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Massey products and Linking

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Abstract

This master's thesis is focussed around investigating Massey products as tools for studying properties of links, in particular the Brunnian property.
In the literature, there are only a few examples of the Massey product being used to study linking, none of which has any emphasis on links with the Brunnian property, except for computations for the Borromean rings.
The result of the work is a number of thorough computations of Massey products in link complements, with the negative conclusion that the Massey product does *not* detect the Brunnian property.

Sammendrag

Denne mastergradsavhandlingen har Massey-produkter som sitt hovedfokus, med det mål å studere egenskaper hos lenker, spesielt den såkalte Brunniske egenskapen.
I litteraturen finnes der bare noen få eksempler hvor Massey-produktet blir brukt til å studere lenker, ingen av disse har imidlertid vekt på den Brunniske egenskapen, bortsett fra noen beregninger gjort for de Borromeiske ringer.
Resultatet av arbeidet er et antall grundige utregninger av Massey-produktet i lenkekomplementer, dog med den negative konklusjon at Massey-produkter *ikke* er i stand til å oppdage den Brunniske egenskapen.

Preface

This thesis is the result of work done under supervision of Prof. Nils Baas and Dr. Andrew Stacey for the course “TMA4900: Master thesis in mathematics” in the spring term of 2012 at NTNU - the Norwegian University of Science and Technology. The workload is considered to be equivalent to one semester of full-term studies.

The initial goal for the work leading up to this thesis was gaining a familiarity with Massey products and how they can detect non-trivial linking properties of first order links and then find a way to produce analogous kinds of products detecting higher order linking, in the sense described in [Baa10].

Surprisingly, it turned out that the triple Massey product does *not* detect the non-trivial linking in the Brunnian 3-link, which is one of the prototypical examples of first order links, so instead of extending a tool that does not seem to do what I had expected, I decided turn the focus to deciding which kind of linking it does detect. This resulted in a quite large number of computations, culminating in the conclusions that higher Massey products detect the linking inherent in the Brunnian *chains* but not in the Brunnian *rings*.

This negative result opens up interesting alleys of investigation, as it seems that new ideas might be necessary to tackle the problem. It also leaves open the question of which specific property or properties of a link the Massey products do detect, as it is not the *Brunnian property*.

All figures in thesis are made using Ti*k*Z, a high level macro language using the lower-level language of `PGF`. I have made extensive use of the package `brunnian` written by my adviser Andrew Stacey, who has also been very helpful in explaining how to use the package for creating Figures 3.5 through to 3.9, as well as providing the code for making Figure 3.2 and the image on the cover.

During the entire endeavour that writing a master thesis is, I have had the pleasure of working alongside my costudent and good friend Roar Bakken Stovner. Being able to spar about mathematical ideas as well as exposition and language with him has improved not only this thesis but also my general appreciation and proficiency in mathematics.

I would like to thank my advisers Nils and Andrew for an interesting thesis problem, stimulating discussions and general help and guidance concerning both small and bigger problems I have encountered during the work. They have also provided helpful and sound career advice.

I have very much enjoyed working on this project and I look forward to continuing this study as a Ph.D. student under their supervision.

Truls Bakkejord Ræder, Trondheim, June 18, 2012

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Table of notations

We will denote the natural numbers, integers, reals and spheres, by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{S}^k$ respectively.

Other notation is summarised below:

$\mathscr{B}$ The Borromean rings

$\mathcal{S}_\bullet(\bullet)$ Singular chain complex

$\mathcal{H}_*(\bullet)$ Singular homology

$\widetilde{\mathcal{H}}_*(\bullet)$ Reduced singular homology

$\mathcal{S}^\bullet(\bullet)$ Singular cochain complex

$\mathcal{H}^*(\bullet)$ Singular cohomology

$\widetilde{\mathcal{H}}^*(\bullet)$ Reduced singular cohomology

$\bullet^*$ Map induced by a contravariant functor

$\bullet_*$ Map induced by a covariant functor

$\smile$ Cup product

$\frown$ Cap product

$\mathcal{J}^*(\bullet, \bullet)$ Massey indeterminacy ideal

$\mathcal{T}\bullet$ Tangent bundle

$\mathcal{N}\bullet$ Normal bundle

Φ Thom class

$\eta_\bullet$ Poincaré dual

$\pitchfork$ Transverse intersection

$\wedge$ Wedge product

$\Omega^\bullet(\bullet)$ de Rham cochain complex

$\mathcal{H}^*_{\mathrm{dR}}(\bullet)$ de Rham cohomology

$\Omega^\bullet_{\mathrm{c}}(\bullet)$ de Rham cochain complex with compact support

$\mathcal{H}^*_{\mathrm{c}}(\bullet)$ de Rham cohomology with compact support

$\Omega^\bullet_{\mathrm{cv}}(\bullet)$ vertically compact de Rham cochain complex

$\mathcal{H}^*_{\mathrm{cv}}(\bullet)$ vertically compact de Rham cohomology

0 Introduction

The aim of this thesis is an investigation into Massey products with particular focus on their use as tools to detect higher order linking, such as is inherent in families of the Brunnian links described in [Baa10].

The sources providing the main inspiration for this thesis are the articles [Baa10] by Baas, [Mas69] by Massey, [UM57] by Massey and Uehara and [O'N79] by O'Neill; and the book [GM81] by Griffiths and Morgan. In addition to this, a number of sources has been used as background material, most notably Bott and Tu's [BT82] and Hatcher's [Hat02].

The parts of this thesis which, to best of the author's knowledge, are novel are contained in Chapter 3, more specifically Section 3.4 to Section 3.8. In these sections we compute Massey products in the complements of links not previously described in the literature.

The contents of the thesis are divided into three chapters, about the necessary algebro-topological tools, the Massey product itself, and computations.

The first chapter introduces both the absolute and relative singular homology and cohomology theories and dualities between them, as well as the additional structure of cup products on cohomology. Furthermore, we compute these in the cases of complements of links in the three-sphere.

The second chapter contains a detailed description of the Massey triple product, including proofs of its well-definedness, naturality and homotopy invariance; its indeterminacy and a sample computation. There is also a discussion on generalisations of the triple product to higher products, as well as some indications of how the Massey products are related to other parts of mathematics.

The third chapter switches focus from singular theories to de Rham theory, with which we compute a number of a triple and higher order products in the complements of the Borromean rings, Brunnian links and Brunnian chains.

1 Singular homology and cohomology

This chapter is devoted to introducing the theory of singular homology and cohomology which is necessary to define the Massey product. We will have a particular focus on the cup product, since it is essential to the definition of the Massey products, but we will also discuss the associated relative theories and duality theorems, which are helpful in the concrete computation of the product.

We will mostly take for granted the part of the machinery that is purely homological-algebraic in nature.

Throughout the thesis we shall use the Borromean rings as a motivating and illustrating example, see Figure 1.1.

1.1 Construction

This section contains a short introduction to singular homology and cohomology. We do this first and foremost in order to fix notation, but also to have a setup for the later sections and an example of a differential graded associative algebra, for which we can define Massey products.

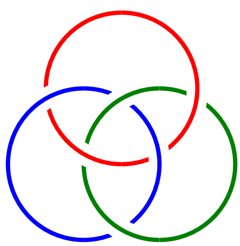


Figure 1.1: The Borromean rings

1.1.1 Singular homology

We start by defining singular simplices and chains on topological spaces and then linear functionals, or cochains, on these chains, and proceed by giving the sets of such chains and cochains more structure.

Definition 1.1.1. The *standard n-simplex* Δ_n is the set of points $\Delta_n = \left\{(x_0, \ldots, x_n) \in \mathbf{R}^{n+1} : x_i \geq 0, \sum x_i = 1\right\}$. ⌟

The low dimensional examples are a point, a line segment, a solid triangle and a solid tetrahedron.

Remark. We could also define this as the affine hull of the points $v_i = (0, \ldots, 1, \ldots, 0) \in \mathbf{R}^{n+1}$, with 1 in the i^{th} place. In this case, we denote the simplex by the alternative notation $[v_0 \cdots v_n]$. Each notation has their own merits, as we will see throughout this chapter.

Definition 1.1.2. Let X a topological space. A *singular n-simplex* on X is a continuous function $c\colon \Delta_n \to X$. ⌟

For $n \in \mathbf{N}$, we denote the free abelian group generated by singular n-simplices on X by $\mathcal{S}_n(X)$, elements of which are called *n-chains.*

To make this collection of groups into a chain complex we define a special kind of function on chains, namely *face maps.* These are maps of degree -1 on the chain complex.

Intuitively the face maps restrict an n-chain to one of its faces, an $(n-1)$-chain, but the covariance forces us to define a map from an $(n-1)$-simplex to an n-simplex.

In the literature, there seems to be some minor technical issues with defining the face maps for the standard simplices which are swept under the carpet, typically that the $(n-1)$-simplices involved are not the standard $(n-1)$-simplex, so here we shall write it out in full detail.

Let $\varphi_i\colon [v_0 \cdots v_{n-1}] \to [v_0 \cdots v_{i-1}\, 0\, v_{i+1} \cdots v_{n-1}]$ be the obvious map of sets preserving the ordering of the vertices.

Furthermore, let $\mathrm{d}'_i\colon \Delta_n \to \Delta_n$ be defined as a map of sets:

$$\mathrm{d}'_i([v_0 \cdots v_n]) = [v_0 \cdots v_{i-1}\, 0\, v_{i+1} \cdots v_n] \subset [v_0 \cdots v_n],$$

by the following pointwise operation for $(t_0, \ldots, t_n) \in \Delta_n$:

$$\mathrm{d}'_i((t_0, \ldots, t_n)) = (t_0 + t_i/n, \ldots, t_{i-1} + t_i/n, 0, t_{i+1} + t_i/n, \ldots, t_n + t_i/n,).$$

We can then form the composition

$$\mathrm{d}'_i \circ \varphi_i \colon \Delta_{n-1} \to \Delta_n$$

which has the correct domain and codomain.

This puts us in a position where we are able to rigorously define the sought-after maps.

Definition 1.1.3. The i^{th} *face* $\mathrm{d}_i c$ of an n-simplex $c\colon \Delta_n \to X$ is the composition $\mathrm{d}_i c = c \circ \mathrm{d}'_i \circ \varphi_i$. We extend to chains by linearity. ⌟

Note. Compositions of face maps are also called face maps.

The key to making $\{\mathcal{S}_n(X)\}_{n\in\mathbf{N}}$ into a chain complex is the *boundary map* ∂, defined using the face maps:

Definition 1.1.4. The *singular boundary map* ∂_n is given by the formula $\partial_n = \sum_{i=0}^n (-1)^i \mathrm{d}_i$, as an operator on n-chains, for $n \in \mathbf{N}$. ⌟

We will most often denote ∂_n by ∂.

Proposition 1.1.1. *The map ∂^2 vanishes, making ∂ a* differential*.*

We call chains in the kernel of ∂ *cycles* and chains in the image of ∂ *boundaries*.

We denote the resulting chain complex $(\mathcal{S}_*(X), \partial)$ by $\mathcal{S}_\bullet(X)$, the *singular chain complex of X*.

Definition 1.1.5. The n^{th} *singular homology of X* is the quotient group

$$\mathcal{H}_n(X) = \frac{\ker \partial_n \colon \mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X)}{\operatorname{im} \partial_{n-1} \colon \mathcal{S}_{n+1}(X) \to \mathcal{S}_n(X)},$$

of cycles modulo boundaries. ⌟

1.1.2 Singular cohomology

We now describe the associated cohomology theory to singular homology. Later, we will make critical use of dualities relating these theories. Another advantage is that the direct sum of the cohomology groups can always be made into a ring, or even a module over a algebra.

With the definitions of the preceding subsection at hand, we can give the following definition of cochains.

We let $\mathcal{S}^n(X) = \hom(\mathcal{S}_n(X), \mathbf{Z})$ and define the *coboundary map* δ to be the dual of ∂, in the following sense: Let u be a cochain and σ a chain of appropriate degrees, then we define δ by the relation:

$$(\delta u)(\sigma) = u(\partial u).$$

The fact that δ is a differential follows directly from the fact that ∂ is a differential.

Definition 1.1.6. The elements of $\mathcal{S}^n(X)$, which are $\mathbf{Z}$-linear maps $u\colon \mathcal{S}_n(X) \to \mathbf{Z}$, are called *singular n-cochains.* ⌟

Analogously to the terminology above, we call cochains in the kernel of δ *cocycles* and cochains in the image of δ *coboundaries.*

We then arrive at the definition of the corresponding

Definition 1.1.7. The n^{th} *singular cohomology of X* is the quotient group

$$\mathcal{H}^n(X) = \frac{\ker \delta^n \colon \mathcal{S}^n(X) \to \mathcal{S}^{n+1}(X)}{\operatorname{im} \delta^{n-1} \colon \mathcal{S}^{n-1}(X) \to \mathcal{S}^n(X)}$$

of cocycles modulo coboundaries. ⌟

Note. We will freely use the terminology of category theory, describing $\mathcal{H}^n(\bullet)$ and $\mathcal{H}_n(\bullet)$ as sequences of functors from some sufficiently nice category of spaces into the category of abelian groups. When we have defined a product, the former sequence will be regarded as a functor to the category of graded rings.

1.2 Cup product

The aim of this section is to define the *cup product* of cohomology classes as well as to emphasise and prove important properties that we will make use of later. We do this by first defining cup product on the level of cochains and then showing that it descends to a product in cohomology, finally we show that it is commutative.

Before proceeding with this section, it will be useful to introduce some additional notation for working with chains. For $i_j \in \{0, \ldots, p\}$ with $j \in \{0, \ldots, k\}$, let $\imath_{i_0,\ldots,i_k} \colon \Delta_k \to \Delta_p$ be the composition of the appropriate map taking the standard k-simplex to the k-simplex $[v_{i_0} \dots v_{i_k}]$ and the inclusion of the standard k-simplex into the standard p-simplex. As in the previous section, we require that the former map in the composition preserve the ordering of the vertices.

For any p-simplex $\sigma\colon \Delta_p \to X$, we can then define another k-simplex $\sigma_{i_0,\dots,i_k}\colon \Delta_k \to X$ as the following composition:

$$\sigma_{i_0,\dots,i_k} = \sigma \circ \imath_{i_0,\dots,i_k},$$

and extend to $\mathcal{S}_\bullet(X)$ by linearity.

In particular, if σ is a p-simplex then $\sigma_{0,\dots,p} = \sigma$.

1.2.1 Definition and basic properties

This subsection introduces the cup product and proves basic yet important properties of the cup product, such as associativity, Leibniz' rule and existence of an identity element.

We will first define cup product on the cochain level.

Definition 1.2.1. Let $u \in \mathcal{S}^p(X)$ and $v \in \mathcal{S}^q(X)$ be cochains. Being cochain, and hence linear functionals, we can define the *cup product* $u \smile v$ of these two cochains by how it acts on chains: We demand that the identity

$$(u \smile v)(\sigma) = u(\sigma_{0,\dots,p}) \cdot_{\mathbf{Z}} v(\sigma_{p,\dots,p+q})$$

hold for each $(p+q)$-chain $\sigma \in \mathcal{S}_{p+q}(X)$. ⌟

Hereafter we will not specify that the product is in $\mathbf{Z}$.

We note that the identity with respect to this multiplication is the cochain $1 \in \hom(\mathcal{S}_*(X), \mathbf{Z})$ taking the value $1 \in \mathbf{Z}$ on all *generating* chains of degree 0.

Since cochains are linear functionals and $\mathbf{Z}$-multiplication is distributive we get distributivity of the cup product for free.

Proving associativity is also easy, as shown by the following computation:

$$
\begin{aligned}
((u \smile v) \smile w)\,(\sigma) &= (u \smile v)(\sigma_{0,\ldots,p+q}) \cdot w(\sigma_{p+q,\ldots,p+q+r}) \\
&= (u(\sigma_{0,\ldots,p}) \cdot v(\sigma_{p,\ldots,p+q})) \cdot w(\sigma_{p+q,\ldots,p+q+r}) \\
&= u(\sigma_{0,\ldots,p}) \cdot (v(\sigma_{p,\ldots,p+q}) \cdot w(\sigma_{p+q,\ldots,p+q+r})) \\
&= u(\sigma_{0,\ldots,p}) \cdot (v \smile w)(\sigma_{p,\ldots,p+q+r}) \\
&= (u \smile (v \smile w))\,(\sigma).
\end{aligned}
$$

Since σ was arbitrary, we have the following identity of cochains:

$$(u \smile v) \smile w = u \smile (v \smile w).$$

We proceed by showing that the product on the level of cochains descends to a $\mathbf{Z}$-bilinear product in cohomology.

To show this we need to establish a formula for the interaction between the differential and the product.

Lemma 1.2.1. *For all cochains $u \in \mathcal{H}^p(X)$ and $v \in \mathcal{H}^q(X)$ the cup product $\smile$ satisfies the Leibniz rule with respect to δ:*

$$\delta(u \smile v) = \delta u \smile v + (-1)^p u \smile \delta v. \tag{1.1}$$

Proof. Each side of Equation 1.1 is a linear functional on chains, so to prove the lemma we show that they act identically. This means that for all chains $\sigma \in \mathcal{S}_{p+q+1}(X)$ we want to show the following:

$$
\begin{aligned}
\delta(u \smile v)(\sigma_{0,\ldots,p+q+1}) = {}& (\delta u)(\sigma_{0,\ldots,p+1}) \cdot v(\sigma_{p+1,\ldots,p+q+1}) \\
&+ (-1)^p u(\sigma_{0,\ldots,p}) \cdot (\delta v)(\sigma_{p,\ldots,p+q+1}).
\end{aligned}
$$

We do this by computing the expressions $((\delta u) \smile v)(\sigma)$, $(u \smile (\delta v))(\sigma)$ and $\delta(u \smile v)(\sigma)$, and then comparing the results.

We start with the first one:

$$
\begin{aligned}
((\delta u) \smile v)(\sigma) &= (\delta u)(\sigma_{0,\ldots,p+1}) \cdot v(\sigma_{p+1,\ldots,p+q+1}) \\
&= u(\partial\sigma_{0,\ldots,p+1}) \cdot v(\sigma_{p+1,\ldots,p+q+1}) \\
&= u\left(\sum_{i=0}^{p+1} (-1)^i \mathrm{d}_i(\sigma_{0,\ldots,p+1})\right) \cdot v\left(\sigma_{p+1,\ldots,p+q+1}\right) \\
&= \sum_{i=0}^{p+1} (-1)^i u\left(\mathrm{d}_i(\sigma_{0,\ldots,p+1})\right) \cdot v\left(\sigma_{p+1,\ldots,p+q+1}\right)
\end{aligned}
$$

and continue with the second one:

$$
\begin{aligned}
(u \smile (\delta v))(\sigma) &= u(\sigma_{0,\ldots,p}) \cdot (\delta v)(\sigma_{p,\ldots,p+q+1}) \\
&= u(\sigma_{0,\ldots,p}) \cdot v(\partial \sigma_{p,\ldots,p+q+1}) \\
&= u(\sigma_{0,\ldots,p}) \cdot v\left(\sum_{j=0}^{q+1} (-1)^j \mathrm{d}_j(\sigma_{p+1,\ldots,p+q+1})\right) \\
&= \sum_{j=0}^{q+1} (-1)^j u(\sigma_{0,\ldots,p}) \cdot v\left(\mathrm{d}_j(\sigma_{p,\ldots,p+q+1})\right) \\
&= \sum_{i=p}^{p+q+1} (-1)^{i-p} u(\sigma_{0,\ldots,p}) \cdot v\left(\mathrm{d}_{i-p}(\sigma_{p,\ldots,p+q+1})\right)
\end{aligned}
$$

Now we look at the terms occurring twice, namely those for which in the first (expression) $i = p+1$ and in the second $i = p$:

$$
\begin{aligned}
&(-1)^{p+1} u\left(\mathrm{d}_{p+1}(\sigma_{0,\ldots,p+1})\right) \cdot v\left(\sigma_{p+1,\ldots,p+q+1}\right) \\
&\quad = (-1)^{p+1} u(\sigma_{0,\ldots,p}) \cdot v(\sigma_{p+1,\ldots,p+q+1}).
\end{aligned} \tag{1.2}
$$

Similarly, multiplying by the sign needed later:

$$
\begin{aligned}
&(-1)^p (-1)^{p-p} u(\sigma_{0,\ldots,p}) \cdot v(\mathrm{d}_{p-p}(\sigma_{p,\ldots,p+q+1})) \\
&\quad = (-1)^p u(\sigma_{0,\ldots,p}) \cdot v((\sigma_{p+1,\ldots,p+q+1})).
\end{aligned} \tag{1.3}
$$

So they differ only by a sign and will cancel when added.

Finally, the third expression is computed as follows:

$$
\begin{aligned}
\delta(u \smile v)(\sigma) &= (u \smile v)(\partial(\sigma)) \\
&= (u \smile v)\left(\sum_{i=0}^{p+q+1} (-1)^i \mathrm{d}_i(\sigma)\right) \\
&= \sum_{i=0}^{p+q+1} (-1)^i (u \smile v)\left(\mathrm{d}_i(\sigma)\right)
\end{aligned} \tag{1.4}
$$

We go on to manipulate the sum of Equation 1.2 and Equation 1.3:

$$
\begin{aligned}
((\delta u) \smile v)(\sigma) + (-1)^p (u \smile (\delta v))(\sigma) \\
&= \sum_{i=0}^{p+1} (-1)^i u \left(\mathrm{d}_i(\sigma_{0,\dots,p+1})\right) \cdot v \left(\sigma_{p+1,\dots,p+q+1}\right) \\
&\quad + \sum_{i=p}^{p+q+1} (-1)^i u(\sigma_{0,\dots,p}) \cdot v \left(\mathrm{d}_{i-p}(\sigma_{p,\dots,p+q+1})\right) \\
&= \sum_{i=0}^{p} (-1)^i u \left(\mathrm{d}_i(\sigma_{0,\dots,p+1})\right) \cdot v \left(\sigma_{p+1,\dots,p+q+1}\right) \\
&\quad + \sum_{i=p+1}^{p+q+1} (-1)^i u(\sigma_{0,\dots,p}) \cdot v \left(\mathrm{d}_{i-p}(\sigma_{p,\dots,p+q+1})\right) \\
&= \sum_{i=0}^{p+q+1} (-1)^i (u \smile v) \left(\mathrm{d}_i(\sigma)\right)
\end{aligned}
\tag{1.5}
$$

We see that Equation 1.4 and Equation 1.5 are equal, hence we have established the Leibniz rule. □

We go to show how to get a product in cohomology from this product on cochains.

Assume that u and v represent the cohomology classes $[u]$ and $[v]$. In particular, u and v are cocycles, so δu and δv are zero. It follows by Leibniz' rule that $\delta(u \smile v)$ is also zero, hence $u \smile v$ is a cocycle.

This allows us to define the cup product of cohomology classes as follows:

Definition 1.2.2. Let $[u]$ and $[v]$ be cohomology classes in $\mathcal{H}^\bullet(X)$, then the cup product is given by $[u] \smile [v] = [u \smile v]$. ⌟

Lemma 1.2.2. *The cup product $\smile$ descends to a well-defined product on $\mathcal{H}^\bullet(X)$, making it a graded ring with unity.*

Proof. We need to check that the product is well-defined, which we do by choosing other representatives for $[u]$ and $[v]$: If u and u' differ by a boundary, say δw, then

$$u \smile v - u' \smile v = (u - u') \smile v = (\delta w) \smile v = \delta(w \smile v),$$

hence $u \smile v$ and $u' \smile v$ coincide in cohomology. The calculation for the second factor is analogous, proving that the product is well-defined on cohomology.

Furthermore, the class of 1, [1], *is* the multiplicative identity on $\mathcal{H}^\bullet(X)$. To prove this, we start by showing that 1 descends to cohomology: The cochain $\delta 1$ acts on elements of $\mathcal{S}_1(X)$ so we pick a generator $\sigma : \Delta_1 \to X$ and calculate the action of $\delta 1$ on it:

$$(\delta 1)(\sigma) = 1(\partial\sigma) = 1(\sigma(1) - \sigma(0)) = 1(\sigma(1)) - 1(\sigma(0)) = 1 - 1 = 0,$$

so 1 is indeed a cocycle.

Now, by definition of 1, it is 1 on every generator of $\mathcal{S}_0(X)$, so

$$(1 \smile u)(\sigma) = 1(\sigma_0) \cdot u(\sigma_{0,\dots,p}) = u(\sigma).$$

This immediately carries over to cohomology:

$$[1] \smile [u] = [1 \smile u] = [u],$$

so [1] is indeed the identity with respect to $\smile$. □

1.2.2 Commutativity

The cup product is *not* commutative on the cochain level. The explanation of this feature is part of the theory of *cohomology operations*, as they are *obstructions*, in a specific way, to the cochain-level commutativity.

However, in the words of [GM81, p.110], "a somewhat grizzly computation" shows that on the level of cohomology, we *do* get graded commutativity.

Theorem 1.2.3. *The cup product is graded commutative.*

Proof. The proof of this theorem is found in for instance [Hat02, p.216-7], but we present a slightly different version, which was given in the course "Algebraic Topology" at the University of Cambridge in the Michaelmas term of 2010.

To prove this result, we will make use of two claims, which we prove later.

Let $\rho\colon \mathcal{S}_p(X) \to \mathcal{S}_p(X)$ be defined by $\rho(\sigma_{0,\dots,p}) = \varepsilon_p \sigma_{p,\dots,0}$, where $\varepsilon_p = (-1)^{p(p+1)/2}$ is the sign obtained by counting the number of transpositions in the permutation $(0, \dots, p) \mapsto (p, \dots, 0)$.

Claim 1. ρ is a *chain map.*

Claim 2. ρ is chain homotopic to the identity map.

Given these two claims, we prove that $u \smile v = (-1)^{pq} v \smile u$ by comparing $\rho^*(u) \smile \rho^*(v)$ to $\rho^*(u \smile v)$. We calculate the former:

$$\begin{aligned}(\rho^*(u) \smile \rho^*(v))(\sigma_{0,\dots,p+q}) &= \rho^*(u)(\sigma_{0,\dots,p}) \cdot \rho^*(v)(\sigma_{p,\dots,p+q}) \\ &= \varepsilon_p u(\sigma_{p,\dots,0}) \cdot \varepsilon_q v(\sigma_{p+q,\dots,p}) \\ &= \varepsilon_p \varepsilon_q v(\sigma_{p+q,\dots,p}) \cdot u(\sigma_{p,\dots,0}) \\ &= \varepsilon_p \varepsilon_q (v \smile u)(\sigma_{p+q,\dots,0}).\end{aligned}$$

In the same way, the latter:

$$(\rho^*(u \smile v))(\sigma_{0,\dots,p+q}) = \varepsilon_{p+q}(u \smile v)(\sigma_{p+q,\dots,0})$$

Since $\rho \simeq \mathrm{id}$, we get $\rho^* = \mathrm{id}$, this means that *in cohomology*, we have

$$\varepsilon_{p+q}(u \smile v) = \varepsilon_p \varepsilon_q (v \smile u). \tag{1.6}$$

Comparing signs:

$$\begin{aligned}\varepsilon_{p+q} &= (-1)^{(p+q)(p+q+1)/2} \\ &= (-1)^{(p^2+pq+p+qp+q^2+q)/2} \\ &= (-1)^{pq}(-1)^{(p^2+p)/2}(-1)^{(q^2+q)/2} \\ &= (-1)^{pq}\varepsilon_p\varepsilon_q.\end{aligned} \tag{1.7}$$

Combining Equation 1.6 and Equation 1.7 gives

$$u \smile v = (-1)^{pq} v \smile u,$$

for $u \in \mathcal{H}^p(X)$ and $v \in \mathcal{H}^q(X)$, which is what we wanted to show.

To finish the proof, we have to prove the claims.

Proof of Claim 1. Recall that a map is a chain map if it commutes with the differential, so we have to show that $\rho\partial = \partial\rho$.

We do this by applying these maps to an arbitrary p-chain σ and compare the results:

$$\begin{aligned}(\rho\partial)(\sigma_{0,\dots,p}) &= \rho\sum_{i=0}^{p}(-1)^i \mathrm{d}_i\sigma_{0,\dots,p}\\ &= \varepsilon_{p-1}\sum_{i=p}^{0}(-1)^{p-i}\mathrm{d}_{p-i}\sigma_{p,\dots,0}\\ &= \varepsilon_{p-1}(-1)^p\sum_{j=0}^{p}(-1)^j \mathrm{d}_j\sigma_{p,\dots,0},\end{aligned}$$

and reversely:

$$\begin{aligned}(\partial\rho)(\sigma_{0,\dots,p}) &= \varepsilon_p\partial(\sigma_{p,\dots,0})\\ &= \varepsilon_p\sum_{j=0}^{p}(-1)^j \mathrm{d}_j\sigma_{p,\dots,0}.\end{aligned}$$

So again we are down to comparing signs:

$$\begin{aligned}(-1)^p\varepsilon_{p-1} &= (-1)^p(-1)^{(p-1)(p-1+1)/2}\\ &= (-1)^{(2p+(p-1)p)/2}\\ &= (-1)^{(p+1)p/2}\\ &= \varepsilon_p,\end{aligned}$$

so $(\rho\partial)(\sigma_{0,\dots,p}) = (\partial\rho)(\sigma_{0,\dots,p})$, which is what we wanted to show. □

Proof of Claim 2. Recall that two chain maps f and g are homotopic if there exists a map h such that $\partial h + h\partial = f - g$. We want to show that ρ is homotopic to id.

For the purpose of this proof we let $\sigma\colon \Delta_p \to X$ be denoted by $[v_0 \cdots v_p]$, and $[v_0,\dots,v_i,w_p,\dots,w_i]$ is the subsimplex of $\Delta_p \times I$ given by the convex hull of the vertices $v_0,\dots,v_i,w_p,\dots,w_i$.

Define the chain map $h^p\colon \mathcal{S}_p(X) \to \mathcal{S}_{p+1}(X)$ by

$$h^p(\sigma) = \sum_{i=0}^{p}\varepsilon_{p-1}\,(\sigma\pi)|_{[v_0,\dots,v_i,w_p,\dots,w_i]}\,,$$

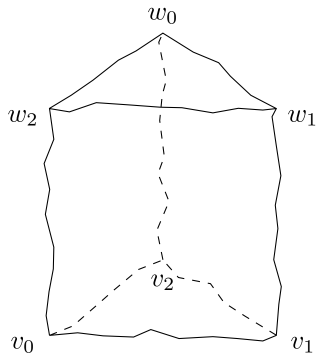


Figure 1.2: $\Delta_p \times I$ with top and bottom oppositely oriented

with $\pi\colon \Delta_p \times [0,1] \to \Delta_p$, and the domain shown in Figure 1.2.

Intuitively, we sum over chains whose orientations successively approach the reversed one.

We will compute ∂h and $h\partial$ separately, find that most terms cancel, and in the end see that we wind up with $\rho - \mathrm{id}$.

$$\begin{aligned}
\partial h^p[v_0 \cdots v_p] &= \sum_{j=0}^{p+1} (-1)^j \mathrm{d}_j \sum_{i=0}^{p} (-1)^i \varepsilon_{p-i}[v_0 \cdots v_i w_p \cdots w_i] \\
&= \sum_{j=0}^{p+1} \sum_{i=0}^{p} (-1)^{i+j} \varepsilon_{p-i} \mathrm{d}_j [v_0 \cdots\cdots v_i w_p \cdots w_i] \\
&= \sum_{j \le i} (-1)^{i+j} \varepsilon_{p-i}[v_0 \cdots \widehat{v_j} \cdots v_i w_p \cdots w_i] \\
&\quad + \sum_{j > i} (-1)^{i+j} \varepsilon_{p-i}[v_0 \cdots v_i w_p \cdots \widehat{w_j} \cdots w_i] \\
&\overset{j \mapsto p+1-j}{=} \sum_{j \le i} (-1)^{i+j} \varepsilon_{p-i}[v_0 \cdots \widehat{v_j} \cdots v_i w_p \cdots w_i] \\
&\quad + \sum_{j \ge i} (-1)^{i+p+1-j} \varepsilon_{p-i}[v_0 \cdots v_i w_p \cdots \widehat{w_j} \cdots w_i]
\end{aligned}$$

Notice that after reindexing, we have two sums with $i = j$. They can be decomposed as follows:

$$\begin{aligned}
\sum_{j=i}(-1)^{i+j}\varepsilon_{p-i}[v_0\cdots\widehat{v_j}\cdots v_i w_p\cdots w_i]& \\
+\sum_{j=i}(-1)^{i+p+1-j}\varepsilon_{p-i}[v_0\cdots v_i w_p\cdots\widehat{w_j}\cdots w_i]& \\
=\varepsilon_p[w_p\cdots w_0]-[v_p\cdots v_0]& \\
+\sum\varepsilon_{p-1}[v_0\cdots v_{i-1}w_p\cdots w_i]& \\
+\sum(-1)^{n+i-1}\varepsilon_{p-1}[v_0\cdots v_i w_p\cdots w_{i+1}],&
\end{aligned}$$

of which the last two terms cancel upon reindexing, say the last term by $i\mapsto i-1$.

In the next calculation, some care is needed in the corner cases.

$$\begin{aligned}
h^p\partial[v_0\cdots v_p] &= h^p\sum_{j=0}^{p}(-1)^j[v_0\cdots\widehat{v_j}\cdots v_p] \\
&= \sum_{i=0}^{p-1}\sum_{j=0}^{p}(-1)^{i+j}\varepsilon_{p-(i-1)}[v_0\cdots\widehat{v_j}\cdots v_i w_p\cdots w_i] \\
&= \sum_{i<j}(-1)^{i+j}\varepsilon_{p-i+1}[v_0\cdots v_i w_p\cdots\widehat{w_j}\cdots w_i] \\
&\quad+\sum_{i>j}(-1)^{i+j}\varepsilon_{p-i+1}[v_0\cdots\widehat{v_j}\cdots v_i w_p\cdots w_i].
\end{aligned}$$

By inspection, we see that the terms for which $i\neq j$ in the sum corresponding to ∂h^p cancel the entire sum corresponding to $h^p\partial$.

Upon relabelling w_i by v_i, this leaves us with the following:

$$\begin{aligned}
h^p\partial[v_0\cdots v_p]+\partial h^p[v_0\cdots v_p] &= \varepsilon_p[v_p\cdots v_0]-[v_p\cdots v_0] \\
&= (\rho-\mathrm{id})[v_0\cdots v_p]
\end{aligned}$$

so ρ and id are chain homotopic, proving Claim 2. □

This completes the proof of commutativity of the cup product. □

1.2.3 Naturality

An important property of any "operation" on cohomology functors is naturality. In this subsection we give a proof that the cup product is indeed natural, making it an (unstable) *cohomology operation*.

Lemma 1.2.4. *The cup product is natural, in the sense that it commutes with homomorphisms induced by maps of spaces. More explicitly, if* $f\colon X \to Y$ *is a continuous function, then for all* $u, v \in \mathcal{H}^*(Y)$ *we have*

$$f^*(u \smile v) = f^*(u) \smile f^*(v) \in \mathcal{H}^*(X)\,.$$

Proof. We first prove the statement on the cochain level, then show that it descends to cohomology.

Let $u \in \mathcal{S}^p(Y)$, $v \in \mathcal{S}^q(Y)$ and $\sigma \in \mathcal{S}_{p+q}(X)$. We then have the following sequence of equalities:

$$\begin{aligned}
(f^*(u \smile v))\,(\sigma) &= (u \smile v)(f_*(\sigma)) \\
&= u(f_*(\sigma)_{0,\ldots,p}) \cdot v(f_*(\sigma)_{p,\ldots,p+q}) \\
&= u(f_*(\sigma_{0,\ldots,p})) \cdot v(f_*(\sigma_{p,\ldots,p+q})) \\
&= (f^*(u))(\sigma_{0,\ldots,p}) \cdot (f^*(v))(\sigma_{p,\ldots,p+q}) \\
&= (f^*(u) \smile f^*(v))(\sigma).
\end{aligned}$$

So the induced map on the cochain level is a ring homomorphism.

We show that the same holds in cohomology:

$$\begin{aligned}
f^*([u] \smile [v]) &= f^*([u \smile v]) \\
&= [f^*(u \smile v)] \\
&= [f^*(u) \smile f^*(v)] \\
&= [f^*(u)] \smile [f^*(v)] \\
&= f^*[u] \smile f^*[v],
\end{aligned}$$

which is what we wanted to show. □

1.3 Relative homology and cohomology

In this section we introduce the *relative* homology and cohomology of pairs of sufficiently nice spaces. We continue focusing on the singular theories, but the discussion is really pure homological algebra, so the same kind of construction goes through in other homology and cohomology theories built from chain complexes.

The primary motivation for this section is that we will make important use of the machinery of relative groups and associated dualities, to be introduced later, when computing Massey products in the coming chapters.

1.3.1 Relative chain complexes

Without further ado, we define the relative singular homology as a quotient of chain complexes.

Definition 1.3.1. The *relative singular chain complex* $\mathcal{S}_\bullet(X, A)$ of the pair (X, A) is the object making the following sequence of chain complexes:

$$0 \to \mathcal{S}_\bullet(A) \to \mathcal{S}_\bullet(X) \to \mathcal{S}_\bullet(X, A) \to 0$$

into a short exact sequence. ⌟

The category $\mathcal{CC}(\mathcal{A})$ of chain complexes of an abelian category $\mathcal{A}$ is also abelian. The category Ab of abelian groups and group homomorphisms is the prototypical abelian category and in abelian categories cokernels always exists, so by abstract nonsense $\operatorname{coker} i^* \colon \mathcal{S}_\bullet(A) \to \mathcal{S}_\bullet(X) \cong \mathcal{S}_\bullet(X, A)$ exists and satisfies the property above. We then get a chain complex, equipped with an induced differential, for free.

Similarly to how we defined the absolute singular cohomology, we now define the relative singular cohomology, beginning with the cochain complex.

Definition 1.3.2. The *relative singular cochain complex* $\mathcal{S}^\bullet(X, A)$ of the pair (X, A) is defined as $\mathcal{S}^\bullet(X, A) = \hom(\mathcal{S}_\bullet(X, A), \mathbf{Z})$. ⌟

By abuse of notation we will use the same notation for the relative differentials as for the absolute differentials.

We can interpret the elements of $\mathcal{S}^\bullet(X, A)$ as cochains whose support lie in $X \setminus A$, in other words, the relative cochains vanish on chains contained in A.

The relative singular homology and cohomology are then defined in the usual way as the homology of these chain complexes.

Definition 1.3.3. The *relative singular homology* of the pair (X, A) is the homology of the chain complex $\{\mathcal{S}_\bullet(X, A), \partial\}$:

$$\mathcal{H}_n(X, A) = \frac{\ker \partial_n \colon \mathcal{S}_n(X, A) \to \mathcal{S}_{n-1}(X, A)}{\operatorname{im} \partial_{n-1} \colon \mathcal{S}_{n+1}(X, A) \to \mathcal{S}_n(X, A)},$$

the group of *relative cycles* modulo *relative boundaries.* ⌟

Definition 1.3.4. The *relative singular cohomology* of the pair (X, A) is the homology of the cochain complex $\{\mathcal{S}^\bullet(X, A), \delta\}$:

$$\mathcal{H}^n(X, A) = \frac{\ker \delta^n \colon \mathcal{S}^n(X, A) \to \mathcal{S}^{n+1}(X, A)}{\operatorname{im} \delta^{n-1} \colon \mathcal{S}^{n-1}(X, A) \to \mathcal{S}^n(X, A)},$$

the group of *relative cocycles* modulo *relative coboundaries.* ⌟

1.3.2 Relative cup product

We now want to extend the cup product in the cohomology of a space to the cohomology of pairs of spaces. We can do this by the same formula as for the absolute cohomology groups, but we have to check that the resulting relative cochain is a relative cochain.

Given the interpretation of $\mathcal{S}^\bullet(X, A)$ as the set of cochains vanishing on chains contained entirely in A, we want to show that the cup product also vanishes on all such chains.

This is in fact easy. From the formula we see that we need to evaluate each of the factors of the product on subsimplices. Both of the subsimplices are obviously entirely contained in A if and only if the full simplex is. This implies that if the chain on which we evaluate the product is contained entirely in A, then the result is zero. In symbols, this is:

$$(u \smile v)(\sigma) = u(\sigma_{0,\ldots,p}) \cdot v(\sigma_{p,\ldots,p+q}) = 0 \cdot 0 = 0.$$

With this trifle out of the way, the properties of the relative cup product are completely analogous to those of the absolute one.

From this discussion, we see that we have a relative cup product as follows:

$$\smile \colon \mathcal{H}^p(X, A) \times \mathcal{H}^q(X, A) \to \mathcal{H}^{p+q}(X, A)\,.$$

There is also a refinement of the above relative product to a slightly more general version:

$$\smile : \mathcal{H}^p(X, A) \times \mathcal{H}^q(X, B) \to \mathcal{H}^{p+q}(X, A \cup B)\,.$$

We will not be needing this and there are a couple of subtleties to defining it, so we will not go into any detail.

1.3.3 Relative orientations

When stating the duality theorems, we will need the notion of orientation for manifolds with possibly non-empty boundary.

Definition 1.3.5. Let M be a compact orientable manifold with boundary ∂M, then a *relative orientation* of $(M, \partial M)$ is a choice of generator $[M, \partial M]$ for $\mathcal{H}_n(M, \partial M) \cong \mathbf{Z}$, we call this generator a *relative fundamental class.* ⌟

1.4 Eilenberg–Steenrod Axioms

Having constructed singular homology and cohomology we claim, without proof, that they satisfy the relevant Eilenberg–Steenrod axioms for pairs of topological spaces.

We will make use of excision and finite additivity; excision in the proof of Alexander duality, and additivity for the computation of the homology and cohomology of the Borromean rings and Brunnian links.

In order to state the axioms, we need to define a couple of general topological terms.

Definition 1.4.1. Let I be the unit interval. Two continuous maps of pairs $f, g\colon (X, A) \to (Y, B)$ are *homotopic as maps of pairs* if there is another map of pairs $F\colon (X \times I, A \times I) \to (Y, B)$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$ and $F(A, t) \subset B$ for all $t \in I$. ⌟

Definition 1.4.2. Let (X, A) be a pair of spaces. A subset U of A is *excisive* if the closure of U is contained in the interior of A. ⌟

1.4.1 In homology

Let $\mathcal{H}_n(\bullet, \bullet) : h\mathcal{CW}^2 \to \mathrm{Ab}$ with $n \in \mathbf{N}$ be a sequence of functors from the category of pairs of spaces with the homotopy type of a CW-complex to the category of abelian groups.

If $\{\mathcal{H}_n(\bullet, \bullet)\}$ satisfies the following axioms, then it is an *ordinary additive homology theory.*

Homotopy The induced map of a homotopy equivalence is the identity.

Excision If U is *excisive* in (X, A), then the induced map of the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ is an isomorphism.

Dimension For a one-point space, the homology is concentrated in the zeroth degree.

Additivity The homology of a disjoint union is the direct sum of the homologies of the components.

Exactness Each pair (X, A) gives a long exact sequence in homology:

$$\cdots \to \mathcal{H}_k(A) \to \mathcal{H}_k(X) \to \mathcal{H}_k(X, A) \to \mathcal{H}_{k-1}(A) \to \cdots .$$

Note. Since we consider the homology groups as a sequence of functors, we can omit the axiom of *naturality.* The same is true for the cohomology groups.

To tie this in with the rest of the chapter, we state the following proposition.

Proposition 1.4.1. *The relative singular homology groups form an ordinary homology theory in the sense of Eilenberg–Steenrod.*

1.4.2 In cohomology

The corresponding axioms for an ordinary cohomology theory is largely similar, save some differences stemming from the *contra*variance of the cohomology functors.

Let $\mathcal{H}^n(\bullet, \bullet) : h\mathcal{CW}^2 \to \mathrm{Ab}$ with $n \in \mathbf{N}$ be a sequence of functors from the category of pairs of spaces with the homotopy type of a CW-complex to the category of abelian groups.

If $\{\mathcal{H}^n(\bullet, \bullet)\}$ satisfies the following axioms, then it is an *ordinary additive cohomology theory.*

Homotopy The induced map of a homotopy equivalence is the identity.

Excision If U is *excisive* in (X, A), then the induced map of the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ is an isomorphism.

Dimension For a one-point space, the cohomology is concentrated in the zeroth degree.

Additivity The cohomology of a disjoint union is the direct product of the cohomologies of the components.

Exactness Each pair (X, A) gives a long exact sequence in cohomology:

$$\cdots \to \mathcal{H}^k(X, A) \to \mathcal{H}^k(X) \to \mathcal{H}^k(A) \to \mathcal{H}^{k+1}(X, A) \to \cdots .$$

We have the analogous proposition.

Proposition 1.4.2. *The relative singular cohomology groups form an ordinary cohomology theory in the sense of Eilenberg–Steenrod.*

1.5 Dualities

When computing Massey products in the complement of a link L we will work in a space which is homotopy equivalent to $\mathbf{S}^3 \setminus L$, but which has the structure of a compact smooth manifold-with-boundary. We obtain such a space by replacing the link components of the complement by non-intersecting regular neighbourhoods of these. The reason for doing this is that then we are able to harness the power that certain duality theorems in geometry gives us.

The material in this section is standard, and found in most books on algebraic topology, e.g. [May99].

1.5.1 Poincaré–Lefschetz duality

Poincaré–Lefschetz duality is a generalisation of the Poincaré duality for closed orientable manifolds to a duality for compact orientable manifolds with boundary, as expressed precisely by the following theorem.

Theorem 1.5.1. *If M is a compact oriented n-manifold with boundary ∂M, then we have isomorphisms*

$$D\colon \mathcal{H}^p(M, \partial M) \to \mathcal{H}_{n-p}(M)$$

and

$$D\colon \mathcal{H}^p(M) \to \mathcal{H}_{n-p}(M, \partial M)\,.$$

Note. When the boundary is empty, this reduces to the classical Poincaré duality, which is why we have omitted mentioning it specifically.

Since the proof of the theorem is standard, we have not included it here, but it can be found in [May99, p.170-171].

Cap product

In order to explicitly describe the above isomorphism, we introduce the *cap product* of homology and cohomology classes of appropriate degrees. This product is closely related to the cup product, indeed many of its properties may be derived from the ones of the cup product. Because of this close relationship, we will not go into the same excruciating level of detail as for the cup product.

We define it on the chain and cochain level, and then claim that it descends to a well-defined product in homology and cohomology.

Definition 1.5.1. Let X be a space and A and B be open subsets of X. Furthermore, let $\sigma \in \mathcal{S}_p(X)$ be chain and $u \in \mathcal{S}^q(X)$ a cochain. Then the *cap product* $\frown\colon \mathcal{S}_p(X) \times \mathcal{S}^q(X) \to \mathcal{S}_{p-q}(X)$ is defined by the following formula:

$$\sigma \frown u = u\,(\sigma_{0,\dots,q}) \cdot \sigma_{q,\dots,p},$$

whenever p is greater than or equal to q. ⌟

Proposition 1.5.2. *The cap product $\frown$ descends to a well-defined* **Z***-bilinear product of homology and cohomology classes:*

$$\frown\colon \mathcal{H}_p(X) \times \mathcal{H}^q(X) \to \mathcal{H}_{p-q}(X)\,.$$

There is also a more general, relative version:

$$\frown\colon \mathcal{H}_p(X, A \cup B) \times \mathcal{H}^q(X, A) \to \mathcal{H}_{p-q}(X, B)\,.$$

The proof can be found in [Hat02, p. 239-240].

By abuse of notation, we will denote both of these by the same symbol.

The relation between the cup and cap product is given in the following proposition[Hat02, p. 249].

Proposition 1.5.3. *Let $u \in \mathcal{H}^p(X, A)$ and $v \in \mathcal{H}^q(X, A)$ be cohomology classes and $\sigma \in \mathcal{H}_{p+q}(X, A)$ be a homology class, then the following identity holds:*

$$u(\sigma \frown v) = (v \smile u)(\sigma).$$

Proof. The proof is merely a short exercise in handling the definitions:

$$\begin{aligned} u(\sigma \frown v) &= u\left(v(\sigma_{0,\dots,q}) \cdot \sigma_{p,\dots,p+q}\right) \\ &= v(\sigma_{0,\dots,q}) \cdot \left(u(\sigma_{p,\dots,p+q})\right) \\ &= (v \smile u)(\sigma). \end{aligned}$$

□

As mentioned earlier, many of the properties of the cap product can be deduced from this relation and properties of the cup product.

Poincaré–Lefschetz duality isomorphism

We are now in a position to describe the isomorphism of the Poincaré–Lefschetz duality theorem explicitly.

Let M be a compact n-manifold with boundary ∂M oriented by the relative fundamental class $[M, \partial M] \in \mathcal{H}_n(M, \partial M)$. Then the isomorphism in Theorem 1.5.1 is given by:

$$[M, \partial M] \frown \bullet \colon \mathcal{H}^p(M, \partial M) \to \mathcal{H}_{n-p}(M)\,.$$

Intersection product

Most of the computations of Massey products in thesis will be done using de Rham cohomology, since this provides a nice and geometric way of understanding them. We will, however, also compute a Massey product in the way described by Massey in his article [Mas69], using singular theories. In order to be able to do this, we will state a theorem regarding the relation between the cup product in cohomology and the *intersection product* in homology.

One reference for the theory of intersection of homology classes is Chapter 13 of Albrecht Dold's "Lecture on Algebraic Topology", [Dol95]. He does it in more generality than we need, since we will only need intersections of homology classes represented of transversely intersecting submanifolds, whereas he does it for arbitrary homology classes of a manifold.

Even the definition of the intersection product $\star$ is involved, and since we will not be needing it, we will content ourselves with giving the following heuristic explanation of how it works:

If χ and ξ are homology classes represented by sufficiently nice cycles, then their intersection product $\chi\star\xi$ is represented by a cycle corresponding to their intersection. In Chapter 3, we will provide examples of "sufficiently nice" cycles and their intersection products, as well as their relation to the wedge product of de Rham cohomology classes.

In Section 2.3, however, we will perform a "classical" calculation of a Massey product, using the singular theories and the relation we the cup product and the intersection product, so we state it as the following proposition, omitting the proof.

Proposition 1.5.4. *If $u, v \in \mathcal{H}^*(M)$ are Poincaré duals of $\chi, \xi \in \mathcal{H}_*(M)$, then $u \smile v$ is the Poincaré dual of $\chi \star \xi$, the intersection product of χ and ξ.*

Interpretation of cup product in terms of linking numbers

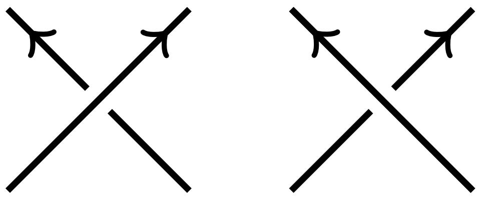
Since the vanishing of cup products is necessary to able to define the Massey product having an interpretation of cup products in link complement in terms of linking numbers of link components is very useful.

To describe how this connection arrises we need to give precise definitions of the linking number.

Linking numbers

We want to define the *linking number* $\mathrm{lk}(\gamma_1, \gamma_2)$ of a 2-component link (γ_1, γ_2) with components γ_1 and γ_2 or, slightly more generally, of two components γ_1 and γ_2 of a link with possibly more than two components.

In order to define it we need to put *orientations*, in the sense of differential topology, on each link component.



(a) $+1$ (b) -1

Figure 1.3: Positive and negative crossings

There are several equivalent ways of defining this linking number, below we will give two of them.

We begin with one defined in terms of knot diagrams, which enables us to quickly calculate it in concrete examples.

Given an orientation on link components, we can assign a sign $\{\pm 1\}$ to each crossing between different components, by the rules described in Figure 1.3. We say that a crossing is *positive* and *negative* in the first and second case, respectively.

Definition 1.5.2. The *linking number* $\mathrm{lk}(\gamma_1, \gamma_2)$ is the sum of signs of the crossings of (γ_1, γ_2) where γ_1 crosses over γ_2. ⌟

The definition is asymmetrical but it can be shown that it does not depend on the ordering of link components.

Note. We will not work explicitly with orientation of links as it will turn out that in all our examples it will be independent of choices of orientation; furthermore, they will all be zero – a necessary condition for being able to define the pertaining Massey products.

Another definition [BT82, p. 229] of the linking number, which is more algebro-topological in nature, is the the following, where we restrict ourselves to submanifolds of dimension 1.

Definition 1.5.3. The *linking number* of two closed, connected 1-submanifolds L_1 and L_2 is given as follows. Choose a generic smooth surface $N \in \mathbf{S}^3$ such that $\partial N = L_1$. The linking number of $\mathrm{lk}(L_1, L_2)$ is then given by

$$\operatorname{lk}(L_1, L_2) = \sum_{N \cap L_2} \pm 1,$$

where the sign at $x \in N \cap L_2$ is determined by whether or not the direct sum $\mathcal{T}_x N \oplus \mathcal{T}_x L_2$ of the tangent spaces of N and L_2 has the same orientation as $\mathcal{T}_x \mathbf{S}^3$, the tangent space of the ambient space. ⌟

Note. We use the word "generic" to avoid having to discuss *transversality* at this point, it will, however, be one of the topics of Section 3.1.

We are then ready to state a proposition regarding the correspondence between cup product and linking numbers.

Proposition 1.5.5. *Let L_1 and L_2 be embedded fattened up circles in $\mathbf{S}^3$, A a chain with boundary L_1, η_{L_2} the Poincaré–Lefschetz dual of L_2 and η_A the dual of A. Then the linking number $\operatorname{lk}(L_1, L_2)$ correspond, under capping with the fundamental class of $\mathbf{S}^3$, to the cup product $\eta_{L_2} \smile \eta_A$.*

A proof of the corresponding statement in de Rham theory is found in [BT82, p.229-34].

The triviality of this cup product then correspond to a linking number which is zero.

1.5.2 Alexander duality

In this section, we state the Alexander duality theorem, which will be useful to compute the homology and cohomology of link complements, as well as giving us a way to compute Massey products, an example of this is given in Section 2.3.

The following statement of the Alexander duality theorem is taken from [Hat02, p. 254].

Theorem 1.5.6. *If K is a compact, locally contractible, non-empty, proper subspace of $\mathbf{S}^n$, then*

$$\widetilde{\mathcal{H}}_i(\mathbf{S}^n \setminus K) \cong \widetilde{\mathcal{H}}^{n-i-1}(K)\,.$$

The main ingredients in the proof are Poincaré duality and excision.

Note. If we wanted to get rid of the assumption that K be locally contractible, then we could replace singular cohomology by Čech cohomology.

1.6 Link complements

In this section we compute the cohomology ring of the complement of the Borromean rings and Brunnian links in $\mathbf{S}^3$. That is, first we compute the additive cohomology of *all* links and then the cup product structure of these two links.

1.6.1 Additive cohomology

Computing the additive cohomology of link complements in $\mathbf{S}^3$ is quite simple given Alexander duality.

From the Alexander duality theorem, it follows that the choice of embedding of K in $\mathbf{S}^n$ does not affect the cohomology of the complement. This means that we can deduce the additive cohomology of the complement of an arbitrary link from Alexander duality and knowledge of the (reduced) homology of the unlinks.

The reduced homology of the disjoint union of k circles is given as follows:

$$\widetilde{\mathcal{H}}_*\left(\sqcup_k S^1\right) = \begin{cases} \mathbf{Z}^{k-1} & \text{if} \quad * = 0 \\ \mathbf{Z}^k & \text{if} \quad * = 1. \end{cases}$$

By Alexander duality this data translates to information about the additive cohomology of the complement $X' = \mathbf{S}^3 \setminus L$ of link L with k components in $\mathbf{S}^3$, which is the case we are interested in:

$$\mathcal{H}^*(X') = \begin{cases} \mathbf{Z}^k & \text{if} \quad * = 1 \\ \mathbf{Z}^{k-1} & \text{if} \quad * = 2. \end{cases}$$

Additionally, we know that since a codimension 2 submanifold cannot separate its ambient manifold, X' is path connected, which is equivalent to $\mathcal{H}^0(X') = \mathbf{Z}$.

Now, we could try to figure out the top cohomology group of this space or we can replace it with a topologically equivalent space. We will do the latter, as hinted to earlier.

We replace the link components by open non-intersecting tubular neighbourhoods, the complement of this space in $\mathbf{S}^3$ can be given the structure of a compact oriented manifold-with-boundary. This space, call it X, is a deformation retract of the original space, so they have the same cohomology.

This allows us to use the Poincaré–Lefschetz duality theorem to compute the top cohomology group: Since X' is path connected, so is X, hence $\mathcal{H}_0(X) \cong \mathbf{Z}$ and $\mathcal{H}_0(X, \partial X) \cong 0$. By Poincaré–Lefschetz duality we have the following isomorphisms:

$$\mathcal{H}_0(X) \cong \mathcal{H}^3(X, \partial X) \cong \mathbf{Z}$$

and

$$\mathcal{H}_0(X, \partial X) \cong \mathcal{H}^3(X) \cong 0.$$

Since there are no torsion phenomena involved, the *universal coefficient theorem for cohomology* gives us a perfect pairing between homology and cohomology groups, resulting in the following isomorphism:

$$\mathcal{H}^i(X) \cong \mathcal{H}_i(X)\,.$$

We also have the corresponding result for relative homology and cohomology:

$$\mathcal{H}^i(X, \partial X) \cong \mathcal{H}_i(X, \partial X)\,.$$

We summarise the results of this section in the following lemma.

Lemma 1.6.1. *Let X be the space obtained by taking the complement in $\mathbf{S}^3$ of open non-intersecting tubular neighbourhoods of a link L with k components. We then have the following homology and cohomology groups:*

$$\begin{aligned}
&\mathcal{H}_0(X) \cong \mathcal{H}^3(X, \partial X) \cong \mathcal{H}_3(X, \partial X) \cong \mathcal{H}^0(X) \cong \mathbf{Z},\\
&\mathcal{H}^1(X) \cong \mathcal{H}_2(X, \partial X) \cong \mathcal{H}^2(X, \partial X) \cong \mathcal{H}_1(X) \cong \mathbf{Z}^k,\\
&\mathcal{H}^2(X) \cong \mathcal{H}_1(X, \partial X) \cong \mathcal{H}^1(X, \partial X) \cong \mathcal{H}_2(X) \cong \mathbf{Z}^{k-1},\\
&\mathcal{H}_0(X, \partial X) \cong \mathcal{H}_3(X) \cong \mathcal{H}^3(X) \cong \mathcal{H}^0(X, \partial X) \cong \mathbf{0}.
\end{aligned}$$

Proof. The seeding, leftmost information is described above.

The first and third isomorphisms in each row come from Poincaré–Lefschetz duality and the second isomorphism in each row comes from a universal coefficients theorem. □

In particular, since X is a compact 3-manifold and $\mathcal{H}_3(X, \partial X) \cong \mathbf{Z}$, X is orientable.

Some generators

There is a particularly nice way of choosing a basis for $\mathcal{H}_1(X, \partial X)$, which we will now describe.

To do this, we take some ordering $L_1, \ldots, L_k$ of the k link components and denote the fattened up versions of these by $\mathbf{L}_1, \ldots, \mathbf{L}_k$. We then denote by μ_{ij} the relative homology class represented by some embedded oriented compact 1-submanifold whose two boundary components lie on ∂X, the first on $\partial \mathbf{L}_i$ and the second on $\partial \mathbf{L}_j$.

The orientation is such that the orientation on the first boundary component is negative and on the second boundary component is positive. In other words, we have a path from $\mathbf{L}_i$ to $\mathbf{L}_j$.

We have the relation $\mu_{ij} = -\mu_{ji} \in \mathcal{H}_1(X, \partial X)$, corresponding to reversing the orientation of the representatives.

From these, the choice of generating set that we will on occasion refer to is $\{\mu_{1j}\}_{j \in \{2,\ldots,k\}}$, which indeed has cardinality $(k-1)$.

We can easily express the other μ_{ij} in terms of these. Indeed, let i be greater than or equal to j, then we have the following relation:

$$\mu_{ij} = \mu_{1j} - \mu_{1i}.$$

1.6.2 Cup product structure

The cup product structure of a link complement can be computed using the result on cup product in terms of linking numbers of cycles discussed in Section 1.5.

Below we will do this computation for the Borromean rings and the Brunnian 3-link, resting assured that the other links we consider will have pairwise zero linking numbers for similar reasons.

1.6.3 Borromean rings

From the discussion above, we know additive cohomology of the space X obtained from the Borromean rings. The content of this subsection is to determine the multiplicative structure, more specifically: making sure that all cup products of 1-cochains vanish, enabling us to define the triple Massey product for these.

Referring to Figure 1.1, we pick a pair of components and choose arbitrary orientations on each of them. There are only two crossings and we immediately see that they must have opposite signs and that the choice of orientations will not change this. We conclude that the linking numbers are zero, hence the cup products are trivial.

1.6.4 Brunnian 3-link

By the remarks in this section, we know the cohomology groups of the complement of the Brunnian 3-link are the same as those of the 3-unlink and the Borromean rings.

The cup product structure is easily calculated by using the linking number interpretation of the cup product. Again, by symmetry considerations it suffices to look at any single pair of link components. Perusing Figure 3.5, we realise that we have to find the signs at four crossings and that these signs are pairwise opposite, so they cancel. Furthermore, they are independent of choices of orientations.

Since the linking numbers are zero, the relevant cup products are trivial.

2 Massey products

In this chapter, we will introduce the Massey product of triples of cohomology classes, with the aim of showing that it detects the non-trivial linking inherent in the Borromean rings and Brunnian 3-link.

It will, however, turn out the Massey products for the Brunnian 3-link are trivial. This is surprising and warrants further investigation into what kind of linking it *does detect*, this is done using the Massey product for de Rham cohomology, which in this case is computationally simpler than the one for singular cohomology.

The Massey product was first introduced in 1957 by Massey and Uehara in [UM57]. In a later article [Mas69], Massey gives interpretations of these products in terms of linking numbers of spheres.

The work in this section can be carried out in the cohomology ring $\mathcal{H}^*(\Gamma)$ of any associative differential graded algebra Γ, where we denote our product, the cup product, by $\smile$.

By abuse of notation, we will also denote the induced product in cohomology by $\smile$.

An advantage with taking this somewhat general starting point is that it makes it possible to discuss the Massey product for other chain complexes with an associative product, such as the relative singular cochain complex or the de Rham complex of differential forms.

The Massey product $\langle u, v, w\rangle$ of cohomology classes $u \in \mathcal{H}^p(\Gamma)$, $v \in \mathcal{H}^q(\Gamma)$ and $w \in \mathcal{H}^r(\Gamma)$ is defined when the cup products $u \smile v$ and $v \smile w$ vanish. When defined, it is an element in the quotient group

$$\mathcal{H}^{p+q+r-1}(\Gamma) \,/\, \big(u \smile \mathcal{H}^{q+r-1}(\Gamma) + \mathcal{H}^{p+q-1}(\Gamma) \smile w\big)\,.$$

2.1 Definition

In this section we give a first attempt on a definition of the triple product, check whether or not it is well-defined, modify it slightly and in the end give a valid definition.

At this stage it will be useful to introduce some notation. If u is a homogeneous cohomology class, then define $\widetilde{u}$ by $\widetilde{u} = (-1)^{\deg u} u$. We use the usual notation for equivalence classes, namely $[a]$ is the equivalence class containing a. Furthermore, u' will be a generic representative of u, so that $[u'] = u$.

First, recall that if a cohomology class is zero, then, on the cochain level, it is the coboundary of some cochain.

In the case where $u \smile v$ and $v \smile w$ vanish, there exist cochains a and b such that the following relations are satisfied:

$$u' \smile v' = \delta a \tag{2.1}$$

$$v' \smile w' = \delta b. \tag{2.2}$$

Note that u', v' and w' are cochains representing the cohomology classes u, v and w, in accordance with the notation introduced above.

Continuing to work on the level of cochains, we define a new cochain

$$z' = a \smile w' - \widetilde{u'} \smile b, \tag{2.3}$$

whose class in cohomology will be the initial attempt of a definition of the triple product. To somewhat anticipate where we will be headed, I will note here that the product will *not* be well-defined without a modification.

2.1.1 Closedness

We need to check that this cochain descends to cohomology, that is, that $\delta z' = 0$. This is a purely mechanical task, using the linearity of the differential, Leibniz' rule and the relations above, but we include it here for completeness.

$$\begin{aligned}
\delta z' &= \delta \left(a \smile w' - \widetilde{u'} \smile b \right) \\
&= \delta \left(a \smile w' \right) - \delta \left(\widetilde{u'} \smile b \right) \\
&= \left(\delta a \smile w' - \widetilde{a} \smile \delta w' \right) - \left(\delta \widetilde{u'} \smile b - u' \smile \delta b \right) \\
&= \left(u' \smile v' \right) \smile w' - u' \smile \left(v' \smile w' \right) \\
&= 0
\end{aligned}$$

The reason why $\delta w'$ and $\delta u'$ are zero is that they represent cohomology classes, so they are closed.

The last equality comes form associativity of the cup product on the cochain level, shown in Section 1.2.

2.1.2 Indeterminacy

The next thing to check is that the product is well-defined on the level of cohomology, that is, independent of any choices made on the cochain level

Normally, this would mean taking different representatives in the definition and checking that the product defined in terms of different representatives coincide in cohomology.

This is not the case for the Massey product. Comparing the products of different representatives, we will find that they differ by cochains that descend to potentially non-trivial cohomology classes. We will, however, see that we have enough control over the *indeterminacies* for the product to be useful.

We will do this in three steps:

Step 1 We add a coboundary to each of u', v' and w', find cochains a' and b' satisfying Equation 2.1 and Equation 2.2 for the altered cup products and compute the difference of the original product with the new one.

Step 2 We show that if we add a cocycle to each of a and b it may result in a non-cohomologous class.

Step 3 We show that the indeterminacies have certain nice properties.

We will make repeated use of the following two forms of the Leibniz rule:

$$\delta(a \smile b) = \delta(a) \smile b + \widetilde{a} \smile \delta(b)$$
$$\delta(a) \smile \delta(b) = \delta(a \smile \delta(b)) = -\delta(\delta\widetilde{a} \smile b).$$

Step 1

We begin by changing u', v', w' as follows:

$$\begin{aligned} u' &\mapsto u' + \delta u_2 \\ v' &\mapsto v' + \delta v_2 \\ w' &\mapsto w' + \delta w_2 \end{aligned}$$

There are cochains bounding the cup products $(u' + \delta u_2) \smile (v' + \delta v_2)$ and $(v' + \delta v_2) \smile (w' + \delta w_2)$. These cochains will differ from a and b by more than a coboundary, as shown by the following computations:

$$\begin{aligned} (u' + \delta u_2) \smile (v' + \delta v_2) &= u' \smile v' + \delta u_2 \smile v' + u' \smile \delta v_2 + \delta u_2 \smile \delta v_2 \\ &= \delta a + \delta(u_2 \smile v') - \widetilde{u_2} \smile \delta v' \\ &\quad + (\delta(\widetilde{u'} \smile v_2) - \delta\widetilde{u'} \smile v_2) + \delta(u_2 \smile \delta(v_2)) \\ &= \delta(a + (u_2 \smile v' + \widetilde{u'} \smile v_2 + u_2 \smile \delta(v_2))) \end{aligned}$$

So we let $a \mapsto a + a_1$, with

$$a_1 = u_2 \smile v' + \widetilde{u'} \smile v_2 + u_2 \smile \delta v_2.$$

Similarly, we let $b \mapsto b + b_1$, with

$$b_1 = v_2 \smile w' + \widetilde{v'} \smile w_2 + (-1)^{\deg v'} \delta v_2 \smile w_2.$$

To sum this up:

$$(u' + \delta u_2) \smile (v' + \delta v_2) = \delta(a + a_1), \tag{2.4}$$
$$(v' + \delta v_2) \smile (w' + \delta w_2) = \delta(b + b_1). \tag{2.5}$$

We can do a considerable amount of simplification before needing the exact form of a_1 and b_1.

Inserting this into the definition of z', we get the following:

$$\begin{aligned} z'' &= (a + a_1) \smile (w' + \delta w_2) - (\widetilde{u' + \delta u_2}) \smile (b + b_1) \\ &= a \smile w' + a_1 \smile w' + (a + a_1) \smile \delta w_2 \\ &\quad - \left(\widetilde{\delta u_2} \smile (b + b_1) + \widetilde{u'} \smile b + \widetilde{u'} \smile b_1\right), \end{aligned}$$

by distributivity.
Next, we use the definition of z' and Leibniz' rule on the two terms $(a+a_1) \smile \delta w_2$ and $\widetilde{\delta u_2} \smile (b+b_1)$ and cancel some signs to get:

$$
\begin{aligned}
z'' = z' &- \widetilde{u'} \smile b_1 + a_1 \smile w' \\
&+ (\delta((\widetilde{a+a_1}) \smile w_2) - \delta(\widetilde{a+a_1}) \smile w_2) \\
&+ (\delta(\widetilde{u_2} \smile (b+b_1)) - u_2 \smile \delta(b+b_1)).
\end{aligned}
$$

We proceed by using Equations (2.4) and (2.5) and collecting terms:

$$
\begin{aligned}
z'' - z' = &-\widetilde{u'} \smile b_1 + a_1 \smile w' \\
&+ (\widetilde{u' + \delta u_2}) \smile (\widetilde{v' + \delta v_2}) \smile w_2 \\
&- u_2 \smile (v' + \delta v_2) \smile (w' + \delta w_2) \\
&+ \delta\left((\widetilde{a+a_1}) \smile w_2 + \widetilde{u_2} \smile (b+b_1)\right).
\end{aligned}
$$

Here it becomes worthwhile to introduce some auxiliary variables to help highlight where something is actually happening, so let

$$c_1' = -(-1)^{\deg u'} b_1$$

and

$$c_3' = (\widetilde{a+a_1}) \smile w_2 + \widetilde{u_2} \smile (b+b_1).$$

With these placeholders, strategically expanding some of the parentheses gives the following:

$$
\begin{aligned}
z'' - z' = u' &\smile c_1' + a_1 \smile w' + \delta c_3' \\
&+ \left(\widetilde{u'} \smile (\widetilde{v' + \delta v_2}) \smile w_2 + \widetilde{\delta u_2} \smile (\widetilde{v' + \delta v_2}) \smile w_2\right) \\
&- (u_2 \smile (v' + \delta v_2) \smile w' + u_2 \smile (v' + \delta v_2) \smile \delta w_2)\,.
\end{aligned}
$$

From which inserting the definition of a_1 and b_1, introducing

$$c_1'' = c_1' + (-1)^{\deg u'} (\widetilde{v' + \delta v_2}) \smile w_2$$

and

$$c_2' = a_1 - u_2 \smile (v' + \delta v_2),$$

and using the (triple) Leibniz' rule

$$\begin{aligned}\delta u_2 \smile (v' + \delta v_2) \smile w_2 = \delta(u_2 \smile (v' + \delta v_2) \smile w_2) \\ - \widetilde{u_2} \smile \delta(v' + \delta v_2) \smile w_2 \\ - \widetilde{u_2} \smile (\widetilde{v' + \delta v_2}) \smile \delta w_2,\end{aligned}$$

where the second term vanishes, gives

$$\begin{aligned}z'' - z' &= u' \smile c_1'' + c_2' \smile w' + \delta c_3' \\ &\quad + (-1)^{\deg u' + \deg v'} \big[\delta(u_2 \smile (v' + \delta v_2) \smile w_2) \\ &\quad - \widetilde{u_2} \smile (\widetilde{v' + \delta v_2}) \smile \delta w_2\big] - u_2 \smile (v' + \delta v_2) \smile \delta w_2.\end{aligned}$$

At this stage, we notice that

$$-(-1)^{\deg u' + \deg v'} \cdot (-1)^{\deg u_2 + \deg v'} = +1$$

and introduce yet another auxiliary variable:

$$c_3 = c_3' + (-1)^{\deg u' + \deg v'} \delta(u_2 \smile (v' + \delta v_2) \smile w_2),$$

to get

$$\begin{aligned}z'' - z' &= u' \smile c_1'' + c_2' \smile w' + \delta c_3 \\ &\quad + u_2 \smile (v' + \delta v_2) \smile \delta w_2 - u_2 \smile (v' + \delta v_2) \smile \delta w_2 \\ &= u' \smile c_1'' + c_2' \smile w' + \delta c_3\end{aligned}$$

For reasons we will see in a moment, we alter c_1'' and c_2' slightly by doing the following

$$c_1 = c_1'' + (-1)^{\deg u'} v_2 \smile w'$$

$$c_2 = c_2' - (-1)^{\deg u'} u' \smile v_2,$$

which amounts to adding and subtracting the same term in $z'' - z'$. Cancelling terms and using these variables, we get the following equation relating z'' and z':

$$z'' - z' = u' \smile c_1 + c_2 \smile w' + \delta c_3$$

where, to sum up, letting $\varepsilon = (-1)^{\deg u'}$, we have

$$\begin{aligned}
c_1 &= \varepsilon \left(-b_1 + (\widetilde{v' + \delta v_2}) \smile w_2 + v_2 \smile w' \right) \\
&= -\varepsilon \left(v_2 \smile w' + \widetilde{v'} \smile w_2 + (-1)^{\deg v'} \delta v_2 \smile w_2 \right) \\
&\quad + \varepsilon \left((\widetilde{v' + \delta v_2}) \smile w_2 + v_2 \smile w' \right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
c_2 &= a_1 - u_2 \smile (v' + \delta v_2) - \varepsilon u' \smile v_2 \\
&= \left(u_2 \smile v' + \widetilde{u'} \smile v_2 + u_2 \smile \delta v_2 \right) \\
&\quad - (u_2 \smile (v' + \delta v_2) + \varepsilon u' \smile v_2) \\
&= 0.
\end{aligned}$$

The above implies that z' and z'' differ only by a coboundary, we conclude that the indeterminacy does *not* arise from choices of representatives of the factors of the product.

Step 2

We do the following transformations and show that the difference between the original and the altered product results in an indeterminacy in cohomology:

$$\begin{aligned}
a &\mapsto a + a_2 \\
b &\mapsto b + b_2,
\end{aligned}$$

where a_2 and b_2 are *cocycles*. This allows us to define the altered product:

$$z''' = (a + a_2) \smile w' - \widetilde{u'} \smile (b + b_2) \tag{2.6}$$

In what follows, we compare z' and z''':

$$\begin{aligned}
z''' - z' &= (a + a_2) \smile w' - \widetilde{u'} \smile (b + b_2) - \left[a \smile w' - \widetilde{u'} \smile b \right] && (2.7) \\
&= a_2 \smile w' - u' \smile b_2. && (2.8)
\end{aligned}$$

Since z' and z''' differ by a cochain which is not necessarily a coboundary their difference is in general not zero in cohomology.

From this we see that the proposed Massey product is indeed not necessarily well-defined.

Note. In Chapter 3 we will see geometric examples of why and how the indeterminacy stems from the choices of a and b.

Step 3

We can resolve this problem by taking the quotient of the cohomology ring by the ideal generated by the possible indeterminacies $u \smile \mathcal{H}^*(\Gamma)$ and $\mathcal{H}^*(\Gamma) \smile w$, then we *will* have a well-defined object: a "coset of the quotient group" in the appropriate degree.

Following [UM57], we define

$$\mathcal{J}^*(u, w) = u \smile \mathcal{H}^*(\Gamma) + \mathcal{H}^*(\Gamma) \smile w, \tag{2.9}$$

giving a graded ideal in the cohomology ring. The fact that this is actually an ideal follows from the commutativity of the cup product on cohomology. We also define the (additive) group $\mathcal{J}^n(u, w)$ as the direct summand of $\mathcal{J}^*(u, w)$ containing the homogeneous elements of degree n:

$$\mathcal{J}^n(u, w) = u \smile \mathcal{H}^{n-p}(\Gamma) + \mathcal{H}^{n-r}(\Gamma) \smile w.$$

We want to show that the two indeterminacies $u' \smile b_2$ and $a_2 \smile w'$ in Equation 2.8 land in $\mathcal{J}^{p+q+r-1}(u, w)$.

Since a_2 and b_2 are cocycles they descend to cohomology. Then, by general properties of the cup product, we get the classes we want:

$$[u' \smile a_2] = u \smile [a_2] \in u \smile \mathcal{H}^{q+r-1}(\Gamma)$$

and

$$[b_2 \smile w'] = [b_2] \smile w \in \mathcal{H}^{p+q-1}(\Gamma) \smile w.$$

From this we can finally give a precise definition of the Massey product.

Definition 2.1.1. The *Massey product*

$$\langle u, v, w \rangle \in \mathcal{H}^{p+q+r-1}(\Gamma) \,/\, \mathcal{J}^{p+q+r-1}(u, w)$$

of cohomology classes $u \in \mathcal{H}^p(\Gamma)$, $v \in \mathcal{H}^q(\Gamma)$ and $w \in \mathcal{H}^r(\Gamma)$ is the coset of the cohomology class represented by $z' = a \smile w' - \widetilde{u'} \smile b$ in the quotient of the cohomology group $\mathcal{H}^{p+q+r-1}(\Gamma)$ by the subgroup $\mathcal{J}^{p+q+r-1}(u, w)$. ⌟

If the indeterminacy is trivial, for instance if *all* cup products vanish, then the product is *strictly defined.*

We will say that the product *contains zero* if there are choices of a and b in z' above such that z' represents the zero class in $\mathcal{H}^*(\Gamma)$. Furthermore, we say that the product *is zero* if it is strictly defined and contains zero.

2.2 Key properties of the triple product

In the original article of Uehara and Massey, they also state the following two properties of the Massey product, leaving proofs to the interested reader.

Throughout this section, we let $f\colon \Gamma_1 \to \Gamma_2$ be a homomorphism of differential graded associative algebras. We also assume that for $u, v, w \in \mathcal{H}^*(\Gamma_1)$, we have $u \smile v = 0 = v \smile w$ so that the Massey product $\langle u, v, w\rangle$ is defined.

2.2.1 Naturality

Naturality means something slightly different from usual in this case.

Proposition 2.2.1. *If u, v, w are homogeneous classes in $\mathcal{H}^*(\Gamma_1)$ then the following inclusion holds:*

$$f^*(\langle u, v, w\rangle) \subset \langle f^*(u), f^*(v), f^*(w)\rangle.$$

Proof. We will first check that the indeterminacy ideals are compatible:

$$\begin{aligned} f^*\left(\mathcal{J}^*(u,w)\right) &= f^*\left(u \smile \mathcal{H}^*(\Gamma_1) + \mathcal{H}^*(\Gamma_1) \smile w\right) \\ &= f^*(u) \smile f^*\left(\mathcal{H}^*(\Gamma_1)\right) + f^*\left(\mathcal{H}^*(\Gamma_1)\right) \smile f^*(w) \\ &\subset f^*(u) \smile \mathcal{H}^*(\Gamma_2) + \mathcal{H}^*(\Gamma_2) \smile f^*(w) \\ &= \mathcal{J}^*\left(f^*(u), f^*(w)\right). \end{aligned}$$

This ensures that $f^*\colon \mathcal{H}^*(\Gamma_1) \to \mathcal{H}^*(\Gamma_2)$ descends to a map

$$f^*\colon \mathcal{H}^*(\Gamma_1) \big/ \mathcal{J}^*(u,w) \to \mathcal{H}^*(\Gamma_2) \big/ \mathcal{J}^*\left(f^*(u), f^*(w)\right).$$

We now look at what happens on the cochain level, making repeated use of the fact that f is a differential graded associative algebra homomorphism, that is, it is a an algebra homomorphism and it commutes with the differential δ.

Let $u', v', w' \in \Gamma_1$ represent homogeneous cohomology classes $u, v, w \in \mathcal{H}^*(\Gamma_1)$. Then the Massey product $\langle u, v, w \rangle$ is represented by $z'_1 = a \smile w' - \widetilde{u'} \smile b$ with a and b cochains so that $\delta_1 a = u \smile v$ and $\delta_2 b = v \smile w$.

Similarly, the Massey product $\langle f^*(u), f^*(v), f^*(w) \rangle$ is represented by $z'_2 = a_2 \smile f^*(w)' - \widetilde{f^*(u)'} \smile b_2$ with $\delta_2 a_2 = f^*(u)' \smile f^*(v)'$ and $\delta_2 b_2 = f^*(v)' \smile f^*(w)'$. Here, the relation between a_2 and a is found as follows: $f(\delta_1 a) = f(u' \smile v') = f(u') \smile f(v')$ and $f(\delta_1 a) = \delta_2(f(a))$, so $\delta_2(f(a)) = f(u') \smile f(v')$, which means that we can choose $a_2 = f(a)$.

The corresponding computation for b and b_2 is completely analogous.

We also get the following sequence of equalities:

$$\begin{aligned} f(z'_1) &= f(a \smile w' - \widetilde{u'} \smile b) \\ &= f(a \smile w') - f(\widetilde{u'} \smile b) \\ &= f(a) \smile f(w') - f(\widetilde{u'}) \smile f(b) \\ &= a_2 \smile f(w') - f(\widetilde{u'}) \smile b_2 = z'_2 \end{aligned}$$

so for the given choices of a, b, a_2 and b_2, $f(z'_1)$ agrees with z'_2.

Since the original choices of a and b were arbitrary, this means that we can always find a_2 and b_2 so that the above holds.

The reason that we get an inclusion and not an equality is that whereas we can find a_2 corresponding to each a, we cannot guarantee that there is an a for each a_2.

This implies that

$$f^*(\langle u, v, w \rangle) \subset \langle f^*(u), f^*(v), f^*(w) \rangle,$$

which is what we wanted to show. □

2.2.2 Homotopy invariance

Homotopy invariance of the Massey product is in fact easy to show given naturality and the proof of naturality presented above.

Proposition 2.2.2. *If $f\colon \Gamma_1 \to \Gamma_2$ is a quasi-isomorphism and u, v, w homogeneous classes in $\mathcal{H}^*(Y)$, then f^* is a ring isomorphism and*

$$f^*(\langle u, v, w\rangle) \cong \langle f^*(u), f^*(v), f^*(w)\rangle.$$

Proof. First, if f^* is an isomorphism the indeterminacy ideals are isomorphic. Also, with the notation as in the proof above, the inverse element of a_2 under the isomorphism correspond to some element which satisfy the same equation as a and similarly for b and b_2. This implies that there is a bijective correspondence of sets, giving

$$\begin{aligned} f^*(\langle u, v, w\rangle) &= \langle f^*(u), f^*(v), f^*(w)\rangle \\ &\cong \langle u, v, w\rangle, \end{aligned}$$

as required. □

The upshot to this discussion is that we can calculate the Massey product using any reasonable construction of cochains complexes and cohomology that we would like. In particular, in the next chapter we will use computations of Massey products in de Rham cohomology to make conclusions about Massey products in singular cohomology with real coefficients.

2.3 Borromean rings computation

Whenever we use it in formulae, we will denote by $\mathscr{B}$ the Borromean rings, possibly embedded in some manifold. To compute the Massey product of the generators of $\mathcal{H}^1(\mathbf{S}^3 \setminus \mathscr{B})$ we follow Massey's article [Mas69] on "Higher order linking numbers" and finish the details left to the reader.

At the time when the article was published the Alexander polynomial of knots and links was the only polynomial invariant, and the Alexander polynomial of the Borromean rings is trivial, so being able to detect the non-triviality of this link using cohomology was an achievement.

2.3.1 Preliminaries

Before initiating the computation, Massey states a theorem helpful in computing the Massey product, we reproduce it here after have introduced the necessary notation and at the same time repeat some of the theory of Section 1.6. Let S_1, S_2 and S_3 be oriented spheres of dimensions d_1, d_2 and d_3 embedded disjointly in $\mathbf{S}^n$ so that the dimensions satisfy

$$1 \leq d_i < n - 2$$

and

$$d_1 + d_2 + d_3 = 2n - 3.$$

Furthermore, we let $X' = S^n \setminus (S_1 \cup S_2 \cup S_3)$ be the complement of the link and denote the Alexander dual of S_i in $\mathcal{H}^{n-d_i-1}(X')$ by w_i.

Note. A three-link embedded in $\mathbf{S}^3$ satisfies all dimension conditions and gives Alexander duals in the first cohomology group of the complement. This means that the Massey product of the w_i's will not be trivial for degree reasons, indeed it will land in the second cohomology group of the complement, which we have seen is non-trivial.

From here on, we diverge slightly from Massey exposition, by taking all embedded spheres to be one-dimensional, in other words, we consider only ordinary links.

Similarly to what we did in Section 1.6, Massey introduces the manifold-with-boundary M which is the complement of regular neighbourhoods of the embedded link. For consistency of notation we will denote it by X and its boundary by ∂X. In the same way as described earlier, X is a deformation retract of X', so their cohomologies are naturally isomorphic.

The last piece of notation needed is the morphism $g^* \colon \mathcal{H}^2(X) \to \mathcal{H}^2(\partial X)$ which fits into the end part of the cohomology exact sequence of the pair:

$$\mathcal{H}^2(X) \to \mathcal{H}^2(\partial X) \to \mathcal{H}^3(X, \partial X) \to 0.$$

From the computations in Section 1.6, we know that these groups are free abelian of ranks two, three and one, respectively; hence g^* must be a monomorphism.

Since we consider X as an oriented manifold, each of its boundary components will also inherit an orientation. We denote these orientation classes by μ_i, for $i \in \{1, 2, 3\}$. This set is a natural basis for $\mathcal{H}^2(\partial X)$.

Since g^* is a monomorphism the elements of its domain are uniquely determined by their image.

This brings us the theorem indicated above, which we quote verbatim.

Theorem 2.3.1. *Let* (i, j, k) *be any permutation of the integers* $(1, 2, 3)$. *Then there exists an integer* m_{ik} *such that*

$$g^*(w_i, w_j, w_k) = m_{ik}(\mu_i - \mu_k).$$

We also state a corollary describing the relation between these m_{ik}'s. For ease of notation, we first define $q_i = n - d_1 - 1$.

Corollary. *The six integers* m_{ik} *thus obtained are all equal in absolute value; to be precise,*

$$(-1)^{q_i q_k} m_{ik} = (-1)^{q_j q_i} m_{ji} = (-1)^{q_k q_j} m_{kj},$$
$$m_{ik} = (-1)^{q_i q_j + q_j q_k + q_k q_i} m_{ki}.$$

Note. In the cases we are considering, all q_i are equal to 1, so all signs end up being -1.

2.3.2 The computation

We now want to do the computation of the singular Massey product of Alexander duals of the link components of the Borromean rings, shown in Figure 1.1.

Massey suggests doing this using duality and intersection theory, and suggest taking what he calls "singular disks" to be cones whose apices are at ∞, regarding $\mathbf{S}^3$ as the one-point compactification of $\mathbf{R}^3$, and whose bases lie on the boundaries of each of the link components.

We number the link components by L_i and denote by D_i a singular disk with L_i as its boundary, as indicated in Figure 2.1. These disks represent relative cycles $[D_i] \in \mathcal{H}_2(\mathbf{S}^3, \mathscr{B})$, from which we get cohomology classes α_i in $\mathcal{H}^1(\mathbf{S}^3 \setminus \mathscr{B})$ by Poincaré–Lefschetz duality.

These are the classes whose Massey product we wish to compute, we state the result of the computation as a proposition.

Proposition 2.3.2. *With the classes* α_i *as above, the triple Massey product* $\langle \alpha_1, \alpha_2, \alpha \rangle \in \mathcal{H}^2(\mathbf{S}^3 \setminus \mathscr{B})$ *is non-trivial.*

Proof. Let a_i be a representative of α_i. By Section 1.6, the cup products $\alpha_1 \smile \alpha_2$ and $\alpha_2 \smile \alpha_3$ vanish, so there exist cochains a and b such $\delta a = a_1 \smile a_2$ and $\delta b = a_2 \smile a_3$.

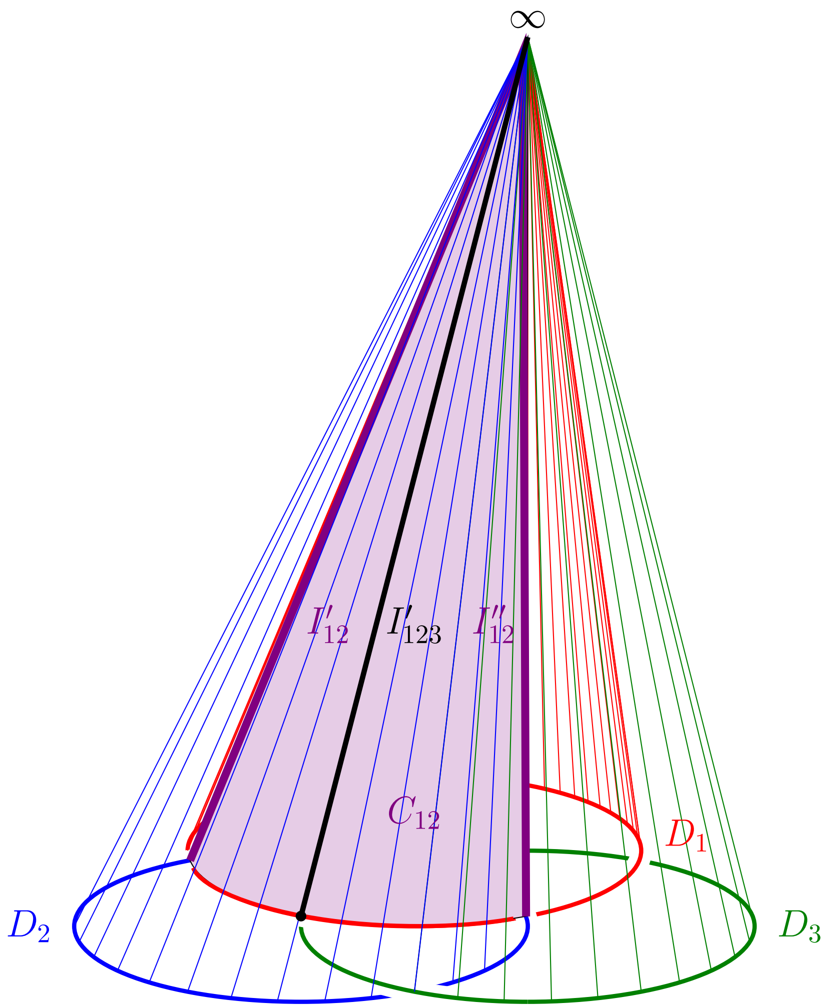


Figure 2.1: Borromean rings for singular Massey computation

Recall that the cochain z' defining the Massey product is given by $z' = a \smile a_3 - \widetilde{a_1} \smile b$. By the intersection theory, $a_1 \smile a_2$ and $a_2 \smile a_3$ correspond to intersections

$$[D_1] \cap [D_2] = [D_1 \cap D_2] = [I_{12}]$$

and

$$[D_2] \cap [D_3] = [D_2 \cap D_3] = [I_{23}]$$

of singular disks.

From here on, we will focus on only one of the terms, the one corresponding to $I_{12} = I'_{12} \cup I''_{12}$. In cohomology, we want to find the cochain a described above, this corresponds to finding a relative class $[C_{12}]$ in $\mathcal{H}_2\big(\mathbf{S}^3, \mathscr{B}\big)$ whose boundary is $[I_{12}]$. In Figure 2.1, this is the area filled with light purple.

We then need to take the intersection product corresponding to the cup product $a \smile a_3$, this is

$$[C_{12}] \cap [D_3] = [C_{12} \cap D_3] = I_{123} = I'_{123} \cup I''_{12},$$

in the figure, this is the union of the black line and the purple line to the right, joining the first and second component of the Borromean rings.

The other term would contribute a term joining the second and third components of the rings, and the middle lines then cancel. This results in a line joining the first and third component of rings. This interval represent the non-zero class μ_{13} in $\mathcal{H}_1\big(\mathbf{S}^3 \setminus \mathscr{B}\big)$, by Poincaré duality, this gives a non-zero class in $\mathcal{H}^2\big(\mathbf{S}^3 \setminus \mathscr{B}\big)$, so the wanted Massey product is non-zero.

In Massey's terminology, this gives $m_{ik} = \pm 1$. □

Note. We will discuss in detail the indeterminacy of the Massey product for the Borromean rings in Section 3.2.

Note. The technology used by Massey differs from the modern one that we will use in the next chapter, where he uses singular disks we will use embedded compact 2-submanifolds.

2.4 Generalisations

There is an extension of the triple product to products of higher *arity*. These are called higher order Massey products and are defined only when certain lower order products vanish.

In this section we will describe these higher products, with some emphasis on the fourfold product, by giving the conditions for when they are defined and the formulae for computing them. All proofs are omitted.

We will avoid explicitly stating the expressions for the indeterminacies of the higher products, since these are ghastly.

Remark. Another generalisation is the *Matric Massey products* of May, introduced in [May69]. This is a different direction of extension of the Massey product than the one we wish to pursue, so we will not go into any greater detail. Examples of calculations of these products in link complements are given in [O'N79].

2.4.1 Fourfold Massey products

We want to define $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$, for α_i in $\mathcal{H}^*(\Gamma)$. This is possible when the two triple products $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$ both contain zero. This is the same type of condition as the vanishing of cup products for triple products.

For reasons that will become clearer in Subsection 2.4.2 on n-fold products we introduce and use a piece of convenient, though seemingly cumbersome, notation inspired by May's article "Matric Massey products" [May69].

Denote by $a_{i-1,i}$ a representative for α_i. Since the triple product is defined, the relevant cup products in cohomology must vanish, and we denote by $a_{i-1,i+1}$ some cochain satisfying $\delta(a_{i-1,i+1}) = \widetilde{a_{i-1,i}} \smile a_{i,i+1}$. With this notation, we can define the triple product as the coset of cohomology classes represented by cochains of the form

$$b_{i-1,i+2} = \widetilde{a_{i-1,i+1}} \smile a_{i+1,i+2} + \widetilde{a_{i-1,i}} \smile a_{i,i+2}.$$

Furthermore, since $\langle \alpha_i, \alpha_{i+1}, \alpha_{i+2} \rangle$ contains zero we can make choices in such a way that we can find a cochain $a_{i-1,i+2}$ satisfying $\delta(a_{i-1,i+2}) = b_{i-1,i+2}$. Then we arrive at the definition of the fourfold product.

Definition 2.4.1. With notation as above, the *fourfold Massey product* $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \in \mathcal{H}^*(\Gamma)$ is defined when $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$ contains zero, and is then represented by the cocycle

$$b_{0,4} = \widetilde{a_{0,3}} \smile a_{3,4} + \widetilde{a_{0,2}} \smile a_{2,4} + \widetilde{a_{0,1}} \smile a_{1,4} \in \Gamma.$$

We say that the product is *strictly defined* if the two triple products have trivial indeterminacies, i.e., they contain *only* zero. ⌟

We will give an example of the computation of a fourfold product in Section 3.5 and Section 3.6.

In the literature I have only managed to find one example of a computation of a non-trivial fourfold Massey product, it is found in the article [O'N79] by Edward J. O'Neill, a doctoral student of Massey's.

Note. In the cases we are interested in, products of degree 1 cohomology classes, the fourfold product will have degree 2.

2.4.2 n-fold Massey products

Despite having seen the formula for the fourfold Massey product, it is not obvious how to extend this to higher products. Defining these will perhaps also make it clearer how the fourfold product is related to the usual Massey product as well as how these fit into a greater framework.

In this subsection, we will give the definition of higher Massey products, as well as necessary and sufficient condition for their being well-defined.

In line with the notation for the triple and fourfold products, we denote the n-fold Massey product of cohomology classes $\alpha_i \in \mathcal{H}^{j_i}(\Gamma)$ by $\langle \alpha_1, \ldots, \alpha_n \rangle$.

Definition 2.4.2. A *defining system* for the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is a collection of cochains $a_{i,j} \in \Gamma$ defined iteratively by the following relations:

$$[a_{i-1,i}] = \alpha_i \tag{2.10}$$

$$\delta(a_{i,j}) = \sum_{i<k<j} \widetilde{a_{0,k}} \smile a_{k,n}, \tag{2.11}$$

where the indices $i \leq j$ runs from 0 through n, except $(i, j) = (0, n)$. ⌟

Note that the sum in Equation 2.10 is empty if $i - j < 2$ holds, which is exactly the case for Equation 2.11. It is in fact consistent even in this case, since then we only get that $a_{i-1,i}$ is a cocycle.

We are now in a position to give the definition.

Definition 2.4.3. The *n-fold Massey product* $\langle \alpha_1, \ldots, \alpha_n \rangle$ of cohomology classes $\alpha_i \in \mathcal{H}^{j_i}(\Gamma)$ is defined when the two $(n-1)$-fold Massey products $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ and $\langle \alpha_1, \ldots, \alpha_n \rangle$ vanish simultaneously. When defined, it the set of classes in $\mathcal{H}^*(\Gamma)$ of the form

$$a_{0,n} = \sum_{0<k<n} \widetilde{a_{0,k}} \smile a_{k,n} \in \Gamma.$$

⌟

Note. As with the fourfold product, n-fold products of degree 1 cohomology classes will have degree 2.

2.5 Context

The triple Massey product is an example of a *higher order cohomology operations*, more specifically, a *secondary cohomology operations.*

To define these, we first need to define ordinary, or primary, *cohomology operations.*

2.5.1 Primary cohomology operations

Definition 2.5.1. Let G_1 and G_2 groups. A *cohomology operation* θ is a natural transformation

$$\theta \colon \mathcal{H}^p(\bullet; G_1) \to \mathcal{H}^q(\bullet; G_2)\,,$$

for some $p, q \in \mathbf{Z}$. ⌟

The most familiar example is the cup product squaring of a cohomology class, which can be seen as an operation $\bullet \smile \bullet \colon \mathcal{H}^p(X; R) \to \mathcal{H}^{2p}(X; R)$ given by $u \mapsto u \smile u$, with $\deg u = p$, for each $p \in \mathbf{Z}$.

Steenrod squares

Another family of cohomology operations are the *Steenrod squares* and *Steenrod reduced p^{th} powers.* These are examples of *stable* cohomology operations, meaning that they commute with the suspension isomorphism.

The Steenrod squares are the ones explaining the non-commutativity of the cup product on the chain level.

Furthermore, the set of Steenrod operations for any given prime can be given the structure of a *Hopf algebra* and the relevant cohomology ring can be made into a module over this algebra, making it an even stronger invariant.

2.5.2 Higher order cohomology operations

The triple Massey product specifically, and the higher Massey products in general, cannot be a cohomology operation in the sense above, since it is not defined for all triples of cohomology classes. Indeed, we require the vanishing of cup products for the triple products, and lower order Massey products for higher Massey products.

Definition 2.5.2. A *secondary cohomology operation* is a natural transformation whose domain is the kernel of some primary cohomology operation and whose codomain is the cokernel of some other primary cohomology operation. ⌟

The higher order operations are defined iteratively:

Definition 2.5.3. An *n-ary cohomology operation* is a natural transformation whose domain is the kernel of some $(n-1)$-ary cohomology operations and whose codomain is the cokernel of some other $(n-1)$-ary cohomology operation. ⌟

We will not prove it, but the Massey products are examples of such operations, as summed up in the following proposition.

Proposition 2.5.1. *For each $n \geq 3$, the n-fold Massey product is an $(n-1)$-fold cohomology operation.*

It is immediate from the vanishing conditions that the triple product is defined only in the kernel of the operation given by cupping with the middle class in the product, but it is not obvious to which cokernel the codomain, the quotient of the cohomology ring by the indeterminacy ideal, correspond to.

2.5.3 Applications

It seems that the most common uses of Massey products, both triple and higher, is a differentials in certain spectral sequences. For instance in the Eilenberg–Moore spectral sequence, which is a generalisation of the Künneth formula and relates the cohomology of a pullback to the cohomology of each the three spaces involved.

This type of application of the product is important enough to warrant a section exclusively on Massey product in the appendix of the book "Complex Cobordism and Stable Homotopy Groups of Spheres" [Rav86] by Douglas Ravenel, which to a large extend uses the technique of spectral sequences to deduce information about the stable homotopy groups of maps of spaces from knowledge of the Steenrod algebras and generalised cohomology modules of the spaces, the pertinent sequence is called the Adams–Novikov spectral sequence.

2.5.4 Toda brackets

Another construction which has similarities with the Massey product is the *Toda bracket*, introduced by Hirosi Toda in [Tod62] to compute stable homotopy groups of spheres. This is a homotopical analogue to the Massey product having as input three composable homotopy classes of maps whose suitable compositions vanish, in the same way as the relevant cup products vanish for Massey products.

More precisely, for the following sequence of maps of spaces:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D,$$

we demand that $g \circ f$ and $h \circ g$ are nullhomotopic. We can then form the product $\langle f, g, h \rangle$, possibly with indeterminacy.

In the same way as for the n-fold Massey product, there exists "higher" Toda brackets with more entries, defined when "lower" Toda brackets vanish.

The author regards Toda brackets as a possible new avenue of research into tools for studying link complements.

3 de Rham computations

In the book "Rational homotopy theory and differential" by Griffiths and Morgan, [GM81], they perform a computation showing that the Massey products of the natural generators of the first de Rham cohomology group of $\mathbf{S}^3 \setminus \mathscr{B}$ is non-trivial.

In this chapter we go through this and a number of other computations, showing a geometric and visually pleasing way of understanding the Massey products.

We begin by presenting a number of theoretical results on which Griffiths and Morgan's exposition rests, such as Thom classes, Thom's isomorphism theorem, tubular neighbourhoods and transversality. We do this assuming basic knowledge of de Rham cohomology and compactly supported de Rham cohomology, denoted $\mathcal{H}^*_{\mathrm{dR}}(\bullet)$ and $\mathcal{H}^*_{\mathrm{c}}(\bullet)$ respectively.

The reader familiar with these notions or only interested in getting an impression of how the computation runs may skip ahead to Section 3.2, since the details are not essential to understanding the arguments.

Recall that by the de Rham theorem the singular cohomology ring $\mathcal{H}^\bullet(\bullet; \mathbf{R})$ with coefficients in $\mathbf{R}$ is isomorphic to the de Rham cohomology ring $\mathcal{H}^\bullet_{\mathrm{dR}}(\bullet)$. By a universal coefficient theorem we then have

$$\mathcal{H}^\bullet_{\mathrm{dR}}(\bullet) \cong \mathcal{H}^\bullet(\bullet; \mathbf{R}) \cong \mathcal{H}^\bullet(\bullet; \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R}.$$

These results paired with Poincaré–Lefschetz duality for singular theories means that singular homology with real coefficients and de Rham cohomology are dual.

Note. In this chapter everything is assumed to be smooth and all submanifolds are embedded, orientable and compact.

Note. The manifolds we are working with have boundaries and we assume that all differential forms vanish on the boundary.

3.1 Technical preliminaries

The goal of this section is to state the Thom isomorphism theorem and define *Thom classes* in de Rham cohomology and explain their relation to Poincaré–Lefschetz duals. Most of the material is found in [BT82], Ch. I, §6.

3.1.1 Vertically compact cohomology

We begin by describing a third cohomology theory on vector bundles, namely *de Rham cohomology with compact support in the vertical direction.*

The name captures the intuition, namely that the restriction of a form to a fibre should have compact support.

Definition 3.1.1. Let $\pi\colon E \to M$ be a vector bundle. An n-form $\omega \in \Omega^n(E)$ is in $\Omega^n_{\mathrm{cv}}(E)$ if $\pi^{-1}(K) \cap \operatorname{supp}(\omega)$ is compact for all compact $K \subset M$. ⌟

In particular, the restriction of a form ω in $\Omega^*_{\mathrm{cv}}(E)$ to any fibre F has compact support, so that $\omega|_F \in \Omega^n_{\mathrm{c}}(F)$.

Having defined these vector spaces for each natural n and using the ordinary de Rham differential, we obtain a cochain complex.

Definition 3.1.2. The homology of the de Rham cochain complex $(\Omega^\bullet_{\mathrm{cv}}(E), \delta)$ is called *the cohomology of E with compact support in the vertical direction.* ⌟

The following theorem relates this cohomology of the vector bundle E to the cohomology of the base manifold M.

Theorem 3.1.1 (Thom isomorphism theorem)**.** *If $\pi\colon E \to M$ is an orientable vector bundle of rank n over a manifold M, then*

$$\mathcal{H}^*_{\mathrm{cv}}(E) \cong \mathcal{H}^{*-n}_{\mathrm{dR}}(M)\,.$$

The proof in [BT82] uses a Mayer-Vietoris argument combined with induction on the cardinality of the *good cover*, so they need to assume M to be of *finite type.*

3.1.2 Thom classes

There are a number of ways of defining Thom classes, but given Thom's isomorphism theorem, there is a particularly simple one.

Definition 3.1.3. Let $\phi\colon \mathcal{H}^*_{\mathrm{dR}}(M) \to \mathcal{H}^{*+n}_{\mathrm{cv}}(E)$ be the Thom isomorphism, then the *Thom class* Φ *of* E is the image of 1 under ϕ. ⌟

We give another characterisation, in the form of a proposition.

Proposition 3.1.2. *The Thom class* Φ *of a rank* n *vector bundle* E *is the unique vertically compact cohomology class restricting to the generator of* $\mathcal{H}^n_{\mathrm{c}}(F)$ *for each fibre* F *of* E*.*

Having defined the Thom class, we can do quite a lot better than merely stating that there is *some* isomorphism in the Thom isomorphism theorem, we can actually give explicit isomorphisms in each direction: From left to right, the map is the so-called *integration along the fibre*, which integrates the class in the fibre direction and from right to left, the map is given by wedging with the Thom class Φ.

The reason we introduced Thom classes was to be able to compute Massey products in the context of de Rham cohomology, and Thom classes allow us to do this by performing the required operations on submanifolds and translating the information back to forms via the notion of Poincaré–Lefschetz duals.

Thom classes and Poincaré–Lefschetz duality

To explain the relation between Thom classes and Poincaré–Lefschetz duals we need some definitions, which we will give in quick succession.

From [BT82] we have the following definition.

Definition 3.1.4. The *Poincaré–Lefschetz dual* of an oriented submanifold N^n in an oriented ambient manifold-with-boundary M^m is the unique cohomology class of any closed $(m-n)$-form η_N satisfying

$$\int_N \omega = \int_M \omega \wedge \eta_N,$$

for all compactly supported n-forms ω on M. ⌟

Definition 3.1.5. The *normal bundle* $\mathcal{N}$ of a submanifold N in M is the vector bundle over N such that the following sequence of vector bundles is exact:

$$0 \to \mathcal{T}N \to \mathcal{T}M \to \mathcal{N} \to 0,$$

meaning that at each point of N, it restricts to a short exact sequence of vector spaces. ⌟

The normal bundle will, in a canonical way, inherit its orientation from the orientation of the tangent bundles of the submanifold and the ambient manifold.

Furthermore, given the following definition, we can state a result on localisation of the support of the Poincaré–Lefschetz dual.

Definition 3.1.6. With N as above, a *tubular neighbourhood* of N is a neighbourhood of N diffeomorphic to the normal bundle of N in M so that N is diffeomorphic to its zero section. ⌟

Proposition 3.1.3 (Localisation principle)**.** *The support of the Poincaré–Lefschetz dual of a submanifold N can be shrunk into any given tubular neighbourhood of N.*

For the above to be useful in computations, we need a result on the existence of tubular neighbourhoods, this is the content of the following theorem.

Theorem 3.1.4 (Tubular neighbourhood theorem)**.** *If N is a compact submanifold in a manifold-with-boundary M then N has a tubular neighbourhood in M.*

Finally we have arrived at the central result in this section, Prop. 6.24. in [BT82].

Lemma 3.1.5. *The Poincaré–Lefschetz dual η_N of an oriented submanifold N in an oriented manifold-with-boundary M and the Thom class Φ of the normal bundle of N in M can be represented by the same form.*

From which the following theorem follows.

Theorem 3.1.6. *The Poincaré–Lefschetz dual of N in M is the Thom class of the normal bundle of N:*

$$\eta_N = j^*(\Phi \wedge 1) = j^*\Phi \in H^{m-n}(M),$$

where $j\colon N \hookrightarrow M$ is the inclusion map.

We will denote by $\widetilde{U_N}$ the Thom class of the normal bundle of the submanifold $N \subset M$ extended by zero outside its support. We suppress the dependence of M, since it will usually be apparent from the context.

3.1.3 Intersection product

Keeping the goal of computing Massey products in mind, we are interested in an operation on submanifolds corresponding to the wedge product of Thom classes of normal bundles.

The wedge product of forms gives a form whose degree is the sum of the degrees of the individual forms. Thinking in terms of the codimensions of corresponding submanifolds, this would mean that the codimensions should add as well. In certain *nice* cases, this is indeed what happens when you take intersections. The key notion here is *transverse intersection* of submanifolds.

Transversality

Intuitively, transverse intersection of two submanifolds should mean that the intersection is not too far from being orthogonal, but its interpretation depends on the sum of dimensions of the submanifolds.

Definition 3.1.7. Two submanifolds N and L in M intersect *transversely* if for each point in their intersection, the (not necessarily direct) sum of their tangent spaces is the tangent space of M at that point. ⌟

We use the symbol $\pitchfork$ to denote transverse intersection of submanifolds. In symbols, the definition then becomes:

$$N \pitchfork L \Leftrightarrow \forall p \in N \cap L : \mathcal{T}_p N + \mathcal{T}_p L = \mathcal{T}_p M.$$

Note. It is important to notice that the condition depends crucially on the (dimension of the) ambient manifold.

As an example, if $\dim N + \dim L < \dim M$, then

$$N \pitchfork L \Leftrightarrow N \cap L = \emptyset,$$

since the dimensions of the tangent spaces does not add up.

The condition that submanifolds intersect transversely in a given ambient manifold is sufficient to ensure that their intersection is again a submanifold. This is a very elementary version of the *transversality theorem*, where the notion of transversality has been generalised from submanifolds to certain maps of manifolds.

Additionally, for transverse intersection of submanifolds we have the following formula relating their codimensions:

$$\operatorname{codim} N + \operatorname{codim} L = \operatorname{codim} N \cap L.$$

We are now ready to express the Thom class of the normal bundle of the transverse intersection of the submanifolds N and L, $\Phi(\mathcal{N}_{N\cap L})$ in terms of the Thom classes $\Phi(\mathcal{N}_N)$ and $\Phi(\mathcal{N}_L)$ of their respective normal bundles :

$$\Phi(\mathcal{N}_{N\cap L}) = \Phi(\mathcal{N}_N \oplus \mathcal{N}_L) = \Phi(\mathcal{N}_N) \wedge \Phi(\mathcal{N}_L)$$

Translating using the notation of Poincaré–Lefschetz duals, we have

$$\eta_{N\cap L} = \Phi(\mathcal{N}_{N\cap L}) = \Phi(\mathcal{N}_N) \wedge \Phi(\mathcal{N}_L) = \eta_N \wedge \eta_L,$$

so by Poincaré–Lefschetz duality the transverse intersection of submanifolds corresponds to wedge product of forms.

3.1.4 Differential equations

The last thing we need to do is figuring out what solving the differential equation $\delta a = b$ corresponds to in terms of submanifolds. This is needed when computing the Massey product $\langle \omega_1, \omega_2, \omega_3 \rangle$, because then we have to solve the coupled differential equations

$$\delta a = \omega_1' \wedge \omega_2' \text{ and } \delta b = \omega_2' \wedge \omega_3'$$

for a and b.

This material is not presented in [BT82], but can be found in [GM81], where it is stated without proof.

We are particularly interested in solving the equation $\delta\gamma = \widetilde{U_N}$ when N is an oriented submanifold of M. The surprisingly simple answer is that if L is a submanifold of M whose boundary is N, then $\gamma = \widetilde{U_L}$ solves the above equation. That is: $\delta\widetilde{U_L} = \widetilde{U_{\partial L}}$, *this* is the key geometric observation making all subsequent computations feasible and therefore important enough to warrant a theorem.

Theorem 3.1.7. *With notation as above, if $N \subset M$ is a submanifold with boundary ∂N then*

$$\delta\widetilde{U_N} = \widetilde{U_{\partial N}}.$$

Before giving the proof, we need the following theorem, which we state without proof.

Theorem 3.1.8 (Smooth collaring theorem)**.** *If N is a manifold with compact boundary ∂N, then there exists a neighbourhood U, with $\partial N \subset U \subset N$, such that that U is diffeomorphic to $\partial N \times [0,1)$.*

Proof of Theorem 3.1.7 . We will prove this theorem using using the smooth collaring theorem and existence of smooth step functions. Let U be the collaring neigbourhood.

In this proof, we regard all forms and classes as belonging to the de Rham cohomology ring of the ambient manifold, smoothly extended by zero outside their supports if necessary.

We begin by defining our step function to some smooth function $\tilde{f} \colon [0,1) \to [0,1]$ with the following properties: $\tilde{f}(0) = 0$, $\tilde{f}(t) = 1$ for t in a neighbourhood of $1 \in [0,1)$ and the derivative satisfies $\tilde{f}'(0.5) = 1$ and is non-negative.

From this, we extend to a function $f \colon N \to [0,1]$ by $f(x) = 1$ for $x \in N \setminus U$ and $\tilde{f}(t)$ on the fibres of the collaring neighbourhood.

We let u be representative for the Thom class of $N \subset M$ and recall that we have required all our differential forms to vanish on the pertinent boundaries.

To go from the 0-form f on N to a form on the normal bundle $p \colon \mathcal{N}_N \to N$ we take the wedge with the form representing the Thom class $u \wedge f$. This form represents the same form as u since f is only zero on the boundary. The differential of this form is $\delta(u \wedge f) = u \wedge \delta f$, supported in the collaring neighbourhood U. Intuitively, this form has non-zero components normal to N from the first factor, and one non-zero component in the direction normal to ∂N and parallel to N.

Now we want to show that the Thom class $U_{\partial N}$ of the normal bundle of the boundary ∂N can be represented by $u \wedge \delta f$.

First, $U_{\partial N}$ is supported in a tubular $(m-n+1)$-neighbourhood of ∂N, so it is of degree $(m-n+1)$. Furthermore ∂N homotopic to $\partial N \times \{0.5\}$ in $\partial N \times [0,1)$, so their Thom classes are the same.

We will use the characterisation of the Thom class given in Proposition 3.1.2. The class U_N restricts to a generator for the vertically compactly cohomology of each fibre of the normal bundle of N and the representative u can then be thought of as bump functions in each of the directions normal to N.

We would like a similar form for the Thom class of the boundary and notice that u is missing the direction parallel to N, but δf has exactly this shape: it is a bump function supported in $\partial N \times [0,1)$. It follows that $u \wedge \delta f$ represent the class restricting to a vertically compact generator on each fibre of the normal bundle of ∂N, seen as a tubular neighbourhood in M.

The identity $\delta U_N = U_{\partial N}$ follows. □

This finishes the technical section.

3.2 The Borromean rings

The aim of this section is to compute a non-trivial Massey product in X, where X is the complement in $\mathbf{S}^3$ of open non-intersecting tubular neighbourhoods of the Borromean rings.

Note. In [GM81] the motivation for doing this computation is to give an example of the *Sullivan minimal model* of differential graded associative algebras.

Of the above technicalities, the most important aspects are the correspondence between Thom classes and submanifolds, wedge products and transverse intersections, and taking the coboundary of forms and taking the geometric boundary of submanifolds; which we will be making repeated use of throughout the rest of this chapter.

It is not immediately obvious that the link in Figure 3.1 is equivalent to the Borromean rings. This is, however, illustrated in [Baa10, p.30], Figure 41. Reproduced here with permission from the creator, see Figure 3.2.

Figure 3.1 is inspired by [GM81, p.156], but I have made an effort to make it easier to "parse" and understand than the original one.

The notation we use for submanifolds and forms is the same as in [GM81] and is illustrated in Figure 3.1. Denote by $\widetilde{U_i}$ the Thom class of the normal bundle of the relative disk D_i extended by zero outside its support in $\mathcal{H}^1_{\mathrm{dR}}(X)$. Furthermore, the interval I_{12} is the intersection of

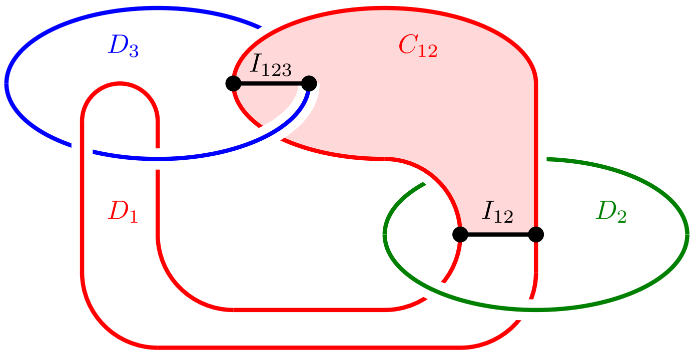
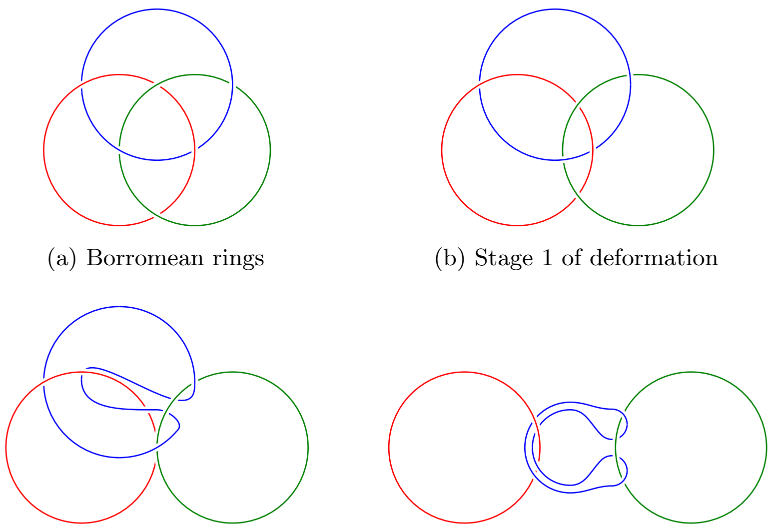


Figure 3.1: The Borromean rings with annotations



(a) Borromean rings

(b) Stage 1 of deformation

(c) Stage 2 of deformation

(d) Brunnian 3-chain

Figure 3.2: Mutation of the Borromean rings from the usual symmetric presentation to a chain-like presentation

D_1 and D_2, the disk-with-boundary C_{12} is the filled in area, and I_{123} is the intersection of C_{12} and D_3.

Theorem 3.2.1. *The following Massey product is non-trivial:*

$$\langle \widetilde{U_1}, \widetilde{U_2}, \widetilde{U_3} \rangle \neq 0 \in \mathcal{H}^2_{\mathrm{dR}}(X)\,.$$

Proof. In order to compute the Massey product, we have to find

$$z' = a \wedge \omega_3 + \omega_1 \wedge b,$$

where a and b are forms satisfying $\omega_1 \wedge \omega_2 = \delta a$ and $\omega_2 \wedge \omega_3 = \delta b$. This means that we need to compute the wedge products $\omega_1 \wedge \omega_2$, $\omega_2 \wedge \omega_3$, $a \wedge \omega_3$ and $b \wedge \omega_1$, and find the solutions to $\omega_1 \wedge \omega_2 = \delta a$ and $\omega_2 \wedge \omega_3 = \delta b$.

The first thing we do is find the forms a and b: The wedge product $\widetilde{U_1} \wedge \widetilde{U_2}$ corresponds to the intersection of disks D_1 and D_2. This is the interval I_{12}. Furthermore, we choose the disk C_{12} as the manifold with boundary I_{12}, so that $\widetilde{U_{C_{12}}} = a$ is a solution to $\delta a = \widetilde{U_1} \wedge \widetilde{U_2}$.

Finding a suitable b is easier. Since we have chosen the disks D_2 and D_3 to have empty intersection, we can also make the corresponding forms have non-intersecting supports. In detail, this is because we can make Poincaré–Lefschetz duals have support in any tubular neighbourhood. The Poincaré–Lefschetz dual of the empty set can be represented by the zero form, hence we can take $b = 0$ to be a solution to $\delta b = U_2 \wedge U_3$.

This gives $z' = a \wedge \widetilde{U_3} + \widetilde{U_1} \wedge b = \widetilde{U_{C_{12}}} \wedge \widetilde{U_3}$, so we are left with computing this one wedge product, corresponding to the intersection of C_{12} and D_3. In Figure 3.1, this intersection is the interval denoted I_{123}, so $z' = \widetilde{U_{I_{123}}}$.

We have to make sure that the cohomology class we get is non-trivial. The interval connects different components of ∂X, hence by a remark made in Section 1.6 it is the generator μ_{13} of $\mathcal{H}_1(X, \partial X)$. By the Poincaré–Lefschetz duality theorem, this means that the cohomology class, which belongs to $\mathcal{H}^2_{\mathrm{dR}}(X)$, representing the product is also non-trivial.

Whence we conclude that $\langle \widetilde{U_1}, \widetilde{U_2}, \widetilde{U_3} \rangle \in \mathcal{H}^2_{\mathrm{dR}}(X)$ is non-zero. □

We conclude this section by making a remark on the indeterminacy in the computation above.

Note. We could have chosen C_{12} differently, namely the other part of D_3. This other choice is the filled area shown in Figure 3.3. This would not

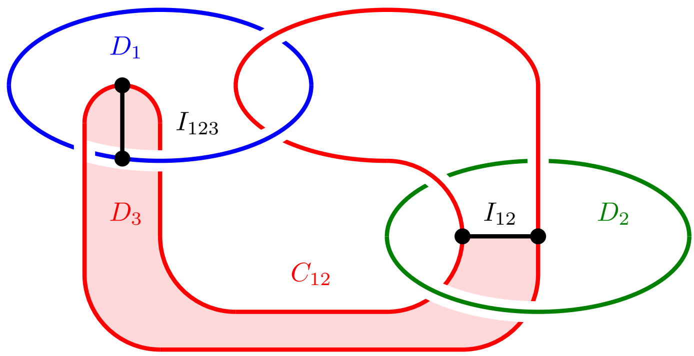


Figure 3.3: The Borromean rings with another choice of C_{12}

have made a difference, since there is a 2-submanifold having these two intervals as its boundary, hence they are homologous, see Figure 3.4.

This shows that this Massey product has trivial indeterminacy.

3.3 The unlink

In [GM81] it is suggested doing the same computation for the unlink with three components. This is actually straight-forward, using the same argument as we did for $\widetilde{U_2} \wedge \widetilde{U_3}$ above we find that both a and b can be taken to be zero. It is possible since for the unlinks we can choose non-intersecting proper disks whose boundaries are the link components.

This implies that the Massey product (of the Thom classes of the disks whose boundaries are the link components) is trivial.

Note. We could have taken any permutation of the factors, and the result would be the same.

3.4 The Brunnian 3-link

The de Rham version of the Massey product for the Brunnian 3-link, shown in Figure 3.5, can actually be computed quite easily in the same way as for the Borromean rings.

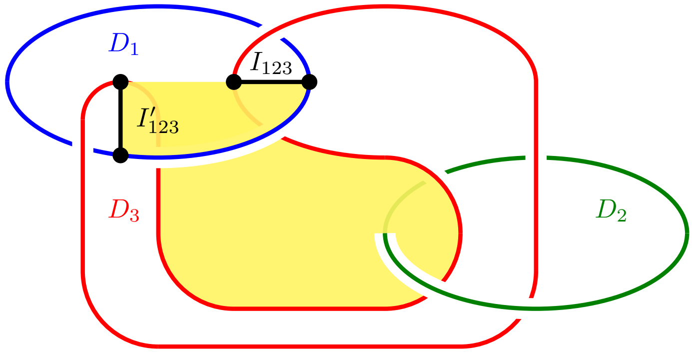


Figure 3.4: The Borromean rings with a bounding relative disk

We repeat the necessary preliminary introduction of notation. Denote by X the complement in $\mathbf{S}^3$ of open non-intersecting tubular neighbourhoods of the link components of the Brunnian 3-link and number the link components counter clockwise, with the leftmost one being number 1. As above, we denote the submanifolds having their boundaries on the fattened up link components by D_i and the Thom classes of the normal bundles of these by $\widetilde{U_i}$.

Theorem 3.4.1. *The Massey product structure on X is trivial.*

Proof. We will focus on the product $\langle \widetilde{U_1}, \widetilde{U_2}, \widetilde{U_3} \rangle \in \mathcal{H}^2_{\mathrm{dR}}(X)$, since the other permutations of $\{1, 2, 3\}$ are equivalent or easier.

To make sure that the Massey product is zero, we have to check that for all the possible choices of a and b in the definition of z', we get something which is trivial in cohomology.

There are 2^2 choices of a and b, corresponding to choices of submanifolds having the intersection of disks as their boundaries. One will make the product trivially trivial, whereas the others will make it non-trivially trivial. We will do the trivial case and one non-trivial case, since one suffices to illustrate the issues we run into.

The problem in the non-trivial case is to ensure that different components of an intersection has opposite orientations. See Figure 3.5.

Again we insert the relevant forms in the definition of z':

$$z' = a \wedge \widetilde{U_3} + \widetilde{U_1} \wedge b = \widetilde{U_{C^*_{12}}} \wedge \widetilde{U_3} + \widetilde{U_1} \wedge \widetilde{U_{C^*_{23}}}.$$

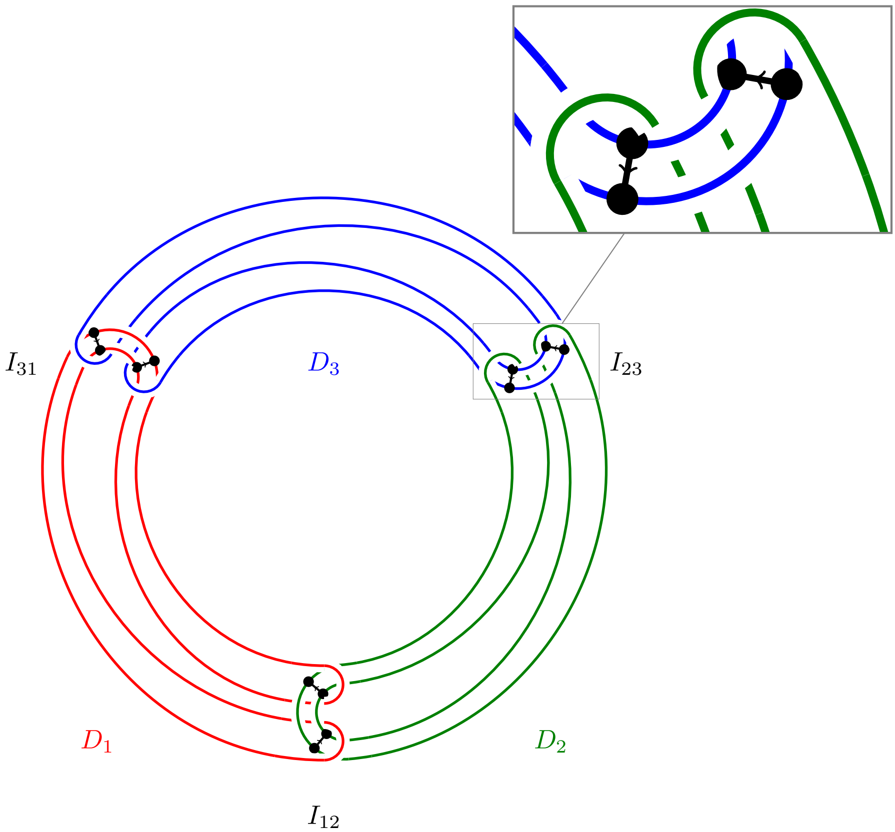


Figure 3.5: The Brunnian 3-link

Let us begin with the easy case: choosing C'_{12} and C'_{23}. Both the intersection of C'_{12} and D_3 and the intersection of C'_{23} and D_1 are empty, so in this case the Massey product is trivial.

We proceed to do one non-trivial case: If we choose C''_{12} and C'_{23}, we take the intersection of C'_{12} and D_3 to obtain

$$C'_{12} \cap D_3 = I_{23} = I'_{23} \sqcup I''_{23}.$$

The other intersection is still empty.

We want to prove that the intervals I'_{23} and I''_{23} have opposite orientations, since then they will cancel in homology.

Recall that we define the orientation of the normal bundle $\mathcal{N}_N$ of a submanifold $N \subset M$ to be so that the direct sum orientation of the Whitney sum $\mathcal{N}_N \oplus \mathcal{T}N$ is the same as the orientation of $\mathcal{T}_M$ restricted to N.

One way of specifying an orientation of C_{12} is choosing to have outward pointing normal vectors on the boundary. Without loss of generality, we can also let the "third direction" be pointing *out* of the paper plane. This leaves us with picking a positive direction on, say, I'_{23}. If we take it to be to the *right* in Figure 3.5, then for the orientation to be coherent the positive direction on I''_{23} needs to be *up* in the figure.

Similarly, if we choose *left* for I'_{23}, then by coherence we need to take *down* for I''_{23}.

Both choices give opposite orientations on I'_{23} and I''_{23}, which means that their union is zero in $\mathcal{H}_1(X, \partial X)$.

This in turn implies that the Massey product is trivial for these choices of a and b.

For the remaining two choices, the product is zero for the same reasons as above. It is the disjoint union of oppositely oriented submanifolds, which cancels in relative homology.

By Poincaré–Lefschetz duality, we conclude that

$$\langle \widetilde{U_1}, \widetilde{U_2}, \widetilde{U_3} \rangle = 0 \in \mathcal{H}^2_{\mathrm{dR}}(X)\,.$$

Furthermore, the cyclic symmetry of link components show that another two of the Massey products are zero. As for non-cyclic permutations of link component, they will again make all products trivial.

From this discussion, we get that the Massey product structure of X is trivial, which is what we wanted to show. $\square$

Since the Massey product structure of the complements of the Borromean rings and the Brunnian 3-link are different the links themselves are not isotopic.

The result that the Massey product structure on the Brunnian 3-link is trivial is both surprising and interesting. One of the initial aims of this thesis was to find tools with which to study the linking and higher order linking structures introduced in [Baa10]. If the Massey product does *not* detect the Brunnian linking, this means that it is probably not the correct tool for the task.

3.5 The Brunnian 4-link

In this section, we continue investigating whether or not we can detect the linking inherent in the family $B(1,n)$ of Brunnian rings described in [Baa10].

We will now calculate the fourfold Massey product of the complement of open non-intersecting tubular neighbourhoods of the Brunnian 4-link in $\mathbf{S}^3$, see Figure 3.6. Again, for ease of notation, we call this compact manifold-with-boundary X.

The recipe for computing the fourfold product in de Rham cohomology is similar to that of the triple product, though with additional operations to perform.

As usual, we denote our submanifolds by D_i, but in order to conform with the notation introduced in Section 2.4.2, we denote the corresponding Thom classes previous denoted $\widetilde{U_i}$ by α_i.

Lemma 3.5.1. *The following fourfold Massey product is trivial:*

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle = \{0\} \subset \mathcal{H}^2_{\mathrm{dR}}(X)\,.$$

Proof. The fourfold product is defined only when the two relevant triple products vanish, hence we need to check this condition first. By symmetry considerations we only need to check this for one of the products, say $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

The intersection of D_1 and D_2 is the boundary of C'_{12} and C''_{12}. The intersection of the C'_{12} and D_3 is empty, so it does not contribute to the product. The intersection of the C''_{12} and D_3 is non-empty, but zero in

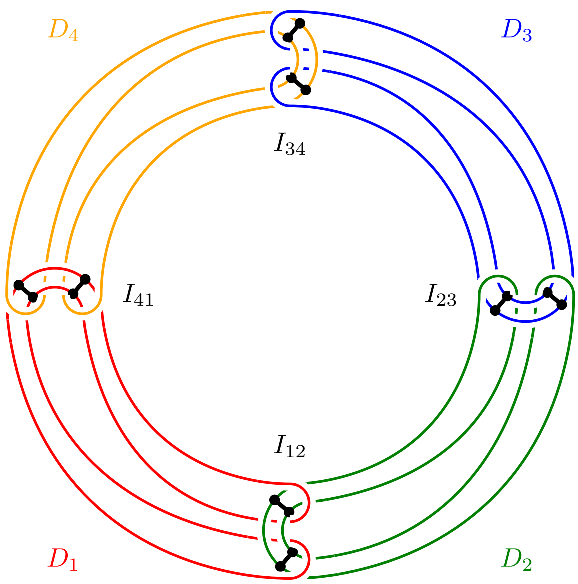


Figure 3.6: The Brunnian 4-link

homology, for the same reason as the one explained in the case of the Brunnian 3-link.

For the other term of z', none of the submanifolds with $D_2 \cap D_3$ as their boundary intersects D_1.

This implies that $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ vanishes strictly, hence the fourfold product $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is strictly defined.

Recall from Subsection 2.4.1, that we defined it as the class of

$$b_{0,4} = \widetilde{a_{0,3}} \wedge a_{3,4} + \widetilde{a_{0,2}} \wedge a_{2,4} + \widetilde{a_{0,1}} \wedge a_{1,4} \in \Omega^\bullet(X).$$

The defining system, as well as correspondences to submanifolds are listed below:

$$\begin{aligned}
[a_{0,1}] &= \alpha_1 \leftrightsquigarrow D_1, \\
[a_{1,2}] &= \alpha_2 \leftrightsquigarrow D_2, \\
[a_{2,3}] &= \alpha_3 \leftrightsquigarrow D_3, \\
[a_{3,4}] &= \alpha_4 \leftrightsquigarrow D_4,
\end{aligned}$$

$$\begin{aligned}
\delta a_{0,2} &= a_{0,1} \wedge a_{1,2} \leftrightsquigarrow \partial C^*_{12} = I_{12}, \\
\delta a_{1,3} &= a_{1,2} \wedge a_{2,3} \leftrightsquigarrow \partial C^*_{23} = I_{23}, \\
\delta a_{2,4} &= a_{2,3} \wedge a_{3,4} \leftrightsquigarrow \partial C^*_{34} = I_{34},
\end{aligned}$$

$$\begin{aligned}
\delta a_{0,3} &= \widetilde{a_{0,2}} \wedge a_{2,3} + \widetilde{a_{0,1}} \wedge a_{1,3} \leftrightsquigarrow \partial C'_{34} = I_{34}, \\
\delta a_{1,4} &= \widetilde{a_{1,3}} \wedge a_{3,4} + \widetilde{a_{1,2}} \wedge a_{2,4} \leftrightsquigarrow \partial C'_{41} = I_{41},
\end{aligned}$$

with * denoting either $'$ or $''$.

Thus we have to take three intersections to do the calculation:

$$\begin{aligned}
C'_{34} \cap D_4 &= I_{34} = I'_{34} \sqcup I''_{34} \\
I_{12} \cap I_{34} &= \emptyset \\
D_1 \cap C'_{41} &= I_{41} = I'_{41} \sqcup I''_{41}.
\end{aligned}$$

As we have seen many times by now, these oriented submanifolds represent the zero homology class in $\mathcal{H}_1(X, \partial X)$. So by Poincaré–Lefschetz duality, $\mathcal{H}_1(X, \partial X) \cong \mathcal{H}^2_{\mathrm{dR}}(X)$, the product is therefore again the trivial class in $\mathcal{H}^2_{\mathrm{dR}}(X)$. $\square$

It is to be expected that n-fold Massey product applied in the same way as above to the Brunnian n-link will vanish for exactly the same reasons. We will study both this problem, and the corresponding one for the Brunnian *n-chain* in Section 3.7 and Section 3.8.

3.6 The Brunnian 5-chain

Our goal for this section is to compute a non-trivial fivefold Massey product, showing that higher Massey products can detect the linking in the family of Brunnian n-chains.

In Figure 3.7, we see the *Brunnian chain* with 5 components. This link belongs to the family of links obtained by adding more components in the middle. This family may be regarded as natural extension of the Borromean rings, as seen in Figure 3.2.

We denote by X the compact manifold-with-boundary obtained by removing open non-intersecting tubular neighbourhoods of each link component in $\mathbf{S}^3$. Looking at the Figure 3.7, let us denote, from left to right, the 2-submanifolds with boundaries on the boundary of each fattened up link component by D_i for i between 1 and 5.

As earlier, we let α_i be the Thom class of the normal bundle of D_i, extended by zero outside its support.

We can then form three different triple products which are not trivially zero, namely those with consecutive and increasing indices. For those with non-consecutive indices the corresponding intersections of submanifolds are empty, leading to trivial products.

In this section we will not explain each step as thoroughly as above.

To ensure that the fivefold product is defined, we have to compute some lower order products. We state the result of the computations as lemmas.

Lemma 3.6.1. *The following Massey product is trivial:*

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 0 \in \mathcal{H}^2_{\mathrm{dR}}(X)\,.$$

Proof. The intersection of D_1 and D_2 is I_{12}. I_{12} is the boundary of C'_{12} and C''_{12}, but $C'_{12} \cap D_3 = I'_{123}$ and $C''_{12} \cap D_3 = I''_{123}$ are homotopic, so the product has no indeterminacy. We can find a submanifold representing a class in $\mathcal{H}_2(X, \partial X)$ whose boundary is I'_{123}, so the homology class it represents is trivial. By duality it follows that the product is zero. $\square$

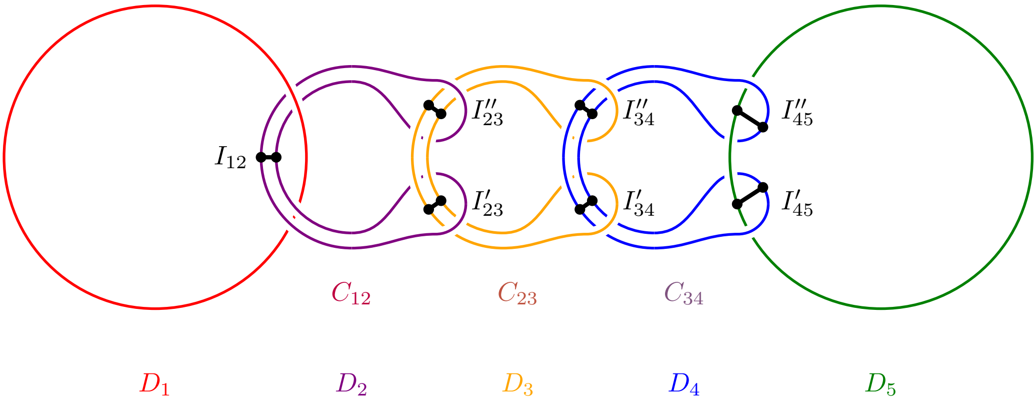


Figure 3.7: The Brunnian 5-chain

Lemma 3.6.2. *The following Massey product is trivial:*

$$\langle \alpha_2, \alpha_3, \alpha_4 \rangle = 0 \in \mathcal{H}^2_{\mathrm{dR}}(X)\,.$$

Proof. The intersection of D_2 and D_3 is I_{23}, which is the boundary of C'_{23} and C''_{23}. $C'_{23} \cap D_4 = \emptyset$ and $C''_{23} \cap D_4 = I_{234} = I'_{234} \sqcup I''_{234}$, but as in the computation for Brunnian 3-link, I'_{234} and I''_{234} cancels in homology, hence the product is trivial. □

Lemma 3.6.3. *The following Massey product is trivial:*

$$\langle \alpha_3, \alpha_4, \alpha_5 \rangle = 0 \in \mathcal{H}^2_{\mathrm{dR}}(X)\,.$$

Proof. The intersection of D_3 and D_4 is I_{34}, which is the boundary of C'_{34} and C''_{34}. $C'_{34} \cap D_4 = \emptyset$ and $C''_{34} \cap D_5 = I_{345} = I'_{345} \sqcup I''_{345}$. We can find a submanifold having I_{345} as its boundary so by the same arguments as before the product is trivial. □

We continue by computing the relevant fourfold products:

Lemma 3.6.4. *The following fourfold Massey product is trivial:*

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle = \{0\} \subset \mathcal{H}^2_{\mathrm{dR}}(X)\,.$$

Proof. By the previous lemmas this fourfold product is well-defined. For the readers' convenience, we recall the expression for the defining cochain:

$$b_{0,4} = \widetilde{a_{0,3}} \wedge a_{3,4} + \widetilde{a_{0,2}} \wedge a_{2,4} + \widetilde{a_{0,1}} \wedge a_{1,4} .$$

One can check that the intersections corresponding to the two last terms of the above expression are zero, so we only need to consider $b_{0,4} = \widetilde{a_{0,3}} \wedge a_{3,4}$. The wedge product corresponds to the intersection $C_{23} \cap D_4 = I_{34}$, which represents the zero class in $\mathcal{H}_1(X, \partial X)$ since it is the boundary of a submanifold representing a class in $\mathcal{H}_1(X, \partial X)$, e.g. C_{34}. □

Lemma 3.6.5. *The following fourfold Massey product is trivial:*

$$\langle \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle = \{0\} \subset \mathcal{H}^2_{\mathrm{dR}}(X) .$$

Proof. Again, by the previous lemmas this fourfold product is well-defined, and the expression for the defining cochain is:

$$b_{1,5} = \widetilde{a_{1,4}} \wedge a_{4,5} + \widetilde{a_{1,3}} \wedge a_{3,5} + \widetilde{a_{1,2}} \wedge a_{2,5} ,$$

and only the first term is not trivially trivial, so $b_{1,5} = \widetilde{a_{1,4}} \wedge a_{4,5}$, corresponding to $(C_{34} \cup C'_{34}) \cap D_5$. This intersection has two components and there exists a 2-submanifold representing a class in $\mathcal{H}_2(X, \partial X)$ and having these as its boundary, oppositely oriented, which means that in $\mathcal{H}_1(X, \partial X)$ they cancel. By the Poincaré–Lefschetz duality theorem the product, which is the corresponding class in $\mathcal{H}^2(X)$, is zero. □

Finally, we will compute the fivefold product, showing that it is non-trivial:

Theorem 3.6.6. *The following fivefold Massey product is non-trivial:*

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle \neq \{0\} \subset \mathcal{H}^2_{\mathrm{dR}}(X) .$$

Proof. By the previous five lemmas the fivefold product is well-defined.
The defining cochain is given as follows:

$$b_{0,5} = \widetilde{a_{0,4}} \wedge a_{4,5} + \widetilde{a_{0,3}} \wedge a_{3,5} + \widetilde{a_{0,2}} \wedge a_{2,5} + \widetilde{a_{0,1}} \wedge a_{1,5} .$$

As before, only the first term corresponds to a non-empty intersection, that of C_{34} and D_5. This intersection, I_{45}, is a 1-submanifold whose boundary components lie on the fourth and fifth components of ∂X. The class μ_{45} represented by I_{45} is non-trivial since it is the difference between the two distinct generators μ_{15} and μ_{14} of $\mathcal{H}_1(X, \partial X)$, as described in Section 1.6.

By the Poincaré–Lefschetz duality theorem the product, which is the corresponding class in $\mathcal{H}^2(X)$, is non-trivial. □

This result is good news, as it provides evidence that higher Massey products detects the non-trivial linking in "higher" Borromean rings, or longer Brunnian chains.

3.7 The Brunnian n-link

In this and the next section we will see that the so-called *higher* Massey products really could be called *longer* products. Indeed, there is nothing in the behaviour of the longer products that separates them from the shorter ones, except the feature of taking more link components into account.

It turns out that the computations of the threefold and fourfold products of Brunnian links captures all the issues we encounter when calculating the n-fold products.

We exemplify such a link by one with $n = 9$ components, Figure 3.8.

We number the link components in the Brunnian n-link in counter clockwise order and let X_n denote the compact manifold-with-boundary obtained by removing non-intersecting open tubular neighbourhoods of an embedding of the Brunnian n-link in $\mathbf{S}^3$. For $i \in \{1, \ldots, n\}$, denote by $\alpha_i \in \mathcal{H}^1_{\mathrm{dR}}(X_n)$ the Thom class of the normal bundle of the 2-submanifold whose boundary lies on the i^{th} component of ∂X_n.

Theorem 3.7.1. *The following n-fold Massey product is trivial:*

$$\langle \alpha_1, \ldots, \alpha_n \rangle = \{0\} \subset \mathcal{H}^2_{\mathrm{dR}}(X_n)\,.$$

Proof. This proof will only be a brief sketch of how the computation runs, since doing it in full would require introducing a lot of notation without a corresponding gain of understanding and insight.

The key technical point is that in the sum of terms defining the cochain $b_{0,n}$ whose equivalence class is the n-fold product only one term

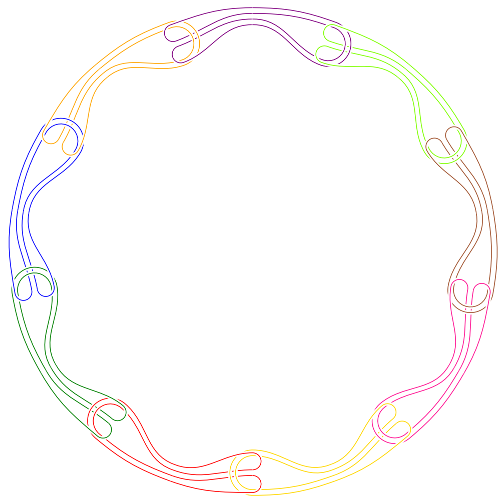


Figure 3.8: The Brunnian 9-link

corresponds to a non-empty intersection of submanifolds, in exactly the same way as for threefold and fourfold products. The intersection it corresponds to can be found iteratively:

- The non-zero term at the k^{th} stage is of the form

$$b_{0,k} = \widetilde{a_{0,k-1}} \wedge a_{k-1,k},$$

corresponding to the intersection of a 2-submanifold of X_n whose boundary is the Poincaré–Lefschetz dual of a representative of the $(k-1)$-fold product of Alexander duals of the $k-1$ first link components, relative to the numbering described above.

This intersection is of the same form as the one described in detail in the computation for the Brunnian 3-link.

We can continue this process up to and including the n^{th} stage, where we end up with a two-component 1-submanifold representing a relative homology class. This class is the zero class by the argument concerning orientations in Section 3.4. $\square$

In the above proof we implicitly assumed that $n \geq 3$ and the proof holds for all such natural n.

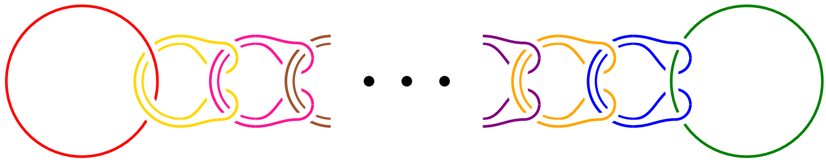


Figure 3.9: The Brunnian n-chain

3.8 The Brunnian n-chain

The computation we are about the perform is closely modelled on the one for the 5-chain; the only difference being that in the more general case of n link components we will need an inductive argument. See Figure 3.9

We number the link components in the Brunnian n-chain from left to right and let X_n denote the compact manifold-with-boundary obtained by removing non-intersecting open tubular neighbourhoods of an embedding of the Brunnian n-chain in $\mathbf{S}^3$. For $i \in \{1, \ldots, n\}$, denote by $\alpha_i \in \mathcal{H}^1_{\mathrm{dR}}(X_n)$ the Thom class of normal bundle of the 2-submanifold whose boundary lies on the k^{th} component of ∂X_n.

Theorem 3.8.1. *The following n-fold Massey product is non-trivial:*

$$\langle \alpha_1, \ldots, \alpha_n \rangle = \{0\} \subset \mathcal{H}^2_{\mathrm{dR}}(X_n)\,.$$

Proof. This proof is analogous to the one in Section 3.7, iteratively realising that only one term in the defining cochain $b_{0,k}$ corresponds to a non-empty intersection of submanifolds and only these can contribute to the product.

Going through the motions, we arrive at the result that the final defining cochain $b_{0,n}$ is equal to $a_{0,n-1} \wedge a_{n-1,n}$, corresponding to the intersection $I_{n-1,n} = C_{n-2,n-1} \cap D_n$. This 1-submanifold represents the relative homology class

$$\mu_{n-1,n} = \mu_{1n} - \mu_{1,n-1} \neq 0 \in \mathcal{H}_1(X, \partial X)\,.$$

By Poincaré–Lefschetz duality, the product in $\mathcal{H}^2_{\mathrm{dR}}(X)$ is non-trivial. □

Note. In the above computations we repeatedly made use of the fact that we could represent relative homology classes by submanifolds. The reader may feel uneasy about when this is actually possible. The question is commonly referred to as "Steenrod's problem", and were studied in detail by René Thom; a discussion of this work is found in Dennis Sullivan's homage [Sul04] to René Thom.

The question is partially, and sufficiently for our use, resolved using Steenrod squares. The result we can use is that for manifolds the smallest degree for which there is a homology class not representable by an embedded compact submanifold is 7. This is clearly sufficient to ensure that classes in the homology of the complement of a link in $\mathbf{S}^3$ is always representable by compact submanifolds.

3.9 Discussion on results of computations

The initial plan for this thesis was to define Massey products and compute them for a number of examples, especially the Borromean rings and the Brunnian 3-link, showing that these exhibit non-trivial Massey products. This would give good evidence that the triple product detects the "Brunnian property". From the results above, this appears not to be the case, which means that another tool, or invariant, is needed to detect this specific property of a link.

In the literature, there are a small number of computations of Massey products in link complements. I have only been able to find such computations in Massey's article [Mas69], where he calculates a product in the complement of the Borromean rings and the complement of a link similar to the Borromean rings; in Griffiths's [GM81], where they also calculate a product in the complement of the Borromean rings; and in O'Neill's [O'N79], where he calculates the fourfold product of a pair of "bracelets"; all cited above.

It was therefore not known whether or not there were non-trivial Massey products in the complement the Brunnian 3-link, although we expected there to be. The result of the computation above is that the Massey products of Brunnian rings are trivial. This means that there is a need for some other set of tools to detect them, this is a possible path to be followed. However, time constraints have prevented us from doing so.

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