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The Fractal Burgers Equation - Theory and Numerics

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Preface

This study marks the completion of my Master of Science degree in applied mathematics at the Norwegian University of Science andTechnology, fulfilling the requirements of the course TMA4900 - Mathematics.

I would like to thank Espen R. Jakobsen for his great help and support throughout this semester. His help has been very useful for my understanding of fractional conservation laws, the topic of this thesis. I would also like to thank Hanna Haaland for her great support and company through my years of studies in Trondheim.

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Abstract

We study the Cauchy problem for the 1-D fractal Burgers equation, which is a non-linear and non-local scalar conservation law used to for instance model overdriven detonation in gases. Properties of classical solutions of this problem are studied using techniques mainly developed for the study of entropy solutions. With this approach we prove several a-priori estimates, using techniques such as Kruzkow doubling of variables. The main theoretical result of this study is a L^1-type contraction estimate, where we show the contraction in time of the positive part of solutions of the fractal Burgers equation. This result is used to show several other a priori estimates, as well as the uniqueness and regularity in time of solutions.

We also solve our Cauchy problem numerically, by proposing, analyzing and implementing one explicit and one implicit-explicit method, both based on finite volume methods. The methods are proved to be monotone, consistent and conservative under suitable CFL conditions. Subsequently, several a priori estimates for the numerical solutions are established. A discussion on how the numerical methods may be implemented efficiently, as well as discussions of some of the numerical results obtained conclude this study.

Sammendrag

Vi studerer Cauchy problemet for den endimensjonale fraksjonelle Burgers ligning, en ikke-lineær og ikke-lokal skalar bevarelseslov som for eksempel brukes til å modellere overdrevet detonasjon i gasser. Egenskaper til klassiske løsninger av dette problemet blir studert ved hjelp av teknikker som i hovedsak er utviklet for studiet av entropiløsninger. Med denne tilnærmingen beviser vi flere a priori estimater, ved hjelp av teknikker som Kruzkow dobling av variabler. Hovedresultatet i dette arbeidet er et L^1-type kontraksjons estimat, der vi viser kontraksjon i tid av den positive delen til løsninger av den fraksjonelle Burgers ligning. Dette resultatet brukes til å vise flere andre a priori estimater, slik at vi kan vise både unikhet og regularitet i tid av løsninger.

Vi løser også vårt Cauchy problem numerisk, ved å introdusere, analysere og implementere en eksplisitt og en implisitt-eksplisitt metode, hvor begge metodene er basert på endelige volum-metoder. Metodene blir vist til å være monotone, konsekvente og konservative, noe som lar oss etablere flere a priori estimat for de numeriske løsningene. Arbeidet avsluttes med å drøfte hvordan disse metodene kan implementeres effektivt, samt diskutere noen av de numeriske resultatene.

Contents

Chapter 1

Introduction

Non-local partial differential equations have received a lot of interest in recent years, thanks to their many applications and interesting mathematical properties. Some of their applications are found in mathematical finance [12], where Levy processes are used to model jumps in e.g. underlying stock prices, and in physical problems such as gas diffusion [11]. In this paper we study the fractal (fractional) Burgers equation, which is a non-local conservation law involved in for instance overdriven detonation in gases [11]. The fractal Burgers equation is considered here in the form of the Cauchy problem

$$\begin{cases} u_t(x,t) + \nabla \cdot f(u(x,t)) = \mathcal{L}(u(x,t)) & (x,t) \in Q_T := \mathbb{R} \times (0,T) \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{1.0.1}$$

where $\alpha \in (1,2)$ and $f = \frac{1}{2}u^2, f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous flux function. The operator $\mathcal{L}$ is a fractional power of order $\alpha/2$ of the Laplace operator. The fractional Laplace operator is formally defined via the Fourier transform. Letting $\mathcal{G}(u) = -\mathcal{L}(u)$ gives

$$\mathcal{F}(\mathcal{G}[u])(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi) \tag{1.0.2}$$

where $\mathcal{F}(\phi)(\xi) = \int_{\mathbb{R}} e^{-2i\phi x\xi}\phi(x)$, which gives us $\mathcal{L}(u) = -(2\pi)^{-\alpha}\left(-\Delta^{\alpha/2}\right)$, as described in [14]. It is much more useful for us to work with an integral representation of the fractional operator, given in [8] as

$$\mathcal{L}(u) := c_\alpha P.V. \int_{|z|>0} \frac{u(x+z) - u(x)}{|z|^{1+\alpha}} dz \tag{1.0.3}$$

where $c_\alpha > 0$ is defined as $c_\alpha = \frac{\alpha\Gamma\left(\frac{1+\alpha}{2}\right)}{2\sqrt{\pi}\pi^\alpha\Gamma\left(1-\frac{\alpha}{2}\right)}$, and Γ is the Euler function. The notation $P.V$ indicates the Cauchy principal value and is defined later in this work.

1.1 Mathematical background

The fractal Burgers equation (1.0.1) was introduced by Biler et. al [5], and has subsequently been studied in great detail. The equation is a generalization of the Cauchy problem for the classical viscous Burgers equation,

$$\begin{cases} u_t(x,t) + \nabla \cdot f(u(x,t)) = \Delta u(x,t) & (x,t) \in Q_T \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{1.1.1}$$

which has applications in areas such as the modeling of gas dynamics and traffic flow. Depending on the value of α, solutions of (1.0.1) should, according to [1, p. 146], share some properties with (1.1.1) and the Cauchy problem of the pure scalar conservation law (the inviscid Burgers equation)

$$\begin{cases} u_t(x,t) + \nabla \cdot f(u(x,t)) = 0 & (x,t) \in Q_T \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{1.1.2}$$

The well-posedness of the Cauchy problem (1.1.2) was established by Kruzkow [19] using the idea of entropy solutions. The introduction of entropy solutions was motivated by the occurrence of shocks in solutions of (1.1.2) in finite time, which can happen even though the initial data is smooth. Moreover, weak solutions can be non-unique (see e.g. [1]). The concept of entropy solutions for fractal conservation laws is thoroughly explained and defined in [1, p.148].

Entropy solutions have proved very useful for the study of fractional conservation laws, including the study of the fractal Burgers equation (1.0.1). Alibaud established the entropy formulation for fractal conservation laws, and showed existence and uniqueness in the L^∞ framework for $\alpha \in (0,1)$ in [1]. Droniou et. al [3] showed that when α is less than 1, the fractal Burgers equation (1.0.1) does not regularize the initial data, and moreover that shocks can occur in finite time, even when the initial data is smooth. The entropy solution framework is therefore particularly useful for the study of the fractal Burgers equation (1.0.1) when $\alpha \in (0,1)$.

However, when $\alpha \in (1,2)$, Droniou et. al [15] showed that (1.0.1) behaves similarly to (1.1.1), and smooth solutions are ensured when the initial data is bounded. The well-posedness of the fractal Burgers equation for $\alpha \in (1,2)$ was first established by Biler et. al [5], while the existence and uniqueness of smooth solutions was shown by Droniou et. al [15]. In this project we focus on the case $\alpha \in (1,2)$ and smooth solutions of (1.0.1). Techniques and ideas from the entropy solution framework are used to study smooth, classical solutions of our Cauchy problem (1.0.1).

There exists a large amount of literature on the numerical treatment of hyperbolic conservation laws, such as the Burgers equation. Finite volume methods are particularly popular for solving hyperbolic conservation laws, and for instance LeVeque's book [21] gives a comprehensive treatment of this topic. The work on numerical methods for non-local conservation laws is however fairly limited, although some recent contributions have been made. Droniou introduced and analyzed a finite difference based discretization of the non-local operator $\mathcal{L}$ in [14], while Jakobsen and Cifani introduced discontinuous Galerkin and vanishing viscosity methods in [7, 10]. In [8, 9] an explicit and an implicit-explicit numerical method were introduced and analyzed in great detail, in a much more general setting than what is needed for our problem (1.0.1). Some of the ideas presented in the papers [8,9,14] are used in our work.

1.2 Project outline

In this work both theoretical and numerical aspects of the fractal Burgers equation, given as the Cauchy problem (1.0.1), are studied. We let $\alpha \in (1, 2)$ and assume the solutions of (1.0.1) to be smooth, such that classical solutions can be studied. After introducing some preliminary results, such as two Kato type of inequalities, we proceed and prove a L^1- type contraction result. Our result is more precise than the standard L^1- contraction result, as contraction of the positive part, and not the absolute value, of solutions is proved. The result is shown by using and adapting techniques from the entropy solution framework. The contraction estimate implies several other results, such as a BV- contraction estimate, a L^∞- estimate as well as a comparison principle. The theoretical part is completed by showing that the classical solutions of (1.0.1) are unique and regular in time.

Continuing, we show how the fractal Burgers equation (1.0.1) may be solved numerically, by introducing an explicit and an implicit-explicit finite volume based method. Both methods are shown to be conservative and monotone when certain CFL conditions hold. We also show that numerical solutions of both methods give a convergent subsequence, of which the limit is a weak solution. In the final section some numerical results are presented, and we give a brief discussion on how the numerical methods may be implemented efficiently. This can be achieved by exploiting the Toeplitz structure of the discretization of the non-local operator $\mathcal{L}$.

Chapter 2

Theory

In this chapter we present and prove some useful a priori results for smooth solutions of the initial value problem (1.0.1). The main result is an L^1- type contraction estimate, which implies uniqueness of solutions, as well as BV-contraction and an L^∞- estimate. Regularity in time is also shown. Some notation and preliminary results are introduced first, as this will be helpful for the remainder of this chapter.

2.1 Notation

Consider the function $(\cdot)^+$ and its first two derivatives:

$$(u)^+ := \begin{cases} u & \text{for } u > 0 \\ 0 & \text{for } u \leq 0 \end{cases} \tag{2.1.1}$$

$$\text{sign}^+(u) := \begin{cases} 1 & \text{for } u > 0 \\ 0 & \text{for } u \leq 0 \end{cases} \quad \text{and} \quad \frac{d}{du}\text{sign}^+(u) = \delta_0(u) \tag{2.1.2}$$

Where $\delta_0(u)$ is the Dirac delta measure, defined by Dirac in [13]. The concept of entropy pairs should also be introduced. Two functions $(\eta(u), q(u))$, where η is convex, is called an entropy pair if

$$q'(u) = f'(u)\eta'(u) \tag{2.1.3}$$

The pair defined by $\eta(u) = (u-k)^+$ and $q(u) = \text{sign}^+(u-k)(f(u)-f(k))$ is used in our work, where $f(u) = \frac{1}{2}u^2$ and $f'(u) = u$.

We refer to the books [22, 24] for a definition of the L^1, L^∞ and BV spaces and their associated norms, which are frequently used in this work.

2.2 Preliminary results

The understanding of the non-local operator $\mathcal{L}$, defined in (1.0.3), is crucial to our work. Some of the most useful properties of the non-local operator $\mathcal{L}$ are collected in the following lemma.

Lemma 1. *Let $\mathcal{L}$ be defined as*

$$\mathcal{L}(u) := c_\alpha P.V. \int_{|z|>0} \frac{u(x+z)-u(x)}{|z|^{1+\alpha}} dz \tag{2.2.1}$$

where $P.V$ denotes the Cauchy principal value, defined in [8] as

$$P.V. \int_{|z|>0} \phi(z)\, dz = \lim_{b\to 0} \int_{b<|z|} \phi(z)\, dz \tag{2.2.2}$$

Also assume that $u \in C^2(\mathbb{R})$. Then observe that the following properties hold:

i) $$\mathcal{L}(u) = \underbrace{c_\alpha P.V. \int_{|z|<r} \frac{u(x+z)-u(x)}{|z|^{1+\alpha}}\, dz}_{=:\, \mathcal{L}_r(u)} + \underbrace{c_\alpha \int_{|z|>r} \frac{u(x+z)-u(x)}{|z|^{1+\alpha}}\, dz}_{=:\, \mathcal{L}^r(u)}$$

ii) $P.V. \int_{|z|<r} \frac{u(x+z)-u(x)}{|z|^{1+\alpha}} = P.V. \int_{|z|<r} \frac{u(x+z)-u(x)-zu'(x)}{|z|^{1+\alpha}}$

iii) $|\mathcal{L}_r(u(x))| \leq \frac{c_\alpha}{2} \|u''\|_{L^\infty(B(x,r))} \int_{|z|<r} \frac{|z|^2}{|z|^{1+\alpha}} < \infty$ *when* $\alpha \in (1,2)$

Proof. i) Property (i) follows from the original definition (2.2.1). Note that $\mathcal{L}^r$ is non-singular, while $\mathcal{L}_r$ is singular.

ii) Property (ii) is a consequence of $P.V. \int_{|z|<r} \frac{z}{|z|^{1+\alpha}}\, dz = 0$, which is a result of the integrand being odd and integrated over a symmetric region around $z = 0$.

iii) Property (iii) is shown by introducing Taylor's formula with integral remainder terms (see [9, p. 16] and [4, Section 7.5]) to obtain the two properties:

$$u(x+z) - u(x) - zu'(x) = \int_0^1 (1-s)u''(x+sz)z^2\, ds \tag{2.2.3}$$

$$u(x+z) - u(x) = \int_0^1 u'(x+sz)z\, ds \tag{2.2.4}$$

Then, by using property (ii) and (2.2.4) we get

$$\begin{aligned}
|\mathcal{L}_r(u(x))| &= c_\alpha \left| \int_{|z|<r} \frac{u(x+z)-u(x)-zu'(x)}{|z|^{1+\alpha}}\,dz \right| \\
&\leq c_\alpha \int_{|z|<r} \int_0^1 \frac{|1-s|\,|u''(x+sz)|\,|z^2|}{|z|^{1+\alpha}}\,ds\,dz \\
&\leq \frac{c_\alpha}{2} \|u''\|_{L^\infty(B(x,r))} \int_{|z|<r} \frac{z^2}{|z|^{1+\alpha}}\,dz
\end{aligned}$$

where $B(x,r)$ is a ball centered around $x = 0$ with radius r. The integral $\int_{|z|<r} \frac{z^2}{|z|^{1+\alpha}}\,dz$ is finite, since it is proportional to the integral $\int_0^r \frac{s^2}{s^{1+\alpha}}\,ds = \int_0^r s^{1-\alpha}\,ds = \frac{1}{2-\alpha} r^{2-\alpha} = \mathcal{O}_r(1)$

□

Lemma 2. *For sufficiently smooth functions φ, ψ the operator*

$$\tilde{\mathcal{L}}^r(\varphi(x,y)) := c_\alpha \int_{|z|>r} \frac{\varphi(x+z,y+z)-\varphi(x,y)}{|z|^{1+\alpha}}\,dz \tag{2.2.5}$$

where c_α is a positive constant, satisfy the following properties:

$$\tilde{\mathcal{L}}^r(\varphi(x-y)) = 0 \tag{2.2.6}$$

$$\tilde{\mathcal{L}}^r(\varphi(x)-\psi(y)) = \mathcal{L}^r(\varphi(x)) - \mathcal{L}^r(\psi(y)) \tag{2.2.7}$$

$$\int \varphi(\tilde{\mathcal{L}}^r\psi) = \int \psi(\tilde{\mathcal{L}}^r\varphi) \tag{2.2.8}$$

where $\mathcal{L}^r$ is defined in lemma 1, property (i)

Proof. Properites (2.2.6) and (2.2.7) are shown by applying the definition of $\tilde{\mathcal{L}}^r$:

$$\tilde{\mathcal{L}}^r(\varphi(x-y)) = c_\alpha \int_{|z|>r} \frac{\varphi(x+z-y-z)-\varphi(x-y)}{|z|^{1+\alpha}}\,dz = 0$$

and

$$\begin{aligned}
\tilde{\mathcal{L}}^r(\varphi(x)-\psi(y)) &= c_\alpha \int_{|z|>r} \frac{\varphi(x+z)-\psi(y+z)-\varphi(x)+\psi(y)}{|z|^{1+\alpha}}\,dz \\
&= \mathcal{L}^r(\varphi(x)) - \mathcal{L}^r(\psi(y))
\end{aligned}$$

The proof that $\tilde{\mathcal{L}}^r$ is self-adjoint is somewhat more complicated. If $\tilde{\mathcal{L}}^r$ is self-adjoint, then

$$\int \varphi(x,y)\tilde{\mathcal{L}}^r(\psi(x,y)) = \int \psi(x,y)\tilde{\mathcal{L}}^r(\varphi(x,y)) \tag{2.2.9}$$

Both sides of (2.2.9) are expanded to see that this relation is equal to

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{|z|>r} \frac{\varphi(x,y)\psi(x+z,y+z)}{|z|^{1+\alpha}}\,dz\,dx\,dy \tag{2.2.10}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{|z|>r} \frac{\psi(x,y)\varphi(x+z,y+z)}{|z|^{1+\alpha}}\,dz\,dx\,dy \tag{2.2.11}$$

and use Fubini's theorem, stated in appendix A, to rewrite the term (2.2.10) as

$$\int_{|z|>r}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\varphi(x,y)\psi(x+z,y+z)}{|z|^{1+\alpha}}\,dx\,dy\,dz \tag{2.2.12}$$

then introduce the change of variables

$$x \to x+z, \quad y \to y+z, \quad z \to -z$$

such that the integral (2.2.12) can be expressed as

$$\int_{|z|>r}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\varphi(x+z,y+z)\psi(x,y)}{|z|^{1+\alpha}}\,dx\,dy\,dz$$

which is equal to the term (2.2.11) (again using Fubini). $\square$

Lemma 3. *Let* $\mathcal{L}^r$ *be defined by* (2.2.1). *Then, for the smooth, convex function* $\eta_k^\delta(u(x))$ *the following inequality holds*

$$\eta_k^{\delta\prime}(u)\mathcal{L}^r(u) \le \mathcal{L}^r(\eta_k^\delta(u)) \tag{2.2.13}$$

Proof. As η^δ is convex, we have that $\eta^{\delta\prime\prime} \ge 0$. We Taylor approximate the function $\eta_k^\delta(u)$ around a point $u+h$ to get

$$\eta_k^\delta(u+h) \approx \eta_k^\delta(u) + h\eta_k^{\delta\prime}(u) + \frac{1}{2}h^2\eta_k^{\delta\prime\prime}(u)$$

giving

$$h\eta_k^{\delta\prime}(u) \le \eta_k^\delta(u+h) - \eta_k^\delta(u) \tag{2.2.14}$$

for some constant h. Now consider $\eta_k^{\delta\prime}(u)\mathcal{L}^r(u)$ and see that this can be expressed as

$$\eta_k^{\delta\prime}(u)\mathcal{L}^r(u) = c_\alpha \int_{|z|>r} \frac{\eta_k^{\delta\prime}(u(x))\,(u(x+z)-u(x))}{|z|^{1+\alpha}}\,dz$$

using (2.2.14) and defining $h := u(x+z) - u(x)$ we get

$$\begin{aligned}
& c_\alpha \int_{|z|>r} \frac{{\eta_k}^{\delta\prime}(u(x))\,(u(x+z)-u(x))}{|z|^{1+\alpha}}\,dz \\
&= c_\alpha \int_{|z|>r} \frac{h{\eta_k}^{\delta\prime}(u(x))}{|z|^{1+\alpha}}\,dz \\
&\leq c_\alpha \int_{|z|>r} \frac{\eta_k^\delta(u(x)+u(x+z)-u(x)) - \eta_k^\delta(u(x))}{|z|^{1+\alpha}}\,dz \\
&= \mathcal{L}^r(\eta_k^\delta(u(x)))
\end{aligned} \tag{2.2.15}$$

□

Lemma 4. *Let $\tilde{\mathcal{L}}^r$ be defined by (2.2.5). Then, for the smooth, convex function $\eta^\delta(u(x)-v(y))$ the following inequality holds*

$$\eta^{\delta\prime}(u(x)-v(y))\tilde{\mathcal{L}}^r(u(x)-v(y)) \leq \tilde{\mathcal{L}}^r(\eta^\delta(u(x)-v(y))) \tag{2.2.16}$$

Proof. The proof of this lemma is very similar to that of lemma 3. Here,

$$\eta^\delta(u-v+h) \approx \eta^\delta(u-v) + h\eta^{\delta\prime}(u-v) + \underbrace{\frac{1}{2}h^2\eta^{\delta\prime\prime}(u-v)}_{\geq 0}$$

such that

$$h\eta^{\delta\prime}(u-v) \leq \eta^\delta(u-v+h) - \eta^\delta(u-v)$$

Now combine this inequality with the definition of $\tilde{\mathcal{L}}^r$ and define $h := u(x+z) - v(y+z) - u(x) + v(y)$ to obtain

$$\begin{aligned}
& \eta^{\delta\prime}(u(x)-v(y))\tilde{\mathcal{L}}^r(u(x)-v(y)) \\
&= c_\alpha \int_{|z|>r} \eta^{\delta\prime}\,(u(x)-v(y))\,\frac{u(x+z)-v(y+z)-u(x)+v(y)}{|z|^{1+\alpha}}\,dz \\
&\leq c_\alpha \int_{|z|>r} \frac{\eta^\delta\,(u(x)-v(y)+u(x+z)-v(y+z)-u(x)+v(y)) - \eta^\delta\,(u(x)-v(y))}{|z|^{1+\alpha}}\,dz \\
&= c_\alpha \int_{|z|>r} \frac{\eta^\delta\,(u(x+z)-v(y+z)) - \eta^\delta\,(u(x)-v(y))}{|z|^{1+\alpha}}\,dz \\
&= \tilde{\mathcal{L}}^r(\eta^\delta(u(x)-v(y)))
\end{aligned}$$

which is the result we wanted to get. □

Lemma 5. *Let $u \in L^1(\mathbb{R})$ and $u = u(x)$ be uniformly continuous. Then,*

$$\lim_{|x|\to\infty} u(x) = 0 \tag{2.2.17}$$

Proof. Assume that u is positive. The lemma is proved using contradiction. Let $\{x_i\}$ be a sequence of points such that $\forall i \; u(x_i) \geq m > 0$ and $x_i \to \infty$. We fix $\epsilon = m/2$ such that $|x - x_i| \leq \delta$ implies $|u(x) - u(x_i)| \leq m/2$. Now, choose a subsequence of $\{x_i\}$ such that $\forall i, j \; d(x_i, x_j) \geq 2\delta$. Let $I_i = [x_i - \delta, x_i + \delta]$ for each member of the subsequence. As a result of our construction of the subsequence, all the intervals I_i must be disjoint. We therefore write

$$\int_{\mathbb{R}} u(x)\,dx = \int_{\bigcup_i I_i} u(x)\,dx + \int_{\mathbb{R}\setminus\bigcup_i I_i} u(x)\,dx$$

As u is positive, the second integral is positive. The first integral is written as

$$\int_{\mathbb{R}} u(x)\,dx = \sum_i \int_{I_i} u(x)\,dx$$

As $u(x_i) \geq m$ and $|u(x) - u(x_i)| \leq m/2$, we get that

$$\int_{I_i} u(x)\,dx \geq m\delta$$

The sum is therefore divergent, so the assumption that u is in $L^1(\mathbb{R})$ is no longer valid, giving us the contradiction that we needed. □

2.3 Main results

The main result in this section is stated in theorem 1. We assume that the solutions u, v of (1.0.1) are smooth enough for us to avoid using the concepts of weak solutions and entropy solutions, and instead treat the solutions as classical solutions. The necessary assumptions are given in points (I) - (III).

In the following let $u = u(x,t)$ and $v = v(x,t)$ be solutions of the initial value problem (1.0.1), with initial data u_0 and v_0. Then, u, v, u_0, v_0 are assumed to satisfy:

(I) $u, v \in L^\infty(0,T; L^1(\mathbb{R})) \cap L^\infty(0,T; BV(\mathbb{R})) \cap C^2(Q_T)$

(II) u, v uniformly continuous, and $u_{xx}, v_{xx} \in L^1(\mathbb{R})$.

(III) $u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$

Assumptions (I) - (III) are assumed to hold for the rest of this work, unless otherwise stated.

2.3.1 A priori estimates

Theorem 1. *Let $u = u(x,t)$ and $v = v(x,t)$ be solutions of the initial value problem* (1.0.1)*, with initial data u_0 and v_0. Let u, v and u_0, v_0 satisfy* (I) - (III)*. Then, for all $t \in (0,T)$*

$$\int_{\mathbb{R}} (u(x,t) - v(x,t))^+ \, dx \leq \int_{\mathbb{R}} (u(x,0) - v(x,0))^+ \, dx \tag{2.3.1}$$

Proof. Let $\eta(u,k) = \text{sign}^+(u-k)$ where k is a constant, and let $\eta'(u,k)$ be the derivative of $(u-k)^+$ with respect to u. The definition of sign$^+$ and its derivative is given in equation (2.1.2). Then observe that the integrand of (2.3.1) is equal to $\eta(u,v)$.

Observe that $\eta'(u,k)$ is discontinuous at $u = k$, and $\lim_{u \to k} \eta''(u,k) = \infty$, which could be problematic for the following analysis. It is more convenient to work with a smooth and bounded approximation to η, expressed as η^δ where $\lim_{\delta \to 0} \eta^\delta = \eta$. We specify $\eta^\delta(u,v), \eta^{\delta\prime}(u,v)$ and $\eta^{\delta\prime\prime}(u,v)$ before continuing. Let

$$\eta^\delta(u,v) = \begin{cases} \sqrt{\delta^2 + (u-v)^2} - \delta & \text{for } u \geq v \\ 0 & \text{for } u < v \end{cases} \tag{2.3.2}$$

$$\eta^{\delta\prime\prime}(u,v) = \begin{cases} \frac{u-v}{\sqrt{\delta^2+(u-v)^2}} & \text{for } u \geq v \\ 0 & \text{for } u < v \end{cases} \tag{2.3.3}$$

$$\eta^{\delta\prime\prime}(u,v) = \begin{cases} \frac{\delta^2}{(\delta^2+(u-v)^2)^{3/2}} & \text{for } u \geq v \\ 0 & \text{for } u < v \end{cases} \tag{2.3.4}$$

Observe that $\eta^{\delta\prime\prime}$ is a δ_0-sequence approximation, and $\lim_{\delta \to 0} \eta^{\delta\prime\prime} \to \delta_0$ as a distribution. The function η^δ is also smooth and convex, thus satisfying the assumptions of lemma 3.

To prove theorem 1 it is sufficient to show that

$$\frac{d}{dt} \int_{\mathbb{R}} (u(x,t) - v(x,t))^+ \ dx \leq 0 \tag{2.3.5}$$

as this implies that the integral $\int_{\mathbb{R}} (u-v)^+ \, dx$ is non-increasing in time. The integral (2.3.5) may be approximated by

$$\frac{d}{dt} \int_{\mathbb{R}} \eta^\delta(u(x,t), v(x,t)) \, dx$$

which, since u, v, η^δ are smooth, is equal to

$$\int_{\mathbb{R}} \frac{d}{dt} \eta^\delta(u(x,t), v(x,t)) \, dx = \int_{\mathbb{R}} \eta^{\delta\prime}(u(x,t), v(x,t))(u_t(x,t) - v_t(x,t)) \, dx =: I \tag{2.3.6}$$

We now take use of the Kruzkow doubling of variables technique introduced in [19], but only double the variables in the spatial domain. This technique was introduced for fractal conservation laws by Alibaud [1], and used in e.g. [8]. We follow [8] and let $u = u(x,t)$ and $v = v(y,t)$, and introduce the smooth function $\phi_\varepsilon(x-y)$. Let ϕ_ε be a smooth mollifier defined by $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$, where $\phi(z)$ is a C^∞ function that satisfy

$$0 \leq \phi(z) \leq 1, \quad \phi(z) = 0 \text{ for } |z| > 1, \quad \phi(-z) = \phi(z), \quad \int_{\mathbb{R}} \phi(z)\,dz = 0$$

This allows us to rewrite integral (2.3.6) as

$$I_\varepsilon := \int_{\mathbb{R}}\int_{\mathbb{R}} {\eta^{\delta}}'(u(x,t),v(y,t))(u_t(x,t) - v_t(y,t))\phi_\varepsilon(x-y)\,dy\,dx \tag{2.3.7}$$

and observe that $I = \lim_{\varepsilon\to 0} I_\varepsilon$. We want to prove that $I_\varepsilon \leq 0$. By using the fact that both $u(x,t)$ and $v(y,t)$ solves equation (1.0.1) the integral (2.3.7) is rewritten as

$$\begin{aligned} I_\varepsilon = \int_{\mathbb{R}}\int_{\mathbb{R}} {\eta^{\delta}}'(u(x,t),v(y,t))\,(\mathcal{L}(u(x,t)) - uu_x(x,t) \\ -\mathcal{L}(v(y,t)) + vv_y(y,t))\,\phi_\varepsilon(x-y)\,dy\,dx \end{aligned} \tag{2.3.8}$$

Now split the integral (2.3.8) into two parts, and write $u(x,t), v(y,t)$ as u, v in the following,

$$I_\varepsilon = \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} {\eta^{\delta}}'(u,v)\,(\mathcal{L}(u) - \mathcal{L}(v))\,\phi_\varepsilon\,dy\,dx}_{(i)} - \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} {\eta^{\delta}}'(u,v)\,(uu_x - vv_y)\,\phi_\varepsilon\,dy\,dx}_{(ii)}$$

and consider the second part (*ii*). Define the entropy flux function q^δ,

$$q^\delta(u,v) = {\eta^{\delta}}'(u,v)(f(u) - f(v)) = \frac{1}{2}{\eta^{\delta}}'(u,v)(u^2 - v^2) \tag{2.3.9}$$

such that $(\eta^\varepsilon(u), q^\varepsilon(u))$ defines an entropy pair, and then differentiate q^δ to get

$$\begin{aligned} {\eta^{\delta}}'(u,v)uu_x &= q^\delta_x - \frac{1}{2}{\eta^{\delta}}''(u,v)(u^2 - v^2)u_x \\ {\eta^{\delta}}'(u,v)vv_y &= -q^\delta_y + \frac{1}{2}{\eta^{\delta}}''(u,v)(u^2 - v^2)v_y \end{aligned}$$

obtaining

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} {\eta^{\delta}}'(u,v)\,(uu_x - vv_y)\phi_\varepsilon\,dy\,dx \qquad (2.3.10) \\ &\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(q^\delta_x + q^\delta_y - {\eta^{\delta}}''(u,v)(f(u) - f(v))(u_x + v_y)\Big)\,\phi_\varepsilon\,dy\,dx \end{aligned}$$

We later argue that the $\eta^{\delta''}$ term is equal to zero, but will keep it for now. By inserting (2.3.10) into (2.3.8) we obtain

$$\begin{aligned} I_\varepsilon = &\int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta'}(u,v)\left(\mathcal{L}(u) - \mathcal{L}(v)\right)\phi_\varepsilon \, dy\, dx \\ &\qquad - \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(q_x^\delta + q_y^\delta - \eta^{\delta''}(u,v)(f(u)-f(v))(u_x+v_y)\Big)\phi_\varepsilon \, dy\, dx \end{aligned} \tag{2.3.11}$$

and then recall that the fractional Laplace operator $\mathcal{L}$ can be split into two parts, $\mathcal{L} = \mathcal{L}_r + \mathcal{L}^r$. Using this (2.3.11) is written as

$$I_\varepsilon = \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta'}(u,v)\left(\mathcal{L}_r(u) - \mathcal{L}_r(v)\right)\phi_\varepsilon \, dy\, dx \tag{2.3.12}$$

$$+ \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta'}(u,v)\left(\mathcal{L}^r(u) - \mathcal{L}^r(v)\right)\phi_\varepsilon \, dy\, dx \tag{2.3.13}$$

$$- \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(q_x^\delta + q_y^\delta - \eta^{\delta''}(u,v)(f(u)-f(v))(u_x+v_y)\Big)\phi_\varepsilon \, dy\, dx \tag{2.3.14}$$

The idea is then to eventually let the singular part (2.3.12) vanish as we take $\lim r \to 0$, while still manipulating the non-singular part (2.3.13). The last part (2.3.14) of I_ε is also kept along to vanish in the end.

First, (2.3.13) is estimated,

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta'}(u,v)\left(\mathcal{L}^r(u) - \mathcal{L}^r(v)\right)\phi_\varepsilon \, dy\, dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta'}(u,v)\tilde{\mathcal{L}}^r(u-v)\phi_\varepsilon \, dy\, dx \\ \underbrace{\leq}_{\text{by (2.2.16)}} \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{\mathcal{L}}^r(\eta^\delta(u,v))\phi_\varepsilon \, dy\, dx &\underbrace{=}_{\text{by (2.2.8)}} \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^\delta(u,v)\tilde{\mathcal{L}}^r(\phi_\varepsilon)\, dy\, dx \end{aligned}$$

from which we get

$$\begin{aligned} I_\varepsilon \leq &\int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta'}(u,v)\left(\mathcal{L}_r(u) - \mathcal{L}_r(v)\right)\phi_\varepsilon \, dy\, dx + \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^\delta(u,v)\tilde{\mathcal{L}}^r(\phi_\varepsilon)\, dy\, dx \\ &\qquad - \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(q_x^\delta + q_y^\delta - \eta^{\delta''}(u,v)(f(u)-f(v))(u_x+v_y)\Big)\phi_\varepsilon \, dy\, dx \end{aligned} \tag{2.3.15}$$

Then, consider each term of equation (2.3.15) separately. The second term of equation (2.3.15) is equal to zero, as by using property (2.2.6) we have that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \eta^\delta(u,v)\tilde{\mathcal{L}}^r(\phi_\varepsilon)\, dy\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^\delta(u,v) \underbrace{\tilde{\mathcal{L}}^r(\phi_\varepsilon(x-y))}_{=0 \text{ by property (2.2.6)}} \, dy\, dx = 0$$

Continuing, consider the first term of equation (2.3.15) and split the term into two parts,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta'}(u,v)\mathcal{L}_r(u)\phi_\varepsilon \, dy\, dx - \int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta'}(u,v)\mathcal{L}_r(v)\phi_\varepsilon \, dy\, dx$$

and consider the first part. We are going to keep δ fixed, while letting $r \to 0$ and argue that the term then goes to zero. But first, observe that

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \eta^{\delta'}(u(x), v(y)) \mathcal{L}_r(u(x)) \phi_\varepsilon(x-y)\, dy\, dx \right| \le \left| \int_{\mathbb{R}} \eta^{\delta'} \mathcal{L}_r(u) * \phi_\varepsilon \, dx \right| \tag{2.3.16}$$

$$\le \|\mathcal{L}_r(u)\|_{L^1(\mathbb{R})} \underbrace{\|\phi_\varepsilon\|_{L^1(\mathbb{R})}}_{=1} = \|\mathcal{L}_r(u)\|_{L^1(\mathbb{R})}$$

where $*$ denotes the convolution operation. Now,

$$\|\mathcal{L}_r(u)\|_{L^1(\mathbb{R})} = c_\alpha \int_{\mathbb{R}} \left| \int_{|z|<r} \frac{u(x+z) - u(x)}{|z|^{1+\alpha}}\, dz \right| dx$$

assume that $r < 1$ and introduce the indicator function $\mathbf{1}_{|z|<r}$ to obtain

$$c_\alpha \int_{\mathbb{R}} \left| \int_{|z|<r} \frac{u(x+z) - u(x)}{|z|^{1+\alpha}}\, dz \right| dx \le c_\alpha \int_{\mathbb{R}} \left| \int_{|z|<1} \frac{u(x+z) - u(x)}{|z|^{1+\alpha}} \mathbf{1}_{|z|<r}\, dz \right| dx \tag{2.3.17}$$

and use property (ii) of lemma 1 to rewrite (2.3.17), such that the integral form of the remainder for the Taylor series $u(x+z) - u(x) - zu'(x)$ can be introduced:

$$\begin{aligned} & c_\alpha \int_{\mathbb{R}} \left| \int_{|z|<1} \frac{u(x+z) - u(x)}{|z|^{1+\alpha}} \mathbf{1}_{|z|<r}\, dz \right| dx \\ =& c_\alpha \int_{\mathbb{R}} \left| \int_{|z|<1} \frac{u(x+z) - u(x) - zu'(x)}{|z|^{1+\alpha}} \mathbf{1}_{|z|<r}\, dz \right| dx \\ \le& c_\alpha \int_{\mathbb{R}} \int_{|z|<r} \int_0^1 \frac{|1-s|\, |u''(x+sz)|\, |z|^2}{|z|^{1+\alpha}} \mathbf{1}_{|z|<r}\, ds\, dz\, dx \end{aligned}$$

Fubini is then used to rewrite, obtaining

$$c_\alpha \int_{\mathbb{R}} \int_{|z|<r} \int_0^1 \frac{|1-s|\, |u''(x+sz)|\, |z|^2}{|z|^{1+\alpha}} \mathbf{1}_{|z|<r}\, ds\, dz\, dx \le \|u''\|_{L^1(\mathbb{R})} \int_{|z|<r} \frac{|z|^2}{|z|^{1+\alpha}}\, dz$$

We have assumed that $u'' \in L^1(\mathbb{R})$, and the integral $\int_{|z|<r} \frac{|z|^2}{|z|^{1+\alpha}}$ was shown in the proof of property (iii) of lemma 1 to be proportional to $r^{2-\alpha}$ which is finite. As we pass r to 0 we get

$$\lim_{r\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \eta^{\delta'}(u, v) \mathcal{L}_r(u) \phi_\varepsilon\, dy\, dx = \lim_{r \to 0} C r^{2-\alpha} = 0$$

And, by the same arguments as above, we also get that

$$\lim_{r\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \eta^{\delta'}(u, v) \mathcal{L}_r(v) \phi_\varepsilon\, dy\, dx = \lim_{r \to 0} D r^{2-\alpha} = 0$$

By these arguments we see that the first term of equation (2.3.15) goes to zero as we let $r \to 0$, while keeping δ and ε fixed.

The next step in the proof is to show that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(q_x^\delta + q_y^\delta \right) \phi_\varepsilon(x-y)\, dy\, dx = 2 \int_{\mathbb{R}} q_x^\delta \, dx \tag{2.3.18}$$

The proof of result A found in appendix A is followed quite closely to show that this holds. We need $q_x^\delta(x,y), q_y^\delta(x,y) \in L^\infty(\mathbb{R}^2)$ as well as q_x^δ, q_y^δ to be continuous almost everywhere and to have compact support in order to use this result. Remember that

$$q_x^\delta = \eta^{\delta\prime}(u,v) u u_x - \frac{1}{2} \eta^{\delta\prime\prime}(u,v)(u^2 - v^2) u_x$$

Now, from assumptions (I) - (III) as well as $\eta^{\delta\prime}, \eta^{\delta\prime\prime}$ being bounded (see definitions (2.3.3), (2.3.4)) we have that $q_x^\delta(x,y) \in L^\infty(\mathbb{R}^2)$. By lemma 5 it is clear that $\lim_{|x| \to 0} u(x) = 0$ and $\lim_{|y| \to 0} v(y) = 0$, implying that q_x^δ has compact support, as $\lim_{|x|,|y| \to 0} q_x^\delta(x,y) = 0$. The function q_x^δ is continuous almost everywhere, as it is assumed that $u, v \in C^2(\mathbb{R})$. The same reasoning also holds for q_y^δ. Therefore, the requirements of result A of appendix A are fulfilled for q_x^δ and q_y^δ. Proceeding, the properties of the smooth mollifier ϕ_ε are used to observe that

$$\begin{aligned} \int_{\mathbb{R}} q_x^\delta(x,y) \phi_\varepsilon(x-y)\, dy - q_x^\delta(x,x) &= \int_{\mathbb{R}} \left(q_x^\delta(x,y) - q_x^\delta(x,x) \right) \phi_\varepsilon(x-y)\, dy \\ &= \int_{|z| \le 1} \left(q_x^\delta(x, x+\varepsilon z) - q_x^\delta(x,x) \right) \phi(z)\, dz \end{aligned}$$

where the last step follows from ϕ having support on $[-1,1]$. From the continuity of q_x^δ we obtain

$$\int_{|z| \le 1} \left(q_x^\delta(x, x+\varepsilon z) - q_x^\delta(x,x) \right) \phi(z)\, dz \to 0 \quad \text{as} \quad \varepsilon \to 0$$

for almost all x. Also,

$$\left| \int_{\mathbb{R}} q_x^\delta(x,y) \phi_\varepsilon(x-y)\, dy \right| \le \|q_x^\delta\|_{L^\infty(\mathbb{R})} < \infty$$

together with the fact that q_x^δ has compact support this implies that we can use the Lebesgue bounded convergence theorem, stated in appendix A, to conclude that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} q_x^\delta(x,y) \phi_\varepsilon(x-y)\, dy\, dx = \int_{\mathbb{R}} q_x^\delta(x,x)\, dx$$

and similarly,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} q_y^\delta(x,y)\phi_\varepsilon(x-y)\,dy\,dx = \int_{\mathbb{R}} q_x^\delta(x,x)\,dx$$

which implies that equation (2.3.18) holds.

Comment 1. *It is important to note that this part can be shown by a different approach also. We could write*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} q_x^\delta \phi_\varepsilon(x,y)\,dy\,dx = \int_{\mathbb{R}} q_x^\delta * \phi_\varepsilon\,dx$$

and then estimate the difference between this term and $\int_{\mathbb{R}} q_x^\delta(x,x)\,dx$, *using the regularity assumptions* (I) - (III) *and for instance Fubini's theorem.*

Continuing, it remains to evaluate the integral $\int_{\mathbb{R}} q_x^\delta\,dx$. Use the definition of q^δ given by equation (2.3.9) to obtain

$$2\int_{\mathbb{R}} q_x^\delta(x)\,dx = \lim_{R\to\infty}\left[{\eta^\delta}'(u,v)(u^2-v^2)\right]_{-R}^{R}$$

As u satisfy the requirements of lemma 5, we conclude that

$$\lim_{R\to\infty}\left[{\eta^\delta}'(u(x),v(x))\left(u(x)^2-v(x)^2\right)\right]_{-R}^{R} = 0$$

The only term of equation (2.3.15) left to consider is the last term, which is

$$-\int_{\mathbb{R}}\int_{\mathbb{R}} {\eta^\delta}''(u(x),v(y))(f(u(x))-f(v(y)))(u_x(x)+v_y(y))\phi_\varepsilon(x-y)\,dy\,dx$$

Again, we turn to result A of appendix A, and the same arguments as given above, in order to pass $\varepsilon\to 0$. The technical details of this is found in appendix B. We conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0}&\int_{\mathbb{R}}\int_{\mathbb{R}} {\eta^\delta}''(u(x),v(y))(f(u(x))-f(v(y)))(u_x(x)+v_y(y))\phi_\varepsilon(x-y)\,dy\,dx\\ &=\int_{\mathbb{R}} {\eta^\delta}''(u(x),v(x))(f(u(x))-f(v(x)))(u_x(x)+v_x(x))\,dx \end{aligned} \tag{2.3.19}$$

Then, using assumptions (I) - (II) the term (2.3.19) is estimated,

$$\begin{aligned}&\left|\int_{\mathbb{R}} {\eta^\delta}''(u(x),v(x))(f(u(x))-f(v(x)))(u(x)_x+v(x)_x)\,dx\right|\\ &\le \frac{1}{2}\left(\|u^2\|_{L^1(\mathbb{R})}+\|v^2\|_{L^1(\mathbb{R})}\right)\left(|u|_{BV(\mathbb{R})}+|v|_{BV(\mathbb{R})}\right)\int_{\mathbb{R}}\left|{\eta^\delta}''(u(x),v(x))\right|\,dx\\ &\le C\int_{\mathbb{R}}\left|{\eta^\delta}''(u(x),v(x))\right|\,dx\end{aligned} \tag{2.3.20}$$

The idea then is to show that the integrand of integral (2.3.20) is monotone in δ, and converging to 0 as δ goes to 0, and then appeal to the Lebesgue monotone convergence theorem presented in appendix A, to show that as $\delta \to 0$ the integral goes to zero. The explicit formula for $\eta^{\delta''}(u,v)$ is introduced to achieve this,

$$\Upsilon_\delta(r) := |\eta^{\delta''}(r)| = \left| \frac{\delta^2}{(\delta^2 + r^2)^{3/2}} \right| = \frac{\delta^2}{(\delta^2 + r^2)^{3/2}}$$

Where $r := u - v$, and $\delta > 0$. Remember that $\eta^\delta(r) = 0$ for $r \leq 0$, so assume that $r > 0$ in the following. Observe that $\lim_{\delta \to 0} \Upsilon_\delta(r) = 0$ when $r \neq 0$, and differentiate Υ_δ with respect to δ to get

$$\frac{\partial}{\partial \delta} \Upsilon_\delta(r) = \frac{2\delta r^2 - \delta^3}{(\delta^2 + r^2)^{\frac{5}{2}}}$$

Set $\delta <$ min(r), then $\frac{\partial}{\partial \delta} \Upsilon_\delta(r) \geq 0$ for all $r > 0$. To summarize, we now have:

$$\lim_{\delta \to 0} \Upsilon_\delta(r) \to 0 \quad \text{for} \quad r \neq 0 \quad \text{and} \quad \frac{\partial}{\partial \delta} \Upsilon_\delta(r) \geq 0 \quad \text{for} \quad r \neq 0 \tag{2.3.21}$$

These properties allows us to use the monotone convergence theorem to conclude the proof. Letting $f_n := \Upsilon_{\delta_n}$ and $\delta_n \geq \delta_{n+1}$ for all $n \geq 1$ as well as $\lim_{n \to \infty} \delta_n = 0$, to get

$$\begin{aligned} f_n(r) &\leq f_{n+1}(r) && \text{for almost every } r, \text{all } n \geq 1 \\ \lim_{n \to \infty} f_n(r) &= 0 && \text{for almost every } r \end{aligned}$$

Then the following holds

$$\lim_{n \to \infty} \int f_n = \int f = \int 0 = 0$$

which is equivalent to

$$\lim_{\delta \to 0} \int_{\mathbb{R}} \Upsilon_\delta(u - v)\, dx = \int_{\mathbb{R}} \lim_{\delta \to 0} \Upsilon_\delta(u - v)\, dx = 0$$

implying that $\lim_{\varepsilon \to 0} I_\varepsilon = I \leq 0$, and the proof of this theorem is completed. □

Corollary 1. *Let $u(x,t)$, $v(x,t)$ be solutions of the initial value problem* (1.0.1) *with initial data u_0, v_0, that satisfy assumptions* (I) - (III). *Then, for all $t \in (0,T)$,*

$$\|u(x,t) - v(x,t)\|_{L^1(\mathbb{R})} \leq \|u_0(x) - v_0(x)\|_{L^1(\mathbb{R})} \tag{2.3.22}$$

Proof. Statement (2.3.22) is equivalent to $\int_{\mathbb{R}} |u(x,t)-v(x,t)|\,dx \leq \int_{\mathbb{R}} |u(x,0)-v(x,0)|\,dx$. Begin by expanding the absolute values of $u-v$ into the positive and negative parts. Note that

$$\int_{\mathbb{R}} |u(x,t)-v(x,t)|\,dx = \int_{\mathbb{R}} (u(x,t)-v(x,t))^+ + (u(x,t)-v(x,t))^-\,dx$$

which, since $(-\phi)^- = (\phi)^+$ is equal to the statement

$$\int_{\mathbb{R}} (u(x,t)-v(x,t))^+ + (v(x,t)-u(x,t))^+\,dx$$

Then use the result of theorem 1 on each of the two terms to see that

$$\begin{aligned}&\int_{\mathbb{R}} (u(x,t)-v(x,t))^+ + (v(x,t)-u(x,t))^+\,dx\\ \leq &\int_{\mathbb{R}} (u(x,0)-v(x,0))^+ + (v(x,0)-u(x,0))^+\,dx\end{aligned}$$

which concludes the proof, since this is equivalent to the statement

$$\int_{\mathbb{R}} |u(x,t)-v(x,t)|\,dx \leq \int_{\mathbb{R}} |u(x,0)-v(x,0)|\,dx$$

□

Comment 2. *The L^1 contraction property* (2.3.22) *implies uniqueness of solutions to* (1.0.1)*, as letting $v(x,t)=0$ in* (2.3.22) *gives*

$$\int_{\mathbb{R}} |u(x,t)|\,dx \leq \int_{\mathbb{R}} |u(x,0)|\,dx$$

Comment 3. *L^1 contraction can be proved directly using the same techniques as in the proof of theorem 1, but redefining $\eta_k(u)$ as $\eta_k(u) = |u-k|$ and then use a suitable smooth approximation to $\eta_k(u)$, for instance $\eta_k^\delta(u) = \sqrt{\delta^2 + (u-k)^2}$.*

Corollary 2. *Let $u(x,t)$, $v(x,t)$ be solutions of the initial value problem* (1.0.1) *with initial data u_0, v_0, that satisfy assumptions* (I) - (III)*. Then, for all $t \in (0,T)$,*

$$\|u_x(x,t)\|_{L^1(\mathbb{R})} \leq \|u_x(x,0)\|_{L^1(\mathbb{R})} \tag{2.3.23}$$

Proof. The proof is structured as follows. We first show that $\lim_{h\to 0} \int_{\mathbb{R}} |u(x+h)-u(x)|\,dx = \int_{\mathbb{R}} |u_x(x)|\,dx$, in order to apply the result of corollary 1 on the integral $\int_{\mathbb{R}} |u(x+h)-u(x)|\,dx$.

Now, use the reverse triangle inequality, $||a|-|b|| \leq |a-b|$ too see that

$$\left|\int_{\mathbb{R}} |u(x+h)-u(x)|\;dx - \int_{\mathbb{R}} h\,|u_x(x)|\;dx\right| \leq \int_{\mathbb{R}} |u(x+h)-u(x)-hu_x(x)|\,dx$$

Then, use the integral remainder term of the Taylor series given in (2.2.4) to write

$$\frac{1}{h}\int_{\mathbb{R}} |u(x+h) - u(x) - hu_x(x)|\,dx \le \frac{1}{h}\int_{\mathbb{R}} \left|\int_0^1 (1-s)u_{xx}(x+sh)h^2\,ds\right|\,dx$$

By assumption (II) we have that $u_{xx} \in L^1(\mathbb{R})$. Therefore,

$$\frac{1}{h}\int_{\mathbb{R}} \left|\int_0^1 (1-s)u_{xx}(x+sh)h^2\,ds\right|\,dx \le h\|u_{xx}\|_{L^1(\mathbb{R})} < \infty$$

and $\lim_{h\to 0} h\|u_{xx}\|_{L^1(\mathbb{R})} \to 0$. By this we have that

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}} |u(x+h) - u(x)|\,dx = \int_{\mathbb{R}} |u_x(x)|\,dx$$

Now, observe that both $u(x+h,t)$ and $u(x,t)$ solve equation (1.0.1) and use the result of corollary 1 to see that

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}} |u(x+h,t) - u(x,t)|\,dx \le \lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}} |u(x+h,0) - u(x,0)|\,dx$$

which implies that

$$\int_{\mathbb{R}} |u_x(x,t)|\,dx \le \int_{\mathbb{R}} |u_x(x,0)|\,dx$$

concluding the proof of the corollary. □

Corollary 3. *Let $u(x,t)$, $v(x,t)$ be solutions of the initial value problem* (1.0.1) *that satisfy assumptions* (I) - (III). *Then, for all $t \in (0,T)$, the following holds for each $x \in \mathbb{R}$:*

$$u(x,0) \le v(x,0) \implies u(x,t) \le v(x,t) \tag{2.3.24}$$

Proof. We already know from theorem 1 that solutions to (1.0.1) satisfy

$$\int_{\mathbb{R}} (v(x,t) - u(x,t))^+\,dx \le \int_{\mathbb{R}} (v(x,0) - u(x,0))^+\,dx$$

which implies

$$\begin{aligned}\int_{\mathbb{R}} (u(x,t) - v(x,t))^-\,dx &= \int_{\mathbb{R}} (v(x,t) - u(x,t))^+\,dx \\ \le \int_{\mathbb{R}} (v(x,0) - u(x,0))^+\,dx &= \int_{\mathbb{R}} (u(x,0) - v(x,0))^-\,dx \end{aligned} \tag{2.3.25}$$

Note that $\int_{\mathbb{R}}(u(x,0) - v(x,0))^-\,dx \le 0$ if $u(x,0) \le v(x,0)$, which implies $(v(x,t) - u(x,t))^+ \le 0$. Since $(\cdot)^+ \ge 0$ this is only the case when $(v(x,t) - u(x,t))^+ = 0$, implying that $u(x,t) \le v(x,t)$. Thus the implication stated in the corollary is obtained. □

Corollary 4. *Let $u(x,t)$, $v(x,t)$ be solutions of the initial value problem* (1.0.1) *with initial data u_0, v_0, that satisfy assumptions* (I) - (III). *Then, for all $t \in (0,T)$,*

$$\|u(x,t)\|_{L^\infty(\mathbb{R})} \leq \|u(x,0)\|_{L^\infty(\mathbb{R})} \tag{2.3.26}$$

Proof. This proof relies on corollary 3. First observe that

$$-\|u_0\|_{L^\infty(\mathbb{R})} \leq u_0 \leq \|u_0\|_{L^\infty(\mathbb{R})}$$

Define $w(x,t) := \|u_0\|_{L^\infty}$, and note that $w(x,t)$ solves equation (1.0.1), as it is just a constant. Then use corollary 3 to see that

$$u(x,t) \leq w(x,t) \quad \text{and} \quad -w(x,t) \leq u(x,t)$$

which implies the result stated in this theorem. □

The a priori estimates obtained so far are summarized in the following theorem:

Theorem 2. *Let $u(x,t)$, $v(x,t)$ be solutions of the initial value problem* (1.0.1) *with initial data u_0, v_0, that satisfy assumptions* (I) - (III). *Then, for all $t \in (0,T)$,*

$$\int_{\mathbb{R}} (u(x,t) - v(x,t))^+ \, dx \leq \int_{\mathbb{R}} (u(x,0) - v(x,0))^+ \, dx \tag{2.3.27}$$

$$\|u(x,t) - v(x,t)\|_{L^1(\mathbb{R})} \leq \|u(x,0) - v(x,0)\|_{L^1(\mathbb{R})} \tag{2.3.28}$$

$$\|u_x(x,t)\|_{L^1(\mathbb{R})} \leq \|u_x(x,0)\|_{L^1(\mathbb{R})} \tag{2.3.29}$$

$$\|u(x,t)\|_{L^\infty(\mathbb{R})} \leq \|u(x,0)\|_{L^\infty(\mathbb{R})} \tag{2.3.30}$$

$$u(x,0) \leq v(x,0) \implies u(x,t) \leq v(x,t) \quad \textit{for each } x \in \mathbb{R}. \tag{2.3.31}$$

2.3.2 Regularity in time

Lemma 6. *Let $u(x,t)$, $v(x,t)$ be solutions of the initial value problem* (1.0.1) *with initial data u_0, v_0, that satisfy assumptions* (I) - (III). *Then, for all $s, t \in (0,T)$,*

$$\|u(x,t) - u(x,s)\|_{L^1(\mathbb{R})} \leq C|t-s|^{\frac{1}{\alpha}} \tag{2.3.32}$$

Proof. The proof is structured as follows. First, convolve u with a mollifier to obtain u_ε, and find an estimate for $\|u_{\varepsilon_t}\|_{L^1(\mathbb{R})}$ using Taylor expansions and integration by parts. By the use of the triangle inequality and the estimate of $\|u_{\varepsilon_t}\|_{L^1(\mathbb{R})}$ a bound for $\|u(x,t) - u(x,s)\|_{L^1(\mathbb{R})}$ in terms of ε is obtained.

Let ρ_ε be a smooth mollifier defined by $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$, where ρ is a C^∞ function that satisfy

$$0 \leq \rho(z) \leq 1, \quad \rho(z) = 0 \text{ for } |z| > 1, \quad \rho(-z) = \rho(z), \quad \int_{\mathbb{R}} \rho(z)\, dz = 0$$

Let $u_\varepsilon = \rho_\varepsilon * u$, where $*$ denotes the convolution between the two functions ρ_ε and u. The convolution is defined by

$$(\rho_\varepsilon * u)(x) = \int_{\mathbb{R}} \rho_\varepsilon(x) u(x-y)\, dy = (u * \rho_\varepsilon)(x)$$

By convolving ρ_ε with u_t and using equation (1.0.1) we get

$$u_{\varepsilon_t} = \rho_{\varepsilon_t} * (uu_x) + \Delta^{\alpha/2} u_\varepsilon$$

Proceeding with the use of Young's inequality (here $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$, which is also a consequence of Fubini) we obtain

$$\begin{aligned} \int_{\mathbb{R}} |u_{\varepsilon_t}|\ dx &\leq \int_{\mathbb{R}} |\rho_\varepsilon * (uu_x)|\ dx + \int_{\mathbb{R}} \left|\Delta^{\alpha/2} u_\varepsilon\right|\ dx \\ &\leq \int_{\mathbb{R}} |\rho_\varepsilon|\ dx \int_{\mathbb{R}} |uu_x|\ dx + \int_{\mathbb{R}} \left|\Delta^{\alpha/2} u_\varepsilon\right|\ dx \end{aligned} \tag{2.3.33}$$

Consider the term $\int_{\mathbb{R}} \Delta^{\alpha/2} u_\varepsilon\, dx$ in the following. Using the definition of $\Delta^{\alpha/2}$ and a change of variables we observe that

$$\begin{aligned} \int_{\mathbb{R}} \Delta^{\alpha/2} u_\varepsilon\, dx &= \int_{\mathbb{R}} \Delta^{\alpha/2} \rho_\varepsilon * u\, dx \\ &= c_\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{|z|>0} \frac{\rho_\varepsilon(x-y+z) - \rho_\varepsilon(x-y)}{|z|^{1+\alpha}} u(y)\, dz\, dy\, dx \end{aligned} \tag{2.3.34}$$

For reasons that will become obvious later on in the proof, the integral (2.3.34) is split into three parts, according to the value of $|z|$, such that

$$\begin{aligned} (\Delta^{\alpha/2} \rho_\varepsilon * u)(x) &= c_\alpha \int_{\mathbb{R}} \int_{0<|z|<\varepsilon} \frac{\rho_\varepsilon(x-y+z) - \rho_\varepsilon(x-y)}{|z|^{1+\alpha}} u(y)\, dz\, dy \\ &+ c_\alpha \int_{\mathbb{R}} \int_{\varepsilon<|z|<1} \frac{\rho_\varepsilon(x-y+z) - \rho_\varepsilon(x-y)}{|z|^{1+\alpha}} u(y)\, dz\, dy \\ &+ c_\alpha \int_{\mathbb{R}} \int_{|z|>1} \frac{\rho_\varepsilon(x-y+z) - \rho_\varepsilon(x-y)}{|z|^{1+\alpha}} u(y)\, dz\, dy \end{aligned} \tag{2.3.35}$$

We then use the property (ii) of lemma 1 to get that

$$\begin{aligned} &\int_{0<|z|<\varepsilon} \frac{\rho_\varepsilon(x-y+z) - \rho_\varepsilon(x-y)}{|z|^{1+\alpha}} \\ &\qquad = \int_{0<|z|<\varepsilon} \frac{\rho_\varepsilon(x-y+z) - \rho_\varepsilon(x-y) - z\rho_\varepsilon'(x-y)}{|z|^{1+\alpha}} \end{aligned}$$

The main idea is then to introduce Taylor's formula with integral remainder terms given in (2.2.3) and (2.2.4), and insert these into equation (2.3.35).

Property (2.2.3) is used to rewrite the first term, while property (2.2.4) is used to rewrite the second term of equation (2.3.35). By doing this we obtain

$$
\begin{aligned}
(\Delta^{\alpha/2}\rho_\varepsilon * u)(x) = c_\alpha \int_{\mathbb{R}}\int_{0<|z|<\varepsilon}\int_0^1 \frac{(1-s)\rho_\varepsilon''(x-y+sz)z^2}{|z|^{1+\alpha}} u(y)\,ds\,dz\,dy \\
+ c_\alpha \int_{\mathbb{R}}\int_{\varepsilon<|z|<1}\int_0^1 \frac{\rho_\varepsilon'(x-y+sz)z}{|z|^{1+\alpha}} u(y)\,ds\,dz\,dy \\
+ c_\alpha \int_{\mathbb{R}}\int_{|z|>1} \frac{\rho_\varepsilon(x-y+z)-\rho_\varepsilon(x-y)}{|z|^{1+\alpha}} u(y)\,dz\,dy
\end{aligned}
\tag{2.3.36}
$$

The next step in the proof is to exploit the fact that $u \in BV(\mathbb{R})$. Using integration by parts on the first two terms of equation (2.3.36), one derivative is moved from ρ_ε to u. By doing this we get that equation (2.3.36) equals

$$
\begin{aligned}
&- c_\alpha \int_{\mathbb{R}}\int_{0<|z|<\varepsilon}\int_0^1 \frac{(1-s)\rho_\varepsilon'(x-y+sz)z^2}{|z|^{1+\alpha}} u'(y)\,ds\,dz\,dy \\
&- c_\alpha \int_{\mathbb{R}}\int_{\varepsilon<|z|<1}\int_0^1 \frac{\rho_\varepsilon(x-y+sz)z}{|z|^{1+\alpha}} u'(y)\,ds\,dz\,dy \\
&+ c_\alpha \int_{\mathbb{R}}\int_{|z|>1} \frac{\rho_\varepsilon(x-y+z)-\rho_\varepsilon(x-y)}{|z|^{1+\alpha}} u(y)\,dz\,dy
\end{aligned}
\tag{2.3.37}
$$

We wish to find an estimate for $\int_{\mathbb{R}} |(\Delta^{\alpha/2}\rho_\varepsilon * u)(x)|\,dx$ and plan to integrate each term of equation (2.3.37) for x over $\mathbb{R}$ to obtain this estimate. Fubini is used to change the order of integration in each of the above terms, which allows us to estimate the integral of the first term of (2.3.37) in the following way:

$$
\begin{aligned}
&c_\alpha \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{0<|z|<\varepsilon}\int_0^1 \left| \frac{(1-s)\rho_\varepsilon'(x-y+sz)z^2}{|z|^{1+\alpha}} u'(y) \right| \,ds\,dz\,dy\,dx \\
\leq &c_\alpha C |u|_{BV(\mathbb{R})} \int_{\mathbb{R}}\int_{0<|z|<\varepsilon} \frac{|\rho_\varepsilon'(x+sz)||z|^2}{|z|^{1+\alpha}}\,dz\,dx
\end{aligned}
$$

Then, following [9, p.18], introduce the variable change $(x,z) \to (\varepsilon x, \varepsilon z)$, as well as the definition of ρ_ε and using Fubini to get:

$$
\begin{aligned}
&c_\alpha C |u|_{BV(\mathbb{R})} \int_{\mathbb{R}}\int_{0<|z|<\varepsilon} \frac{|\rho_\varepsilon'(x+sz)||z|^2}{|z|^{1+\alpha}}\,dz\,dx \\
\leq &C\varepsilon^{1-\alpha} \int_{\mathbb{R}} |\rho'|\,dx \int_{0<|z|<1} \frac{|z|^2}{|z|^{1+\alpha}} \\
\leq &C\varepsilon^{1-\alpha} |\rho|_{BV(\mathbb{R})} \int_{0<|z|<1} \frac{|z|^2}{|z|^{1+\alpha}} \leq C\varepsilon^{1-\alpha}
\end{aligned}
$$

where C is some appropriate constant that may change from term to term. Similar estimates (using the same change of variables and Fubini) are performed for the integral of the second term of equation (2.3.37) to obtain

$$
\begin{aligned}
&c_\alpha \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\varepsilon<|z|<1}\int_0^1 \frac{|\rho_\varepsilon(x-y+sz)-\rho_\varepsilon(x-y)|\,|z|}{|z|^{1+\alpha}}\,|u'(y)|\;ds\,dz\,dy\,dx \\
\le & C\varepsilon^{1-\alpha}|u|_{BV(\mathbb{R})}\|\rho\|_{L^1(\mathbb{R})}\int_{1<|z|<\frac{1}{\varepsilon}}\frac{|z|}{|z|^{1+\alpha}}\,dz \le C\varepsilon^{1-\alpha}
\end{aligned}
$$

The integral $\int_{1<|z|<\frac{1}{\varepsilon}}\frac{|z|}{|z|^{1+\alpha}}\,dz$ is finite, as it is proportional to (using polar coordinates) the integral $\int_{1<s<\frac{1}{\varepsilon}} s^{-\alpha}\,ds$ which is calculated directly (assuming $\varepsilon<1$).

$$
\int_{1<s<\frac{1}{\varepsilon}} s^{-\alpha}\,ds = \frac{1}{1-\alpha}s^{1-\alpha}\Big|_1^{\frac{1}{\varepsilon}} = \frac{1}{1-\alpha}\left(\varepsilon^{-1+\alpha}-1^{1-\alpha}\right)
$$

This result is finite, since $\alpha\in(1,2)$. Finally the last term of equation (2.3.37) is estimated,

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{|z|>1}&\frac{\rho_\varepsilon(x-y+z)-\rho_\varepsilon(x-y)}{|z|^{1+\alpha}}u(y)\,dz\,dy\,dx \\
&\le C\|\rho_\varepsilon\|_{L^1(\mathbb{R})}\|u\|_{L^1(\mathbb{R})}\int_{|z|>1}\frac{1}{|z|^{1+\alpha}}\,dz \le C
\end{aligned}
$$

Combine these three estimates to get

$$
\int_{\mathbb{R}} |(\Delta^{\alpha/2}\rho_\varepsilon * u)(x)|\,dx \le C(\varepsilon^{1-\alpha}+1) \tag{2.3.38}
$$

which we insert into equation (2.3.33) to get

$$
\begin{aligned}
\int_{\mathbb{R}} |u_{\varepsilon_t}|\;dx &\le \int_{\mathbb{R}}|\rho_\varepsilon|\;dx\int_{\mathbb{R}}|uu_x|\;dx + \int_{\mathbb{R}}\left|\Delta^{\alpha/2}u_\varepsilon\right|\;dx \\
&\le \|\rho_\varepsilon\|_{L^1(\mathbb{R})}\|u\|_{L^1(\mathbb{R})}\,|u|_{BV(\mathbb{R})} + C(\varepsilon^{1-\alpha}+1) \le C(\varepsilon^{1-\alpha}+1)
\end{aligned} \tag{2.3.39}
$$

The usefulness of the estimate (2.3.39) will soon become apparent. Consider the integral $\int_{\mathbb{R}}|u(x,t)-u(x,s)|\,dx$ and use the triangle inequality such that

$$
\begin{aligned}
\int_{\mathbb{R}} &|u(x,t)-u(x,s)|\,dx \\
&\le \underbrace{\int_{\mathbb{R}}|u(x,t)-u_\varepsilon(x,t)|\,dx}_{(i)} + \underbrace{\int_{\mathbb{R}}|u_\varepsilon(x,t)-u_\varepsilon(x,s)|\,dx}_{(ii)} \\
&+ \underbrace{\int_{\mathbb{R}}|u_\varepsilon(x,s)-u(x,s)|\,dx}_{(iii)}
\end{aligned}
$$

Then observe that integral (ii) can be rewritten,

$$\int_{\mathbb{R}} |u_\varepsilon(x,t) - u_\varepsilon(x,s)|\, dx \leq \int_s^t \int_{\mathbb{R}} |u_{\varepsilon t}|\, dx\, dt \leq C|t-s|(1+\varepsilon^{1-\alpha})$$

where inequality (2.3.39) is used. Moreover, the integrals (i) and (iii) may be rewritten. For instance,

$$\int_{\mathbb{R}} |u_\varepsilon(x,s) - u(x,s)|\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\varepsilon(x-y)|u_\varepsilon(y,s) - u(x,s)|\, dy\, dx \leq \varepsilon |u|_{BV(\mathbb{R})}$$

By combining these results we conclude that

$$\int_{\mathbb{R}} |u(x,t) - u(x,s)|\, dx \leq C\varepsilon + C|t-s|(1+\varepsilon^{1-\alpha}) =: h(\varepsilon) \tag{2.3.40}$$

valid for all $\varepsilon > 0$. To choose the optimal ε, we solve $h'(\varepsilon) = 0$ for ε. Then, $C + C(1-\alpha)|t-s|\varepsilon^{-\alpha} = 0$ which gives us $\varepsilon = C\left(|t-s|\right)^{1/\alpha}$. Inserting this back into inequality (2.3.40) we obtain

$$\begin{aligned} &\int_{\mathbb{R}} |u(x,t) - u(x,s)|\, dx \leq C|t-s|^{1/\alpha} + C|t-s|\left(1 + |t-s|^{\frac{1-\alpha}{\alpha}}\right) \\ &= C\left(|t-s|^{1/\alpha} + |t-s|\right) \end{aligned}$$

If $|t-s| <$ this estimate reduces to $C|t-s|^{\frac{1}{\alpha}}$ by an appropriate choice of C. If on the other hand $|t-s| > 1$, we choose $C \geq \frac{2\|u\|_{L^1(\mathbb{R})}}{T}$. To conclude, choose the largest of these two constants, such that the estimate stated in the this lemma holds for all $t, s > 0$. $\square$

Chapter 3

Numerical analysis

3.1 The numerical method

We introduce two different, but similar, numerical schemes designed to find the numerical solution of (1.0.1). The numerical solution, denoted as U_i^n, is found on an uniform grid. The grid is defined by discretizing in space, letting $x_i = i\Delta x$, with $i \in \mathbb{Z}$, and in time by letting $t_n = n\Delta t$, with $n = 0, ..., N$, such that $T = N\Delta t$. The explicit Euler (EE) method is used to discretize in time, a finite volume method (FVM) to discretize the flux function f, while the fractional Laplace operator is approximated in a manner similar to that of papers [8, 14].

3.1.1 The flux function

There exists many different finite volume methods designed to solve nonlinear conservation laws numerically. Some desirable properties of a FVM method are given in [8], and they are (the numerical flux function is denoted by F):

- *Consistent*, $F(u, u) = f(u)$.
- *Lipschitz continous*, $|F(a, b) - F(c, c)| \leq L_F \text{max}(|a - c|, |b - c|)$.
- *Non-decreasing w.r.t. the first argument*, $\frac{\partial}{\partial a} F(a, b) \geq 0$.
- *Non-increasing w.r.t. the second argument*, $\frac{\partial}{\partial b} F(a, b) \leq 0$.

The popular Engquist-Osher method is chosen to approximate the flux function. With $f(u) = \frac{1}{2}u^2$ we get the following approximation

$$F(a, b) = \frac{1}{2}\left(f(a) + f(b) - \int_a^b |s| ds\right) \tag{3.1.1}$$

which, by [18, p. 112], can be written as

$$F(a,b) = \frac{1}{2}\left(\max(a,0)^2 + \min(b,0)^2\right) \tag{3.1.2}$$

With this choice of F consistency is achieved, and we see that F is non-decreasing in the first argument, and non-increasing in the second argument:

$$\frac{\partial}{\partial a}F(a,b) = \begin{cases} a & \text{for } a \geq 0 \\ 0 & \text{for } a < 0 \end{cases} \quad \text{and} \quad \frac{\partial}{\partial b}F(a,b) = \begin{cases} 0 & \text{for } b \geq 0 \\ b & \text{for } b < 0 \end{cases}$$

The Lipschitz continuity of F is commented on shortly.

3.1.2 The fractional Laplace operator

The numerical treatment of the fractional Laplace operator is now considered. We are working with a piecewise constant numerical solution $\bar{u}(x,t)$, which (following [8]) for the explicit scheme (3.1.7) is defined as:

$$\bar{u}(x,t) = U_i^n \quad \text{for all} \quad (x,t) \in [x_i, x_i + \Delta x) \times [t_n, t_{n+1}) \tag{3.1.3}$$

while for the implicit-explicit scheme to be defined in (3.1.8) $\bar{u}(x,t)$ is defined as:

$$\bar{u}(x,t) = U_i^{n+1} \quad \text{for all} \quad (x,t) \in [x_i, x_i + \Delta x) \times (t_n, t_{n+1}] \tag{3.1.4}$$

where U is the numerical solution of either (3.1.7) or (3.1.8). In both cases $i \in \mathbb{Z}$ and $n = 0, ..., N$. Using the definition (3.1.3), this allows us to approximate the fractional Laplace operator in the following way:

$$c_\alpha \int_{|z|>0} \frac{\bar{u}(x+z,t) - \bar{u}(x,t)}{|z|^{1+\alpha}} dz \approx c_\alpha \sum_{j \neq 0} G_j \left(U_{i+j}^n - U_i^n\right)$$

where

$$G_j = \int_{x_i - \Delta x/2}^{x_i + \Delta x/2} |z|^{-(1+\alpha)} dz \tag{3.1.5}$$

Note that the spatial domain of the fractional operator is discretized in the same way as for the flux function, with an uniform grid, $x_i = i\Delta x$ and $i \in \mathbb{Z}$. The singularity is cut from the fractional operator, which is the contribution

$$R = c_\alpha \int_{-\Delta x/2}^{\Delta x/2} \frac{\bar{u}(z,t) - \bar{u}(0,t)}{|z|^{1+\alpha}} dz \tag{3.1.6}$$

If the solution $\bar{u}$ had been smooth (C^2), the contribution R would have gone to zero as $\Delta x \to 0$. However, smooth solutions are not the case here, so by leaving out R an error is introduced to the numerical solution. Whether the non-local operator (3.1.5) is evaluated at time $t = t_n$ or $t = t_{n+1}$ determines if the explicit or the implicit-explicit scheme is used.

3.1.3 Properties of the numerical method

As presented, the explicit scheme is:

$$\begin{cases} U_i^{n+1} & = U_i^n - \frac{\Delta t}{\Delta x}(F(U_i^n, U_{i+1}^n) - F(U_{i-1}^n, U_i^n)) + c_\alpha \Delta t \sum_{j\neq 0} G_j(U_{i+j}^n - U_i^n) \\ U_i^0 & = \frac{1}{\Delta x}\int_{x_i-\Delta x/2}^{x_i+\Delta x/2} u_0(x)dx \end{cases} \tag{3.1.7}$$

while the implicit-explicit scheme is given as:

$$\begin{cases} U_i^{n+1} & = U_i^n - \frac{\Delta t}{\Delta x}(F(U_i^n, U_{i+1}^n) - F(U_{i-1}^n, U_i^n)) + c_\alpha \Delta t \sum_{j\neq 0} G_j(U_{i+j}^{n+1} - U_i^{n+1}) \\ U_i^0 & = \frac{1}{\Delta x}\int_{x_i-\Delta x/2}^{x_i+\Delta x/2} u_0(x)dx \end{cases} \tag{3.1.8}$$

both schemes holding for $i \in \mathbb{Z}$ and all $n \in [0, .., N]$. The operator F is given by (3.1.2) and G is given by (3.1.5).

We assume that the initial data u_0 is bounded, i.e. $u_0(x) \in L^\infty(\mathbb{R})$, which implies that the numerical solutions of the methods (3.1.7) - (3.1.8) are bounded as well. Corollary 2 showed that when $-\|u_0\|_{L^\infty(\mathbb{R})} \leq u_0(x) \leq \|u_0\|_{L^\infty(\mathbb{R})}$ almost everywhere, then $u(x, t_n) \in [-\|u_0\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^\infty(\mathbb{R})}]$. This fact ensures the Lipschitz continuity of F, as since f is bounded and therefore Lipschitz, F is Lipschitz. In the following we let $L_F := \|u_0\|_{L^\infty(\mathbb{R})}$.

The remainder of the numerical analysis depends on these schemes being monotone and conservative. These two properties are proved in the following two lemmas.

Lemma 7. *The numerical method* (3.1.7) *is monotone when*

$$2L_F\frac{\Delta t}{\Delta x} + c_\alpha \Delta t \left(\frac{2}{\Delta x}\right)^\alpha \int_{|z|>1} |z|^{-(1+\alpha)} \leq 1 \tag{3.1.9}$$

while a sufficient requirement the numerical method (3.1.8) *to be monotone is*

$$\frac{2L_F\Delta t}{\Delta x} \leq 1 \tag{3.1.10}$$

Proof. The proof for the scheme (3.1.7) is presented first. Monotonicity is in [21, p. 245] defined as $\frac{\partial U_i^{n+1}}{\partial U_j^n} \geq 0$ for all $j \in \mathbb{Z}$. First consider the case when $j = i - 1$. Then,

$$\frac{\partial U_i^{n+1}}{\partial U_{i-1}^n} = \begin{cases} U_{i-1} + c_\alpha \Delta t G_{-1} & \text{when } U_{i-1} \geq 0 \\ c_\alpha \Delta t G_{-1} & \text{when } U_{i-1} < 0 \end{cases}$$

which is greater than or equal to zero, since $c_\alpha, \Delta x, \Delta t > 0$, and all $G_i \geq 0$, $(i \neq 0)$. The case for $j = i+1$ is very similar, with a similar positive result. For all other cases, except for $i = j$, we get G_j which is positive. For the case $i = j$ we get

$$\frac{\partial U_i^{n+1}}{\partial U_i^n} = 1 - \frac{\Delta t}{\Delta x}|U_i^n| - c_\alpha \Delta t \sum_{j \neq 0} G_j$$

for this to be non-negative, we must require that

$$\frac{\Delta t}{\Delta x}|U_i^n| + c_\alpha \Delta t \sum_{j \neq 0} G_j \leq 1$$

which is equivalent to

$$\frac{\Delta t}{\Delta x}|U_i^n| + c_\alpha \Delta t \int_{|z| > \Delta x/2} |z|^{-(1+\alpha)} \leq 1$$

The integral is rewritten using the substitution $z := \frac{2z}{\Delta x}$, and the Lipschitz constant L_F,

$$2L_F \frac{\Delta t}{\Delta x} + c_\alpha \Delta t \left(\frac{2}{\Delta x}\right)^\alpha \int_{|z|>1} |z|^{-(1+\alpha)} \leq 1$$

showing that the the lemma holds for the method (3.1.7). The proof for the implicit-explicit method is simpler, but similar, and is omitted. □

Lemma 8. *The numerical methods* (3.1.7) - (3.1.8) *are conservative, i.e.*

$$\sum_{k \in \mathbb{Z}} U_k^{n+1} = \sum_{k \in \mathbb{Z}} U_k^n$$

Proof. Again, the proofs for the two methods are very similar, and we choose to only present the proof for the explicit method (3.1.7). We begin the proof by summing over all indices $k \in \mathbb{Z}$ on both sides of (3.1.7), to get

$$\sum_{k \in \mathbb{Z}} U_k^{n+1} = \sum_{k \in \mathbb{Z}} U_k^n - \frac{\Delta t}{\Delta x} \underbrace{\sum_{k \in \mathbb{Z}} \left(F(U_k^n, U_{k+1}^n) - F(U_{k-1}^n, U_k^n)\right)}_{(i)}$$
$$+ c_\alpha \Delta t \underbrace{\sum_{k \in \mathbb{Z}} \sum_{j \neq 0} G_j \left(U_{k+j}^n - U_k^n\right)}_{(ii)}$$

If the terms (i) and (ii) are equal to zero, then the proof is completed. Consider (i) first and observe that this is a telescoping series. As we are summing over all indices, $k \in \mathbb{Z}$, the indices can be shifted in either term, to get that (i) is equal to zero. A similar argument can be used for the (ii) term, if we first ensure that the above series (ii) is absolutely convergent. For this to be the case, $\sum_k |U_k^n|$ and $\sum_{j\neq 0} G_j$ must be finite for all $n = 0, ..., N$. We estimate $\sum_k |U_k^{n+1}|$ using the numerical method (3.1.7) and some of the properties of the numerical flux F,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} |U_k^{n+1}| \\
&\leq \sum_{k\in\mathbb{Z}} \left(|U_k^n| + L_F \frac{\Delta t}{\Delta x} \left(|U_k^n - U_{k-1}^n| + |U_{k+1}^n - U_k^n| \right) + \Delta t \sum_{j\neq 0} G_j |U_{k+j}^n - U_k^n| \right) \\
&\leq \sum_{k\in\mathbb{Z}} \left(|U_k^n| + L_F \frac{\Delta t}{\Delta x} \left(|U_k^n| + 2|U_{k-1}^n| + |U_{k+1}^n| \right) + \Delta t \sum_{j\neq 0} G_j |U_{k+j}^n - U_k^n| \right) \\
&\leq \left(1 + 4L_F \frac{\Delta t}{\Delta x} + 2\Delta t \sum_{j\neq 0} G_j \right) \sum_{k\in\mathbb{Z}} |U_k^n| \qquad (3.1.11)
\end{aligned}$$

Both $\sum_k |U_k^n|$ and $\sum_{j\neq 0} G_j$ must be bounded for this estimate to be bounded. When deriving the CFL condition (3.1.3) we showed that

$$\sum_{j\neq 0} G_j = \int_{|z|>\Delta x/2} |z|^{-(1+\alpha)} = \left(\frac{2}{\Delta x} \right)^\alpha \int_{|z|>1} |z|^{-(1+\alpha)} < \infty \qquad (3.1.12)$$

since α is positive. Also, having assumed that $u_0(x) \in L^1(\mathbb{R})$, implies that $\bar{u}(x,0) \in L^1(\mathbb{R})$ which implies that $\sum_k |U_k^0| < \infty$. By iterating with (3.1.11) it is clear that $\sum_k |U_k^n|$ is bounded for all $n = 0, ..., N$.

With this in order, we can use Fubini to change the order of summation in the term (ii) to obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \sum_{j\neq 0} G_j \left(U_{k+j}^n - U_k^n \right) &= \sum_{j\neq 0} G_j \sum_{k\in\mathbb{Z}} \left(U_{k+j}^n - U_k^n \right) \\
&= \sum_{j\neq 0} G_j \left(\sum_{k\in\mathbb{Z}} U_{k+j}^n - \sum_{k\in\mathbb{Z}} U_k^n \right) = 0
\end{aligned}$$

Comment 4. *This proof holds for* (3.1.8) *also, as the sum for the G_j terms is zero regardless of what time they are evaluated at. Also, the estimate* (3.1.11) *can be adapted pretty easily to show that the method* (3.1.8) *is conservative.*

□

3.2 Convergence to a weak solution

Theorem 3. *Let $\bar{u}(x,t)$ denote the numerical solution computed with method* (3.1.7) *or method* (3.1.8), *and assume that $\bar{u}$ converges to a function u as $\Delta x, \Delta t \to 0$, in*

$L^1(\mathbb{R})$. *Then,* $u(x,t)$ *is a weak solution to equation* (1.0.1).

Proof. The proof is first performed for the explicit scheme (3.1.7), and then the slight differences a proof for the implicit-explicit scheme (3.1.8) would involve are commented on. Our strategy is to multiply the numerical scheme (3.1.7) with a piecewise constant approximation to the smooth function ϕ, and then sum over all indices $j \in \mathbb{Z}$ and all $n = 0, ..., N$. We want to show that this construction converges (letting $\Delta x, \Delta t \to 0$) to the weak solution of the fractional Burgers equation, as defined in [2, Definition 2.2]:

$$\int_0^T \int_{\mathbb{R}} \left(u\partial_t \phi + \frac{u^2}{2}\partial_x \phi + u\mathcal{L}(\phi) \right) + \int_{\mathbb{R}} u_0 \phi(0) = 0 \tag{3.2.1}$$

where we must require that $u_0 \in L^\infty(\mathbb{R})$ and $\phi \in C_c^\infty((0,T) \times \mathbb{R})$. Introduce ϕ_j^n as the piecewise constant approximation to the smooth test function $\phi \in C_c^\infty$ in each cell $[x_j, x_j + \Delta x) \times [t_n, t_{n+1})$, multiply it with the numerical scheme (3.1.7), and then sum over all $j \in \mathbb{Z}$ and $n = 0, ..., N$ to get

$$\begin{aligned} \underbrace{\sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} (U_j^{n+1} - U_j^n)\phi_j^n}_{(i)} + \frac{\Delta t}{\Delta x} \underbrace{\sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} \left(F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n) \right) \phi_j^n}_{(ii)} \\ - c_\alpha \Delta t \underbrace{\sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} \sum_{k\neq 0} G_k (U_{k+j}^n - U_j^n)\phi_j^n}_{(iii)} = 0 \end{aligned} \tag{3.2.2}$$

We rewrite the terms $(i) - (iii)$ using summation by parts. Start with (i) to get (using $\phi^n = 0$ for $n \geq N$)

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} (U_j^{n+1} - U_j^n)\phi_j^n &= -\sum_{j\in\mathbb{Z}} \phi_j^0 U_j^0 + \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} \phi_j^n \left(U_j^{n+1} - U_j^n \right) + \sum_{j\in\mathbb{Z}} \phi_j^0 U_j^1 \\ &= -\sum_{j\in\mathbb{Z}} \phi_j^0 U_j^0 - \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} \phi_j^n U_j^n + \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} \phi_j^{n-1} U_j^n \\ &= -\sum_{j\in\mathbb{Z}} \phi_j^0 U_j^0 - \sum_{j\in\mathbb{Z}} \sum_{n=1}^{N} \left(\phi_j^n - \phi_j^{n-1} \right) U_j^n \end{aligned}$$

A similar operation is performed with the (ii) term to obtain

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\left(F(U_j^n,U_{j+1}^n)-F(U_{j-1}^n,U_j^n)\right)\phi_j^n \\
&\qquad=\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}F(U_j^n,U_{j+1}^n)\phi_j^n-\underbrace{\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}F(U_{j-1}^n,U_j^n)\phi_j^n}_{\text{substitute } j\to j+1} \\
&\qquad=\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}(\phi_j^n-\phi_{j+1}^n)F(U_j^n,U_{j+1}^n)=-\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}(\phi_{j+1}^n-\phi_j^n)F(U_j^n,U_{j+1}^n)
\end{aligned}
$$

and finally consider the third term (iii),

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\sum_{k\neq 0}G_k(U_{k+j}^n-U_j^n)\phi_j^n&=\underbrace{\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\sum_{k\neq 0}G_kU_{k+j}^n\phi_j^n}_{\text{substitute } j\to j+k,k\to -k}-\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\sum_{k\neq 0}G_kU_j^n\phi_j^n \\
&=\underbrace{\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\sum_{k\neq 0}G_kU_j^n\phi_{j+k}^n}_{\text{using symmetry, } G_{-k}=G_k}-\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\sum_{k\neq 0}G_kU_j^n\phi_j^n \\
&=\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\sum_{k\neq 0}G_k(\phi_{j+k}^n-\phi_j^n)U_j^n
\end{aligned}
$$

Combine all three terms $(i)-(iii)$ to write (3.2.2) as

$$
\begin{aligned}
&\Delta t\Delta x\sum_{j\in\mathbb{Z}}\sum_{n=1}^{N}\frac{\phi_j^n-\phi_j^{n-1}}{\Delta t}U_j^n+\Delta t\Delta x\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\frac{\phi_{j+1}^n-\phi_j^n}{\Delta x}F(U_j^n,U_{j+1}^n) \\
&+c_\alpha\Delta t\Delta x\sum_{j\in\mathbb{Z}}\sum_{n=0}^{N}\sum_{k\neq 0}G_k(\phi_{j+k}^n-\phi_j^n)U_j^n+\Delta x\sum_{j\in\mathbb{Z}}\phi_j^0U_j^0=0 \qquad (3.2.3)
\end{aligned}
$$

The proof is completed when it is shown that

$$\lim_{\Delta x\to 0} \Delta x \sum_{j\in\mathbb{Z}} \phi_j^0 U_j^0 = \int_{-\infty}^{\infty} u_0 \phi(0) \tag{3.2.4}$$

$$\lim_{\Delta x,\Delta t\to 0} \Delta t\Delta x \sum_{j\in\mathbb{Z}} \sum_{n=1}^{N} \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} U_j^n = \int_0^T \int_{-\infty}^{\infty} u \partial_t \phi \tag{3.2.5}$$

$$\lim_{\Delta x,\Delta t\to 0} \Delta t\Delta x \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} \frac{\phi_{j+1}^n - \phi_j^n}{\Delta x} F(U_j^n, U_{j+1}^n) = \int_0^T \int_{-\infty}^{\infty} \frac{u^2}{2} \partial_x \phi \tag{3.2.6}$$

$$\lim_{\Delta x,\Delta t\to 0} c_\alpha \Delta t\Delta x \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} \sum_{k\neq 0} G_k(\phi_{j+k}^n - \phi_j^n) U_j^n = \int_0^T \int_{-\infty}^{\infty} u\mathcal{L}(\phi) \tag{3.2.7}$$

The Lax-Wendroff theorem, given in [18, Theorem 3.4], can be applied to prove that the local terms (3.2.4) - (3.2.6) hold. In particular, for the term (3.2.6) the proof of [18, Theorem 3.4] is followed. Using the consistency and Lipschitz continuity of F we show

$$\Delta t\Delta x \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} |F(U_j^n, U_{j+1}^n) - f(U_j^n)| = \Delta t\Delta x \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} |F(U_j^n, U_{j+1}^n) - F(U_j^n, U_j^n)|$$

$$\leq \Delta t\Delta x L_F \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} |U_{j+1}^n - U_j^n| \leq \Delta t\Delta x L_F \sum_{n=1}^{N} \text{T.V.}(\bar{u}(x,0)) = \Delta x T L_F \text{T.V.}(U^0)$$

where [18, Theorem 3.6] and the assumption that the total variation at time $t = 0$, $\sum_{j\in\mathbb{Z}} |U_{j+1}^0 - U_j^0|$, is finite is used to say that $\text{T.V.}(U^n) \leq \text{T.V.}(U^0)$. With this we have shown that

$$\lim_{\Delta x\to 0} \Delta t\Delta x \sum_{n=1}^{N} \sum_{j\in\mathbb{Z}} |F(U_j^n, U_{j+1}^n) - f(U_j^n)| = 0$$

Continuing, the results of appendix A and theorem A.8 of [18] are needed to conclude that when $\Delta x, \Delta t \to 0$ the Riemann sums converge to their respective integrals, as a result of the assumption that $\bar{u}$ converges to u in the L^1-norm, the bounded variation of $\bar{u}$ and the smoothness of the test function ϕ. From this the statements (3.2.4) - (3.2.6) hold. It remains to show that statement (3.2.7) holds.

The left hand side of (3.2.7) can be interpreted as the fractional Laplace operator acting on a piecewise constant interpolation $\bar{\phi}$ of the smooth function

ϕ. Then,

$$
\begin{aligned}
&c_\alpha \Delta t \Delta x \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} \sum_{k\neq 0} G_k(\phi_{j+k}^n - \phi_j^n) U_j^n \\
&= c_\alpha \int_0^T \int_{-\infty}^{\infty} \bar{u} \int_{|z|>\frac{\Delta x}{2}} \frac{\bar{\phi}(x+z) - \bar{\phi}(x)}{|z|^{1+\alpha}} \, dz \, dx \, dt
\end{aligned}
$$

As the fractional Laplace operator is only well defined for functions in C^2, it could be smart to rewrite

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \bar{u} \int_{|z|>\frac{\Delta x}{2}} \frac{\bar{\phi}(x+z) - \bar{\phi}(x)}{|z|^{1+\alpha}} \, dz \, dx \\
&= \int_{-\infty}^{\infty} \bar{u} \int_{|z|>\frac{\Delta x}{2}} \frac{\phi(x+z) - \phi(x)}{|z|^{1+\alpha}} \, dz \, dx \\
&\qquad + \int_{-\infty}^{\infty} \bar{u} \int_{|z|>\frac{\Delta x}{2}} \frac{(\bar{\phi}-\phi)(x+z) - (\bar{\phi}-\phi)(x)}{|z|^{1+\alpha}} \, dz \, dx \\
&= \int_{-\infty}^{\infty} \bar{u} \int_{|z|>0} \frac{\phi(x+z) - \phi(x)}{|z|^{1+\alpha}} \, dz \, dx \\
&\qquad + \int_{-\infty}^{\infty} \bar{u} \int_{|z|>\frac{\Delta x}{2}} \frac{(\bar{\phi}-\phi)(x+z) - (\bar{\phi}-\phi)(x)}{|z|^{1+\alpha}} \, dz \, dx \\
&\qquad - \int_{-\infty}^{\infty} \bar{u} \int_{|z|<\frac{\Delta x}{2}} \frac{\phi(x+z) - \phi(x)}{|z|^{1+\alpha}} \, dz \, dx
\end{aligned}
$$

But, our problem in the following is to show that the two last terms in this construction go to zero as $\Delta x \to 0$, such that equation (3.2.7) holds. Thankfully, we can resort to the paper by Droniou [14]. In section 3.4 of this paper Droniou establishes that equation (3.2.7) holds, for the same methods (3.1.7) - (3.1.8). The proof is very technical, and therefore beyond the scope of this work. □

Comment 5. *This proof holds for the implicit-explicit method* (3.1.8) *also, as the only difference compared to the proof above is the details regarding the term* (3.2.7). *The arguments regarding term* (3.2.7) *given in [14, Section 3.4] also hold for the term*

$$
\lim_{\Delta x, \Delta t \to 0} c_\alpha \Delta t \Delta x \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N} \sum_{k\neq 0} G_k(\phi_{j+k}^{n+1} - \phi_j^{n+1}) U_j^{n+1}
$$

Comment 6. *A similar, but more complicated proof can be made to show the convergence to an unique entropy solution for numerical solutions of* (3.1.7) *or* (3.1.8). *This proof is presented in [8].*

3.3 A priori estimates

Theorem 4. *For solutions U, V of either the explicit or the implicit-explicit schemes* (3.1.7) - (3.1.8), *with the corresponding definitions of $\bar{u}, \bar{v}$ stated in equations* (3.1.3) - (3.1.4), *the following a priori estimates hold for $t \in (0, T)$:*

$$(\bar{u}(x,t) - \bar{v}(x,t))^+ \leq (\bar{u}(x,0) - \bar{v}(x,0))^+ \tag{3.3.1}$$

$$\|\bar{u}(x,t) - \bar{v}(x,t)\|_{L^1(\mathbb{R})} \leq \|\bar{u}(x,0) - \bar{v}(x,0)\|_{L^1(\mathbb{R})} \tag{3.3.2}$$

$$\|\bar{u}(x,t)\|_{L^\infty(\mathbb{R})} \leq \|\bar{u}(x,0)\|_{L^\infty(\mathbb{R})} \tag{3.3.3}$$

$$|\bar{u}(x,t)|_{BV(\mathbb{R})} \leq |\bar{u}(x,0)|_{BV(\mathbb{R})} \tag{3.3.4}$$

Proof. Again, the proof for the case of the implicit scheme (3.1.8) is very similar to that of the explicit scheme (3.1.7), and is omitted. Further details regarding the proof of the a priori estimates for the implicit-explicit scheme can be found in [9, p. 12 - 13].

We begin by proving estimate (3.3.1) and (3.3.2) for the explicit scheme (3.1.7), and follow the details of the proof of theorem 3.6 in [18] to show that the first requirement of the Crandall-Tartar lemma given in [18] and appendix A is satisfied, which can be used to show that the other requirements hold.

Let $\Omega = \mathbb{R}$ and D be the set of all piecewise constant functions in $L^1(\Omega)$, specified on the grid for (3.1.7). Also, let $T(U^0) = U^n$. As the numerical method (3.1.7) is conservative we have that $\sum_k U_k^n = \sum_k U_k^0$, implying that $\int_\Omega T(U^0) = \int_\Omega U^n = \int_\Omega U^0$. Also, monotonicity (when the CFL condition (3.1.3) holds) implies that

$$U^0 \leq V^0 \implies U^n \leq V^n$$

which implies result (3.3.1), as following the proof of Crandall-Tartar, with the notation $U \vee V = \max\{U, V\}$ we get that $T(U^0 \vee V^0) - T(U^0) \geq 0 \implies T(U^0) - T(V^0) \leq T(U^0 \vee V^0) - T(V^0)$ and also $(T(U^0) - T(V^0))^+ \leq T(U^0 \vee V^0) - T(V^0)$. Integrate to get

$$\begin{aligned} \int_\Omega \left(T(U^0) - T(V^0)\right)^+ &\leq \int_\Omega \left(T(U^0 \vee V^0) - T(V^0)\right) \\ &= \int_\Omega \left(U^0 \vee V^0 - V^0\right) = \int_\Omega (U^0 - V^0)^+ \end{aligned}$$

The fact that (3.3.1) holds, implies (3.3.2), as then

$$\begin{aligned}\int_\Omega \left|T(U^0) - T(V^0)\right| &= \int_\Omega \left(T(U^0) - T(V^0)\right)^+ + \int_\Omega \left(T(V^0) - T(U^0)\right)^+ \\ &\leq \int_\Omega \left(U^0 - V^0\right)^+ + \int_\Omega \left(V^0 - U^0\right)^+ \\ &= \int_\Omega \left|U^0 - V^0\right|\end{aligned}$$

The proof of statement (3.3.3) for the explicit scheme is essentially the same as the proof of lemma 4.5 in [8], and it is a consequence of the monotonicity of the numerical methods. Let $s = \sup_k |U_k^n|$, and define $U^n \equiv -s$. Then,

$$U_k^{n+1} \geq -s - \frac{\Delta t}{\Delta x} \underbrace{(F(-s,-s) - F(-s,-s))}_{=0} + c_\alpha \Delta t \sum_{j \neq 0} G_j \underbrace{(-s+s)}_{=0} = -s$$

Similarly, letting $U^n \equiv s$ gives $U^{n+1} \leq s$, allowing us to conclude that statement (3.3.3) holds.

Statement (3.3.4) is proved in a similar fashion as corollary 2. Expand the left hand side of inequality (3.3.4) and use the fact that both $\bar{u}(x,t)$ and $\bar{u}(x+\Delta x, t)$ are numerical solutions of (3.1.7), and use statement (3.3.2) to get

$$\begin{aligned}|\bar{u}(x,t)|_{BV(\mathbb{R})} &= \frac{1}{\Delta x} \sum_k |\bar{u}(x_k + \Delta x, t) - \bar{u}(x_k, t)| \\ &\leq \frac{1}{\Delta x} \sum_k |\bar{u}(x_k + \Delta x, 0) - \bar{u}(x_k, 0)| = |\bar{u}(x,0)|_{BV(\mathbb{R})}\end{aligned}$$

□

3.4 Compactness

We refer to [9, Lemma 5.4] for the proof of regularity in time for solutions of both the explicit and the implicit-explicit numerical methods (3.1.7) - (3.1.8). The proof is similar to the proof of lemma 2.3.2, but more involved. It does require that the a priori estimates presented in theorem 4 hold. Then, for a numerical solution U of either scheme (3.1.7) or scheme (3.1.8), where $\bar{u}$ is the appropriate interpolation of U, the following holds for all $s, t \in (0,T)$:

$$\|\bar{u}(x,t) - \bar{u}(x,s)\|_{L^1(\mathbb{R})} \leq (|s-t| + \Delta t)^{\frac{1}{\alpha}} \tag{3.4.1}$$

when $\alpha \in (1,2)$. We then refer [18, Theorem 3.8] which is based on Kolmogorov's compactness theorem [18, Theorem A.5] as well as Helly's theorem [18, Corollary A.7] to conclude that there exists subsequences for both methods (3.1.7) - (3.1.8), converging to some function u. By theorem 3 we know that the function u is a weak solution of the fractal Burgers equation (1.0.1), and by [9, Theorem 5.5] we can conclude that the function u inherits all the a priori estimates of $\bar{u}$.

Comment 7. *The paper [9] does in fact show that the function u found by either of the numerical schemes* (3.1.7) - (3.1.8) *is the unique entropy solution of the fractal Burgers equation* (1.0.1).

3.5 Local truncation error

We would like to determine the local truncation error l^n of the numerical methods (3.1.7) - (3.1.8). The explicit method (3.1.7) can be written on the form $U^{n+1} = \mathcal{F}(U^n)$. The local truncation error is in [18, p. 69] formally defined as

$$l^n := \frac{1}{\Delta t}\left(u(x,t_{n+1}) - \mathcal{F}(u(x,t_n))\right) \tag{3.5.1}$$

which is the local error introduced by the numerical method at each step in time. The truncation error is in our case

$$l^n = \frac{1}{\Delta t}\Bigg(u(x,t+\Delta t) - u(x,t) + \frac{\Delta t}{\Delta x}\left(f(u(x,t)) - f(u(x-\Delta x,t))\right) \\ -c_\alpha \Delta t \sum_{j\neq 0} G_j\left(u(x+j\Delta x,t) - u(x,t)\right)\Bigg)$$

The Engquist-Osher FVM method used to discretize the Burgers' flux term is according to [16] of first order. We therefore focuse on the non-local operator, and use (3.1.12) to write

$$l^n = \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t) - c_\alpha\left(\left(\frac{2}{\Delta x}\right)^\alpha \int_{|z|>1} |z|^{-(1+\alpha)}\right)\sum_{j\neq 0}\left(u(x+j\Delta x,t) - u(x,t)\right)$$

Continuing to manipulate this term, and by assuming that u is sufficiently smooth Taylor expansions are used to write

$$\begin{aligned} &c_\alpha\left(\left(\frac{2}{\Delta x}\right)^\alpha \int_{|z|>1} |z|^{-(1+\alpha)}\right)\sum_{j\neq 0}\left(u(x+j\Delta x,t) - u(x,t)\right) \\ =&c_\alpha\left(\left(\frac{2}{\Delta x}\right)^\alpha \int_{|z|>1} |z|^{-(1+\alpha)}\right)\sum_{j\neq 0}\left(\underbrace{j\Delta x u_x(x,t)}_{=0\text{ by symmetry}} + \frac{j^2\Delta x^2}{2}u_{xx}(x,t) + \mathcal{O}(\Delta x^3)\right) \\ =&\mathcal{O}((\Delta x)^{-\alpha})\sum_{j\neq 0}\left(\frac{j^2\Delta x^2}{2}u_{xx}(x,t) + \mathcal{O}(\Delta x^3)\right) = \mathcal{O}((\Delta x)^{2-\alpha}) \end{aligned}$$

From this we conclude that our numerical method (3.1.7) is of order $2-\alpha$ in space and of order one in time. This indicates that the rate of convergence is very low as α gets close to 2.

Comment 8. *According to [20] we can in general not expect the rate of converge to be larger than* $1/2$ *when the solution includes discontinuities or shocks.*

Chapter 4

Implementation and results

We implement the numerical methods (3.1.7) and (3.1.8) in MATLAB, and illustrate and discuss some of the results obtained. We show how the non-local operator $\mathcal{L}$ can be implemented as a Toeplitz operator, making the numerical computations more efficient for both the explicit and the implicit-explicit methods. Some comments regarding the approximate rates of convergence, and the case when $\alpha \approx 2$ are also made.

4.1 Numerical implementation

To implement the numerical schemes (3.1.7) - (3.1.8), some assumptions and simplifications needs to be made. First of all, even though the non-local operator $\mathcal{L}$ acts on the whole domain $\mathbb{R}$, we need to limit our numerical grid to a bounded domain. We have therefore chosen to work on the bounded domain $\Omega_R := \{(x,t) : |x| \leq R, 0 \leq t \leq T\}$, where R is some appropriate positive constant, chosen such that the solution U is approximately constant when $|x| > R$. Droniou discussed the choice of R in [14].

The use of Ω_R transforms our non-local problem into a local problem. This will introduce some form of error to the numerical solution, which we do not account for in the following. This issue was also discussed in [10]. For simplicity, we impose Dirichlet boundary conditions and let the numerical solution $\bar{u}$ be equal to zero outside of the domain Ω, i.e. $\bar{u}|_{\Omega^C} = 0$.

It turns out that the discretization of the fractional Laplace operator leads to a symmetric Toeplitz operator, which can be expressed as a matrix specified by only one row or one column. In this setting it is convenient to write our numerical methods on vector form. We can express the numerical method

(3.1.7) as

$$\begin{cases} U^{n+1} & = U^n - \Delta t D^- F(U^n) + c_\alpha \Delta t B U^n \\ U_i^0 & = \frac{1}{\Delta x}\int_{x_i-\Delta x/2}^{x_i+\Delta x/2} u_0(x)dx \end{cases} \tag{4.1.1}$$

while the implicit-explicit scheme (3.1.8) can be given as:

$$\begin{cases} U^{n+1} & = U^n - \Delta t D^- F(U^n) + c_\alpha \Delta t B U^{n+1} \\ U_i^0 & = \frac{1}{\Delta x}\int_{x_i-\Delta x/2}^{x_i+\Delta x/2} u_0(x)dx \end{cases} \tag{4.1.2}$$

Here, D^- is the backwards difference operator, $D^- U_i^n = \frac{1}{\Delta x}(U_i^n - U_{i-1}^n)$. The Toeplitz matrix B can be specified just by defining the row vector R, which if for instance $i = -2, -1, 0, 1, 2$ ($N = 5$) can be given as

$$R := \begin{bmatrix} -\sum_i G_i & G_1 & G_2 & 0 & 0 \end{bmatrix}$$

where G_i is defined by (3.1.5). The symmetry of the matrix is a consequence of $G_{-i} = G_i$ for all $i \neq 0$. The $N \times N$ matrix B can then be created in `MATLAB` by the command $B =$ `toeplitz(R)`, which for this example leads to

$$B = \begin{bmatrix} -\sum_i G_i & G_1 & G_2 & 0 & 0 \\ G_1 & -\sum_i G_i & G_1 & G_2 & 0 \\ G_2 & G_1 & -\sum_i G_i & G_1 & G_2 \\ 0 & G_2 & G_1 & -\sum_i G_i & G_1 \\ 0 & 0 & G_2 & G_1 & -\sum_i G_i \end{bmatrix}$$

The explicit method (4.1.1) requires the matrix multiplication BU^n to be performed. This multiplication does normally require $\mathcal{O}(n^2)$ time for a vector U of length $\mathcal{O}(n)$. But, we can exploit the Toeplitz structure of the matrix B and use the Fast Fourier Transform (FFT) to perform the matrix multiplication, using only $\mathcal{O}(n\ \text{log}n)$ time. We refer to [17, p. 193] for further details regarding this, but will sketch how this procedure can be performed. The first step is to transform the $N \times N$ Toeplitz matrix B into a $2N \times 2N$ circulant Toeplitz matrix C_{2N} (see [17] for a definition and the procedure). We can use FFT to diagonalize a circulant Toeplitz matrix C to get

$$C = F^* \text{diag}(Fa) F$$

where $a = [a_0, a_1, ..., a_N]$ are the elements of the first column of C, F is the Fourier matrix $F(j,k) = \frac{1}{n} e^{-(j-1)(k-1)2\pi i/n}$ and $*$ denotes the complex conjugate. The matrix multiplication $y = Cx$ can then be performed in $\mathcal{O}(n\ \text{log}n)$ time by performing the following procedure:

i) $f = Fx$

ii) $g = Fa$

iii) $z^T = [f_1 g_1, f_2 g_2, ..., f_N g_N]$

iv) $y = F^* z$

The Toeplitz matrix B also needs less memory than an unstructured matrix for storage, as B only has $\mathcal{O}(n)$ variables, instead of $\mathcal{O}(n^2)$ variables.

The implicit-explicit method (4.1.2) should also exploit the fact that B is a Toeplitz matrix. The book [6] is a good reference on different iterative solvers for Toeplitz matrices, most of which are based on Preconditioned Conjugate Gradient (PCG) methods. We chose to use the code given in [6, Appendix A], and adapt the code to solve our system

$$(I - c_\alpha \Delta t B)\, U^{n+1} = U^n - \Delta t D^- F(U^n)$$

where we note that the matrix $(I - c_\alpha \Delta t B)$ also is a Toeplitz matrix.

The choice of the best preconditioner (see [6] for further details) for the PCG method is a difficult one. Based on numerical tests of 11 different preconditioners, we chose to use the de la Vallee Poussin kernel (see [6, p. 40]) as a preconditioner, as this provided the fastest computations.

The positive constant c_α which was given in chapter 1 as

$$c_\alpha = \frac{\alpha \Gamma(\frac{1+\alpha}{2})}{2\pi^{\frac{1}{2}+\alpha}\Gamma(1-\frac{\alpha}{2})}$$

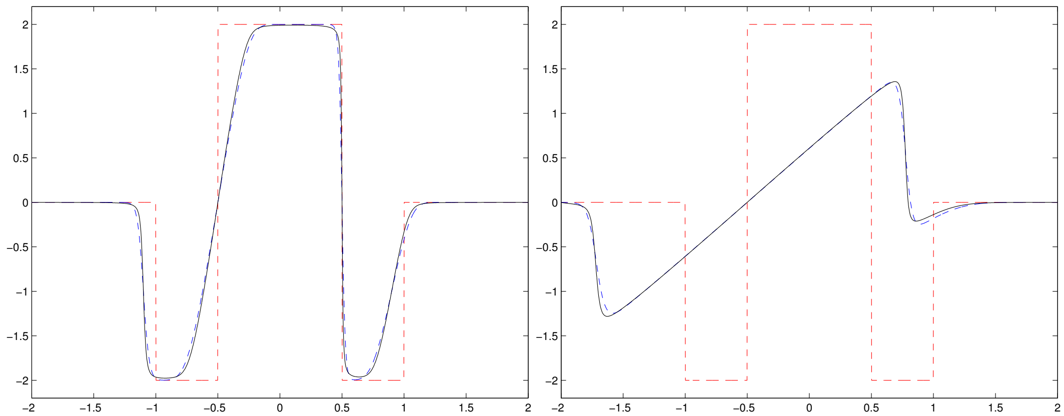
for $\alpha \in (0, 2)$ must also be calculated. As suggested in [7], we evaluate this expression numerically using the built in MATLAB gamma function Γ.

4.2 Numerical results

Some numerical solutions of (1.0.1) obtained by the use of scheme (4.1.1) are presented in figures 4.1 - 4.4. All of the results seem to exhibit the qualitative properties that we expected to see. Each solution is convected with a speed $f'(u) = u$, and the non-local operator smoothens the solution in time, when $\alpha \in (1, 2)$.

Figure 4.1 illustrates how the solution for the initial data u_0 from time $T = 0.1$ to time $T = 0.8$. The solution is smoothened as time increases, starting from the piecewise constant initial data u_0. The solution of (1.0.1) for $\alpha = 1.8$ is also very close to the solution of (4.2.1), which is discussed in section 4.2.2.

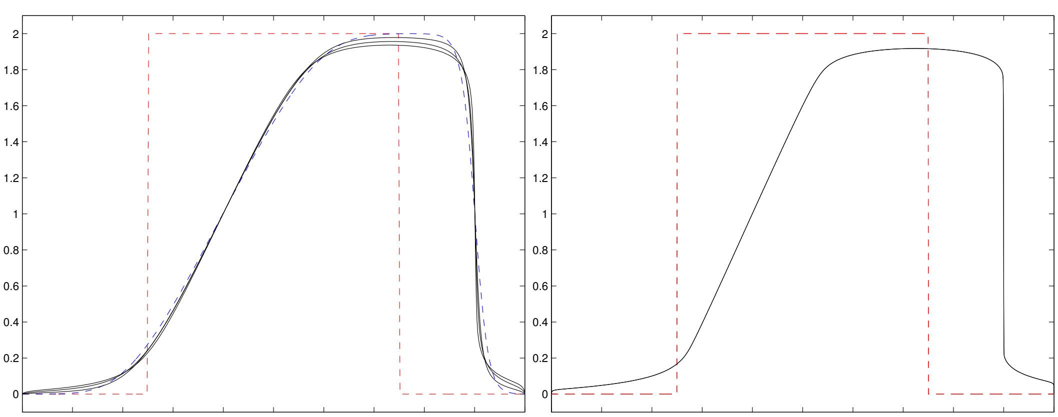
Figure 4.2 illustrates solutions for various values of α. Figure 4.2a compares the solutions for three different values of α with the solution of equation



(a) $\alpha = 1.8, \quad T = 0.1$ (b) $\alpha = 1.8, \quad T = 0.8$

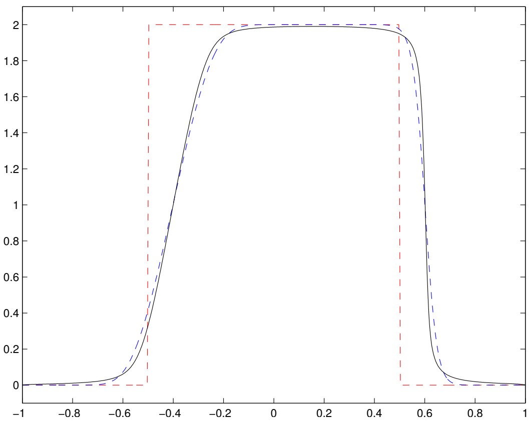
Figure 4.1: *Numerical solutions of* (1.0.1) *shown in black, with initial data in red,* $u_0(x) = -sign(x-1) + sign(x-1/2) - sign(x+1/2) + sign(x+1)$, $\Delta x = 1/320$. *The numerical solution of* (4.2.1) *is plotted in blue.*

(4.2.1). One might be able to spot how the numerical solutions converge towards the solution of (4.2.1), as α increases towards 2. Figure 4.2b illustrates that when $\alpha \in (0, 1)$ we can get discontinuities or shocks in the solution, and in this case we get a shock located at $x \approx 0.8$.

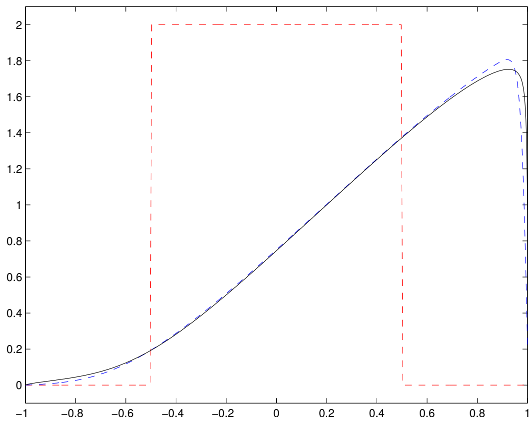


(a) $\alpha = 1.1, 1.4$ & $1.7, \quad T = 0.3$ (b) $\alpha = 0.5, \quad T = 0.3$

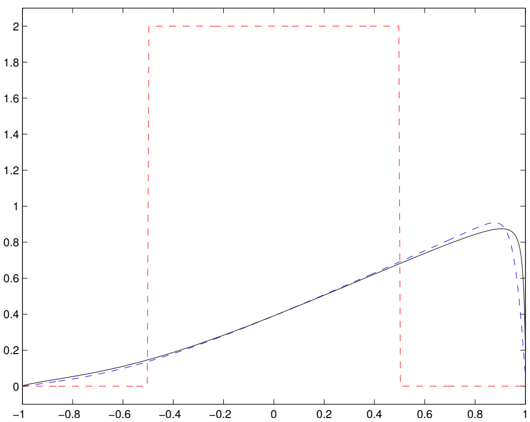
Figure 4.2: *Numerical solutions of* (1.0.1) *for various values of* α*, with initial data in red,* $u_0(x) = -sign(x - 1/2) + sign(x + 1/2)$, $\Delta x = 1/320$. *The numerical solution of* (4.2.1) *is plotted in blue.*



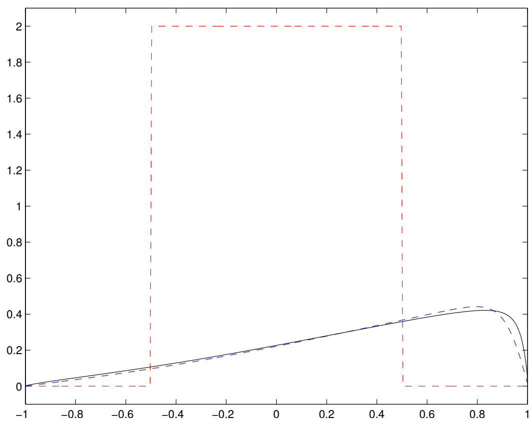
(a) $T = 0.1$



(b) $T = 0.7$



(c) $T = 1.5$



(d) $T = 3.0$

Figure 4.3: *Numerical solutions of* (1.0.1) *shown in black, with initial data in red,* $u_0(x) = -sign(x - 1/2) + sign(x + 1/2)$, *fixed* $\alpha = 1.5$, $\Delta x = 1/320$. *The numerical solution of* (4.2.1) *is plotted in blue.*

4.2.1 Rates of convergence

It is of interest to check whether the theory developed regarding the local truncation error (cf. section 3.5) and the use of the CFL-condition (3.1.3) holds true for the numerical simulations. We follow [10] (choosing $p = 1$, implying that we use the L^1- norm) and measure the error as

$$E_{\Delta x} := \|\bar{u}_{\Delta x}(x,T) - \tilde{u}_{\Delta x}(x,T)\|_{L^1(\mathbb{R})}$$

where $\tilde{u}(x,T)$ is the numerical solution at $t = T$, calculated using $\Delta x = 1/2560$. The relative error is defined as

$$R_{\Delta x} := \frac{E_{\Delta x}}{\|\tilde{u}_{\Delta x}(x,T)\|_{L^1(\mathbb{R})}}$$

and the approximate rate of convergence is

$$r_{\Delta x} := \frac{\log(E_{\Delta x}) - \log(E_{\Delta x/2})}{\log(2)}$$

According to the estimate of the local truncation error, presented in section 3.5, we would expect the rate of convergence to be approximately $2-\alpha$ when $\alpha \in (1,2)$. We have calculated the approximate rates of convergence for the result shown in figure 4.3 for $T = 0.3$, and these rates are shown in table 4.1. The values of $r_{\Delta x}$ suggest that the rate is close to 1, which does not agree with our theoretical results. We are not aware of why this interesting result occurs, further work and simulations are therefore needed to investigate this matter.

Table 4.1: *Details regarding the error $E_{\Delta x}$, the relative error $R_{\Delta x}$ and the approximate rate of convergence $r_{\Delta x}$ for the numerical solution illustrated in figure 4.3 for $T = 0.1$.*

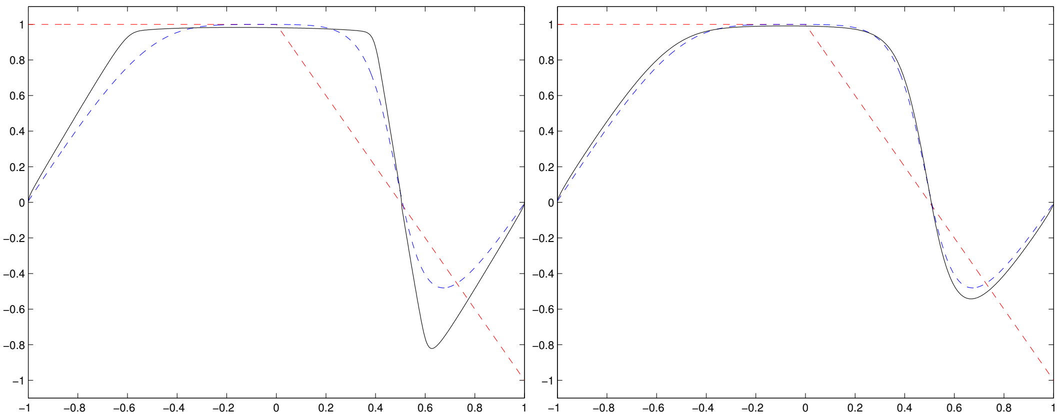
Δx	$E_{\Delta x}$	$R_{\Delta x}$	$r_{\Delta x}$
$1/10$	0.2982	0.1492	1.1636
$1/20$	0.1331	0.0666	1.0780
$1/40$	0.0631	0.0316	0.7928
$1/80$	0.0364	0.0182	1.2580
$1/160$	0.0152	0.0076	1.0519
$1/320$	0.0073	0.0037	1.5312
$1/640$	0.0025	0.0013	-

4.2.2 The case $\alpha \approx 2$

According to several papers, e.g. [14], we can expect numerical solutions of (1.0.1) to be similar to the numerical solution of the problem

$$\begin{cases} u_t(x,t) + \nabla \cdot f(u(x,t)) = \frac{1}{4\pi^2}\Delta u(x,t) & (x,t) \in Q_T \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{4.2.1}$$

when $\alpha \approx 2$. The factor $1/4\pi^2$ is included to compensate for the factor c_α when $\alpha \to 2$, as suggested in [14]. Equation (4.2.1) is solved by a central finite difference scheme, using the same boundary conditions as for the non-local problem (1.0.1). Solutions of the two different equations are plotted in figures 4.1 - 4.4. The convergence towards solutions of (4.2.1) when $\alpha \to 2$ in (1.0.1) is illustrated in particular in figure 4.4, as well as in figure 4.2a.



(a) $\alpha = 0.1, \quad T = 0.4$ (b) $\alpha = 1.7, \quad T = 0.4$

Figure 4.4: *Numerical solutions of* (1.0.1) *shown in black, with initial data in red,* $u_0(x) = |x-1| - |x|$, *fixed* $T = 0.4$, $\Delta x = 1/320$. *The numerical solution of* (4.2.1) *is plotted in blue.*

It is worth mentioning that when α is close to 2 the CFL condition (3.1.3) puts a severe condition on the choice of Δt for the explicit scheme (4.1.1). It is therefore much more efficient to use the implicit-explicit scheme (4.1.2), which has a less restrictive CFL condition.

Chapter 5

Discussion, conclusions and further work

5.1 Discussion and conclusion

In this study theoretical and numerical aspects of the 1-D fractal Burgers equation (1.0.1) are studied. We establish a priori estimates as well as the regularity in time, for smooth, classical solutions. Traditionally, solutions of the fractal Burgers equation have been studied in the framework of entropy solutions. By assuming sufficient regularity the concept of entropy solutions is avoided in this study. This allows us to prove several a priori estimates in a new and interesting manner. Many mathematical techniques developed for the study of entropy solutions are here used and adapted to work for the study of classical solutions.

The main theoretical result of our current study, stated in theorem 1, is a L^1-type contraction estimate that shows how the positive part of the difference between two solutions is contractive in time. The proof of this theorem is rather technical, and several assumptions on the regularity of the classical solutions must be made in order to complete the proof. However, the technical challenges help to showcase some of the more intriguing properties of the non-local operator. The integral representation of the fractional derivative given in (1.0.3) allows us to establish two Kato type of inequalities, as well as proving several other useful properties of the non-local operator. These findings are essential for the completion of the theoretical part of this work.

One might wonder how reasonable it is to assume the existence of classical solutions of the fractal Burgers equations. When $\alpha \in (1, 2)$ Droniou et. al. [15] established that the fractal Burgers equation has smooth solutions, as long as the initial data is bounded. Smooth solutions are however not ensured when $\alpha \in (0,1)$, which is our main reason for assuming $\alpha \in (1,2)$. Most of the theoretical and numerical results presented in this work are however valid for $\alpha \in (0,2)$, even though the assumption of smooth solutions makes most sense when $\alpha \in (1,2)$.

Moreover, an effort is also made to study numerical aspects of the fractal Burgers equation. We propose two different numerical schemes, one explicit and one implicit-explicit scheme. The schemes are constructed by using the Engquist-Osher finite volume method to approximate the Burgers flux term, while the non-local operator is approximated similarly to the approximations given in the papers [8, 9, 14]. The numerical methods are under suitable CFL conditions proved to be monotone, conservative and consistent. The CFL condition of the explicit method is more restrictive than that of the implicit-explicit method, and especially demanding when α is close to 2.

Several a priori estimates for the numerical solutions are proved by using the Crandall-Tartar lemma. It is also established that there exists a convergent subsequence produced by both numerical methods, of which the limit is a weak solution of (1.0.1). The numerical analysis is completed by finding the local truncation error of the numerical methods.

The numerical results illustrate some of the qualitative properties of the fractal Burgers equation. The regularizing effect of the non-local operator resemble that of the Laplace operator when α approaches the value 2, and the solutions of the fractal Burgers equation (1.0.1) resemble the solutions of the classical viscous Burgers equation (1.1.1).

The numerical methods are implemented in MATLAB, where an effort is made to implement the methods efficiently. We exploit the Toeplitz structure of the non-local operator to speed up matrix computations. A fast Fourier transform algorithm is used to perform the matrix multiplications needed for the explicit method, while a preconditioned conjugate gradient method is used to iteratively solve the nonlinear system of the implicit-explicit method. Interestingly, we do not obtain the rates of convergence that were expected, as the numerical methods converge much faster than what the numerical analysis predicts. Why this is the case is not known, but our approach of transforming the non-local problem into a local problem by working on a bounded domain could perhaps be a contributing factor.

5.2 Further work

The mathematical theory of non-local partial differential equations and fractional conservation laws in particular has been the subject of a great deal of research in recent years. The current study has provided further insight into the properties of the fractal Burgers equation, but several interesting questions are still unanswered. A natural continuation of our work with classical solutions would be to study how other properties of the fractal Burgers equation can be shown, without using the concept of entropy solutions. It would be useful for the completeness of our work to verify that the solutions for $\alpha \in (1, 2)$ are smooth, as shown in [15], while the solutions for $\alpha \in (0, 1)$ can develop discontinuities, as shown in [3].

The numerical results illustrate that when α is close to 2, solutions of the fractal Burgers and the viscous Burgers equation are very similar. It would be interesting to confirm this more rigorously, showing the convergence of solutions when $\alpha \to 2$.

Regarding the numerical aspects of solving fractional conservation laws, several questions are left unanswered. It would be interesting to show how numerical solutions of the fractal Burgers converge towards weak solutions of the viscous Burgers equation when α approaches the value 2. One could also conduct similar experiments to study the weak solutions when $\alpha \to 1$ and $\alpha \to 0$. The issue concerning the rates of convergence should also be investigated, as current numerical results do not agree with the predictions of the numerical analysis.

It would also be interesting to implement an operator splitting method to solve our equation. Using this method one could exploit that the Burgers flux term and the non-local fractional term demand different CFL requirements when treated separately. Perhaps one could first solve the inviscid Burgers equation on a coarse grid and then the fractional heat equation on a fine grid. By combining the two numerical solutions afterwards, one could obtain the solution of the fractional Burgers equation.

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Appendix A

Useful theorems

Monotone convergence theorem (Corollary 1.0 - [24])

Suppose $\{f_n\}$ is a sequence of non-negative measurable functions with

$$\begin{aligned} f_n(x) &\leq f_{n+1}(x) && \text{for almost every } x, \text{all } n \geq 1 \\ \lim_{n\to\infty} f_n(x) &= f(x) && \text{for almost every } x \end{aligned}$$

Then

$$\lim_{n\to\infty} \int f_n = \int f$$

Dominated convergence theorem (Theorem 1.13 - [24])

Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ for almost every x, as n tends to infinity. If $|f_n(x)| \leq g(x)$, where g is integrable, then

$$\int |f_n - f| \to 0 \quad \text{as } n \to \infty$$

and consequently

$$\int f_n \to \int f \quad \text{as } n \to \infty$$

Bounded convergence theorem (Proposition - [23])

Suppose $\{f_n\}$ is a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and almost all x. If $f_n(x) \to f(x)$ for almost every x in E, as n tends to infinity, then

$$\lim_{n\to\infty} \int_E f_n \to \int_E f$$

Result A (Lemma 2.9 - [18])

Let $\omega(\sigma)$ be a C^∞ function such that

$$0 \leq \omega(\sigma) \leq 1, \quad \operatorname{supp} \omega \subseteq [-1,1], \quad \omega(-\sigma) = \omega(\sigma), \quad \int_{-1}^{1} \omega(\sigma)\, d\sigma = 1.$$

Then, let $h \in L^\infty(\mathbb{R})$ with compact support. Assume that for almost all $x_0 \in \mathbb{R}$ the function $h(x,y)$ is continuous at (x_0, y_0). Then

$$\lim_{\varepsilon \to 0} \int \int h(x,y)\omega_\varepsilon(x-y)\, dy\, dx = \int h(x,x)\, dx$$

Proof. Observe first that

$$\begin{aligned} &\int h(x,y)\omega_\varepsilon(x-y)\, dy - h(x,x) \\ &\qquad \int (h(x,y) - h(x,x))\, \omega_\varepsilon(x-y)\, dy \\ &\qquad \int_{|z|\leq 1} (h(x, x+\varepsilon z) - h(x,x))\, \omega(z)\, dz \to 0 \text{ as } \varepsilon \to 0 \end{aligned}$$

for almost all x, using the continuity of h. Furthermore,

$$\left| \int h(x,y)\omega_\varepsilon(x-y)\, dy \right| \leq \|h\|_{L^\infty}$$

and hence we can use Lebesgue's bounded convergence theorem to obtain the result. $\square$

Crandall-Tartar (Lemma 2.12 - [18])

Let D be a subset of $L^1(\Omega)$, where Ω is some measure space. Assume that if ϕ and ψ are in D, then also max(ϕ, ψ) is in D. Assume furthermore that there is a map $T : D \to L^1(\Omega)$ such that

$$\int_\Omega T(\phi) = \int_\Omega \phi, \quad \phi \in D$$

Then the following statements are equivalent for all $\phi, \psi \in D$

i) If $\phi \leq \psi$, then $T(\phi) \leq T(\psi)$

ii) $\int_\Omega (T(\phi) - T(\psi))^+ \leq \int_\Omega (\phi - \psi)^+$

iii) $\int_\Omega |T(\phi) - T(\psi)| \leq \int_\Omega |\phi - \psi|$

Fubini's theorem (Theorem 3.1 - [24])

(Adapted slightly for our setting).
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function. If $f \in L^1(\mathbb{R}^2)$, then the integrals

$$\int_{\mathbb{R}} f(a,y)\,dy \quad \text{and} \quad \int_{\mathbb{R}} f(x,a)\,dx$$

are defined for almost all $a \in \mathbb{R}$, the functions

$$x \to \int_{\mathbb{R}} f(a,y)\,dy \quad \text{and} \quad y \to \int_{\mathbb{R}} f(x,a)\,dx$$

are measurable, and

$$\int_{\mathbb{R}^2} f = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\,dy\,dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\,dx\,dy$$

Appendix B

One result

This section contains a proof needed in section 2.3. The proof is based on the proof of result A given in appendix A. We want to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta^{\delta''}(u(x),v(y))(f(u(x))-f(v(y)))(u_x(x)+v_y(y))\phi_\varepsilon(x-y)\,dy\,dx \\
&= \int_{\mathbb{R}} \eta^{\delta''}(u(x),v(x))(f(u(x))-f(v(x)))(u_x(x)+v_x(x))\,dx \qquad \text{(B.0.1)}
\end{aligned}$$

First observe the following properties of $h(x,y) := \eta^{\delta''}(u(x),v(y))(f(u(x)) - f(v(y)))(u_x(x)+v_y(y))$. From assumptions (I) - (III) we have that $h(x,y) \in L^\infty(\mathbb{R})$. We also see that $h(x,y)$ is continuous, as $u,v \in C^2(\mathbb{R})$ implies that both f and u_x, v_y are continuous. Finally, we can use lemma 5 to see that $\lim_{|x|,|y|\to 0} h(x,y) = 0$, as $\lim_{|x|\to 0} u(x) = 0$ and $\lim_{|y|\to 0} v(y) = 0$, which in other words means that h has compact support. Then, observe that

$$\begin{aligned}
&\int_{\mathbb{R}} \eta^{\delta''}(u(x),v(y))(f(u(x))-f(v(y)))u_x(x)\phi_\varepsilon(x-y)\,dy \\
&\quad - \eta^{\delta''}(u(x),v(x))(f(u(x))-f(v(x)))u_x(x) \\
&= \int_{\mathbb{R}} \eta^{\delta''}(u(x),v(y))(f(u(x))-f(v(y)))u_x(x) \\
&\quad - \eta^{\delta''}(u(x),v(x))(f(u(x))-f(v(x)))u(x)_x\phi_\varepsilon(x-y)\,dy \\
&= \int_{|z|\le 1} \eta^{\delta''}(u(x),v(x+\varepsilon z))(f(u(x))-f(v(x+\varepsilon z)))u_x(x) \\
&\quad - \eta^{\delta''}(u(x),v(x))(f(u(x))-f(v(x)))u_x(x)\phi(z)\,dy
\end{aligned}$$

which follows from the compact support of ϕ in $[-1,1]$. Then, use the continuity of $h(x,y)$ to get

$$\begin{aligned}
&\int_{|z|\le 1} \eta^{\delta''}(u(x),v(x+\varepsilon z))(f(u(x))-f(v(x+\varepsilon z)))u_x(x) \\
&\qquad -\eta^{\delta''}(u(x),v(x))(f(u(x))-f(v(x)))u_x(x)\phi(z)\,dy \to 0 \quad \text{as} \quad \varepsilon\to 0
\end{aligned}$$

And observe that the following integral is bounded:

$$\int_{\mathbb{R}} \eta^{\delta''}(u(x), v(y))(f(u(x)) - f(v(y)))u_x(x)\phi_\varepsilon(x-y)\,dy$$

as $h(x, y)$ is bounded and has compact support. The same argument follows for the similar v_y term. We can conclude in the same manner as in result A given in appendix A by referring to the Lebesgue bounded convergence theorem, found in appendix A, to see that equation (B.0.1) holds.