NTNU – Trondheim
Norwegian University of
Science and Technology

Adeles in Mathematical Physics

Erik Makino Bakken

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Supervisor: Trond Digernes, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Preface

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Erik Makino Bakken

Abstract

In this thesis, quantum mechanics is studied over the p-adic numbers and adeles. In particular the harmonic oscillator is investigated. There is no Hamiltonian in the p-adic and adelic case for this model, but an analogous theory is studied. In addition, expectation values for some operators in the simplest ground state are calculated. Necessary background information about p-adic numbers, adeles, topological groups and quantum mechanics is given.

Sammendrag

Denne oppgaven handler om kvantemekanikk over p-adiske tall og adeler. Hovedfokus har vært på den harmoniske oscillatoren. For oscillatoren er det ingen Hamilton-operator i det p-adiske og adeliske tilfellet, men en analog teori er gjennomgått. I tillegg så er forventningsverdier funnet for noen operatorer i den enkleste grunntilstanden. En gjennomgang av p-adiske tall, adeler, topologiske grupper og kvantemekanikk er gitt.

Contents

Chapter 1

Introduction

The standard for quantum mechanics is to study wave functions from $\mathbb{R}^n$ to $\mathbb{C}$. A problem with the current model over the real numbers is what happens at the Planck length, which is approximately $1.6 \cdot 10^{-35}$m. This is a very small size, even compared to a proton which has a diameter which is approximately $1.6 \cdot 10^{-15}$m. The Planck length is the smallest length which is possible to measure. So what happens under the Planck length? As is discussed by Volovich in his paper "Number Theory as the Ultimate Physical Theory" ([14]), the archimedean axiom [1] becomes questionable when dealing with lengths under the Planck length.

Over the history of physics one has accepted more and more of what can be considered less intuitive models. That the space we live in can be anything different from Euclidean space was unthinkable a couple of centuries ago. Now it is standard to work with a four-dimensional manifold as space-time. For a long time one has taken for granted that the space consists of real numbers. The possibility of using a p-adic space was first noted by Vladimirov and Volovich in 1983. The p-adic numbers, denoted by $\mathbb{Q}_p$, and the real numbers share the property of being fields which are completions of the rational numbers. It is convenient to have a model which is based on $\mathbb{Q}$ since all physical results are rational numbers. One can go further and consider adelic space, such that the real and p-adic numbers are treated simultaneously. The adeles, denoted by $\mathbb{A}$, are unfortunately just a ring and not a field, but it is a locally compact abelian group.

There are several paths to go from a real model to a p-adic and adelic model. In this thesis, I have chosen to follow Weyl's formulation of quantum mechanics. In this model the functions go from the adeles or p-adic numbers to the complex numbers. In the p-adic model, mass, position, time and so on become p-adic, and similarly these quantities become adelic in the adelic model. A problem is that $\mathbb{Q}_p$ is not an ordered field, and one cannot talk about before and after. That the adeles are not a field makes things even harder to interpret. Another path is for instance

[1] The archimedean axiom states that if you have a large and a small line segment, then if the small segment is added enough times, the length will surpass the larger line segment.

to let the time stay real. One can also consider models where the functions are p-adic valued.

The harmonic oscillator will be the model which is investigated. The harmonic oscillator appears often and is very important. It is also a simple model such that the eigenvalues and eigenvectors can be found for the Hamiltonian.

In Chapter 2, p-adic numbers are introduced. Locally compact abelian groups are studied in Chapter 3. In Chapter 4, the adeles are investigated. The treatment of integration theory and Fourier transform follows Tate's thesis [6]. Chapter 5 gives an introduction to quantum mechanics. In particular it contains the mathematical formula of the Feynman path integral given in [13]. The one-dimensional harmonic oscillator over real numbers, p-adic numbers and adeles is investigated in Chapter 6. Finally in Chapter 7, one obtains eigenvalues and eigenfunctions for the evolution operator for the harmonic oscillator, which is analogous to finding the eigenvalues and eigenvectors for the Hamiltonian. The treatment of the p-adic harmonic oscillator follows [5], while the treatment of adelic harmonic oscillator follows [7].

The purpose of this thesis is to go thoroughly through all the necessary background knowledge needed for p-adic and adelic quantum mechanics, as well as investigating the work of Vladimirov, Volovich and Zelenov on the p-adic harmonic oscillator, and the work of Dragovich on the adelic harmonic oscillator.

Chapter 2

The p-adic Numbers

The field of p-adic numbers, denoted by $\mathbb{Q}_p$, were first described by Kurt Hensel in 1897. Even though one says the field of p-adic numbers, there are actually several fields. For each prime p, one gets one field $\mathbb{Q}_p$. So for instance if $p = 3$, one gets the 3-adic numbers $\mathbb{Q}_3$. When we give a statement about $\mathbb{Q}_p$, it means that the statement is true for all primes p. Each $\mathbb{Q}_p$ is a completion of the rational numbers $\mathbb{Q}$ with respect to a different norm than the usual one.

2.1 Construction of $\mathbb{Q}_p$ by Analysis

There are several ways to give a construction of $\mathbb{Q}_p$. This section will be about the construction of $\mathbb{Q}_p$ by analysis.

First it will be necessary to define an absolute value, and then define the p-adic absolute value which will extend to be an absolute value on $\mathbb{Q}_p$.

Definition 2.1.1. (Absolute value) An absolute value on a field, $\mathbb{k}$, is a function $|\cdot| : \mathbb{k} \longrightarrow \mathbb{R}^+$ which satisfies

(i) $|x| = 0$ if and only if $x = 0$

(ii) $|xy| = |x||y| \quad \forall x, y \in \mathbb{k}$

(iii) $|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{k}$.

The absolute value is in addition called non-archimedean if it satisfies

$$|x + y| \leq \max\{|x|, |y|\} \qquad \forall x, y \in \mathbb{k}. \tag{2.1.1}$$

Definition 2.1.2. (The p-adic absolute value and valuation) Let $x = \frac{a}{b}$ be a rational number different from zero. One can factorize such that $x = p^k \frac{a'}{b'}$ with $p \nmid a'b'$. Then the p-adic absolute value on $\mathbb{Q}$ is

$$|x|_p = p^{-k} \tag{2.1.2}$$

and the p-adic valuation on $\mathbb{Q}$ is:

$$v_p(x) = k \tag{2.1.3}$$

For $x = 0$, $v_p(x) = \infty$, and with the usual conventions on handling ∞, $|x|_p = 0$

Here are a few examples of the absolute value.

Example $|7|_7 = \frac{1}{7}$, $\quad |162|_3 = |2 \cdot 3^4|_3 = 3^{-4}$, $\quad |3^{-5} \cdot 2^{-5}|_5 = 1$

It is worth noting that the p-adic absolute value actually is non-archimedean. The next two lemmas follow easily from the definition.

Lemma 2.1.1. *The p-adic valuation satisfies*

(i) $v_p(xy) = v_p(x) + v_p(y) \quad \forall x, y \in \Bbbk$

(ii) $v_p(x + y) \geq \min\{v_p(x), v_p(y)\} \quad \forall x, y \in \Bbbk$.

Lemma 2.1.2. *The p-adic absolute value $|\cdot|_p$ is a non-archimedean absolute value.*

As it is of no interest, the trivial absolute value will often be excluded from the theorems.

Definition 2.1.3. The trivial absolute value is an absolute value such that $|0| = 0$ and $|x| = 1$ for $x \neq 0$.

The real numbers, $\mathbb{R}$, is the completion of $\mathbb{Q}$ with respect to the usual absolute value, denoted by $|\cdot|_\infty$. On the other hand, $\mathbb{Q}_p$ is obtained by completing $\mathbb{Q}$ with respect to the p-adic absolute value, $|\cdot|_p$. It is worth to note that the completion actually is necessary since $\mathbb{Q}$ is not complete with respect to $|\cdot|_p$.

Lemma 2.1.3. *$\mathbb{Q}$ is not complete with respect to $|\cdot|_p$.*

The proof of the lemma can be found in [1].

The idea for the completion process is to add all the limits which are missing. This is done by looking at all Cauchy sequences in $\mathbb{Q}$ with respect to $|\cdot|_p$, and divide out all such sequences which are converging to zero. This will be our definition of $\mathbb{Q}_p$.

Definition 2.1.4. Define C as the set of all Cauchy sequences in $\mathbb{Q}$ with respect to $|\cdot|_p$. Define N to be the set of all such sequences which converge to zero.

C will be a ring with pointwise addition and multiplication. It can be shown that N is a maximal ideal of C. Now we are ready to define $\mathbb{Q}_p$.

Definition 2.1.5. (The p-adic numbers) We define $\mathbb{Q}_p$ to be

$$\mathbb{Q}_p = C/N. \tag{2.1.4}$$

Note that $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, by letting $x \in \mathbb{Q}$ go to the Cauchy sequence which is constantly x. The difference between two such different sequences will be a constant different from zero, and thus it will not tend to zero.

Theorem 2.1.4.

(i) *The p-adic absolute value on $\mathbb{Q}$ extends to $\mathbb{Q}_p$.*

(ii) *$\mathbb{Q}_p$ has $\mathbb{Q}$ as a dense subset.*

(iii) *$\mathbb{Q}_p$ is a complete field.*

Proof.
(i) Let $x \in \mathbb{Q}_p$ and let (x_n) be a coset representative (which is a Cauchy sequence) for x. One defines the p-adic absolute value on $\mathbb{Q}_p$ by

$$|x|_p = \lim_{n\to\infty} |x_n|_p. \tag{2.1.5}$$

There are a few things that have to be checked. It has to independent of the choice of coset representative and the limit must exist. Furthermore it clearly coincides with the absolute value defined on $\mathbb{Q}$. Finally, one has to check that it is a non-archimedean absolute value.

(ii) Let $x \in \mathbb{Q}_p$ and $\epsilon > 0$. We want to show that there exists an element in $\mathbb{Q}$ such that the distance to x is less than ϵ. Choose a Cauchy sequence (x_n) which represents x. Since it is Cauchy there exists an N such that $|x_n - x_m|_p < \epsilon/2$ for $n, m \geq N$. Let (x_N) be the sequence constantly equal to x_N. Then

$$|x - (x_N)|_p = \lim_{n\to\infty} |x_n - x_N|_p < \epsilon \tag{2.1.6}$$

which proves the claim.

(iii) Since C is a unital commutative ring and N is maximal ideal in the ring, $\mathbb{Q}_p$ is a field. Since $\mathbb{Q}$ is not complete, we went to the completion to get $\mathbb{Q}_p$. To show that it actually is complete, let $(x_n)_n$ be a Cauchy sequence of p-adic numbers. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, there exists a sequence of rational numbers $(y_n)_n$ such that $\lim_{n\to\infty} |x_n - y_n|_p = 0$. We have that

$$|y_n - y_m| \leq |y_n - x_n|_p + |x_n - x_m|_p + |x_m - y_m|_p \tag{2.1.7}$$

by the triangle inequality (the non-archimedean property is not needed here). This proves that $(y_n)_n$ is Cauchy since $(x_n)_n$ is Cauchy and $\lim_{n\to\infty} |x_n - y_n| = 0$. But then $(y_n)_n$ is an element in $\mathbb{Q}_p$, and by the choice of $(y_n)_n$, $x_n \to (y_n)_n$ (To make it clear, the sequence in $\mathbb{Q}_p$, $(x_n)_n$, converges to $(y_n)_n$ seen as an element in $\mathbb{Q}_p$). This proves that $\mathbb{Q}_p$ is complete. $\square$

The properties in Theorem 2.1.4 are also true for $\mathbb{R}$ with the usual absolute value. This similarity makes some of the analysis on $\mathbb{Q}_p$ quite similar to the analysis on $\mathbb{R}$. But as we shall see later, there are many properties which are very different.

One could ask why one is looking at these special absolute values. Actually, the only non-trivial absolute values on $\mathbb{Q}$ are, up to equivalence, the usual absolute value and the p-adic ones. This makes it more natural to study these structures.

Definition 2.1.6. (Equivalence of absolute values) Two absolute values are said to be equivalent on a field $\mathbb{k}$ if they define the same topology on $\mathbb{k}$.

Now Ostrowski's theorem can be stated. The proof is found in [1].

Theorem 2.1.5. *(Ostrowski) Each non-trivial absolute value on $\mathbb{Q}$ is either equivalent to $|\cdot|_p$ for some p or it is equivalent to the usual absolute value.*

It is standard convention to write $\mathbb{Q}_\infty$ for $\mathbb{R}$ and $|\cdot|_\infty$ as the usual absolute value. One can think of $\mathbb{Q}_p$ as studying $\mathbb{Q}$ "locally around p", and one thinks of $\mathbb{R}$ as studying $\mathbb{Q}$ "locally around ∞", and one often refers to the prime ∞.

2.2 Further Properties of $\mathbb{Q}_p$

In this section we will establish some important properties of $\mathbb{Q}_p$, but also some results which are meant to illustrate how the numbers behave. Some of the results will be rather unintuitive at first, but it will be clearer later on.
First we will look at the product formula. This will be the first example of what is called an adelic formula. There will be more about adeles later.

Theorem 2.2.1. *(Product Formula) For every $x \in \mathbb{Q}$,*

$$\prod_p |x|_p = 1, \qquad p = \infty, 2, 3, 5, 7, ... \tag{2.2.1}$$

where $|x|_\infty$ is the usual absolute value of x.

Proof. We can factorize x as $\pm p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$. Then the absolute values will all be 1 when p gets large, and hence the product is well defined. Clearly $|x|_{p_i} = p^{-n_i}$ and $|x|_\infty = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, and the result follows. $\square$

Lemma 2.2.2. *Let x,y be elements in $\mathbb{Q}_p$. If $|x|_p \neq |y|_p$ then*

$$|x + y|_p = \max\{|x|_p, |y|_p\}. \tag{2.2.2}$$

Proof. Assume $|y|_p > |x|_p$. That $|x + y|_p \leq |y|_p$ follows immediately from the non-archimedean property. For the other inequality: $|y|_p = |y + x - x|_p \leq \max \{|y+x|_p, |x|_p\} = |y+x|_p$ where one gets the last equality from the fact that choosing $|x|_p$ as maximum would contradict the assumption first made in the proof. The proof for $|y|_p < |x|_p$ is similar. $\square$

This lemma is very useful and is seen in many proofs. The next lemma is important as well.

Lemma 2.2.3. *All triangles in $\mathbb{Q}_p$ are isosceles.*

Proof. First the claim has to be explained. A triangle will be created by three points, x, y and z, and the length of the sides will be $|x-y|_p$, $|x-z|_p$ and $|y-z|_p$. We know that

$$|x-y|_p = |(x-z)+(z-y)|_p. \tag{2.2.3}$$

If $|x-z|_p$ and $|z-y|_p$ are equal, we are done. If not, by Lemma 2.2.2, $|x-y|_p$ will be equal to the longest of the two sides, and we are done. □

Now we will look at open and closed balls and the topological properties of $\mathbb{Q}_p$.

Definition 2.2.1. (Open and closed ball) Let $a \in \mathbb{Q}_p$, and r be a positive real number. Then define the open ball around a with radius r to be

$$B(a,r) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}, \tag{2.2.4}$$

and the closed ball around a with radius r to be

$$\bar{B}(a,r) = \{x \in \mathbb{Q}_p : |x-a|_p \leq r\}. \tag{2.2.5}$$

Notice that the bar over $B(a,r)$ does not mean closure. It is in fact not true that the closure of $B(a,r)$ is $\bar{B}(a,r)$. It will be shown later that the open ball is a closed set.

Lemma 2.2.4. *Every point in an open ball is the center of the open ball, i.e. if $b \in B(a,r)$ then*

$$B(a,r) = B(b,r). \tag{2.2.6}$$

The statement holds true for closed balls as well.

Proof. The proof will only be given for an open ball. Let $x \in B(a,r)$. Then

$$|b-x|_p \leq \max\{|b-a|_p, |a-x|_p\} < r, \tag{2.2.7}$$

so $B(a,r) \subset B(b,r)$. The reverse inclusion is clear by a similar argument. □

Corollary 2.2.5. *Two balls are either disjoint or contained in one another. That is, if $r \leq s$ are two real numbers, and b and a are two p-adic numbers, then either $B(b,r) \cap B(a,s) = \emptyset$ or $B(b,r) \subset B(a,s)$. The statement holds true for closed balls as well.*

Proof. The proof will only be given for an open ball. If they are not disjoint, then take $x \in B(a,s) \cap B(b,r)$. Then

$$B(b,r) = B(x,r) \subset B(x,s) = B(a,s). \tag{2.2.8}$$

□

The p-adic integers play a special role in this theory, and will be used quite frequently.

Definition 2.2.2. (The p-adic integers) The p-adic integers, $\mathbb{Z}_p$ are defined as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \tag{2.2.9}$$

Before the topological properties are stated comes an important theorem about a representation of the p-adic numbers. This is the way of thinking about these numbers when doing calculations.

Theorem 2.2.6. *Any y in $\mathbb{Q}_p$ can be represented uniquely on the form:*

$$y = a_n p^n + a_{n+1} p^{n+1} + a_{n+2} p^{n+2} + \ldots \tag{2.2.10}$$

where $0 \leq a_i < p$ and $n \in \mathbb{Z}$. If $y \neq 0$, then one can assume that a_n is non-zero, and then we have that $|y|_p = p^{-n}$.

Proof. The proof follows [1]. First we will show that every element in $\mathbb{Z}_p$ can be written as $\sum_{i=0}^{\infty} a_i p^i$. Let $x \in \mathbb{Z}_p$. The idea of the proof will be to show that there is a Cauchy sequence (α_n) converging to x such that $|\alpha_{n+1} - \alpha_n|_p \leq p^{-(n+1)}$ and $0 \leq \alpha_n < p^{n+1}$, and that the sequence satisfying these properties is unique. From this sequence one can show that x has the unique representation that is stated in the theorem. Choose an integer $n \geq 0$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ there is an element $a/b \in \mathbb{Q}$ (written in lowest terms) such that $|x - a/b|_p \leq p^{-(n+1)}$. Since $|a/b|_p \leq 1$ by the non-archimedean property of the absolute value, p does not divide b. Now choose a rational integer z such that $bz \equiv 1 \pmod{p^{n+1}}$. Then

$$|az - x|_p \leq \max\{|az - \frac{a}{b}|_p, |\frac{a}{b} - x|_p\} \leq p^{-(n+1)} \tag{2.2.11}$$

Now let α_n be the unique integer less than p^{n+1} such that $\alpha_n \equiv az \pmod{p^{n+1}}$. Then $|x - \alpha_n|_p \leq p^{-(n+1)}$. Now $0 \leq \alpha_n < p^{n+1}$ and $|\alpha_{n+1} - \alpha_n|_p \leq p^{-(n+1)}$ follows from the non-archimedean property again. The sequence is clearly Cauchy and converges to x. Now $0 \leq \alpha_0 < p$ and call this integer a_0. Furthermore $0 \leq \alpha_1 < p^2$ and $|\alpha_1 - \alpha_0|_p < p^{-1}$. Hence $\alpha_1 = a_1 p + a_0$ where a_1 is an integer satisfying $0 \leq a_1 < p$. Doing this inductively one gets the desired sum $x = \sum_{i=0}^{\infty} a_i p^i$. Notice that a_i are unique since the α_i are unique and that the sum converges since the partial sums are just the α_i which is a Cauchy sequence converging to x. Since every $y \in \mathbb{Q}_p$ is equal to x/p^n for some $x \in \mathbb{Z}_p$ of absolute value 1, and some $n \in \mathbb{Z}$, y can be written as $y = \sum_{i=-n}^{\infty} a_i p^i$. $\square$

This is certainly easier to work with than the definition. The representation certainly looks similar to the decimal expansion for the real numbers. The difference is that the "carry" goes to the right (higher power of p) and not to the left. Addition and multiplication are analogous to what is done for Laurent series which are finite to the left, but with carry. For instance the sum of $a_0 + a_1 p + a_2 p^2 + \ldots$ and $b_2 p^2 + b_3 p^3 + \ldots$ will be $c_0 + c_1 p + c_2 p^2 + c_3 p^3 + \ldots$, where $c_0 = a_0$, $c_1 = a_1$, $c_2 = a_2 + b_2 \mod p$, $c_3 = ((a_3 + b_3) + (a_2 + b_2 - c_2)) \mod p$, and so on. The product of these two numbers is $d_0 + d_1 p + d_2 p^2 + d_3 p^3 + \ldots$, where $d_0 = 0$, $d_1 = 0$, $d_2 = a_0 b_2 \mod p$, $d_3 = ((a_0 b_3 + a_1 b_2) + (a_0 b_2 - d_2)) \mod p$. Notice the extra terms $a_2 + b_2 - c_2$ and $a_0 b_2 - d_2$ which one does not get with Laurent series. Let us now see how this representation can be useful.

Example Let $x = \sum_{i=n}^{\infty} a_i p^i$ and $y = \sum_{i=m}^{\infty} b_i p^i$ with a_n and b_m non-zero. Now it is not that hard to show that $|x + y|_p = \max\{|x|_p, |y|_p\}$ if $|x|_p \neq |y|_p$. That $|x|_p \neq |y|_p$ just means that $n \neq m$ by Theorem 2.2.6. Assume $n < m$. Since the carry goes to the right, the first term in $x+y$ is still $a_n p^n$ and $|x+y|_p = |x|_p$. The case $m < n$ is similar. Note that this is just to see it from another point of view. The first proof of this lemma was the "right" proof since it also works for arbitrary fields with a non-archimedean absolute value, and the result is actually used for $\mathbb{Q}$ when one extends the absolute value to $\mathbb{Q}_p$.

The next example will be stated as a lemma. It will be important for the topological properties of $\mathbb{Q}_p$.

Lemma 2.2.7. *Every open ball is a closed and open set (which is called a clopen set).*

Proof. Let $a \in \mathbb{Q}_p$, let r be a positive real number and let $B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$ be the open ball of radius r around a. Notice that since the absolute value only takes discrete values by Theorem 2.2.6, $B(a, r)$ is just $\{x \in \mathbb{Q}_p : |x-a|_p \leq p^{-k}\} = \bar{B}(a, p^{-k})$ where p^{-k} is the biggest power of p that is strictly smaller than r. This is a closed ball, and hence a closed set. $\square$

Example That all points in an open ball is the center of the ball can be hard to understand. When one uses Theorem 2.2.6 it is easier to see. This example will follow the notation from Lemma 2.2.7. It is easy to see that $B(a, r)$ consists of all numbers that have the same coefficients as a up to the p^k term, that is $\{x \in \mathbb{Q}_p : x - a \in p^k\mathbb{Z}_p\}$. To say that two p-adic numbers are equivalent mod p^k if they have the same coefficients up to the p^k term is clearly an equivalence relation. Now it is apparent that the choice of center does not matter, since all elements in the ball are equivalent with respect to this equivalence relation.

The absolute value is non-archimedean has a huge impact on the topology. We know that $\mathbb{R}$ is a connected Hausdorff space, but we will see that $\mathbb{Q}_p$ is not only disconnected, but totally disconnected.

Lemma 2.2.8. *$\mathbb{Q}_p$ is totally disconnected and Hausdorff.*

Proof. Since $\mathbb{Q}_p$ is a metric space, it is Hausdorff. Now assume that a set X contains two distinct points, x and y in $\mathbb{Q}_p$. Their distance is $|x-y|_p = r$. The ball $B(x, r/2)$ in $\mathbb{Q}_p$ is also open and closed in the subspace topology when intersected with X. The complement of this set in X is also open and closed. Both these sets are non-empty and do not intersect, thus X is disconnected. Hence, the connected components of $\mathbb{Q}_p$ must consist of just one point which means that $\mathbb{Q}_p$ is totally disconnected. $\square$

Now comes a result which will be very important for the integration theory.

Theorem 2.2.9. *$\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is locally compact.*

Proof. That $\mathbb{Q}_p$ is locally compact means that every point of $\mathbb{Q}_p$ has a compact neighborhood. Note that to prove that $\mathbb{Q}_p$ is locally compact it is enough to prove that the closed unit ball of 0 is compact (translation is a homeomorphism and continuous maps take compact sets to compact sets). So what is left to prove is that $\mathbb{Z}_p$ is compact. To prove that it is compact, we have to prove that it is complete and totally bounded. It is a closed subset of $\mathbb{Q}_p$ which is complete, hence it is complete. Now to prove that it is totally bounded, let $\epsilon > 0$, and p^{-k} be the highest power of p strictly less than ϵ. Remember that any $x \in \mathbb{Z}_p$ can be written as

$$x = a_0 + a_1 p + a_2 p^2 + ... \tag{2.2.12}$$

Also remember that two numbers having distance less than or equal p^{-k} means that the first k coefficients are the same. There are p^k possible combinations for the first k coefficients. So take p^k ϵ-balls with each of these combinations as center. This will clearly cover all of $\mathbb{Z}_p$ and we are done. □

2.3 Elementary Functions on $\mathbb{Q}_p$

On the real numbers we have the functions $\sin x$, $\cos x$, $\ln x$ and e^x. We want to define the p-adic analog of these functions. We will define these functions by power series, and the power series will look identical to the real ones. To be able to do this, we first have to develop some results about convergence of p-adic series.

We will start with two lemmas which do not hold in $\mathbb{R}$.

Lemma 2.3.1. *A sequence (a_n) in $\mathbb{Q}_p$ converges if and only if*

$$\lim_{n\to\infty} |a_{n+1} - a_n|_p = 0. \tag{2.3.1}$$

Proof. We have that for $m > n$,

$$|a_m - a_n|_p \leq \max\{|a_m - a_{m-1}|_p, ..., |a_{n+1} - a_n|_p\}. \tag{2.3.2}$$

This shows that if equation (2.3.1) holds, then (a_n) is Cauchy, and hence convergent. □

Lemma 2.3.2. *The absolute value of the elements in a Cauchy sequence (a_n) in $\mathbb{Q}_p$, not converging to 0, will eventually be constant.*

Proof. Since the sequence is not converging to 0, then there exists an $\epsilon > 0$ and an N such that $|a_n|_p > \epsilon$ for $n > N$. Since it is Cauchy, there exists an M such that $|a_n - a_m| < \epsilon$ for $n, m > M$. But then if $n, m > \max\{N, M\}$,

$$|a_n|_p = |a_n - a_m + a_m|_p = |a_m|_p \tag{2.3.3}$$

by Lemma 2.2.2. □

Lemma 2.3.3. *Let (a_n) be a sequence in $\mathbb{Q}_p$. Then the infinite series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$. Furthermore, in this case we get that*

$$|\sum_{n=0}^{\infty} a_n|_p \leq \max_n\{|a_n|_p\}. \tag{2.3.4}$$

Proof. That the sum converges follows immediately from the previous lemma. The inequality follows from the fact that the absolute value of the elements in a sequence, not converging to 0, will eventually be constant. This reduces the inequality to the finite case which follows from induction. □

We want to look at functions on the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{2.3.5}$$

and determine when the series converges. We know that the series converges for those x which satisfy $\lim_{n\to\infty} |a_n x^n|_p = 0$. The next proposition will be useful to determine for which x that condition is satisfied. The proof is pretty straightforward and will be omitted here (it is found in [1]).

Proposition 2.3.4. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and define $\rho = (\limsup \sqrt[n]{|a_n|_p})^{-1}$ with the usual convention when dealing with ∞.*

(i) If $\rho = 0$, then the power series converges only for $x = 0$.

(ii) If $\rho = \infty$, then the power series converges for all $x \in \mathbb{Q}_p$.

(iii) If $0 < \rho < \infty$ and $\lim_{n\to\infty} |a_n|_p \rho^n = 0$, then the power series converges for x if and only if $|x|_p \leq \rho$.

(iv) If $0 < \rho < \infty$ and $\lim_{n\to\infty} |a_n|_p \rho^n \neq 0$, then the power series converges for x if and only if $|x|_p < \rho$.

Let us now define the functions. We will first define the logarithm by

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}. \tag{2.3.6}$$

After a few calculations we get that $\sqrt[n]{|a_n|_p} \to 1$ as $n \to \infty$, and thus $\rho = 1$. It is clear that the power series diverges for $|x|_p = 1$ since $|a_n x^n|_p = |a_n|_p = |\frac{1}{n}|_p$, so the power series defining $\ln(1+x)$ converges if and only if $|x|_p < 1$.

The results about power series in x are of course true for power series in $x - \alpha$ too. Now we can define the logarithm.

Definition 2.3.1. (Logarithm Function) The p-adic logarithm is defined as

$$\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \tag{2.3.7}$$

which is defined only for $x \in \mathbb{Z}_p$ such that $|x-1|_p < 1$.

We continue to use the power series which are used in $\mathbb{R}$ to define the functions.

Definition 2.3.2. (Exponential Function) The p-adic exponential function is defined as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{2.3.8}$$

Definition 2.3.3. (Trigonometric Functions) The p-adic trigonometric functions are defined as

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \tag{2.3.9}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \tag{2.3.10}$$

$$\tan x = \frac{\sin x}{\cos x}. \tag{2.3.11}$$

Lemma 2.3.5. *The region of convergence for* $\sin x$*,* $\cos x$*,* $\tan x$ *and* e^x *is* $\{x : |x|_p < p^{-\frac{1}{p-1}}\}$.

Proof. A proof will only be given for e^x. Recall that we write $|x|_p = p^{-v_p(x)}$. We want to calculate $v_p(n!)$. It is easily seen that

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \leq \sum_{i=1}^{\infty} \frac{n}{p^i} = \frac{n}{p-1}. \tag{2.3.12}$$

Then $|\frac{1}{n!}|_p \leq p^{\frac{n}{p-1}}$ so we know that the series for e^x converges when $|x|_p < p^{-1/(p-1)}$. To prove divergence when $|x|_p = p^{-1/(p-1)}$, we will look at the terms in the sum when $n = p^m$. Then

$$v_p(n!) = p^{m-1} + p^{m-2} + ... + 1 = \frac{p^m - 1}{p-1}, \tag{2.3.13}$$

and this gives us

$$v_p\left(\frac{x}{p^m!}\right) = \frac{1}{p-1}. \tag{2.3.14}$$

The power series for e^x clearly diverges, and the result follows. $\square$

This is very different from the real numbers, where functions like e^x converge for all x. In the real case $|\frac{1}{n!}|$ goes to 0 as n goes to infinity. However, in the p-adic case $|\frac{1}{n!}|_p$ will go to infinity as n goes to infinity, and the region of convergence is then expected to be much smaller. Also note that the region of convergence for these functions is $\{x : |x|_p \leq p^{-1}\}$ for $p \geq 3$ since the absolute value does not take any values between p^{-1} and $p^{-\frac{1}{p-1}}$ (and it is $\{x : |x|_p \leq p^{-2}\}$ for $p = 2$).

Definition 2.3.4. We define G_p to be the additive group where the exponential function is defined,

$$G_p = \begin{cases} \{x \in \mathbb{Q}_p : |x|_p \leq p^{-1}\} & \text{if } p \neq 2, \\ \{x \in \mathbb{Q}_p : |x|_p \leq p^{-2}\} & \text{if } p = 2. \end{cases} \tag{2.3.15}$$

Definition 2.3.5. We define L_p to be the multiplicative group where the logarithm function is defined,

$$L_p = \{x \in \mathbb{Q}_p : 1 - x \in G_p\}. \tag{2.3.16}$$

It can be shown that $\ln(ab) = \ln(a) + \ln(b)$ and $e^{x+y} = e^x e^y$ $(x, y \in G_p)$ as we expect from the logarithm and exponential. The p-adic logarithm is the analog of the natural logarithm on the real numbers. From [5] we get the next results.

Lemma 2.3.6. *The function e^x is an isomorphism from G_p to L_p with $\ln x$ as the inverse function.*

Lemma 2.3.7.

$$|e^x|_p = 1, \qquad |\sin x|_p = |x|_p, \qquad |\cos x|_p = 1, \qquad x \in G_p. \tag{2.3.17}$$

Proof. For instance, let us prove that $|\sin x|_p = |x|_p$, or that $|\frac{\sin x}{x}|_p = 1$. We have that

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}. \tag{2.3.18}$$

By using a bit stronger approximation than we did in Lemma 2.3.5, one can show that $|\frac{x^{n-1}}{n!}|_p < 1$. Now, by Lemma 2.3.3, the series for $\frac{\sin x}{x}$ is 1 minus a p-adic number of absolute value less than 1, and the result follows. $\square$

Definition 2.3.6. (Legendre Symbol) Let p be an odd prime, and a an integer. Then the Legendre symbol is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p \text{ and } a \not\equiv 0 \pmod{p} \\ -1 & \text{if } a \text{ is a quadratic non-residue} \pmod{p} \\ 0 & \text{if } a \equiv 0 \pmod{p} \end{cases} \tag{2.3.19}$$

It can be shown that an equivalent definition is that $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$ where $\left(\frac{a}{p}\right)$ is in the set $\{-1, 0, 1\}$. It can also be shown that the Legendre symbol is multiplicative, that is,

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right). \tag{2.3.20}$$

Lemma 2.3.8. *Let $a = p^{\gamma(a)}(a_0 + a_1 p + ...)$ be a p-adic number, where $0 \leq a_i < p$ and $a_0 \neq 0$. Then the equation*

$$x^2 = a \tag{2.3.21}$$

has a solution for $p \neq 2$ if and only if *(i)* $\gamma(a)$ *is even.*

(ii) $\left(\frac{a_0}{p}\right) = 1,$

and a solution for $p = 2$ *if and only if*

(i) $\gamma(a)$ *is even.*

(ii) $a_1 = a_2 = 0.$

Proof. We will prove the lemma for $p \neq 2$. Let x be on the standard form $x = p^{\gamma(x)}(x_0 + x_1 p + ...)$. Then $x^2 = a$ becomes

$$p^{2\gamma(x)}(x_0 + x_1 p + ...)^2 = p^{\gamma(a)}(a_0 + a_1 p + ...). \tag{2.3.22}$$

Then we immediately see that $\gamma(a)$ must be even and that $a_0 \equiv x_0^2 \pmod{p}$. Conversely, assume that $\gamma(a)$ is even and that $(\frac{a_0}{p}) = 1$. Then we can choose $\gamma(x) = \frac{1}{2}\gamma(a)$. Also, there exists an x_0 such that $x_0^2 \equiv a_0 \pmod{p}$. From the above equation we also get that

$$2x_0 x_j + N_j \equiv a_j \pmod{p}, \tag{2.3.23}$$

where N_j is an integer which is only a function of $x_0, x_1, ..., x_{j-1}$. The equation has a unique solution x_j for each j. This proves the lemma. □

Lemma 2.3.9. *The functions*

$$\cos x, \qquad \frac{\sin x}{x}, \tag{2.3.24}$$

are squares of p*-adic functions on* G_p*.*

Proof. From the proof in Lemma 2.3.7 we get that $\gamma = 0$ and $a_0 = 1$ for $\frac{\sin x}{x}$. The result follows from Lemma 2.3.8. The proof for $\cos x$ is similar. □

Definition 2.3.7. (p-adic Units) The p-adic units, $\mathbb{Z}_p^\times$, are defined to be the invertible elements in $\mathbb{Z}_p$, which are

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p : |x|_p = 1\}. \tag{2.3.25}$$

The p-adic units form a multiplicative group. The proof of the next lemma is found in [1].

Lemma 2.3.10. *Let* V *be the set of roots of unity. For all* p*,* V *is a subset of* $\mathbb{Z}_p^\times$*. For* $p \neq 2$ *this is a cyclic group of order* $p - 1$*, and for* $p = 2$ *this is a cyclic group of order* 2*. Furthermore* L_p *is a subset of* $\mathbb{Z}_p^\times$*, and we have the isomorphism* $\mathbb{Z}_p^\times \cong V \times L_p$*.*

The next corollary then follows immediately.

Corollary 2.3.11. *Every p-adic number, z, can be represented uniquely as*

$$z = p^{\gamma}\epsilon^{k}e^{a}, \tag{2.3.26}$$

where $\gamma \in \mathbb{Z}$, ϵ is a generator for the group V, k is in the additive group $\{0, 1, ..., p-2\}$ for $p \neq 2$ and in $\{0, 1\}$ for $p = 2$, and $a \in G_p$.

Lemma 2.3.12. *The equation $x^2 = -1$ has a solution in $\mathbb{Q}_p$ if and only if $p \equiv 1$ (mod 4).*

Proof. Obviously $|x|_p = 1$. From Lemma 2.3.8 this reduces to if $\left(\frac{-1}{p}\right) = 1$. From the alternative definition of the Legendre symbol, we get that

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ -1 & p \equiv 3 \pmod 4 \end{cases} \tag{2.3.27}$$

□

An element that satisfies this equation is sometimes denoted by i as we do in the real numbers. One should keep in mind the next definition as it will be frequently used in the integration theory.

Definition 2.3.8. (Fractional Part) The fractional part of a p-adic number, $x = \sum_{i=n}^{\infty} a_i p^i$, is defined as

$$\{x\} = \sum_{i=n}^{-1} a_i p^i \tag{2.3.28}$$

when $|x| > 1$ and defined as 0 when $|x| \leq 1$.

2.4 A Useful Function

The function, λ_p, which will be defined here is very useful since it occurs in the Gaussian integrals which will be defined later.

Let x be a p-adic number different from zero. Recall that it can be written on the form

$$x = p^{\gamma}(a_0 + a_1 p + a_2 p^2 + ...), \tag{2.4.1}$$

where $\gamma \in \mathbb{Z}$ and $0 \leq a_i < p$ ($a_0 \neq 0$).

We will also denote the multiplicative group of p-adic numbers by $\mathbb{Q}_p^*$. Since $\mathbb{Q}_p$ is a field, $\mathbb{Q}_p^*$ consists of all p-adic numbers except for 0.

Definition 2.4.1. Define the function $\lambda_p : \mathbb{Q}_p^* \to \mathbb{C}$ as

$$\lambda_2(a) = \begin{cases} \frac{1}{\sqrt{2}}(1 + (-1)^{a_1} i) & \text{if } \gamma \text{ is even} \\ \frac{1+i}{\sqrt{2}} i^{a_1}(-1)^{a_2} & \text{if } \gamma \text{ is odd} \end{cases} \tag{2.4.2}$$

$$\lambda_p(a) = \begin{cases} 1 & \text{if } \gamma \text{ is even} \\ \left(\frac{a_0}{p}\right) & \text{if } \gamma \text{ is odd and } p \equiv 1 \pmod 4 \\ i\left(\frac{a_0}{p}\right) & \text{if } \gamma \text{ is odd and } p \equiv 3 \pmod 4 \end{cases} \tag{2.4.3}$$

where γ, a_0, a_1 and a_2 are given in equation (2.4.1), and p in equation (2.4.3) is not equal to 2. Also define the function $\lambda_\infty : \mathbb{R}^* \to \mathbb{C}$ as

$$\lambda_\infty(a) = \frac{1}{\sqrt{2}}(1 - \text{sign } a), \tag{2.4.4}$$

where $\mathbb{R}^*$ is the multiplicative group of real numbers.

The next two lemmas are found in [5].

Lemma 2.4.1. *Some important properties are*

$$\begin{aligned} &|\lambda_p(a)|_p = 1, \qquad \lambda_p(a)\lambda_p(-a) = 1, \qquad a \neq 0, \\ &\lambda_p(ac^2) = \lambda_p(a), \qquad a, c \neq 0. \end{aligned} \tag{2.4.5}$$

Proof. Let us prove that $\lambda_p(a)\lambda_p(-a) = 1$. If γ is even, it is trivially true. If γ is odd, then by Lemma 2.3.12, $\lambda_p(a) = \lambda_p(-a)$ if $p \equiv 1 \pmod 4$, and $\lambda_p(a) = -\lambda_p(-a)$ if $p \equiv 3 \pmod 4$. □

Lemma 2.4.2.

$$\lambda_p(a)\lambda_p(b) = \lambda_p(a+b)\lambda_p(\frac{1}{a} + \frac{1}{b}), \qquad a, b, a+b \in \mathbb{Q}_p^*. \tag{2.4.6}$$

Lemma 2.4.3. *We have the adelic product*

$$\prod_\nu \lambda_\nu(a) = 1, \qquad a \in \mathbb{Q}^*, \tag{2.4.7}$$

where $\nu = \infty, 2, 3, 5, ...$

Proof. The product converges for all $a \in \mathbb{Q}^*$ since $\lambda_\nu(a)$ is eventually 1. Since

$$\lambda_\nu(a)\lambda_\nu(-a) = 1, \qquad \lambda_\nu(ac^2) = \lambda_\nu(a) \tag{2.4.8}$$

it is enough to prove the lemma for a of the form

$$a = 2^\alpha p_1 p_2 \cdots p_n, \tag{2.4.9}$$

where α is 0 or 1. We will only prove the lemma for $\alpha = 0$ since the case $\alpha = 1$ is similar, but with longer calculations. So let $\alpha = 0$. One gets that

$$\lambda_{p_j}(a) = \begin{cases} \left(\frac{\prod_{k\neq j} p_k}{p_j}\right) & p_j \equiv 1 \pmod 4 \\ i\left(\frac{\prod_{k\neq j} p_k}{p_j}\right) & p_j \equiv 3 \pmod 4). \end{cases} \tag{2.4.10}$$

One also gets that $\lambda_p(a) = 1$ if $p \neq 2$ and $p \neq p_j$, and that $\lambda_\infty(a) = \exp(-i\frac{\pi}{4})$. To calculate $\lambda_2(a)$, let l denote the number of primes in the set $\{p_1, ..., p_n\}$ which are of the form $4N+3$. Note that the product of k primes of the form $4N+3$ is equal to an integer of the form $4N+3$ if k is odd, and of the form $4N+1$ if k is even. This gives that $\lambda_2(a) = \frac{1}{\sqrt{2}}(1+(-1)^l i)$. By the quadratic reciprocity law

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \tag{2.4.11}$$

for odd primes p and q, we get that

$$\begin{aligned}\prod_{1\le j\le n}\left(\frac{\prod_{k\neq j} p_k}{p_j}\right) &= \prod_{1\le j\le n}\left(\frac{p_k}{p_j}\right) = \prod_{1\le j<k\le n}\left(\frac{p_k}{p_j}\right)\left(\frac{p_j}{p_k}\right) \\ &= \prod_{1\le j<k\le n}(-1)^{\frac{p_k-1}{2}\frac{p_j-1}{2}} = (-1)^{\frac{l(l-1)}{2}}.\end{aligned} \tag{2.4.12}$$

The last equation follows from the fact that $\frac{p_k-1}{2}\frac{p_j-1}{2}$ is odd only if both p_k and p_j are of the form $4N+3$, and this will happen $l(l-1)/2$ times. Now we can combine the results to get

$$\prod_\nu \lambda_\nu(a) = \exp(-i\frac{\pi}{4})\frac{1}{\sqrt{2}}(1+i(-1)^l)i^l(-1)^{\frac{l(l-1)}{2}} = 1 \tag{2.4.13}$$

by just checking the cases $l = 4k, 4k+1, 4k+2, 4k+3$. This proves the result. $\square$

Chapter 3

Locally Compact Abelian Groups

In this chapter we will give an introduction to locally compact abelian groups and integration on these groups. At the end we will do integration on $\mathbb{Q}_p$.

3.1 Locally Compact Abelian Groups and the Haar Measure

Definition 3.1.1. (Topological Group) A topological group is a group G with a topology such that $x \mapsto x^{-1}$ is a continuous operation from G to G, and $(x, y) \mapsto xy$ is a continuous operation from $G \times G$ to G.

Definition 3.1.2. (Compact Group) A compact group is a topological group whose topology is compact.

Definition 3.1.3. (Locally Compact Group) A locally compact group is a topological group whose topology is locally compact.

In this thesis we will assume that all compact and locally compact groups have a topology which is Hausdorff.

A locally compact abelian group is a locally compact group with an abelian group operation. These definitions are important because the p-adic numbers and the adeles are locally compact abelian groups.

Locally compact groups have a measure called the Haar measure. Since $\mathbb{Q}_p$ is a locally compact abelian group, it has a Haar measure. This measure will give rise to an integration theory on $\mathbb{Q}_p$.

Definition 3.1.4. (Outer and Inner Regular Measure) Let X be a set and let E be a Borel subset of X. Let μ be a Borel measure on X. Then μ is called outer regular on E if

$$\mu(E) = \inf\{\mu(O) : O \supset E, O \text{ open}\} \tag{3.1.1}$$

and called inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}. \tag{3.1.2}$$

Definition 3.1.5. (Radon Measure) A Radon measure on a set X is a Borel measure, μ, such that that μ is finite on all compact sets, inner regular on all open sets and outer regular on all Borel sets.

Now the Haar measure can be defined along with a very important theorem which asserts that every locally compact group has a Haar measure.

Definition 3.1.6. (Haar Measure) Let G be a locally compact group. Also let E be a Borel set. A Borel measure, μ, on G is left-invariant if $\mu(xE) = \mu(E)$ for all $x \in G$. Similarly it is right-invariant if $\mu(Ex) = \mu(E)$. A left(right) Haar measure is a non-zero left(right)-invariant Radon measure on G.

An example of a Haar measure is the Lebesgue measure on $\mathbb{R}$ restricted to the Borel sets.

Theorem 3.1.1. *Every locally compact group G has a left Haar measure. Moreover, a left Haar measure is unique up to a positive scalar.*

The proof of the theorem can be found in [2]. A direct consequence of this is that $\mathbb{Q}_p$ has a unique left Haar measure up to a constant. Notice that the left Haar measure will also be a right Haar measure since $\mathbb{Q}_p$ is abelian. We will refer to it as the Haar measure.

3.2 The Pontryagin Dual Group

Definition 3.2.1. (Character) Let G be a locally compact abelian group. A character on G is a continuous homomorphism from G to $\mathbb{T}$ which is the multiplicative group of complex numbers of absolute value 1.

Definition 3.2.2. (Pontryagin Dual Group) Let G be a group. The set of characters of G under pointwise multiplication is called the Pontryagin dual group and is denoted by $\hat{G}$. The topology on $\hat{G}$ will be the compact-open topology, viewing $\hat{G}$ as a subset of all continuous functions from G to $\mathbb{T}$. This is the topology where convergence is given as uniform convergence on compact sets.

Lemma 3.2.1. *If G is a locally compact abelian group. Then $\hat{G}$ is a locally compact abelian group.*

An important class of functions is the class of integrable functions.

Definition 3.2.3. (Integrable Function or L^1-function) A measurable function f is integrable on a locally compact group G if

$$||f||_1 = \int_G |f(g)| d\mu(g) < \infty. \tag{3.2.1}$$

The space $L^1(G)$ is the set of integrable functions, where one identifies functions which are equal almost everywhere.

Definition 3.2.4. (Square-integrable Function or L^2-function) Similarly to the definition of $L^1(G)$, we define $L^2(G)$ to be the set of measurable functions on G such that

$$||f||_2^2 = \int_G |f(g)|^2 d\mu(g) < \infty, \tag{3.2.2}$$

where two functions are the same if they are equal almost everywhere. This is a Hilbert space with inner product

$$\langle f, h \rangle = \int_G f(g)\overline{h(g)} d\mu(g), \qquad f, h \in L^2(G). \tag{3.2.3}$$

The Fourier transform is a very useful tool.

Definition 3.2.5. (Fourier Transform) Let G be a locally compact abelian group, and let $f \in L^1(G)$. Then the Fourier transform $\mathcal{F}$ takes f to a function $\mathcal{F}f$ on $\hat{G}$ given by

$$\mathcal{F}f(\xi) = \int_G f(g)\overline{\xi(g)} d\mu(g), \qquad \xi \in \hat{G}. \tag{3.2.4}$$

We will often write $\hat{f}$ instead of $\mathcal{F}f$.

Theorem 3.2.2. *(Plancherel) The Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to a unitary isomorphism from $L^2(G)$ to $L^2(G)$.*

The above theorem is found in [3].

Proposition 3.2.3. *Let G be a compact group and let μ be its Haar measure normalized such that $\mu(G) = 1$. Then $\hat{G}$ form an orthonormal basis in $L^2(G)$.*

Proof. Let $\chi \in \hat{G}$. We know that $\int_G \chi(g)\overline{\chi(g)} d\mu(g) = 1$ since $\overline{\chi(g)}\chi(g) = 1$. Now let $\eta \in \hat{G}$ be different from χ. Then there exists an element h such that $\chi\eta^{-1}(h) \neq 1$. Then

$$\begin{aligned} \int_G \chi\overline{\eta}(g) d\mu(g) &= \int_G \chi\eta^{-1}(g) d\mu(g) \\ &= \chi\eta^{-1}(h) \int_G \chi\eta^{-1}(g-h) d\mu(g). \end{aligned} \tag{3.2.5}$$

We will make the substitution $g' = g - h$. By the invariance of the Haar measure we get that the integral equals

$$\chi\eta^{-1}(h) \int_G \chi\eta^{-1}(g') d\mu(g'). \tag{3.2.6}$$

Since $\chi\eta^{-1}(h) \neq 1$, we get that $\int_G \chi\eta^{-1}(g) d\mu(g) = 0$, which proves that the characters form an orthonormal set. Now, let $f \in L^2(G)$. If

$$\int_G f(g)\overline{\chi(g)} d\mu(g) = \hat{f}(\chi) = 0 \tag{3.2.7}$$

for all $\chi \in \hat{G}$, then we get that $f = 0$ by Theorem 3.2.2. This proves the proposition. $\square$

Proposition 3.2.4. *If G is compact, then $\hat{G}$ is discrete.*

The proof is found in [3].

Finally we will state a big theorem called the Pontryagin duality theorem. We will not need it, but it is included here for completeness. A proof is given in [3].

Theorem 3.2.5. *(Pontryagin Duality Theorem) Let G be a locally compact abelian group. Define Φ_x to be the element in the double dual of G acting as*

$$\Phi_x(\xi) = \xi(x). \tag{3.2.8}$$

Then $\Phi : G \to \hat{\hat{G}}$, $x \mapsto \Phi_x$, is a topological and algebraic isomorphism.

3.3 Integration on $\mathbb{Q}_p$

All functions on the p-adic numbers will be complex valued.

The Haar measure is unique up to a scalar. Then the Haar measure μ which satisfies $\mu(\mathbb{Z}_p) = 1$ is unique, and this is the Haar measure which will be used. By Lebesgue theory this gives a p-adic integral. So with this measure we have

$$\int_{\mathbb{Z}_p} d\mu(x) = 1 \tag{3.3.1}$$

Then it is possible to find the volume of balls of radius p^k which we will denote by B_k. Note that $B_k = p^{-k}\mathbb{Z}_p$. A p-adic integer x can be written as $x = a_0 + a_1 p + ...$. For $0 \leq a, b < p-1$, let $A = \{x \in \mathbb{Z}_p : a_0 = a\}$ and $B = \{x \in \mathbb{Z}_p : a_0 = b\}$. By translation invariance, $\mu(A) = \mu(B)$. This gives that $\mu(A) = \mu(p\mathbb{Z}_p) = 1/p$. Doing this inductively one gets that

$$\mu(B_{-k}) = p^{-k}, \tag{3.3.2}$$

where B_{-k} is the ball of radius p^{-k} and k is a positive integer. One can do this in the same way for balls of radius p^k, with k a positive integer, by noticing that all numbers on the form $a_{-1}p^{-1} + a_0 + a_1 p + ...$, with a_{-1} fixed, differ from $a_0 + a_1 p + ...$ by a constant $a_{-1}p^{-1}$. Again by induction one gets

$$\mu(B_k) = p^k. \tag{3.3.3}$$

To sum it up, $\mu(B_k) = p^k$ for $k \in \mathbb{Z}$. By translation invariance, the result extends to balls around an arbitrary element. Since two balls are either disjoint or contained in one another, an open set is a disjoint union of open balls. Since every open set is a disjoint union of balls, we know the measure of open sets. Finally, the Haar measure on Borel sets is determined by outer regularity of the Haar measure.

Lemma 3.3.1. *The measure μ_c given by $\mu_c(X) = \mu(cX)$ for a Borel set X in $\mathbb{Q}_p$ is also a Haar measure on $\mathbb{Q}_p$. Furthermore we have that*

$$\mu_c(X) = \mu(cX) = |c|_p \mu(X). \tag{3.3.4}$$

Proof. We see that for $a, c \in \mathbb{Q}_p$ and $X \subset \mathbb{Q}_p$

$$\mu_c(a + X) = \mu(ca + cX) = \mu(cX) = \mu_c(X) \tag{3.3.5}$$

so it is a (left) Haar measure. From Theorem 3.1.1 the Haar measure is unique up to a constant so $\mu_c(X) = f(c)\mu(X)$. Let the p-adic number c have absolute value p^{-n}. Since c can be written uniquely as up^n, $|u|_p = 1$ and $u\mathbb{Z}_p = \mathbb{Z}_p$ we get that

$$\mu_c(\mathbb{Z}_p) = \mu(c\mathbb{Z}_p) = \mu(p^n u\mathbb{Z}_p) = \mu(p^n\mathbb{Z}_p) = |c|_p\mu(\mathbb{Z}_p). \tag{3.3.6}$$

Hence, $f(c) = |c|_p$ and

$$\mu_c(X) = \mu(cX) = |c|_p\mu(X) \tag{3.3.7}$$

which was what we wanted. □

To know the integral over a "circle" in $\mathbb{Q}_p$ will be useful. By defining $S_k = \{x \in \mathbb{Q}_p : |x| = p^k\}$ one gets that

$$\int_{S_k} d\mu(x) = p^k - p^{k-1} \tag{3.3.8}$$

by noticing that S_k is the difference between B_k and B_{k-1}.

Lemma 3.3.2. *The dual group of $\mathbb{Q}_p$ is $\mathbb{Q}_p$.*

Proof. In this proof we will follow [3]. The main goal of the proof is to show that every character on $\mathbb{Q}_p$ can be written as

$$\chi_u(x) = e^{2\pi i\{ux\}} \tag{3.3.9}$$

with $u \in \mathbb{Q}_p$. In particular we define

$$\chi_p(x) = e^{2\pi i\{x\}}. \tag{3.3.10}$$

Recall that $\{a_{-n}p^{-n} + a_{-n+1}p^{-n+1} + ... + a_{-1}p^{-1} + a_0 + a_1p + ...\} = a_{-n}p^{-n} + a_{-n+1}p^{-n+1} + ... + a_{-1}p^{-1}$. The isomorphism (topological and algebraic) from $\mathbb{Q}_p$ to $\hat{\mathbb{Q}}_p$ is then given by $u \mapsto \chi_u$. One can show that $\{x + y\} = \{x\} + \{y\} - N$ where N is 1 or 0. Using this, it is not hard to show that $\chi_u(x) = e^{2\pi i\{ux\}}$ is a character on $\mathbb{Q}_p$ as an additive group, and that the map $u \mapsto e^{2\pi i\{ux\}}$ is a group homomorphism. It is also clear that the map is injective.

Now we will prove that all characters are of the form $\chi_u(x) = e^{2\pi i\{ux\}}$. Let χ be a character. Since a character is continuous and maps 0 to 1, there exists a ball B_k such that χ maps B_k into $\{z \in \mathbb{T} : |z - 1| < 1\}$. Since B_k is a subgroup of $\mathbb{Q}_p$, $\{z \in \mathbb{T} : |z - 1| < 1\}$ must be a subgroup of $\mathbb{T}$, and hence is the set $\{1\}$. This shows that there exists a ball B_k such that χ is equal to 1 on this ball.

Since χ is a homomorphism, if one knows the values it takes on the numbers $\{p^k\}$ where $k \in \mathbb{Z}$, then one knows how it acts on finite sums like $a_kp^k + ... + a_np^n$.

Since it is continuous, one also knows its values on infinite sums, and hence all p-adic numbers, namely as

$$\chi(\lim_{n\to\infty}\sum_{i=k}^{n} a_i p^i) = \lim_{n\to\infty}\chi(\sum_{i=k}^{n} a_i p^i). \tag{3.3.11}$$

Let $\tilde{\chi}$ be a character which is 1 on p^k for $k \geq 0$, but is not equal to 1 on p^{-1} (such a character exists, for instance $e^{2\pi i\{x\}}$). We want to prove that there is a sequence $(c_n)_0^\infty$ where $c_n \in \{0, 1, ..., p-1\}$ for $n > 0$ and $c_0 \in \{1, 2, ..., p-1\}$ such that $\tilde{\chi}(p^{-k}) = e^{2\pi i \sum_{j=1}^{k} c_{k-j}p^{-j}}$ for $k > 0$. One has that

$$\tilde{\chi}(p^{-(k+1)})^p = \tilde{\chi}(p^{-k}). \tag{3.3.12}$$

With $k = 0$, we get that

$$\tilde{\chi}(p^{-1})^p = 1, \tag{3.3.13}$$

so $\tilde{\chi}(p^{-1})$ is a pth root of unity, and thus $\tilde{\chi}(p^{-1}) = e^{2\pi i c_0 p^{-1}}$ for some $c_0 \in \{1, 2, ..., p-1\}$. Proceeding inductively, by equation (3.3.12), one gets that $\tilde{\chi}(p^{-k}) = e^{2\pi i \sum_{j=1}^{k} c_{k-j}p^{-j}}$ for some sequence (c_n) which satisfies what we claimed. Now we want to show that there exists a $u \in \mathbb{Q}_p$, with $|u|_p = 1$, such that $\tilde{\chi} = \chi_u$ with χ_u as in equation (3.3.9). Define $u = \sum_{j=0}^{\infty} c_j p^j$. Then $|u|_p = 1$ and for $k > 0$

$$\begin{aligned}\tilde{\chi}(p^{-k}) &= e^{2\pi i \sum_{j=1}^{k} c_{k-j}p^{-j}} = e^{2\pi i \sum_{j=-k}^{-1} c_{k+j}p^{j}} \\ &= \chi_p(\sum_{j=-k}^{\infty} c_{k+j}p^j) = \chi_p(p^{-k}u) = \chi_u(p^{-k}).\end{aligned} \tag{3.3.14}$$

So to finally prove that all characters are on the form $\chi_u(x) = e^{2\pi i\{ux\}}$, let ξ be an arbitrary character different from 1. Then there is an integer k such that $\xi(p^j) = 1$ for $j \geq k$ and $\xi(p^{k-1}) \neq 1$. Let $\eta(x) = \xi(p^k x)$. Then $\eta(x) = \chi_y(x)$ for some $y \in \mathbb{Q}_p$ with $|y|_p = 1$. But then

$$\xi(x) = \eta(p^{-k}x) = \chi_y(p^{-k}x) = \chi_{p^{-k}y}(x), \tag{3.3.15}$$

which proves that all characters are on the given form.

Finally we must show that $u \mapsto e^{2\pi i\{ux\}}$ is a homeomorphism. A neighbourhood base for 0 in $\mathbb{Q}_p$ is the set $\{B_k\}$ with $k \in \mathbb{Z}$. A neighbourhood base for 1 in $\hat{\mathbb{Q}}_p$ is the set

$$\tilde{N}(K, U) = \{\chi \in \hat{\mathbb{Q}}_p : \chi(K) \subset U\}, \tag{3.3.16}$$

where K is a compact set and U is a neighbourhood of 1. Since all compact sets are contained in a ball B_k which also is compact, we can use the set

$$N(j, k) = \{\chi \in \hat{\mathbb{Q}}_p : |\chi(x) - 1| < j^{-1} \text{ for } |x| \leq p^k\}, \tag{3.3.17}$$

where j runs through the positive integers and $k \in \mathbb{Z}$. Note that it is enough to look at neighbourhoods around the identities since translation is a homeomorphism in a topological group. $\chi_p(B_k)$ is equal to $\{1\}$ if $k \le 0$, and equal to the set of p^kth roots of unity if $k > 0$. Thus, the image of B_k is contained in $\{z \in \mathbb{T} : |z-1| < j^{-1}\}$ if and only if $k \le 0$. Similarly one gets that $\chi_u(B_k)$ is equal to $\{1\}$ if $|u|_p \le p^{-k}$, and equal to the set of p^{l-k}th roots of unity if $|u|_p = p^l > p^k$. Thus, $\chi_u \in N(j,k)$ if and only if $|u|_p \le p^{-k}$, and this proves that the map is a homeomorphism. $\square$

Lemma 3.3.3. *The integral $\int_{S_k} e^{2\pi i\{x\}} d\mu(x)$ is equal to -1 for $k = 1$ and equal to 0 when $k > 1$.*

Proof. The integral will first be calculated for $k = 1$. Here b_{-1} will be the notation for the p^{-1} coefficient for a p-adic number. The notation $b_{-1} = a_{-1}$ in the integral means that the integration goes over all p-adic numbers of absolute value p^{-1} where the p^{-1} coefficient is a_{-1}.

$$\begin{aligned}\int_{S_1} e^{2\pi i\{x\}} d\mu(x) &= \sum_{a_{-1}=1}^{p-1} \int_{b_{-1}=a_{-1}} e^{2\pi i a_{-1}/p} d\mu(x) \\ &= \sum_{a_{-1}=1}^{p-1} e^{2\pi i a_{-1}/p} = -1\end{aligned}$$

by using the fact that it is a geometric series.

For $k > 1$ one gets by a similar argument

$$\int_{S_k} e^{2\pi i\{x\}} d\mu(x) = \sum_{b_{-1},\ldots,b_{-k}} e^{2\pi i b_{-1}p^{-1}+\ldots+b_{-k}p^{-k}} = 0, \tag{3.3.18}$$

where the sum is a sum over all p-adic numbers of absolute value p^k and where b_{-i} varies from 0 to $p-1$ (b_{-k} is of course bigger than 0). The sum becomes zero because if one fixes b_{-2}, ... , b_{-k} and sums over b_{-1} it becomes zero. $\square$

Definition 3.3.1. (Tate-Gel'fand-Graev p-adic Gamma Function) The p-adic gamma function [4] (also called the Tate-Gel'fand-Graev p-adic gamma function) is defined as

$$\Gamma(s) = \int_{\mathbb{Q}_p} \chi_p(x)|x|_p^{s-1} d\mu(x) \tag{3.3.19}$$

where $\chi_p(x) = e^{2\pi i\{x\}}$, and s is a complex number.

By $|x|_p^{s-1}$, we mean $e^{(s-1)\ln(|x|_p)}$, where ln is the real logarithm. This can be written in terms of elementary functions. We will now calculate this integral to show an example of how integration is done over the p-adic numbers. It is convenient to calculate the integral over $\mathbb{Z}_p$ and $\mathbb{Q}_p \setminus \mathbb{Z}_p$ separately. This is because χ_p is equal to 1 in $\mathbb{Z}_p$. Also notice that $|x|_p$ is constant on each S_k (by definition of S_k). So

$$\begin{aligned}\int_{\mathbb{Z}_p} \chi_p(x)|x|_p^{s-1} d\mu(x) &= \sum_{k=0}^{\infty} \int_{S_{-k}} |x|_p^{s-1} d\mu(x) \\ &= \sum_{k=0}^{\infty} p^{-k(s-1)} \int_{S_{-k}} d\mu(x)\end{aligned}$$

Now by using equation (3.3.8) we get

$$\begin{aligned}\sum_{k=0}^{\infty} p^{-k(s-1)} \int_{S_{-k}} d\mu(x) &= \sum_{k=0}^{\infty} p^{-k(s-1)}(p^{-k} - p^{-k-1}) = \frac{p-1}{p} \sum_{k=0}^{\infty} p^{-sk} \\ &= \frac{p-1}{p} \frac{1}{1-p^{-s}}\end{aligned}$$

if $\operatorname{Re} s > 0$. The other part of the integral becomes by similar reasoning

$$\int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \chi_p(x)|x|_p^{s-1} d\mu(x) = \sum_{k=1}^{\infty} p^{k(s-1)} \int_{S_k} e^{2\pi i \{x\}} d\mu(x). \tag{3.3.20}$$

This sum is equal to $-p^{s-1}$ by Lemma 3.3.3.

By adding up all the parts one gets if $\operatorname{Re} s > 0$,

$$\begin{aligned}\Gamma(s) = \int_{\mathbb{Q}_p} \chi_p(x)|x|_p^{s-1} d\mu(x) &= \frac{p-1}{p} \frac{1}{1-p^{-s}-p^{s-1}} \\ &= \frac{1-p^{s-1}}{1-p^{-s}}.\end{aligned} \tag{3.3.21}$$

It is very important to know how to compute the Gaussian integrals. These integrals are time consuming to calculate, but they are done in detail in [5].

Theorem 3.3.4. *For $a \neq 0$ and $p \neq 2$,*

$$\int_{B_\gamma} \chi_p(ax^2 + bx) d\mu(x) = \begin{cases} p^\gamma \Omega(p^\gamma |b|_p), & |a|_p p^{2\gamma} \leq 1, \\ \lambda_p(a)|a|_p^{-1/2} \chi_p(-\frac{b^2}{4a}) \Omega(p^{-\gamma}|\frac{b}{2a}|_p), & |a|_p p^{2\gamma} > 1, \end{cases} \tag{3.3.22}$$

where $\Omega(|x|) = 1$ if $|x|_p \leq 1$ and $\Omega(|x|) = 0$ if $|x|_p > 1$.

Theorem 3.3.5. *For $a \neq 0$,*

$$\int_{B_\gamma} \chi_2(ax^2 + bx) d\mu(x) = \begin{cases} 2^\gamma \Omega(2^\gamma |b|_2), & |a|_2 2^{2\gamma} \leq 1, \\ \lambda_2(a)|2a|_2^{-1/2} \chi_2(-\frac{b^2}{4a}) \delta(|b|_2 - 2^{1-\gamma}), & |a|_2 2^{2\gamma} = 2, \\ \lambda_2(a)|2a|_2^{-1/2} \chi_2(-\frac{b^2}{4a}) \Omega(2^\gamma |b|_2), & |a|_2 2^{2\gamma} = 4, \\ \lambda_2(a)|2a|_2^{-1/2} \chi_2(-\frac{b^2}{4a}) \Omega(2^{-\gamma}|\frac{b}{2a}|_2), & |a|_2 2^{2\gamma} \geq 8, \end{cases} \tag{3.3.23}$$

where $\delta(0) = 1$ and δ is zero otherwise.

Letting $\gamma \to \infty$, one gets the next theorem.

Theorem 3.3.6. *For all p,*

$$\int_{\mathbb{Q}_p} \chi_p(ax^2 + bx)d\mu(x) = \lambda_p(a)|2a|_p^{-1/2}\chi_p(-\frac{b^2}{4a}), \qquad a \neq 0. \tag{3.3.24}$$

3.4 The Fourier Transform on $\mathbb{Q}_p$

First comes the definition of a test function. These are functions which have a well defined Fourier transform, and the set of test functions is invariant under the Fourier transform. This is the analog of the Schwartz functions on the real numbers.

Definition 3.4.1. (Test Function or Schwartz-Bruhat Function) A test function, ϕ, on $\mathbb{Q}_p$ is a function which is locally constant with compact support. To be locally constant means that there is an $m \in \mathbb{Z}$ such that for each $x \in \mathbb{Q}_p$, $\phi(x+t) = \phi(x)$ if $|t| \leq p^m$. This is also called a Schwartz-Bruhat function. The space of test functions is denoted by $\mathcal{D}(\mathbb{Q}_p)$

To have compact support on $\mathbb{Q}_p$ is equivalent to there being an $n \in \mathbb{Z}$ such that $\phi(x) = 0$ if $|x|_p \geq p^n$. This is the same since a set is compact in $\mathbb{Q}_p$ if and only if it is closed and bounded in $\mathbb{Q}_p$. The proof of this is found in [5].

Definition 3.4.2. (Fourier Transform) The Fourier transform of an integrable function ϕ is defined as

$$\hat{\phi}(u) = \int_{\mathbb{Q}_p} \chi_u(x)\phi(x)d\mu(x), \qquad u \in \mathbb{Q}_p \tag{3.4.1}$$

where χ_u is as given in equation (3.3.9).

Notice here that we have used the identification between $\mathbb{Q}_p$ and $\hat{\mathbb{Q}}_p$ given as $u \mapsto \chi_u$.

Lemma 3.4.1. *The Fourier transform of a p-adic test function is again a p-adic test function.*

Proof. Let $\phi(x)$ be zero when $|x|_p \geq p^n$ and $\phi(x+t) = \phi(x)$ for $|t| \leq p^m$. To prove that $\hat{\phi}(x)$ is compactly supported we call the integration variable for y, and do the substitution $y = x + t$ where $|t|_p = p^m$ to obtain

$$\hat{\phi}(u) = \chi_u(t) \int_{\mathbb{Q}_p} \chi_u(x)\phi(x)d\mu(x) = \chi_u(t)\hat{\phi}(u). \tag{3.4.2}$$

If $|u|_p > p^{-m}$ then $\chi_u(t) \neq 1$ so that $\hat{\phi}(u) = 0$.
Since $\phi(x) = 0$ for $|x|_p \geq p^n$

$$\hat{\phi}_p(u) = \int_{|x|_p < p^n} \chi_u(x)\phi(x)d\mu(x). \tag{3.4.3}$$

Then

$$\hat{\phi}_p(u+t) = \int_{|x|_p<p^n} \chi_t(x)\chi_u(x)\phi(x)d\mu(x) \tag{3.4.4}$$

which is equal to $\hat{\phi}(u)$ for $|t|_p \leq p^{-n}$ because then $\chi_t(x)$ is constantly equal to 1. This proves the lemma. □

It can be shown that the space of test functions is dense in the Hilbert space $L^2(\mathbb{Q}_p)$ in the L^2-norm. The inner product on $L^2(\mathbb{Q}_p)$ is of course given by

$$\langle \phi, \psi \rangle = \int_{\mathbb{Q}_p} \phi(x)\overline{\psi(x)}d\mu(x), \qquad \phi, \psi \in L^2(\mathbb{Q}_p). \tag{3.4.5}$$

We know from Theorem 3.2.2 that the Fourier transform extends to $L^2(\mathbb{Q}_p)$. The Fourier transform on $L^2(\mathbb{Q}_p)$ is given by

$$\hat{\phi}(u) = \lim_{\gamma\to\infty} \int_{B_\gamma} \phi(x)\chi_u(x)d\mu(x), \tag{3.4.6}$$

where $\phi \in L^2(\mathbb{Q}_p)$ and the limit is in the L^2-sense.

Chapter 4

The Adeles

In this chapter we will look at the adeles. The adeles are in some sense the product of all $\mathbb{Q}_p$ where p ranges over all primes and ∞, and $\mathbb{Q}_\infty = \mathbb{R}$. This will be a way to look at all the completions of $\mathbb{Q}$ at the same time, such that no $\mathbb{Q}_p$ is special. The notation will be as follows: When indexing with p, it will denote all primes, and not include ∞. Indexing with ν will give all primes and ∞.

4.1 Introduction to the Adeles

Definition 4.1.1. (Restricted Direct Product) Let Λ be an indexing set, and let $\{G_\lambda\}$ be a family of locally compact abelian groups, where $\lambda \in \Lambda$. For all but a finite number of λ let H_λ be an subgroup of G_λ which is open and compact. Call the subset of Λ where there is no H_λ for Λ'. Let G be the group consisting of all sequences $(g_\lambda)_\lambda$ where $g_\lambda \in G_\lambda$ for all λ and $g_\lambda \in H_\lambda$ for all but a finite number of λ. We get the topology of G by letting the basis of the topology be $\prod_\lambda U_\lambda$ where U_λ is an open set in G_λ for each λ and $U_\lambda = H_\lambda$ for all but a finite number of λ. With this topology, G is called the restricted direct product of $(G_\lambda)_\lambda$ with respect to $(H_\lambda)_\lambda$. The restricted direct product will be written as

$$G = \prod_\lambda{}' G_\lambda. \tag{4.1.1}$$

Lemma 4.1.1. *Let S be a finite subset of Λ where $\Lambda' \subset S$. Define G_S to be*

$$G_S = \prod_{\lambda \in S} G_\lambda \times \prod_{\lambda \notin S} H_\lambda. \tag{4.1.2}$$

Then G_S is an open subset of X (it is in the basis of the topology).

The product topology on G is different from the topology that it was given. The subspace topology on G_S however, is the same as the product topology on G_S. A set in the standard basis for the product topology is a product of open sets (some are in a H_λ, and some are in a G_λ) where all but a finite number of sets are H_λ. A basic open set in the subspace topology is a basic open set in G intersected with G_S. This set will be a product of open sets where all but a finite numbers is equal to H_λ. Now it is easily seen that these two topologies coincide.

Lemma 4.1.2. *If all G_λ are locally compact abelian groups and all H_λ are compact and open, then the restricted product of $(G_\lambda)_\lambda$ with respect to $(H_\lambda)_\lambda$ is a locally compact abelian group.*

Proof. Let S be a finite subset of Λ containing Λ'. Then the infinite product in G_S ($\prod_{\lambda \notin S} H_\lambda$) is compact by Tychonoff's theorem. Hence G_S is locally compact because it is a finite product of locally compact sets. Since $G = \bigcup_S G_S$ and each G_S is open, G is locally compact. What is left to prove is that the group operations are continuous. We will prove that addition is continuous, and then the proof of continuity of inversion will be similar. First we want to prove that addition is continuous on G_S. Let x_α and y_β be nets in G_S. We want to prove that if $x_\alpha \to x$ and $y_\beta \to y$, then $x_\alpha + y_\beta \to x + y$. Since G_S has the product topology, one needs to show that $x_\alpha^\lambda + y_\beta^\lambda \to x^\lambda + y^\lambda$ for each λ. But this is true since each G_λ is a topological group. Define the function $\phi : G \times G \to G$ by $\phi(a, b) = a + b$. This is the function we want to prove is continuous. To prove that it is continuous, take an open set O in G. Clearly $G \times G = \bigcup_{S,T} G_S \times G_T$, and since $G_S \times G_T \subset G_{S\cup T} \times G_{S\cup T}$, $G \times G = \bigcup_S G_S \times G_S$. Then we have that $\phi^{-1}(O) = \bigcup_S [\phi^{-1}(O) \cap (G_S \times G_S)]$. Finally since $\phi^{-1}(O) \cap (G_S \times G_S) = \phi^{-1}(O \cap G_S) \cap (G_S \times G_S)$, we have that $\phi^{-1}(O) = \bigcup_S [\phi^{-1}(O \cap G_S) \cap (G_S \times G_S)]$. Since $\phi^{-1}(O \cap G_S) \cap (G_S \times G_S)$ is open in $G_S \times G_S$, and hence in $G \times G$, ϕ is continuous.

□

Definition 4.1.2. (Adeles) Let $(G_\lambda)_\lambda$ consist of the additive groups $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p. Furthermore, let H_λ be $\mathbb{Z}_p$ for all primes p (remember that $\mathbb{Z}_p$ is a compact and open set). The adeles are then defined to be the restricted direct product of $(G_\lambda)_\lambda$ with respect to $(H_\lambda)_\lambda$ and are denoted by $\mathbb{A}$. In other words, x is an adele if it is an element of $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times ...$

$$x = (x_\infty, x_2, ..., x_p, ...) \tag{4.1.3}$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ for each p, and $|x_p|_p \leq 1$ for all but a finite number of x_p.

The adeles form a ring with pointwise addition and multiplication, and it is called the adele ring. It is not a field because not every element has an inverse (for instance if one component is 0).

Definition 4.1.3. (Principal Adeles) There is an inclusion

$$\mathbb{Q} \hookrightarrow \mathbb{A}, \qquad r \mapsto (r, r, r, ...). \tag{4.1.4}$$

The element $(r, r, r, ...)$ is an adele since eventually $|r|_p = 1$. The image of this inclusion is called the ring of principal adeles. It can be shown that $\mathbb{Q}$ is discrete in $\mathbb{A}$.

By Theorem 2.2.1 we see that for every principal adele $r \neq 0$ we have

$$|r|_{\mathbb{A}} = 1, \tag{4.1.5}$$

where $|r|_{\mathbb{A}} = \prod_{\nu} |r|_{\nu}$.

An additive character on the adeles is a continuous homomorphism to the unit circle in $\mathbb{C}$. The character which will be used here is given by

$$\chi_{\mathbb{A}}(x) = \prod_{\nu} \chi_{\nu}(x_{\nu}) = \prod_{\nu} \exp\left(2\pi i \{x_{\nu}\}_{\nu}\right) \tag{4.1.6}$$

Here $\{x_\infty\}_\infty$ means $-x_\infty$ such that $\exp(2\pi i\{x_\infty\}_\infty)$ will be $\exp(-2\pi i x_\infty)$. Again, all the factors except for a finite number will be one since $\{x\} = 0$ for $x \in \mathbb{Z}_p$, and the product converges. We will show that this additive character is continuous. We know that each $\exp(2\pi i\{x_\nu\}_\nu)$ is continuous on $\mathbb{Q}_\nu$, and then it will be continuous on each G_S in the product topology, and hence in the subspace topology. The product of all these functions will be a continuous function on G_S since all but a finite number of functions are constantly equal to 1, and a finite product of continuous functions is a continuous function. Since it is continuous on each G_S in the subspace topology and G_S are open sets, it is continuous on the whole space G.

Lemma 4.1.3. *For a principal adele $r = (r, r, ...)$, the additive character $\chi_{\mathbb{A}}$ is equal to 1. In other words we have the adelic relation*

$$\chi_{\mathbb{A}}(r) = \prod_{\nu} \exp\left(2\pi i\{r\}_{\nu}\right) = 1 \tag{4.1.7}$$

Proof. The rational number r can be written as $r = N p_1^{-\alpha_1} p_2^{-\alpha_2} \cdots p_k^{-\alpha_k}$ where N is an integer and α_i are positive integers. Now r can be written in the form

$$r = \frac{N_1}{p^{\alpha_1}} + \frac{N_2}{p^{\alpha_2}} + ... + \frac{N_k}{p^{\alpha_k}} + M \tag{4.1.8}$$

where M is an integer and $1 \leq N_i < p^{\alpha_i}$ for $i = 1...k$. Since $\{r\}_p = \frac{N_i}{p^{\alpha_i}}$ for $p = p_i$ and 0 otherwise,

$$\sum_{p} \{r\}_p = r - M. \tag{4.1.9}$$

Since $\{r\}_\infty = -r$ by the definition above, $\sum_\nu \{r\}_\nu = -M$, and the result follows. □

Let S be a finite set such that it contains the coordinate for $\mathbb{Q}_\infty$ in the restricted direct product. Define $\mathbb{A}_S$ to be

$$\mathbb{A}_S = \prod_{\nu \in S} \mathbb{Q}_\nu \times \prod_{\nu \notin S} \mathbb{Z}_\nu. \tag{4.1.10}$$

A useful result is that a sequence of adeles a_n converges to $a \in \mathbb{A}_S$ if and only if it converges component-wise and that a_n eventually is in $\mathbb{A}_S$.

4.2 The Haar Measure on $\mathbb{A}$

The additive structure of the adele ring is a locally compact group, and thus it has a Haar measure. The restriction that $|x_p|_p \leq 1$ for all but a finite number of x_p was used to make the space locally compact.

It would be nice if the Haar measure would be (with the right scaling) just $\mu = \prod_\nu \mu_\nu$ where each μ_ν is the Haar measure on $\mathbb{Q}_\nu$. This is not the case, but it will be the product of the measures in some sense. For the next theorem about the Haar measure on $\mathbb{A}$ we will need two results from the section on Radon measures in [2].

Theorem 4.2.1. *If X and Y are second countable spaces and μ and ν are Radon measures on X and Y respectively, then $\mu \times \nu$ is a Radon measure on $X \times Y$.*

Theorem 4.2.2. *Let $(X_\alpha)_{\alpha \in A}$ be a collection of compact Hausdorff spaces, and for each α let μ_α be a Radon measure on X_α such that $\mu_\alpha(X_\alpha) = 1$. For $\alpha_1, ..., \alpha_n \in A$, let $\pi_{(\alpha_1,...,\alpha_n)}$ be the projection $\pi_{(\alpha_1,...,\alpha_n)}(x) = (x_{\alpha_1}, ..., x_{\alpha_n})$. Then there is a unique Radon measure μ on $X = \prod_\alpha X_\alpha$ such that for any $\alpha_1, ..., \alpha_n \in A$ and any Borel set E in $\prod_{i=1}^n X_{\alpha_i}$, we have that*

$$\mu \circ \pi^{-1}_{(\alpha_1,...,\alpha_n)}(E) = (\mu_{\alpha_1} \times \cdots \times \mu_{\alpha_n})(E).^1 \tag{4.2.1}$$

Theorem 4.2.3. *Let S be a finite set such that it contains the coordinate for $\mathbb{Q}_\infty$ in the restricted direct product. Define $\mathbb{A}_S$ to be*

$$\mathbb{A}_S = \prod_{\nu \in S} \mathbb{Q}_\nu \times \prod_{\nu \notin S} \mathbb{Z}_\nu. \tag{4.2.2}$$

Then the Haar measure on $\mathbb{A}$ restricted to $\mathbb{A}_S$, denoted by μ_S, is equal to the product of the measures μ_ν where the measures are scaled such that $\mu_p(\mathbb{Z}_p) = 1$ and $\mu_\infty([0,1]) = 1$.

Proof. We will look at $\mathbb{A}_S$ as a finite product of the compact group $\prod_{\nu \notin S} \mathbb{Z}_\nu$ and the groups $\mathbb{Q}_\nu$ for $\nu \in S$. Then we can use Theorem 4.2.2 to get a Radon measure on $\prod_{\nu \notin S} \mathbb{Z}_\nu$, and then use Theorem 4.2.1 to get a Radon measure μ'_S on $\mathbb{A}_S$. Since the measures μ_ν are left invariant, so is μ'_S, and hence it is a left Haar measure on $\mathbb{A}_S$. Let $\tilde{\mu}$ be a Haar measure on $\mathbb{A}$, and denote the restriction of $\tilde{\mu}$ to $\mathbb{A}_S$ by $\tilde{\mu}_S$. Since the two measures are both Haar measures on $\mathbb{A}_S$, the two measures only differ by a scalar. Then choose the measure μ on $\mathbb{A}$ such that the restriction of μ to $\mathbb{A}_S$ is equal to μ'. Call the restricted measure μ_S. The question now is if μ is independent of the choice of the set S. We will show that it is. So let T be a finite set such that $S \subset T$. By the same procedure for the set T as we did for S, we get

[1] In Folland's book, the symbol $\hat{\times}$ is used instead of $\times$ because the product is actually a Radon product(which is different in the general case), but in our case all the spaces are second countable and the measures are σ-finite, and then the Radon product coincides with the normal product. For more, see chapter 7.4 in the book.

a measure μ_T. Now we will define $\mathbb{A}_T$ in a similar fashion as for $\mathbb{A}_S$. So define it as

$$\mathbb{A}_T = \prod_{\nu \in T} \mathbb{Q}_\nu \times \prod_{\nu \notin T} \mathbb{Z}_\nu, \tag{4.2.3}$$

and then we can see that $\mathbb{A}_S$ is a subgroup of $\mathbb{A}_T$. The restriction of μ_T to $\mathbb{A}_S$ is again a Haar measure, and it is easily seen that it must be equal to μ_S (just evaluate them on $[0,1] \times \prod_\nu \mathbb{Z}_\nu$). Hence the measure μ is the same for the bigger set T. Then let U be another finite set that contains the index for $\mathbb{Q}_\infty$. By what we have just seen, we will choose the same measure μ for the sets S and $S \cup U$ to get the product of the measures. Restricting $\mu_{S\cup U}$ to $\mathbb{A}_U$ gives the product of the measures, and hence the choice of μ is independent of the set S. $\square$

Because of this result one often writes

$$\mu = \mu_\infty \mu_2 \mu_3 \cdots . \tag{4.2.4}$$

or $\mu = \prod_\nu \mu_\nu$. We will use this measure in the next sections, and we will keep the scaling that $\mu_p(\mathbb{Z}_p) = 1$ and $\mu_\infty([0,1]) = 1$.

4.3 Integration on the Adeles

All the functions defined over the adeles will be complex valued unless something else is explicitly stated. We will look at integration of special types of function on the form $f(a) = \prod_\nu f_\nu(a_\nu)$ where a is an adele. We can then make use of Theorem 4.2.3.

Lemma 4.3.1. *For a function $f(a)$ on $\mathbb{A}$ we have that if f is a real non-negative measurable function or is in $L^1(\mathbb{A})$, then*

$$\int_{\mathbb{A}} f(a) d\mu(a) = \lim_S \int_{\mathbb{A}_S} f(a) d\mu(a) \tag{4.3.1}$$

where the limit is taken over larger and larger finite sets of indices S(it is a limit of the net where inclusion of sets is the binary relation). In the case of a real non-negative measurable function, the integrals are allowed to take the value ∞.

Proof. We know that for such functions,

$$\int_{\mathbb{A}} f(a) d\mu(a) = \lim_K \int_K f(a) d\mu(a) \tag{4.3.2}$$

where the limit is taken over larger and larger compact sets K in $\mathbb{A}$. Since every compact set is contained in some $\mathbb{A}_S$ (since the sets $\mathbb{A}_T$ are open sets covering $\mathbb{A}$), we have that

$$\lim_K \int_K f(a) d\mu(a) \le \lim_S \int_{\mathbb{A}_S} f(a) d\mu(a) \tag{4.3.3}$$

when f is a positive measurable function. The other inequality (when f is a positive measurable function) comes from the fact that

$$\lim_S \int_{\mathbb{A}_S} f(a)d\mu(a) \leq \int_{\mathbb{A}} f(a)d\mu(a). \tag{4.3.4}$$

If f is in $L^1(\mathbb{A})$, then one can write it as $f = f_1 - f_2 + if_3 - if_4$ where f_i is a positive measurable function for each i, and the result follows. $\square$

Lemma 4.3.2. *Let S be a finite set of indices containing the index for $\mathbb{Q}_\infty$. For each ν define a continuous function $f_\nu \in L^1(\mathbb{Q}_\nu)$ such that for $\nu \notin S$, $f_\nu(a_\nu) = 1$ on $\mathbb{Z}_\nu$. Define the function f on $\mathbb{A}$ to be $f(a) = \prod_\nu f_\nu(a_\nu)$. Then*

(i) The function f is continuous on $\mathbb{A}$.

(ii) We have

$$\int_{\mathbb{A}_S} f(a)d\mu(a) = \prod_{\nu\in S} \left[\int_{\mathbb{Q}_\nu} f_\nu(a_\nu)d\mu_\nu(a_\nu)\right]. \tag{4.3.5}$$

Proof.
(i) The proof is similar to the proof that the characteristic function $\chi_{\mathbb{A}}$ is continuous. It comes from the fact that f is a finite product of continuous functions on each $\mathbb{A}_T$ where T contains the index for $\mathbb{Q}_\infty$.

(ii) We have that

$$\begin{aligned}\int_{\mathbb{A}_S} f(a)d\mu(a) &= \int_{\mathbb{A}_S} f(a)d\mu_S(a)\\ &= \prod_{\nu\in S}\left[\int_{\mathbb{Q}_\nu} f_\nu(a_\nu)d\mu_\nu(a_\nu)\right]\prod_{\nu\notin S}\left[\int_{\mathbb{Z}_\nu} f_\nu(a_\nu)d\mu_\nu(a_\nu)\right] \\ &= \prod_{\nu\in S}\left[\int_{\mathbb{Q}_\nu} f_\nu(a_\nu)d\mu_\nu(a_\nu)\right]\end{aligned} \tag{4.3.6}$$

since $f(a) = \prod_\nu f_\nu(a_\nu)$ and $\mu = \prod_\nu \mu_\nu$. One gets the last line from the fact that all the integrals when $\nu \notin S$ are equal to 1. $\square$

Theorem 4.3.3. *Let the notation be as in the preceding lemma and define*

$$\prod_\nu \left[\int_{\mathbb{Q}_\nu} |f_\nu(a_\nu)|d\mu_\nu(a_\nu)\right] = \lim_S \left\{\prod_{\nu\in S}\left[\int_{\mathbb{Q}_\nu} |f_\nu(a_\nu)|d\mu_\nu(a_\nu)\right]\right\}. \tag{4.3.7}$$

If

$$\prod_\nu \left[\int_{\mathbb{Q}_\nu} |f_\nu(a_\nu)|d\mu_\nu(a_\nu)\right] < \infty, \tag{4.3.8}$$

then $f \in L^1(\mathbb{A})$ and

$$\int_{\mathbb{A}} f(a)d\mu(a) = \prod_\nu \left[\int_{\mathbb{Q}_\nu} f_\nu(a_\nu)d\mu_\nu(a_\nu)\right]. \tag{4.3.9}$$

Proof. Since f is a continuous function, it is measurable, and hence $|f|$ is measurable. Since $|f(a)|$ and $|f_\nu(a_\nu)|$ satisfy the conditions of the two preceding lemmas,

$$\begin{aligned}\int_{\mathbb{A}} |f(a)| d\mu(a) &= \lim_S \int_{\mathbb{A}_S} |f(a)| d\mu(a) \\ &= \lim_S \left\{ \prod_{\nu \in S} \left[\int_{\mathbb{Q}_\nu} |f_\nu(a_\nu)| d\mu_\nu(a_\nu) \right] \right\} < \infty.\end{aligned} \tag{4.3.10}$$

Applying the two lemmas again (but now using the other condition in Lemma 4.3.1) on the function $f(a)$ itself gives the result. □

4.4 Fourier Transform on the Adeles

Lemma 4.4.1. *Let G be the restricted direct product of locally compact abelian groups G_λ with respect to the subgroups H_λ. Then $\hat{G}$ is isomorphic, topologically and algebraically, to the restricted direct product of the $\hat{G}_\lambda$'s, that is*

$$\hat{G} \cong \prod_\lambda{}' \hat{G}_\lambda. \tag{4.4.1}$$

Here the product is with respect to $H_\lambda^\perp \subset \hat{G}_\lambda (\lambda \notin \Lambda)$ where $H_\lambda^\perp$ is the subgroup consisting of all characters in $\hat{G}_\lambda$ that are equal to 1 on H_λ. The subgroups $H_\lambda^\perp$ are open and compact such that they satisfy the conditions for the restricted direct product.

The proof is found in [6].

Lemma 4.4.2. *$\mathbb{A}$ is self-dual. That is, $\hat{\mathbb{A}} \cong \mathbb{A}$ as groups and topological spaces.*

Proof. From Lemma 4.4.1 we have that $\hat{\mathbb{A}} \cong \prod_\nu' \hat{\mathbb{Q}}_\nu$ with respect to $\mathbb{Z}_\nu^\perp$ for $\nu \neq \infty$. From Lemma 3.3.2 we have an isomorphism from $\mathbb{Q}_p$ to $\hat{\mathbb{Q}}_p$ given by $\xi \mapsto \psi_p(\xi)$, where $[\psi_p(\xi)](x) = \exp(2\pi i \{\xi x\})$. We also know that an isomorphism from $\mathbb{R}$ to $\hat{\mathbb{R}}$ is given by $\xi \mapsto \psi_\infty(\xi)$, where $[\psi_\infty(\xi)](x) = \exp(-2\pi i \xi x)$. Then for an adele $a = (a_\nu)_\nu$ we have the isomorphism from $\mathbb{A}$ to $\hat{\mathbb{A}}$, namely

$$a \mapsto (\psi_\nu(a_\nu))_\nu. \tag{4.4.2}$$

It is easy to see that it is an algebraic isomorphism. To see that it is a topological isomorphism, one just has to note that $\psi_p(\mathbb{Z}_p) = \mathbb{Z}_p^\perp$ and $\psi_p^{-1}(\mathbb{Z}_p^\perp) = \mathbb{Z}_p$. This is because an element in the base of the topology of $\mathbb{A}$ is of the form $\prod_\nu O_\nu$ where O_ν are open for all ν and equal to $\mathbb{Z}_p$ for all but a finite number number of p, and an element in the base of the topology of $\hat{\mathbb{A}}$ is of the form $\prod_\nu O_\nu$ where O_ν are open for all ν and equal to $\mathbb{Z}_p^\perp$ for all but a finite number number of p.

□

Next comes a class of functions called Schwartz-Bruhat functions. This space is in many ways analogous to the Schwartz functions.

Definition 4.4.1. (Schwartz-Bruhat functions) The Schwartz-Bruhat functions on $\mathbb{A}$ are the functions which are finite linear combinations of functions $\Phi_{\mathbb{A}}$ which satisfy

1. $\Phi_{\mathbb{A}}(x) = \prod_\nu \phi_\nu(x_\nu)$
2. $\phi_\infty(x_\infty)$ is function on $\mathbb{R}$ which is infinitely differentiable and such that the function and all its derivatives decrease faster than any power of $|x_\infty|$ as $|x_\infty| \to \infty$. This is called a Schwartz function.
3. $\phi_p(x_p)$ is a Schwartz-Bruhat function on $\mathbb{Q}_p$. In other words, it is a p-adic test function.
4. $\phi_p(x_p) = \Omega(|x_p|_p)$ for all but a finite number of p.

The functions $\Phi_{\mathbb{A}}(x)$ are called elementary functions. The space of Schwartz-Bruhat functions is often denoted by $\mathcal{S}(\mathbb{A})$.

The last condition will make the product in the first condition converge. We want to define the Fourier transform on the group of adeles.

Definition 4.4.2. (Fourier Transform on the Adeles) The adelic Fourier transform of a function $f \in L^1(\mathbb{A})$ is defined to be

$$\hat{f}(u) = \int_{\mathbb{A}} \chi_{\mathbb{A}}(ux) f(x) d\mu(x), \qquad u \in \mathbb{A}. \tag{4.4.3}$$

Lemma 4.4.3. *Let S be a finite set of indices containing the index for $\mathbb{Q}_\infty$. For each ν define a continuous function $f_\nu \in L^1(\mathbb{Q}_\nu)$ such that for $\nu \notin S$, $f_\nu(a_p) = \Omega(|a_p|_p)$. Define the function f on $\mathbb{A}$ to be $f(a) = \prod_\nu f_\nu(a_\nu)$. Then*

$$\hat{f}(u) = \prod_\nu \hat{f}_\nu(u_\nu). \tag{4.4.4}$$

Proof. Note that all $f_\nu(x_\nu) \exp(2\pi i \{u_\nu x_\nu\}_\nu)$ satisfy the conditions in Theorem 4.3.3 and that proves the equation $\hat{f}(u) = \prod_\nu \hat{f}_\nu(u_\nu)$. □

The Fourier transform acts extra nicely on Schwartz-Bruhat functions.

Lemma 4.4.4. *The Fourier transform of a Schwartz-Bruhat function is a Schwartz-Bruhat function.*

Proof. By the linearity of the integral, it suffices to show that the Fourier transform of an elementary function is an elementary function. Let $f(x)$ be an elementary function. Then

$$\hat{f}(u) = \prod_\nu \hat{f}_\nu(u_\nu). \tag{4.4.5}$$

such that it satisfies the first condition.
We will now show that the Fourier transform of $\Omega(|x_p|_p)$ is $\Omega(|x_p|_p)$. Since $\Omega(|x_p|_p)$

is zero outside of $\mathbb{Z}_p$ we just integrate over $\mathbb{Z}_p$. Let u_p be a p-adic number where $|u_p| = p^n$. Then by the substitution $y = u_p x$ we get

$$\hat{\gamma}_p(u_p) = p^{-n} \int_{p^{-n}\mathbb{Z}_p} \chi(y) d\mu(y). \tag{4.4.6}$$

If $|u_p|_p \leq 1$, then χ is constantly equal to 1 so the integral is p^n. If $|u_p|_p > 1$, then the integral is zero by Lemma 3.3.3.
That the Fourier transform of f_∞ will be a Schwartz function is known from Fourier analysis on the real line.
Finally, from Lemma 3.4.1, the set of p-adic test functions is invariant under the Fourier transform. This proves the lemma. □

Theorem 4.4.5. *(Fourier Inversion Theorem) Let G be a locally compact abelian group, and let χ be the character in the Fourier transform. There exists a Haar measure on $\hat{G}$, denoted by $\hat{\mu}$, such that for all $f \in L^1(G)$ such that $\hat{f} \in L^1(\hat{G})$, we have for almost all x in G*

$$f(x) = \int_{\hat{G}} \overline{\chi(ux)} \hat{f}(u) d\hat{\mu}(u). \tag{4.4.7}$$

If f is continuous, it holds for all $x \in G$.

The proof of this theorem is found in [3]. The measure $\hat{\mu}$ on $\hat{G}$ is called the dual measure of μ.

If μ is the Haar measure on the adeles, we have that μ is self dual ($\hat{\mu} = \mu$). To see this, we will first prove that the measure on each $\mathbb{Q}_p$ is self dual.

Lemma 4.4.6. *The measure μ_p on $\mathbb{Q}_p$, scaled such that $\mu_p(\mathbb{Z}_p) = 1$, is self dual with respect to the isomorphism from $\mathbb{Q}_p$ to $\hat{\mathbb{Q}}_p$ given by $[\psi_p(\xi)](x) = \exp(2\pi i\{\xi x\})$.*

Proof. Let $f(x) = \Omega(|x_p|_p)$. The dual measure is a positive constant c_p times μ_p. Then by the inversion theorem used on f, we get

$$f(x) = c_p \int_{\mathbb{Z}_p} \exp(-2\pi i\{ux\}) d\mu(x). \tag{4.4.8}$$

For $x \in \mathbb{Z}_p$, the left hand side is 1, and the right hand side is c_p. For $|x| > 1$ both sides are 0 by Lemma 3.3.3. Hence c_p must be equal to 1. □

Lemma 4.4.7. *The Haar measure μ on $\mathbb{A}$ is self dual with respect to the isomorphism in Lemma 4.4.2.*

Proof. We know that $\hat{\mu} = c\mu$ for some scalar positive c since $\mathbb{A}$ is self dual. Let f be a non-zero Schwartz-Bruhat function, $f = \prod_\nu f_\nu(x_\nu)$. By the inversion theorem,

$$f(x) = c \int_{\mathbb{A}} \overline{\chi_{\mathbb{A}}(ux)} \hat{f}(u) d\mu(u). \tag{4.4.9}$$

Then by Lemma 4.4.3,

$$\int_{\mathbb{A}} \overline{\chi_{\mathbb{A}}(ux)} \hat{f}(u) d\mu(u) = \prod_{\nu} \int_{\mathbb{Q}_\nu} \overline{\chi_\nu(u_\nu x_\nu)} \hat{f}_\nu(u_\nu) d\mu_\nu(u_\nu). \tag{4.4.10}$$

By the inversion theorem on each $\mathbb{Q}_\nu$ this becomes

$$\prod_{\nu} \int_{\mathbb{Q}_\nu} \overline{\chi_\nu(u_\nu x_\nu)} \hat{f}_\nu(u_\nu) d\mu_\nu(u_\nu) = \prod_{\nu} f_\nu(x_\nu) \tag{4.4.11}$$

since each μ_ν is self dual. We are then left with the equation

$$f(x) = c \prod_{\nu} f_\nu(x_\nu), \tag{4.4.12}$$

so $c = 1$ since f is non-zero. □

Then we have that the inverse Fourier transform is given by

$$f(x) = \int_{\mathbb{A}} \overline{\chi_{\mathbb{A}}(ux)} \hat{f}(u) d\mu(u), \qquad x \in \mathbb{A}. \tag{4.4.13}$$

It can be shown that the space of Schwartz-Bruhat functions is dense in the Hilbert space $L^2(\mathbb{A})$ in the L^2-norm. The inner product on $L^2(\mathbb{A})$ is of course given by

$$\langle f, g \rangle = \int_{\mathbb{A}} f(x) \overline{g(x)} d\mu(x), \qquad f, g \in L^2(\mathbb{A}). \tag{4.4.14}$$

The Fourier transform extends to $L^2(\mathbb{A})$ by Theorem 3.2.2.

4.5 Orthonormal Basis for $L^2(\mathbb{A})$

We will look at how an orthonormal basis for the Hilbert space $L^2(\mathbb{A})$ looks like. This is done by showing that $L^2(\mathbb{A})$ is an infinite tensor product of the Hilbert spaces $L^2(\mathbb{Q}_\nu)$. In this section we will follow [12].

We begin by defining an infinite tensor product of Hilbert spaces.

Definition 4.5.1. (Stabilizing Sequence) Let $(\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of Hilbert spaces. A sequence $(e^n)_{n \in \mathbb{N}}$ where $e^n \in \mathcal{H}_n$ is called a stabilizing sequence if $||e^n|| = 1$ for all $n \in \mathbb{N}$.

Now, let $(\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of separable Hilbert spaces with a stabilizing sequence (e^n), and with orthonormal bases $(e^n_k)_{k \in \mathbb{N}}$ such that $e^n_1 = e^n$. Let $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive integer, and let Λ be the set of all α such that α_n eventually is equal to 1. Then define the formal product

$$e_\alpha = e^1_{\alpha_1} \otimes e^2_{\alpha_2} \otimes \cdots, \tag{4.5.1}$$

where $\alpha \in \Lambda$. Notice that since α_n eventually is one, $e^n_{\alpha_n}$ is eventually $e^n_1 = e^n$.

The infinite tensor product of the Hilbert spaces $(\mathcal{H}_n)$ with respect to the stabilizing sequence (e^n), denoted by $\bigotimes_{e,n} \mathcal{H}_n$, is defined to be the Hilbert space which has the set $\{e_\alpha\}_{\alpha\in\Lambda}$ as an orthonormal basis by definition. All elements in $\bigotimes_{e,n} \mathcal{H}_n$ are thus of the form

$$f = \sum_{\alpha\in\Lambda} f_\alpha e_\alpha, \tag{4.5.2}$$

where (f_α) is a sequence of complex numbers such that $\sum_{\alpha\in\Lambda} |f_\alpha|^2 < \infty$. The inner product of two elements $f = \sum_{\alpha\in\Lambda} f_\alpha e_\alpha$ and $g = \sum_{\alpha\in\Lambda} g_\alpha e_\alpha$ is

$$\langle f, g\rangle = \sum_{\alpha\in\Lambda} f_\alpha \overline{g}_\alpha. \tag{4.5.3}$$

Let X_n be a closed subspace of $\mathcal{H}_n$, and let $(e^n_k)_{k\in\mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}_n$ such that $(e^n_k)_{k\in\mathbb{N}}$ is an orthonormal basis for X_n, and $e^n_1 = e^n$ such that the stabilizing sequence (e^n) lies in (X_n). Define the spaces

$$\mathcal{H}_e = \bigotimes_{e,n} \mathcal{H}_n, \tag{4.5.4}$$

and

$$\mathcal{H}^l_e = \bigotimes_{n=1}^{l} \mathcal{H}_n \otimes \bigotimes_{e,n>l} X_n = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_l \otimes X_{l+1} \otimes X_{l+2} \otimes \cdots . \tag{4.5.5}$$

Lemma 4.5.1. *The spaces $\mathcal{H}_e$ and $\mathcal{H}^l_e$ satisfy*

$$\mathcal{H}_e = \overline{\bigcup_{l\geq 1} \mathcal{H}^l_e} \tag{4.5.6}$$

Proof. Since $\mathcal{H}^l_e \subset \mathcal{H}_e$ for all l, we have that $\mathcal{H}_e \supset \overline{\bigcup_{l\geq 1} \mathcal{H}^l_e}$. For the converse, we see that the element

$$e_\alpha = e^1_{\alpha_1} \otimes e^2_{\alpha_2} \otimes \cdots \otimes e^k_{\alpha_k} \otimes e^{k+1} \otimes e^{k+2} \otimes \cdots \tag{4.5.7}$$

in $\mathcal{H}_e$ is also in $\mathcal{H}^k_e$. It follows that $\mathcal{H}_e \subset \overline{\bigcup_{l\geq 1} \mathcal{H}^l_e}$ which proves the lemma. $\square$

Our goal is to show that $L^2(\mathbb{A}) = \bigotimes_{e,\nu} L^2(\mathbb{Q}_\nu)$ for some stabilizing sequence (e^n). The measures on the spaces we work with will be the measures obtained in Section 4.2. Remember that to define the integral for Schwartz-Bruhat functions we first did it for the set $\mathbb{A}_S = \prod_{\nu\in S} \mathbb{Q}_\nu \times \prod_{\nu\notin S} \mathbb{Z}_\nu$. We will use a similar strategy here. Define

$$\mathbb{A}_n = \mathbb{R} \times \prod_{p\leq p_n} \mathbb{Q}_p \times \prod_{p>p_n} \mathbb{Z}_p, \tag{4.5.8}$$

where p_n is the nth prime.

Theorem 4.5.2. *We have that*

$$\bigotimes_{e,p} L^2(\mathbb{Z}_p) \cong L^2(\prod_p \mathbb{Z}_p), \tag{4.5.9}$$

where $e^p(x) = \Omega(|x|_p) = 1$ for all primes p.

From this theorem we get the next corollary.

Corollary 4.5.3. *We have that*

$$L^2(\mathbb{R}) \otimes \bigotimes_{p \leq p_n} L^2(\mathbb{Q}_p) \otimes \bigotimes_{e, p > p_n} L^2(\mathbb{Z}_p) \cong L^2(\mathbb{A}_n), \tag{4.5.10}$$

where $e^p = \Omega(|x|_p) = 1$ for all primes $p > p_n$.

Theorem 4.5.4. *The space $L^2(\mathbb{A})$ is isomorphic to the infinite tensor product of $L^2(\mathbb{Q}_\nu)$, that is*

$$L^2(\mathbb{A}) \cong \bigotimes_{e,\nu} L^2(\mathbb{Q}_\nu), \tag{4.5.11}$$

where the elements in the stabilizing sequence (e^p) are given as $e^p(x) = \Omega(|x|_p)$, and e^∞ is any element in the orthonormal basis for $L^2(\mathbb{R})$.

Proof. By Lemma 4.5.1 and Corollary 4.5.3, we get that

$$\bigotimes_{e,\nu} L^2(\mathbb{Q}_\nu) \cong \overline{\bigcup_n L^2(\mathbb{A}_n)}. \tag{4.5.12}$$

What is left to show is that $\overline{\bigcup_n L^2(\mathbb{A}_n)} = L^2(\mathbb{A})$. The inclusion $\overline{\bigcup_n L^2(\mathbb{A}_n)} \subset L^2(\mathbb{A})$ is obvious. For the converse, since the Schwartz-Bruhat functions are dense in $L^2(\mathbb{A})$ an element $f \in L^2(\mathbb{A})$ is a limit of Schwartz-Bruhat functions f_i. Each f_i is an element in $L^2(\mathbb{A}_n)$ for some n. Hence the limit must be in $\overline{\bigcup_n L^2(\mathbb{A}_n)}$, and thus $\overline{\bigcup_n L^2(\mathbb{A}_n)} \supset L^2(\mathbb{A})$. This proves the theorem. □

Now for each $L^2(\mathbb{Q}_\nu)$ fix an orthonormal basis $(e_k^\nu)_{k\in\mathbb{N}}$ such that $(e_1^\nu)_\nu$ is the stabilizing sequence in Theorem 4.5.4. Then the orthonormal basis for $L^2(\mathbb{A})$ consists of elements

$$e_\alpha = e^\infty_{\alpha_\infty} \otimes e^2_{\alpha_2} \otimes \cdots, \tag{4.5.13}$$

where $\alpha = (\alpha_n)$ ranges over all sequences of positive integers which eventually become 1. So all elements are of the form

$$e_\alpha = e^\infty_{\alpha_\infty} \otimes e^2_{\alpha_2} \otimes e^3_{\alpha_3} \otimes \cdots \otimes e^k_{\alpha_{p_k}} \otimes e^{p_{k+1}} \otimes e^{p_{k+2}} \otimes \cdots, \tag{4.5.14}$$

where $e^p(x) = \Omega(|x|_p)$. It is evaluated on an adele $x = (x_\infty, x_2, ...)$ by

$$e_\alpha(x) = e^\infty_{\alpha_\infty}(x_\infty) e^2_{\alpha_2}(x_2) \cdots e^{p_k}_{\alpha_{p_k}}(x_k) \Omega(|x_{k+1}|_p) \Omega(|x_{k+2}|_p) \cdots. \tag{4.5.15}$$

Notice that it is a finite product.

Chapter 5

Quantum Mechanics

5.1 Some Classical Mechanics

Before one enters the world of quantum mechanics, one needs to understand some basic facts from classical mechanics. One can say that the main contributor to classical mechanics was Sir Isaac Newton. After Newton we have had two major reformulations, the Lagrangian formulation after Joseph Louis Lagrange, and the Hamiltonian formulation after William Rowan Hamilton.

The position of a particle is denoted by $\mathbf{q}$, and the momentum of a particle with mass m is given by $\mathbf{p} = m\mathbf{v}$. The total energy of a particle moving in a potential V(which is only a function of the position), considered as a function of position and momentum, is called the Hamiltonian and is given by

$$H(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}). \tag{5.1.1}$$

Here $T = \frac{\mathbf{p}^2}{2m}$ is the kinetic energy of the particle.

Generally, the position $\mathbf{q}$ is a point on a manifold M, and the momentum is a cotangent vector in the cotangent space $T^*_{\mathbf{q}}M$. We will simplify this, and let $M = \mathbb{R}^n$. The space of all possible positions of a particle is called the configuration space. According to Newtonian mechanics, if one knows all the forces acting on a particle, then the motion of the particle is completely determined by the momentum and position at an initial time. Having the position and momentum at a certain time gives us the state of the particle. The space of all possible pairs of position and momentum ($\mathbb{R}^n \times \mathbb{R}^n$) is thus called the state space (or phase space). We will later refer to this as the classical state space or classical phase space.

The Lagrangian of a particle is the function

$$L(\mathbf{q}, \mathbf{v}) = m\mathbf{v}^2 - H(\mathbf{q}, m\mathbf{v}) = \frac{m\mathbf{v}^2}{2} - V(\mathbf{q}) = T(\mathbf{v}) - V(\mathbf{q}). \tag{5.1.2}$$

Let $\gamma(t)$ be a smooth path in the configuration space, parametrized by the time t, such that $\gamma(t_0) = \mathbf{q_0}$ and $\gamma(t_1) = \mathbf{q_1}$. Then the action functional S is defined as

$$S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \gamma'(t))dt. \tag{5.1.3}$$

An important principle in classical mechanics is the principle of least action. It states that a path $\gamma(t)$ describes the motion of the particle if and only if it is a critical point of the action functional S. A critical point is where the action is stationary to first order. Note that the name principle of least action is a bit unfortunate as the action does not need to be a minimum. We will later see a quantum mechanical analog to the action integral called the Feynman path integral.

5.2 Introduction

In this section a few terms from quantum mechanics will be introduced, and the theory will be given for a particle moving in $\mathbb{R}^n$. As we will see, quantum mechanics is quite different from classical mechanics. It explains how small particles behave, that is, systems on an atomic scale. There are three concepts which are very important in quantum mechanics, and those are observables, states and the dynamics of a system. In quantum mechanics a particle in the space $\mathbb{R}^n$ is described by a complex valued wave function $\psi(\mathbf{q}, t)$, where $\mathbf{q} \in \mathbb{R}^n$ and $t \in \mathbb{R}$ is the time. For a fixed time t, $|\psi(\mathbf{q}, t)|^2$ can be interpreted as the probability density function for the position of the particle at time t, so the integral of this function is 1. This probabilistic view of the particle is at the heart of quantum mechanics, and is quite different from classical mechanics where a particle is at a given place at a given time. Position is something we can measure, or in other words, something we can observe. Thus, measurable quantities will be called observables.

The state space, or phase space, for a particle is a complex separable Hilbert space $\mathcal{H}$, and the possible states of the quantum system are represented by unit vectors in $\mathcal{H}$. The state of a quantum system at a given time is described by a wave function. From the probabilistic nature of the observables, we want to be able take their expectations. An observable will correspond to an operator on $\mathcal{H}$.

Definition 5.2.1. (Expectation of an Observable) The expectation of an observable which corresponds to the operator A in the state ψ is given as

$$E_\psi[A] = \langle A\psi, \psi \rangle. \tag{5.2.1}$$

Physicists often prefer the inner product to be linear in the second argument, but we will not follow this convention. The operators corresponding to position and momentum are important. We will not show the derivation of these operators. Q_j is the position operator for the jth coordinate and is densely defined on $L^2(\mathbb{R}^n)$ by

$$Q_j\psi(\mathbf{q}) = q_j\psi(\mathbf{q}), \tag{5.2.2}$$

and P_j is the momentum operator for the jth coordinate and is defined densely on $L^2(\mathbb{R}^n)$ by

$$P_j\psi(\mathbf{q}) = -i\hbar\frac{\partial}{\partial q_j}\psi(\mathbf{q}). \tag{5.2.3}$$

These operators correspond to classical position and momentum. Now we can for instance calculate the expectation of the jth coordinate of a particle with wave function ψ. The expectation at time t becomes

$$E_\psi[Q_j] = \int_{\mathbb{R}^n} q_j|\psi(\mathbf{q},t)|^2 d^n\mathbf{q}. \tag{5.2.4}$$

With the wave function, we can also find the probability of finding a particle inside a region in space. The probability of finding a particle in the state $\psi(\mathbf{q},t)$, in a region S, is

$$\int_S |\psi(\mathbf{q},t)|^2 d^n\mathbf{q} \tag{5.2.5}$$

Another important concept is the momentum space. The momentum space is $\mathbb{R}^n$ just as the configuration space, except that the space now consists of all possible momenta and not positions. The two spaces are closely related by the Fourier transform. Define the Fourier transform on $L^2(\mathbb{R}^n)$ as

$$\mathcal{F}\psi(\mathbf{p}) = \hat{\psi}(\mathbf{p}) = (2\pi\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{q}}\psi(\mathbf{q})d^n\mathbf{q}, \tag{5.2.6}$$

which is understood in the L^2-sense. It is a unitary operator with inverse

$$\mathcal{F}^{-1}\hat{\psi}(\mathbf{q}) = (2\pi\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{q}}\psi(\mathbf{p})d^n\mathbf{p}, \tag{5.2.7}$$

and the Parseval formula is

$$\int_{\mathbb{R}^n} |\hat{\psi}(\mathbf{p})|d^n\mathbf{p} = \int_{\mathbb{R}^n} |\psi(\mathbf{q})|d^n\mathbf{q}. \tag{5.2.8}$$

Then from integration by parts,

$$\mathcal{F}P_j\psi(\mathbf{p}) = (2\pi\hbar)^{-n/2}\frac{\hbar}{i}\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{q}}\frac{\partial}{\partial q_j}\psi(\mathbf{q})d^n\mathbf{q} = p_j\mathcal{F}\psi(\mathbf{p}). \tag{5.2.9}$$

We can then define $\hat{P}_j = \mathcal{F}P_j\mathcal{F}^{-1}$ which acts as multiplication by p_j, and this is analogous with Q_j acting as multiplication by q_j in the configuration space. Similarly $\hat{Q}_j = \mathcal{F}Q_j\mathcal{F}^{-1}$ acts as $i\hbar\frac{\partial}{\partial q_j}$. If the wave function in configuration space is ψ, then $\hat{\psi}$ is the wave function in the momentum space. There is no new physics in this, it is just a different representation of the same physical system. As an example, the probability that the momentum of a particle in a state ψ is in the set S at time t is

$$\int_S |\hat{\psi}(\mathbf{p},t)|^2 d^n\mathbf{p}. \tag{5.2.10}$$

We can not expect all the operators to be bounded. For instance the position operator is not bounded. Since the operators are unbounded, they will be defined on a smaller space. Let $D(A)$ denote the domain of the operator A. $D(A)$ is a linear subspace of $\mathcal{H}$, and since it is convenient for the operator to be defined for most states, $D(A)$ will also be assumed to be dense in $\mathcal{H}$.

Definition 5.2.2. (Adjoint Operator) Let A be an operator on the Hilbert space $\mathcal{H}$ with dense domain $D(A)$. The domain of the adjoint of A is the set of all $\psi \in \mathcal{H}$ such that the map $\phi \mapsto \langle \psi, A\phi \rangle$ ($\phi \in D(A)$) extends to a bounded linear functional on all of $\mathcal{H}$. Let $\tilde{\psi}$ be the element corresponding to the functional such that $\langle \tilde{\psi}, \phi \rangle = \langle \psi, A\phi \rangle$. The adjoint of A, A^*, is the operator which satisfies $A^*\psi = \tilde{\psi}$.

Note that A^* is well defined since $D(A)$ is dense.

Definition 5.2.3. (Symmetric Operator) Let A be an operator on the Hilbert space $\mathcal{H}$ with dense domain $D(A)$. Then it is called symmetric (or Hermitian) if

$$\langle \psi, A\phi \rangle = \langle A\psi, \phi \rangle \tag{5.2.11}$$

for all $\phi, \psi \in D(A)$.

For a symmetric operator, clearly $D(A) \subset D(A^*)$.

Definition 5.2.4. (Self-Adjoint Operator) Let A be a symmetric operator on $\mathcal{H}$. A is called a self-adjoint operator if $D(A^*) = D(A)$.

Since the operator is symmetric, A and A^* will also coincide on their domain. We then write $A = A^*$. Another important class of operators are the essentially self-adjoint operators.

Definition 5.2.5. (Essentially Self-Adjoint Operator) Let A be a symmetric operator on $\mathcal{H}$. A is called an essentially self-adjoint operator if it has a unique self-adjoint extension.

Now that we are done with general definitions, let us go back to operators associated with an observable. When doing measurements one wants results from the real numbers. Hence, it is convenient to assume that our operators are symmetric so they satisfy

$$\langle \psi, A\psi \rangle = \langle A\psi, \psi \rangle, \tag{5.2.12}$$

which is the expectation of A in the state ψ. We will not go into deeper detail, but we will require the operators corresponding to an observable to be self-adjoint.

We can of course also calculate the variance of the observables.

Definition 5.2.6. (Mean Square Deviation of an Observable) The mean square deviation of an observable corresponding to the operator A, in the state ψ, is

$$\Delta_\psi(A)^2 = E_\psi[A^2] - E_\psi[A]^2 = ||(A - E_\psi[A])\psi||^2. \tag{5.2.13}$$

Note that $\Delta_\psi(A)^2 = 0$ if and only if ψ is an eigenvector of A and $E_\psi[A]$ is the corresponding eigenvalue. When doing measurements of an observable, one can only obtain the eigenvalues of the operator corresponding to the observable. This is a reason why finding the eigenvalues is very important. If the set of eigenvalues is discrete, the observable can only obtain discrete value. This can for instance happen for the energy operator which is another surprising and unintuitive result in quantum mechanics.

Let us now turn to the dynamics of a quantum system. We want to know how the system changes over time. A state $\psi(\mathbf{q}, t)$ will change as

$$\psi(\mathbf{q}, t) = U(t)\psi(\mathbf{q}, 0). \tag{5.2.14}$$

There are certain properties $U(t)$ should have, based on the mathematical framework and on physical experiments. Since it should take states to states,

$$||U(t)\psi|| = ||\psi||. \tag{5.2.15}$$

Moreover, it is reasonable to assume that it is linear and continuous in the strong operator topology. Now we again need to state a few definitions.

Definition 5.2.7. (Unitary Representation) A unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$ is a homomorphism $G \mapsto U(\mathcal{H})$, where $U(\mathcal{H})$ is the set of unitary operators on $\mathcal{H}$.

Definition 5.2.8. (Strongly Continuous Unitary Representation) A strongly continuous unitary representation is unitary representation such that $x \mapsto U(x)f$ is continuous for all $f \in \mathcal{H}$.

Definition 5.2.9. (Strongly Continuous One-Parameter Unitary Group) Let U be strongly continuous unitary representation of G. Then the family of operators $U(x), x \in G$ is called a strongly continuous one-parameter unitary group.

We require U to be a strongly continuous unitary representation of $G = \mathbb{R}$ on $\mathcal{H}$. We call U (or $U(t)$) the evolution operator or time evolution.

The Hamilton operator H should also be mentioned. The Hamilton operator is the self-adjoint operator which corresponds to the energy of the system. In quantum mechanics we will look at Hamiltonians of the form $H = \frac{1}{2m}\mathbf{P}^2 + V(\mathbf{Q})$, where $\mathbf{P}^2 = P_1^2 + ... + P_n^2$, $\mathbf{Q} = (Q_1, ..., Q_n)$, V is the potential and m is the mass of the particle. In classical mechanics in the case where a particle is moving in space with no potential (V = 0), $H = T = \frac{p^2}{2m}$. In quantum mechanics, $H = \frac{1}{2m}\mathbf{P}^2$. In both classical physics and quantum mechanics, this is called a free particle.

An important equation in quantum mechanics is the (time dependent) Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{q}, t) = H\psi(\mathbf{q}, t), \tag{5.2.16}$$

where $\hbar = \frac{h}{2\pi}$ with h being Planck's constant[1]. This differential equation describes how the state of the system changes with time.

One can ask what the connection is between $U(t)$ and H. Equation (5.2.14) is the (weak) solution of equation (5.2.16) with $U(t) = e^{-iHt/\hbar}$ (we will look closer at this in Section 5.4). A theorem of Stone [9] gives the relation between self-adjoint operators and strongly continuous one-parameter unitary groups.

Theorem 5.2.1. *(Stone) Let $U(t)$ be a strongly continuous one-parameter unitary group. Then there exists a unique self-adjoint operator A such that*

$$U(t) = e^{-itA}. \tag{5.2.17}$$

Conversely, let A be a self-adjoint operator. Then $\{U(t) = e^{-itA}\}$ is a strongly continuous one-parameter unitary group.

Another important concept in quantum mechanics is Heisenberg's uncertainty principle. First define $[A, B] = AB - BA$.

Theorem 5.2.2. *(Heisenberg's Uncertainty Principle) Let A and B be two symmetric operators. Then*

$$\Delta_\psi(A)\Delta_\psi(B) \geq \frac{1}{2}|E_\psi([A, B])| \tag{5.2.18}$$

for all $\psi \in D(AB) \cap D(BA)$.

As a corollary, we get that

$$\Delta_\psi(P_j)\Delta_\psi(Q_j) \geq \frac{1}{2}\hbar \tag{5.2.19}$$

for all $\psi \in D(P_jQ_j) \cap D(Q_jP_j)$. So, if one measures the position of a particle very well, one knows little about its momentum, and conversely, if one knows the momentum very well, one knows little about the position. The error is not due to bad measurement equipment, but due to the nature of quantum mechanics.

5.3 Weyl System

An equivalent way to describe quantum mechanics, is to describe it by a Weyl system which is Hermann Weyl's formulation of quantum mechanics. In the previous section, we saw Heisenberg's uncertainty relation, which comes from the fact that

$$[Q_j, P_j] = i\hbar. \tag{5.3.1}$$

We would like to describe this relation in another way. In this section we will work in one dimension and denote by Q and P the position and momentum operator. Define

$$U(x) = e^{-ixQ}, \qquad V(\xi) = e^{-i\xi P}, \tag{5.3.2}$$

[1] h is approximately equal to $6.63 \cdot 10^{-34}$ Js (joule seconds).

where $x, \xi \in \mathbb{R}$. U and V are strongly continuous unitary representations.

These operators will act in the following way.

$$(U(x)f)(t) = e^{-itx}\phi(t), \qquad (V(\xi)f)(t) = \phi(t - \hbar\xi). \tag{5.3.3}$$

The operators satisfy

$$U(x)V(y) = e^{i\hbar xy}V(y)U(x), \qquad x, y \in \mathbb{R}. \tag{5.3.4}$$

Under suitable conditions on P and Q this equation is equivalent to Heisenberg's commutation relation. The pair (U, V) form what is called a Weyl system.

Definition 5.3.1. (Weyl System, first definition) Let G be a locally compact abelian group. Let R be a strongly continuous unitary representation of G, and S a strongly continuous unitary representation of $\hat{G}$. The pair (R, S) is called a Weyl system if it satisfies

$$R(x)S(\chi) = \chi(x)^{-1}S(\chi)R(x), \qquad x \in G, y \in \hat{G}. \tag{5.3.5}$$

Notice that in equation (5.3.4), the pair becomes a Weyl system from the fact that $\mathbb{R}$ is isomorphic to $\hat{\mathbb{R}}$.

Furthermore we can define W on $\mathbb{R} \times \mathbb{R}$ as

$$W(x, y) = e^{-(i\hbar/2)xy}U(x)V(y), \tag{5.3.6}$$

It satisfies the equation

$$W(x, y)W(x', y') = e^{i\hbar/2(xy' - x'y)}W(x + x', y + y'), \tag{5.3.7}$$

where $x, y, x', y' \in \mathbb{R}$. Now we will give a second definition of a Weyl system, using W instead of U and V.

Definition 5.3.2. (Weyl System, second definition) Let G be a separable locally compact abelian group. A Weyl system on $G \times \hat{G}$ is a pair $(\mathcal{H}, W)$ where $\mathcal{H}$ is a Hilbert space and W is a strongly continuous function from $G \times \hat{G}$ to the unitary operators on $\mathcal{H}$ such that

$$W(x)W(y) = m(x, y)W(x + y), \tag{5.3.8}$$

where m is a Borel function on $G \times \hat{G}$ to the complex numbers of absolute value one, and it is called the multiplier of W.

The above equation is called the Weyl relation. Notice here that W is not a representation if $m \neq 1$, but it is instead called a projective representation.

5.4 Feynman Path Integral

Feynman's path integral formulation is another equivalent way to describe quantum mechanics. It is the quantum mechanical analog of the action principle in classical mechanics.

It can be shown that when the Hamiltonian is of the form $H = \frac{\mathbf{P}^2}{2m} + V(\mathbf{Q})$ and V is bounded on compact subsets of $\mathbb{R}^n$ and bounded from below, then $\psi(\mathbf{q}', t) = U(t)\psi(\mathbf{q})$ is a weak solution of time-dependent Schrödinger equation, and the time evolution $U(t)$ is on the form

$$\psi(\mathbf{q}', t) = U(t)\psi(\mathbf{q}) = \int_{\mathbb{R}^n} \mathcal{K}_t(\mathbf{q}', \mathbf{q})\psi(\mathbf{q})d^n\mathbf{q}, \tag{5.4.1}$$

where $\psi \in L^2(\mathbb{R}^n)$. $\mathcal{K}_t^{(\infty)}$ is called the kernel or propagator, and is in general a distribution with initial condition $\mathcal{K}_0^{(\infty)} = \delta(\mathbf{q} - \mathbf{q}')$ where δ is Dirac's delta function. The equation also has to be understood in the L^2-sense. That is,

$$U(t)\psi(\mathbf{q}) = \lim_{R\to\infty} \int_{|\mathbf{q}|\leq R} \mathcal{K}_t(\mathbf{q}', \mathbf{q})\psi(\mathbf{q})d^n\mathbf{q}, \tag{5.4.2}$$

where the limit is in the L^2-norm. Also note that $U(t)$ has been extended to act on distributions.

Since $U(t' + t) = U(t')U(t)$, we get that

$$\psi(\mathbf{q}', t') = U(t' - t)\psi(\mathbf{q}, t) = \int_{\mathbb{R}^n} \mathcal{K}_{t'-t}(\mathbf{q}', \mathbf{q})\psi(\mathbf{q}, t)d^n\mathbf{q}. \tag{5.4.3}$$

The physical interpretation of $|\mathcal{K}_{t'-t}(\mathbf{q}', \mathbf{q})|^2$ is that it gives the probability density for the probability of a particle being at position $\mathbf{q}'$ at time t' given that it was at position $\mathbf{q}$ at time t.

The kernel is obtained for a free particle by solving the Schrödinger equation in the momentum space, which is

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(\mathbf{p}, t) = \frac{1}{2m}\mathbf{p}^2\psi(\hat{\mathbf{p}}, t), \qquad \hat{\psi}(\mathbf{p}, 0) = \hat{\psi}(p). \tag{5.4.4}$$

The solution is

$$\hat{\psi}(\mathbf{p}, t) = \exp{(-\frac{i\mathbf{p}^2}{2m\hbar}t)}\hat{\psi}(\mathbf{p}). \tag{5.4.5}$$

In the configuration space this becomes

$$\psi(\mathbf{q}', t') = (2\pi\hbar)^{-n/2}\int_{\mathbb{R}^n} \exp\Big(\frac{i}{\hbar}(\mathbf{q}' \cdot \mathbf{p} - \frac{1}{2m}\mathbf{p}^2)(t' - t)\Big)\hat{\psi}(\mathbf{p}, t)d^n\mathbf{p}. \tag{5.4.6}$$

One obtains the kernel

$$\mathcal{K}_{t'-t}(\mathbf{q}', \mathbf{q}) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(\mathbf{p}(\mathbf{q}'-\mathbf{q}) - \frac{\mathbf{p}^2}{2m}(t'-t))}d^n\mathbf{p}. \tag{5.4.7}$$

By the Fresnel integral (Lemma 5.4.1), one gets

$$\mathcal{K}_{t'-t}(\mathbf{q}',\mathbf{q}) = \Big(\frac{m}{2\pi i\hbar(t'-t)}\Big)^{n/2} e^{\frac{im}{2\hbar(t'-t)}(\mathbf{q}'-\mathbf{q})^2}, \tag{5.4.8}$$

where $i^{n/2} = e^{-\frac{i\pi n}{4}}$ and $T > 0$.

Lemma 5.4.1. *(Fresnel Integral) For a real number $a \neq 0$,*

$$\int_{-\infty}^{\infty} e^{iax^2} = e^{\frac{\pi i \operatorname{sgn}(a)}{4}} \sqrt{\frac{\pi}{|a|}}. \tag{5.4.9}$$

where the left side has to be interpreted as the principal value of the integral.

The lemma can be proved by contour integration.

Further on we will assume the space to be one dimensional ($\mathbb{R}$). Now that the kernel is found for a free particle, one could hope that the kernel for a Hamiltonian $H = H_0 + V$, where H_0 is the Hamiltonian for a free particle and V is the potential, is obtained by using $e^{-\frac{i}{\hbar}tH} = e^{-\frac{i}{\hbar}tH_0}e^{-\frac{i}{\hbar}tV}$. This is however not the case since H_0 and V do not commute. Fortunately we have the Lie-Kato-Trotter product formula.

Theorem 5.4.2. *(Lie-Kato-Trotter product formula) Let A and B be two self-adjoint operators on $\mathcal{H}$ such that $A+B$ is essentially self-adjoint on $D(A)\cap D(B)$. Then*

$$e^{i(A+B)}\psi = \lim_{n\to\infty}(e^{\frac{i}{n}A}e^{\frac{i}{n}B})^n\psi \tag{5.4.10}$$

for all $\psi \in \mathcal{H}$.

We will assume that the Hamiltonian is essentially self-adjoint on $D(H_0)\cap D(V)$, so the Lie-Kato-Trotter product formula can be used. Denote the time difference $t' - t$ by T, and let $\Delta t = T/n$. Then

$$e^{-\frac{i}{\hbar}TH} = \lim_{n\to\infty}(e^{-\frac{i\Delta t}{\hbar}H_0}e^{-\frac{i\Delta t}{\hbar}V})^n. \tag{5.4.11}$$

in the strong operator topology. From the kernel for $e^{-\frac{i\Delta t}{\hbar}H_0}$, the kernel for $e^{-\frac{i\Delta t}{\hbar}H_0}e^{-\frac{i\Delta t}{\hbar}V}$ then becomes

$$\mathcal{K}_{t'-t}(q',q) = \sqrt{\Big(\frac{m}{2\pi i\hbar\Delta t}\Big)}e^{\frac{i}{\hbar}(\frac{m}{2\Delta t}(q'-q)^2 - V(q)\Delta t)}. \tag{5.4.12}$$

Note that

$$\begin{aligned}
&e^{-\frac{i(t_2-t_1)}{\hbar}H_0}e^{-\frac{i(t_2-t_1)}{\hbar}V}e^{-\frac{i(t_1-t_0)}{\hbar}H_0}e^{-\frac{i(t_1-t_0)}{\hbar}V}\psi(q,t)\\
&= \int_{\mathbb{R}}\mathcal{K}_{t_2-t_1}(q_2,q_1)\int_{\mathbb{R}}\mathcal{K}_{t_1-t_0}(q_1,q_0)\psi(q_0,t_0)dqdq_1\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{K}_{t_2-t_1}(q_2,q_1)\mathcal{K}_{t_1-t_0}(q_1,q_0)dq_1\psi(q_0,t_0)dq_0
\end{aligned} \tag{5.4.13}$$

where Fubini is used for the last equality. The kernel for $(e^{-\frac{i\Delta t}{\hbar}H_0}e^{-\frac{i\Delta t}{\hbar}V})^2$ is then

$$\int_{\mathbb{R}} \mathcal{K}_{t_2-t_1}(q_2,q_1)\mathcal{K}_{t_1-t_0}(q_1,q_0)dq_1. \tag{5.4.14}$$

Continuing inductively one obtains the kernel for $(e^{-\frac{i\Delta t}{\hbar}H_0}e^{-\frac{i\Delta t}{\hbar}V})^n$ which is

$$\int_{\mathbb{R}^{n-1}} \prod_{k=0}^{n-1} \mathcal{K}_{\Delta t}(q_{k+1},q_k) \prod_{k=1}^{n-1} dq_k, \tag{5.4.15}$$

where $q_0 = q$ (obtained at time t) and $q_n = q'$ (obtained at time t').

Finally the kernel for $U(t'-t)$ is obtained by taking the limit,

$$\begin{aligned}\mathcal{K}_{t'-t}(q',q) &= \lim_{n\to\infty}\Big(\frac{m}{2\pi i\hbar\Delta t}\Big)^{n/2} \\ &\cdot \int_{\mathbb{R}^{n-1}} \exp\Big(\frac{i}{\hbar}\sum_{k=0}^{n-1}\big[\frac{m}{2}\Big(\frac{q_{k+1}-q_k}{\Delta t}\Big)^2 - V(q_k)\big]\Delta t\Big)\prod_{k=1}^{n-1} dq_k\end{aligned} \tag{5.4.16}$$

The convergence is in the distributional sense, such that the integral $\int_{\mathbb{R}^{n-1}}$ is to be understood as $n-1$ integrals of the form $\lim_{R\to\infty}\int_{|q_k|\le R}$ where the limit is in the L^2-norm.

Now we will look at the physical interpretation and see why it is called a path integral. If there is a smooth path $\gamma(t)$ such that $\gamma(t_k) = q(t_k) = q_k$, then we get that,

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\big[\frac{m}{2}\Big(\frac{q_{k+1}-q_k}{\Delta t}\Big)^2 - V(q_k)\big]\Delta t = S(\gamma), \tag{5.4.17}$$

where

$$S(\gamma) = \int_t^{t'} \frac{1}{2}m\dot{q}^2(\tau) - V(q(\tau))d\tau = \int_t^{t'} L(q(\tau),\dot{q}(\tau))d\tau \tag{5.4.18}$$

is the classical action.

The integral can then be interpreted as integration over all smooth paths which start in q at time t and end in q' at time t'. We then write the kernel as a Feynman path integral

$$\mathcal{K}_{t'-t}(q',q) = \int_{P(\mathbb{R})_{q,t}^{q',t'}} e^{\frac{i}{\hbar}S(\gamma)}\mathfrak{D}q, \tag{5.4.19}$$

where $P(\mathbb{R})_{q,t}^{q',t'}$ denotes the set of all smooth paths which start in q at time t and end in q' at time t'. Notice the difference between classical mechanics where the particle follows a path, and quantum mechanics where one takes a weighted sum over all possible paths. This is in accordance with the fact that one deals with randomness and uncertainty in quantum mechanics. The "measure" $\mathfrak{D}q$ given by

$$\mathfrak{D}q = \lim_{n\to\infty}(\frac{m}{2\pi i\hbar\Delta t})^{n/2}\prod_{k=1}^{n-1} dq_k \tag{5.4.20}$$

is not measure; because if it were, it would be a complex measure, but with a total variation which diverges, and this is impossible. The mathematical meaning of the path integral is equation (5.4.16). Notice that the integrand has absolute value 1, so one only gets convergence if the phases cancel each other out. When the classical action S is almost constant, the phases will almost be the same, and will not cancel. Because we divide by a small number $\hbar$, the phase oscillates very fast when S changes, and the phases will tend to cancel each other out. Recall the principle of least action from classical mechanics which states that a path describes the motion of the particle if and only if it is a critical point of the action functional S. Thus, the main contribution to the integral will be the paths which are close to the classical path. Later we will find the kernel for $U(t)$ for the harmonic oscillator in one dimension.

Chapter 6

One-Dimensional Harmonic Oscillator

We will look at the harmonic oscillator over the real numbers, p-adic numbers and adeles. Most of the results in the section about the oscillator over the real numbers will not be proved since they are well known and because it only is meant as motivation for the p-adic and adelic oscillator.

6.1 Introduction to the Classical Harmonic Oscillator

In this section we will give a brief introduction to the harmonic oscillator in one dimension. In classical mechanics, the harmonic oscillator is characterized by something which oscillates around an equilibrium. It could for instance be a simple pendulum. We will describe the harmonic oscillator by the Hamiltonian. For the harmonic oscillator (in one dimension), the Hamiltonian for a particle becomes

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega q^2, \tag{6.1.1}$$

where ω is the angular frequency and q is the displacement from the equilibrium. One can see that the first term is the kinetic energy, while the second term is the potential energy. One wants to know how the system changes over time. The equations of motion are given as

$$p(t) = mq'(t), \qquad p'(t) = -m\omega^2 q(t), \tag{6.1.2}$$

with initial conditions $q(0) = q$ and $p(0) = p$.

By differentiating the left equation and putting it in the right equation one gets

$$q''(t) + \omega^2 q(t) = 0. \tag{6.1.3}$$

This is easily solved, and with the initial conditions one obtains

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = T_t \begin{pmatrix} q \\ p \end{pmatrix}, \tag{6.1.4}$$

where

$$T_t = \begin{pmatrix} \cos\omega t & (m\omega)^{-1}\sin\omega t \\ -m\omega\sin\omega t & \cos\omega t \end{pmatrix}. \tag{6.1.5}$$

This can be referred to as the time evolution in the phase space. Even though this was done in $\mathbb{R}$, the equations (6.1.4) and (6.1.5) are also true in $\mathbb{Q}_p$ but in the p-adic case, all the quantities are p-adic. That is, p, m, q, ω and t are p-adic and cos and sin are the p-adic cosine and sine functions. Note that in the p-adic case $\omega t \in G_p$ so the sine and cosine functions are well defined. An important relation which is easy to show is that $T_t T_{t'} = T_{t+t'}$.

6.2 Real Quantum Oscillator

As already mentioned, the Hamiltonian is the operator which corresponds to the total energy of the system. The Hamiltonian in this case is the operator H given by

$$H = \frac{1}{2}\frac{P^2}{m} + \frac{1}{2}m\omega Q^2. \tag{6.2.1}$$

We wish to find the eigenvectors and eigenvalues of the operator. The equation

$$H\psi(q) = E\psi(q), \tag{6.2.2}$$

where E is the (energy) eigenvalue, becomes

$$\frac{\hbar^2}{2m}\frac{d^2}{dq^2}\psi(q) + (E - \frac{m\omega^2}{2})\psi(q) = 0. \tag{6.2.3}$$

This equation is the (time independent) Schrödinger equation for the harmonic oscillator. We will simplify by setting $m = \omega = h = 1$. Then we get

$$\frac{d^2}{dq^2}\psi(q) + (E - \frac{1}{2})\psi(q) = 0. \tag{6.2.4}$$

The equation has a non-trivial solution if and only if $E = E_n$, where

$$E_n = \frac{1}{2\pi}(n + 1/2), \tag{6.2.5}$$

and the solution for $E = E_n$ is then

$$\psi_n(q) = \frac{2^{1/4}}{2^n n!}e^{-\pi q^2} H_n(q\sqrt{2\pi}), \tag{6.2.6}$$

where H_n are the Hermite polynomials given by

$$H_n(q) = (-1)^n e^{q^2}\frac{d^n}{dq^n}e^{-q^2}. \tag{6.2.7}$$

Earlier we saw that we can describe quantum mechanics by using a Weyl system. An equivalent formulation of quantum mechanics is given by the triple

$$(L^2(\mathbb{R}), W(z), U(t)). \tag{6.2.8}$$

Here W is the projective unitary representation of $\mathbb{R} \times \mathbb{R}$ acting on $L^2(\mathbb{R})$ by

$$W(z)\psi(x) = \chi_\infty(kq/2 + kq)\psi(x+q), \tag{6.2.9}$$

where $z = (q,p) \in \mathbb{R} \times \mathbb{R}$ is a point in the classical phase space. Notice here that we used a different scaling for W compared to what we did in the section on Weyl systems. It satisfies the Weyl relation

$$W(z)W(z') = \chi_\infty\left(\frac{1}{2}B(z,z')\right)W(z+z'), \tag{6.2.10}$$

where $z = (q,p), z' = (q',p') \in \mathbb{R} \times \mathbb{R}$ and $B(z,z') = -pq' + qp'$. In other words, $(L^2(\mathbb{R}), W(z))$ defines a Weyl system.

U is the time evolution operator from $\mathbb{R}$ to $U(L^2(\mathbb{R}))$ (the unitary operators on $L^2(\mathbb{R})$) and is defined as

$$U(t)\psi(x) = \int_{\mathbb{R}} \mathcal{K}_t(x,y)\psi(y)dy. \tag{6.2.11}$$

We wish to find the kernel $\mathcal{K}_t(x,y)$ by using the Feynman path integral. The important step in finding it is the next proposition taken from [13].

Proposition 6.2.1. *Let A be real symmetric non-degenerate $n \times n$ matrix. Then*

$$\int_{\mathbb{R}^n} e^{\frac{i}{2}(A\mathbf{q})\cdot\mathbf{q} + i\mathbf{p}\cdot\mathbf{q}} d^n\mathbf{q} = \exp\left(\frac{\pi i n}{4}\right)\sqrt{\frac{(2\pi)^n}{\det A}}\exp\left(\frac{i}{2}(A^{-1}\mathbf{p})\cdot\mathbf{p}\right),$$

where the integral is understood in the distributional sense as $\lim_{R\to\infty}\int_{|\mathbf{q}|\le R}$.

Remember the notation $T = t' - t$ and $\Delta t = T/n$. For this proposition we will not use $m = \omega = h = 1$.

Proposition 6.2.2. *The kernel $\mathcal{K}$ for the harmonic oscillator is given as*

$$\mathcal{K}_T(q',q) = \sqrt{\frac{m\omega}{2\pi\hbar\sin\omega T}}\exp\left(\frac{im\omega}{2\hbar\sin\omega T}((q^2 + q'^2)\cos\omega T - 2qq')\right),$$

for $\frac{\pi\nu}{\omega} = T_\nu < T < T_{n+1} = \frac{\pi(\nu+1)}{\omega}$, $\nu \in \mathbb{N}$. In the limit $T \to T_\nu$ the kernel is $e^{-\frac{\pi i\nu}{2}}\delta(q-q')$ for even ν and $e^{-\frac{\pi i \nu}{2}}\delta(q+q')$ for odd ν.

Proof. We want to use Proposition 6.2.1 on equation (5.4.16). Starting with the expression in equation (5.4.16) with $V = \frac{m\omega Q^2}{2}$ and by the substitution $\tilde{q}_k =$

$\sqrt{\frac{m}{\hbar\Delta t}}q_k$ we get

$$\begin{aligned}
&\mathcal{K}_T(q', q) \\
&= \Big(\frac{m}{2\pi i\hbar\Delta t}\Big)^{n/2} \int_{\mathbb{R}^{n-1}} \exp\Big(\frac{i}{\hbar}\sum_{k=0}^{n-1}\big[\frac{m}{2}\Big(\frac{q_{k+1}-q_k}{\Delta t}\Big)^2 - \frac{m\omega q_k^2}{2}\big]\Delta t\Big)\prod_{k=1}^{n-1} dq_k \\
&= \sqrt{\frac{m}{(2\pi i)^n\hbar\Delta t}} \int_{\mathbb{R}^{n-1}} \exp\Big(\frac{i}{2}\sum_{k=0}^{n-1}\Big(\tilde{q}_{k+1}-\tilde{q}_k\Big)^2 - \epsilon\tilde{q}_k^2\Big)\prod_{k=1}^{n-1} d\tilde{q}_k,
\end{aligned}$$

where $\epsilon = \omega\Delta t$.

With the tridiagonal $(n-1)\times(n-1)$ matrix

$$A_{n-1} = \begin{pmatrix} 2-\epsilon^2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2-\epsilon^2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2-\epsilon^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2-\epsilon^2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2-\epsilon^2 \end{pmatrix}, \tag{6.2.12}$$

one gets that

$$\sum_{k=0}^{n-1}((\tilde{q}_{k+1}-\tilde{q}_k)^2 - \epsilon^2\tilde{q}_k^2) = (A_{n-1}\mathbf{q})\cdot\mathbf{q} + 2\mathbf{p}\cdot\mathbf{q} + \tilde{q_0}^2 - \epsilon^2\tilde{q_0}^2 + \tilde{q}_n^2, \tag{6.2.13}$$

where $\mathbf{q} = (\tilde{q}_1, \tilde{q}_2, ..., \tilde{q}_{n-1})$ and $\mathbf{p} = (-\tilde{q}_0, 0, ..., 0, -\tilde{q}_n)$ are $n-1$ dimensional vectors.

Then by Proposition 6.2.1,

$$\begin{aligned}
&\sqrt{\frac{m}{(2\pi i)^n\hbar\Delta t}} \int_{\mathbb{R}^{n-1}} \exp\Big(\frac{i}{2}\sum_{k=0}^{n-1}\Big(\tilde{q}_{k+1}-\tilde{q}_k\Big)^2 - \epsilon\tilde{q}_k^2\Big)\prod_{k=1}^{n-1} d\tilde{q}_k \\
&= \sqrt{\frac{m}{2\pi i\hbar\Delta t\det A_{n-1}}}\exp\Big(\frac{im}{2\hbar\Delta t}(q^2 - \epsilon^2 q^2 + q'^2 - (A_{n-1}^{-1}\mathbf{p})\cdot\mathbf{p})\Big).
\end{aligned} \tag{6.2.14}$$

We will not go into detail on how to do the rest of the calculations (see [13]). For large n one obtains

$$\det A_{n-1} = \frac{\sin\omega T}{\omega\Delta t}(1 + O(n^{-1})), \tag{6.2.15}$$

and also that A_{n-1} has ν negative eigenvalues when $T_\nu < T < T_{\nu+1}$. Furthermore one sees that the corner elements in A_{n-1}^{-1} are just given by determinants of A_{n-1} and A_{n-2}. One gets the kernel

$$\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\Big(\frac{im\omega}{2\hbar\sin\omega T}\Big[(q^2+q'^2)\cos\omega T - 2qq'\Big]\Big). \tag{6.2.16}$$

When $T_\nu < T < T_{\nu+1}$, we get what we wanted.

The result for $T \to T_\nu$ is obtained by

$$\lim_{t\to 0} \frac{1}{\sqrt{2\pi t}} e^{\frac{i(x-y)^2}{2t}} = e^{\frac{\pi i}{4}} \delta(x-y). \tag{6.2.17}$$

□

When $m = \omega = h = 1$,

$$\mathcal{K}_T(q', q) = \sqrt{\frac{1}{\sin \omega T}} \exp\Big(\frac{i\pi}{\sin T}((q^2 + q'^2)\cos T - 2qq')\Big). \tag{6.2.18}$$

It can be shown that

$$U(t)W(z) = W(T_t z)U(t) \tag{6.2.19}$$

which shows a relation between quantum and classical time evolution.

It can also be shown that

$$U(t)\psi_n(x) = \chi_\infty(E_n t)\psi_n(x), \tag{6.2.20}$$

where $E_n = \frac{1}{2\pi}(n + 1/2)$ as in equation (6.2.5) and $\psi_n(q) = \frac{2^{1/4}}{2^n n!} e^{-\pi q^2} H_n(q\sqrt{2\pi})$ as in equation (6.2.6). We have found the same eigenvalues as we get in the Schrödinger equation by using a Weyl system. The strategy to find eigenvalues in the p-adic and adelic case will then be to formulate quantum mechanics by Weyl's formulation.

The states corresponding to the 0 eigenvalue are called ground states. The ground state is given by

$$\psi_0(q) = 2^{1/4} e^{-\pi q^2}, \tag{6.2.21}$$

and it is invariant under the Fourier transform.

The expectation of an observable A in the state ψ_0 will be denoted by

$$\langle A \rangle = \langle A\psi_0, \psi_0 \rangle. \tag{6.2.22}$$

When an expectation is taken of an expression in q, it means that it is the expectation of the operator which acts as multiplication with that expression in q. Then for instance the expectation of the position operator is written $\langle q \rangle$, and the expectation is $\int_{\mathbb{R}} q|\psi_0(q)|^2 dq$. Similarly an expression in p means multiplication with that expression in the momentum space. Since the ground state is invariant under the Fourier transform, we can use our results from the configuration space. Now we will do some calculations. By symmetry

$$\langle q \rangle = \langle p \rangle = 0. \tag{6.2.23}$$

For $\operatorname{Re} s > -1$ one gets by a simple substitution,

$$\langle |q|^s \rangle = \langle |p|^s \rangle = \sqrt{2}\Gamma\left(\frac{s+1}{2}\right)(2\pi)^{-(s+1)/2}, \tag{6.2.24}$$

where Γ is the regular complex gamma function given by

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \qquad \operatorname{Re} z > 0. \tag{6.2.25}$$

Then we get that

$$\begin{aligned}\Delta q = \Delta p = \langle q^2\rangle^{1/2} &= (\sqrt{2}\Gamma(-3/2)(2\pi)^{-3/2})^{1/2} \\ &= (\sqrt{2}\frac{\sqrt{\pi}}{2}(2\pi)^{-3/2})^{1/2} = \frac{1}{2\sqrt{\pi}},\end{aligned} \tag{6.2.26}$$

and by similar calculations,

$$\Delta|p| = \Delta|q| = \frac{1}{2\sqrt{\pi}}(1 - \frac{2}{\pi})^{1/2}. \tag{6.2.27}$$

6.3 p-adic Quantum Oscillator

For simplicity, we will continue to use $m = \omega = h = 1$. As already mentioned, a Weyl system can be used for other locally compact abelian groups than $\mathbb{R}$. To define p-adic quantum mechanics we will use the idea of a Weyl system. When one wants to generalize the harmonic oscillator to the p-adic numbers, one might try to define it as $H = \frac{1}{2}\frac{P^2}{m} + \frac{1}{2}m\omega Q^2$ as in the real case, but where Q is given as $Q\psi(x) = x\psi(x)$. When x is a p-adic number and ψ is complex-valued, this makes no sense. One can instead define Q^2 directly as $Q^2\psi(x) = |x|^2\psi(x)$ which would be an analog of the operator on the real numbers. As stated above, we will instead use Weyl systems. p-adic quantum mechanics will be given by the triple

$$(L^2(\mathbb{Q}_p), W(z), U(t)). \tag{6.3.1}$$

The projective unitary representation W on $L^2(\mathbb{Q}_p)$ is given by

$$W(z)\psi(x) = \chi_p(\frac{pq}{2} + px)\psi(x+q), \tag{6.3.2}$$

where $z = (q, p)$ is a point in the classical phase space. It satisfies the Weyl relation

$$W(z)W(z') = \chi_p(\frac{1}{2}B(z, z'))W(z+z'), \tag{6.3.3}$$

where $z = (q, p)$, $z' = (q', p')$ and $B(z, z') = -pq' + qp'$ just as in the real case.

The p-adic evolution operator U is a function on G_p given as

$$U(t)\psi(x) = \int_{\mathbb{Q}_p} \mathcal{K}_t(x, y)\psi(y)d\mu_p(y) \tag{6.3.4}$$

where $\psi \in L^2(\mathbb{Q}_p)$ and $\mathcal{K}_t$ is the kernel for the harmonic oscillator and is given as

$$\mathcal{K}_t(x, y) = \lambda_p(2t)|t|^{-1/2}\chi_p\left(\frac{xy}{\sin t} - \frac{x^2+y^2}{2\tan t}\right) \qquad t \in G_p, t \neq 0 \tag{6.3.5}$$

$$\mathcal{K}_0(x,y) = \delta_p(x-y) \tag{6.3.6}$$

where δ_p is the p-adic Dirac delta function. A derivation of the propagator is given in [15] by a p-adic analog of the Feynman path integral.

The equation has to be understood in the L^2-sense since the integral is generally not convergent for a L^2-function. The process of extending $U(t)$ from $L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ to $L^2(\mathbb{Q}_p)$ is similar to what was done for the Fourier transform. Since $U(t)$ preserves the L^2-norm, the extension is unique.

Theorem 6.3.1. *$U(t)$ is strongly continuous unitary representation of the group G_p on the Hilbert space $L^2(\mathbb{Q}_p)$. Moreover, it maps $\mathcal{D}(\mathbb{Q}_p)$ (the space of test functions) into itself.*

Proof. We have that

$$U(t)\psi(x) = \chi_p(-\frac{x^2}{2\tan t})\frac{\lambda_p(2t)}{|t|_p^{1/2}}\mathcal{F}\Big[\psi(y)\chi_p(-\frac{y^2}{2\tan t})\Big]\Big|_{\frac{x}{\sin t}}, \tag{6.3.7}$$

where $\mathcal{F}$ is the p-adic Fourier transform. This is a composition of four unitary operators which map $\mathcal{D}(\mathbb{Q}_p)$ into itself. When $t=0$ we get that $U(t)\psi(x) = \psi(x)$. What is left to prove is

$$U(t+t') = U(t)U(t'). \tag{6.3.8}$$

We will prove equation (6.3.8) for $p \geq 3$. The case $p=2$ is similar. We will also not look at the case where t, t' or $t+t'$ are equal to zero as it will be just a simpler case. Let $\psi \in \mathcal{D}(\mathbb{Q}_p)$ and assume that $\psi(x) = 0$ for $|x| > p^N$ and $U(t)\psi(x) = 0$ for $|x| > p^M$. Then

$$\begin{aligned}
&U(t)U(t')\psi(x)\\
&= \int_{|y|_p \leq p^M} \mathcal{K}_t(x,y) \int_{|z|_p\leq p^N} \mathcal{K}_{t'}(y,z)\psi(z)d\mu_p(z)d\mu_p(y)\\
&= \int_{|z|_p\leq p^N} \psi(z) \int_{|y|_p\leq p^M} \mathcal{K}_t(x,y)\mathcal{K}_{t'}(y,z)d\mu_p(y)d\mu_p(z)
\end{aligned} \tag{6.3.9}$$

by Fubini's theorem.

Now by Theorem 3.3.6 we get that

$$\begin{aligned}
&\int_{|z|_p\leq p^N} \psi(z) \int_{|y|_p\leq p^M} \mathcal{K}_t(x,y)\mathcal{K}_{t'}(y,z)d\mu_p(y)d\mu_p(z)\\
&= \frac{\lambda_p(2t)\lambda_p(2t')}{|tt'|^{1/2}}\chi_p(-\frac{x^2}{2\tan t})\int_{|z|_p\leq p^N}\psi(z)\chi_p(-\frac{z^2}{2\tan(t')})\\
&\cdot \int_{|y|_p\leq p^M}\chi_p(-y^2(\frac{1}{2\tan t}+\frac{1}{2\tan t'})+y(\frac{x}{\sin t}+\frac{z}{\sin t'}))d\mu_p(y)d\mu_p(z)
\end{aligned} \tag{6.3.10}$$

We will first calculate the inner integral. To do this, define

$$a = -\frac{1}{2\tan t} - \frac{1}{2\tan t'}, \qquad b = \frac{x}{\sin t} + \frac{z}{\sin t'} \tag{6.3.11}$$

By the fact that $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$, $|\tan(\alpha)|_p = |\alpha|_p$ and Lemma 2.2.2, we get that

$$|a|_p = |\frac{\tan(t+t')(1-\tan(t)\tan(t'))}{\tan(t)\tan(t')}|_p. \tag{6.3.12}$$

By similar reasoning

$$|\frac{b}{2a}p^M|_p = p^{-M}|\frac{x\sin(t')+z\sin(t)}{t+t}|_p. \tag{6.3.13}$$

The left part in equation (6.3.9) will not change if one takes a larger M. Since one can choose M as big as one wants, we get by choosing M sufficiently big and by Theorem 3.3.4,

$$\begin{aligned}&\int_{|y|_p\le p^M} \chi_p(-y^2(\frac{1}{2\tan t}+\frac{1}{2\tan t'})+y(\frac{x}{\sin t}+\frac{z}{\sin t'}))d\mu_p(y)\\ &= \lambda_p(a)|\frac{tt'}{t+t'}|_p^{1/2}\chi_p(-\frac{b^2}{4a})\end{aligned} \tag{6.3.14}$$

It follows from Lemma 2.4.2, $\lambda(\alpha\beta^2) = \lambda(\alpha)$, $\lambda_p(\alpha)\lambda_p(-\alpha) = 1$, and Lemma 2.3.9, that

$$\begin{aligned}&\lambda_p(-\frac{1}{2\tan t}-\frac{1}{2\tan t'}) = \lambda_p(-\frac{\tan(t+t')(1-\tan(t)\tan(t'))}{2\tan(t)\tan(t')})\\ &= \lambda_p(-\frac{t+t'}{2tt'}) = \lambda_p(-\frac{2t+2t'}{2t2t'}) = \frac{\lambda_p(2t+2t')}{\lambda_p(2t)\lambda_p(2t')}\end{aligned} \tag{6.3.15}$$

We also have that

$$\begin{aligned}&-\frac{b^2}{4a} = (\frac{x}{\sin t}+\frac{z}{\sin t'})^2(\frac{2}{\tan t}+\frac{2}{\tan t'})^{-1}\\ &= -\frac{x^2+z^2}{2\tan(t+t')}+\frac{2xz}{2\sin(t+t')}+\frac{x^2}{2\tan t}+\frac{z^2}{2\tan t'}\end{aligned} \tag{6.3.16}$$

by using the equation $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$. Then we get the result

$$\begin{aligned}&U(t)U(t')\psi(x)\\ &= \frac{\lambda_p(2t+2t')}{|t+t'|_p^{1/2}}\int_{\mathbb{Q}_p}\chi_p(-\frac{x^2+z^2}{2\tan(t+t')}+\frac{xz}{\sin(t+t')})\psi(z)d\mu_p(z)\\ &= U(t+t')\psi(x).\end{aligned} \tag{6.3.17}$$

The result extends to L^2-functions. □

From this calculation we also get that

$$\mathcal{K}_{t+t'}(x,y) = \int_{\mathbb{Q}_p}\mathcal{K}_t(x,z)\mathcal{K}_{t'}(z,y)d\mu_p(z). \tag{6.3.18}$$

It can be shown that the operators $W_p(z)$ and $U_p(t)$ satisfy

$$U(t)W(z) = W(T_t z)U(t). \tag{6.3.19}$$

Chapter 7 is devoted to finding the eigenvalues and eigenfunctions of the evolution operator $U(t)$. The only result needed for now is that the simplest ground state (there are several ground states) is

$$\psi_{00}(x) = \Omega(|x|_p). \tag{6.3.20}$$

Note that it is invariant under the Fourier transform. To avoid confusion x is position, k is momentum and p is a prime number.

The expectation of an observable A in the state ψ_{00} will be denoted by

$$\langle A\rangle = \langle A\psi_{00}, \psi_{00}\rangle. \tag{6.3.21}$$

The mean square deviation is given by

$$\Delta A = (\langle A^2\rangle - \langle A\rangle^2)^{1/2}, \tag{6.3.22}$$

just like in the real case.

Note that $\langle x\rangle$ and $\langle k\rangle$ are not defined, while $\langle |x|_p\rangle$ and $\langle |k|_p\rangle$ are. We get that

$$\langle |x|_p^s\rangle = \langle |k|_p^s\rangle = \frac{1-p^{-1}}{1-p^{-s-1}}, \qquad \operatorname{Re} s > -1, \tag{6.3.23}$$

by the calculations which were done on the p-adic Γ-function in section 3.3. Furthermore we get that

$$\begin{aligned}\Delta|k|_p = \Delta|x|_p &= \left(\frac{1-p^{-1}}{1-p^{-3}} - \frac{(1-p^{-1})^2}{(1-p^{-2})^2}\right)^{1/2} \\ &= \left(\frac{1-p^{-1}}{1-p^{-3}}\right)^{1/2}\left(1-\frac{(1-p^{-1})(1-p^{-3})}{(1-p^{-2})^2}\right)^{1/2}.\end{aligned} \tag{6.3.24}$$

6.4 Adelic Quantum Oscillator

When we are dealing with real and p-adic functions at the same time, there will be an extra index p, ∞ or ν to make it clear which functions we are using. To define an adelic quantum mechanics we will again use a Weyl system. The Hilbert space will be $L^2(\mathbb{A})$. Then we have the projective unitary representation of $\mathbb{A}\times\mathbb{A}$, denoted by W, together with the adelic evolution operator U. So we have the triple

$$(L^2(\mathbb{A}), W(z), U(t)). \tag{6.4.1}$$

The operator $W(z)$ is defined as

$$W(z)\psi(x) = \chi_{\mathbb{A}}(pq/2 + px)\psi(x+q), \tag{6.4.2}$$

where $z = (q, p)$ is in the adelic classical phase space and $\psi \in L^2(\mathbb{A})$. One often writes $W(z) = \prod_\nu W_\nu(z_\nu)$, where W_∞ is the W we used in the real case and W_p is the W we used in the p-adic case, since this is how it acts on Schwartz-Bruhat functions. It satisfies the Weyl relation

$$W(z)W(z') = \chi_{\mathbb{A}}(\frac{1}{2}B(z, z'))W(z + z'), \tag{6.4.3}$$

where $z = (q, p)$ and $z' = (q', p')$ are in the adelic classical phase space and $B(z, z') = -pq' + qp'$.

The evolution operator U is defined as

$$U(t)\psi(x) = \int_{\mathbb{A}} \mathcal{K}_t(x, y)\psi(y)d\mu(y), \tag{6.4.4}$$

where $\mathcal{K}_t$ is defined as

$$\mathcal{K}_t(x, y) = \prod_\nu K^{(\nu)}_{t_\nu}(x_\nu, y_\nu), \qquad t \neq 0 \tag{6.4.5}$$

$$\mathcal{K}_0(x, y) = \delta(x_\infty - y_\infty)\delta_2(x_2 - y_2)\delta_3(x_3 - y_3)\cdots, \tag{6.4.6}$$

and $K^{(\nu)}_{t_\nu}$ are given by equation (6.2.18) and (6.3.5).

It is easily seen that $\mathcal{K}_t(x, y)$ makes no sense as a function, even when $t \neq 0$, and must be seen as a distribution. The next definition shows how $U(t)$ acts on elementary functions.

Definition 6.4.1. For an elementary function ψ, $U(t)$ is given as

$$U(t)\psi(x) = \prod_\nu \int_{\mathbb{Q}_\nu} \mathcal{K}^{(\nu)}_t(x_\nu, y_\nu)\psi^\nu(y_\nu)d\mu_\nu(y_\nu). \tag{6.4.7}$$

One can then write $U(t) = \prod_\nu U_\nu(t_\nu)$, where $U_\infty(t_\infty)$ and $U_p(t_p)$ are the real and p-adic evolution operators respectively. One of course has to check that the infinite product converges. Also note that the time t must be in

$$G_{\mathbb{A}} = \mathbb{R} \times G_2 \times G_3 \times \cdots . \tag{6.4.8}$$

Lemma 6.4.1. *For large enough p,*

$$\int_{\mathbb{Q}_p} \mathcal{K}^{(p)}_{t_p}(x_p, y_p)\psi^p(y_p)d\mu_p(y_p) = 1, \tag{6.4.9}$$

where $\psi(y) = \prod_\nu \psi^\nu(y_\nu)$ is an elementary function (see section 4.4).

Proof. If $t_p = 0$, then the result is easily shown. So assume that $t_p \neq 0$. Since x is a fixed adele, $|x_p|_p \leq 1$ for $p > N_1$ for some natural number N_1 . We also have that ψ^p is the characteristic function on $\mathbb{Z}_p$ for $p > N_2$ for some natural number N_2 since ψ is an elementary function. Then for $p > \max\{N_1, N_2\}$ we have

$$
\begin{aligned}
&\int_{\mathbb{Q}_p} \mathcal{K}^{(p)}_{t_p}(x_p, y_p)\psi^p(y_p)d\mu_p(y_p) = \int_{\mathbb{Z}_p} \mathcal{K}^{(p)}_{t_p}(x_p, y_p)d\mu_p(y_p) \\
&= \lambda_p(2t_p)|t_p|_p^{-1/2}\chi_p(-\frac{x_p^2}{2\tan t_p}) \int_{\mathbb{Z}_p} \chi_p(-\frac{y_p^2}{2\tan t_p} + \frac{x_p y_p}{\sin t_p})d\mu_p(y_p) \\
&= \lambda_p(2t_p)\lambda_p(-\frac{1}{2\tan t_p})|t_p|_p^{-1/2}|-\frac{1}{2\tan t_p}|^{-1/2} \\
&\cdot \chi_p\Big(-\frac{x_p^2}{2\tan t_p}\Big)\chi_p\Big(\frac{x_p^2\tan t_p}{2(\sin t_p)^2}\Big)\Omega\Big(|-\frac{x_p\tan t_p}{\sin t_p}|\Big),
\end{aligned}
\tag{6.4.10}
$$

by Theorem 3.3.4.
First notice that since $|\sin t_p| = |t_p|$, $|\cos t_p| = 1$ and $|x_p|_p \leq 1$, we have that

$$|t_p|_p^{-1/2}|-\frac{1}{2\tan t_p}|^{-1/2} = 1, \tag{6.4.11}$$

$$\Omega\Big(|-\frac{x_p\tan t_p}{\sin t_p}|\Big) = 1 \tag{6.4.12}$$

Furthermore we have that

$$\lambda_p(2t_p)\lambda_p(-\frac{1}{2\tan t_p}) = 1 \tag{6.4.13}$$

since $\lambda_p(ac^2) = \lambda(a)$ and $\lambda(a)\lambda(-a) = 1$, and by Lemma 2.3.9. Finally we have to look at

$$
\begin{aligned}
&\chi_p\Big(-\frac{x_p^2}{2\tan t_p}\Big)\chi_p\Big(\frac{x_p^2\tan t_p}{2(\sin t_p)^2}\Big) = \chi_p\Big(\frac{x_p^2}{2\sin t_p}(\frac{1}{\cos t_p} - \cos t_p)\Big) \\
&= \chi_p(x_p^2\tan t_p/2) = 1,
\end{aligned}
\tag{6.4.14}
$$

since $|x_p|_p \leq 1$ and $|\tan t_p/2|_p < 1$. This proves equation (6.4.9). □

The lemma proves that the product in Definition 6.4.1 makes sense. Since $U_\infty(t+t') = U_\infty(t)U_\infty(t')$ and $U_p(t+t') = U_p(t)U_p(t')$, it follows that

$$U(t+t') = U(t)U(t'). \tag{6.4.15}$$

Similarly to what is done earlier, $U(t)$ extends uniquely to $L^2(\mathbb{A})$. $U(t)$ is a unitary operator since $U_\nu(t_\nu)$ are unitary operators. What is left to prove is that it is strongly continuous. Since $\prod_\nu U_\nu(t_\nu)\psi^\nu(x_\nu)$ actually is a finite product when ψ is an elementary function, and since each $U_\nu(t_\nu)$ is strongly continuous, $U(t)$ is strongly continuous on the space of Schwartz-Bruhat functions ($\mathcal{S}(\mathbb{A})$). Let $\psi \in L^2(\mathbb{A})$. Since $\mathcal{S}(\mathbb{A})$ is dense in $L^2(\mathbb{Q}_p)$, there is a function $\tilde{\psi} \in \mathcal{S}(\mathbb{A})$ such that $||\psi - \tilde{\psi}|| < \epsilon/3$. Let t_n be a sequence of adeles converging to the adele t. There exists a positive integer N such that for $n > N$, $||U(t)\tilde{\psi} - U(t_n)\tilde{\psi}|| < \epsilon/3$. Then for $n > N$

$$\begin{aligned} &||U(t)\psi - U(t_n)\psi|| \leq ||U(t)\psi - U(t)\tilde{\psi}|| + ||U(t)\tilde{\psi} - U(t_n)\tilde{\psi}|| \\ &+ ||U(t_n)\tilde{\psi} - U(t_n)\psi|| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned} \tag{6.4.16}$$

since U is a unitary representation. Thus, U is a strongly continuous unitary representation of $G_{\mathbb{A}}$ on $L^2(\mathbb{A})$. As in the real and p-adic case, we get that

$$U(t)W(z) = W(T_t z)U(t), \tag{6.4.17}$$

where $W(T_t z) = \prod_\nu W_\nu(T_{t_\nu}^\nu z_\nu)$ and T_t^ν are the real and p-adic T_t from equation (6.1.5).

The eigenstates and eigenvalues are found in Section 7.4. The simplest ground state is given by

$$\psi_{00}(x) = \psi_0^{(\infty)}(x_\infty) \prod_p \psi_{00}^{(p)}(x_p) = 2^{1/4} e^{-\pi x_\infty^2} \prod_p \Omega(|x_p|_p), \tag{6.4.18}$$

which is invariant under the adelic Fourier transform.

The expectation of an observable A in the simplest ground state is denoted by

$$\langle A \rangle = \langle A\psi_{00}, \psi_{00} \rangle \tag{6.4.19}$$

Analogously we also define the mean square deviation as

$$\Delta A = (\langle A^2 \rangle - \langle A \rangle^2)^{1/2}. \tag{6.4.20}$$

We will use x as position, k as momentum and p as a prime. We want to find $\langle |x|^s \rangle$ and $\langle |k|^s \rangle$ where

$$|x| = \prod_\nu |x_\nu|_\nu, \qquad |k| = \prod_\nu |k_\nu|_\nu. \tag{6.4.21}$$

These products do not always converge. Therefore one instead computes $\langle |x|^s \rangle$ and $\langle |k|^s \rangle$ as limits of $|x|_{(p_n)}^s$ and $|k|_{(p_n)}^s$ which are given as

$$|x|_{(p_n)}^s = |x_\infty|_\infty^s \prod_{p=2}^{p_n} |x_p|_p^s, \qquad |k|_{(p_n)}^s = |k_\infty|_\infty^s \prod_{p=2}^{p_n} |k_p|_p^s, \tag{6.4.22}$$

where s is a complex number such that $\operatorname{Re} s > -1$ and p_n denotes the nth prime. This does not solve the mathematical problem that the products do not converge, but it may give an answer which makes sense physically.

By equation (6.3.23) and (6.2.24) we get that

$$\langle |k|_{(p_n)}^s \rangle = \langle |x|_{(p_n)}^s \rangle = \sqrt{2}\Gamma\left(\frac{s+1}{2}\right)(2\pi)^{-\frac{s+1}{2}} \prod_{p=2}^{p_n} \frac{1-p^{-1}}{1-p^{-s-1}}, \tag{6.4.23}$$

where $\operatorname{Re} s > -1$. One also gets that

$$\begin{aligned}\Delta|k|_{(p_n)} &= \Delta|x|_{(p_n)} \\ &= \left(\frac{1}{4\pi}\prod_{p=2}^{p_n}\frac{1-p^{-1}}{1-p^{-3}}\right)^{\frac{1}{2}}\left(1-\frac{2}{\pi}\prod_{p=2}^{p_n}\frac{(1-p^{-1})(1-p^{-3})}{(1-p^{-2})^2}\right)^{\frac{1}{2}}.\end{aligned} \tag{6.4.24}$$

The limit $\lim_{n\to\infty}\prod_{k=0}^{n}\frac{1}{1-p_k^{-s}}$ is equal to the Riemann zeta function $\zeta(s)$ if $\operatorname{Re} s > 1$. So for $\operatorname{Re} s > 0$,

$$\lim_{n\to\infty}\prod_{k=0}^{n}\frac{1}{1-p_k^{-(s+1)}} = \zeta(s+1). \tag{6.4.25}$$

Mertens' Theorem ([11]) states that

$$\lim_{n\to\infty}\frac{1}{\ln n}\prod_{p_k\le n}\frac{1}{1-\frac{1}{p_k}} = e^{\gamma}, \tag{6.4.26}$$

where γ is the Euler-Mascheroni constant[1].

Since the Riemann zeta function is convergent for $\operatorname{Re} s > 0$, we get that

$$\langle|k|^s\rangle = \langle|x|^s\rangle = \lim_{n\to\infty}\prod_{p_k\le n}\frac{1-\frac{1}{p}}{1-\frac{1}{p_k^{(s+1)}}} = 0 \tag{6.4.27}$$

for $\operatorname{Re} s > 0$.

Similarly

$$\Delta|k| = \Delta|x| = 0. \tag{6.4.28}$$

An infinite product is said to converge if the sequence of partial products is convergent to a limit not equal to 0. Even though the limit in our case does not satisfy the condition, we will still interpret the answer as 0.

[1] $\gamma \approx 0.5772$

Chapter 7

Spectral Analysis of the Evolution Operator

As in the real case we want to find the eigenvectors of the p-adic evolution operator and the corresponding eigenvalues. The problem is not trivial because the propagator is not so easy to deal with. We will begin by finding the eigenvalues, then the dimensions of the eigenspaces, and finally the eigenvectors. To do this we first need some theory from harmonic analysis.

7.1 Eigenvalues and Eigenspaces

Let G be a compact abelian group. From Proposition 3.2.4 we know that $\hat{G}$ is discrete. We will enumerate the characters from an index set I, so we can write the set of characters as $\hat{G} = \{\chi_\alpha, \alpha \in I\}$. We want to split the Hilbert space $L^2(G)$ as an orthogonal sum $L^2(G) = \bigoplus_{\alpha \in I} H_\alpha$, and define the projection on each H_α. The projections are defined as vector-valued integrals, so we will first need some theory on this subject.

Definition 7.1.1. (Vector-Valued Integral) Let Ξ be a locally convex topological vector space, and let Ξ^* be the space of continuous linear functionals on Ξ. Furthermore, let (X, μ) be a measure space. A function $F : X \to \Xi$ is called weakly integrable if $\phi \circ F \in L^1(X, \mu)$ for all $\phi \in \Xi^*$. If F is weakly integrable and there exists an element v in Ξ such that

$$\phi(v) = \int \phi \circ F(x) d\mu(x) \tag{7.1.1}$$

for all $\phi \in \Xi^*$, then v is called the integral of F and we write

$$v = \int F(x) d\mu(x) \tag{7.1.2}$$

Lemma 7.1.1. *Let Ξ and (X, μ) be as in the above definition. Let $F : X \to \Xi$ be weakly integrable, and assume that $v = \int F(x)d\mu(x)$ exists. Let Ξ' be another locally convex topological vector space and let $T : \Xi \to \Xi'$ be a continuous linear map. Then $T \circ F$ is weakly integrable and*

$$T \int F(x)d\mu(x) = \int T \circ F(x)d\mu(x). \tag{7.1.3}$$

Proof. We know that $\phi \circ T \in \Xi^*$ if $\phi \in (\Xi')^*$ which shows that $T \circ F$ is weakly integrable. Since we have assumed that

$$\psi(v) = \int \psi \circ F(x)d\mu(x) \tag{7.1.4}$$

for all $\psi \in \Xi^*$, we get that

$$\phi \circ T(v) = \int \phi \circ T \circ F(x)d\mu(x) \tag{7.1.5}$$

for all $\phi \in (\Xi')^*$. This proves the lemma. □

One wants to know when the element v exists, and if it in this case is unique. The next lemma is taken from [8].

Lemma 7.1.2. *Let Ξ be a locally convex topological vector space. Given two distinct vectors x, y in Ξ, there exists a continuous linear functional χ such that $\chi(x) \neq \chi(y)$.*

The next Corollary is a direct consequence of the lemma.

Corollary 7.1.3. *If the vector $v = \int F(x)d\mu(x)$ exists, then it is unique.*

Existence is harder to show, and we will need a theorem from [3].

Theorem 7.1.4. *Let Ξ be a Banach space and let μ be a Radon measure on the locally compact Hausdorff space X. If g is a scalar-valued function in $L^1(X, \mu)$ and $H : X \to \Xi$ is bounded and continuous, then $\int gH(x)d\mu(x)$ exists and belongs to the closed linear span of the range of H, and*

$$\left|\left| \int gH(x)d\mu(x) \right|\right| \leq \sup_{x \in X} ||H(x)|| \int |g(x)|d\mu(x). \tag{7.1.6}$$

Let U be a strongly continuous unitary representation of the group G, and let $\mathcal{H}$ be the corresponding Hilbert space. The projection operator P_α is defined as

$$P_\alpha = \int_G \overline{\chi_\alpha(g)} U(g)d\mu(g). \tag{7.1.7}$$

Existence of this integral is not immediate from Theorem 7.1.4 since U only is assumed to be continuous in the strong operator topology, and $B(\mathcal{H})$ (bounded linear operators on $\mathcal{H}$) is not a Banach space with this topology. However, we can use the theorem to define the integral pointwise. For each $v \in \mathcal{H}$, the function $g \mapsto U(g)v$ is a continuous and bounded function from G to $\mathcal{H}$. So equation (7.1.7) actually means that

$$P_\alpha v = \int_G \overline{\chi_\alpha(g)} U(g) v d\mu(g), \qquad \forall v \in \mathcal{H}. \tag{7.1.8}$$

Again, by Theorem 7.1.4 we have that

$$||P_\alpha v|| \leq \sup_{g \in G}\{||U(g)v||\} \int_G 1 d\mu(g) = ||v||, \tag{7.1.9}$$

since $U(g)$ is a unitary operator. Hence, P_α is bounded. We need to show that $P_\alpha^* = P_\alpha$ and $P_\alpha P_\beta = P_\alpha \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker delta. We will show that $P_\alpha^* = P_\alpha$. By the definition of the vector-valued integral, and by the substitution $h = g^{-1}$ we have that

$$\begin{aligned}
&\langle v, P_\alpha w\rangle = \overline{\langle P_\alpha w, v\rangle} = \overline{\int_G \overline{\chi_\alpha(g)} \langle U(g)w, v\rangle d\mu(g)} \\
&= \int_G \chi_\alpha(g) \overline{\langle U(g)w, v\rangle} d\mu(g) = \int_G \overline{\chi_\alpha(h)} \langle U(h)v, w\rangle d\mu(h) \\
&= \langle P_\alpha v, w\rangle
\end{aligned} \tag{7.1.10}$$

It is not hard to show that $P_\alpha u = 0 \ \ \forall \alpha \in I \Rightarrow u = 0$, and we have that

$$\begin{aligned}
U(h)P_\alpha v &= \int_G \overline{\chi_\alpha(g)} U(g+h) v d\mu(g) \\
&= \int_G \overline{\chi_\alpha(g'-h)} U(h') v d\mu(h') \\
&= \chi_\alpha(h) \int_G \overline{\chi_\alpha(g')} U(h') v d\mu(h') = \chi_\alpha(h) P_\alpha v.
\end{aligned} \tag{7.1.11}$$

Then we have that $\mathcal{H}$ can we written as the orthogonal sum

$$\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha, \tag{7.1.12}$$

where $\mathcal{H}_\alpha = P_\alpha \mathcal{H}$. Finally we can write

$$U(g) = \sum_{\alpha \in I} \chi_\alpha(g) P_\alpha. \tag{7.1.13}$$

Our next goal will be to find the dimensions of the spaces $\mathcal{H}_\alpha$ for the case when U is the evolution operator on G_p and where $\mathcal{H} = L^2(G)$. By what we just showed, all eigenvalues of $U(g)$ must be of the form $\chi_\alpha(g)$, and they are eigenvalues if the dimension of $\mathcal{H}_\alpha$ is bigger than 0. We need a result from [5] which says that the characters of G_p are $\chi_p(\alpha t)$ where for $p \neq 2$

$$\alpha = 0 \text{ or } \alpha = p^{-\gamma}(\alpha_0 + \alpha_1 p + ... + \alpha_{\gamma-2} p^{\gamma-2}), \tag{7.1.14}$$

where $\gamma = 2, 3, 4, ...$, $\alpha_0 \neq 0$, $0 \leq \alpha_i < p$. For $p = 2$

$$\alpha = 0 \text{ or } \alpha = 2^{-\gamma}(1 + \alpha_1 2 + ... + \alpha_{\gamma-3}2^{\gamma-3}), \tag{7.1.15}$$

where $\gamma = 3, 4, 5, ...$ and $0 \leq \alpha_i < 2$. The set of these α will for each p be denoted by I_p.

Notice that adding higher terms of p to α will not change the character. An α which is in I_p is thus a special choice of α.

We will also use the Haar measure on $\mathbb{Q}_p$, so

$$\mu_p(G_p) = \begin{cases} 1/p & p \neq 2, \\ 1/4 & p = 2. \end{cases} \tag{7.1.16}$$

The projection must then be normalized, so we define it as

$$P_\alpha = \mu_p(G_p)^{-1} \int_{G_p} \chi(-\alpha t)U(t)d\mu(t). \tag{7.1.17}$$

For the next proposition, we have to define the trace of an operator.

Definition 7.1.2. (Trace) Let $\mathcal{H}$ be a separable Hilbert space, and let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$. Then the trace of a bounded linear operator A on $\mathcal{H}$ is defined as

$$\mathrm{Tr}(A) = \sum_{i=0}^{\infty} \langle Ae_i, e_i \rangle. \tag{7.1.18}$$

If A is a positive element in the C^*-algebra of bounded linear operators, then the above sum is independent of the choice of basis, and converges (including ∞).

Proposition 7.1.5. *For all α in I_p,*

$$\dim H_\alpha = \mathrm{Tr}\, P_\alpha. \tag{7.1.19}$$

This is seen by choosing a suitable basis. Now we can state the main theorem of this section.

Theorem 7.1.6. *The spaces H_α have the following dimensions: If $p \equiv 1 \pmod 4$, then dim $H_\alpha = \infty$ for all $\alpha \in I_p$. If $p \equiv 3 \pmod 4$, then*

$$\dim H_\alpha = \begin{cases} 1, & \alpha = 0, \\ p+1, & |\alpha|_p = p^\gamma \textit{ and } \gamma \neq 0 \textit{ is even}, \\ 0, & \textit{else}. \end{cases} \tag{7.1.20}$$

If $p = 2$, then

$$\dim H_\alpha = \begin{cases} 2, & \alpha = 0 \textit{ or } |\alpha|_2 = 2^3, \\ 4, & |\alpha|_2 \geq 2^4 \textit{ and } \alpha_1 = 1, \\ 0, & \textit{else}. \end{cases} \tag{7.1.21}$$

Proof. The proof will only be given for $p \neq 2$. The case $p = 2$ is similar. We will prove the theorem by using Proposition 7.1.5. So the goal will be to calculate $\mathrm{Tr}(P_\alpha)$. It can be shown that if A is a positive operator and (T_n) is a sequence of positive elements converging strongly to the identity operator, then

$$\lim_{n\to\infty} \mathrm{Tr}(T_n A T_n) = \mathrm{Tr}(A). \tag{7.1.22}$$

Define ω_n on $L^2(\mathbb{Q}_p)$ by

$$\omega_n \psi(x) = \Omega(p^{-n}|x|_p)\psi(x). \tag{7.1.23}$$

This operator converges strongly to the identity operator, and we get that

$$\lim_{n\to\infty} \mathrm{Tr}(\omega_n P_\alpha \omega_n) = \mathrm{Tr}(P_\alpha). \tag{7.1.24}$$

The next goal is then to calculate $\mathrm{Tr}(\omega_n P_\alpha \omega_n)$ and to take the limit. By Lemma 7.1.1 it becomes

$$\mathrm{Tr}(\omega_n P_\alpha \omega_n) = \frac{1}{\mu_p(G_p)} \int_{G_p} \chi_p(-\alpha t)\, \mathrm{Tr}(\omega_n U(t) \omega_n) d\mu_p(t). \tag{7.1.25}$$

Now we will look closer at $\mathrm{Tr}(\omega_n U(t)\omega_n)$. When K is a compact set, μ is a measure on K and A is an integral operator on K with kernel $\mathcal{K}(x,x)$, then $\mathrm{Tr}\, A = \int_K \mathcal{K}(x,x) d\mu(x)$ if $\mathcal{K}(x,x)$ is continuous on $K \times K$. One then gets

$$\begin{aligned} \mathrm{Tr}(\omega_n U(t)\omega_n) &= \int_{|x|_p \le p^n} \Omega(p^{-n}|x|_p) \mathcal{K}_t^{(p)}(x,x) \Omega(p^{-n}|x|_p) d\mu_p(x) \\ &= \int_{|x|_p \le p^n} \mathcal{K}_t^{(p)}(x,x) d\mu_p(x) = \frac{\lambda_p(2t)}{|t|_p^{1/2}} \int_{|x|_p \le p^n} \chi_p(\tan\left(\frac{t}{2}\right) x^2) d\mu_p(x). \end{aligned} \tag{7.1.26}$$

By Theorem 3.3.4 with $a = \tan\left(\frac{t}{2}\right)$ we get that

$$\int_{|x|_p \le p^n} \chi_p(\tan\left(\frac{t}{2}\right) x^2) d\mu_p(x) = \begin{cases} p^n & |a|_p \le p^{-2n}, \\ \lambda_p(a)|2a|_p^{-1/2} & |a|_p > p^{-2n}. \end{cases} \tag{7.1.27}$$

By Lemma 2.3.7 and 2.3.9 this can be written as

$$\int_{|x|_p \le p^n} \chi_p(\tan\left(\frac{t}{2}\right) x^2) d\mu_p(x) = \begin{cases} p^n & |t|_p \le p^{-2n}, \\ \lambda_p(2t)|t|_p^{-1/2} & |t|_p > p^{-2n}. \end{cases} \tag{7.1.28}$$

Then we get that

$$\begin{aligned} \mathrm{Tr}(\omega_n P_\alpha \omega_n) &= p^{n+1} \int_{|t|_p \le p^{-2n}} \frac{\lambda_p(2t)}{|t|_p^{1/2}} \chi_p(-\alpha t) d\mu_p(t) \\ &+ p \int_{|t|_p > p^{-2n}} \frac{\lambda_p(2t)^2}{|t|_p} \chi_p(-\alpha t) d\mu_p(t) \end{aligned} \tag{7.1.29}$$

By some calculations, one gets that the first term becomes

$$p^{n+1}\int_{|t|_p\le p^{-2n}}\frac{\lambda_p(2t)}{|t|_p^{1/2}}\chi_p(-\alpha t)d\mu_p(t)=p, \tag{7.1.30}$$

and that the second term, denoted by J, becomes

$$J=\begin{cases}(p-1)(2n-1) & \text{if } \alpha=0, p\equiv 1(\text{mod } 4)\\ (p-1)(2n-N)-1 & \text{if } |\alpha|_p=p^N, 2\le N\le 2n, p\equiv 1(\text{mod } 4)\\ 1-p & \text{if } \alpha=0, p\equiv 3(\text{mod } 4)\\ (-1)^N\frac{p+1}{2}-\frac{p-1}{2} & \text{if } |\alpha|_p=p^N, 2\le N\le 2n, p\equiv 3(\text{mod } 4).\end{cases} \tag{7.1.31}$$

Letting $n\to\infty$ one gets the desired result.

□

Corollary 7.1.7. *The eigenvalues of the p-adic evolution operator $U(t)$ are of the form $\chi_p(\alpha t)$ where*

$$\alpha=0 \ \textit{or}\ \alpha=p^{-\gamma}(\alpha_0+\alpha_1 p+...+\alpha_{\gamma-2}p^{\gamma-2}), \tag{7.1.32}$$

where $\alpha_0\neq 0$, $0\le\alpha_i<p$ and $\gamma=2,3,4,...$ for $p\equiv 1$ (mod 4), $\gamma=2,4,6,...$ for $p\equiv 1$ (mod 4) . For $p=2$

$$\alpha=0 \ \textit{or}\ \alpha=2^{-\gamma}(1+\alpha_1 2+...+\alpha_{\gamma-3}2^{\gamma-3}), \tag{7.1.33}$$

where $0\le\alpha_i<2$, $\gamma=3,4,5,...$ and $\alpha_1=1$ for $|\alpha|_p\ge 2^4$. The set of these α will for each p be denoted by J_p.

When there are several eigenvectors corresponding to the same eigenvalue, it is called degeneracy. When this is the case for the Hamiltonian, it is called energy degeneracy. It means that two or more different states are possible for the same energy level.

The problem of finding the eigenfunctions for the eigenvalues is done in three parts. It is done for $p=2$, $p\equiv 1$ (mod 4) and $p\equiv 3$ (mod 4). We will not look at the case $p=2$.

7.2 The Case $p\equiv 1$ (mod 4)

By Lemma 2.3.12 there exists an element τ in $\mathbb{Q}_p$ such that $\tau^2=-1$ when $p\equiv 1$ (mod 4). The analysis of the eigenfunctions will be greatly simplified by the operator $\mathfrak{I}$, which for a p-adic Schwartz-Bruhat function f, is defined as

$$\mathfrak{I}[f](x)=\int_{\mathbb{Q}_p}\chi_p(\tau x^2-\frac{\tau}{2}z^2+2xz)f(z)d\mu_p(z). \tag{7.2.1}$$

It extends to L^2-functions by the usual process.

Proposition 7.2.1. *$\mathfrak{I}$ is a unitary operator on $L^2(\mathbb{Q}_p)$ and maps $\mathcal{D}(\mathbb{Q}_p)$ to itself.*

Proof. We have that

$$\mathfrak{I}[f](x) = \chi_p(\tau x^2)\mathcal{F}\Big[f(z)\chi_p(-\frac{\tau}{2}z^2)\Big]\Bigg|_{2x}, \tag{7.2.2}$$

where $\mathcal{F}$ is the p-adic Fourier transform. This is a composition of four unitary operators which map $\mathcal{D}(\mathbb{Q}_p)$ to itself. $\square$

Theorem 7.2.2. *For all $f \in L^2(\mathbb{Q}_p)$, we have that*

$$U(t)\mathfrak{I}[f](x) = \mathfrak{I}[f(e^{-\tau t}z)](x), \tag{7.2.3}$$

where $t \in G_p$ and $p \equiv 1$ (mod 4).

A full proof of this theorem is found in [5]. It mostly consists of the same calculations and techniques which were used to prove that U is a unitary representation. This theorem will simplify the problem greatly. Let us begin by finding the eigenvectors corresponding to the ground state. To solve

$$U(t)\psi = \psi, \tag{7.2.4}$$

we use the fact that $\mathfrak{I}$ is unitary to write $\psi = \mathfrak{I}[f]$, and this reduces to

$$f(e^{-\tau t}z) = f(z), \qquad t \in G_p. \tag{7.2.5}$$

By Lemma 2.3.11, we have that a p-adic number z can be written as $z = p^\gamma \epsilon^k e^a$, so then $e^{-\tau t}z = p^\gamma \epsilon^k e^{a-\tau t}$. Thus, we have to solve

$$f(p^\gamma \epsilon^k e^{a-\tau t}) = f(p^\gamma \epsilon^k e^a). \tag{7.2.6}$$

A general solution of this equation is the set of all functions in $L^2(\mathbb{Q}_p)$ which are on the form $f(z) = f(\gamma, k)$. Furthermore we see that this is equivalent to f being on the form $f(z) = f(|z|_p, z_0)$ where $z = p^{-\gamma}(z_0 + z_1 p + ...)$ since $|e^a - 1|_p < 1$. We then get that all eigenvectors corresponding to the ground state are of the form

$$\psi(x) = \int_{\mathbb{Q}_p} \chi_p(\tau x^2 - \frac{\tau}{2}z^2 + 2xz)f(|z|_p, z_0)d\mu_p(z). \tag{7.2.7}$$

It is not straightforward to find the eigenvectors corresponding to the ground state explicitly. By using

$$\begin{aligned}
&\int_{\mathbb{Q}_p} \chi_p(\tau x^2 - \frac{\tau}{2}z^2 + 2xz)f(|z|_p, z_0)d\mu_p(z) \\
&= \sum_{-\infty<\gamma<\infty} \sum_{1\leq k\leq p-1} f(p^{-\gamma}, k) \\
&\cdot \int_{|z|_p = p^\gamma, z_0 = k} \chi_p(\tau x^2 - \frac{\tau}{2}z^2 + 2xz)d\mu_p(z),
\end{aligned} \tag{7.2.8}$$

and

$$\begin{aligned}
&\int_{|z|_p=p^\gamma, z_0=k} \chi_p(\tau x^2 - \frac{\tau}{2}z^2 + 2xz) d\mu_p(z) \\
&= \chi_p(\tau x^2 + 2p^{-\gamma}kx)p^{\gamma-1}\Omega(p^{\gamma-1}|x|_p), \qquad \gamma \leq 0.
\end{aligned} \tag{7.2.9}$$

we can get some eigenvectors corresponding to the ground state. If we choose $f(|z|_p, z_0) = \Omega(|z|_p)$ we get $\psi_0(x) = \Omega(|x|_p)$. If $f(|z|_p, z_0) = \delta(p^\gamma - |z|_p)$ (where $\gamma = 1, 2, ...$) we get $\psi_\gamma = \chi_p(-\tau x^2)\delta(p^\gamma - |x|_p)$ (where $\gamma = 1, 2, ...$), where $\delta(0) = 1$ and $\delta(x) = 0$ for $x \neq 0$. Thus, the dimension of the eigenspace is infinite, in accordance with Theorem 7.1.6.

The other eigenfunctions are found from the equation

$$U(t)\psi_\alpha(x) = \chi_p(\alpha t)\psi_\alpha(x), \tag{7.2.10}$$

where α is a non-zero element in I_p and $t \in G_p$. Again, by using $\psi_\alpha = \Im[f_\alpha]$, we get

$$f_\alpha(e^{-\tau t}z) = \chi_p(\alpha t)f_\alpha(z), \qquad t \in G_p. \tag{7.2.11}$$

We know that the general solution of equation (7.2.5) is any function in $L^2(\mathbb{Q}_p)$ such that $\phi(z) = \phi(|z|_p, z_0)$. Now write $\phi(z) = f(z)\chi_p(-\alpha\tau a)$. By inserting this function in equation (7.2.5), we see that $\phi(z)$ satisfies equation (7.2.5) if and only if $f(z)$ satisfies equation (7.2.11). Then $f(z)$ satisfies equation (7.2.11) if and only if $f(z) = \phi(|z|_p, z_0)\chi_p(\alpha\tau a)$ where $\phi \in L^2(\mathbb{Q}_p)$. Thus, the general solution of equation (7.2.11) is

$$f_\alpha(z) = \phi_\alpha(|z|_p, z_0)\chi_p(\alpha\tau a). \tag{7.2.12}$$

Then we get the excited states

$$\psi_\alpha = \int_{\mathbb{Q}_p} \chi_p(\tau x^2 - \frac{\tau}{2}z^2 + 2xz + \alpha\tau a)\phi_\alpha(|z|_p, z_0)d\mu_p(z). \tag{7.2.13}$$

These are all the eigenvectors, but they are not on a particularly nice form. For the adelic oscillator, these eigenvectors will be useful, but it will be important to know whether they are in $\mathcal{D}(\mathbb{Q}_p)$. It turns out that we can pick an orthonormal basis for H_α consisting of Schwartz-Bruhat functions. From [10], we get the next theorem.

Theorem 7.2.3. *Let $\mathcal{H}$ be a separable Hilbert space. Then any dense subspace of $\mathcal{H}$ contains an orthonormal basis for $\mathcal{H}$.*

Since the set of p-adic Schwartz-Bruhat functions is a dense subspace of $L^2(\mathbb{Q}_p)$, the set of Schwartz-Bruhat functions in H_α is a dense subspace of H_α, and we can thus choose an orthonormal basis for H_α consisting of Schwartz-Bruhat functions. We will choose an orthonormal basis for $L^2(\mathbb{Q}_p)$ given by $\psi_{\alpha_p\beta_p}$ where α_p is the eigenvalue and β_p corresponds to the energy degeneracy.

7.3 The Case $p \equiv 3 \pmod 4$

The case $p \equiv 3 \pmod 4$ is harder than the case $p \equiv 1 \pmod 4$, because there is no x such that $x^2 = -1$, so we can not use the same trick as we did in the other case. Instead we will first calculate the eigenvectors for a different Weyl system, and obtain the eigenvectors for the oscillator through a unitary operator. In this section we will view the classical phase space $\mathbb{Q}_p \times \mathbb{Q}_p$ as the quadratic extension $\mathbb{Q}_p(\sqrt{-1})$. We will write i for $\sqrt{-1}$, and write any element $z \in \mathbb{Q}_p(\sqrt{-1})$ as $z = x + iy$, where $x, y \in \mathbb{Q}_p$. Continuing the analogy to the complex numbers, we define $\overline{z} = x - iy$. The norm of z is defined as $||z|| = \max\{|x|_p, |y|_p\}$.

Define $T = \{e^{it}, t \in G_p\}$. We also define the bigger group Y to be $\{z \in \mathbb{Q}_p(\sqrt{-1}) : z\overline{z} = 1\}$. The above definitions will be used to get what is called the polar decomposition of the elements in $\mathbb{Q}_p(\sqrt{-1})$.

Fix an element $\epsilon \in \mathbb{Q}_p(\sqrt{-1})$ such that $\epsilon\overline{\epsilon} = -1$. Also define $\mathbb{Q}_{p+} = \{x \in \mathbb{Q}_p : \left(\frac{x_0}{p}\right) = 1\}$.

Lemma 7.3.1. *For $p \equiv 3 \pmod 4$ all $z \in \mathbb{Q}_p^*(\sqrt{-1})$ can be written on the form*

$$z = r\epsilon^k c^n e^{i\tau}, \tag{7.3.1}$$

where r is a non-zero element in $\mathbb{Q}_{p+}$, $k \in \{0, 1\}$, c is a generator of the cyclic group of order $p+1$ denoted by Z_{p+1}, $n \in \{0, 1, ..., p\}$ and $\tau \in G_p$.

Proof. It can be shown that Y is isomorphic to $Z_{p+1} \times T$. Let $z = x + iy$ be an element in $\mathbb{Q}_p(\sqrt{-1})$. Then $z\overline{z} = x^2 + y^2$ is either a square or -1 times a square. To see this, first notice that if a number $a \neq 0$ in the finite field of p elements, $\mathbb{F}_p$, is not a square, then -1 times a will be a square since -1 is not a square when $p \equiv 3 \pmod 4$. Then γ in Lemma 2.3.8 must be even, and by the same lemma, we see that $x^2 + y^2$ or $-1(x^2 + y^2)$ must be a square. Assume that $z\overline{z}$ is not a square. Then $-z\overline{z}$ is a square and define $r = \sqrt{-z\overline{z}}$, where r is defined to be the square root which is in $\mathbb{Q}_{p+}$. Then

$$\left(\frac{z}{r\epsilon}\right)\overline{\left(\frac{z}{r\epsilon}\right)} = 1, \tag{7.3.2}$$

so that $\frac{z}{r\epsilon} = c^n e^{it}$. The case where $z\overline{z}$ is a square is similar. □

This decomposition is called the polar decomposition. The next lemma contains some important properties of r, k and n as functions of z.

Lemma 7.3.2. *Let $z, z' \in \mathbb{Q}_p^*(\sqrt{-1})$ such that $||z|| \geq p$ and $||z'|| \leq 1$. Then*

(i) $|r(z + z') - r(z)|_p \leq 1$.

(ii) $k(z + z') = k(z)$.

(iii) $n(z + z') = n(z)$.

Let $\delta_{k,n}$ be the Kronecker delta,

$$\delta_{k,n} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases} \tag{7.3.3}$$

Define $\delta^{\epsilon}_m(z) = \delta^{\epsilon}_m(r\epsilon^k c^n e^{i\tau}) = \delta_{m,k}$ for $m = 0, 1$, and define $\delta^c_\theta(z) = \delta^c_\theta(r\epsilon^k c^n e^{i\tau}) = \delta_{\theta,n}$ for $\theta = 0, 1, ..., p$. It is easily checked that for $a \neq b$,

$$\operatorname{supp} \delta^{\epsilon}_a(z) \cap \operatorname{supp} \delta^{\epsilon}_b(z) = \emptyset, \qquad \operatorname{supp} \delta^c_a(z) \cap \operatorname{supp} \delta^c_b(z) = \emptyset. \tag{7.3.4}$$

Now we will look at a different Weyl system. The Hilbert space is

$$L^{\chi}_2 = \{\phi \in L^2(\mathbb{Q}_p \times \mathbb{Q}_p) : \phi(z + z') = \chi_p(\tilde{B}(z, z'))\phi(z)\}, \tag{7.3.5}$$

where $z, z' \in \mathbb{Q}_p \times \mathbb{Q}_p, ||z'|| \leq 1$, and $\tilde{B}(x + iy, x' + iy') = -yx' + xy'$. We define the unitary operator $\tilde{W}(z)$ by $\tilde{W}(z)\phi(w) = \chi_p(\tilde{B}(z, w))\phi(w - z)$, where $\phi \in L^{\chi}_2$ and $z, w \in \mathbb{Q}_p \times \mathbb{Q}_p$. This gives us the Weyl system $(L^{\chi}_2, W(z))$. An analog to the evolution operator is $\tilde{U}(t)$ which for $t \in G_p$ and $\phi \in L^{\chi}_2$ is given by $\tilde{U}(t)\phi(w) = \phi(e^{-it}w)$. In addition it satisfies $\tilde{U}(t)\tilde{W}(z) = \tilde{W}(e^{it}z)\tilde{U}(t)$. Similarly to what we did for our original Weyl system, this Weyl system can also be split into an orthogonal sum of eigenspaces with the same eigenvalues as in the original Weyl system. We will later see a unitary operator between the Hilbert spaces which takes eigenvectors corresponding to the eigenvalue $\chi_p(\alpha t)$ to eigenvectors corresponding to the eigenvalue $\chi_p(-\alpha t)$, and this shows that the eigenspaces for the two Weyl systems have the same dimensions. We want to find the solution of $\tilde{U}(t)\phi = \chi_p(\alpha t)\phi$, which becomes

$$\phi(e^{-it}z) = \chi_p(\alpha t)\phi(z), \qquad \phi(z + z') = \chi_p(\tilde{B}(z, z'))\phi(z),\ ||z'|| \leq 1. \tag{7.3.6}$$

Theorem 7.3.3. *The eigenvector corresponding to the ground state for $\tilde{U}(t)$ on L^{χ}_2 is*

$$\phi_0(z) = \Omega(||z||), \tag{7.3.7}$$

and the $p + 1$ eigenvectors for the eigenvalue $\chi(\alpha t)$, $\alpha \in J_p \setminus \{0\}$, are given by

$$\phi^n_\alpha(z) = \delta^{\epsilon}_m(z)\delta^c_n(z)\Omega(|r(z) - a|_p)\chi_p(-\alpha\tau(z)), \tag{7.3.8}$$

where

$$m = \frac{1}{2}\left(1 + \left(\frac{\alpha_0}{p}\right)\right), \qquad a = \sqrt{(-1)^{m+1}\alpha} \tag{7.3.9}$$

and $n = 0, 1, ..., p$.

Proof. First it is easily seen that ϕ_0 is a solution of equation (7.3.6). So from now assume that $\alpha \neq 0$. From equation (7.3.4) it follows that the $p + 1$ eigenvectors are orthogonal. Since we know the dimensions of the eigenspaces, it is sufficient to check if the functions satisfy equation (7.3.6). That ϕ^n_α satisfy the first equation in equation (7.3.6) follows from the fact that multiplication with e^{-it} just sends τ to

$\tau - t$. Also notice that since $|\alpha|_p \geq p^2$ (since $\alpha \in J_p \setminus 0$), $|a|_p \geq p$. Putting ϕ_α^n in the second equation in equation (7.3.6) yields by Lemma 7.3.2,

$$\begin{aligned}&\delta_m^\epsilon(z)\delta_n^c(z)\Omega(|r(z)-a|_p)\chi_p(-\alpha\tau(z+z'))\\&=\chi_p(\tilde{B}(z,z'))\delta_m^\epsilon(z)\delta_n^c(z)\Omega(|r(z)-a|_p)\chi_p(-\alpha\tau(z)).\end{aligned} \tag{7.3.10}$$

This equation is equivalent to

$$\chi_p(\alpha\Delta\tau + \tilde{B}(z,z')) = 1 \tag{7.3.11}$$

with $z = r(z)\epsilon^m c^n e^{i\tau(z)}$, $z+z' = r(z+z')\epsilon^m c^n e^{i\tau(z+z')}$, $|r(z)-a|_p \leq 1$ and $\Delta\tau = \tau(z+z') - \tau(z)$. Since $|a|_p \geq p$, this implies that $|r(z)|_p = |a|_p$. Since $\tilde{B}$ is linear in the second argument and conjugate linear in the first, one gets that

$$\begin{aligned}\tilde{B}(z,z') &= \tilde{B}(z,z+z')\\&= r(z)r(z+z')(\epsilon)^m(\bar{\epsilon})^m c^n\bar{c}^n\tilde{B}(e^{i\tau(z)},e^{i\tau(z+z')})\\&= r(z)r(z+z')(-1)^m\sin\Delta\tau,\end{aligned} \tag{7.3.12}$$

where the last equation follows from $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

It is important to show that certain expressions are smaller than 1 in absolute value since then χ_p of that expression is equal to 1. It is not hard to show that

$$|a\Delta\tau|_p \leq 1. \tag{7.3.13}$$

Since $|r(z)-a|_p \leq 1$, $|r(z+z')-a|_p \leq 1$, $|a\Delta\tau|_p \leq 1$ and $|r(z)\Delta\tau|_p \leq 1$, we get that

$$\begin{aligned}&\chi_p(\alpha\Delta\tau + \tilde{B}(z,z')) = \chi_p(\alpha\Delta\tau + r(z)r(z+z')(-1)^m\sin\Delta\tau)\\&= \chi_p(\alpha\Delta\tau + a^2(-1)^m\sin\Delta\tau).\end{aligned} \tag{7.3.14}$$

One can easily show that $|\sin\Delta\tau - \Delta\tau|_p \leq |\frac{\Delta\tau}{p}|_p$ which together with equation (7.3.13) gives

$$\chi_p((\alpha + (-1)^m a^2)\Delta\tau) = 1. \tag{7.3.15}$$

This equation holds for all $\Delta\tau \in G_p$ if the equation

$$\alpha + (-1)^m a^2 = 0 \tag{7.3.16}$$

is satisfied. Notice that the choice of m makes $(-1)^{m+1}\alpha$ a square. The theorem is proved.

□

Theorem 7.3.4. *Let ψ_0 be the eigenvector corresponding to the ground state for the Weyl system $(L^2(\mathbb{Q}_p), W)$. Then the $n+1$ eigenfunctions for $U(t)$ corresponding to the eigenvalue $\chi_p(-\alpha t)$ $(\alpha \neq 0)$ are given as*

$$\psi_\alpha^n = \int_{G_p}\chi_p(-\alpha t')W(a\epsilon^m c^n e^{it'})\psi_0 d\mu_p(t'), \tag{7.3.17}$$

and for $\alpha = 0$, $\psi_0(x) = \Omega(|x|_p)$.

Proof. Define the operator $S : L_2^\chi \to L^2(\mathbb{Q}_p)$ by

$$S\phi = \int_{\mathbb{Q}_p \times \mathbb{Q}_p} \langle \phi, \tilde{W}(z)\phi_0 \rangle W(z)\psi_0 dz \tag{7.3.18}$$

which can be shown to be unitary. It can also be shown that

$$S\phi_\alpha^n = \text{const} \cdot \int_{G_p} \chi_p(-\alpha t')W(a\epsilon^m c^n e^{it'})\psi_0 d\mu_p(t'). \tag{7.3.19}$$

Since $U(t)W(z) = W(T_t z)U(t)$ we get that

$$\begin{aligned} U(t)\psi_\alpha^n &= \int_{G_p} \chi_p(-\alpha t')U(t)W(a\epsilon^m c^n e^{it'})\psi_0 d\mu_p(t') \\ &= \int_{G_p} \chi_p(-\alpha t')W(a\epsilon^m c^n T_t e^{it'})U(t)\psi_0 d\mu_p(t'). \end{aligned} \tag{7.3.20}$$

Since $T_t e^{it'} = e^{i(t'-t)}$ and $U(t)\psi_0 = \psi_0$, from the substitution $s = t' - t$, we get that

$$\begin{aligned} &\int_{G_p} \chi_p(-\alpha t')W(a\epsilon^m c^n T_t e^{it'})U(t)\psi_0 d\mu_p(t') \\ &= \int_{G_p} \chi_p(-\alpha t')W(a\epsilon^m c^n e^{i(t'-t)})\psi_0 d\mu_p(t') \\ &= \int_{G_p} \chi_p(-\alpha(s+t))W(a\epsilon^m c^n e^{is})\psi_0 d\mu_p(s) = \chi_p(-\alpha t)\psi_\alpha^n. \end{aligned} \tag{7.3.21}$$

Since ψ_α^n are just a factor times $S\phi_\alpha^n$, and since S is a unitary operator, we get that the eigenvectors are $p+1$ different eigenvectors. □

It is easily checked that these eigenvectors are Schwartz-Bruhat functions. As in the previous section we write $\psi_{\alpha_p\beta_p}$ for the eigenvectors, where α_p is the corresponding eigenvalue and β_p is the energy degeneracy.

7.4 Eigenvalues and Eigenvectors on the Adeles

To find eigenvalues and eigenvectors of the evolution operator, we will use our results from the real and p-adic oscillator. Define

$$\psi_{\alpha\beta}(x) = \frac{2^{1/4}}{2^n n!} e^{-\pi x_\infty^2} H_n(x_\infty \sqrt{2\pi}) \prod_p \psi_{\alpha_p\beta_p}(x_p). \tag{7.4.1}$$

Here α and β are adelic indices

$$\alpha = (n, \alpha_2, \alpha_3, ...), \qquad \beta = (0, \beta_2, \beta_3, ...), \tag{7.4.2}$$

with the restriction that $\psi_{\alpha_p \beta_p} = \Omega(|x|_p)$ for all but a finite number of p. In the previous sections we saw that all the p-adic eigenvectors $\psi_{\alpha_p \beta_p}$ are p-adic Schwartz-Bruhat functions. This is also the case for $p = 2$ even though we did not show it. Thus, $\psi_{\alpha\beta}$ are adelic Schwartz-Bruhat functions. It is easily seen that these functions are eigenfunctions of the adelic evolution operator, and one obtains

$$U(t)\psi_{\alpha\beta}(x) = \chi(Et)\psi_{\alpha\beta}(x), \tag{7.4.3}$$

where E is the adele $(E_n, \alpha_2, \alpha_3, ...)$. The adele E can be interpreted as the energy value. The set of eigenvectors form an orthonormal basis for $L^2(\mathbb{A})$ by Section 4.5.

Chapter 8

Concluding Remarks

The first chapters cover p-adic numbers, topological groups, adeles and quantum mechanics. These chapters should cover the necessary knowledge needed to study p-adic and adelic quantum mechanics. The last two chapters give an analysis of the one-dimensional harmonic oscillator in the p-adic and adelic case. One cannot immediately define a Hamiltonian in these cases, since multiplication of a p-adic number or an adele with a complex number is not well-defined. There are several ways to define an analog, and this thesis has used Weyl's formulation of quantum mechanics. From this formulation, eigenvalues and eigenvectors are obtained for the time evolution operator $U(t)$. In contrast to the real case, the eigenvalues are degenerate, and for $p \equiv 1 \pmod 4$, there is even an infinite degeneration. In the adelic case, the degeneration is infinite for all eigenvalues.

It is still too early to say if the p-adic or adelic model is the "correct" model for the universe. One of the problems with the real model is that it does not work under the Planck scale. One sees that the uncertainty in an analog of the position and momentum operator in the simplest ground state is 0 (see equation (6.4.28)), which may or may not be good news.

In future work one can consider other models than the harmonic oscillator. A still open problem is to find a good relationship between the p-adic and the real model. For instance that one can obtain eigenvalues for the real model by knowing the eigenvalues for all the p-adic numbers. There may be models where the p-adic cases are simpler such that solving these cases will solve real case. The adelic model is some sort of relationship between these numbers, but it does not solve the problem. The methods used in this thesis for the adelic case were based on knowing the solutions to the real and p-adic cases.

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