, then there is an action  $\hat{\alpha}$  of  $\widehat{G}$  on  $A \rtimes_{\alpha} G$  such that

$$(A \rtimes_{\alpha} G) \rtimes_{\hat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G)).$$

Motivated by the goal of extending this result to nonabelian groups, one introduces coactions, so that if  $G$  is abelian, then a coaction of  $G$  on a  $C^*$ -algebra  $A$  is an action of  $\widehat{G}$  on  $A$ .

Moreover, an action  $\alpha$  of  $G$  on  $A$  may be identified with a map

$$\tilde{\alpha} : A \rightarrow M(A \otimes C_0(G)) \cong C_b(G, M(A)), \quad \tilde{\alpha}(x)(a) = \alpha_x(a), \quad x \in A, a \in G.$$

Inspired by this fact, one says that a coaction of  $G$  on  $A$  is an injective nondegenerate homomorphism  $\delta : A \rightarrow M(A \otimes C^*(G))$  satisfying

$$\begin{aligned} \overline{\text{span}}\{\delta(A)(1 \otimes C^*(G))\} &= A \otimes C^*(G) \\ (\delta \otimes i) \circ \delta &= (i \otimes \delta_G) \circ \delta, \end{aligned}$$

where the coaction  $\delta_G$  of  $G$  on  $C^*(G)$  is given by  $C^*(G) \rightarrow M(C^*(G) \otimes C^*(G))$ ,  $a \mapsto a \otimes a$ . The associated (co-)crossed product of  $(A, G, \delta)$  is a  $C^*$ -algebra  $A \rtimes_{\delta} G$  whose representations are the same as the covariant representations of  $(A, G, \delta)$  (see [9, Appendix A.5]).

In Paper V [11, Appendix], we consider an injective homomorphism  $\varphi : H \rightarrow G$ , its integrated form  $\pi_{\varphi} : C^*(H) \rightarrow M(C^*(G))$ , and the coaction of  $G$  on  $C^*(H)$  defined by  $\delta = (\overline{\text{id} \otimes \pi_{\varphi}}) \circ \delta_H$ .

### 3.3 Twisted $C^*$ -dynamical systems

To unify the constructions of twisted group  $C^*$ -algebras and crossed products, we now consider twisted  $C^*$ -dynamical systems. Since we only need this construction in the unital case, we will assume that the  $C^*$ -algebras are unital and that the groups are discrete. The more general construction of separable twisted  $C^*$ -dynamical systems is nicely treated by Packer and Raeburn [23].

A (unital) twisted  $C^*$ -dynamical system is a quadruple  $(A, G, \alpha, \omega)$  consisting of a unital  $C^*$ -algebra  $A$ , a discrete group  $G$ , and maps  $\alpha : G \rightarrow \text{Aut } A$  and  $\omega : G \times G \rightarrow \mathcal{U}(A)$  satisfying

$$\begin{aligned}\alpha_a \alpha_b &= \text{Ad}(\omega(a, b)) \alpha_{ab} \\ \omega(a, b) \omega(ab, c) &= \alpha_a(\omega(b, c)) \omega(a, bc) \\ \omega(e, e) &= 1_A\end{aligned}$$

for all  $a, b, c \in G$ , and from this it is easily deduced that

$$\omega(a, e) = \omega(e, b) = 1_A, \quad \alpha_e = \text{id}_A, \quad \omega(a, a^{-1}) = \alpha_a(\omega(a^{-1}, a)).$$

Twisted  $C^*$ -dynamical systems coming from discrete groups were introduced by Zeller-Meier [27] in the case where  $\omega$  is central-valued, and then in more generality by Busby and Smith [4].

A covariant representation of a twisted  $C^*$ -dynamical system  $(A, G, \alpha, \omega)$  is a pair  $(\pi, U)$  consisting of a representation  $\pi$  of  $A$  on a Hilbert space  $\mathcal{H}$  and a map  $U : G \rightarrow \mathcal{U}(\mathcal{H})$  satisfying

$$\begin{aligned}U(a)U(b) &= \pi(\omega(a, b))U(ab) \\ \pi \circ \alpha_a &= \text{Ad}(U(a))\pi\end{aligned}$$

for all  $a, b \in G$ .

We equip the Banach space  $\ell^1(G, A)$  with twisted convolution and involution given by

$$\begin{aligned}(f * g)(a) &= \sum_{b \in G} f(b) \alpha_b(g(b^{-1}a)) \omega(b, b^{-1}a) \\ f^*(a) &= \omega(a, a^{-1})^* \alpha_a(f(a^{-1}))^*\end{aligned}$$

and denote the resulting Banach  $*$ -algebra by  $\ell^1(A, G, \alpha, \omega)$ . There is 1-1 correspondence between covariant representations of  $(A, G, \alpha, \omega)$  and representations of  $\ell^1(A, G, \alpha, \omega)$ . In particular, every covariant representation  $(\pi, U)$  of a twisted  $C^*$ -dynamical system  $(A, G, \alpha, \omega)$  induces a representation, denoted by  $\pi \times U$ , of  $\ell^1(A, G, \alpha, \omega)$  defined by

$$(\pi \times U(f)\xi)(a) = \sum_{b \in G} \pi(\alpha_{a^{-1}}(f(b))) (U(b)\xi)(a).$$

Define now a  $C^*$ -norm on  $\ell^1(A, G, \gamma, \omega)$  by

$$\begin{aligned} \|f\|_{\max} &= \sup\{\|\pi(f)\| : \pi \text{ is a representation of } \ell^1(A, G, \alpha, \omega)\} \\ &= \sup\{\|\pi \times U(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha, \omega)\}. \end{aligned}$$

The full twisted crossed product  $A \rtimes_{(\alpha, \omega)} G$  is the completion of  $\ell^1(A, G, \gamma, \omega)$  with respect to  $\|\cdot\|_{\max}$ , that is, the enveloping  $C^*$ -algebra of  $\ell^1(A, G, \gamma, \omega)$ .

The following example was one of the motivations for studying twisted crossed products in the first place and is essential when constructing induced representations.

**Example 3.2.** Let  $H$  be a normal subgroup of a group  $G$  with quotient group  $K = G/H$ , that is, we have a short exact sequence of groups

$$e \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow e.$$

Let  $\sigma$  be a multiplier of  $G$  and  $\sigma'$  the restriction to  $H$ . Then we may decompose  $C^*(G, \sigma)$  into a twisted crossed product ([20] based on [23, Theorem 4.1])

$$C^*(G, \sigma) \cong C^*(H, \sigma') \rtimes_{(\alpha, \omega)} K,$$

where

$$\begin{aligned} \alpha_a(i_H(b)) &= i_G(n(a))i_G(b)i_G(n(a))^* \\ \omega(a, b) &= i_G(n(a))i_G(n(b))i_G(n(ab))^* \end{aligned}$$

for a normalized section  $n$  for the quotient map  $G \rightarrow K$ .

This decomposition becomes an ordinary crossed product if and only if the sequence splits and  $\sigma = 1$  when restricted to  $K$ .

Let  $\pi$  be a representation of  $A$  on a Hilbert space  $\mathcal{H}$  and define the covariant representation  $(\tilde{\pi}, \lambda)$  on  $\ell^2(G, \mathcal{H})$  by

$$\begin{aligned} (\tilde{\pi}(x)\xi)(a) &= \pi(\alpha_{a^{-1}}(x))\xi(a) \\ (\lambda(b)\xi)(a) &= \pi(\omega(a^{-1}, b))\xi(b^{-1}a) \end{aligned}$$

for  $a, b \in G$ ,  $\xi \in \ell^2(G, \mathcal{H})$ , and  $x \in A$ , and set  $\text{Ind } \pi = \tilde{\pi} \times \lambda$ . For  $f \in \ell^1(A, G, \alpha, \omega)$ , define

$$\|f\|_{\text{red}} = \sup\{\|\text{Ind } \pi(f)\| : \pi \text{ is a representation of } A\} = \|\text{Ind } \rho(f)\|$$

for some faithful representation  $\rho$  of  $A$ . The completion of  $\ell^1(A, G, \alpha, \omega)$  with respect to  $\|\cdot\|_{\text{red}}$  is the reduced twisted crossed product of  $A$  by  $G$  and is denoted by  $A \rtimes_{(\alpha, \omega), r} G$ . If  $G$  is amenable, then it is well known that the full and reduced twisted crossed products are isomorphic.

Moreover,  $\text{Ind } \pi$  is faithful on  $A \rtimes_{(\alpha, \omega), r} G$  if and only if  $\{\pi \circ \alpha_a\}_{a \in G}$  is separating for  $A$ , as explained in [20] based on [27, Théorème 4.11]. The next results can be deduced from Mackey's work [13], but is shown in [20] and [2, Appendix] as well. First, the following are equivalent:

- (i)  $\text{Ind } \pi$  is irreducible,
- (ii)  $\pi$  is irreducible and the stabilizer group  $\{a \in G \mid \pi \circ \alpha_a \simeq \pi\}$  is trivial,
- (iii)  $[\pi] \in \widehat{A}$  is a free point for the natural action of  $G$  on  $\widehat{A}$ , that is,  $[\pi] \neq [\pi \circ \alpha_a]$  for all  $a \neq e$ .

Furthermore, suppose that  $\pi_1$  and  $\pi_2$  are irreducible representations such that  $\pi_1 \circ \alpha_a \not\cong \pi_2$  for all  $a \in G$ . Then  $\text{Ind } \pi_1 \not\cong \text{Ind } \pi_2$ .

## 4 Some aspects of structure and classification

The structure and classification theory for  $C^*$ -algebras are vast subjects, and we will only mention a few aspects here. One of the most interesting topics is investigation of the ideal structure (and especially simplicity) which is central in all of the included papers.

Moreover,  $K$ -theory plays a major role in the classification program for  $C^*$ -algebras. Even though  $K$ -theory is not dealt with directly in any of the included papers, we consider the properties of being nuclear (Paper III and V) and purely infinite (Paper V), since these notions are very useful for classification by  $K$ -theory.

Representation theory is also central in all of the included papers, and this motivates the study of Morita equivalence of  $C^*$ -algebras.

### 4.1 Ideals

By an ideal of a  $C^*$ -algebra we will always mean a closed (and thus self-adjoint) two-sided ideal. As usual, a  $C^*$ -algebra is *simple* if it contains no proper nontrivial ideals, and *prime* if any pair of nonzero ideals has nonzero intersection. A  $C^*$ -algebra with a faithful irreducible representation is called *primitive*. In general, primitivity is a property between simplicity and primeness. Obviously, every simple  $C^*$ -algebra is primitive, and it is not difficult to see that every primitive  $C^*$ -algebra is prime. Conversely, every prime and separable  $C^*$ -algebra is primitive by a result of Dixmier [6], that is, the notions of primeness and primitivity are equivalent for separable  $C^*$ -algebras. There are rather few examples of prime nonprimitive  $C^*$ -algebras (the first was presented by Weaver [24]). It is also well known that every prime  $C^*$ -algebra has trivial center, so that we have the

following:

$$\text{simplicity} \implies \text{primitivity} \implies \text{primeness} \implies \text{trivial center}$$

Moreover, a von Neumann algebra is a *factor* if it has trivial center, or equivalently, if it contains no proper nontrivial weakly closed ideals. If  $A$  is a concrete unital  $C^*$ -algebra, and  $A''$  is a factor, then  $A$  is prime. Hence, a von Neumann algebra is a factor if and only if it is prime (as a  $C^*$ -algebra).

Following Dixmier [7], a  $C^*$ -algebra  $A$  is *antiliminary* if  $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$  for all, or equivalently, some faithful irreducible representations  $\pi$  of  $A$ .

Let  $A$  be a separable unital  $C^*$ -algebra. Then, by [7],  $A$  is primitive and antiliminary if and only if the pure state space of  $A$  is weak\*-dense in the state space of  $A$ .

## 4.2 Kirchberg algebras

A  $C^*$ -algebra is called *nuclear* if the identity map, as a completely positive map, approximately factors through matrix algebras. Equivalently,  $A$  is nuclear if  $A \otimes_{\min} B \cong A \otimes_{\max} B$  for all  $C^*$ -algebras  $B$ , or yet equivalently, if  $A''$  is an injective von Neumann algebra.

A simple  $C^*$ -algebra  $A$  is *purely infinite* if and only if every hereditary  $C^*$ -subalgebra of  $A$  contains an infinite projection. In the separable case, this is the same as saying that every corner  $xAx^*$  of  $A$  contains an infinite projection.

A *Kirchberg algebra* is a separable, simple, nuclear, purely infinite  $C^*$ -algebra in the UCT class (meaning  $KK$ -equivalent with a commutative  $C^*$ -algebra). Moreover, Kirchberg algebras are classifiable by  $K$ -theory, and it is therefore of interest to show that  $C^*$ -algebras coming from for example  $C^*$ -dynamical systems belong to this class.

## 4.3 Morita equivalence

Let  $A$  and  $B$  be  $C^*$ -algebras. Then  $A$  and  $B$  are *Morita equivalent* if there exists an  $A - B$ -imprimitivity bimodule. That is, if there is an  $A - B$ -bimodule  $E$  which is simultaneously a full left Hilbert  $A$ -module under an  $A$ -valued inner product  ${}_A\langle \cdot, \cdot \rangle$  and a full right Hilbert  $B$ -module under a  $B$ -valued inner product  $\langle \cdot, \cdot \rangle_B$  such that

$${}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$$

for all  $\xi, \eta, \zeta \in E$ .

One important feature of a Morita equivalence is that it gives a functorial correspondence between the representations of the algebras. In particular, the spectrum and primitive ideal spaces are homeomorphic.

If  $A$  and  $B$  are separable, they are Morita equivalent if and only if they are stably isomorphic.

#### 4.4 Group $C^*$ -algebras and crossed products

The trivial representation  $\iota$  of any locally compact group  $G$  on  $\mathbb{C}$  given by  $\iota(g) = 1$  for all  $g \in G$ , induces a representation of  $C^*(G)$  on  $\mathbb{C}$ . Hence,  $C^*(G)$  will always have an ideal of codimension 1, called the augmented ideal. That is,  $C^*(G)$  is never simple, unless  $G$  is trivial. Therefore, primitive and prime group  $C^*$ -algebras may be considered as the building blocks for the class of group  $C^*$ -algebras.

The problem of determining whether a group  $C^*$ -algebra is primitive seems hard in general. For example, primitivity of the group  $C^*$ -algebra of the group  $\mathbb{F}_2 \times \mathbb{F}_2$  can be related to Connes' embedding problem [2, Remark 2.2]. In [17], Murphy gives some conditions and examples of primitive group  $C^*$ -algebras.

On the other hand, the reduced group  $C^*$ -algebra  $C_r^*(G)$  can be simple for nontrivial  $G$ . Much work is done in the area of determining the class of  $C^*$ -simple groups, see e.g. de la Harpe [5]. Simplicity of  $C_r^*(G)$  is in general unrelated to primitivity of  $C^*(G)$ .

Also, the full twisted group  $C^*$ -algebra  $C^*(G, \sigma)$  may be simple when  $G$  is amenable. For example, by the work of Kleppner, it is known that if  $G$  is abelian, then  $\text{Prim } C^*(G, \sigma)$  is homeomorphic with  $\widehat{S_\sigma}$ , where

$$S_\sigma = \{a \in G \mid \sigma(a, b) = \sigma(b, a) \text{ for all } b \in G\},$$

and  $C^*(G, \sigma)$  is simple if  $S_\sigma$  is trivial.

Moreover, if  $G$  is discrete, then  $C^*(G, \sigma)$  is simple and nuclear if and only if  $C_r^*(G, \sigma)$  is simple and nuclear, and for this to hold, we must have that  $G$  is amenable and  $\sigma$  is nontrivial. This gives another motivation for considering the twisted case. However, our main focus is to study primeness and primitivity of the full and reduced twisted group  $C^*$ -algebras corresponding to discrete groups.

Furthermore, we have the following (references given in [11]):

- If  $(A, G, \alpha, \omega)$  is a twisted  $C^*$ -dynamical system with  $A$  nuclear and  $G$  amenable, then  $A \rtimes_{(\alpha, \omega)} G$  is nuclear.
- Let  $(A, G, \alpha)$  be a  $C^*$ -dynamical system with  $A = C_0(X)$  commutative and  $G$  amenable and discrete so that the action of  $G$  on  $X$  is topologically free. Then  $A \rtimes_\alpha G$  is simple if the action of  $G$  on  $X$  is minimal and  $A \rtimes_\alpha G$  is purely infinite if the action of  $G$  on  $X$  is locally contractive.
- If  $(A, G, \alpha)$  is a  $C^*$ -dynamical system with  $A$  commutative and  $G$  amenable and discrete, then  $A \rtimes_\alpha G$  belongs to the UCT class.



Finally, as an application of “Green’s symmetric imprimitivity theorem”, we get the following (see e.g. [26, p. 126]). Suppose that  $K$  and  $H$  are closed subgroups of a locally compact group  $G$ . Let  $K$  act by left multiplication on  $G$  and let  $H$  act by right multiplication on  $G$ . Then  $C_0(K \backslash G) \rtimes_{\text{rt}} H$  is Morita equivalent to  $C_0(G/H) \rtimes_{\text{lt}} K$ .

## 5 Overview of the thesis

### Paper I and II

In [2], we study the projective special linear groups  $\text{PSL}(n, \mathbb{Z})$  for  $n \geq 2$ . The main result is [2, Theorem 2.3], which says that  $C^*(\text{PSL}(2, \mathbb{Z}))$  is primitive, and also antiliminary. The proof of this result uses the techniques mentioned in the end of Section 3.3 (and Example 3.2), namely we construct a faithful irreducible representation through an inducing process [2, Theorem 2.1 and Appendix]. In [17] Murphy mentions that he knows no example of an icc group whose full group  $C^*$ -algebra is nonprimitive. When  $n \geq 3$ , we show that  $C^*(\text{PSL}(n, \mathbb{Z}))$  is nonprimitive so that for example  $\text{SL}(3, \mathbb{Z})$  provides such an example.

The main result of [1] is [1, Theorem 1.2], where we show that  $C^*(G_1 * G_2)$  is primitive whenever  $G_1$  and  $G_2$  are countable discrete amenable groups such that  $|G_1 - 1| \cdot |G_2 - 1| \geq 2$ . Since  $\text{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ , this is a generalization of the previous paper, and the technique of the proof is again an application of [2, Theorem 2.1 and Appendix]. However, the argument turns out to be combinatorially harder in this case. Moreover, in [1, Lemma 3.2, Corollary 3.3, and Corollary 3.4] we give conditions to ensure that  $C^*(G_1 * G_2)$  is antiliminary whenever  $G_1$  and  $G_2$  satisfy the conditions of [1, Theorem 1.2].

Finally, we remark that in a recent preprint by Dykema and Torres-Ayala [8], related results are shown with a different approach.

### Paper III

Let  $G$  be an arbitrary discrete group and  $\sigma$  a multiplier of  $G$ . The aim of [19] is to generalize results of Murphy [17] and Kleppner [12] and give precise conditions for primeness of  $C_r^*(G, \sigma)$ . Following [12], an element  $a$  of  $G$  is called  $\sigma$ -regular if  $\sigma(a, b) = \sigma(b, a)$  whenever  $b$  commutes with  $a$ . Moreover,  $\sigma$ -regularity is a property of conjugacy classes, and we will say that  $(G, \sigma)$  satisfies *condition K* if every nontrivial  $\sigma$ -regular conjugacy class of  $G$  is infinite. The main result is [19, Theorem 2.7], which says that condition K on  $(G, \sigma)$  is equivalent with primeness of  $C_r^*(G, \sigma)$ . Also, [19, Corollary 2.8] gives that condition K on  $(G, \sigma)$  is necessary for primeness of  $C^*(G, \sigma)$ .

In the final sections, we consider the cases where  $G = G_1 \times G_2$  and where  $G = G_1 * G_2$ . The direct product is in general harder to handle since a multiplier  $\sigma$  of  $G$  does not necessarily decompose nicely in this case, but has a “cross-term” as discussed in Section 2.2. In the free product case, we obtain with [1, Theorem 1.2] a generalization of [19, Theorem 4.1] to the twisted case.

*Remark.* Significant parts of [19, Section 2], especially [19, Lemma 2.2 and 2.4], were already obtained in [20], although rewritten here.

## Paper IV

In [21], we study the free nilpotent groups of class 2 and rank  $n$ , denoted by  $G(n)$ . These groups may be considered as generalized Heisenberg groups with higher-dimensional center. Motivated by Packer [22], we compute the second cohomology group of  $G(n)$  and give explicit formulas for the multipliers in [21, Theorem 2.7], by applying techniques of Mackey [14, Section 9]. Then we give conditions for simplicity of the twisted group  $C^*$ -algebras  $C^*(G(n), \sigma)$  in [21, Section 4]. We also describe  $C^*(G(n), \sigma)$  in terms of generators and relations in [21, Theorem 3.1], and as a continuous field over  $\mathbb{T}^{\frac{1}{2}n(n-1)}$  with the noncommutative  $n$ -tori as fibers in [21, Theorem 1.1].

## Paper V

Inspired by the work of Cuntz and Li on ring  $C^*$ -algebras, we give a crossed product construction of a family of  $C^*$ -algebras  $\overline{\mathcal{Q}}$  associated with the  $a$ -adic numbers. We show that these algebras are nonunital Kirchberg algebras in the UCT class [11, Corollary 2.8].

The  $a$ -adic numbers are locally compact abelian groups that appear as Hausdorff completions of additive subgroups of  $\overline{\mathbb{Q}}$ , and the most commonly studied examples are the  $p$ -adic numbers  $\mathbb{Q}_p$ .

The main result is [11, Theorem 4.1] which says that  $\overline{\mathcal{Q}}$  is Morita equivalent with a crossed product  $C^*$ -algebra coming from an  $ax + b$ -action on  $\mathbb{R}$  of a certain subgroup of  $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$ . The proof uses “Green’s symmetric imprimitivity theorem” and relies especially on two additional results, a duality result for groups [11, Theorem 3.3], and a “subgroup of dual group theorem” that we prove in a more general setting, for coactions, in [11, Appendix].

*Remark.* The main results of [11] are also summarized in a preprint for a conference proceedings paper [18].

## References

- [1] E. Bédos and T. Omland. Primitivity of some full group  $C^*$ -algebras. *Banach J. Math. Anal.*, 5(2):44–58, 2011.
- [2] E. Bédos and T. Omland. The full group  $C^*$ -algebra of the modular group is primitive. *Proc. Amer. Math. Soc.*, 140(4):1403–1411, 2012.
- [3] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.
- [4] R. C. Busby and H. A. Smith. Representations of twisted group algebras. *Trans. Amer. Math. Soc.*, 149:503–537, 1970.
- [5] P. de la Harpe. On simplicity of reduced  $C^*$ -algebras of groups. *Bull. Lond. Math. Soc.*, 39(1):1–26, 2007.
- [6] J. Dixmier. Sur les  $C^*$ -algèbres. *Bull. Soc. Math. France*, 88:95–112, 1960.
- [7] J. Dixmier. *Les  $C^*$ -algèbres et leurs représentations*. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris, 1964.
- [8] K. J. Dykema and F. Torres-Ayala. Primitivity of unital full free products of residually finite dimensional  $C^*$ -algebras. arXiv:1201.4892, Jun 2012.
- [9] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn. A categorical approach to imprimitivity theorems for  $C^*$ -dynamical systems. *Mem. Amer. Math. Soc.*, 180(850):viii+169, 2006.
- [10] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*. Die Grundlehren der mathematischen Wissenschaften, Bd. 115. Academic Press Inc., Publishers, New York, 1963.
- [11] S. Kaliszewski, T. Omland, and J. Quigg. Cuntz-Li algebras from  $a$ -adic numbers. arXiv:1211.4806, Nov 2012.
- [12] A. Kleppner. The structure of some induced representations. *Duke Math. J.*, 29:555–572, 1962.
- [13] G. W. Mackey. Borel structure in groups and their duals. *Trans. Amer. Math. Soc.*, 85:134–165, 1957.
- [14] G. W. Mackey. Unitary representations of group extensions. I. *Acta Math.*, 99:265–311, 1958.

- 
- [15] K. McClanahan.  $KK$ -groups of twisted crossed products by groups acting on trees. *Pacific J. Math.*, 174(2):471–495, 1996.
- [16] C. C. Moore. Extensions and low dimensional cohomology theory of locally compact groups. I, II. *Trans. Amer. Math. Soc.*, 113:40–63, 1964.
- [17] G. J. Murphy. Primitivity conditions for full group  $C^*$ -algebras. *Bull. London Math. Soc.*, 35(5):697–705, 2003.
- [18] T. Omland.  $C^*$ -algebras associated with  $a$ -adic numbers. Proceedings of the “Operator Algebra and Dynamics, NordForsk Network Closing Conference” (Springer), to appear.
- [19] T. Omland. Primeness and primitivity conditions for twisted group  $C^*$ -algebras. *Math. Scand.*, to appear.
- [20] T. Omland. On primitivity and primeness of twisted group  $C^*$ -algebras corresponding to discrete groups. Master’s thesis, University of Oslo, Dec 2007.
- [21] T. Omland.  $C^*$ -algebras generated by projective representations of free nilpotent groups. arXiv:1301.2942, Jan 2013.
- [22] J. A. Packer.  $C^*$ -algebras generated by projective representations of the discrete Heisenberg group. *J. Operator Theory*, 18(1):41–66, 1987.
- [23] J. A. Packer and I. Raeburn. Twisted crossed products of  $C^*$ -algebras. *Math. Proc. Cambridge Philos. Soc.*, 106(2):293–311, 1989.
- [24] N. Weaver. A prime  $C^*$ -algebra that is not primitive. *J. Funct. Anal.*, 203(2):356–361, 2003.
- [25] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [26] D. P. Williams. *Crossed products of  $C^*$ -algebras*, volume 134 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.
- [27] G. Zeller-Meier. Produits croisés d’une  $C^*$ -algèbre par un groupe d’automorphismes. *J. Math. Pures Appl. (9)*, 47:101–239, 1968.

# Paper I

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The full group  $C^*$ -algebra of the modular group is  
primitive

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# THE FULL GROUP $C^*$ -ALGEBRA OF THE MODULAR GROUP IS PRIMITIVE

ERIK BÉDOS AND TRON Å. OMLAND

*This paper is dedicated to the memory of Gerard J. Murphy*

## Abstract

We show that the full group  $C^*$ -algebra of  $\mathrm{PSL}(n, \mathbb{Z})$  is primitive when  $n = 2$ , and not primitive when  $n \geq 3$ . Moreover, we show that there exists an uncountable family of pairwise inequivalent, faithful irreducible representations of  $C^*(\mathrm{PSL}(2, \mathbb{Z}))$ .

## 1 Introduction

Simple and, more generally, primitive and prime  $C^*$ -algebras may be considered as building blocks of the theory, playing a somewhat similar role as factors do within the theory of von Neumann algebras. If we restrict ourselves to separable  $C^*$ -algebras, as we always do in this paper, primitivity is equivalent to primeness (see for example [17]), and we will therefore refer to primitivity for both notions. Now, given some class of separable  $C^*$ -algebras, one natural task is to investigate which members of this class are simple or primitive.

An interesting family of separable  $C^*$ -algebras consists of the group  $C^*$ -algebras associated with countable discrete groups. We recall that such a group  $G$  is called  $C^*$ -simple if its reduced group  $C^*$ -algebra  $C_r^*(G)$  is simple. As the full group  $C^*$ -algebra  $C^*(G)$  is simple only when  $G$  is trivial, this terminology is not ambiguous. The class of  $C^*$ -simple groups has received a lot of attention during the last decades and the reader may consult [8] for a recent, comprehensive review. It is also well known (see [15, 14]) that  $C_r^*(G)$  is primitive if and only if  $G$  is icc (that is, every nontrivial conjugacy class in  $G$  is infinite) if and only if the group von Neumann algebra of  $G$  is a factor.

On the other hand, the problem of determining when  $C^*(G)$  is primitive seems hard in general. A necessary condition is that  $G$  is icc [14], and this condition is also sufficient when  $G$  is assumed to be amenable, as  $C^*(G)$  is then isomorphic to  $C_r^*(G)$ . We note in passing that this problem is quite different from the one of determining the class of groups having a faithful irreducible unitary representation, which contains many other groups besides all icc groups (see [3]).

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Until a few years ago, the only known nonamenable icc groups having a primitive full group  $C^*$ -algebra were nonabelian free groups, as originally shown by H. Yoshizawa [21] and rediscovered later by M. D. Choi [5, 6]. Then primitivity of  $C^*(G)$  was established when  $G = G_1 * G_2$  is the free product of two countable subgroups  $G_1$  and  $G_2$  satisfying at least one of the following assumptions:

- (i)  $G_1 = \mathbb{Z} * \mathbb{Z}$  or  $G_1 = \mathbb{Z} * \mathbb{Z}_2$  ( $G_2$  being then any group).
- (ii)  $G_1$  is nontrivial and free, and  $G_2$  is nontrivial and amenable.
- (iii)  $G_1$  is nonabelian and free, and  $C^*(G_2)$  admits no nontrivial projections.

Case (i) is due to N. Khatthou [9, Théorèmes 2 et 3], while (ii) and (iii) are due to G. J. Murphy [14, Theorems 3.3 and 3.4].

In [8, Problem 25], P. de la Harpe raises the problem of finding other (nonamenable icc) groups having a primitive full group  $C^*$ -algebra. One may especially wonder whether this property holds for any group  $G$  which is the free product of two nontrivial groups, where at least one of them has more than two elements (as the infinite dihedral group  $\mathbb{Z}_2 * \mathbb{Z}_2$  is not icc). The simplest case for which the answer is unknown is the modular group  $\mathrm{PSL}(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ , and our main result in this paper is that  $C^*(\mathrm{PSL}(2, \mathbb{Z}))$  is indeed primitive (cf. Theorem 2.3).

An outline of our proof is as follows. Let  $H$  be the kernel of the canonical homomorphism from  $G = \mathbb{Z}_2 * \mathbb{Z}_3$  onto  $\mathbb{Z}_2 \times \mathbb{Z}_3$ . Then  $H \simeq \mathbb{Z} * \mathbb{Z}$ . Exploiting a certain phase-action of the circle group  $\mathbb{T}$  on  $C^*(H)$ , we then show how a faithful irreducible representation of  $C^*(H)$  may be picked so that it induces a representation of  $C^*(G)$  which is also faithful and irreducible. Moreover, we show that there exists an uncountable family of pairwise inequivalent, irreducible faithful representations of  $C^*(G)$ . A similar idea was used by Murphy in his proof of [14, Theorem 3.3], where he considers certain semidirect products of nonabelian free groups by amenable groups. However, in our case, the exact sequence  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3 \rightarrow 1$  does not split, so we have to decompose  $C^*(G)$  as a *twisted* crossed product of  $C^*(H)$  by  $\mathbb{Z}_2 \times \mathbb{Z}_3$  and use results of J. A. Packer and I. Raeburn from [16]. Actually, when  $H$  is a normal subgroup of a group  $G$ , we give a criterion ensuring that primitivity of  $C^*(H)$  passes over to  $C^*(G)$  (see Theorem 2.1), and use it to deduce Theorem 2.3.<sup>1</sup>

Murphy mentions in [14] that he knows of no example of an icc group whose full group  $C^*$ -algebra is not primitive, but that it is unlikely that such groups do not exist. Now it is almost immediate (cf. Proposition 2.5) that  $C^*(G)$  is not primitive whenever  $G$  is a nontrivial group having Kazhdan's property (T). As

<sup>1</sup>In a recent paper (*Banach J. Math. Anal.*, **5**(2), 44-58, 2011), we use this criterion to show that  $C^*(G)$  is primitive whenever  $G$  is the free product of two nontrivial *amenable* groups where at least one of them has more than two elements. The proof is combinatorially much more involved than in the case of the modular group.



there are many nontrivial icc groups having property (T), such as  $G = \mathrm{PSL}(n, \mathbb{Z})$  for any integer  $n \geq 3$  (see [4]), this confirms that the full group  $C^*$ -algebra of an icc group is not necessarily primitive. Moreover, as it is known that  $\mathrm{PSL}(n, \mathbb{Z})$  is always  $C^*$ -simple (see [1, 2]), this also illustrates that  $C^*$ -simplicity of  $G$  does not imply that  $C^*(G)$  is primitive.

## 2 On primitivity of full group $C^*$ -algebras and the modular group

We use standard notation and terminology in operator algebras; see for example [7, 17, 6]. All Hilbert spaces are assumed to be complex. By a representation of a  $C^*$ -algebra  $A$ , we always mean a  $*$ -homomorphism  $\pi: A \rightarrow B(\mathcal{H})$  into the bounded operators  $B(\mathcal{H})$  on some Hilbert space  $\mathcal{H}$ . We use the same symbol  $\simeq$  to denote unitary equivalence of operators on Hilbert spaces, (unitary) equivalence of representations of a  $C^*$ -algebra and  $*$ -isomorphism between  $C^*$ -algebras.

All the groups we consider are assumed to be countable and discrete. If  $G$  is such a group, we let  $e_G$ , or just  $e$ , denote its unit. When  $G$  acts on a nonempty set  $X$  and  $x \in X$ , we say that  $x$  is a *free point* (for the action of  $G$ ) whenever  $g \cdot x \neq x$  for all  $g \in G$ ,  $g \neq e$ .

Let  $A$  be a separable  $C^*$ -algebra and  $\widehat{A}$  denote the set of (unitary) equivalence classes of nonzero irreducible representations of  $A$ . Set

$$\widehat{A}^\circ = \{[\pi] \in \widehat{A} \mid \pi \text{ is faithful}\}.$$

This set is clearly well-defined, and it is nonempty if and only if  $A$  is primitive.

Assume now that a group  $G$  has a normal subgroup  $H$  such that  $A = C^*(H)$  is primitive and set  $K = G/H$ . Then  $K$  acts on  $\widehat{A}^\circ$  in a natural way.

To see this, let  $n: K \rightarrow G$  be a normalized section for the canonical homomorphism  $p$  from  $G$  onto  $K$  (so  $n(e_K) = e_G$  and  $p \circ n$  gives the identity map on  $K$ ).

Let  $\alpha: K \rightarrow \mathrm{Aut}(A)$  and  $u: K \times K \rightarrow A$  be determined by

$$\begin{aligned} \alpha_k(i_H(h)) &= i_H(n(k)hn(k)^{-1}), \quad k \in K, h \in H, \\ u(k, l) &= i_H(n(k)n(l)n(kl)^{-1}), \quad k, l \in K, \end{aligned}$$

where  $i_H$  denotes the canonical injection of  $H$  into  $A$ .

Then  $(\alpha, u)$  is a twisted action of  $K$  on  $A$  (see [16] or the Appendix); especially, we have

$$\alpha_k \alpha_l = \mathrm{Ad}(u(k, l)) \alpha_{kl}, \quad k, l \in K,$$

where, as usual,  $\mathrm{Ad}(v)$  denotes the inner automorphism implemented by some unitary  $v$  in  $A$ .

This twisted action  $(\alpha, u)$  clearly induces an action of  $K$  on  $\widehat{A}$  given by

$$k \cdot [\pi] = [\pi \circ \alpha_{k^{-1}}].$$

By restriction, we get the *natural action* of  $K$  on  $\widehat{A}^\circ$ , which is easily seen to be independent of the choice of normalized section  $n$  for  $p$ .

The following result holds:

**Theorem 2.1.** *Assume that a group  $G$  has a normal subgroup  $H$  such that*

- (a)  $A = C^*(H)$  is primitive,
- (b)  $K = G/H$  is amenable,
- (c) the natural action of  $K$  on  $\widehat{A}^\circ$  has a free point.

Then  $C^*(G)$  is primitive.

*Proof.* We use the notation introduced above and recall that Packer and Raeburn have shown (see [16, Theorem 4.1]) that  $C^*(G)$  may be decomposed as the twisted crossed product associated with  $(\alpha, u)$ :

$$C^*(G) \simeq A \times_{\alpha, u} K.$$

Let  $[\pi] \in \widehat{A}^\circ$  be a free point for the natural action of  $K$ . This means that

$$\pi \circ \alpha_k \not\equiv \pi \text{ for all } k \in K, k \neq e.$$

Now, this condition implies that the induced regular representation  $\text{Ind } \pi$  of  $A \times_{\alpha, u} K$  is irreducible. Indeed, as  $G$  is discrete, this could be deduced from [11] (see the discussion in [18, Introduction]; see also [12, 13, 19]). For completeness, we give a proof in the Appendix (cf. Corollary 3.2(a)).

Further, as  $K$  is amenable, [16, Theorem 3.1] gives that  $\text{Ind } \pi$  is faithful. Altogether, it follows that  $C^*(G)$  has a faithful, irreducible representation, as desired.  $\square$

*Remark 2.2.* Assume that  $G$  has a normal subgroup  $H$  and  $K = G/H$ . It would be interesting to find more general conditions than those given in Theorem 2.1 ensuring that  $C^*(G)$  is primitive. However, even for the case where  $G$  is the direct product of  $H$  and  $K$ , this is a nontrivial problem. Murphy has shown in [14, Theorem 2.5] that  $C^*(H \times K)$  is primitive whenever  $C^*(H)$  is primitive and  $K$  is amenable and icc. But when for example  $\mathbb{F}$  is a free nonabelian group, it is unknown whether  $C^*(\mathbb{F} \times \mathbb{F})$  is primitive or not. Note that if it should happen that  $C^*(\mathbb{F} \times \mathbb{F})$  is *not* primitive, this would imply that

$$C^*(\mathbb{F}) \otimes_{\max} C^*(\mathbb{F}) \not\cong C^*(\mathbb{F}) \otimes_{\min} C^*(\mathbb{F}).$$

Thus, when  $\mathbb{F}$  has infinitely many generators, this would solve negatively an open problem of E. Kirchberg, which is known to be equivalent to Connes' famous embedding problem (see [10]).

**Theorem 2.3.** *Set  $G = \mathrm{PSL}(2, \mathbb{Z})$ . Then  $C^*(G)$  is primitive. Moreover, there exists an uncountable family of pairwise inequivalent, irreducible faithful representations of  $C^*(G)$ .*

*Proof.* Write  $G = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 = 1 \rangle$  and let  $H$  denote the kernel of the canonical homomorphism  $p$  from  $G$  onto  $K = \mathbb{Z}_2 \times \mathbb{Z}_3 (\simeq \mathbb{Z}_6)$ .

Then  $H$  is freely generated as a group by  $x_1 = abab^2$  and  $x_2 = ab^2ab$  (see e.g. [20, I.1.3, Proposition 4]).

Set  $A = C^*(H)$ . Using [21] or [5], we may pick  $[\pi] \in \widehat{A}^\circ$ . Set

$$U_1 = i_H(x_1), \quad V_1 = \pi(U_1), \quad U_2 = i_H(x_2), \quad V_2 = \pi(U_2),$$

so  $V_1, V_2$  are unitary operators on the separable Hilbert space  $\mathcal{H}_\pi$  on which  $\pi$  acts. As shown in the proof [5, Theorem 6], we may and do assume that  $V_2$  is diagonal relative to some orthonormal basis for  $\mathcal{H}_\pi$ , with (distinct) diagonal entries given by some  $\mu_j \in \mathbb{T}$ ,  $j \in \mathbb{N}$ .

For each  $\lambda \in \mathbb{T}$ , let  $\gamma_\lambda$  be the  $*$ -automorphism of  $A$  determined by

$$\gamma_\lambda(U_1) = U_1, \quad \gamma_\lambda(U_2) = \lambda U_2,$$

and set  $\pi_\lambda = \pi \circ \gamma_\lambda$ . Clearly,  $[\pi_\lambda] \in \widehat{A}^\circ$ .

We will show that we can pick  $\lambda \in \mathbb{T}$  such that  $[\pi_\lambda]$  is a free point for the natural action of  $K$  on  $\widehat{A}^\circ$ . As  $K$  is amenable, the primitivity of  $C^*(G)$  will then follow from Theorem 2.1. To pick  $\lambda$ , we proceed as follows.

As a normalized section for  $p: G \rightarrow K$ , we choose  $n: K \rightarrow G$  given by

$$n(i, j) = a^i b^j, \quad i \in \{0, 1\}, \quad j \in \{0, 1, 2\}.$$

For each  $k = (i, j) \in K$  we let  $\alpha_k$  be the  $*$ -automorphism of  $A$  used to define the natural action of  $K$  on  $\widehat{A}^\circ$ .

It is clear that  $[\pi_\lambda]$  will be a free point for this action of  $K$  if for each  $k \in K$ ,  $k \neq (0, 0)$ , we have

$$(\pi_\lambda \circ \alpha_k)(U_r) \neq \pi_\lambda(U_r) \text{ for } r = 1 \text{ or } r = 2.$$

Some elementary computations give:

$$\pi_\lambda(U_1) = V_1, \quad \pi_\lambda(U_2) = \lambda V_2;$$

$$\begin{aligned}
&\text{when } k = (0, 1) : (\pi_\lambda \circ \alpha_k)(U_2) = V_1^*; \\
&\text{when } k = (0, 2) : (\pi_\lambda \circ \alpha_k)(U_1) = (\lambda V_2)^*; \\
&\text{when } k = (1, 0) : (\pi_\lambda \circ \alpha_k)(U_2) = (\lambda V_2)^*; \\
&\text{when } k = (1, 1) : (\pi_\lambda \circ \alpha_k)(U_2) = V_1; \\
&\text{when } k = (1, 2) : (\pi_\lambda \circ \alpha_k)(U_1) = \lambda V_2.
\end{aligned}$$

It follows that  $[\pi_\lambda]$  will be a free point whenever

$$(*) \quad V_1 \not\cong \lambda V_2, \quad V_1 \not\cong (\lambda V_2)^*, \quad \lambda V_2 \not\cong (\lambda V_2)^*.$$

Define

$$\begin{aligned}
\Omega_1 &= \{\lambda \in \mathbb{T} \mid V_1 \simeq \lambda V_2\}, \\
\Omega_2 &= \{\lambda \in \mathbb{T} \mid V_1 \simeq (\lambda V_2)^*\},
\end{aligned}$$

and

$$\Omega_3 = \{\lambda \in \mathbb{T} \mid \lambda V_2 \simeq (\lambda V_2)^*\}.$$

As the point spectrum of  $V_2$  is given by  $\sigma_p(V_2) = \{\mu_j \mid j \in \mathbb{N}\} \subseteq \mathbb{T}$ , the sets  $\Omega_1$ ,  $\Omega_2$ , and  $\Omega_3$  are all countable.

Indeed, if  $\Omega_1$  was uncountable, then, as  $\sigma_p(V_1) = \lambda \sigma_p(V_2)$  for all  $\lambda \in \Omega_1$ ,  $\sigma_p(V_1)$  would also be uncountable; as  $\mathcal{H}_\pi$  is separable, this is impossible. In the same way, we see that  $\Omega_2$  must be countable. Finally, if  $\Omega_3$  were uncountable, then the equality

$$\lambda\{\mu_j \mid j \in \mathbb{N}\} = \bar{\lambda}\{\bar{\mu}_j \mid j \in \mathbb{N}\}$$

would hold for uncountably many  $\lambda$ 's in  $\mathbb{T}$ , and this is easily seen to be impossible.

Hence, the set  $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$  is countable. Especially,  $\Omega \neq \mathbb{T}$  and  $(*)$  holds for every  $\lambda$  in the complement  $\Omega^c$  of  $\Omega$  in  $\mathbb{T}$ . Thus, we have shown that  $C^*(G)$  is primitive.

To prove the second assertion, we consider  $\lambda, \lambda' \in \Omega^c$ , so  $\text{Ind } \pi_\lambda$  and  $\text{Ind } \pi_{\lambda'}$  are irreducible and faithful. A well-known argument (adapted to our twisted setting; see Corollary 3.2(b) in the Appendix) gives that  $\text{Ind } \pi_\lambda$  and  $\text{Ind } \pi_{\lambda'}$  will be inequivalent whenever

$$\pi_\lambda \circ \alpha_j \not\cong \pi_{\lambda'} \quad \text{for all } j \in K.$$

Using our previous computations, we see that this will hold whenever

$$\begin{aligned}
V_1 &\not\cong \lambda V_2, & V_1 &\not\cong (\lambda V_2)^*, \\
V_1 &\not\cong \lambda' V_2, & V_1 &\not\cong (\lambda' V_2)^*, \\
\lambda V_2 &\not\cong \lambda' V_2, & (\lambda V_2)^* &\not\cong \lambda' V_2.
\end{aligned}$$

The first four conditions are satisfied since  $\lambda, \lambda' \in \Omega^c$ . Set

$$\Omega_\lambda = \{\omega \in \mathbb{T} \mid \lambda V_2 \simeq \omega V_2 \text{ or } (\lambda V_2)^* \simeq \omega V_2\}.$$

Then  $\Omega_\lambda$  is countable (arguing as in the first part of the proof), so  $\Omega \cup \Omega_\lambda$  is countable. Hence, if we assume, as we may, that  $\lambda' \in (\Omega \cup \Omega_\lambda)^c$ , then all six conditions above are satisfied, and it follows that  $\text{Ind } \pi_\lambda$  and  $\text{Ind } \pi_{\lambda'}$  are inequivalent, irreducible and faithful.

Proceeding inductively, we may produce in this way a countably infinite family of pairwise inequivalent, irreducible faithful representations of  $C^*(G)$ . In fact, even an uncountable family of such representations does exist. Indeed, observe that  $\text{Ind } \pi_\lambda$  is an *essential* representation of  $C^*(G)$ ; that is, its range contains no compact operators other than zero. Otherwise, the irreducible representations  $\text{Ind } \pi_\lambda$  and  $\text{Ind } \pi_{\lambda'}$  would have to be equivalent since they have the same kernel (cf. [7, Corollaire 4.1.10]). As  $C^*(G)$  is separable, the claim then follows from [7, Compléments 4.7.2].  $\square$

*Remark 2.4.* Let  $G = \text{PSL}(2, \mathbb{Z})$ . As we have seen in the above proof,  $C^*(G)$  has a faithful irreducible representation which is essential. Hence,  $C^*(G)$  is antiliminary (cf. [7, Compléments 9.5.4]). Since  $C^*(G)$  is also primitive (and therefore prime), it follows that the pure state space of  $C^*(G)$  is weak\*-dense in the state space of  $C^*(G)$  (cf. [7, Lemme 11.2.4]). This is also true when  $G$  is a nonabelian free group; in fact, this is precisely what Yoshizawa proves in [21] when  $G = \mathbb{F}_2$ .

Our next observation is quite obvious and surely known to specialists.

**Proposition 2.5.** *Let  $G$  be a group with Kazhdan's property (T) (see e.g. [4]) and assume that  $C^*(G)$  is primitive. Then  $G$  is trivial.*

*Proof.* Set  $A = C^*(G)$ . We endow the primitive ideal space  $\text{Prim}(A)$  of  $A$  with its Jacobson (hull-kernel) topology and  $\widehat{A}$  with the weakest topology making the canonical map from  $\widehat{A}$  onto  $\text{Prim}(A)$  continuous. Since  $A$  is primitive, we may pick  $[\pi_0] \in \widehat{A}^\circ$ . As  $\{0\}$  is dense in  $\text{Prim}(A)$ ,  $\{[\pi_0]\}$  is dense in  $\widehat{A}$ .

Now let  $\pi_1$  denote the representation of  $A$  associated with the trivial one-dimensional unitary representation of  $G$ . Property (T) means that  $[\pi_1]$  is isolated in  $\widehat{A}$ ; i.e.  $\{[\pi_1]\}$  is open in  $\widehat{A}$ . Thus we must have  $[\pi_1] = [\pi_0]$ . Specifically,  $\pi_1$  must be faithful, which implies that  $G$  is trivial.  $\square$

**Corollary 2.6.** *Set  $G = \text{PSL}(n, \mathbb{Z})$ ,  $n \geq 3$ . Then  $G$  is icc, but  $C^*(G)$  is not primitive.*

*Proof.* As it is well known that  $G$  is icc and has property (T) (see [4]), this follows from Proposition 2.5.  $\square$

Moreover, as  $\mathrm{PSL}(n, \mathbb{Z})$  is always  $C^*$ -simple (cf. [1, 2]), this result also shows that  $C^*$ -simplicity of a group  $G$  does not imply that  $C^*(G)$  is primitive.

### 3 Appendix

We prove here a couple of results about induced representations of discrete twisted crossed products, which we could not find explicitly in the literature in the form needed for our purposes.

Let  $(A, K, \alpha, u)$  be a twisted  $C^*$ -dynamical system as considered by Packer and Raeburn [16], where  $A$  is a unital  $C^*$ -algebra,  $K$  is a discrete group with unit  $e$ , and  $(\alpha, u)$  is a twisted action of  $K$  on  $A$ ; this means that  $\alpha$  is a map from  $K$  into  $\mathrm{Aut}(A)$ , the group of  $*$ -automorphisms of  $A$ , and  $u$  is a map from  $K \times K$  into  $\mathcal{U}(A)$ , the unitary group of  $A$ , satisfying

$$\begin{aligned}\alpha_k \alpha_l &= \mathrm{Ad}(u(k, l)) \alpha_{kl}, \\ u(k, l) u(kl, m) &= \alpha_k(u(l, m)) u(k, lm), \\ u(k, e) &= u(e, k) = 1\end{aligned}$$

for all  $k, l, m \in K$ . (To avoid technicalities, we assume that  $A$  is unital; otherwise, one has to assume that the 2-cocycle  $u$  takes value in the multiplier algebra of  $A$ ).

The full twisted crossed product  $A \times_{\alpha, u} K$  may then be considered as the enveloping  $C^*$ -algebra of the Banach  $*$ -algebra  $\ell^1(A, K, \alpha, u)$ , which consists of the Banach space  $\ell^1(K, A)$  equipped with product and involution given by

$$\begin{aligned}(f * g)(l) &= \sum_{k \in K} f(k) \alpha_k(g(k^{-1}l)) u(k, k^{-1}l), \quad f, g \in \ell^1(K, A), l \in K, \\ f^*(l) &= u(l, l^{-1})^* \alpha_l(f(l^{-1}))^*, \quad f \in \ell^1(K, A), l \in K.\end{aligned}$$

We let  $i_K$  and  $i_A$  denote the canonical injections of  $K$  and  $A$  into  $A \times_{\alpha, u} K$ , respectively.

Let now  $\pi$  be a nondegenerate representation of  $A$  on some Hilbert space  $\mathcal{H} = \mathcal{H}_\pi$  and let  $\pi_\alpha$  be the associated representation of  $A$  on  $\mathcal{H}_K = \ell^2(K, \mathcal{H})$  defined by

$$(\pi_\alpha(a)\xi)(k) = \pi(\alpha_{k^{-1}}(a))\xi(k), \quad a \in A, \xi \in \mathcal{H}_K, k \in K.$$

For every  $k \in K$ , let  $\lambda_u(k)$  be the unitary operator on  $\mathcal{H}_K$  given by

$$(\lambda_u(k)\xi)(l) = \pi(u(l^{-1}, k))\xi(k^{-1}l), \quad k, l \in K, \xi \in \mathcal{H}_K.$$

(Note that we follow [22] here; the right-hand version is used in [16]).

The pair  $(\pi_\alpha, \lambda_u)$  is then a covariant representation of  $(A, K, \alpha, u)$ , that is,

$$\begin{aligned}\pi_\alpha(\alpha_k(a)) &= \text{Ad}(\lambda_u(k))(\pi_\alpha(a)), \quad k \in K, a \in A; \\ \lambda_u(k)\lambda_u(l) &= \pi_\alpha(u(k, l))\lambda_u(kl), \quad k, l \in K.\end{aligned}$$

This covariant representation induces a nondegenerate representation  $\text{Ind } \pi$  of  $A \times_{\alpha, u} K$  on  $\mathcal{H}_K$  determined by

$$(\text{Ind } \pi)(f) = \sum_{k \in K} \pi_\alpha(f(k))\lambda_u(k), \quad f \in \ell^1(K, A),$$

that is, by

$$(\text{Ind } \pi)(i_A(a)) = \pi_\alpha(a), \quad (\text{Ind } \pi)(i_K(k)) = \lambda_u(k), \quad a \in A, k \in K.$$

For each  $k \in K$ , let  $\mathcal{H}_k$  denote the copy of  $\mathcal{H}$  in  $\mathcal{H}_K$  given by

$$\mathcal{H}_k = \{\xi \in \mathcal{H}_K \mid \xi(l) = 0 \text{ for all } l \in K, l \neq k\},$$

giving us the natural direct sum decomposition  $\mathcal{H}_K = \bigoplus_{k \in K} \mathcal{H}_k$ .

Assume now that  $\pi'$  is a nondegenerate representation of  $A$  on  $\mathcal{H}'$  and denote by  $(\pi'_\alpha, \lambda'_u)$  the associated covariant representation of  $(A, K, \alpha, u)$  on  $\mathcal{H}'_K$ .

Let  $T \in B(\mathcal{H}_K, \mathcal{H}'_K)$ . Denote by  $[T_{k,l}]_{k,l \in K}$  the matrix of  $T$  with respect to the natural direct sum decompositions of  $\mathcal{H}_K$  and  $\mathcal{H}'_K$ , and identify each  $T_{k,l}$  as an element in  $B(\mathcal{H}, \mathcal{H}')$ .

Hence, if  $\eta \in \mathcal{H}$  and  $k, l \in K$ , then  $T_{k,l}\eta = (T\eta_l)(k)$ , where  $\eta_l \in \mathcal{H}_K$  is given by  $\eta_l(k) = \eta$  when  $k = l$ , and  $\eta_l(k) = 0$  otherwise.

Some tedious (but straightforward) computations give:

$$\begin{aligned}(1) \quad & (T\pi_\alpha(a))_{k,l} = T_{k,l}\pi(\alpha_{l^{-1}}(a)), \quad (\pi'_\alpha(a)T)_{k,l} = \pi'(\alpha_{k^{-1}}(a))T_{k,l}, \\ (2) \quad & (T\lambda_u(j))_{k,l} = T_{k,jl}\pi(u(l^{-1}j^{-1}, j)), \quad (\lambda'_u(j)T)_{k,l} = \pi'(u(k^{-1}, j))T_{j^{-1}k,l},\end{aligned}$$

**Proposition 3.1.** *Assume  $\pi$  and  $\pi'$  are irreducible, and  $\pi \circ \alpha_j \not\cong \pi'$  for all  $j \in K, j \neq e$ . Let  $T \in B(\mathcal{H}_K, \mathcal{H}'_K)$  intertwine  $\text{Ind } \pi$  and  $\text{Ind } \pi'$ . Then  $T_{k,k}$  intertwines  $\pi$  and  $\pi'$  for all  $k \in K$ . Further,  $T$  is decomposable; that is,  $T_{k,l} = 0$  for all  $k \neq l$  in  $K$ .*

*Proof.* We first note that  $T\pi_\alpha(a) = \pi'_\alpha(a)T$  for all  $a \in A$ . Using (1), we then get

$$(3) \quad T_{k,l}\pi(\alpha_{l^{-1}}(a)) = \pi'(\alpha_{k^{-1}}(a))T_{k,l} \text{ for all } k, l \in K, a \in A.$$

Letting  $l = k$ , this clearly implies that  $T_{k,k}$  intertwines  $\pi$  and  $\pi'$  for all  $k \in K$ . Assume now that  $k \neq l$ . Using (3) with  $a = \alpha_k(b)$ , we get

$$(4) \quad T_{k,l}(\pi \circ \text{Ad}(u(l^{-1}, k)) \circ \alpha_{l^{-1}k})(b) = (\pi' \circ \text{Ad}(u(k^{-1}, k)))(b)T_{k,l} \text{ for all } b \in A.$$

From the assumption, we have  $\pi' \not\cong \pi \circ \alpha_{l^{-1}k}$ . Hence, it follows that  $\pi \circ \text{Ad}(u(l^{-1}, k) \circ \alpha_{l^{-1}k})$  and  $\pi' \circ \text{Ad}(u(k^{-1}, k))$  are irreducible and inequivalent. But (4) says that  $T_{k,l}$  intertwines these two representations of  $A$ , and we can therefore conclude that  $T_{k,l} = 0$ .  $\square$

The following corollary is due to Zeller-Meier in the case where  $u$  takes values in the center of  $A$  (see [22, Propositions 3.8 and 4.4]). Part (a) could be deduced from [19, Theorem], but as we also need part (b), we prove both.

**Corollary 3.2.** (a) *Ind  $\pi$  is irreducible whenever  $\pi$  is irreducible and the stabilizer subgroup  $K_\pi = \{k \in K \mid \pi \circ \alpha_k \simeq \pi\}$  is trivial.*

(b) *Assume that  $\pi$  and  $\pi'$  both are irreducible. Then  $\text{Ind } \pi \not\cong \text{Ind } \pi'$  whenever  $\pi \circ \alpha_j \not\cong \pi'$  for all  $j \in K$ .*

*Proof.* (a) Suppose that  $\pi$  is irreducible and  $K_\pi$  is trivial. Let  $T \in B(\mathcal{H}_K)$  lie in the commutant of  $(\text{Ind } \pi)(A \times_{\alpha, u} K)$ . Using Proposition 3.1 with  $\pi' = \pi$ , it follows that  $T$  is decomposable and  $T_{k,k} \in \pi(A)'$  for all  $k \in K$ . As  $\pi$  is irreducible, this gives that  $T_{k,k} \in \mathbb{C}I_{\mathcal{H}}$  for all  $k \in K$ . Further, we have  $T\lambda_u(j) = \lambda_u(j)T$  for all  $j \in K$ . Hence, using (2), we get

$$\begin{aligned} \pi(u(k^{-1}, kl^{-1}))T_{k,k} &= T_{k,k}\pi(u(k^{-1}, kl^{-1})) = (T\lambda_u(kl^{-1}))_{k,k} \\ &= (\lambda_u(kl^{-1})T)_{k,l} = \pi(u(k^{-1}, kl^{-1}))T_{l,l}, \end{aligned}$$

which implies that  $T_{k,k} = T_{l,l}$  for all  $k, l \in K$ . Altogether, this means that  $T$  is a scalar multiple of the identity operator on  $\mathcal{H}_K$ . Hence we have shown that  $\text{Ind } \pi$  is irreducible, as desired.

(b) Assume that  $\pi$  and  $\pi'$  both are irreducible and  $\pi \circ \alpha_j \not\cong \pi'$  for all  $j \in K$ . Let  $T \in B(\mathcal{H}_K, \mathcal{H}'_K)$  intertwine  $\text{Ind } \pi$  and  $\text{Ind } \pi'$ . It follows from Proposition 3.1 that  $T_{k,l} = 0$  for all  $k, l \in K$ ,  $k \neq l$ , and that  $T_{k,k}$  intertwine  $\pi$  and  $\pi'$  for all  $k \in K$ . As  $\pi \not\cong \pi'$  by assumption, we also have  $T_{k,k} = 0$  for all  $k \in K$ . Hence,  $T = 0$ . This shows that  $\text{Ind } \pi \not\cong \text{Ind } \pi'$ , as desired.  $\square$

Actually, both implications converse to those stated in (a) and (b) of Corollary 3.2 also hold (as in [22]). However, since we don't need these in this paper, we skip the proofs.

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## References

- [1] M. E. B. Bekka, M. G. Cowling, and P. de la Harpe. Simplicity of the reduced  $C^*$ -algebra of  $\mathrm{PSL}(n, \mathbb{Z})$ . *Internat. Math. Res. Notices*, (7):285ff., approx. 7 pp. (electronic), 1994.
- [2] M. E. B. Bekka, M. G. Cowling, and P. de la Harpe. Some groups whose reduced  $C^*$ -algebra is simple. *Inst. Hautes Études Sci. Publ. Math.*, (80):117–134 (1995), 1994.
- [3] M. E. B. Bekka and P. de la Harpe. Irreducibly represented groups. *Comment. Math. Helv.*, 83(4):847–868, 2008.
- [4] M. E. B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.
- [5] M. D. Choi. The full  $C^*$ -algebra of the free group on two generators. *Pacific J. Math.*, 87(1):41–48, 1980.
- [6] K. R. Davidson.  *$C^*$ -algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.
- [7] J. Dixmier. *Les  $C^*$ -algèbres et leurs représentations*. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris, 1964.
- [8] P. de la Harpe. On simplicity of reduced  $C^*$ -algebras of groups. *Bull. Lond. Math. Soc.*, 39(1):1–26, 2007.
- [9] N. Khattou. Deux propriétés de la  $C^*$ -algèbre maximale de certains produits libres. In *Comptes rendus de la première rencontre maroco-andalouse sur les algèbres et leurs applications (Tétouan, 2001)*, pages 54–63. Univ. Abdelmalek Essaâdi. Fac. Sci. Tétouan, Tétouan, 2003.
- [10] E. Kirchberg. On nonsemisplit extensions, tensor products and exactness of group  $C^*$ -algebras. *Invent. Math.*, 112(3):449–489, 1993.
- [11] G. W. Mackey. Borel structure in groups and their duals. *Trans. Amer. Math. Soc.*, 85:134–165, 1957.
- [12] G. W. Mackey. Unitary representations of group extensions. I. *Acta Math.*, 99:265–311, 1958.
- [13] G. W. Mackey. Induced representations and normal subgroups. In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, pages 319–326. Jerusalem Academic Press, Jerusalem, 1961.

- [14] G. J. Murphy. Primitivity conditions for full group  $C^*$ -algebras. *Bull. London Math. Soc.*, 35(5):697–705, 2003.
- [15] J. A. Packer. Twisted group  $C^*$ -algebras corresponding to nilpotent discrete groups. *Math. Scand.*, 64(1):109–122, 1989.
- [16] J. A. Packer and I. Raeburn. Twisted crossed products of  $C^*$ -algebras. *Math. Proc. Cambridge Philos. Soc.*, 106(2):293–311, 1989.
- [17] G. K. Pedersen.  *$C^*$ -algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press Inc., London, 1979.
- [18] J. Quigg. On the irreducibility of an induced representation. *Pacific J. Math.*, 93(1):163–179, 1981.
- [19] J. Quigg. On the irreducibility of an induced representation. II. *Proc. Amer. Math. Soc.*, 86(2):345–348, 1982.
- [20] J.-P. Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.
- [21] H. Yoshizawa. Some remarks on unitary representations of the free group. *Osaka Math. J.*, 3:55–63, 1951.
- [22] G. Zeller-Meier. Produits croisés d’une  $C^*$ -algèbre par un groupe d’automorphismes. *J. Math. Pures Appl. (9)*, 47:101–239, 1968.

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# Paper II

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Primitivity of some full group  $C^*$ -algebras

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# PRIMITIVITY OF SOME FULL GROUP $C^*$ -ALGEBRAS

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## Abstract

We show that the full group  $C^*$ -algebra of the free product of two nontrivial countable amenable discrete groups, where at least one of them has more than two elements, is primitive. We also show that in many cases, this  $C^*$ -algebra is antiliminary and has an uncountable family of pairwise inequivalent, faithful irreducible representations.

## 1 Introduction

Let  $G$  denote a countable discrete group. It is known that  $C^*(G)$ , the full group  $C^*$ -algebra of  $G$ , is primitive in a number of cases [17, 3, 11, 8, 10, 1]. Especially, this is true for many groups which have a free product decomposition satisfying various conditions: see [8, 10, 1]. These results suggest that  $C^*(G)$  should be primitive whenever  $G$  is the free product of two nontrivial countable discrete groups  $G_1$  and  $G_2$ , where at least one of them has more than two elements. In this note, we show that this is indeed the case when both  $G_1$  and  $G_2$  are also assumed to be amenable.

This applies for example when  $G_1$  and  $G_2$  are both finite with  $|G_1| \geq 2$  and  $|G_2| \geq 3$ . This case is not covered by any of the papers cited above, except when  $G_1 = \mathbb{Z}_2$  and  $G_2 = \mathbb{Z}_3$ , i.e.  $G$  is the modular group  $\mathrm{PSL}(2, \mathbb{Z})$ , for which primitivity of  $C^*(G)$  was shown in [1]. The reader should consult [10] and [1] for more information around the problem of determining when the full group  $C^*$ -algebra of a countable discrete group is primitive.

Our proof will rely on the following result from [1]:

**Theorem 1.1.** *Assume that a group  $G$  has a normal subgroup  $H$  such that*

- (i)  $C^*(H)$  is primitive,
- (ii)  $K = G/H$  is amenable,
- (iii) the natural action of  $K$  on  $\widehat{C^*(H)}^\circ$  has a free point.

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Then  $C^*(G)$  is primitive.

We recall here what condition (iii) means. Set  $A = C^*(H)$ . Then the set

$$\widehat{A}^\circ = \{[\pi] \in \widehat{A} \mid \pi \text{ is faithful}\}$$

is nonempty since  $A$  is assumed to be primitive. The natural action of  $K = G/H$  on  $\widehat{A}^\circ$  is defined as follows.

Let  $n: K \rightarrow G$  be a normalized section for the canonical homomorphism  $p$  from  $G$  onto  $K$ . Let  $\alpha: K \rightarrow \text{Aut}(A)$  and  $u: K \times K \rightarrow A$  be given by

$$\begin{aligned} \alpha_k(i_H(h)) &= i_H(n(k)hn(k)^{-1}), \quad k \in K, h \in H, \\ u(k, l) &= i_H(n(k)n(l)n(kl)^{-1}), \quad k, l \in K, \end{aligned}$$

where  $i_H$  denotes the canonical injection of  $H$  into  $A$ . Then  $(\alpha, u)$  is a twisted action of  $K$  on  $A$  (cf. [12]), which induces an action of  $K$  on  $\widehat{A}^\circ$  given by

$$k \cdot [\pi] = [\pi \circ \alpha_{k^{-1}}].$$

This action is independent of the choice of normalized section for  $p$  and called the natural action of  $K$  on  $\widehat{A}^\circ$ . Finally, we recall that  $[\pi] \in \widehat{A}^\circ$  is a free point for this action whenever we have  $k \cdot [\pi] \neq [\pi]$  for all  $k \in K$ ,  $k \neq e$ .

Throughout this paper, we let  $G_1$  and  $G_2$  be two nontrivial countable discrete groups and assume that at least one of them has more than two elements. Further, we let  $G = G_1 * G_2$  denote the free product of  $G_1$  and  $G_2$ . It is well known that  $G$  is icc and nonamenable. Section 2 is devoted to the proof of our main result in this paper:

**Theorem 1.2.** *Assume moreover that  $G_1$  and  $G_2$  are both amenable. Then  $C^*(G)$  is primitive.*

In the final section (Section 3), we discuss the problem of deciding when  $C^*(G)$  is antiliminary and has an uncountable family of pairwise inequivalent, faithful irreducible representations.

As will be evident from its proof, the annoying amenability assumption in Theorem 1.2 is due to the amenability assumption on  $K$  in Theorem 1.1. Now, if one replaces this assumption on  $K$  by requiring that the twisted action of  $K$  on  $C^*(H)$  is amenable in the sense that the full and the reduced crossed products of  $C^*(H)$  by this action agree, then Theorem 1.1 still holds. An interesting problem is whether one can find condition(s) other than the amenability of  $K$  ensuring that this more general requirement is satisfied.

## 2 Proof of Theorem 1.2

We let  $e_1$  (resp.  $e_2$ ) denote the unit of  $G_1$  (resp.  $G_2$ ) and set  $G'_1 = G_1 \setminus \{e_1\}$ ,  $G'_2 = G_2 \setminus \{e_2\}$ . We let  $X \subset G$  denote the set of commutators given by

$$X = \{[a, b] = aba^{-1}b^{-1} \in G \mid a \in G'_1, b \in G'_2\}.$$

As is well known (see e.g. [14]),  $X$  is free and generates the kernel  $H$  of the canonical homomorphism  $p$  from the free product  $G = G_1 * G_2$  onto the direct product  $K = G_1 \times G_2$ . The map  $(a, b) \mapsto [a, b]$  is then a bijection between  $G'_1 \times G'_2$  and  $X$ , and  $H$  is isomorphic to the free group  $\mathbb{F}_{|X|}$  with  $|X|$  generators.

As  $|X| = |G'_1| \cdot |G'_2| \geq 2$ ,  $A = C^*(H)$  is primitive (cf. [17, 3]). Further, as  $G_1$  and  $G_2$  are both assumed to be amenable,  $K$  is amenable.

Now let  $\pi$  be a faithful irreducible representation of  $A$  acting on a (necessarily separable) Hilbert space  $\mathcal{H}_\pi$ . For each function  $\lambda: X \rightarrow \mathbb{T}$ , we let  $\gamma_\lambda$  denote the \*-automorphism of  $A$  determined by

$$\gamma_\lambda(i_H(x)) = \lambda(x)i_H(x), \quad x \in X,$$

and set  $\pi_\lambda = \pi \circ \gamma_\lambda$ . Clearly, each  $\pi_\lambda$  is also faithful and irreducible, i.e.  $[\pi_\lambda] \in \widehat{A}^\circ$ .

The burden of the proof is to establish the following:

**Proposition 2.1.** *There exist  $[\pi] \in \widehat{A}^\circ$  and  $\lambda: X \rightarrow \mathbb{T}$  such that  $[\pi_\lambda]$  is a free point for the natural action of  $K$  on  $\widehat{A}^\circ$ .*

Once we have proven this proposition, the primitivity of  $C^*(G)$  then clearly follows from Theorem 1.1 and the proof of Theorem 1.2 will therefore be finished.

*Proof of Proposition 2.1.* As a normalized section  $n: K \rightarrow G$  for  $p$ , we choose

$$n(a, b) = ab, \quad a \in G_1, b \in G_2.$$

We have to show that some faithful irreducible representation  $\pi$  of  $A$  and some  $\lambda: X \rightarrow \mathbb{T}$  may be chosen so that

$$\pi_\lambda \circ \alpha_k \not\equiv \pi_\lambda$$

for all nontrivial  $k \in K$ .

Clearly, to show that this condition holds, it suffices to show that for each nontrivial  $k \in K$ , there exists some  $x \in X$  (depending on  $k$ ) such that

$$(\pi_\lambda \circ \alpha_k)(i_H(x)) \not\equiv \pi_\lambda(i_H(x)). \quad (2.1)$$

To show this, we will use following fact:

Assume  $x_0 \in X$  is fixed. Then, as follows from Choi's proof [3] (see [10, Proof of Theorem 3.2]), we may choose a faithful irreducible representation  $\pi = \pi_{x_0}$  of  $A$  such that for each  $x \neq x_0$  in  $X$  the unitary operator  $\pi(i_H(x))$  is diagonal relative to some orthonormal basis of  $\mathcal{H}_\pi$  (which depends on  $x$ ). We will call such a representation for a *Choi representation of  $A$  associated to  $x_0$* .

Our choice of  $x_0$ , and thereby of  $\pi = \pi_{x_0}$ , will depend on the possible existence of elements of order 2 in  $G_1$  or  $G_2$ .

We will also use repeatedly the following elementary fact (already used in [10] and in [1]):

Assume  $\mathcal{H}$  is a separable Hilbert space. Let  $U$  and  $V$  be unitary operators on  $\mathcal{H}$  and assume that  $U$  is diagonal relative to some orthonormal basis of  $\mathcal{H}$ . Then the sets

$$\{\mu \in \mathbb{T} \mid \mu U \simeq V\} \text{ and } \{\mu \in \mathbb{T} \mid \mu U \simeq (\mu U)^*\}$$

are both countable.

Consider some faithful irreducible representation  $\pi$  of  $A$  and  $\lambda: X \rightarrow \mathbb{T}$ . When  $a \in G'_1$ ,  $b \in G'_2$ , so  $[a, b] \in X$ , we let  $U(a, b)$  ( $= U_\pi(a, b)$ ) denote the unitary operator on  $\mathcal{H}_\pi$  given by  $U(a, b) = \pi(i_H([a, b]))$ . Further, we set  $\lambda(a, b) = \lambda([a, b])$ . Thus we have

$$\pi_\lambda(i_H([a, b])) = \lambda(a, b)U(a, b). \quad (2.2)$$

Some straightforward calculations give the following identities which we will use in the sequel:

$$\begin{aligned} \pi_\lambda(\alpha_{(a,b)}(i_H([a^{-1}, b^{-1}])) &= \lambda(a, b)U(a, b) \\ \pi_\lambda(\alpha_{(a,e_2)}(i_H([a^{-1}, b])) &= (\lambda(a, b)U(a, b))^* \\ \pi_\lambda(\alpha_{(e_1,b)}(i_H([a, b^{-1}])) &= (\lambda(a, b)U(a, b))^* \end{aligned} \quad (2.3)$$

$$\begin{aligned} \pi_\lambda(\alpha_{(a,b)}(i_H([a^{-1}c, b^{-1}])) &= \lambda(a, b)U(a, b)(\lambda(c, b)U(c, b))^* \\ \pi_\lambda(\alpha_{(a,b)}(i_H([c, b^{-1}])) &= \lambda(a, b)U(a, b)(\lambda(ac, b)U(ac, b))^* \end{aligned} \quad (2.4)$$

$$\begin{aligned} \pi_\lambda(\alpha_{(a,e_2)}(i_H([a^{-1}c, b])) &= \lambda(c, b)U(c, b)(\lambda(a, b)U(a, b))^* \\ \pi_\lambda(\alpha_{(a,e_2)}(i_H([c, b])) &= \lambda(ac, b)U(ac, b)(\lambda(a, b)U(a, b))^* \end{aligned} \quad (2.5)$$

whenever  $a \in G'_1$ ,  $b \in G'_2$ , and  $c \in G'_1 \setminus \{a, a^{-1}\}$ .

We will show how to pick  $\pi$  and  $\lambda$  such that (2.1) holds. It turns out that the possible existence of elements of order 2 in  $G_1$  or  $G_2$  complicates the argument. Set

$$P = \{s \in G'_1 \mid s^2 \neq e_1\}, \quad S = G'_1 \setminus P, \quad Q = \{t \in G'_2 \mid t^2 \neq e_2\}, \text{ and } T = G'_2 \setminus Q.$$

Hence, we have

$$G_1 = \{e_1\} \sqcup P \sqcup S \text{ and } G_2 = \{e_2\} \sqcup Q \sqcup T.$$



We divide our discussion into three separate cases.

**Case 1.** *Both  $P$  and  $Q$  are nonempty.*

We pick  $p_0 \in P$ ,  $q_0 \in Q$  and set  $x_0 = [p_0^{-1}, q_0^{-1}] \in X$ . Then we let  $\pi = \pi_{x_0}$  be a Choi representation of  $A$  associated to  $x_0$ , and set  $U(a, b) = U_\pi(a, b)$  for each  $x = [a, b] \in X$ . It remains to define  $\lambda: X \rightarrow \mathbb{T}$  so that (2.1) holds for each nontrivial  $k \in K$ . We introduce the following notation.

Assume that  $a \in G'_1$ ,  $b \in G'_2$ ,  $p \in P$ ,  $q \in Q$ ,  $s \in S$ , and  $t \in T$ . Then we set

$$\begin{aligned}\Omega(a, b) &= \{\mu \in \mathbb{T} \mid \mu U(a, b) \simeq U(a^{-1}, b^{-1})\}, \\ \Omega_1(p) &= \{\mu \in \mathbb{T} \mid \mu U(p, q_0) \simeq U(p^{-1}, q_0)^*\}, \\ \Omega_2(q) &= \{\mu \in \mathbb{T} \mid \mu U(p_0, q) \simeq U(p_0, q^{-1})^*\}, \\ \Omega_1(s) &= \{\mu \in \mathbb{T} \mid \mu U(s, q_0) \simeq (\mu U(s, q_0))^*\}, \\ \Omega_2(t) &= \{\mu \in \mathbb{T} \mid \mu U(p_0, t) \simeq (\mu U(p_0, t))^*\}.\end{aligned}$$

Note that if  $(a, b) \neq (p_0^{-1}, q_0^{-1})$ , then  $\Omega(a, b)$  is countable (as  $U(a, b)$  is then diagonalizable). Similarly,  $\Omega_1(p)$ ,  $\Omega_2(q)$ ,  $\Omega_1(s)$ , and  $\Omega_2(t)$  are countable.

To ease our notation, we will define  $\lambda$  on  $G'_1 \times G'_2$  and identify it with the function on  $X$  given by  $\lambda([a, b]) = \lambda(a, b)$ ,  $a \in G'_1$ ,  $b \in G'_2$ . We will first define  $\lambda$  on  $P \times Q$ .

Let  $P = \sqcup_{i \in I} \{p_i, p_i^{-1}\}$  and  $Q = \sqcup_{j \in J} \{q_j, q_j^{-1}\}$  be enumerations of  $P$  and  $Q$ , where the index set  $I$  (resp.  $J$ ) is a (finite or infinite) set of successive integers starting from 0. For each  $i \in I$  and  $j \in J$ , we set

$$\lambda(p_i^{-1}, q_j) = \lambda(p_i^{-1}, q_j^{-1}) = 1.$$

Now let  $i \in I$  and  $j \in J$ . Using (2.2) and (2.3), we see that (2.1) will hold for

$$\begin{aligned}k &= (p_i, q_j^{-1}) \text{ and } k = (p_i^{-1}, q_j) \text{ if } \lambda(p_i, q_j^{-1})U(p_i, q_j^{-1}) \not\simeq U(p_i^{-1}, q_j); \\ k &= (p_i, q_j) \text{ and } k = (p_i^{-1}, q_j^{-1}) \text{ if } \lambda(p_i, q_j)U(p_i, q_j) \not\simeq U(p_i^{-1}, q_j^{-1}); \\ k &= (p_i, e_2) \text{ and } k = (p_i^{-1}, e_2) \text{ if } \lambda(p_i, q_0)U(p_i, q_0) \not\simeq U(p_i^{-1}, q_0)^*; \\ k &= (e_1, q_j) \text{ and } k = (e_1, q_j^{-1}) \text{ if } \lambda(p_0, q_j)U(p_0, q_j) \not\simeq (\lambda(p_0, q_j^{-1})U(p_0, q_j^{-1}))^*.\end{aligned}$$

For each  $i \in I$  and  $j \in J$ , we therefore pick

$$\lambda(p_i, q_j^{-1}) \in \mathbb{T} \setminus \Omega(p_i, q_j^{-1}).$$

Next, for each  $i \in I$ ,  $i \neq 0$ , and  $j \in J$ ,  $j \neq 0$ , we pick

$$\begin{aligned}\lambda(p_i, q_j) &\in \mathbb{T} \setminus \Omega(p_i, q_j), \\ \lambda(p_i, q_0) &\in \mathbb{T} \setminus (\Omega(p_i, q_0) \cup \Omega_1(p_i)), \\ \lambda(p_0, q_j) &\in \mathbb{T} \setminus (\Omega(p_0, q_j) \cup \overline{\lambda(p_0, q_j^{-1})\Omega_2(q_j)}).\end{aligned}$$

Finally, we pick

$$\lambda(p_0, q_0) \in \mathbb{T} \setminus (\Omega(p_0, q_0) \cup \Omega_1(p_0) \cup \overline{\lambda(p_0, q_0^{-1})} \Omega_2(q_0)).$$

All these choices are possible as all the involved  $\Omega$ 's are countable. After having done this,  $\lambda$  is defined on  $P \times Q$  and we know that (2.1) will hold for all  $k \in (P \times Q) \cup (P \times \{e_2\}) \cup (\{e_1\} \times Q)$ .

This means that if both  $S$  and  $T$  happen to be empty, then  $\lambda$  is defined on the whole of  $X$  and (2.1) holds for every nontrivial  $k \in K$ , as desired.

*We assume from now on and until the end of Case 1 that  $S$  is nonempty.*

Consider  $s \in S$ . For each  $j \in J$  we set  $\lambda(s, q_j^{-1}) = 1$ . Using (2.2) and (2.3), we see that (2.1) will hold for

$$\begin{aligned} k = (s, q_j) \text{ and } k = (s, q_j^{-1}) \text{ if } \lambda(s, q_j)U(s, q_j) &\not\cong U(s, q_j^{-1}); \\ k = (s, e_2) \text{ if } \lambda(s, q_0)U(s, q_0) &\not\cong (\lambda(s, q_0)U(s, q_0))^*. \end{aligned}$$

For each  $j \in J$ ,  $j \neq 0$ , we therefore pick  $\lambda(s, q_j) \in \mathbb{T} \setminus \Omega(s, q_j)$ . We also pick  $\lambda(s, q_0) \in \mathbb{T} \setminus (\Omega(s, q_0) \cup \Omega_1(s))$ . Again, these choices are possible as all the involved  $\Omega$ 's are countable.

Following this procedure for every  $s \in S$ , we achieve that  $\lambda$  is defined on  $G'_1 \times Q$  in such a way that (2.1) will hold for all

$$k \in (G'_1 \times (\{e_2\} \cup Q)) \cup (\{e_1\} \times Q).$$

If  $T$  happens to be empty, this means that  $\lambda$  is defined on the whole of  $X$  and (2.1) holds for every nontrivial  $k$  in  $K$ , as desired.

*Finally, we assume from now on and until the end of Case 1 that  $T$  is also nonempty.*

Consider  $t \in T$ . For each  $i \in I$  we set  $\lambda(p_i^{-1}, t) = 1$ . Using (2.2) and (2.3), we see that (2.1) will hold for

$$\begin{aligned} k = (p_i, t) \text{ and } k = (p_i^{-1}, t) \text{ if } \lambda(p_i, t)U(p_i, t) &\not\cong U(p_i^{-1}, t); \\ k = (e_1, t) \text{ if } \lambda(p_0, t)U(p_0, t) &\not\cong (\lambda(p_0, t)U(p_0, t))^*. \end{aligned}$$

For each  $i \in I$ ,  $i \neq 0$ , we pick  $\lambda(p_i, t) \in \mathbb{T} \setminus \Omega(p_i, t)$ . We also pick  $\lambda(p_0, t) \in \mathbb{T} \setminus (\Omega(p_0, t) \cup \Omega_2(t))$ . Once again, these choices are possible as all the involved  $\Omega$ 's are countable. By doing this for every  $t \in T$ , we achieve that  $\lambda$  is defined on  $(G'_1 \times G'_2) \setminus (S \times T)$  and (2.1) will hold for all

$$k \in (G'_1 \times (\{e_2\} \cup Q)) \cup ((\{e_1\} \cup P) \times G'_2).$$

It remains to define  $\lambda$  on  $S \times T$  in a way which ensures that (2.1) also will hold for all  $k \in S \times T$ .

Let  $t \in T$ . We will below describe how to define  $\lambda$  on  $S \times \{t\}$  in a way which ensures that (2.1) will hold for all  $k \in S \times \{t\}$ . By following this procedure for each  $t \in T$ , the proof in Case 1 will then be finished.

It is now appropriate to partition  $S$  as  $S = S' \sqcup S''$ , where

$$S' = \{s \in S \mid sp_0 \in P\} \text{ and } S'' = \{s \in S \mid sp_0 \in S\}.$$

Assume that  $s \in S'$ . Using (2.2) and (2.4), we see that (2.1) will hold for

$$k = (s, t) \text{ if } \lambda(s, t)U(s, t)(\lambda(p_0, t)U(p_0, t))^* \neq \lambda(sp_0, t)U(sp_0, t).$$

Note that  $\lambda(sp_0, t)$  is already defined since  $sp_0 \in P$ . Further, as  $\lambda(sp_0, t)U(sp_0, t)$  is diagonalizable, the set

$$\Omega'(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(sp_0, t)U(sp_0, t)) \simeq U(s, t)(\lambda(p_0, t)U(p_0, t))^*\}$$

is countable. We can therefore pick  $\lambda(s, t) \in \mathbb{T} \setminus \overline{\Omega'(s, t)}$ . If  $S'$  is nonempty, we can do this for each  $s \in S'$  and  $\lambda$  will then be defined on  $S' \times \{t\}$  in such a way that (2.1) will hold for every  $k \in S' \times \{t\}$ . If  $S''$  is empty, then  $S'$  has to be nonempty and the proof of Case 1 is then finished.

Assume now that  $S''$  is nonempty and consider  $s \in S''$ , so  $(sp_0)^2 = e_1$ . One easily checks that this implies that  $sp_0^n = p_0^{-n}s$  for all  $n \in \mathbb{Z}$ . It is then almost immediate that  $S''(s) = \{sp_0^n \mid n \in \mathbb{Z}\}$  is a subset of  $S''$ .

Furthermore, if  $\tilde{s} \in S'' \setminus S''(s)$ , then  $S''(s)$  and  $S''(\tilde{s})$  are disjoint. Hence, as  $S''$  is countable, we may pick a countable family  $\{s_l\}_{l \in L}$  of distinct elements in  $S''$  such that  $S'' = \sqcup_{l \in L} S''(s_l)$ .

Consider  $l \in L$ . To ease notation we write  $s = s_l$ . We are going to define  $\lambda$  on  $S''(s) \times \{t\}$  in such a way that (2.1) will hold for every  $k \in S''(s) \times \{t\}$ . By doing this for each  $l \in L$ ,  $\lambda$  will then be defined on  $S'' \times \{t\}$  and (2.1) will hold for every  $k \in S'' \times \{t\}$ . Since  $S \times \{t\} = (S' \times \{t\}) \sqcup (S'' \times \{t\})$ , the proof of Case 1 will then be finished.

For each  $n \in \mathbb{Z}$ , using (2.2) and (2.4) (with  $a = sp_0^n$ ,  $b = t$ , and  $c = sp_0^{n \pm 1}$ ), we see that (2.1) will hold for

$$\begin{aligned} k = (sp_0^n, t) \text{ if } & \lambda(sp_0^n, t)U(sp_0^n, t)(\lambda(sp_0^{n+1}, t)U(sp_0^{n+1}, t))^* \neq \lambda(p_0, t)U(p_0, t) \\ & \text{or } \lambda(sp_0^n, t)U(sp_0^n, t)(\lambda(sp_0^{n-1}, t)U(sp_0^{n-1}, t))^* \neq \lambda(p_0^{-1}, t)U(p_0^{-1}, t). \end{aligned}$$

Suppose first that  $p_0$  is aperiodic, so  $S''(s) = \sqcup_{n \in \mathbb{Z}} \{sp_0^n\}$ . We first set  $\lambda(s, t) = 1$ . Then, for each  $m \in \mathbb{N}$ , we do inductively the following two steps:

i) Define

$$\Omega^m(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(p_0, t)U(p_0, t)) \simeq \lambda(sp_0^{m-1}, t)U(sp_0^{m-1}, t)U(sp_0^m, t)^*\}$$

(which is countable) and pick  $\lambda(sp_0^m, t) \in \mathbb{T} \setminus \Omega^m(s, t)$ .

ii) Define  $\Omega^{-m}(s, t)$  as the set

$$\{\mu \in \mathbb{T} \mid \mu(\lambda(p_0^{-1}, t)U(p_0^{-1}, t)) \simeq \lambda(sp_0^{-m+1}, t)U(sp_0^{-m+1}, t)U(sp_0^{-m}, t)^*\}$$

(which is countable) and pick  $\lambda(sp_0^{-m}, t) \in \mathbb{T} \setminus \Omega^{-m}(s, t)$ . Once this inductive process is finished,  $\lambda$  is defined on  $S''(s) \times \{t\}$  and we know that (2.1) holds for every  $k = (sp_0^{\pm(m-1)}, t)$ ,  $m \in \mathbb{N}$ , i.e. for every  $k \in S''(s) \times \{t\}$ , as desired.

Assume now that  $p_0$  is periodic with period  $N$ . Note that  $N \geq 3$  since  $p_0 \in P$ . The aperiodic case has to be modified as follows.

Again, we first set  $\lambda(s, t) = 1$ . Then, for each  $m = 1, \dots, N-2$ , we define inductively

$$\Omega^m(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(p_0, t)U(p_0, t)) \simeq \lambda(sp_0^{m-1}, t)U(sp_0^{m-1}, t)U(sp_0^m, t)^*\}$$

(which is countable) and pick  $\lambda(sp_0^m, t) \in \mathbb{T} \setminus \Omega^m(s, t)$ . This ensures that (2.1) holds for each  $k = (sp_0^{m-1}, t)$ ,  $m = 1, \dots, N-2$ .

We also define

$$\Omega^{N-1}(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(p_0, t)U(p_0, t)) \simeq \lambda(sp_0^{N-2}, t)U(sp_0^{N-2}, t)U(sp_0^{N-1}, t)^*\}$$

(which is countable). If we pick  $\lambda(sp_0^{N-1}, t)$  outside  $\Omega^{N-1}(s, t)$ , then (2.1) will hold for  $k = (sp_0^{N-2}, t)$ . However, we want to pick  $\lambda(sp_0^{N-1}, t)$  so that (2.1) also holds for  $k = (sp_0^{N-1}, t)$ . Now, using (2.2) and (2.4) (with  $a = sp_0^{N-1}$ ,  $b = t$ , and  $c = s$ ), we see that (2.1) will hold for  $k = (sp_0^{N-1}, t)$  if

$$\lambda(p_0, t)U(p_0, t) \not\simeq \lambda(sp_0^{N-1}, t)U(sp_0^{N-1}, t)U(s, t)^*.$$

Hence, we define

$$\Omega_N(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(p_0, t)U(p_0, t)) \simeq U(sp_0^{N-1}, t)U(s, t)^*\}$$

(which is countable) and pick

$$\lambda(sp_0^{N-1}, t) \in \mathbb{T} \setminus (\Omega^{N-1}(s, t) \cup \overline{\Omega_N(s, t)}).$$

This choice does ensure that (2.1) holds both for  $k = (sp_0^{N-2}, t)$  and  $k = (sp_0^{N-1}, t)$ .

Hence,  $\lambda$  is defined on  $S''(s) \times \{t\}$  and (2.1) holds for every  $k \in S''(s) \times \{t\}$ . This finishes the proof of Case 1.

**Case 2.** *Either  $P$  is nonempty and  $Q$  is empty, or  $P$  is empty and  $Q$  is nonempty.*

Clearly, it suffices to consider the first alternative. We then pick  $p_0 \in P$ ,  $t_0 \in T$  and set  $x_0 = [p_0^{-1}, t_0] \in X$ . We let  $\pi = \pi_{x_0}$  be a Choi representation of  $A$  associated to  $x_0$  and set  $U(a, b) = U_\pi(a, b)$  for each  $x = [a, b] \in X$ .

Our proof that  $\lambda: X \rightarrow \mathbb{T}$  may be defined so that (2.1) holds for each nontrivial  $k \in K$  is quite similar to our proof of Case 1, but some care is required and some repetitions seem unavoidable in our presentation.

For  $p \in P$ ,  $s \in S$ , and  $t \in T$ , we now set

$$\begin{aligned}\Omega(p, t) &= \{\mu \in \mathbb{T} \mid \mu U(p, t) \simeq U(p^{-1}, t)\}, \\ \Omega_1(p) &= \{\mu \in \mathbb{T} \mid \mu U(p, t_0) \simeq U(p^{-1}, t_0)^*\}, \\ \Omega_1(s) &= \{\mu \in \mathbb{T} \mid \mu U(s, t_0) \simeq (\mu U(s, t_0))^*\}, \\ \Omega_2(t) &= \{\mu \in \mathbb{T} \mid \mu U(p_0, t) \simeq (\mu U(p_0, t))^*\}.\end{aligned}$$

Note that if  $(p, t) \neq (p_0^{-1}, t_0)$ , then  $\Omega(p, t)$  is countable. On the other hand,  $\Omega_1(p)$  is countable when  $a \neq p_0^{-1}$ , while  $\Omega_1(s)$  and  $\Omega_2(t)$  are always countable.

Let  $P = \sqcup_{i \in I} \{p_i, p_i^{-1}\}$  be an enumeration of  $P$ , where  $I$  is a (finite or infinite) set of successive integers starting from 0. First, we set  $\lambda(p_i^{-1}, t) = 1$  for all  $i \in I$  and  $t \in T$ .

Let  $i \in I$  and  $t \in T$ . Using (2.2) and (2.3), we see that (2.1) will hold for

$$\begin{aligned}k &= (p_i, t) \text{ and } k = (p_i^{-1}, t) \text{ if } \lambda(p_i, t)U(p_i, t) \not\simeq U(p_i^{-1}, t); \\ k &= (p_i, e_2) \text{ and } k = (p_i^{-1}, e_2) \text{ if } \lambda(p_i, t_0)U(p_i, t_0) \not\simeq U(p_i^{-1}, t_0)^*; \\ k &= (e_1, t) \text{ if } \lambda(p_0, t)U(p_0, t) \not\simeq (\lambda(p_0, t)U(p_0, t))^*.\end{aligned}$$

Therefore, for each  $i \in I$ ,  $i \neq 0$ , and  $t \in T$ ,  $t \neq t_0$ , we pick

$$\begin{aligned}\lambda(p_i, t) &\in \mathbb{T} \setminus \Omega(p_i, t), \\ \lambda(p_i, t_0) &\in \mathbb{T} \setminus (\Omega(p_i, t_0) \cup \Omega_1(p_i)), \\ \lambda(p_0, t) &\in \mathbb{T} \setminus (\Omega(p_0, t) \cup \Omega_2(t)).\end{aligned}$$

Finally, we pick

$$\lambda(p_0, t_0) \in \mathbb{T} \setminus (\Omega(p_0, t_0) \cup \Omega_1(p_0) \cup \Omega_2(t_0)).$$

These choices ensure that  $\lambda$  is defined on  $P \times T$  and (2.1) will hold for all  $k \in (P \times (T \cup \{e_2\})) \cup (\{e_1\} \times T)$ .

This means that if  $S$  happens to be empty,  $\lambda$  is defined on the whole of  $X$  and (2.1) holds for every nontrivial  $k$  in  $K$ , as desired.

*We assume from now on and until the end of Case 2 that  $S$  is nonempty.*

Consider  $s \in S$ . Using (2.2) and (2.3), we see that (2.1) will hold for

$$k = (s, e_2) \text{ if } \lambda(s, t_0)U(s, t_0) \not\approx (\lambda(s, t_0)U(s, t_0))^*.$$

We will therefore pick  $\lambda(s, t_0)$  in a subset of  $\mathbb{T} \setminus \Omega_1(s)$ . But which subset will depend on whether  $s$  belongs to  $S'$  or  $S''$ , where

$$S' = \{s \in S \mid sp_0 \in P\} \text{ and } S'' = \{s \in S \mid sp_0 \in S\}$$

(using the same notation as in Case 1).

Assume that  $s \in S'$  and  $t \in T$ . As in Case 1, (2.1) will hold for

$$k = (s, t) \text{ if } \lambda(s, t)U(s, t)(\lambda(p_0, t)U(p_0, t))^* \not\approx \lambda(sp_0, t)U(sp_0, t).$$

Again, we set

$$\Omega'(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(sp_0, t)U(sp_0, t)) \simeq U(s, t)(\lambda(p_0, t)U(p_0, t))^*\}.$$

If  $t = t_0$ , then we pick  $\lambda(s, t_0) \in \mathbb{T} \setminus (\Omega_1(s) \cup \overline{\Omega'(s, t_0)})$ . Otherwise, we pick  $\lambda(s, t) \in \mathbb{T} \setminus \overline{\Omega'(s, t)}$ .

If  $S'$  is nonempty, we can do this for every  $s \in S'$  and every  $t \in T$ . This ensures that  $\lambda$  is defined on  $S' \times T$  and that (2.1) will hold for every  $k \in S' \times (T \cup \{e_2\})$ . Hence, if  $S''$  is empty, then  $S'$  has to be nonempty and the proof of Case 2 is finished.

Assume now that  $S''$  is nonempty. As in Case 1, we then pick a countable family  $\{s_l\}_{l \in L}$  of distinct elements in  $S''$  such that  $S'' = \sqcup_{l \in L} S''(s_l)$ , where  $S''(s) = \{sp_0^n \mid n \in \mathbb{Z}\}$  for  $s \in S''$ .

Consider  $l \in L$ ,  $t \in T$  and set  $s = s_l$ . If  $t = t_0$ , then we pick  $\lambda(s, t_0) \in \mathbb{T} \setminus \Omega_1(s)$ . Otherwise, we set  $\lambda(s, t) = 1$ .

Let  $n \in \mathbb{Z}$ . As in Case 1, (2.1) will hold for

$$\begin{aligned} k = (sp_0^n, t) \text{ if } & \lambda(sp_0^n, t)U(sp_0^n, t)(\lambda(sp_0^{n+1}, t)U(sp_0^{n+1}, t))^* \not\approx \lambda(p_0, t)U(p_0, t) \\ & \text{or } \lambda(sp_0^n, t)U(sp_0^n, t)(\lambda(sp_0^{n-1}, t)U(sp_0^{n-1}, t))^* \not\approx \lambda(p_0^{-1}, t)U(p_0^{-1}, t). \end{aligned}$$

Suppose first that  $p_0$  is aperiodic, so  $S''(s) = \sqcup_{n \in \mathbb{Z}} \{sp_0^n\}$ . Then, for each  $m \in \mathbb{N}$ , we proceed inductively and do the following two steps:

i) Define

$$\Omega^m(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(p_0, t)U(p_0, t)) \simeq \lambda(sp_0^{m-1}, t)U(sp_0^{m-1}, t)U(sp_0^m, t)^*\}.$$

If  $t = t_0$ , then we pick  $\lambda(sp_0^m, t_0) \in \mathbb{T} \setminus (\Omega_1(sp_0^m) \cup \Omega^m(s, t_0))$ . Otherwise, we pick  $\lambda(sp_0^m, t) \in \mathbb{T} \setminus \Omega^m(s, t)$ .

ii) Define  $\Omega^{-m}(s, t)$  as the set

$$\{\mu \in \mathbb{T} \mid \mu(\lambda(p_0^{-1}, t)U(p_0^{-1}, t)) \simeq \lambda(sp_0^{-m+1}, t)U(sp_0^{-m+1}, t)U(sp_0^{-m}, t)^*\}.$$

If  $t = t_0$ , then we pick  $\lambda(sp_0^{-m}, t_0) \in \mathbb{T} \setminus (\Omega_1(sp_0^{-m}) \cup \Omega^{-m}(s, t_0))$ . Otherwise, we pick  $\lambda(sp_0^{-m}, t) \in \mathbb{T} \setminus \Omega^{-m}(s, t)$ .

Assume next that  $p_0$  is periodic with period  $N \geq 3$ . Then for each  $m = 1, \dots, N-2$ , proceeding inductively, we define

$$\Omega^m(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(p_0, t)U(p_0, t)) \simeq \lambda(sp_0^{m-1}, t)U(sp_0^{m-1}, t)U(sp_0^m, t)^*\}.$$

If  $t = t_0$ , we pick  $\lambda(sp_0^m, t_0) \in \mathbb{T} \setminus (\Omega_1(sp_0^m) \cup \Omega^m(s, t_0))$ . Otherwise, we pick  $\lambda(sp_0^m, t) \in \mathbb{T} \setminus \Omega^m(s, t)$ .

We also define

$$\Omega^{N-1}(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(p_0, t)U(p_0, t)) \simeq \lambda(sp_0^{N-2}, t)U(sp_0^{N-2}, t)U(sp_0^{N-1}, t)^*\}.$$

As in Case 1, (2.1) will hold for  $k = (sp_0^{N-1}, t)$  if

$$\lambda(p_0, t)U(p_0, t) \not\simeq \lambda(sp_0^{N-1}, t)U(sp_0^{N-1}, t)U(s, t)^*.$$

So we define

$$\Omega_N(s, t) = \{\mu \in \mathbb{T} \mid \mu(\lambda(p_0, t)U(p_0, t)) \simeq U(sp_0^{N-1}, t)U(s, t)^*\}.$$

Now, if  $t = t_0$ , then we pick

$$\lambda(sp_0^{N-1}, t_0) \in \mathbb{T} \setminus (\Omega_1(sp_0^{N-1}) \cup \Omega^{N-1}(s, t_0) \cup \overline{\Omega_N(s, t_0)}).$$

Otherwise, we pick

$$\lambda(sp_0^{N-1}, t) \in \mathbb{T} \setminus (\Omega^{N-1}(s, t) \cup \overline{\Omega_N(s, t)}).$$

Under both alternatives ( $p_0$  being aperiodic or not), these processes ensure that  $\lambda$  is defined on  $S''(s) \times \{t\}$  and that (2.1) will hold for every  $k \in S''(s) \times (\{t\} \cup \{e_2\})$ .

After having done this for every  $s = s_l$ ,  $l \in L$  and every  $t \in T$ ,  $\lambda$  is defined on  $S'' \times T$  and we know that (2.1) will hold for every  $k \in S'' \times (T \cup \{e_2\})$ .

Altogether, this means that  $\lambda$  is defined on the whole of  $G'_1 \times G'_2$  and (2.1) holds for every nontrivial  $k \in K$ . This finishes the proof of Case 2.

**Case 3.** *Both  $P$  and  $Q$  are empty.*

This means that  $G'_1 = S$  and  $G'_2 = T$ , i.e. all nontrivial elements in  $G_1$  and  $G_2$  have order 2, so both groups are abelian. As one of them is assumed to have more than two elements, we may assume that  $|G_1| \geq 4$  and  $|G_2| \geq 2$ .

We pick  $s_0 \in S$ ,  $t_0 \in T$  and set  $x_0 = [s_0, t_0] \in X$ . Next, we let  $\pi = \pi_{x_0}$  be a Choi representation of  $A$  associated to  $x_0$  and set  $U(a, b) = U_\pi(a, b)$  for each  $(a, b) \in S \times T = G'_1 \times G'_2$ .

Now, since  $S$  is countable, it is not difficult to see that we may find a family  $\{s_l\}_{l \in L}$  of distinct elements in  $S \setminus \{s_0\}$  such that

$$S = \{s_0\} \sqcup \left( \bigsqcup_{l \in L} \{s_l, s_0 s_l\} \right),$$

where  $L$  is a (finite or infinite) set of successive integers starting from 1.

Let  $t \in T$ . Set  $\lambda(s_0, t) = 1$  and  $\lambda(s_l, t) = 1$  for each  $l \in L$ ,  $l \geq 2$ . Using (2.2) and (2.3), we see that (2.1) will hold for

$$k = (e_1, t) \text{ if } \lambda(s_1, t)U(s_1, t) \not\approx (\lambda(s_1, t)U(s_1, t))^*.$$

Hence, we set  $\Omega(t) = \{\mu \in \mathbb{T} \mid \mu U(s_1, t) \simeq (\mu U(s_1, t))^*\}$ , which is countable, and pick

$$\lambda(s_1, t) \in \mathbb{T} \setminus \Omega(t).$$

Consider now  $l \in L$ . Using (2.2), (2.3), (2.4), and (2.5), we see that (2.1) will hold for

$$\begin{aligned} k = (s_0, t) \text{ and } k = (s_l, e_2) \text{ if } \lambda(s_0 s_l, t)U(s_0 s_l, t) \not\approx U(s_0, t)(\lambda(s_l, t)U(s_l, t))^*; \\ k = (s_0, e_2) \text{ and } k = (s_l, t) \text{ if } \lambda(s_0 s_l, t)U(s_0 s_l, t) \not\approx \lambda(s_l, t)U(s_l, t)U(s_0, t)^*; \\ k = (s_0, t) \text{ and } k = (s_0 s_l, e_2) \text{ if } \lambda(s_l, t)U(s_l, t) \not\approx U(s_0, t)(\lambda(s_0 s_l, t)U(s_0 s_l, t))^*; \\ k = (s_0 s_l, t) \text{ and } k = (s_0, e_2) \text{ if } \lambda(s_l, t)U(s_l, t) \not\approx \lambda(s_0 s_l, t)U(s_0 s_l, t)U(s_0, t)^*. \end{aligned}$$

For each  $l \in L$ , we therefore set

$$\begin{aligned} \Omega_1(l, t) &= \{\mu \in \mathbb{T} \mid \mu U(s_l s_0, t) \simeq U(s_0, t)(\lambda(s_l, t)U(s_l, t))^*\}, \\ \Omega_2(l, t) &= \{\mu \in \mathbb{T} \mid \mu U(s_0 s_l, t) \simeq \lambda(s_l, t)U(s_l, t)U(s_0, t)^*\}, \\ \Omega_3(l, t) &= \{\mu \in \mathbb{T} \mid \mu(\lambda(s_l, t)U(s_l, t)) \simeq U(s_0, t)U(s_0 s_l, t)^*\}, \\ \Omega_4(l, t) &= \{\mu \in \mathbb{T} \mid \mu(\lambda(s_l, t)U(s_l, t)) \simeq U(s_l s_0, t)U(s_0, t)^*\}. \end{aligned}$$

All these sets are countable. Hence, for each  $l \in L$ , we can pick

$$\lambda(s_0 s_l, t) \in \mathbb{T} \setminus (\Omega_1(l, t) \cup \Omega_2(l, t) \cup \Omega_3(l, t) \cup \overline{\Omega_4(l, t)}).$$

We have thereby defined  $\lambda$  on  $S \times \{t\}$  in such a way that (2.1) will hold for every  $k \in (G_1 \times \{t\}) \sqcup (S \times \{e_2\})$ . By doing this for each  $t \in T$ ,  $\lambda$  is defined on  $S \times T = G'_1 \times G'_2$  and (2.1) holds for every nontrivial  $k \in K$ . This finishes the proof of Case 3 (and thereby the proofs of Proposition 2.1 and Theorem 1.2).

□



### 3 Some further aspects

We believe that if  $G$  is a countable group such that  $C^*(G)$  is primitive, then  $C^*(G)$  is antiliminary and has an uncountable family of pairwise inequivalent, irreducible faithful representations. It is not difficult to see that this true in the case where  $G$  is nontrivial, icc and amenable (see below). As pointed out in [1], this also holds when  $G = \mathbb{Z}_2 * \mathbb{Z}_3$ . The argument was based on the following observation, which goes back to the work of J. Glimm and J. Dixmier in the sixties. We recall that a representation of a  $C^*$ -algebra is called *essential* whenever its range contains no compact operators other than zero.

**Proposition 3.1.** *Let  $A$  be a primitive separable  $C^*$ -algebra and consider the set  $\widehat{A}^\circ = \{[\pi] \in \widehat{A} \mid \pi \text{ is faithful}\}$ . Then the following conditions are equivalent:*

- (i)  $|\widehat{A}^\circ| > 1$ .
- (ii) *Every faithful irreducible representation of  $A$  is essential.*
- (iii)  *$A$  has a faithful irreducible representation which is essential.*
- (iv)  $\widehat{A}^\circ$  *is uncountable.*

*Moreover, if  $A$  satisfies any of these conditions, then  $A$  is antiliminary.*

*Proof.* The implications (ii)  $\Rightarrow$  (iii) and (iv)  $\Rightarrow$  (i) are trivial. The implication (i)  $\Rightarrow$  (ii) follows from [4, Corollaire 4.1.10], while (iii)  $\Rightarrow$  (iv) follows from [4, Compléments 4.7.2]. The final assertion follows from [4, Compléments 9.5.4].  $\square$

For completeness we mention that there is another way to show that a unital separable  $C^*$ -algebra is primitive and antiliminary. Indeed, using that primitivity and primeness are equivalent notions for separable  $C^*$ -algebras (see e.g. [13]), one deduces that a separable unital  $C^*$ -algebra  $A$  is primitive and antiliminary if and only if the pure state space of  $A$  is weak\*-dense in the state space of  $A$  (cf. [4, Lemme 11.2.4 and Compléments 11.6.6]). H. Yoshizawa showed in [17] that the right-hand side of this equivalence holds when  $A = C^*(\mathbb{F}_2)$ .

Now let  $G = G_1 * G_2$  be as in Theorem 1.2. It is conceivable that one might be able to check that condition (i) in Proposition 3.1 holds for  $A = C^*(G)$  by following the line of proof used in [1] when  $G = \mathbb{Z}_2 * \mathbb{Z}_3$ . However, in light of our proof of Theorem 1.2, the necessary combinatorics will certainly be very messy. We will instead use the following well-known lemma to check that condition (ii) holds for  $A = C^*(G)$  in many cases.

**Lemma 3.2.** *Let  $A$  be a primitive, unital, infinite-dimensional  $C^*$ -algebra. Assume that  $A$  contains no nontrivial projections or that  $A$  has a faithful tracial state. Then  $A$  satisfies condition (ii) in Proposition 3.1.*

*Proof.* For completeness, we give the proof. Let  $\pi$  be a faithful irreducible representation of  $A$  acting on a Hilbert space  $\mathcal{H}$  and let  $\mathcal{K}$  denote the compact operators on  $\mathcal{H}$ . Note that  $\mathcal{H}$  is infinite-dimensional since  $\pi(A)$  is infinite-dimensional.

Assume first that  $A$  contains no nontrivial projections. Since  $\pi$  is faithful,  $\pi(A)$  contains no nontrivial projections. Hence,  $\pi(A) \cap \mathcal{K} = \{0\}$  (otherwise we would have  $\mathcal{K} \subset \pi(A)$  by irreducibility, and  $\pi(A)$  would contain all finite-dimensional projections), so  $\pi$  is essential.

Assume now that  $A$  has a faithful tracial state  $\tau$ . Assume (for contradiction) that  $\pi(A) \cap \mathcal{K} \neq \{0\}$ . Then  $\mathcal{K} \subset \pi(A)$ . As is well known, when  $\mathcal{H}$  is infinite-dimensional, the only bounded trace on  $\mathcal{K}$  is the zero map. Hence, the restriction of  $\tau$  to  $\mathcal{K}$  must be zero. But  $\mathcal{K}$  contains nontrivial projections and evaluation of  $\tau$  on any of these does not give zero since  $\tau$  is faithful. This gives a contradiction, and it follows that  $\pi$  is essential.  $\square$

**Corollary 3.3.** *Let  $G = G_1 * G_2$  satisfy the assumptions of Theorem 1.2. Assume also that  $G_1$  and  $G_2$  are both torsion-free. Then  $C^*(G)$  has no nontrivial projections. Moreover, it is antiliminary and has an uncountable family of pairwise inequivalent, irreducible faithful representations.*

*Proof.* The first assertion is mentioned by G. J. Murphy [10, p. 703], where he refers to [5] and [9] for a proof. It seems to us that this is somewhat unprecise. We therefore provide an alternative way to prove this assertion:

Since  $G_1$  and  $G_2$  are amenable,  $G$  has the Haagerup property ([2, Proposition 6.2.3]). Hence, as shown by N. Higson and G. Kasparov in [7],  $G$  satisfies the Baum-Connes conjecture. As  $G$  is easily seen to be torsion-free,  $G$  also satisfies the Kadison-Kaplansky conjecture (see e.g. [16]), i.e. the reduced group  $C^*$ -algebra  $C_r^*(G)$  contains no nontrivial projections.

Moreover, as shown by J.-L. Tu in [15], any group having the Haagerup property is  $K$ -amenable. It follows that the homomorphism  $\lambda_*$  from  $K_0(C^*(G))$  to  $K_0(C_r^*(G))$  induced by the canonical map  $\lambda: C^*(G) \rightarrow C_r^*(G)$  is an isomorphism. It is then straightforward to check that this implies that  $C^*(G)$  has no nontrivial projections.

Now, Theorem 1.2 says that  $C^*(G)$  is primitive. The second assertion follows therefore from Proposition 3.1 in combination with the first assertion and Lemma 3.2.  $\square$

To our knowledge, the class of countable discrete groups which are such that their full group  $C^*$ -algebras have a *faithful* tracial state has not been much studied. Clearly, it does contain all countable amenable groups (as the full and the reduced group  $C^*$ -algebras agree for such groups, and the canonical tracial state on the reduced algebra is always faithful). Hence, if  $H$  is nontrivial, icc and amenable, then  $C^*(H)$  is primitive (cf. [10, 11]) and Lemma 3.2 may be applied.

Our assertion at the beginning of this section follows then from Proposition 3.1. On the other hand, this class also contains all free groups with countably many generators. This fact is due to Choi [3, Corollary 9] and may be put in a somewhat more general framework as follows.

We first recall that a  $C^*$ -algebra is called *residually finite-dimensional* (RFD) if it has a separating family of finite-dimensional representations (see e.g. [6]). Clearly, any abelian or finite-dimensional  $C^*$ -algebra is RFD. If  $\mathbb{F}$  is a free group on countably many generators, then  $C^*(\mathbb{F})$  is RFD (cf. [3, Theorem 7]). Moreover, the class of RFD  $C^*$ -algebras is closed under free products (see [6, Theorem 3.2]). Finally, any unital RFD  $C^*$ -algebra has a faithful tracial state (see the proof of [3, Corollary 9]). Hence, we get:

**Corollary 3.4.** *Consider  $G = G_1 * G_2$ , where at least one of the  $G_i$ 's has more than two elements, and assume that  $G_1$  (resp.  $G_2$ ) is abelian or finite. Then  $C^*(G)$  is RFD, antiliminary and has an uncountable family of pairwise inequivalent, irreducible faithful representations.*

*Proof.* It follows from Theorem 1.2 that  $C^*(G)$  is primitive. Moreover,  $C^*(G) = C^*(G_1) * C^*(G_2)$  is RFD since  $C^*(G_1)$  and  $C^*(G_2)$  are RFD. Hence,  $C^*(G)$  has a faithful tracial state, and the assertion follows from Proposition 3.1 combined with Lemma 3.2.  $\square$

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## References

- [1] E. Bédos and T. Omland. The full group  $C^*$ -algebra of the modular group is primitive. *Proc. Amer. Math. Soc.*, 140(4):1403–1411, 2012.
- [2] P.-A. Cherix, M. G. Cowling, P. Jolissaint, P. Julg, and A. Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001.
- [3] M. D. Choi. The full  $C^*$ -algebra of the free group on two generators. *Pacific J. Math.*, 87(1):41–48, 1980.
- [4] J. Dixmier. *Les  $C^*$ -algèbres et leurs représentations*. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris, 1964.
- [5] K. J. Dykema and P. de la Harpe. Some groups whose reduced  $C^*$ -algebras have stable rank one. *J. Math. Pures Appl. (9)*, 78(6):591–608, 1999.

- [6] R. Exel and T. A. Loring. Finite-dimensional representations of free product  $C^*$ -algebras. *Internat. J. Math.*, 3(4):469–476, 1992.
- [7] N. Higson and G. Kasparov. Operator  $K$ -theory for groups which act properly and isometrically on Hilbert space. *Electron. Res. Announc. Amer. Math. Soc.*, 3:131–142 (electronic), 1997.
- [8] N. Khattou. Deux propriétés de la  $C^*$ -algèbre maximale de certains produits libres. In *Comptes rendus de la première rencontre maroco-andalouse sur les algèbres et leurs applications (Tétouan, 2001)*, pages 54–63. Univ. Abdelmalek Essaâdi. Fac. Sci. Tétouan, Tétouan, 2003.
- [9] R. Ji and S. Pedersen. Certain full group  $C^*$ -algebras without proper projections. *J. Operator Theory*, 24(2):239–253, 1990.
- [10] G. J. Murphy. Primitivity conditions for full group  $C^*$ -algebras. *Bull. London Math. Soc.*, 35(5):697–705, 2003.
- [11] J. A. Packer. Twisted group  $C^*$ -algebras corresponding to nilpotent discrete groups. *Math. Scand.*, 64(1):109–122, 1989.
- [12] J. A. Packer and I. Raeburn. Twisted crossed products of  $C^*$ -algebras. *Math. Proc. Cambridge Philos. Soc.*, 106(2):293–311, 1989.
- [13] G. K. Pedersen.  *$C^*$ -algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press Inc., London, 1979.
- [14] J.-P. Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1980.
- [15] J.-L. Tu. La conjecture de Baum-Connes pour les feuilletages moyennables. *K-Theory*, 17(3):215–264, 1999.
- [16] A. Valette. The conjecture of idempotents: a survey of the  $C^*$ -algebraic approach. *Bull. Soc. Math. Belg. Sér. A*, 41(3):485–521, 1989.
- [17] H. Yoshizawa. Some remarks on unitary representations of the free group. *Osaka Math. J.*, 3:55–63, 1951.

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# Paper III

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Primeness and primitivity conditions for twisted  
group  $C^*$ -algebras

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# PRIMENESS AND PRIMITIVITY CONDITIONS FOR TWISTED GROUP $C^*$ -ALGEBRAS

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## Abstract

For a multiplier (2-cocycle)  $\sigma$  on a discrete group  $G$  we give conditions for which the twisted group  $C^*$ -algebra associated with the pair  $(G, \sigma)$  is prime or primitive. We also discuss how these conditions behave on direct products and free products of groups.

## Introduction

In this paper,  $G$  will always denote a discrete group with identity  $e$ . The full group  $C^*$ -algebra associated with  $G$ ,  $C^*(G)$  is simple only if  $G$  is trivial, but other aspects of its ideal structure are of interest. Recall that a  $C^*$ -algebra is called *primitive* if it has a faithful irreducible representation and *prime* if nonzero ideals have nonzero intersection. Primeness of a  $C^*$ -algebra is in general a weaker property than primitivity. However, according to a result of Dixmier [9], the two notions coincide for separable  $C^*$ -algebras.

Furthermore, recall what the *icc property* means for  $G$  — that every nontrivial conjugacy class is infinite, and its importance comes to light in the following theorem.

**Theorem A.** *The following are equivalent:*

- (i)  $G$  has the *icc property*.
- (ii) The group von Neumann algebra  $W^*(G)$  is a factor.
- (iii) The reduced group  $C^*$ -algebra  $C_r^*(G)$  is prime.

The equivalence (i)  $\Leftrightarrow$  (ii) is a well-known result of Murray and von Neumann [19], while (i)  $\Leftrightarrow$  (iii) is proved by Murphy [18]. Murphy also shows that the *icc property* is a necessary condition for primeness of  $C^*(G)$ . Therefore, for the class of discrete groups, primeness and, in the countable case, primitivity, may be regarded as  $C^*$ -algebraic analogs of factors. The theorem gives as a corollary that

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if  $G$  is countable and amenable, then primitivity of  $C^*(G)$  is equivalent with the icc property of  $G$ . Moreover, since amenability of  $G$  implies injectivity of  $W^*(G)$ , this is also equivalent to  $W^*(G)$  being the hyperfinite  $\text{II}_1$  factor if  $G$  is nontrivial, according to Connes [8].

In the present paper, Theorem A will be adapted to a twisted setting where pairs  $(G, \sigma)$  consisting of a group  $G$  and a multiplier  $\sigma$  on  $G$  are considered. We will show that the reduced twisted group  $C^*$ -algebra  $C_r^*(G, \sigma)$  is prime if and only if every nontrivial  $\sigma$ -regular conjugacy class of  $G$  is infinite, and say that the pair  $(G, \sigma)$  satisfies *condition K* if it possesses this property. It was first introduced by Kleppner [13], who proves that this property is equivalent to the fact that the twisted group von Neumann algebra  $W^*(G, \sigma)$  is a factor. The main part of our proof is to show that  $(G, \sigma)$  satisfies condition K if and only if  $C_r^*(G, \sigma)$  has trivial center, and this argument is, of course, inspired by the mentioned works of Kleppner and Murphy. As a corollary, we get that primeness of the full twisted group  $C^*$ -algebra  $C^*(G, \sigma)$  implies condition K on  $(G, \sigma)$ . The converse is not true in general, but at least holds if  $G$  is amenable, as the full and reduced twisted group  $C^*$ -algebras then are isomorphic. Thus, if  $G$  is countable and amenable, condition K on  $(G, \sigma)$  is equivalent to primitivity of  $C^*(G, \sigma)$  by applying Dixmier's result. This fact is also explained by Packer [21] with a different approach. On the other hand, no examples of nonprimitive, but prime twisted group  $C^*$ -algebras are known, so it is not clear whether we need the countability assumption on  $G$ .

In the last two sections we will investigate primeness and primitivity of the twisted group  $C^*$ -algebras of  $(G, \sigma)$  when  $G = G_1 \times G_2$  and when  $G = G_1 * G_2$ . The free product case turns out to be easier to handle in general, since the corresponding  $C^*$ -algebra always decomposes into a free product of the two components. For direct products, however, the multiplier  $\sigma$  on  $G$  can have a 'cross-term' which makes a  $C^*$ -algebra decomposition into tensor products impossible.

## 1 Twisted group $C^*$ -algebras

Let  $G$  be a group and  $\mathcal{H}$  a nontrivial Hilbert space. The projective unitary group  $PU(\mathcal{H})$  is the quotient of the unitary group  $U(\mathcal{H})$  by the scalar multiples of the identity, that is,

$$PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}1_{\mathcal{H}}.$$

A projective unitary representation  $G$  is a homomorphism  $G \rightarrow PU(\mathcal{H})$ . Every lift of a projective representation to a map  $U : G \rightarrow U(\mathcal{H})$  must satisfy

$$U(a)U(b) = \sigma(a, b)U(ab) \tag{1.1}$$



for all  $a, b \in G$  and some function  $\sigma : G \times G \rightarrow \mathbb{T}$ . From the associativity of  $U$  and by requiring that  $U(e) = 1_{\mathcal{H}}$ , the identities

$$\begin{aligned}\sigma(a, b)\sigma(ab, c) &= \sigma(a, bc)\sigma(b, c) \\ \sigma(a, e) &= \sigma(e, a) = 1\end{aligned}\tag{1.2}$$

must hold for all elements  $a, b, c \in G$ .

**Definition.** Any function  $\sigma : G \times G \rightarrow \mathbb{T}$  satisfying (1.2) is called a *multiplier* on  $G$ , and any map  $U : G \rightarrow U(\mathcal{H})$  satisfying (1.1) is called a  $\sigma$ -*projective unitary representation* of  $G$  on  $\mathcal{H}$ .

The lift of a homomorphism  $G \rightarrow PU(\mathcal{H})$  up to  $U$  is not unique, but any other lift is of the form  $\beta U$  for some function  $\beta : G \rightarrow \mathbb{T}$ . Therefore, two multipliers  $\sigma$  and  $\tau$  are said to be *similar* if

$$\tau(a, b) = \beta(a)\beta(b)\overline{\beta(ab)}\sigma(a, b)$$

for all  $a, b \in G$  and some  $\beta : G \rightarrow \mathbb{T}$ . Note that we must have  $\beta(e) = 1$  for this to be possible. We say that  $\sigma$  is *trivial* if it is similar to 1 and call  $\sigma$  *normalized* if  $\sigma(a, a^{-1}) = 1$  for all  $a \in G$ .

Moreover, the set of similarity classes of multipliers on  $G$  is an abelian group under pointwise multiplication. This group is the *Schur multiplier* of  $G$  and will henceforth be denoted by  $\mathcal{M}(G)$ . Also, we remark that multipliers are often called *2-cocycles on  $G$  with values in  $\mathbb{T}$* , and that the Schur multiplier of  $G$  coincides with the second cohomology group  $H^2(G, \mathbb{T})$ .

Let  $\sigma$  be a multiplier on  $G$ . We will briefly explain how the operator algebras associated with the pair  $(G, \sigma)$  are constructed and refer to Zeller-Meier [24] for further details. First, the Banach  $*$ -algebra  $\ell^1(G, \sigma)$  is defined as the set  $\ell^1(G)$  together with twisted convolution and involution given by

$$\begin{aligned}(f *_\sigma g)(a) &= \sum_{b \in G} f(b)\sigma(b, b^{-1}a)g(b^{-1}a) \\ f^*(a) &= \overline{\sigma(a, a^{-1})f(a^{-1})}\end{aligned}$$

for elements  $f, g$  in  $\ell^1(G)$ , and is equipped with the usual  $\|\cdot\|_1$ -norm.

**Definition.** The full twisted group  $C^*$ -algebra  $C^*(G, \sigma)$  is the universal enveloping algebra of  $\ell^1(G, \sigma)$ . Moreover, the canonical injection of  $G$  into  $C^*(G, \sigma)$  will be denoted by  $i_{(G, \sigma)}$  or just  $i_G$  if no confusion arise.

For  $a$  in  $G$ , let  $\delta_a$  be the function on  $G$  defined by

$$\delta_a(b) = \begin{cases} 1 & \text{if } b = a, \\ 0 & \text{if } b \neq a. \end{cases}$$

Then the set  $\{\delta_a\}_{a \in G}$  is an orthonormal basis for  $\ell^2(G)$  and generates  $\ell^1(G, \sigma)$ , so that for all  $a$  in  $G$ ,  $i_{(G, \sigma)}(a)$  is the image of  $\delta_a$  in  $C^*(G, \sigma)$ . The *left regular  $\sigma$ -projective unitary representation*  $\lambda_\sigma$  of  $G$  on  $B(\ell^2(G))$  is given by

$$(\lambda_\sigma(a)\xi)(b) = (\delta_a *_\sigma \xi)(b) = \sigma(a, a^{-1}b)\xi(a^{-1}b).$$

Note in particular that

$$\lambda_\sigma(a)\delta_b = \delta_a *_\sigma \delta_b = \sigma(a, b)\delta_{ab}$$

for all  $a, b \in G$ . Moreover, the integrated form of  $\lambda_\sigma$  on  $\ell^1(G, \sigma)$  is defined by

$$\lambda_\sigma(f) = \sum_{a \in G} f(a)\lambda_\sigma(a).$$

**Definition.** The reduced twisted group  $C^*$ -algebra and the twisted group von Neumann algebra of  $(G, \sigma)$ ,  $C_r^*(G, \sigma)$  and  $W^*(G, \sigma)$  are, respectively, the  $C^*$ -algebra and the von Neumann algebra generated by  $\lambda_\sigma(\ell^1(G, \sigma))$ , or equivalently by  $\lambda_\sigma(G)$ .

If  $\tau$  is similar with  $\sigma$ , then in all three cases, the operator algebras associated with  $(G, \tau)$  and  $(G, \sigma)$  are isomorphic.

Moreover, there is a natural one-to-one correspondence between the representations of  $C^*(G, \sigma)$  and the  $\sigma$ -projective unitary representations of  $G$ . In particular,  $\lambda_\sigma$  extends to a  $*$ -homomorphism of  $C^*(G, \sigma)$  onto  $C_r^*(G, \sigma)$ . If  $G$  is amenable, then  $\lambda_\sigma$  is faithful. However, it is not known whether the converse holds unless  $\sigma$  is trivial.

Following the work of Kleppner [13], an element  $a$  in  $G$  is called  *$\sigma$ -regular* if  $\sigma(a, b) = \sigma(b, a)$  whenever  $b$  commutes with  $a$ , or equivalently if

$$U(a)U(b) = U(b)U(a)$$

for all  $b$  commuting with  $a$  whenever  $U$  is a  $\sigma$ -projective unitary representation of  $G$ . If  $\sigma$  and  $\tau$  are similar multipliers on  $G$ , it is easily seen that  $a$  in  $G$  is  $\sigma$ -regular if and only if it is  $\tau$ -regular. Furthermore, if  $a$  is  $\sigma$ -regular, then  $cac^{-1}$  is  $\sigma$ -regular for all  $c$  in  $G$ , and thus the notion of  $\sigma$ -regularity makes sense for conjugacy classes [13, Lemma 3]. The following theorem may now be deduced from [13, Lemma 4].

**Theorem B.** *Let  $\sigma$  be a multiplier on  $G$ . Then the following are equivalent:*

- (i) *Every nontrivial  $\sigma$ -regular conjugacy class of  $G$  is infinite.*
- (ii)  *$W^*(G, \sigma)$  is a factor.*

**Definition.** We say that the pair  $(G, \sigma)$  satisfies *condition K* if (i) is satisfied.

If  $G$  has the icc property, then  $(G, \sigma)$  always satisfies condition K. If  $G$  is abelian, or more generally, if all the conjugacy classes of  $G$  are finite, then  $(G, \sigma)$  satisfies condition K only if there are no nontrivial  $\sigma$ -regular elements in  $G$ .

**Example 1.1.** For  $n \geq 2$ , let  $\mathbb{Z}_n$  denote the cyclic group of order  $n$ . Then  $\mathcal{M}(\mathbb{Z}_n \times \mathbb{Z}_n) \cong \mathbb{Z}_n$  and its elements may be represented by multipliers  $\sigma_k$  given by

$$\sigma_k((a_1, a_2), (b_1, b_2)) = e^{2\pi i \frac{k}{n} a_2 b_1}$$

for  $0 \leq k \leq n-1$ . An element  $a = (a_1, a_2)$  in  $\mathbb{Z}_n \times \mathbb{Z}_n$  is  $\sigma_k$ -regular if and only if both  $ka_1$  and  $ka_2$  belong to  $n\mathbb{Z}$ . Therefore,  $(\mathbb{Z}_n \times \mathbb{Z}_n, \sigma_k)$  satisfies condition K if and only if  $k$  and  $n$  are relatively prime, in which case we have

$$C^*(\mathbb{Z}_n \times \mathbb{Z}_n, \sigma_k) \cong C_r^*(\mathbb{Z}_n \times \mathbb{Z}_n, \sigma_k) = W^*(\mathbb{Z}_n \times \mathbb{Z}_n, \sigma_k) \cong M_n(\mathbb{C}).$$

**Example 1.2.** It is well known that  $\mathcal{M}(\mathbb{Z}^n) \cong \mathbb{T}^{\frac{1}{2}n(n-1)}$  and that the multipliers are, up to similarity, determined by

$$\sigma_\theta(a, b) = e^{2\pi i \sum_{1 \leq i < j \leq n} a_i t_{ij} b_j}$$

for  $\theta = (t_{12}, t_{13}, \dots, t_{n-1, n})$  in  $[0, 1)^{\frac{1}{2}n(n-1)}$ . Note that the  $C^*$ -algebras associated with the pair  $(\mathbb{Z}^n, \sigma_\theta)$ ,  $C^*(\mathbb{Z}^n, \sigma_\theta) \cong C_r^*(\mathbb{Z}^n, \sigma_\theta)$ , are the noncommutative  $n$ -tori when  $\theta$  is nonzero.

Furthermore,  $(\mathbb{Z}^n, \sigma_\theta)$  satisfies condition K if there are no nontrivial  $\sigma_\theta$ -regular elements in  $\mathbb{Z}^n$ , that is, if there for all  $a$  in  $\mathbb{Z}^n$  exists  $b$  in  $\mathbb{Z}^n$  such that

$$\sigma_\theta(a, b) \overline{\sigma_\theta(b, a)} = e^{2\pi i \sum_{1 \leq i < j \leq n} t_{ij} (a_i b_j - b_i a_j)} \neq 1.$$

For  $n = 2$  and  $3$  we can give a good description of this property. Indeed,  $(\mathbb{Z}^2, \sigma_\theta)$  satisfies condition K if and only if  $\theta$  is irrational, and  $(\mathbb{Z}^3, \sigma_\theta)$  satisfies condition K if and only if

$$\dim \mathbb{Q}_\theta = 3 \text{ or } 4,$$

where  $\mathbb{Q}_\theta$  denotes the vector space over  $\mathbb{Q}$  spanned by  $\{1, t_{12}, t_{13}, t_{23}\}$ .

For  $n \geq 4$ , the situation is more complicated. In particular, condition K on  $(\mathbb{Z}^n, \sigma_\theta)$  does not only depend on the dimension of  $\mathbb{Q}_\theta$ . For example, if  $t_{12} = t_{34}$  is some irrational number in  $[0, 1)$  and  $t_{ij} = 0$  elsewhere, then  $\dim \mathbb{Q}_\theta = 2$ , and  $(\mathbb{Z}^4, \sigma_\theta)$  satisfies condition K. On the other hand, if  $t_{12} = t_{23} = t_{34} = 1 - t_{14}$  is some irrational number in  $[0, 1)$  and  $t_{13} = t_{24} = 0$ , then  $\dim \mathbb{Q}_\theta = 2$ , but it can be easily checked that  $(1, 1, 1, 1)$  in  $\mathbb{Z}^4$  is  $\sigma_\theta$ -regular.

**Example 1.3.** For each natural number  $n \geq 2$  let  $G(n)$  be the group with presentation

$$G(n) = \langle u_i, v_{jk} : [v_{jk}, v_{lm}] = [u_i, v_{jk}] = e, [u_j, u_k] = v_{jk} \rangle$$

for  $1 \leq i \leq n$ ,  $1 \leq j < k \leq n$  and  $1 \leq l < m \leq n$ . The group  $G(n)$  is sometimes called the *free nilpotent group of class 2 and rank  $n$* .

In a separate work<sup>2</sup>, we will calculate the multipliers of  $G(n)$  and show that

$$\mathcal{M}(G(n)) \cong \mathbb{T}^{\frac{1}{3}(n-1)n(n+1)}.$$

Note that  $G(2)$  is isomorphic with the discrete Heisenberg group and this case is already investigated by Packer [20].

To describe our result in the case of  $G(3)$ , we first remark that  $G(3)$  is isomorphic to the group with elements  $a = (a_1, a_2, a_3, a_4, a_5, a_6)$ , where all entries are integers, and with multiplication defined by

$$a \cdot b = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + a_1 b_2, a_5 + b_5 + a_1 b_3, a_6 + b_6 + a_2 b_3).$$

For every  $\mu$  in  $\mathbb{T}^8$ , the element  $[\sigma_\mu]$  in  $\mathcal{M}(G(3))$  may be represented by

$$\begin{aligned} \sigma_\mu(a, b) = & \mu_{13}^{b_6 a_1 + b_3 a_4} \mu_{22}^{b_5 a_2 + b_3(a_4 - a_1 a_2)} \\ & \cdot \mu_{11}^{b_4 a_1 + \frac{1}{2} b_2 a_1(a_1 - 1)} \mu_{21}^{a_2(b_4 + a_1 b_2) + \frac{1}{2} a_1 b_2(b_2 - 1)} \\ & \cdot \mu_{12}^{b_5 a_1 + \frac{1}{2} b_3 a_1(a_1 - 1)} \mu_{32}^{a_3(b_5 + a_1 b_3) + \frac{1}{2} a_1 b_3(b_3 - 1)} \\ & \cdot \mu_{23}^{b_6 a_2 + \frac{1}{2} b_3 a_2(a_2 - 1)} \mu_{33}^{a_3(b_6 + a_2 b_3) + \frac{1}{2} a_2 b_3(b_3 - 1)} \end{aligned}$$

where  $\mu_{ij} \in \mathbb{T}$ .

The pair  $(G(3), \sigma_\mu)$  satisfies condition K if and only if  $G(3)$  has no nontrivial central  $\sigma_\mu$ -regular elements, that is, if for all  $c = (0, 0, 0, c_1, c_2, c_3)$  in  $Z(G(3)) = \mathbb{Z}^3$  there exists  $a$  in  $G(3)$  such that  $\sigma_\mu(a, c) \overline{\sigma_\mu(c, a)} \neq 1$ .

Set  $\mu_{31} = \mu_{13} \overline{\mu_{22}}$ . One can then show that this holds if and only if for each nontrivial  $c$  in  $\mathbb{Z}^3$  there is some  $i = 1, 2$  or  $3$  such that

$$\prod_{1 \leq j \leq 3} \mu_{ij}^{c_j} \neq 1.$$

## 2 Primeness and primitivity

Henceforth, we fix a group  $G$  and a multiplier  $\sigma$  on  $G$ . Consider the right regular  $\overline{\sigma}$ -projective unitary representation  $\rho_{\overline{\sigma}}$  of  $G$  on  $B(\ell^2(G))$  defined by

$$(\rho_{\overline{\sigma}}(a)\xi)(c) = (\xi *_{\overline{\sigma}} \delta_a^*)(c) = \overline{\sigma(c, a)} \xi(ca).$$

<sup>2</sup> $C^*$ -algebras generated by projective representations of free nilpotent groups, arXiv:1301.2942, submitted for publication.

To simplify notation in what follows, we write just  $\bar{\rho}$  and  $\lambda$  for  $\rho_{\bar{\sigma}}$  and  $\lambda_{\sigma}$ . It is straightforward to see that  $\lambda(a)$  commutes with  $\bar{\rho}(b)$  for all  $a, b$  in  $G$ , that is,  $W^*(G, \sigma)$  is contained in  $\bar{\rho}(G)'$ . In fact, it is well known that  $W^*(G, \sigma) = \bar{\rho}(G)'$ . Moreover,

$$(\lambda(a)\bar{\rho}(a)\xi)(c) = \overline{\sigma(a^{-1}, c)}\sigma(a^{-1}ca, a^{-1})\xi(a^{-1}ca) \quad (2.1)$$

for all  $a, c \in G$  and all  $\xi \in \ell^2(G)$ . In particular,

$$\lambda(a)\bar{\rho}(a)\delta_e = \bar{\rho}(a)\lambda(a)\delta_e = \delta_e \quad (2.2)$$

for all  $a \in G$ .

*Remark 2.1.* The vector  $\delta_e$  is clearly cyclic for  $W^*(G, \sigma)$ . It is also separating. Indeed, if  $x\delta_e = 0$ , then

$$x\delta_a = x\lambda(a)\delta_e = x\bar{\rho}(a)^*\delta_e = \bar{\rho}(a)^*x\delta_e = 0$$

for all  $a \in G$ . Moreover, the state  $\varphi$  given by  $\varphi(x) = \langle x\delta_e, \delta_e \rangle$  is a faithful trace on  $W^*(G, \sigma)$ . Thus,  $W^*(G, \sigma)$  is finite and is therefore a  $\text{II}_1$  factor whenever  $G$  is infinite and  $(G, \sigma)$  satisfies condition K, according to Theorem B.

**Lemma 2.2.** *Let  $T$  be an operator in  $W^*(G, \sigma)$  and set  $f_T = T\delta_e$ . Then the following are equivalent:*

- (i)  $T$  belongs to the center of  $W^*(G, \sigma)$ .
- (ii)  $f_T(aca^{-1}) = \sigma(a, c)\overline{\sigma(aca^{-1}, a)}f_T(c)$  for all  $a, c \in G$ .

Moreover,  $f_T$  can be nonzero only on the finite conjugacy classes.

*Proof.* The operator  $T$  belongs to the center of  $W^*(G, \sigma)$  if and only if  $T = \lambda(a)T\lambda(a)^*$  for all  $a \in G$ . Since, by Remark 2.1,  $\delta_e$  is separating for  $W^*(G, \sigma)$ , this is equivalent to  $f_T = \lambda(a)T\lambda(a)^*\delta_e$  for all  $a \in G$ . By (2.2) we have

$$\lambda(a)T\lambda(a)^*\delta_e = \lambda(a)T\bar{\rho}(a)\delta_e = \lambda(a)\bar{\rho}(a)T\delta_e = \lambda(a)\bar{\rho}(a)f_T$$

for all  $a \in G$ . Thus  $T$  belongs to the center if and only if  $f_T = \lambda(a)\bar{\rho}(a)f_T$  for all  $a \in G$  and the desired equivalence now follows from (2.1). If a function  $f$  satisfies (ii), then  $|f|$  is constant on conjugacy classes. Since  $f_T$  belongs to  $\ell^2(G)$ , it can be nonzero only on the finite conjugacy classes.  $\square$

*Remark 2.3.* Lemma 2.2 is proved in [13, Theorem 1]. However, the proof provided above is shorter. Lemma 2.4 below is proved in [13, Lemma 2] in the case where  $C$  is a single point. Also, note that we do not restrict to normalized multipliers as in [13].

**Lemma 2.4.** *Let  $C$  be a conjugacy class of  $G$ . Then following are equivalent:*

- (i)  $C$  is  $\sigma$ -regular.
- (ii) There is a function  $f : G \rightarrow \mathbb{C}$  satisfying:

1.  $f(c) \neq 0$  for all  $c \in C$ .
2.  $f(aca^{-1}) = \sigma(a, c)\overline{\sigma(aca^{-1}, a)}f(c)$  for all  $c \in C$  and all  $a \in G$ .

Moreover,  $f$  can be chosen in  $\ell^2(G)$  if and only if  $C$  is finite.

*Proof.* (ii)  $\Rightarrow$  (i): Suppose  $c$  belongs to  $C$  and that  $a$  commutes with  $c$ . Then there is a function  $f : G \rightarrow \mathbb{C}$  satisfying  $0 \neq f(c) = \sigma(a, c)\overline{\sigma(c, a)}f(c)$ . Hence  $\sigma(a, c) = \sigma(c, a)$ , so  $c$  is  $\sigma$ -regular.

(i)  $\Rightarrow$  (ii): This clearly holds if  $C$  is trivial, so suppose  $C$  is nontrivial and  $\sigma$ -regular and fix an element  $c$  in  $C$ . Define a function  $f : G \rightarrow \mathbb{C}$  by

$$f(x) = \begin{cases} \sigma(a, c)\overline{\sigma(aca^{-1}, a)} & \text{if } x \in C, \quad x = aca^{-1} \text{ for some } a \in G \\ 0 & \text{if } x \notin C \end{cases}$$

First we show that  $f$  is well-defined, so assume  $aca^{-1} = bcb^{-1}$ , and note that

$$\begin{aligned} \sigma(a^{-1}, aca^{-1})\sigma(ca^{-1}, b) &= \sigma(a^{-1}, aca^{-1}b)\sigma(aca^{-1}, b) \\ &= \sigma(a^{-1}, bc)\sigma(bcb^{-1}, b). \end{aligned}$$

As  $c$  is  $\sigma$ -regular and commutes with  $a^{-1}b$ ,  $\sigma(a^{-1}b, c) = \sigma(c, a^{-1}b)$ . Thus

$$\begin{aligned} \sigma(c, a^{-1})\sigma(ca^{-1}, b) &= \sigma(c, a^{-1}b)\sigma(a^{-1}, b) \\ &= \sigma(a^{-1}, b)\sigma(a^{-1}b, c) = \sigma(a^{-1}, bc)\sigma(b, c). \end{aligned}$$

Together, we get from these equations that

$$\sigma(a^{-1}, aca^{-1})\sigma(b, c) = \sigma(c, a^{-1})\sigma(bcb^{-1}, b). \quad (2.3)$$

Finally, the two identities

$$\begin{aligned} \sigma(a^{-1}, aca^{-1})\sigma(ca^{-1}, a) &= \sigma(a^{-1}, ac)\sigma(aca^{-1}, a) \\ \sigma(c, a^{-1})\sigma(ca^{-1}, a) &= \sigma(a^{-1}, a) = \sigma(a^{-1}, ac)\sigma(a, c) \end{aligned}$$

give that

$$\sigma(a^{-1}, aca^{-1})\sigma(a, c) = \sigma(c, a^{-1})\sigma(aca^{-1}, a). \quad (2.4)$$

Combining (2.3) and (2.4) we get that

$$\sigma(a, c)\overline{\sigma(aca^{-1}, a)} = \sigma(b, c)\overline{\sigma(bcb^{-1}, b)}.$$

Hence  $f$  is well-defined, so  $f(aca^{-1}) = f(bcb^{-1})$ .

It is easily seen that  $|f(x)| = 1$  for all  $x$  in  $C$ . In fact, if  $f$  is any function satisfying (ii), then  $|f|$  must be constant and nonzero on  $C$ , hence  $f$  belongs to  $\ell^2(G)$  if and only if  $C$  is finite.

In particular,  $f(c) = 1$  in our case, so  $f$  satisfies part 2 of (ii) for the chosen  $c$  in  $C$ . It remains to show that  $f$  satisfies part 2 of (ii) for all other  $x$  in  $C$ . Suppose  $x$  is an element of  $C$ , that is, there exists  $b$  in  $G$  such that  $x = bcb^{-1}$ . Note first that

$$f(x) = f(bcb^{-1}) = \sigma(b, c)\overline{\sigma(bcb^{-1}, b)} = \sigma(b, c)\overline{\sigma(x, b)}.$$

Next,

$$\begin{aligned} \sigma(axa^{-1}, a)\sigma(ax, b)\sigma(ab, c) &= \sigma(axa^{-1}, ab)\sigma(a, b)\sigma(ab, c) \\ &= \sigma(axa^{-1}, ab)\sigma(a, bc)\sigma(b, c), \end{aligned}$$

which, since  $xb = bc$ , gives that

$$\begin{aligned} \sigma(a, x)\overline{\sigma(x, b)} &= \sigma(a, xb)\overline{\sigma(ax, b)} = \sigma(a, bc)\overline{\sigma(ax, b)} \\ &= \sigma(axa^{-1}, a)\sigma(ab, c)\overline{\sigma(axa^{-1}, ab)\sigma(b, c)}. \end{aligned}$$

Hence

$$\begin{aligned} f(axa^{-1}) &= f(abc b^{-1} a^{-1}) = \sigma(ab, c)\overline{\sigma(abc b^{-1} a^{-1}, ab)} \\ &= \sigma(ab, c)\overline{\sigma(axa^{-1}, ab)} = \sigma(a, x)\overline{\sigma(axa^{-1}, a)}\sigma(b, c)\overline{\sigma(x, b)} \\ &= \sigma(a, x)\overline{\sigma(axa^{-1}, a)}f(x). \end{aligned}$$

□

Before stating the main theorem, we recall two results which are part of the folklore of operator algebras. The first can be shown as sketched in the proof of [18, Proposition 2.3], while the second is a rather easy consequence of Urysohn's Lemma. Remark that together these two results imply that if  $A$  is von Neumann algebra, then  $A$  is prime if and only if it is a factor.

**Proposition 2.5.** *If  $A$  is a concrete unital  $C^*$ -algebra and its bicommutant  $A''$  is a factor, then  $A$  is prime.*

**Proposition 2.6.** *Every prime  $C^*$ -algebra has trivial center.*

**Theorem 2.7.** *The following conditions are equivalent:*

- (i)  $(G, \sigma)$  satisfies condition K.

(ii)  $W^*(G, \sigma)$  is a factor.

(iii)  $C_r^*(G, \sigma)$  is prime.

(iv)  $C_r^*(G, \sigma)$  has trivial center.

*Proof.* For completeness, we include the few lines required to prove (i)  $\Rightarrow$  (ii): Suppose  $(G, \sigma)$  satisfies condition K and let  $T$  belong to the center of  $W^*(G, \sigma)$ . By Lemma 2.2 and Lemma 2.4,  $f_T$  can be nonzero only on the finite  $\sigma$ -regular conjugacy classes, hence on  $\{e\}$ . So  $T\delta_e = f_T(e)\delta_e$ , thus  $T = f_T(e)I$  as  $\delta_e$  is separating for  $W^*(G, \sigma)$  by Remark 2.1.

The implications (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) follow from Proposition 2.5 and 2.6.

(iv)  $\Rightarrow$  (i): Suppose  $C$  is a finite nontrivial  $\sigma$ -regular conjugacy class of  $G$ . Let  $f$  be a function satisfying part (ii) of Lemma 2.4 and define the operator  $T = \sum_{c \in C} f(c)\lambda(c)$ . Then  $T$  belongs to the center of  $C_r^*(G, \sigma)$ . Indeed,

$$\begin{aligned} \lambda(a)T\lambda(a)^* &= \sum_{c \in C} f(c)\lambda(a)\lambda(c)\lambda(a)^* \\ &= \sum_{c \in C} f(c)\sigma(a, c)\overline{\sigma(aca^{-1}, a)}\lambda(aca^{-1}) \\ &= \sum_{b \in aCa^{-1}} f(a^{-1}ba)\sigma(a, a^{-1}ba)\overline{\sigma(b, a)}\lambda(b) \\ &= \sum_{b \in C} f(a^{-1}ba)\overline{\sigma(a^{-1}, b)}\sigma(a^{-1}ba, a^{-1})\lambda(b) \\ &= \sum_{b \in C} f(b)\lambda(b) = T \end{aligned}$$

for all  $a \in G$ , where the identity (2.4) is used to get the fourth equality.  $\square$

The proof of the following corollary goes along the same lines as the one given in [18, Proposition 2.1] in the untwisted case.

**Corollary 2.8.** *If  $C^*(G, \sigma)$  is prime, then  $(G, \sigma)$  satisfies condition K.*

*Proof.* Observe that the set  $\{\lambda(a)\}_{a \in G}$  is linear independent in  $C_r^*(G, \sigma)$ , and the universal property of  $C^*(G, \sigma)$  ensures that there is a surjective  $*$ -homomorphism  $C^*(G, \sigma) \rightarrow C_r^*(G, \sigma)$  mapping  $i_G(a)$  to  $\lambda(a)$ . Hence,  $\{i_G(a)\}_{a \in G}$  is also linear independent and has dense span in  $C^*(G, \sigma)$ .

Therefore, the result follows by replacing  $i_G$  with  $\lambda$  in the proof of Theorem 2.7, and repeating the argument for (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i) word by word.  $\square$



*Remark 2.9.* In general, the center of  $C^*(G, \sigma)$  is not easily determined.

However, a slightly stronger version of Corollary 2.8 is known in the untwisted case. If  $C^*(G)$  has trivial center, then  $G/H$  is icc whenever  $H$  is a normal subgroup of  $G$  satisfying Kazhdan's property T (see e.g. [14]).

**Corollary 2.10** ([21, Proposition 1.4]). *Assume  $G$  is countable and amenable. Then the following conditions are equivalent:*

- (i)  $(G, \sigma)$  satisfies condition K.
- (ii)  $C^*(G, \sigma)$  is primitive.

*Proof.* If  $(G, \sigma)$  satisfies condition K, then  $C_r^*(G, \sigma)$  is prime by Theorem 2.7. As  $G$  is countable,  $C_r^*(G, \sigma)$  is separable and hence primitive by Dixmier's result. Now, the amenability of  $G$  implies that  $C^*(G, \sigma) \cong C_r^*(G, \sigma)$ , so  $C^*(G, \sigma)$  is also primitive. Finally, (ii) always implies (i) by Corollary 2.8.  $\square$

*Remark 2.11.* Condition K on  $(G, \sigma)$  does not imply primeness or primitivity of  $C^*(G, \sigma)$  in general. To see this, let  $G = \text{SL}(3, \mathbb{Z})$  and  $\sigma = 1$ . Then,  $G$  is countable, icc and satisfies Kazhdan's property T. In particular,  $G$  is nonamenable. As explained in [4, Proposition 2.5],  $C^*(G)$  is not primitive.

On the other hand, I don't know any example of an uncountable and amenable group such that (i) holds, but not (ii).

*Remark 2.12.* If  $G$  is countable and nilpotent, then condition K on  $(G, \sigma)$  is actually equivalent to simplicity of  $C^*(G, \sigma)$  [21, Proposition 1.7]. The same is also true if  $G$  is finite.

However, this does not hold for all countable, amenable groups. For example, if  $G$  is the group of all finite permutations on a countably infinite set, then  $G$  is countable, amenable and icc, so  $C^*(G)$  is primitive and nonsimple.

*Remark 2.13.* Note that  $C_r^*(\text{SL}(3, \mathbb{Z}))$  is known to be simple [5], so Remark 2.11 and 2.12 show that primitivity of a full twisted group  $C^*$ -algebra is in general unrelated to simplicity of the corresponding reduced twisted group  $C^*$ -algebra.

**Proposition 2.14.** *The following conditions are equivalent:*

- (i)  $G$  is amenable.
- (ii)  $C^*(G, \sigma)$  is nuclear.
- (iii)  $C_r^*(G, \sigma)$  is nuclear.
- (iv)  $W^*(G, \sigma)$  is injective.

*Proof.* This is well known in the untwisted case. The result in the twisted case appeared in a preprint by Bédos and Conti [2], but was left out in the final version. For the convenience of the reader we repeat the argument. First, (i)  $\Rightarrow$  (ii) follows from [22, Corollary 3.9]. The implication (ii)  $\Rightarrow$  (iii) holds since every quotient of a nuclear  $C^*$ -algebra is nuclear. Moreover, the von Neumann algebra generated by a nuclear  $C^*$ -algebra is injective, hence (iii)  $\Rightarrow$  (iv). Finally, if  $W^*(G, \sigma)$  is injective, it has a hypertrace and thus  $G$  is amenable by [1, Corollary 1.7], so (iv)  $\Rightarrow$  (i).  $\square$

According to [8], all injective  $\text{II}_1$  factors acting on a separable Hilbert space are isomorphic to the hyperfinite  $\text{II}_1$  factor. Hence, we get the following corollary.

**Corollary 2.15.** *Assume  $G$  is countably infinite. Then the following conditions are equivalent:*

- (i)  $G$  is amenable and  $(G, \sigma)$  satisfies condition  $K$ .
- (ii)  $C^*(G, \sigma)$  is nuclear and primitive.
- (iii)  $C_r^*(G, \sigma)$  is nuclear and primitive.
- (iv)  $W^*(G, \sigma)$  is the hyperfinite  $\text{II}_1$  factor.

### 3 Direct products

Let  $G_1$  and  $G_2$  be two groups. A function  $f : G_1 \times G_2 \rightarrow \mathbb{T}$  is called a *bihomomorphism* if

$$f(a_1 b_1, a_2) = f(a_1, a_2) f(b_1, a_2) \quad \text{and} \quad f(a_1, a_2 b_2) = f(a_1, a_2) f(a_1, b_2)$$

for all  $a_1, b_1 \in G_1$  and  $a_2, b_2 \in G_2$ . Let  $B(G_1, G_2)$  denote the set of bihomomorphisms  $G_1 \times G_2 \rightarrow \mathbb{T}$ . This is a group under pointwise multiplication and is isomorphic with  $\text{Hom}(G_1, \text{Hom}(G_2, \mathbb{T}))$ .

It is well known (see e.g. [15]) that the Schur multiplier of  $G_1 \times G_2$  decomposes as

$$\mathcal{M}(G_1 \times G_2) \cong \mathcal{M}(G_1) \oplus \mathcal{M}(G_2) \oplus B(G_1, G_2).$$

We will only need to know the following. Let  $(\sigma_1, \sigma_2, f)$  be a triple where  $\sigma_1$  and  $\sigma_2$  are multipliers on  $G_1$  and  $G_2$ , respectively, and  $f$  belongs to  $B(G_1, G_2)$ . Then we can define a multiplier  $\sigma$  on  $G_1 \times G_2$  by

$$\sigma((a_1, a_2), (b_1, b_2)) = \sigma_1(a_1, b_1) \sigma_2(a_2, b_2) f(b_1, a_2) \quad (3.1)$$

for  $a_1, b_1 \in G_1$  and  $a_2, b_2 \in G_2$ , and it can be shown that every multiplier on  $G_1 \times G_2$  is similar to such a  $\sigma$ . When  $\sigma$  is a multiplier on  $G_1 \times G_2$ , we let  $\sigma_1$  be the multiplier on  $G_1$  defined by

$$\sigma_1(a_1, b_1) = \sigma((a_1, e), (b_1, e))$$

for  $a_1, b_1 \in G_1$  and call it the restriction of  $\sigma$  to  $G_1$ . Similarly we can define the restriction  $\sigma_2$  of  $\sigma$  to  $G_2$ .

Henceforth, we fix two groups  $G_1$  and  $G_2$ , multipliers  $\sigma_1$  on  $G_1$  and  $\sigma_2$  on  $G_2$ , and a bihomomorphism  $f$  in  $B(G_1, G_2)$ . We set  $G = G_1 \times G_2$  and let  $\sigma$  be the multiplier on  $G$  defined by (3.1). Moreover, we write  $\sigma = \sigma_1 \times \sigma_2$  if  $f = 1$ .

It is convenient to record the identity

$$\sigma(a, b)\overline{\sigma(b, a)} \cdot f(a_1, b_2)\overline{f(b_1, a_2)} = \sigma_1(a_1, b_1)\overline{\sigma_1(b_1, a_1)} \cdot \sigma_2(a_2, b_2)\overline{\sigma_2(b_2, a_2)} \quad (3.2)$$

which follows directly from (3.1). Note also that  $C$  is a conjugacy class of  $G$  if and only if  $C = C_1 \times C_2$  where  $C_1$  and  $C_2$  are conjugacy classes of  $G_1$  and  $G_2$ , respectively.

**Proposition 3.1.** *The following are equivalent:*

- (i)  $C_r^*(G, \sigma)$  is prime.
- (ii) For every finite nontrivial conjugacy class  $C$  of  $G$ , there exist  $a = (a_1, a_2)$  in  $C$  and  $b = (b_1, b_2)$  in  $G$  such that at least one of these conditions hold:

1.  $a_1b_1 = b_1a_1$  and  $f(b_1, a_2) \neq \overline{\sigma_1(a_1, b_1)}\sigma_1(b_1, a_1)$ .
2.  $a_2b_2 = b_2a_2$  and  $f(a_1, b_2) \neq \sigma_2(a_2, b_2)\overline{\sigma_2(b_2, a_2)}$ .

*Proof.* Suppose that condition (ii) does not hold. Then there is a finite nontrivial conjugacy class  $C$  such that both 1. and 2. fail for all  $a$  in  $C$  and  $b$  in  $G$ . Hence,  $f(b_1, a_2) = \overline{\sigma_1(a_1, b_1)}\sigma_1(b_1, a_1)$  and  $f(a_1, b_2) = \sigma_2(a_2, b_2)\overline{\sigma_2(b_2, a_2)}$  whenever  $a = (a_1, a_2)$  is in  $C$ ,  $b = (b_1, b_2)$  in  $G$  and  $b$  commutes with  $a$ . Then  $C$  is  $\sigma$ -regular by (3.2), and therefore  $(G, \sigma)$  does not satisfy condition K, that is,  $C_r^*(G, \sigma)$  is not prime by Theorem 2.7. Thus, (i)  $\Rightarrow$  (ii).

Conversely, assume that  $(G, \sigma)$  does not satisfy condition K and let  $C = C_1 \times C_2$  be a finite nontrivial  $\sigma$ -regular conjugacy class of  $G$ . If  $b_1$  in  $G_1$  commutes with  $a_1$  in  $C_1$ , then  $(b_1, e)$  commutes with  $(a_1, a_2)$  for every  $a_2$  in  $C_2$ . Hence, the  $\sigma$ -regularity of  $C$  and identity (3.2) give that

$$f(b_1, a_2) = \overline{\sigma_1(a_1, b_1)}\sigma_1(b_1, a_1)$$

whenever  $a$  belongs to  $C$  and  $b_1$  in  $G_1$  commutes with  $a_1$ . Similarly,

$$f(a_1, b_2) = \sigma_2(a_2, b_2)\overline{\sigma_2(b_2, a_2)}$$

whenever  $b_2$  in  $G_2$  commutes with  $a_2$ . It follows that for all  $a$  in  $C$  and  $b$  in  $G$ , both 1. and 2. fail to hold, hence condition (ii) is not satisfied.  $\square$

*Remark 3.2.* Let  $G_1$  and  $G_2$  be abelian and assume that  $\sigma_1$  and  $\sigma_2$  are trivial. Condition (ii) of Proposition 3.1 then says that for all nontrivial  $(a_1, a_2)$  in  $G$  there exists  $(b_1, b_2)$  in  $G$  such that  $f(a_1, b_2) \neq 1$  or  $f(b_1, a_2) \neq 1$ . If this holds,  $\sigma$  is called nondegenerate and it was first shown by Slawny [23, Theorem 3.7] that  $C^*(G, \sigma) \cong C_r^*(G, \sigma)$  is simple in this case.

**Lemma 3.3.** *Let  $a = (a_1, a_2)$  be an element in  $G$ . If two of the following conditions hold, then all three hold:*

- (i)  $a$  is  $\sigma$ -regular.
- (ii)  $a_i$  is  $\sigma_i$ -regular for both  $i = 1$  and  $2$ .
- (iii)  $f(a_1, b_2) = f(b_1, a_2)$  whenever  $b = (b_1, b_2)$  commutes with  $a$ .

Moreover, (iii) is equivalent to:

- (iv)  $f(a_1, b_2) = f(b_1, a_2) = 1$  whenever  $b = (b_1, b_2)$  commutes with  $a$ .

*Proof.* Suppose that (ii) holds and pick any  $b = (b_1, b_2)$  in  $G$ . Then it follows readily from (3.2) that (i) holds if and only if (iii) holds.

Next, assume that (iii) holds and let  $b = (b_1, b_2)$  commute with  $a$ . Then  $b' = (b_1, e)$  also commutes with  $a$ , so  $1 = f(a_1, e) = f(b_1, a_2)$ . Similarly, we get  $f(a_1, b_2) = 1$  and thus (iv) holds.

Suppose finally that (i) and (iii) hold and pick an element  $b = (b_1, b_2)$  in  $G$  that commutes with  $a$ . As (iv) also holds, we have that  $f(b_1, a_2) = 1$ . By applying (3.2) with  $b' = (b_1, e)$ , we see that  $a_1$  is  $\sigma_1$ -regular. Similarly,  $f(a_1, b_2) = 1$  and  $a_2$  is  $\sigma_2$ -regular.  $\square$

**Corollary 3.4.** *Let  $C = C_1 \times C_2$  be a conjugacy class of  $G$ . Suppose there is some  $a = (a_1, a_2)$  in  $C$  such that  $f(a_1, b_2) = f(b_1, a_2)$  whenever  $b = (b_1, b_2)$  commutes with  $a$ . Then the following are equivalent:*

- (i)  $C$  is a finite nontrivial  $\sigma$ -regular conjugacy class of  $G$ .
- (ii)  $C_i$  is a finite  $\sigma_i$ -regular conjugacy class of  $G_i$  for both  $i = 1$  and  $2$  and at least one of  $C_1$  and  $C_2$  is nontrivial.

**Corollary 3.5.** *Suppose both  $C_r^*(G_1, \sigma_1)$  and  $C_r^*(G_2, \sigma_2)$  are prime. Let  $a = (a_1, a_2)$  be such that  $f(a_1, b_2) = f(b_1, a_2)$  whenever  $b = (b_1, b_2)$  commutes with  $a$ . Then at most one of the following two conditions hold:*

- (i)  $a$  is  $\sigma$ -regular.
- (ii)  $a$  belongs to a finite nontrivial conjugacy class of  $G$ .

**Corollary 3.6.** *Suppose  $f(a_1, b_2) = f(b_1, a_2)$  whenever  $a = (a_1, a_2)$  is  $\sigma$ -regular and  $b = (b_1, b_2)$  commutes with  $a$ . Then  $C_r^*(G, \sigma)$  is prime if both  $C_r^*(G_1, \sigma_1)$  and  $C_r^*(G_2, \sigma_2)$  are prime.*

*Remark 3.7.* In general, primeness of  $C_r^*(G, \sigma)$  does not imply primeness of either  $C_r^*(G_1, \sigma_1)$  or  $C_r^*(G_2, \sigma_2)$ . For example, if  $G_1 = G_2 = \mathbb{Z}$ , then  $C_r^*(G, \sigma)$  can be simple even if both  $\sigma_1$  and  $\sigma_2$  are trivial.

Also,  $C_r^*(G, \sigma)$  can be nonprime even if both  $C_r^*(G_1, \sigma_1)$  and  $C_r^*(G_2, \sigma_2)$  are simple. To see this, let  $G_1 = G_2 = \mathbb{Z}^2$  and consider the case described in the last part of Example 1.2.

**Proposition 3.8.** *Suppose  $f(c_1, c_2) = 1$  whenever  $c_i$  belongs to a finite conjugacy class of  $G_i$  for either  $i = 1$  or  $2$ . Then  $C_r^*(G, \sigma)$  is prime if and only if both  $C_r^*(G_1, \sigma_1)$  and  $C_r^*(G_2, \sigma_2)$  are prime.*

*In particular, this holds when  $\sigma = \sigma_1 \times \sigma_2$ .*

*Proof.* Suppose  $C_r^*(G, \sigma)$  is prime and  $C_1$  is a finite  $\sigma_1$ -regular conjugacy class of  $G_1$ . Then  $C_1 \times \{e\}$  is  $\sigma$ -regular by Corollary 3.4 so  $C_1 = \{e\}$  and hence  $C_r^*(G_1, \sigma_1)$  is prime. Similarly we get that  $C_r^*(G_2, \sigma_2)$  is prime.

The converse follows directly from Corollary 3.5.  $\square$

*Remark 3.9.* Assume that  $\sigma = \sigma_1 \times \sigma_2$ . Then  $C_r^*(G, \sigma)$  is simple if and only if both  $C_r^*(G_1, \sigma_1)$  and  $C_r^*(G_2, \sigma_2)$  are simple. Indeed, note that the map  $\lambda_\sigma(a_1, a_2) \mapsto \lambda_{\sigma_1}(a_1) \otimes \lambda_{\sigma_2}(a_2)$  induces an isomorphism

$$C_r^*(G, \sigma) \cong C_r^*(G_1, \sigma_1) \otimes_{\min} C_r^*(G_2, \sigma_2).$$

The result now follows from the fact that a spatial tensor product of two  $C^*$ -algebras is simple if and only if both involved  $C^*$ -algebras are simple (see [12, 11.5.5-6]).

The only positive result on primitivity so far in this paper concerns countable, amenable groups. However, Corollary 2.10 relies on Dixmier's result that is not constructive in the sense that it does not give a procedure to construct an explicit faithful irreducible representation.

In some cases, one may construct faithful irreducible representations of  $C^*(G, \sigma)$  through an inducing process on representations of  $C^*(G_1, \sigma_1)$ .

**Theorem 3.10.** *Assume that  $G_2$  is amenable. Suppose there exists a faithful irreducible representation  $\pi$  of  $C^*(G_1, \sigma_1)$  such that for any given nontrivial  $a_2$  in  $G_2$ , there exists  $a_1$  in  $G_1$  such that*

$$f(a_1, a_2)\pi(i_{G_1}(a_1)) \neq \pi(i_{G_1}(a_1)).$$

*Then  $C^*(G, \sigma)$  is primitive.*

*Proof.* Recall that there is a twisted action  $(\alpha, \omega)$  of  $G_2$  on  $A = C^*(G_1, \sigma_1)$  satisfying (see e.g. [24])

$$\begin{aligned}\alpha_{a_2}(i_{G_1}(a_1)) &= \overline{f(a_1, a_2)} i_{G_1}(a_1), \\ \omega(a_2, b_2) &= \sigma_2(a_2, b_2).\end{aligned}$$

Hence, there is also a natural action of  $G_2$  on the set  $\widehat{A}^0$  of equivalence classes of faithful irreducible representations of  $A$  given by

$$a_2 \cdot [\psi] = [\psi \circ \alpha_{a_2^{-1}}].$$

For any given nontrivial  $a_2$  in  $G_2$ , the assumptions on  $\pi$  gives that

$$\pi \circ \alpha_{a_2^{-1}}(i_{G_1}(a_1)) = f(a_1, a_2) \pi(i_{G_1}(a_1)) \neq \pi(i_{G_1}(a_1))$$

for some  $a_1$  in  $G_1$ . Hence

$$a_2 \cdot [\pi] \neq [\pi]$$

for all nontrivial  $a_2$  in  $G_2$ . In other words,  $[\pi]$  is a free point for this action. The conclusion follows from [4, Theorem 2.1].  $\square$

**Example 3.11.** Let  $G = \mathbb{F}_2 \times \mathbb{Z}$  and let  $u, v$  be the generators of  $\mathbb{F}_2$ . Since  $\mathcal{M}(\mathbb{F}_2) = \mathcal{M}(\mathbb{Z}) = \{1\}$ , every multiplier on  $G$  is, up to similarity, determined by a bichomomorphism  $f : \mathbb{F}_2 \times \mathbb{Z} \rightarrow \mathbb{T}$ . Moreover,  $f$  is determined by its values on the generators, that is, by  $f(u, 1)$  and  $f(v, 1)$ . Let  $\sigma$  be the multiplier on  $G$  defined by these two numbers, say  $\mu$  and  $\nu$ . We remark that

$$C^*(G, \sigma) \cong C^*(\mathbb{F}_2) \rtimes_{\alpha} \mathbb{Z}$$

where  $\alpha$  is determined by  $\alpha_k(i_{\mathbb{F}_2}(x)) = \overline{f(x, k)} i_{\mathbb{F}_2}(x)$  for  $x \in \mathbb{F}_2$  and  $k \in \mathbb{Z}$ .

Assume  $\mu$  is nontorsion and let  $A = C^*(\mathbb{F}_2)$  sit inside  $B(\mathcal{H})$  for some separable Hilbert space  $\mathcal{H}$ . Let  $U = i_{\mathbb{F}_2}(u)$  and  $V = i_{\mathbb{F}_2}(v)$  be the two unitaries in  $B(\mathcal{H})$  generating  $A$ . Now, proceeding as Choi in [7, Lemma 4], there is an operator  $D$  for which  $U - D$  is compact and such that the following hold; with respect to a suitable basis on  $\mathcal{H}$ ,  $D$  is diagonal with diagonal entries  $\{z_i\}_{i=1}^{\infty}$  satisfying  $|z_i| = 1$  for all  $i$ ,  $z_1 = 1$ ,  $z_i \neq z_j$  if  $i \neq j$  and  $z_i \notin \{\mu^k : k \in \mathbb{Z}\}$  when  $i \geq 2$ .

Using [7, Lemma 5], we can find a compact perturbation  $E$  of  $V$  which is a unitary operator having no common nontrivial invariant subspace with  $D$ . Then, as explained in [7, Theorem 6], the map  $U \mapsto D$ ,  $V \mapsto E$  defines a faithful and irreducible representation  $\pi$  of  $A$  on  $\mathcal{H}$ .

Now we have

$$\pi \circ \alpha_{k^{-1}}(U) = f(u, k) \pi(U) = \mu^k \pi(U) \neq \pi(U)$$

for all  $k$  in  $\mathbb{Z}$ . Indeed, this holds as the point spectrum of  $\pi(U) = D$  is different from the point spectrum of  $\pi(\alpha_{k-1}(U)) = \mu^k D$  by construction.

A similar argument also holds if  $\nu$  is nontorsion. Hence, we get from Theorem 3.10 that  $C^*(G, \sigma)$  is primitive if either  $\mu$  or  $\nu$  is nontorsion.

On the other hand, if  $(G, \sigma)$  satisfies condition K, then at least one of  $\mu$  and  $\nu$  must be nontorsion, so this is also a necessary condition for primitivity of  $C^*(G, \sigma)$ . Indeed, condition (ii) of Proposition 3.1 does not hold if both  $\mu$  and  $\nu$  are torsion.

**Proposition 3.12.** *Assume that  $\sigma = \sigma_1 \times \sigma_2$  and that both  $C^*(G_1, \sigma_1)$  and  $C^*(G_2, \sigma_2)$  are primitive. Then  $C^*(G, \sigma)$  is primitive if at least one of  $G_1$  and  $G_2$  is amenable.*

*Proof.* Without loss of generality we may assume that  $G_1$  is amenable. Then  $C^*(G_1, \sigma_1)$  is nuclear by Proposition 2.14 so the minimal and maximal tensor products of  $C^*(G_1, \sigma_1)$  and  $C^*(G_2, \sigma_2)$  coincide. According to [11, Section 3], there is a unique isomorphism

$$C^*(G, \sigma) \rightarrow C^*(G_1, \sigma_1) \otimes C^*(G_2, \sigma_2)$$

given by  $i_G(a_1, a_2) \mapsto i_{G_1}(a_1) \otimes i_{G_2}(a_2)$ .

For  $i = 1, 2$ , let  $\pi_i$  be a faithful irreducible representation of  $C^*(G_i, \sigma_i)$  on  $\mathcal{H}_i$ . Then  $\pi = \pi_1 \otimes \pi_2$  is a representation of  $C^*(G, \sigma)$  on  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ , which is faithful by [17, Theorem 6.5.1] and irreducible by [11, Section 2]. Hence  $C^*(G, \sigma)$  is primitive.  $\square$

*Remark 3.13.* Primitivity of  $C^*(G, \sigma)$  is in general difficult to decide. For example, let  $\mathbb{F}$  be a free nonabelian group, then it is unknown whether  $C^*(\mathbb{F} \times \mathbb{F})$  is primitive (see [4, Remark 2.2] for a brief discussion).

## 4 Free products

In some sense, free products are easier to treat than direct products, since the Schur multiplier decomposes nicely. Indeed, let  $G_1$  and  $G_2$  be two groups. Then we have that (see e.g. [6, page 51])

$$\mathcal{M}(G_1 * G_2) \cong \mathcal{M}(G_1) \oplus \mathcal{M}(G_2). \quad (4.1)$$

Let  $\sigma_1$  be a normalized multiplier on  $G_1$  and  $\sigma_2$  a normalized multiplier on  $G_2$ . Following [16, Section 5], we will explain how to obtain a normalized free product multiplier  $\sigma_1 * \sigma_2$  on  $G_1 * G_2$ .

Every nontrivial element  $x$  in  $G_1 * G_2$  can be uniquely written as a reduced word  $x = x_1 x_2 \cdots x_n$ , for which the letters with odd index belong to  $G_i$  and the letters with even index belong to  $G_j$  for  $i \neq j$ . Define the length function as

$l(x) = l(x_1x_2 \cdots x_n) = n$  and  $l(e) = 0$ . If  $l(x), l(y) \leq 1$ , we write  $x \perp y$  if either  $x = e$  or  $y = e$  or else if  $x$  is in  $G_i$  and  $y$  is in  $G_j$  for  $i \neq j$ .

Let  $s(x)$  and  $r(x)$  denote the first and last letter of a nontrivial word  $x$  and set  $s(e) = r(e) = e$ . For a pair of words  $(x, y)$ , we say that the pair is reduced if  $r(x) \neq s(y)^{-1}$ .

When  $(x, y)$  is not reduced, let  $w$  be the longest word such that  $r(xw^{-1}) \perp s(w)$  and  $r(w^{-1}) \perp s(wy)$ . Set  $x_w = xw^{-1}$  and  $y_w = wy$ , so that  $x = x_w w$  and  $y = w^{-1}y_w$ . Let  $(x, y)_w = (x_w, y_w)$  be the reduction of  $(x, y)$  and note in particular that  $x_w y_w = xy$ .

If the pair  $(x, y)$  is reduced, then we set  $(x, y)_w = (x, y)$ .

Define now the multiplier  $\tau$  on  $G_1 * G_2$  by

$$\tau(x, y) = \tau((x, y)_w) = \begin{cases} \sigma_1(r(x_w), s(y_w)) & \text{if } r(x_w), s(y_w) \in G_1 \setminus \{e\}, \\ \sigma_2(r(x_w), s(y_w)) & \text{if } r(x_w), s(y_w) \in G_2 \setminus \{e\}, \\ 1 & \text{if } r(x_w) \perp s(y_w), \end{cases}$$

and note that this definition coincides with the one explained in [16, Section 5]. Furthermore, let

$$X = \{[a, b] = aba^{-1}b^{-1} : a \in G_1 \setminus \{e\}, b \in G_2 \setminus \{e\}\}$$

and recall that the free nonabelian group on  $X$ , denoted  $\mathbb{F}_X$ , may be identified with the normal subgroup of  $G_1 * G_2$  generated by  $X$ .

Moreover, define a function  $\beta : G_1 * G_2 \rightarrow \mathbb{T}$  by  $\beta(x) = 1$  if  $x \notin \mathbb{F}_X$ , while for nontrivial  $x = q_1^{p_1} \cdots q_n^{p_n}$  in  $\mathbb{F}_X$ , where  $q_i$  belongs to  $X$  and  $p_i$  is an integer, we set

$$\beta(x) = \beta(q_1^{p_1} \cdots q_n^{p_n}) = \begin{cases} \tau(q_1^{p_1}, q_2^{p_2}) \tau(q_2^{p_2}, q_3^{p_3}) \cdots \tau(q_{n-1}^{p_{n-1}}, q_n^{p_n}) & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases}$$

Now define the multiplier  $\sigma$  on  $G_1 * G_2$  by

$$\sigma(x, y) = \beta(x)\beta(y)\overline{\beta(xy)}\tau(x, y).$$

We write  $\sigma = \sigma_1 * \sigma_2$  and note that  $\sigma \sim \tau$ ,  $\sigma|_{G_i \times G_i} = \sigma_i$  and  $\sigma|_{\mathbb{F}_X \times \mathbb{F}_X} = 1$ .

On the other hand, if  $\sigma$  is a normalized multiplier on  $G_1 * G_2$ , we can define the restriction  $\sigma_1$  on  $G_1$  by

$$\sigma_1(x, y) = \begin{cases} \sigma(x, y) & \text{if } x, y \in G_1 \setminus \{e\}, \\ 1 & \text{if } x \text{ or } y = e. \end{cases}$$

Similarly, we can define the restriction  $\sigma_2$  of  $\sigma$  to  $G_2$ . Next, define the function  $\beta : G_1 * G_2 \rightarrow \mathbb{T}$  by  $\beta(x) = 1$  if  $l(x) \leq 1$  and else

$$\beta(x) = \beta(x_1 \cdots x_n) = \sigma(x_1, x_2)\sigma(x_1x_2, x_3) \cdots \sigma(x_1 \cdots x_{n-1}, x_n).$$



Then  $\sigma$  is similar to  $\sigma_1 * \sigma_2$  through  $\beta$ .

Remark that every multiplier is similar to a normalized one. Therefore, every multiplier on  $G_1 * G_2$  is similar to  $\sigma_1 * \sigma_2$  for some normalized multipliers  $\sigma_1$  on  $G_1$  and  $\sigma_2$  on  $G_2$ .

We are now ready to prove the twisted version of [3, Theorem 1.2].

**Theorem 4.1.** *Assume  $G = G_1 * G_2$ , where  $G_1$  and  $G_2$  are countable and amenable and  $(|G_1| - 1)(|G_2| - 1) \geq 2$ , and let  $\sigma$  be a multiplier on  $G$ . Then  $C^*(G, \sigma)$  is primitive.*

*Proof.* We may assume that  $\sigma = \sigma_1 * \sigma_2$  where  $\sigma_1$  and  $\sigma_2$  are normalized multipliers on  $G_1$  and  $G_2$ , respectively, and that  $\sigma|_{\mathbb{F}_X \times \mathbb{F}_X} = 1$ . The proof is only a slight modification of the proof of [3, Theorem 1.2], so we just point out what needs to be adjusted in this proof and use the notation therein. First, recall that there is a twisted action  $(\alpha, \omega)$  of  $(G_1 * G_2)/\mathbb{F}_X \cong G_1 \times G_2$  on  $H = \mathbb{F}_X$ . Straightforward calculations give that

$$\alpha_{(c,d)}(i_H([a, b])) = \begin{cases} i_H(cd[a, b]d^{-1}c^{-1}) \cdot \sigma_2(d, b) & \text{if } d \neq e \\ i_H(cd[a, b]d^{-1}c^{-1}) \cdot \sigma_1(c, a) & \text{if } d = e \end{cases}$$

for  $a, c \in G_1$  and  $b, d \in G_2$ . Hence the expressions in the equations [3, (2.3),(2.4)] remain unchanged, so it is enough to reconsider [3, Case 3]. More straightforward calculations give that the conditions at the bottom of [3, page 54] must be replaced with:

$$\begin{aligned} &k = (s_0, t) \text{ and } k = (s_l, e_2) \text{ if} \\ &\quad \lambda(s_0s_l, t)U(s_0s_l, t) \not\cong \sigma_1(s_l, s_0s_l)U(s_0, t)(\lambda(s_l, t)U(s_l, t))^* ; \\ &k = (s_0, e_2) \text{ and } k = (s_l, t) \text{ if} \\ &\quad \lambda(s_0s_l, t)U(s_0s_l, t) \not\cong \sigma_1(s_0, s_0s_l)\lambda(s_l, t)U(s_l, t)U(s_0, t)^* ; \\ &k = (s_0, t) \text{ and } k = (s_0s_l, e_2) \text{ if} \\ &\quad \lambda(s_l, t)U(s_l, t) \not\cong \sigma_1(s_0s_l, s_l)U(s_0, t)(\lambda(s_0s_l, t)U(s_0s_l, t))^* ; \\ &k = (s_0s_l, t) \text{ and } k = (s_0, e_2) \text{ if} \\ &\quad \lambda(s_l, t)U(s_l, t) \not\cong \sigma_1(s_0, s_l)\lambda(s_0s_l, t)U(s_0s_l, t)U(s_0, t)^* . \end{aligned}$$

Now it is easily seen that the rest of the proof works with appropriate modifications.  $\square$

*Remark 4.2.* Theorem 4.1 is not surprising. In fact, I am not aware of any pair  $(G, \sigma)$  such that  $C^*(G)$  is primitive, but  $C^*(G, \sigma)$  is nonprimitive.

*Remark 4.3.* Let  $G = G_1 * G_2$ , let  $\sigma$  be a multiplier on  $G$  and assume  $\sigma = \sigma_1 * \sigma_2$ . Then it is known that (see [16, Section 5])

$$C^*(G, \sigma) = C^*(G_1, \sigma_1) * C^*(G_2, \sigma_2).$$

**Example 4.4.** As explained in Example 1.1 we have that for each natural number  $n$ , there exists a multiplier  $\sigma_k$  on  $\mathbb{Z}_n \times \mathbb{Z}_n$  such that  $C^*(\mathbb{Z}_n \times \mathbb{Z}_n, \sigma_k) \cong M_n(\mathbb{C})$ . One immediate consequence of Theorem 4.1 is that

$$M_j(\mathbb{C}) * M_k(\mathbb{C})$$

is primitive for all  $j, k \geq 2$ . More generally, it has recently been shown [10] that  $F_1 * F_2$  is primitive whenever  $F_1$  and  $F_2$  are finite-dimensional  $C^*$ -algebras and  $(\dim F_1 - 1)(\dim F_2 - 1) \geq 2$ .

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## References

- [1] E. Bédos. Notes on hypertraces and  $C^*$ -algebras. *J. Operator Theory*, 34(2):285–306, 1995.
- [2] E. Bédos and R. Conti. On twisted Fourier analysis and convergence of Fourier series on discrete groups. *J. Fourier Anal. Appl.*, 15(3):336–365, 2009.
- [3] E. Bédos and T. Omland. Primitivity of some full group  $C^*$ -algebras. *Banach J. Math. Anal.*, 5(2):44–58, 2011.
- [4] E. Bédos and T. Omland. The full group  $C^*$ -algebra of the modular group is primitive. *Proc. Amer. Math. Soc.*, 140(4):1403–1411, 2012.
- [5] M. E. B. Bekka, M. G. Cowling, and P. de la Harpe. Simplicity of the reduced  $C^*$ -algebra of  $\mathrm{PSL}(n, \mathbb{Z})$ . *Internat. Math. Res. Notices*, (7):285ff., approx. 7 pp. (electronic), 1994.
- [6] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.
- [7] M. D. Choi. The full  $C^*$ -algebra of the free group on two generators. *Pacific J. Math.*, 87(1):41–48, 1980.
- [8] A. Connes. Classification of injective factors. Cases  $\mathrm{II}_1$ ,  $\mathrm{II}_\infty$ ,  $\mathrm{III}_\lambda$ ,  $\lambda \neq 1$ . *Ann. of Math. (2)*, 104(1):73–115, 1976.

- 
- [9] J. Dixmier. Sur les  $C^*$ -algèbres. *Bull. Soc. Math. France*, 88:95–112, 1960.
- [10] K. J. Dykema and F. Torres-Ayala. Primitivity of unital full free products of residually finite dimensional  $C^*$ -algebras. arXiv:1201.4892, Jun 2012.
- [11] A. Guichardet. *Tensor products of  $C^*$ -algebras: Finite tensor products*, volume 12 of *Lecture notes series*. Aarhus Universitet, Matematisk Institut, 1969.
- [12] R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. II*, volume 100 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.
- [13] A. Kleppner. The structure of some induced representations. *Duke Math. J.*, 29:555–572, 1962.
- [14] V. Losert. On the center of group  $C^*$ -algebras. *J. Reine Angew. Math.*, 554:105–138, 2003.
- [15] G. W. Mackey. Unitary representations of group extensions. I. *Acta Math.*, 99:265–311, 1958.
- [16] K. McClanahan.  $KK$ -groups of twisted crossed products by groups acting on trees. *Pacific J. Math.*, 174(2):471–495, 1996.
- [17] G. J. Murphy.  *$C^*$ -algebras and operator theory*. Academic Press Inc., Boston, MA, 1990.
- [18] G. J. Murphy. Primitivity conditions for full group  $C^*$ -algebras. *Bull. London Math. Soc.*, 35(5):697–705, 2003.
- [19] F. J. Murray and J. von Neumann. On rings of operators. IV. *Ann. of Math. (2)*, 44:716–808, 1943.
- [20] J. A. Packer.  $C^*$ -algebras generated by projective representations of the discrete Heisenberg group. *J. Operator Theory*, 18(1):41–66, 1987.
- [21] J. A. Packer. Twisted group  $C^*$ -algebras corresponding to nilpotent discrete groups. *Math. Scand.*, 64(1):109–122, 1989.
- [22] J. A. Packer and I. Raeburn. Twisted crossed products of  $C^*$ -algebras. *Math. Proc. Cambridge Philos. Soc.*, 106(2):293–311, 1989.
- [23] J. Slawny. On factor representations and the  $C^*$ -algebra of canonical commutation relations. *Comm. Math. Phys.*, 24:151–170, 1972.

- [24] G. Zeller-Meier. Produits croisés d'une  $C^*$ -algèbre par un groupe d'automorphismes. *J. Math. Pures Appl. (9)*, 47:101–239, 1968.

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# Paper IV

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$C^*$ -algebras generated by projective  
representations of free nilpotent groups

*Tron Ånen Omland*

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# Paper V

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Cuntz-Li algebras from  $a$ -adic numbers

*S. Kaliszewski, Tron Omland, and John Quigg*

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