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Computing Almost Split Sequences

An algorithm for computing almost split sequences of finitely generated modules over a finite dimensional algebra

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Abstract

An artin algebra Λ over a commutative, local, artinian ring R was fixed, and with this foundation some topics from representation theory were discussed. A series of functors of module categories were defined, and almost split sequences were introduced along with some results. An isomorphism $\omega_{\delta,X} : D\delta^* \to \delta_*(D\,\mathrm{Tr}(X))$ of Γ-modules for an artin R-algebra Γ was constructed. The isomorphism $\omega_{\delta,X}$ was applied to a special case, yielding a deterministic algorithm for computing almost split sequences in the case that R is a field.

Norsk sammendrag

En artinsk algebra Λ over en kommutativ, lokal ring R ble fiksert, og med dette som utgangspunkt ble endel emner fra representasjonsteori diskutert. En rekke funktorer over modulkategorier ble definert, og nesten splitt-eksakte følger ble introdusert sammen med noen resultater. En isomorfi $\omega_{\delta,X} : D\delta^* \to \delta_*(D\,\mathrm{Tr}(X))$ av Γ-moduler for en artinsk R-algebra Γ ble konstruert. Isomorfien $\omega_{\delta,X}$ ble anvendt ved et spesialtilfelle, og dette ga opphav til en deterministisk algoritme for å regne ut nesten splitt-eksakte følger i tilfellet at R er en kropp.

Preface

This thesis is the final part of my master's degree in Industrial Mathematics at the Norwegian University of Science and Technology (NTNU). The thesis was written from the 23rd of January to the 18th of June. I would like to thank my supervisor Øyvind Solberg for fantastic help and feedback along the way. I would also like to thank my classmates for encouragement in the most challenging phases.

Contents

Chapter 1

Introduction

The aim of this thesis is to develop a method for computing almost split sequences.

Chapter 2 will serve as an introduction to the theory which will be required for our work. A series of topics will be discussed, and important result will be stated and demonstrated.

In Chapter 3 we will embark on the task of designing an algorithm for computing almost split sequences. Our approach will be divided into two main steps:

In Section 3.1 we will apply a variety of results from Chapter 2 in order to design an isomorphism which depends on certain parameters.

In Section 3.2 we will fix these parameters. Together with prior results, the isomorphism we then get will suggest a connection between identity homomorphisms and almost split sequences which motivates a deterministic algorithm for computing the latter. We will complete Chapter 3 with a presentation of this algorithm, along with a demonstration of its correctness.

Chapter 2

Background Theory

In this chapter we will traverse a series of topics from representation theory. Section 2.1 will provide a review of basic category theory. In Section 2.2 we will give the definition of artinian rings, artinian modules and artin algebras, and, at this, set the framework for this thesis. In the subsequent sections a multitude of functors will be presented together with a survey of their respective features. A formal definition along with basic properties of almost split sequences will be presented in Section 2.9. In section 2.10 we will study a certain algebra arising from prior investigation, and we will make useful observations regarding its top and socle.

2.1 Category theory

In this thesis we shall mainly focus on modules over artin algebras, but before we introduce the framework in which we will be working most of the time, we recall the following concepts from category theory:

Definition 1.

(i) A *category* $\mathcal{C}$ consists of the following:

- a class $\mathrm{Ob}(\mathcal{C})$ of objects,
- for X, $Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of morphisms from X to Y,
- for X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, a binary operation

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Z) \\ (g,f) &\mapsto gf \end{aligned}$$

such that the following holds:

* For all X, Y, Z, $W \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(Z,W)$, then

$$(fg)h = f(gh).$$

* For all $X \in \mathrm{Ob}(\mathcal{C})$, then there is an *identity morphism* $1_X \in \mathrm{Hom}_{\mathcal{C}}(X,X) := \mathrm{End}_{\mathcal{C}}(X)$ such that for all $Y \in \mathrm{Ob}(\mathcal{C})$, then

$$f 1_X = 1_X$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, and

$$1_X f = f$$

for all $f \in \mathrm{Hom}_C(Y,X)$.

For a morphism set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ where X, $Y \in \mathrm{Ob}(\mathcal{C})$, then X is called the *source object of* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, and Y is called the *target object of* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

(ii) An *additive category* is a category $\mathcal{C}$ where

- $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an abelian group for all X, $Y \in \mathrm{Ob}(\mathcal{C})$, such that the following holds:

$$j(f+g)h = jfh + jgh$$

for all W, X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$ and $h \in \mathrm{Hom}_{\mathcal{C}}(W,X)$, f, $g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $j \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$,

- there is a zero object $0 \in \mathrm{Ob}(\mathcal{C})$ such that

$$|\,\mathrm{Hom}_{\mathcal{C}}(X,0)| = |\,\mathrm{Hom}_{\mathcal{C}}(0,X)| = 1$$

for all $X \in \mathrm{Ob}(\mathcal{C})$,

- for all X, $Y \in \mathrm{Ob}(\mathcal{C})$ there is $X \oplus Y \in \mathrm{Ob}(\mathcal{C})$ together with $\iota_X \in \mathrm{Hom}_{\mathcal{C}}(X, X\oplus Y)$, $\iota_Y \in \mathrm{Hom}_{\mathcal{C}}(Y, X\oplus Y)$, $\pi_X \in \mathrm{Hom}_{\mathcal{C}}(X\oplus Y, X)$ and $\pi_Y \in \mathrm{Hom}_{\mathcal{C}}(X\oplus Y, Y)$ such that the following holds:

$$\pi_X \iota_X = 1_X,$$

$$\pi_Y \iota_Y = 1_Y,$$

$$\pi_Y \iota_X = 0,$$

$$\pi_X \iota_Y = 0$$

and

$$\iota_X \pi_X + \iota_Y \pi_Y = 1_{X\oplus Y}.$$

The ι's and π's are called *inclusions* and *projections*.

(iii) Let $\mathcal{C}$ be an additive category, and let $X, Y \in \mathrm{Ob}(\mathcal{C})$. Suppose $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

- A *kernel of* f is an object $\mathrm{Ker}(f) \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $\iota_f \in \mathrm{Hom}_C(\mathrm{Ker}(f), X)$ such that the following holds:
  * $f\iota_f = 0$,

* For all $T \in \mathrm{Ob}(\mathcal{C})$ and $t \in \mathrm{Hom}_{\mathcal{C}}(T, X)$ such that $ft = 0$, then there exists unique $s \in \mathrm{Hom}_{\mathcal{C}}(T, \mathrm{Ker}(f))$ such that

$$t = \iota_f s.$$

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\iota_f"] & X \arrow[r, "f"] & Y \\
T \arrow[u, dashed, "s"] \arrow[ur, "t"'] & &
\end{tikzcd}$$

- A *cokernel of* f is an object $\mathrm{Cok}(f) \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $\pi_f \in \mathrm{Hom}_C(Y, \mathrm{Cok}(f))$ such that the following holds:
  * $\pi_f f = 0$,
  * For all $T \in \mathrm{Ob}(\mathcal{C})$ and $t \in \mathrm{Hom}_{\mathcal{C}}(Y, T)$ such that $tf = 0$, then there exists unique $s \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Cok}(f), T)$ such that

$$t = s\pi_f.$$

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "\pi_f"] \arrow[dr, "t"'] & \mathrm{Cok}(f) \arrow[d, dashed, "s"'] \\
& & T
\end{tikzcd}$$

(iv) An *abelian category* is an additive category $\mathcal{C}$ such that for all X, $Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, then f has a kernel and a cokernel, and moreover,

$$\mathrm{Cok}(\iota_f) \simeq \mathrm{Ker}(\pi_f).$$

(v) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant (contravariant) functor*

$$F : \mathcal{C} \to \mathcal{D}$$

consists of a map

$$F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and, for all X, $Y \in \mathrm{Ob}(\mathcal{C})$, a map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

$$(\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X)))$$

such that the following statements hold:

- $F(1_X) = 1_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$,

- If X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, then
$$F(gf) = F(g)F(f)$$
$$(F(gf) = F(f)F(g)).$$

(vi) Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, and let F be a covariant (contravariant) functor
$$F : \mathcal{C} \to \mathcal{D}.$$
Moreover, let
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
be an exact sequence in $\mathcal{C}$. (We assume that the notion of an exact sequence is familiar to the reader.) We say that F is

- *left exact* if
$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$
$$\left(0 \longrightarrow F(C) \longrightarrow F(B) \longrightarrow F(A) \right)$$
is an exact sequence in $\mathcal{D}$,
- *right exact* if
$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$
$$\left(F(C) \longrightarrow F(B) \longrightarrow F(A) \longrightarrow 0 \right)$$
is an exact sequence in $\mathcal{D}$,
- *exact* if
$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$
$$\left(0 \longrightarrow F(C) \longrightarrow F(B) \longrightarrow F(A) \longrightarrow 0 \right)$$
is an exact sequence in $\mathcal{D}$.

(vi) Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let F and G be covariant (contravariant) functors from $\mathcal{C}$ to $\mathcal{D}$.

- A *natural transformation* $\alpha : F \to G$ consists of, for all $X \in \mathrm{Ob}(\mathcal{C})$, a morphism $\alpha_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$, such that:
For any X, $Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ then
$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

$$\left(\begin{array}{c}\begin{tikzcd} F(Y) \arrow[r, "\alpha_Y"] \arrow[d, "F(f)"] & G(Y) \arrow[d, "G(f)"] \\ F(X) \arrow[r, "\alpha_X"] & G(X) \end{tikzcd}\end{array}\right)$$

commutes. We may denote the natural transformation by $\{\alpha_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ as well as by α, and we equivalently say that α_X is natural in X.

– We say that F and G are *naturally isomorphic* if there exists a natural transformation

$$\alpha : F \to G$$

such that, for all $X \in \mathrm{Ob}(\mathcal{C})$, then α_X is an isomorphism. In this case, we write

$$F \underset{\text{nat.}}{\simeq} G.$$

We often write $X \in \mathcal{C}$ in stead of $X \in \mathrm{Ob}(\mathcal{C})$ for an object X of a category $\mathcal{C}$. The following lemma states that inverses of and compositions of natural transformations, are in turn natural.

Lemma 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories.*

(i) *Let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$, and let*

$$\{\alpha_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))\}_{X \in \mathcal{C}} : F \to G$$

be a natural transformation. Moreover, suppose that for any $X \in \mathcal{C}$ there is $\varphi_X \in \mathrm{Hom}_{\mathcal{D}}(G(X), F(X))$ such that

$$\alpha_X \varphi_X = 1_{G(X)}$$

and

$$\varphi_X \alpha_X = 1_{F(X)}.$$

Then $\{\varphi_X\}_{X \in \mathcal{C}}$ is a natural transformation $G \to F$.

(ii) *Let F, G and H be functors from $\mathcal{C}$ to $\mathcal{D}$. Let $\alpha : F \to G$ and $\beta : G \to H$ be natural transformations. Then $\beta\alpha : F \to H$ defined by*

$$(\beta\alpha)_X := \beta_X \circ \alpha_X$$

is a natural transformation.

Proof.

(i) Suppose F and G are covariant (contravariant) functors. By the definition of a natural transformation, then $\{\varphi_X\}_{X \in \mathcal{C}} : G \to F$ is a natural transformation if and only if the following diagram commutes:

$$\begin{tikzcd} G(X) \arrow[r, "\varphi_X"] \arrow[d, "G(f)"] & F(X) \arrow[d, "F(f)"] \\ G(Y) \arrow[r, "\varphi_Y"] & F(Y) \end{tikzcd} \left(\begin{tikzcd} G(Y) \arrow[r, "\varphi_Y"] \arrow[d, "G(f)"] & F(Y) \arrow[d, "F(f)"] \\ G(X) \arrow[r, "\varphi_X"] & F(X) \end{tikzcd} \right) \tag{2.1}$$

That is, we must show that

$$F(f)\varphi_X = \varphi_Y G(f) \tag{2.2}$$
$$(F(f)\varphi_Y = \varphi_X G(f)) \tag{2.3}$$

for all $X \in \mathcal{C}$. Since $\{\alpha_X\}_{X\in\mathcal{C}}$ is a natural transformation, then

$$G(f)\alpha_X = \alpha_Y F(f)$$
$$(G(f)\alpha_X = \alpha_Y F(f))$$

for all $X \in \mathcal{C}$. Composing with φ_Y (φ_X) from the left and φ_X (φ_Y) from the right, we get

$$\varphi_Y G(f)\alpha_X\varphi_X = \varphi_Y\alpha_Y F(f)\varphi_X$$
$$(\varphi_X G(f)\alpha_Y\varphi_Y = \varphi_X\alpha_X F(f)\varphi_Y,)$$

hence (2.2) ((2.3)) holds.

(ii) We assume F, G and H are covariant functors. Let X, $Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Then

$$(\beta\alpha)_Y F(f) = \beta_Y\alpha_Y F(f) = \beta_Y G(f)\alpha_X = H(f)\beta_X\alpha_X = H(f)(\beta\alpha)_X,$$

hence the following diagram is commutative:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, dashed, "\alpha_X"] \arrow[dd, bend right, "(\beta\alpha)_X"'] & F(Y) \arrow[d, dashed, "\alpha_Y"'] \arrow[dd, bend left, "(\beta\alpha)_Y"] \\ G(X) \arrow[r, "G(f)"] \arrow[d, dashed, "\beta_X"] & G(Y) \arrow[d, dashed, "\beta_Y"'] \\ H(X) \arrow[r, "H(f)"] & H(Y) \end{tikzcd}$$

Thus $\beta\alpha$ is a natural transformation. The proof is similar if F, G and H are contravariant.

$\square$

We will now look at two important types of functors, namely the covariant and the contravariant hom functors.

Lemma 3. *Let $\mathcal{C}$ be a category, and let $X \in \mathcal{C}$.*

(i) *There is a covariant functor*

$$\operatorname{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Set}$$

defined by

$$\operatorname{Hom}_{\mathcal{C}}(X, -)(Y) := \operatorname{Hom}_{\mathcal{C}}(X, Y)$$

for $Y \in \mathcal{C}$, and for any Y, $Z \in \mathcal{C}$ and $f \in \operatorname{Hom}_{\mathcal{C}}(Y, Z)$, then

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(X, -)(f) : \operatorname{Hom}_{\mathcal{C}}(X, Y) &\to \operatorname{Hom}_{\mathcal{C}}(X, Z) \\ g &\mapsto fg. \end{aligned}$$

(ii) *There is a contravariant functor*

$$\operatorname{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \to \mathrm{Set}$$

defined by

$$\operatorname{Hom}_{\mathcal{C}}(-, X)(Y) := \operatorname{Hom}_{\mathcal{C}}(Y, X)$$

for $Y \in \mathcal{C}$, and for any Y, $Z \in \mathcal{C}$ and $f \in \operatorname{Hom}_{\mathcal{C}}(Y, Z)$, then

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(-, X)(f) : \operatorname{Hom}_{\mathcal{C}}(Z, X) &\to \operatorname{Hom}_{\mathcal{C}}(Y, X) \\ g &\mapsto gf. \end{aligned}$$

Proof.

(i) It is evident that $\operatorname{Hom}_{\mathcal{C}}(X, Y) \in \mathrm{Set}$, and that

$$\operatorname{Hom}_{\mathcal{C}}(X, -)(1_Y) = [g \mapsto 1_Y g = g] = 1_{\operatorname{Hom}_{\mathcal{C}}(X,-)(Y)}$$

for all $Y \in \mathcal{C}$. For Y, Z and $W \in \mathcal{C}$, $f_1 \in \operatorname{Hom}_{\mathcal{C}}(Y, Z)$ and $f_2 \in \operatorname{Hom}_{\mathcal{C}}(Z, W)$, then

$$\begin{aligned} (\operatorname{Hom}_{\mathcal{C}}(X, -)(f_2 f_1))(g) &= (f_2 f_1)g \\ &= f_2(f_1 g) \\ &= (\operatorname{Hom}_{\mathcal{C}}(X, -)(f_2))(f_1 g) \\ &= \operatorname{Hom}_{\mathcal{C}}(X, -)(f_2) \operatorname{Hom}_{\mathcal{C}}(X, -)(f_1)(g) \end{aligned}$$

for all $g \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, hence

$$\operatorname{Hom}_{\mathcal{C}}(X, -)(f_2 f_1) = \operatorname{Hom}_{\mathcal{C}}(X, -)(f_2) \operatorname{Hom}_{\mathcal{C}}(X, -)(f_1).$$

(ii) Similar to (i).

$\square$

We hereby introduce a compact, yet informative way of writing the resulting morphism when applying a hom functor to a morphism.

Definition 4. Let $\mathcal{C}$ be a category, and let $X \in \mathcal{C}$. We denote $\mathrm{Hom}_{\mathcal{C}}(X,-)(f)$ by $(f \circ -)_X$, since it takes a morphism and composes it with f from the left hand side. Similarly, we denote $\mathrm{Hom}_{\mathcal{C}}(X,-)(f)$ by $(- \circ f)_X$, since it takes a morphism and composes it with f from the right hand side.

We observe that the hom functors are in fact left exact. That is, in the environment where left exactness is defined, namely for for abelian categories.

Lemma 5. *If $\mathcal{C}$ is an abelian category and $X \in \mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(X,-)$ and $\mathrm{Hom}_{C}(-,X)$ are left exact functors*

$$\mathcal{C} \to \mathrm{Ab}\,.$$

Proof. By the very definition of an abelian category, then $\mathrm{Hom}_{\mathcal{C}}(X,-)$ and $\mathrm{Hom}_{C}(-,X)$ are functors

$$\mathcal{C} \to \mathrm{Ab}\,.$$

We show the left exactness of $\mathrm{Hom}_{\mathcal{C}}(X,-)$. Let

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

be an exact sequence in $\mathcal{C}$. We need to show that

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,A) \xrightarrow{(f\circ -)_X} \mathrm{Hom}_{\mathcal{C}}(X,B) \xrightarrow{(g\circ -)_X} \mathrm{Hom}_{\mathcal{C}}(X,C)$$

is an exact sequence in Ab.

We first show that $(f \circ -)_X$ is a monomorphism. In Ab, this is the same as being injective. Suppose $g \in \mathrm{Hom}_{\mathcal{C}}(X,A)$ such that

$$(f \circ -)_X(g) = fg = 0.$$

Then since f is a monomorphism, it follows that $g = 0$.

For $h \in \mathrm{Hom}_{\mathcal{C}}(X,B)$, then

$$(f \circ -)_X(g \circ -)_X(h) = (f \circ -)_X(gh) = fgh = (fg \circ -)_X(h),$$

hence

$$(f \circ -)_X(g \circ -)_X = (fg \circ -)_X.$$

That is,

$$\mathrm{Im}((f \circ -)_X) \subseteq \mathrm{Ker}((g \circ -)_X).$$

We finally show that

$$\mathrm{Ker}((g \circ -)_X) \subseteq \mathrm{Im}((f \circ -)_X.$$

Let $h \in \mathrm{Ker}((g \circ -)_X)$. Then $gh = 0$, and since f is the kernel of g, then h factors through f. That is, there is $j \in \mathrm{Hom}_{\mathcal{C}}(X, A)$ such that

$$h = fj = (f \circ -)_X(j),$$

hence $h \in \mathrm{Im}((f \circ -)_X)$.

By similar arguments, $\mathrm{Hom}_{\mathcal{C}}(-, X)$ is a left exact functor. □

Definition 6. We let

(i) Set := the category of sets, where the morphisms are maps,

(ii) Ab := the category of abelian groups, where the morphisms are abelian group homomorphisms,

(iii) $\mathrm{Mod}(S)$:= the category of left S-modules for a ring S, where the morphisms are S-module homomorphisms.

Note that the categories of (ii) and (iii) of the above definition are abelian categories. The abelian group structure on the hom sets originates from the abelian group structure on the objects themselves; in particular, from that on the target objects: For example, for X, $Y \in$ Ab and f, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, then

$$(f + g)(x) := f(x) + g(x)$$

for all $x \in X$.

Suppose $\mathcal{C}$ is an abelian category, and consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r, "\iota_f"] \arrow[d, dashed, "u_{\mathrm{Ker}}"] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "\pi_f"] \arrow[d, "v"] & \mathrm{Cok}(f) \arrow[r] \arrow[d, dashed, "v_{\mathrm{Cok}}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(g) \arrow[r, "\iota_g"] & C \arrow[r, "g"] & D \arrow[r, "\pi_g"] & \mathrm{Cok}(g) \arrow[r] & 0
\end{tikzcd} \tag{2.4}$$

It can be shown that there exist unique $\mathcal{C}$-homomorphisms $u_{\mathrm{Ker}} \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Ker}(f), \mathrm{Ker}(g))$ and $v_{\mathrm{Cok}} \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Cok}(f), \mathrm{Cok}(g))$ such that the above diagram commutes.

Definition 7. The *kernel map of u* and the *cokernel map of v (with respect to Diagram 2.4)* are the $\mathcal{C}$-homomorphisms $u_{\mathrm{Ker}} \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Ker}(f), \mathrm{Ker}(g))$ and $v_{\mathrm{Cok}} \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Cok}(f), \mathrm{Cok}(g))$ making Diagram 2.4 commutative. We will stick to the subscripts Ker and Cok for kernel maps and cokernel maps throughout this thesis.

The fact that u_{Ker} and v_{Cok} from Definition 7 are dependent on all of Diagram 2.4 and not only on u and v, respectively, suggests a more clarifying notation for these morphisms. However, whenever this notation is used it will be clear from the context which diagram the kernel or cokernel morphism originates from, and we will omit specifying this explicitly.

We now turn our attention to module categories. For a ring S then the left S^{op}-modules are the right S-modules. Throughout this thesis we will be referring to the left S-modules as merely S-modules, and to the right S-modules as S^{op}-modules. Also, for a homomorphism set $\mathrm{Hom}_{\mathrm{Mod}(S)}(A,B)$ where A, $B \in \mathrm{Mod}(S)$, we will instead write $\mathrm{Hom}_S(A,B)$.

The hom functors conveniently commute with direct sums, as stated by Lemma 8.

Lemma 8. *Let $\oplus_{i=1}^n X_i$ be a direct sum in $\mathrm{Mod}(S)$ for some ring S, with given inclusions*

$$\nu_i : X_i \to \oplus_{i=1}^n X_i$$

and projections

$$\rho_i : \oplus_{i=1}^n X_i \to X_i$$

for $1 \leq i \leq n$. Then the following holds for any $Y \in \mathrm{Mod}(S)$.

(i) *There is an isomorphism of sets*

$$\begin{aligned} \xi : \mathrm{Hom}_S(\oplus_{i=1}^n X_i, Y) &\to \oplus_{i=1}^n \mathrm{Hom}_S(X_i, Y) \\ f &\mapsto \{f\nu_i\}_{i=1}^n, \end{aligned}$$

whose inverse is given by

$$\begin{aligned} \xi^{-1} : \oplus_{i=1}^n \mathrm{Hom}_S(X_i, Y) &\to \mathrm{Hom}_S(\oplus_{i=1}^n X_i, Y) \\ \{f_i\}_{i=1}^n &\mapsto \sum_{i=1}^n f_i\rho_i. \end{aligned}$$

(ii) *If either $Y = S$ or $X_i = S$ for $1 \leq i \leq n$, then the homomorphism sets in question are S^{op}-modules, and ξ and ξ^{-1} are isomorphisms of S^{op}-modules.*

Proof.

(i) Suppose $f \in \mathrm{Hom}_S(\oplus_{i=1}^n X_i, Y)$. Then

$$\xi^{-1}\xi(f) = \xi^{-1}(\{f\nu_i\}_{i=1}^n) = \sum_{i=1}^n f\nu_i\rho_i = f \underbrace{\sum_{i=1}^n \nu_i\rho_i}_{=1_{\oplus_{i=1}^n X_i}} = f,$$

hence

$$\xi^{-1}\xi = 1_{\mathrm{Hom}_S(\oplus_{i=1}^n X_i, Y)}.$$

Suppose $\{f_i\}_{i=1}^n \in \oplus_{i=1}^n \mathrm{Hom}_S(X_i, Y)$. Then

$$\xi\xi^{-1}(\{f_i\}_{i=1}^n) = \xi\left(\sum_{i=1}^n f_i\rho_i\right) = \left\{\left(\sum_{i=1}^n f_i\rho_i\right)\nu_j\right\}_{j=1}^n = \left\{\sum_{i=1}^n f_i(\rho_i\nu_j)\right\}_{j=1}^n.$$

Note that $\rho_i \nu_j = 1_{X_j}$ for $i = j$ and 0 otherwise, hence

$$\sum_{i=1}^{n} f_i(\rho_i \nu_j) = f_j,$$

implying that

$$\xi \xi^{-1} = 1_{\oplus_{i=1}^{n} \mathrm{Hom}_S(X_i, Y)}.$$

(ii) We leave this as an exercise.

$\square$

We can identify a module over a ring S with the set of S-module homomorphisms from S to the module, because of the following result.

Lemma 9. *Let S be a ring, and let $M \in \mathrm{Mod}(S)$. Then*

(i) $\mathrm{Hom}_S(S, M)$ *is an S-module with the following multiplication*

$$S \times \mathrm{Hom}_S(S, M) \to \mathrm{Hom}_S(S, M) :$$

For $s \in S$ and $f \in \mathrm{Hom}_S(S, M)$, then

$$(s \cdot f)(a) := f(as)$$

for $a \in S$.

(ii) *The map*

$$\begin{aligned} \xi_M : \mathrm{Hom}_S(S, M) &\to M \\ f &\mapsto f(1_S) \end{aligned}$$

is an isomorphism of S-modules.

(iii) *In the case that $M = S$, then*

$$\xi_S : \mathrm{End}_S(S) \to S^{\mathrm{op}}$$

is a ring isomorphism.

Proof.

(i) For $s \in S$, we must check that $sf \in \mathrm{Hom}_S(S, M)$. For s', a, $a' \in S$, then

$$\begin{aligned} (sf)(s'a + a') &= f((s'a + a')s) \\ &= f(s'(as)) + f(a's) \\ &= s' f(as) + f(a's) \\ &= s'(sf)(a) + (sf)(a'). \end{aligned}$$

Let s, $s' \in S$ and f, $f' \in \mathrm{Hom}_S(S, M)$. It is obvious that

$$(s + s')f = sf + s'f$$

and

$$s(f + f') = sf + sf'.$$

For any $a \in S$, then

$$((ss')f)(a) = f(a(ss')) = f((as)s') = (s'f)(as) = (s(s'f))(a),$$

hence

$$(ss')f = s(s'f).$$

(ii) We first show that ξ_M is an S-module homomorphism. Let $s \in S$ and f, $f' \in \mathrm{Hom}_S(S, M)$. Then

$$\xi_M(sf + g) = (sf + g)(1_S) = sf(1_S) + g(1_S) = s\xi_M(f) + \xi_M(g).$$

We now show that that ξ_M is bijective. If $\xi_M(f) = 0$, then $f(1_S) = 0$, implying that

$$f(s) = sf(1_S) = 0$$

for all $s \in S$, hence $f = 0$. Then ξ_M is injective.

For any $m \in M$, we leave it up to the reader to check that

$$f_m(s) := sm$$

for $s \in S$ defines an S-module homomorphism $f_m \in \mathrm{Hom}_S(S, M)$. Then

$$\xi_M(f_m) = f_m(1_S) = m.$$

Thus ξ_M is surjective.

(iii) Let f_1, $f_2 \in \mathrm{End}(S)$. Then

$$\begin{aligned}
\xi_S(f_1 f_2) &= (f_1 f_2)(1_S) \\
&= f_1(f_2(1_S)) \\
&= f_1(f_2(1_S)1_S) \\
&= f_2(1_S)f_1(1_S) \\
&= \xi_S(f_2)\xi_S(f_1),
\end{aligned}$$

hence ξ_S is a ring isomorphism.

□

We finally make a useful observation regarding exactness of sequences in $\mathrm{Mod}(S)$ (for a ring S):

Lemma 10. *Let S be a ring, and let*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "\alpha"', "\simeq"] & C \arrow[d, "\beta"', "\simeq"] & & \\ & & B' \arrow[r, "u"] & C' \arrow[r, "v"] & D' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of exact rows in $\mathrm{Mod}(S)$*. Then*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{v\beta} D' \longrightarrow 0$$

is an exact sequence of S-modules.

Proof. The exactness in A and B follows from the exactness of the given diagram. Since v and β are epimorphisms then so is the composition $v\beta$, so the sequence is exact in D'. Since $(v\beta)g = (vu)\alpha = 0$, then $\mathrm{Im}(g) \subseteq \mathrm{Ker}(v\beta)$. We need to show that $\mathrm{Ker}(v\beta) \subseteq \mathrm{Im}(g)$.

Let $c \in \mathrm{Ker}(v\beta)$, that is, $v\beta(c) = 0$. Then $\beta(c) \in \mathrm{Ker}(v) = \mathrm{Im}(u)$, so there is $b' \in B'$ such that

$$u(b') = \beta(c).$$

Then $\alpha^{-1}(b') \in B$ such that

$$\beta g \alpha^{-1}(b') = u(b') = \beta(c).$$

Since β is a monomorphism, this implies that

$$g(\alpha^{-1}(b')) = c,$$

hence $c \in \mathrm{Im}(g)$. □

The reader should be familiar with the notions of projective and injective modules as well as the material traversed in this section. For a definition of these concepts along with some basic results, we refer to [1, Ch. 5]. Note especially the relation between projective and injective modules and exactness of hom functors of Proposition 16.9.

2.2 Our framework

In this thesis we will be studying modules over a fixed R-algebra Λ,[1] where R is a given ring. A lot of the results will depend on certain properties exhibited by R and Λ, which we will dedicate this section to be familiarized with. First of all, we shall assume that R is commutative. We now give the definition of the second condition which we will have on R.

Definition 11. A ring R is *local* if it has a unique maximal ideal.

[1] An R-algebra will be defined formally in Definition 16(i).

It is well-known that a ring R is local if and only if its non-units form an ideal. This equivalent definition is used in [2, Ch. 1].

Note that any factor of a local ring is in turn local.

Lemma 12. *If R is a local ring then R/J is a local ring for any proper ideal J in R.*

Proof. Let R be a local ring, and suppose J is a proper ideal in R. Let $\underline{m}$ denote the maximal ideal in R. Then $\underline{m}/J$ is an ideal in R/J, and we claim that it is the unique maximal ideal.

Suppose Y is a non-trivial, proper ideal in R/J. Then Y is of the form

$$Y = X/J,$$

where X is an ideal in R such that

$$J \subsetneq X \subsetneq R.$$

Since $\underline{m}$ is the maximal ideal in R then

$$X \subseteq \underline{m},$$

hence

$$Y = X/J \subseteq \underline{m}/J.$$

□

The last condition on our ring R will be that is is an artinian ring. This property is defined below along with the similar concept of noetherianness.

Definition 13.

(i) A *left (right) artinian ring* R is a ring such that any descending chain of left (right) ideals stabilizes. That is, given a descending chain

$$R = I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$$

of left (right) ideals in R, then there is $N \in \mathbb{N}$ such that $I_n = I_m$ for all m, $n \geq N$.

(ii) A *left (right) noetherian ring* R is a ring such that any ascending chain of left (right) ideals stabilizes. That is, given an ascending chain

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq ...$$

of left (right) ideals in R, then there is $N \in \mathbb{N}$ such that $I_n = I_m$ for all m, $n \geq N$.

The following definition is analogous to Definition 13, but for R-modules.

Definition 14.

(i) An *artinian R-module M* is an R-module M such that any descending chain

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$$

of R-sumbodules of M stabilizes. That is, there is $N \in \mathbb{N}$ such that $M_n = M_m$ for all $m,\, n \geq N$.

(ii) A *noetherian R-module M* is an R-module M such that any ascending chain

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$$

of R-sumbodules of M stabilizes. That is, there is $N \in \mathbb{N}$ such that $M_n = M_m$ for all $m,\, n \geq N$.

The following lemma is useful in situations where it is desirable to derive artinianness or noetherianness for an R-module.

Lemma 15. *Let*

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

be an exact sequence of R-modules. Then the two following statements are equivalent.

(i) *B is an artinian (noetherian) R-module.*

(ii) *A and C are artinian (noetherian) R-modules.*

Proof. We will prove the lemma for artinian R-modules. The proof in the case of noetherian R-modules is similar.

(i) $\Rightarrow$ (ii) : Suppose B is an artinian R-module.

We first show that C is an artinian R-module. Let

$$C = C_0 \supseteq C_1 \supseteq C_2 \supseteq ... \tag{2.5}$$

be a descending chain of R-submodules of C. We need to show that (2.5) stabilizes. Note that (2.5) induces a descending chain

$$B = g^{-1}(C_0) \supseteq g^{-1}(C_1) \supseteq g^{-1}(C_2) \supseteq ... \tag{2.6}$$

of R-submodules of B. We claim that if $g^{-1}(C_n) = g^{-1}(C_m)$, then $C_n = C_m$. Assume $g^{-1}(C_n) = g^{-1}(C_m)$, and let $c \in C_n$. Since g is onto C, there is $b \in B$ such that $g(b) = c$. Hence $b \in g^{-1}(C_n)$, then by hypothesis $b \in g^{-1}(C_m)$. This means that $g(b) \in C_m$. Thus $C_n \subseteq C_m$. By symmetry, we conclude that $C_n = C_m$.

Since B is an artinian R-module, we know that (2.6) stabilizes. That is, there is $N \in \mathbb{N}$ such that $g^{-1}(C_n) = g^{-1}(C_m)$ for all $m,\, n \geq N$. Then by the above result, $C_n = C_m$ for all $m,\, n \geq N$, thus (2.5) stabilizes.

We now show that A is an artinian R-module. Let

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq ... \tag{2.7}$$

be a descending chain of R-submodules of A. We need to show that (2.7) stabilizes. Note that (2.7) induces a descending chain

$$B = f(A_0) \supseteq f(A_1) \supseteq f(A_2) \supseteq ... \tag{2.8}$$

of R-submodules of B. We claim that if $f(A_n) = f(A_m)$, then $A_n = A_m$. Assume $f(A_n) = f(A_m)$, and let $a \in A_n$. Then $f(a) \in f(A_n)$, thus by hypothesis, $f(a) \in f(A_m)$. Then there is $a' \in A_m$ such that $f(a') = f(a)$. Since f is injective, then $a = a'$, thus $a \in A_m$. Then we have shown that $A_n \subseteq A_m$, hence, by symmetry, $A_n = A_m$.

Again, since B is an artinian R-module, we know that (2.8) stabilizes. That is, there is $M \in \mathbb{N}$ such that $f(A_n) = f(A_m)$, and thus $A_n = A_m$, for all m, $n \geq M$. Hence (2.7) stabilizes. Then A and C are proven to be artinian R-modules.

(ii) $\Rightarrow$ (i): Suppose A and C are artinian R-modules. Let

$$B = B_0 \supseteq B_1 \supseteq B_2 \supseteq ... \tag{2.9}$$

be a descending chain of R-submodules of B. We need to show that (2.9) stabilizes. As in the first part of the proof, we consider the induced descending chains

$$C = g(B_0) \supseteq g(B_1) \supseteq g(B_2) \supseteq ... \tag{2.10}$$

$$A = f^{-1}(B_0) \supseteq f^{-1}(B_0) \supseteq f^{-1}(B_0) \supseteq \tag{2.11}$$

of R-submodules of A and C, respectively. Since A and C are artinian, then (2.10) and (2.11) stabilize; there is $N \in \mathbb{N}$ such that $g(B_n) = g(B_m)$ and $f^{-1}(B_n) = f^{-1}(B_m)$ for all m, $n \geq N$. We claim that $B_n = B_m$ for all m, $n \geq N$.

Consider some fixed m, $n \geq N$, and suppose $B_m \neq B_n$. Since the B_i's arise from a descending chain, it is clear that one of these submodules is contained in the other. Without loss of generality we assume $B_n \subseteq B_m$. Suppose $b \in B_m \backslash B_n$. Then

$$g(b) \in g(B_m) = g(B_n),$$

thus there is $b' \in B_n$ such that

$$g(b') = g(b).$$

Then

$$b - b' \in \mathrm{Ker}(g) = \mathrm{Im}(f), \tag{2.12}$$

Moreover, $b' \in B_n \subseteq B_m$, so

$$b - b' \in B_m. \tag{2.13}$$

By (2.12) there is $a \in A$ such that $f(a) = b - b'$, and by (2.13),

$$a \in f^{-1}(B_m) = f^{-1}(B_n).$$

Then $f(a) = b - b' \in B_n$, so

$$b = (b - b') + b' \in B_n,$$

contradicting the assumption. We conclude that $B_m = B_n$ for all m, $n \geq N$, thus (2.9) stabilizes. This completes the proof; B is proven to be an artinian R-module.

□

A special case of an artinian R-module is an artin R-algebra. Consider the following definition.

Definition 16.

(i) An *R-algebra* Λ is a ring which is also an R-module, such that the following holds: For all α, β, $\lambda \in \Lambda$ and r, $s \in R$, then

$$(r\alpha + s\beta)\lambda = r(\alpha\lambda) + s(\beta\lambda)$$

and

$$\alpha(r\beta + s\lambda) = r(\alpha\beta) + s(\alpha\lambda).$$

(ii) An *artin R-algebra* is an R-algebra which is finitely generated as R-module.

Informally, the length of an S-module M (over some ring S) is obtained by considering all ways of writing descending chains of S-submodules of M where all containments are proper, and taking the length of the longest such. For a more technical definition, see [2, Ch. 1].

It can be shown that a module over an artin R-algebra is finitely generated if and only if it has finite length, and this again occurs if and only if the module is noetherian. We shall implicitly use this result throughout this thesis, as we from now on let Λ denote a fixed artin R-algebra. Note that Λ^{op} is then also an artin R-algebra, hence a result involving Λ and Λ^{op} is generally still valid if these two rings are interchanged. Particularly, a functor (with no parameters from Λ or Λ^{op})

$$\text{mod}(\Lambda) \to \text{mod}(\Lambda^{\text{op}})$$

is also a functor

$$\text{mod}(\Lambda^{\text{op}}) \to \text{mod}(\Lambda).$$

Definition 17. Let S be a ring.

(i) For any $M \in \text{Mod}(S)$, we let $l_S(M)$ denote the length of M as S-module.

(ii) We let mod(S) be the full subcategory of Mod(S) whose objects are the S-modules of finite length;

$$\mathrm{Ob}(\mathrm{mod}(S)) := \{M \in \mathrm{Ob}(\mathrm{Mod}(S)) \mid l_S(M) < \infty\}.$$

Note that mod(S) is also an abelian category, because the lengths of the kernel and the cokernel of a morphism in mod(S) must also be of finite length. Whenever we apply a hom functor in this thesis, it will be from one of the abelian categories which we have seen, hence, by Lemma 5, it will always be left exact.

2.2.1 Some useful results

In Lemma 19 we will see that the artin R-algebra Λ is also an artinian ring, but first we need to establish some relations between modules and homomorphisms of R and Λ in our current framework.

The next lemma is considered basic knowledge, and will occasionally be applied without reference. Note that for any ring S, then the identity map

$$1_S : S \to S$$

in $\mathrm{Hom}_S(S, S)$ can be regarded as multiplication by the identity element of S itself, so we will use the same notation 1_S for the identity element of S.

Lemma 18. *The following statements are true:*

(i) $\mathrm{Mod}(\Lambda) \subseteq \mathrm{Mod}(R)$,

(ii) $\mathrm{Hom}_\Lambda(A, B) \subseteq \mathrm{Hom}_R(A, B)$ *for all* $A,\ B \in \mathrm{Mod}(\Lambda)$,

(iii) $\mathrm{mod}(\Lambda) \subseteq \mathrm{mod}(R)$.

Proof.

(i) Let $M \in \mathrm{Mod}(\Lambda)$. Then M is an R-module under the following binary operation:

$$\begin{aligned} R \times M &\to M \\ (r, m) &\mapsto \underbrace{(r \cdot 1_\Lambda)}_{\in \Lambda} m. \end{aligned} \tag{2.14}$$

(ii) Let $A,\ B \in \mathrm{Mod}(\Lambda)$, and suppose $f \in \mathrm{Hom}_\Lambda(A, B)$. Then for $a_1,\ a_2 \in A$ and $\lambda \in \Lambda$,

$$f(\lambda a_1 + a_2) = \lambda f(a_1) + f(a_2).$$

Thus for all $r \in R$, we see that

$$f(ra_1 + a_2) \overset{(2.14)}{=} f(\underbrace{(r1_\Lambda)}_{\in \Lambda} a_1 + a_2) = (r1_\Lambda) f(a_1) + f(a_2) \overset{(2.14)}{=} rf(a_1) + f(a_2).$$

(iii) If $M \in \text{mod}(\Lambda)$, then for some $n \in \mathbb{N}$ there exists a Λ-module epimorphism

$$g : \Lambda^n \to A.$$

By (ii), g is also an R-module epimorphism.

Recall that Λ is an artin algebra, that is, Λ is finitely generated as an R-module. Then for some $m \in \mathbb{N}$ there exists an R-module epimorphism

$$h : R^m \to \Lambda,$$

and thus there is an R-module epimorphism

$$h^n : R^{mn} \to \Lambda^n.$$

By composing g with h^n we get an R-module homomorphism

$$gh^n : R^{mn} \to A,$$

thus A is finitely generated as an R-module.

□

Note that since R is a commutative ring, Lemma 18(i) implies that any Λ^{op}-module is an R-Λ-bimodule. With Lemma 15 and Lemma 18(i) at hand, we are ready to show that Λ is an artinian ring.

Lemma 19. *The artin R-algebra Λ is an artinian ring.*

Proof. Note that an R-submodule of R is the same as an ideal in R, so since R is an artinian ring then R is also an artinian R-module. We will now proceed as follows.

I) We first show that R^n is an artinian R-module.

II) We use I) to show that Λ is an artinian R-module.

III) Finally we show that II) implies that Λ is an artinian ring.

I) We show that R^n is an artinian R-module by induction on n. Suppose the statement holds for $n = k - 1$. Consider the R-module epimorphism

$$\begin{aligned} g_k : R^k &\to R^{k-1} \\ \{r_i\}_{i=1}^k &\mapsto \{r_i\}_{i=2}^k. \end{aligned}$$

We see that $\text{Ker}(g_k) = R$, thus

$$0 \longrightarrow R \longrightarrow R^k \xrightarrow{g_k} R^{k-1} \longrightarrow 0$$

is an exact sequence of R-modules. By Lemma 15, since R and R^{k-1} are artinian R-modules, then so is R^k.

II) Since Λ is a finitely generated R-algebra, we know that Λ can be written as

$$\Lambda = R\lambda_1 + R\lambda_2 + ... + R\lambda_n \tag{2.15}$$

for some finite subset $\{\lambda_i\}_{i=1}^n \subseteq \Lambda$. We claim that the maps

$$\begin{aligned} f_{\lambda_i} : R &\to \Lambda \\ r &\mapsto r\lambda_i. \end{aligned}$$

for $i \in \{1, ..., n\}$ are R-module homomorphisms. For s, r_1, $r_2 \in R$ then

$$f_{\lambda_i}(sr_1 + r_2) := (sr_1 + r_2)\lambda_i = sr_1\lambda_i + r_2\lambda_i = sf_{\lambda_i}(r_1) + f_{\lambda_i}(r_2).$$

Then

$$[f_{\lambda_1}, f_{\lambda_2}, ..., f_{\lambda_n}] : R^n \to \Lambda$$

is also an R-module homomorphism, and by (2.15) it is onto Λ. Again, by Lemma 15, then Λ is an artinian R-module.

III) Let

$$\Lambda = I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$$

be a descending chain of ideals in Λ. That is, $I_i \in \mathrm{Mod}(\Lambda)$ for all $i \geq 0$. Then by Lemma 18(i), all I_i's are R-modules, and since Λ is an artinian R-module, the chain must stabilize.

□

We often wish to show that some functor F goes from $\mathrm{mod}(S)$ to $\mathrm{mod}(S')$ for rings S and S'. Then in addition to assigning S-module structure to the codomain of F, we need to show that the resulting S-module is of finite length. For this, the following two lemmas are useful.

Lemma 20. *Let A, $B \in \mathrm{mod}(S)$ for a ring S, and suppose $f \in \mathrm{Hom}_S(A, B)$. Then $\mathrm{Cok}(f) \in \mathrm{mod}(S)$.*

Proof. For some $n \in \mathbb{N}$, there is an S-epimorphism

$$S^n \longrightarrow B \longrightarrow 0.$$

Then by composing with the canonical projection from B onto $\mathrm{Cok}(f)$, we get an S-epimorphism

$$R^n \longrightarrow \mathrm{Cok}(f) \longrightarrow 0.$$

□

If a Λ-module is finitely generated as R-module, is is also finitely generated as Λ-module:

Lemma 21. *If $M \in \mathrm{Mod}(\Lambda) \cap \mathrm{mod}(R)$, then $M \in \mathrm{mod}(\Lambda)$.*

Proof. Recall that

$$rm := (r1_\Lambda)m$$

defines the R-module structure on M. Suppose

$$\{m_1, m_2, ..., m_n\} \subseteq M$$

generates M as R-module. Then any $m \in M$ can be written as

$$m = \sum_{i=1}^{n} r_i m_i = \sum_{i=1}^{n} \underbrace{(r_i 1_\Lambda)}_{\in \Lambda} m_i \subseteq \sum_{i=1}^{n} \Lambda m_i,$$

implying that M is finitely generated as Λ-module. $\square$

In addition to being abelian groups, the homomorphism sets (in our current situation) admit miscellaneous module structures depending on their respective source and target objects. Here we will give a few examples. It might seem redundant to include both (i) and (ii) of the following lemma, but we find it useful in order to get a better grasp on how these structures interact.

Lemma 22.

(i) *Let* $M \in \mathrm{Mod}(\Lambda)$*, and let* $N \in \mathrm{Mod}(R)$*. Then* $\mathrm{Hom}_R(M,N)$ *is a* Λ^{op}*-module with multiplication*

$$\mathrm{Hom}_R(M,N) \times \Lambda \to \mathrm{Hom}_R(M,N)$$

defined by

$$(h\lambda)(m) := h(\lambda m) \tag{2.16}$$

for all $m \in M$*.*

(ii) *Let* $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$*, and let* $N \in \mathrm{Mod}(R)$*. Then* $\mathrm{Hom}_R(M,N)$ *is a* Λ*-module with multiplication*

$$\Lambda \times \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N)$$

defined by

$$(\lambda h)(m) := h(m\lambda) \tag{2.17}$$

for all $m \in M$*.*

(iii) *Let* M*,* $N \in \mathrm{Mod}(\Lambda)$*. Then* $\mathrm{Hom}_\Lambda(M,N)$ *is an* R*-module with multiplication*

$$R \times \mathrm{Hom}_\Lambda(M,N) \to \mathrm{Hom}_\Lambda(M,N)$$

defined by

$$(rh)(m) := h(rm) = r(h(m)) \tag{2.18}$$

for all $m \in M$*.*

(iv) *Let M, $N \in \mathrm{Mod}(R)$. Then* $\mathrm{Hom}_R(M,N)$ *is an R-module with multiplication*

$$R \times \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N)$$

defined by

$$(rh)(m) := h(rm) = r(h(m)) \tag{2.19}$$

for all $m \in M$.

(v) *Let M, $N \in \mathrm{Mod}(\Lambda)$. Then* $\mathrm{Hom}_\Lambda(M,N)$ *is an R-submodule of* $\mathrm{Hom}_R(M,N)$.

Proof.

(i) We first show that $h\lambda \in \mathrm{Hom}_R(M,I)$ for any $h \in \mathrm{Hom}_R(M,I)$ and $\lambda \in \Lambda$. Let $r \in R$ and m_1, $m_2 \in M$. Then

$$\begin{aligned}(h\lambda)(rm_1 + m_2) &= h(\lambda(rm_1 + m_2)) \\ &= h(\lambda(rm_1) + \lambda m_2) \\ &= h(r(\lambda m_1) + \lambda m_2) \\ &= rh(\lambda m_1) + h(\lambda m_2) \\ &= r(h\lambda)(m_1) + (h\lambda)(m_2).\end{aligned}$$

We now show that

$$h(\lambda_1\lambda_2) = (h\lambda_1)\lambda_2$$

for all $h \in \mathrm{Hom}_R(M,I)$ and λ_1, $\lambda_2 \in \Lambda$. For any $m \in M$, then

$$\begin{aligned}(h(\lambda_1\lambda_2))(m) &= h((\lambda_1\lambda_2)m) \\ &= h(\lambda_1(\lambda_2 m)) \\ &= (h\lambda_1)(\lambda_2 m) \\ &= ((h\lambda_1)\lambda_2)(m).\end{aligned}$$

We leave it up to the reader to check that the distributive laws hold for this action of Λ on $\mathrm{Hom}_R(M,I)$.

(ii) Similar to (i). Besides, if $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ then we can interpret M as a left Λ^{op}-module as well as a right Λ-module, and apply (i). Then if * denotes the multiplication

$$\Lambda^{\mathrm{op}} \times M \to M$$

and $h \in \mathrm{Hom}_R(M,N)$, $m \in M$ and $\lambda \in \Lambda$, we have

$$(\lambda h)(m) = (h * \lambda)(m) \overset{\text{(i)}}{=} h(\lambda * m) = h(m\lambda).$$

(iii) Let $r \in R$ and $h \in \mathrm{Hom}_\Lambda(M,N)$. By Lemma 18(ii) then $h \in \mathrm{Hom}_R(M,N)$, so

$$h(r \cdot m) = r \cdot h(m)$$

for all $m \in M$. For any $s \in R$ and $m \in M$, then

$$(rh)(sm) = r(h(sm)) = r(sh(m)) = s(rh(m)) = s(rh)(m),$$

so $rh \in \mathrm{Hom}_R(M, N)$. It is evident that $1_R \cdot h = h$. For associativity of multiplication of $\mathrm{Hom}_R(M, N)$ with R, we recall that R is a commutative ring, and we leave it up to the reader to verify that the following distributive laws are satisfied:

- $(r_1 + r_2)h = r_1h + r_2h$ for all r_1, $r_2 \in R$ and $h \in \mathrm{Hom}_R(M, N)$,
- $r(h_1 + h_2) = rh_1 + rh_2$ for all $r \in R$ and h_1, $h_2 \in \mathrm{Hom}_R(M, N)$,
- $rh \in \mathrm{Hom}(M, N)$ for all $r \in R$ and $h \in \mathrm{Hom}_R(M, N)$.

(iv) Similar to (iii).

(v) We have from (iii) and (iv) that $\mathrm{Hom}_\Lambda(M, N)$ and $\mathrm{Hom}_R(M, N)$ are both R-modules with the same multiplicative structure, and by Lemma 18(ii), $\mathrm{Hom}_\Lambda(M, N) \subseteq \mathrm{Hom}_R(M, N)$.

□

We have seen that given an abelian category $\mathcal{C}$ and an object $X \in \mathcal{C}$, then $\mathrm{Hom}_\mathcal{C}(X, -)$ and $\mathrm{Hom}_\mathcal{C}(-, X)$ are functors

$$\mathcal{C} \to \mathrm{Ab}\,.$$

In light of Lemma 22, it is interesting to ask whether $\mathrm{Hom}_\mathcal{C}(X, -)$ and $\mathrm{Hom}_\mathcal{C}(-, X)$ can be regarded as functors to module categories if we let their domain $\mathcal{C}$ be some module category. The answer is yes, there are multiple examples of where this occurs. Unfortunately it is too tedious to go through all of them, but demonstrations for a few cases will be carried out in the proofs of Proposition 29 and Proposition 40.

There are a series of result which are valid for finitely generated modules. It is therefore of significant whether a hom set with some module structure is finitely generated as such.

Lemma 23. *Let A, $B \in \mathrm{mod}(R)$. Then*

(i) $\mathrm{Hom}_R(A, B) \in \mathrm{mod}(R)$.

(ii) $\mathrm{Hom}_\Lambda(A, B) \in \mathrm{mod}(R)$.

Proof.

(i) For some $n \in \mathbb{N}$, there is an R-module homomorphism from R^n onto A, giving rise to the following exact sequence of R-modules:

$$R^n \longrightarrow A \longrightarrow 0 \tag{2.20}$$

We apply $\mathrm{Hom}_R(-, B)$ to (2.20). It follows from Lemma 22(iv) that the following exact sequence is of R-modules:

$$0 \longrightarrow \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(R^n, B) \tag{2.21}$$

Let $S = R$, $X_i = R$ for $i \in \{1, ..., n\}$, and $Y = B$. Then since R is commutative, by Lemma 8 there is an isomorphism of R-modules

$$\mathrm{Hom}_R(R, B)^n \to \mathrm{Hom}_R(R^n, B).$$

This result together with Lemma 9(ii) now implies that $\mathrm{Hom}_R(R^n, B) \simeq B^n$ as R-modules, and since B is finitely generated as an R-module then so is B^n. Thus B^n is noetherian, and by applying Lemma 15 to (2.21) we see that $\mathrm{Hom}(A, B)$ is a noetherian R-module. By [1, Proposition 10.9, Ch. 3], all submodules of noetherian R-modules are finitely generated as R-modules, thus $\mathrm{Hom}_R(A, B) \subseteq \mathrm{mod}(R)$.

(ii) By Lemma 22(v), $\mathrm{Hom}_\Lambda(A, B)$ is an R-submodule of $\mathrm{Hom}_R(A, B)$, thus since $\mathrm{Hom}_R(A, B)$ is a noetherian R-module, then $\mathrm{Hom}_\Lambda(A, B)$ is a finitely generated R-module.

□

The previous result has the following convenient consequence:

Proposition 24. *Let* $X \in \mathrm{mod}(\Lambda)$*. Then*

(i) $\mathrm{End}_\Lambda(X)$ *is an artin* R*-algebra.*

(ii) *If* $I \in \mathrm{End}_\Lambda(X)$ *is an ideal, then* $\mathrm{End}_\Lambda(X)/I$ *is an artin algebra.*

Proof.

(i) Lemma 18 implies that $X \in \mathrm{mod}(R)$, and then, by Lemma 23(ii), $\mathrm{End}_\Lambda(X)$ is a finitely generated R-module. Moreover, composition of morphisms defines an R-algebra structure on $\mathrm{End}_\Lambda(X)$; it is easy to see that the map

$$\begin{aligned} \mathrm{End}_\Lambda(X) \times \mathrm{End}_\Lambda(X) &\to \mathrm{End}_\Lambda(X) \\ (f, g) &\mapsto f \circ g \end{aligned}$$

is R-bilinear. Thus $\mathrm{End}_\Lambda(X)$ is an artin R-algebra.

(ii) Since $\mathrm{End}_\Lambda(X)$ is an artin algebra, then by definition, for some $n \in \mathbb{N}$ there is an R-module epimorphism

$$R^n \to \mathrm{End}_\Lambda(X).$$

By composing this R-epimorphism with the canonical projection from $\mathrm{End}_\Lambda(X)$ onto $\mathrm{End}_\Lambda(X)/I$, we see that $\mathrm{End}_\Lambda(X)/I$ is also an artin R-algebra.

$\square$

In Chapter 3 we will fix an element $X \in \mathrm{mod}(\Lambda)$, and then we will construct some isomorphisms with the usage of diagrams in $\mathrm{mod}(R)$ obtained from applying functors – studied in this section – to objects in $\mathrm{mod}(\Lambda)$, $\mathrm{mod}(\Lambda^{\mathrm{op}})$ and $\mathrm{mod}(R)$, respectively. As it turns out, the R-modules of greatest interest (namely the domains and codomains of the mentioned isomorphisms) are also endoved with either Γ-module or Γ^{op}-module structure for a factor Γ of $\mathrm{End}_\Lambda(X)$, and, more importantly, these structures are preserved by the isomorphisms. That is, we will be constructing isomorphisms of Γ-modules and of Γ^{op}-modules. The artinianness of Γ and Γ^{op} will then be a great advantage, since any result which is derived for Λ in this section is evidently also valid for Γ and Γ^{op}. [2]

flytte? Moreover, we will see in Section 2.10 that with certain conditions on $X \in \mathrm{mod}(\Lambda)$, then Γ (and similarly Γ^{op}) is a local ring whose factor modulo its radical is a simple Γ-module, a property which will be of great importance for our work in Section 3.2.2.

2.3 The dual

In this section we will study an important exact hom functor, obtained from a special injective R-module. Consider the following definition.

Definition 25. Let S be a ring.

(i) Let M, $N \in \mathrm{Mod}(S)$, and suppose $M \subseteq N$. We say that N is an *essential extension of M* if

$$X \cap N \neq 0$$

for all submodules $X \subseteq M$.

(ii) An *injective envelope I of $N \in \mathrm{Mod}(S)$* is an injective S-module I together with a monomorphism ι of S-modules

$$\iota : M \to I,$$

where I is an essential extension of $\mathrm{Im}(\iota)$.

For a ring S it can be shown that there exists an injective envelope, unique up to isomorphism in $\mathrm{Mod}(S)$, of any S-module. Since R is a commutative ring, its localness indicates the existence of a unique maximal ideal, from now denoted by $\underline{m}$. We let I be the injective envelope of the simple R-module given by $K := R/\underline{m}$.

Definition 26. The *dual* is the hom functor

$$D := \mathrm{Hom}_R(-, I).$$

[2]Our only assumption on Λ is that it is an artin R-algebra.

Note that the above definition is a little imprecise since we have not specified the domain and codomain of the dual D. This is because we wish to vary these categories. It turns out that the dual D has an interesting property when regarded as a functor between certain module categories, namely that of being a duality.

Definition 27. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A contravariant functor

$$F : \mathcal{C} \to \mathcal{D}$$

is called a *duality* if there exists a functor

$$G : \mathcal{D} \to \mathcal{C}$$

such that

$$GF \underset{\text{nat.}}{\simeq} 1_{\mathcal{C}}$$

and

$$FG \underset{\text{nat.}}{\simeq} 1_{\mathcal{D}}.$$

We shall see that for $F := D$ and for appropriate choices for $\mathcal{C}$ and $\mathcal{D}$, then D is a duality with $G = D$. In order to prove this we will need the following lemma.

Lemma 28. *There is an isomprphism of R-modules*

$$\mathrm{Hom}_K(K, I) \simeq K.$$

Proof. Let ν denote the inclusion of K into its injective envelope I, and consider the exact sequence of R-modules given by

$$0 \longrightarrow K \xrightarrow{\nu} I \xrightarrow{\mu} X \longrightarrow 0,$$

where μ is the cokernel of ν. It can be shown that $\mathrm{Hom}_K(K, -)$ is a (left exact) functor from $\mathrm{Mod}(R)$ to $\mathrm{Mod}(R)$, yielding the following exact sequence of R-modules.

$$0 \longrightarrow \mathrm{Hom}_R(K, K) \xrightarrow{(\nu \circ -)_K} \mathrm{Hom}_R(K, I) \xrightarrow{(\mu \circ -)_K} \mathrm{Hom}_R(K, X).$$

We claim that $(\mu \circ -)_I = 0$. Let $f \in \mathrm{Hom}_R(K, I)$. If $f = 0$, then $(\mu \circ -)_I(f) = \mu f$ is obviously 0. Assume $f \neq 0$. Then since

$$\mathrm{Ker}(f) \subseteq K$$

and K is a simple R-module, then $\mathrm{Ker}(f) = 0$, that is, f is injective. Since $\mathrm{Im}(f) \subseteq I$ is a nonzero submodule, and I is an essential extension of $\mathrm{Im}(\nu)$, then

$$\mathrm{Im}(f) \cap \mathrm{Im}(\nu) \neq 0.$$

Furthermore, $\mathrm{Im}(f)$ and $\mathrm{Im}(\nu)$ are both simple since they are isomorphic to K, and since $\mathrm{Im}(f) \cap \mathrm{Im}(\nu)$ is a nonzero submodule of them both, then

$$\mathrm{Im}(f) = \mathrm{Im}(f) \cap \mathrm{Im}(\nu) = \mathrm{Im}(\nu).$$

Then for any $x \in K$, we have

$$f(x) = \nu(x')$$

for some $x' \in K$, hence

$$\mu f(x) = \mu\nu(x') = 0.$$

Thus the composition

$$\mu f = 0$$

for any $f \in \mathrm{Hom}_R(K, I)$, that is, $(\mu \circ -)_I = 0$, implying that

$$(\nu \circ -)_I : \mathrm{Hom}_R(K, K) \to \mathrm{Hom}_R(K, I)$$

is an isomorphism of R-modules.

We now show that

$$\mathrm{Hom}_R(K, K) \simeq K$$

as R-modules. Consider the canonical R-module epimorphism

$$R \longrightarrow K \longrightarrow 0.$$

We apply the left exact functor $\mathrm{Hom}_R(-, K)$, and get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(K, K) \longrightarrow \mathrm{Hom}_R(R, K)$$

of R-modules. By Lemma 9(ii), then

$$\mathrm{Hom}_R(R, K) \simeq K.$$

Since K is simple and $\mathrm{Hom}_R(K, K) \neq 0$, then the image of the inclusion of $\mathrm{Hom}_R(K, K)$ into $\mathrm{Hom}_R(R, K)$ must be all of $\mathrm{Hom}_R(R, K)$, hence

$$\mathrm{Hom}_R(K, K) \simeq K$$

as claimed. This completes the proof. $\square$

We now have all the results required in order to prove the following convenient results for the dual D.

Proposition 29.

(i) *We can regard D as an exact, contravariant functor*

  (a) $D : \mathrm{mod}(R) \to \mathrm{mod}(R)$,

  (b) $D : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{\mathrm{op}})$.

(ii) *For any $M \in \mathrm{mod}(R)$ then*

$$l_R(DM) = l_R(M).$$

(iii) *The functor D is a duality in either of these three cases.*

Proof. We know that D is a contravariant, left exact functor to the category of abelian groups in either of these three cases, and because I is injective then D is exact. What needs to be proven, is that D is also a functor to the claimed codomains.

(i) (a) Let $M \in \mathrm{mod}(R)$. Then by Lemma 23(i), $\mathrm{Hom}_R(M, I) \in \mathrm{mod}(R)$. We must check that, for any M, $M' \in \mathrm{mod}(R)$ and $h \in \mathrm{Hom}_R(M, M')$, then $Dh \in \mathrm{Hom}_R(DM', DM)$. Suppose $f' \in DM'$ and $r \in R$. Then

$$\begin{aligned} Dh(rf')(m) &= (- \circ h)_I(rf')(m) \\ &= (rf'h)(m') \\ &\overset{(2.19)}{=} r(f'h)(m') \\ &= r(- \circ h)_I(f')(m) \\ &= rDh(f')(m) \end{aligned}$$

for all $m \in M$, hence

$$Dh(rf') = rDh(f').$$

We leave it up to the reader to check that

$$Dh(f_1' + f_2') = Dh(f_1') + Dh(f_2') \tag{2.22}$$

for all f_1', $f_2' \in DM'$.

(b) Suppose $M \in \mathrm{mod}(\Lambda)$. In Lemma 22(i) we saw that $\mathrm{Hom}_R(M, I) \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$. Also, $\mathrm{Hom}_R(M, I)$ is finitely generated as R-module by Lemma 23(i), hence by Lemma 21 it follows that $\mathrm{Hom}_R(M, I) \in \mathrm{mod}(\Lambda^{\mathrm{op}})$. We must also show that D takes Λ-module homomorphisms to Λ^{op}-module homomorphisms. Let M, $M' \in \mathrm{mod}(\Lambda)$ and $h \in \mathrm{Hom}_\Lambda(M, M')$. Then for $f' \in DM'$ and $\lambda \in \Lambda$, we have

$$\begin{aligned} Dh(f'\lambda)(m) &= (- \circ h)_I(f'\lambda)(m) \\ &= (f'\lambda)(\underbrace{h(m)}_{\in M'}) \\ &= f'(\lambda h(m)) \\ &= f'(h(\lambda m)) \\ &= (- \circ h)_I(f')(\lambda m) \\ &= \underbrace{Dh(f')}_{\in \mathrm{Hom}_R(M,I)}(\lambda m) \\ &= (Dh(f')\lambda)(m) \end{aligned}$$

for all $m \in M$, hence

$$Dh(f'\lambda) = Dh(f')\lambda.$$

By Lemma 18 then 2.22 also holds for f_1', $f_2' \in DM'$ in this case, and Dh is thus a Λ^{op}-module homomorphism.

(ii) Let $M \in \text{mod}(R)$. We will show that

$$l_R(DM) = l_R(M) \tag{2.23}$$

by induction on $l_R(M)$.

Suppose $l_R(M) = 1$. Then M is a simple R-module, and since R is a local ring, then R has a unique maximal ideal. This implies that R has only one simple module, thus

$$M \simeq K$$

as R-modules. Then by Lemma (28), we see that

$$\text{Hom}_R(M, I) \simeq M$$

as R-modules. It follows that

$$l_R(DM) = l_R(\text{Hom}_R(M, I)) = l_R(M) = 1.$$

Now suppose (2.23) holds for all R-modules M of length $n-1$ for some $n \in \mathbb{N}$, and let M' be an R-module of length n. Then M' has a submodule M of length $n-1$, and the cokernel of the inclusion of M into M' is simple and thus isomorphic to K. Hence

$$0 \longrightarrow M \longrightarrow M' \longrightarrow K \longrightarrow 0$$

is an exact sequence R-modules. By applying D, which was shown in to (i)(a) to be an exact functor $\text{mod}(R) \to \text{mod}(R)$, we get the following exact sequence of R-modules:

$$0 \longrightarrow DK \longrightarrow DM' \longrightarrow DM \longrightarrow 0.$$

Then by [2, Proposition 1.3, Ch. 1], we have that

$$l_R(DM') = \underbrace{l_R(DK)}_{=1} + \underbrace{l_R(DM)}_{=n-1} = n.$$

Thus (2.23) holds when $l_R(M) = n$ for all $n \in \mathbb{N}$.

(iii) (a) We claim that $\alpha : 1_{\text{mod}(R)} \to D^2$ defined by

$$\begin{aligned} \alpha_M : M &\to \text{Hom}_R(\text{Hom}_R(M, I), I) \\ m &\mapsto [f \mapsto f(m)] \end{aligned}$$

for $M \in \text{mod}(R)$ is a natural transformation of functors. Given a fixed $M \in \text{mod}(R)$, it is clear that $\alpha_M(m) \in \text{Hom}_R(\text{Hom}_R(M, I), I)$ for all

$m \in M$: Suppose $f_1,\ f_2 \in \mathrm{Hom}_R(M, I)$ and $r \in R$, then

$$\begin{aligned} \alpha_M(m)(rf_1 + f_2) &= (rf_1 + f_2)(m) \\ &= rf_1(m) + f_2(m) \\ &= r\alpha_M(f)(f_1) + \alpha_M(m)(f_2). \end{aligned}$$

Moreover, for $m_1,\ m_2 \in M$ and $r \in R$, then

$$\begin{aligned} \alpha_M(rm_1 + m_2)(f) &= f(rm_1 + m_2) \\ &= rf(m_1) + f(m_2) \\ &= r\alpha_M(m)(f_1) + \alpha_M(m_2)(f) \\ &= (r\alpha_M(m_1) + \alpha_M(m_2))(f) \end{aligned}$$

for all $f \in \mathrm{Hom}_R(M, I)$, thus

$$\alpha_M(rm_1 + m_2)(f) = r\alpha_M(m_1) + \alpha_M(m_2),$$

and α_M is an R-module homomorphism.
We now show that α is a natural transformation. Let $M,\ M' \in \mathrm{mod}(R)$, and let $h \in \mathrm{Hom}_R(M, M')$. Then

$$\begin{aligned} D^2(h) &= \mathrm{Hom}_R(-, I)(\mathrm{Hom}_R(-, I)(h)) \\ &= \mathrm{Hom}_R(-, I)((- \circ h)_I) \\ &= (- \circ (- \circ h)_I)_I. \end{aligned}$$

We must show that

$$\begin{tikzcd} M \arrow[r, "\alpha_M"] \arrow[d, "h"] & \mathrm{Hom}_R(\mathrm{Hom}_R(M, I), I) \arrow[d, "(- \circ (- \circ h)_I)_I"] \\ M' \arrow[r, "\alpha_{M'}"] & \mathrm{Hom}_R(\mathrm{Hom}_R(M', I), I) \end{tikzcd}$$

is a commutative diagram. Suppose $m \in M$. Consider Diagram 2.24.

$$\begin{tikzcd} m \arrow[r, mapsto] \arrow[d, mapsto] & {[f \mapsto f(m)]} \arrow[d, mapsto, dashed, "?"] \\ h(m) \arrow[r, mapsto] & {[f' \mapsto f'h(m)]} \end{tikzcd} \tag{2.24}$$

We must show that

$$(- \circ (- \circ h)_I)_I([f \mapsto f(m)]) = [f' \mapsto f'h(m)].$$

For $f' \in \mathrm{Hom}_R(\mathrm{Hom}_R(M', I), I)$, then

$$\begin{aligned}(- \circ (- \circ h)_I)_I([f \mapsto f(m)])(f') &= [f \mapsto f(m)] \circ (- \circ h)_I(f') \\ &= [f \mapsto f(m)](f'h) \\ &= f'h(m).\end{aligned}$$

Thus α is a natural transformation.

Finally, we show that α_M is an isomorphism of R-modules for any $M \in \mathrm{mod}(R)$.

We begin by demonstrating the injectiveness of α_M. Let $M \in \mathrm{mod}(R)$, and suppose $m \in M$ is a nonzero element. We must show that $\alpha_M(m) \neq 0$, that is, that

$$\alpha_M(m)(f) = f(m) \neq 0$$

for some $f \in \mathrm{Hom}_R(M, I)$. Consider the map

$$\begin{aligned}\hat{f} : Rm &\to I \\ rm &\mapsto r + \underline{m}.\end{aligned}$$

Note that $r + \underline{m} \in R/\underline{m} = K \subseteq I$. We show that $\hat{f}$ is well-defined: If $rm = r'm$, then $(r - r')m = 0$, and since m is nonzero this means that $r - r'$ is a non-unit in R. Recall that $\underline{m}$ is the ideal in R generated by all the non-units. Hence $r - r' \in \underline{m}$, implying that $r + \underline{m} = r' + \underline{m}$. It is easy to see that $\hat{f}$ is an R-module homomorphism. Moreover, since $Rm \subseteq M$ is a submodule, then the inclusion

$$\iota : Rm \to M$$

is an R-module monomorphism. Thus by the lifting property of an injective module, there exists an R-module homomorphism $f : M \to I$ such that

$$f\iota = \hat{f} :$$

$$\begin{tikzcd} Rm \arrow[r, "\iota"] \arrow[d, "\hat{f}"'] & M \arrow[dl, dashed, "\exists f"] \\ I & \end{tikzcd}$$

Then

$$f(m) = f\iota(m) = \hat{f}(m) = \hat{f}(1_R m) = 1_R + \underline{m} \neq 0,$$

and α_M is shown to be injective. By (ii) then $l_R(M) = l_R(D^2M)$, hence by [2, Proposition 1.4, Ch. 1], α_M is an isomorphism of R-modules.

(b) Let $M \in \mathrm{mod}(\Lambda)$, and consider the R-module isomorphism

$$\alpha_M : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, I), I)$$

used in the proof of (iii)(a). We claim that α_M is also a homomorphism of Λ-modules.
Let $m \in M$ and $\lambda \in \Lambda$. Then for $f \in \mathrm{Hom}_R(M, I)$,

$$\alpha_M(\lambda m)(f) = f(\lambda m) = (f\lambda)(m)$$

because of the Λ^{op}-module structure on $\mathrm{Hom}_R(M, I)$ of Lemma 22(i). Moreover, the Λ-module structure on $\mathrm{Hom}_R(\underbrace{\mathrm{Hom}_R(M, I)}_{\in \mathrm{mod}(\Lambda^{\mathrm{op}})}, I)$ (Lemma 22(ii)) now implies that

$$(\lambda\alpha_M(m))(f) = \alpha_M(m)(f\lambda) = (f\lambda)(m)$$

for $f \in \mathrm{Hom}_R(M, I)$. Hence

$$\alpha_M(\lambda m) = \lambda\alpha_M(m).$$

Also, $\alpha_M(m_1 + m_2)$ is obviously equal to $\alpha_M(m_1) + \alpha_M(m_2)$ for all $m_1, m_2 \in M$, thus α_M is a Λ-module homomorphism. Since α_M was shown to be bijective in the proof of (iii)(a), it follows that α_M is an isomorphism of Λ-modules. It was also shown in (iii)(a) that α_M is natural in M. This completes the proof.

(c) Similar to (iii)(b).

□

Most of the investigation of Chapter 3 will be carried out without any more assumptions on R than the ones presented in Section 2.2, [3] but in the very last section we will add the condition that R be a field. The motivation for making this restriction is that the finitely generated R-modules then become finite dimensional R-vector spaces, and by choosing an R-basis of an R-vector space V we have a method for obtaining a set of elements of the dual space DV – which even turns out to form an R-basis of DV. This procedure will be explained in this section. Recall that K denotes the field $R/\underline{m}$, where $\underline{m}$ is the maximal ideal in R.

Lemma 30. *In the case that R is a field, then $R = K$ and the functor D of Definition 27 is given by*

$$D = \mathrm{Hom}_K(-, K),$$

and it is a functor

$$\mathrm{mod}(K) \to \mathrm{mod}(K).$$

[3] R is a commutative, local and artinian ring.

Proof. If R is a field, then the maximal ideal $\underline{m} = 0$, thus

$$R = R/\underline{m} = K.$$

Since R is injective as R-module, then R is its own injective envelope, [4] hence

$$I = R = K$$

in this case. It now follows from Proposition 29(i)(a) that D is a functor

$$\mathrm{mod}(K) \to \mathrm{mod}(K).$$

□

Note that when equipped with a K-basis for a K-vector space V, we can immediately obtain elements of DV in the following manner: For an element of the given basis then the action of extracting the K-coefficient of this particular basis element from any $v \in V$, forms a K-module homomorphism from V to K. (We leave it to the reader to check that this is true.)

Definition 31. Let $V \in \mathrm{mod}(K)$, and let

$$\mathcal{B}_V := \{v_1, v_2, ..., v_l\}$$

be a K-basis of V. Let

$$d_{\mathcal{B}_V} : \mathcal{B}_V \to DV$$

be the mapping given by

$$d_{\mathcal{B}_V}(v_i)(v_j) := \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} . \tag{2.25}$$

We will now see that for a K-vector space V then the $d_{\mathcal{B}_V}(v_i)$'s of Definition 31 form a K-basis of DV.

Proposition 32. *Let $V \in \mathrm{mod}(K)$, and let*

$$\mathcal{B}_V := \{v_1, v_2, ..., v_l\}$$

be a K-basis of V. Let

$$d\mathcal{B}_V := \{d_{\mathcal{B}_V}(v_1), d_{\mathcal{B}_V}(v_2), ..., d_{\mathcal{B}_V}(v_l)\}.$$

Then $d\mathcal{B}_V$ is a K-basis of $D(V)$.

[4] Any nonzero submodule of R evidently has nonzero intersection with R.

Proof. We first show that $d\mathcal{B}_V$ is linearly independent. Suppose $\alpha_1, ..., \alpha_l \in K$ such that $\sum_{i=1}^{l} \alpha_i d_{\mathcal{B}_V}(v_i) = 0$. This means that $\sum_{i=1}^{l} \alpha_i d_{\mathcal{B}_V}(v_i)(v) = 0$ for all $v \in V$. In particular,

$$\sum_{i=1}^{l} \alpha_i \ d_{\mathcal{B}_V}(v_i)(v_j) = 0$$

for $j \in \{1, ..., l\}$. Note that $\sum_{i=1}^{l} \alpha_i \ d_{\mathcal{B}_V}(v_i)(v_j)$ is also equal to α_j. Thus $d\mathcal{B}_V$ is linearly independent.

We now show that any $f \in D(V)$ can be written as a K-linear combination of elements of $d\mathcal{B}_V$. Note that

$$d_{\mathcal{B}_V}(v_i)\left(\sum_{j=1}^{l} \alpha_j v_j\right) = \sum_{j=1}^{l} \alpha_j d_{\mathcal{B}_V}(v_i)(v_j) = \alpha_i.$$

For any element $v = \sum_{j=1}^{l} \alpha_j v_j$ in V, then

$$\begin{aligned} f(v) &= f\left(\sum_{j=1}^{l} \alpha_j v_j\right) \\ &= \sum_{j=1}^{l} \alpha_j f(v_j) \\ &= \sum_{j=1}^{l} d_{\mathcal{B}_V}(v_j)(v) f(v_j) \\ &= \sum_{j=1}^{l} f(v_j) d_{\mathcal{B}_V}(v_j)(v) \\ &= \left(\sum_{j=1}^{l} f(v_j) d_{\mathcal{B}_V}(v_j)\right)(v), \end{aligned}$$

hence $f = \sum_{j=1}^{l} f(v_j) d_{\mathcal{B}_V}(v_j)$.

(The K-coefficients of f with respect to the K-basis $d\mathcal{B}_V$ of $D(V)$ are thus obtained by evaluating f in the elements of $\mathcal{B}_V$.) □

Definition 33. Let $V \in \mathrm{mod}(K)$, and let

$$\mathcal{B}_V := \{v_1, v_2, ..., v_l\}$$

be a K-basis of V. The K-basis $d\mathcal{B}_V$ of $D(V)$ is called the *dual basis of* $D(V)$ *(with respect to* $\mathcal{B}_V$*)*.

Given an isomorphism of two vector spaces, then a K-basis for one of the vector spaces corresponds bijectively to a K-basis for the other through application of the isomorphism. Moreover, since D is a functor, D induces an isomorphism of the respective dual spaces. The following lemma states that we are at liberty to apply $d_{\mathcal{B}}$ (where $\mathcal{B}$ is either the K-basis of the one vector space or the other) before or after applying the appropriate isomorphism, without changing the outcome.

Lemma 34. *Let V, $W \in \operatorname{mod}(K)$, and suppose $\xi \in \operatorname{Hom}_K(V, W)$ is an isomorphism. Let*

$$\mathcal{B}_V := \{v_1, v_2, ..., v_l\}$$

be a K-basis of V, and let

$$\xi(\mathcal{B}_V) := \{\xi(v_1), \xi(v_2), ..., \xi(v_l)\}$$

denote the corresponding K-basis of W. Then

$$(D\xi^{-1})(d_{\mathcal{B}_V}(v_j)) = d_{\xi(\mathcal{B}_V)}(\xi(v_j))$$

for all $1 \leq j \leq l$.

Proof. Let $1 \leq j \leq l$, and suppose $\sum_{i=1}^{l} \alpha_i \xi(v_i) \in W$. Then

$$\begin{aligned}
(D\xi^{-1})(d_{\mathcal{B}_V}(v_j)) \left(\sum_{i=1}^{l} \alpha_i \xi(v_i) \right) &= d_{\mathcal{B}_V}(v_j) \circ \xi^{-1} \left(\sum_{i=1}^{l} \alpha_i \xi(v_i) \right) \\
&= d_{\mathcal{B}_V}(v_j) \left(\sum_{i=1}^{l} \alpha_i v_i \right) \\
&= \alpha_j \\
&= d_{\xi(\mathcal{B}_V)}(\xi(v_j)) \left(\sum_{i=1}^{l} \alpha_i \xi(v_i) \right),
\end{aligned}$$

hence

$$(D\xi^{-1})(d_{\mathcal{B}_V}(v_j)) = d_{\xi(\mathcal{B}_V)}(\xi(v_j)).$$

□

2.4 Projective covers and minimal projective presentations

In this section we will introduce the notion of a minimal projective presentation. This is an important topic, as a minimal projective presentation will be the point of departure for many of the computations in the sequel.

Definition 35.

(i) We let $\mathcal{P}(\Lambda)$ denote the category whose objects are the finitely generated and projective Λ-modules, and

$$\operatorname{Hom}_{\mathcal{P}(\Lambda)}(P, P') := \operatorname{Hom}_{\Lambda}(P, P')$$

for any P, $P' \in \mathcal{P}(\Lambda)$.

(ii) Let $A,\ B \in \mathrm{Mod}(\Lambda)$. An *essential epimorphism* $t \in \mathrm{Hom}_\Lambda(A, B)$ is a Λ-module epimorphism such that the following holds. If $M \in \mathrm{Mod}(\Lambda)$ and $u \in \mathrm{Hom}_\Lambda(M, A)$ such that the composition tu is an epimorphism, then u is also an epimorphism.

(iii) Let $X \in \mathrm{Mod}(\Lambda)$. A *projective cover of* X is a projective Λ-module P together with an essential epimorphism

$$t : P \to X.$$

We will be referring to both P and t as the projective cover of X depending on the situation; it should be clear from the context whether we are referring to the object or to the morphism.

The existence of projective covers of finitely generated Λ-modules is essential for our further work. For this, we refer to [2, Theorem 4.2, Ch. 1]. Although this theorem already states the facts of the following lemma, we include it here for completeness.

Lemma 36. *Let* $X \in \mathrm{mod}(\Lambda)$.

(i) *Suppose* P *is a projective cover of* X*. Then* $P \in \mathcal{P}(\Lambda)$*. That is,* P *is a finitely generated* Λ*-module.*

(ii) *Suppose*

$$P \xrightarrow{t} X \longrightarrow 0$$

and

$$P' \xrightarrow{t'} X \longrightarrow 0$$

are projective covers of X*. Then there exists an isomorphism* $h \in \mathrm{Hom}_\Lambda(P, P')$ *such that the following diagram commutes:*

$$\begin{tikzcd} P \arrow[dd, "h"'] \arrow[dr, "t"] & \\ & X \\ P' \arrow[ur, "t'"] & \end{tikzcd}$$

Proof.

(i) There is an essential epimorphism $t \in \mathrm{Hom}_\Lambda(P, X)$, and an epimorphism $u \in \mathrm{Hom}_\Lambda(\Lambda^n, X)$ for some $n \in \mathbb{N}$.

$$\begin{tikzcd}
 & \Lambda^n \arrow[dl, dashed, "\exists v"'] \arrow[d, "u"] & \\
P \arrow[r, "t"] & X \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

Then by the lifting property of the projective Λ-module Λ^n, there is an epimorphism $v \in \mathrm{Hom}_\Lambda(\Lambda^n, P)$ such that

$$tv = u.$$

Then tv is an epimorphism, and since t is an essential epimorphism then v is an epimorphism. Hence $P \in \mathcal{P}(\Lambda)$.

(ii) Consider the following diagram:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "h"'] \arrow[d, "t"] & \\
P' \arrow[r, "t'"'] \arrow[ur, dashed, "h'"'] & X \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

Because of the lifting property of P and P', there exist $h \in \mathrm{Hom}_\Lambda(P, P')$ and $h' \in \mathrm{Hom}_\Lambda(P', P)$ such that

$$t'h = t.$$

and

$$th' = t'$$

Then, since t' is an essential epimorphism and $t'h$ is an epimorphism, h is an epimorphism. Moreover,

$$t'(hh') = (t'h)h' = th' = t',$$

and by the same arguments as above, hh' is an epimorphism. Since $l_\Lambda(P') < \infty$, then hh' is an isomorphism, implying that h is a monomorphism. Then

$$h : P' \to P$$

is an isomorphism of Λ-modules.

$\square$

Recall that any homomorphism of finitely generated Λ-modules has a kernel in $\mathrm{mod}(\Lambda)$ which is unique up to isomorphism.

Definition 37. (i) For $X \in \mathrm{mod}(\Lambda)$, we let $P(X)$ denote the projective cover of X, and we let $\Omega_\Lambda(X)$ denote the kernel of the projective cover map t. That is, $\Omega_\Lambda(X)$ is defined by the exactness of the following sequence:

$$0 \longrightarrow \Omega_\Lambda(X) \xrightarrow{\iota} P(X) \xrightarrow{t} X \longrightarrow 0$$

(ii) Let $X \in \mathrm{mod}(\Lambda)$. A *minimal projective presentation of* X is an exact sequence

$$P_1 \xrightarrow{s} P_0 \xrightarrow{t} X \longrightarrow 0$$

where

$$P_0 := P(X)$$

is the projective cover of X and

$$P_1 := P(\Omega_\Lambda(X))$$

is the projective cover of $\Omega_\Lambda(X)$, and s is the composition of the inclusion ι of $\Omega_\Lambda(X)$ into $P(X)$ and the projective cover w from $P(\Omega_\Lambda(X))$ onto $\Omega_\Lambda(X)$:

$$\begin{tikzcd}
P(\Omega_\Lambda(X)) \arrow[rr, "s"] \arrow[dr, dashed, "w"] & & P(X) \arrow[r, "t"] & X \arrow[r] & 0 \\
& \Omega_\Lambda(X) \arrow[ur, dashed, "\iota"] & & &
\end{tikzcd}$$

Note that for $X \in \mathrm{mod}(\Lambda)$, the existence of a minimal projective presentation follows directly from the existence of projective covers. We now show that minimal projective presentations are unique up to isomorphism.

Lemma 38. *Let*

$$P_1 \xrightarrow{s} P_0 \xrightarrow{t} X \longrightarrow 0$$

and

$$P_1' \xrightarrow{s'} P_0' \xrightarrow{t'} X \longrightarrow 0$$

be two minimal projective presentations of X. Then there are isomorphisms $h_0 \in \mathrm{Hom}_\Lambda(P_0, P_0')$ and $h_1 \in \mathrm{Hom}_\Lambda(P_1, P_1')$ such that the following diagram is commutative.

$$\begin{tikzcd}
P_1 \arrow[r, "s"] \arrow[d, dashed, "h_1"] & P_0 \arrow[r, "t"] \arrow[d, dashed, "h_0"] & X \arrow[r] \arrow[d, equal] & 0 \\
P_1' \arrow[r, "s'"] & P_0' \arrow[r, "t'"] & X \arrow[r] & 0
\end{tikzcd}$$

Proof. Consider the inclusions

$$\iota : \mathrm{Ker}(t) \to P_0$$

and

$$\iota' : \mathrm{Ker}(t') \to P_0',$$

and the projections

$$\pi : P_1 \to \mathrm{Im}(s) = \mathrm{Ker}(t)$$

and

$$\pi' : P_1' \to \mathrm{Im}(s') = \mathrm{Ker}(t').$$

$$\begin{tikzcd}
P_1 \arrow[rr, "s"] \arrow[dr, "\pi"] \arrow[dd, dashed, "h_1"'] & & P_0 \arrow[r, "t"] \arrow[dd, "\simeq"', "h_0"] & X \arrow[r] \arrow[dd, equal] & 0 \\
& \mathrm{Ker}(t) \arrow[ur, "\iota"] \arrow[dd, dashed, "\simeq"', "u"] & & & \\
P_1' \arrow[rr, "s'"] \arrow[dr, "\pi'"] \arrow[dd, dashed, "h_1'"'] & & P_0' \arrow[r, "t'"] \arrow[dd, "\simeq"', "h_0'"] & X \arrow[r] \arrow[dd, equal] & 0 \\
& \mathrm{Ker}(t') \arrow[ur, "\iota'"] \arrow[dd, dashed, "\simeq"', "u'"] & & & \\
P_1 \arrow[rr, "s"] \arrow[dr, "\pi"] & & P_0 \arrow[r, "t"] & X \arrow[r] & 0 \\
& \mathrm{Ker}(t) \arrow[ur, "\iota"] & & &
\end{tikzcd}$$

The existence of the isomorphisms $h_0 \in \mathrm{Hom}_\Lambda(P_0, P_0')$ and $h_0' \in \mathrm{Hom}_\Lambda(P_0', P_0)$

follows from Lemma 36(ii). By [3, Lemma 3.32 (Five Lemma), Ch. 3], there are isomorphisms $u \in \mathrm{Hom}_\Lambda(\mathrm{Ker}(t), \mathrm{Ker}(t'))$ and $u' \in \mathrm{Hom}_\Lambda(Ker(t'), \mathrm{Ker}(t))$ such that

$$h_0\iota = \iota' u$$

and

$$h_0'\iota' = \iota u'.$$

Since π' is an epimorphism and P_1 is projective, there is $h_1 \in \mathrm{Hom}_\Lambda(P_1, P_1')$ such that

$$u\pi = \pi' h_1.$$

Moreover, $u\pi$ is an epimorphism and π' is an essential epimorphism, hence h_1 is also an epimorphism. Similarly, there is an epimorphism $h_1' \in \mathrm{Hom}_\Lambda(P_1', P_1)$ such that

$$u'\pi' = \pi h_1'.$$

The composition $h_1'h_1$ is an epimorphism, and then, since P_1 has finite length (by Lemma 36(i)), $h_1'h_1$ is an isomorphism. This again implies that h_1 is a monomorphism, and therefore an isomorphism. □

2.5 The $()^*$-functor

In this brief section we will derive some results for $\mathrm{Hom}_\Lambda(-, \Lambda)$. Because this functor will be frequently applied in the next section, we find it practical to assign to it the following abbreviating notation:

Definition 39. We let the Hom-functor $\mathrm{Hom}_\Lambda(-, \Lambda)$ be denoted by $()^*$.

We make the following observations about the $()^*$-functor.

Proposition 40. *We can regard $()^*$ as a contravariant, left exact functor*

(i) $\mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{\mathrm{op}})$,

(ii) $\mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda^{\mathrm{op}})$.

Proof. We have seen that $\mathrm{Hom}_\Lambda(-, \Lambda)$ is a contravariant, left exact functor

$$\mathrm{mod}(\Lambda) \to \mathrm{Ab}\,.$$

(i) Let $M \in \mathrm{mod}(\Lambda)$. By Lemma 23(ii), then $\mathrm{Hom}_\Lambda(M, \Lambda) \in \mathrm{mod}(R)$. We claim that $\mathrm{Hom}_\Lambda(M, \Lambda) \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ with the following multiplication

$$\mathrm{Hom}_\Lambda(M, \Lambda) \to \mathrm{Hom}_\Lambda(M, \Lambda) :$$

$$(f\lambda)(m) := f(m)\lambda.$$

For this we need to verify the following statements:

I) $f\lambda \in \mathrm{Hom}_\Lambda(M, \Lambda)$ for all $f \in \mathrm{Hom}_\Lambda(M, \Lambda)$ and $\lambda \in \Lambda$.

II) The multiplication with Λ is associative.

III) $()^*$ takes Λ-module homomorphisms to Λ^{op}-module homomorphisms.

In case that this holds, then $\mathrm{Hom}_\Lambda(M,\Lambda) \in \mathrm{Mod}(\Lambda^{\mathrm{op}}) \cap \mathrm{mod}(R)$, thus, by Lemma 21, $\mathrm{Hom}_\Lambda(M,\Lambda) \in \mathrm{mod}(\Lambda^{\mathrm{op}})$. We now show that I) through III) hold.

I) Let $f \in \mathrm{Hom}_\Lambda(M,\Lambda)$ and $\lambda \in \Lambda$. Then

$$\begin{aligned}(f\lambda)(\alpha m_1 + m_2) &= f(\alpha m_1 + m_2)\lambda \\ &= (\alpha f(m_1) + f(m_2))\lambda \\ &= \alpha f(m_1)\lambda + f(m_2)\lambda \\ &= \alpha(f\lambda)(m_1) + (f\lambda)(m_2)\end{aligned}$$

for all m_1, $m_2 \in M$ and $\alpha \in \Lambda$, thus $f\lambda$ is indeed an element of $\mathrm{Hom}_\Lambda(M,\Lambda)$.

II) Let $f \in \mathrm{Hom}_\Lambda(M,\Lambda)$ and λ_1, $\lambda_2 \in \Lambda$. Then

$$\begin{aligned}(f(\lambda_1\lambda_2))(m) &= f(m)(\lambda_1\lambda_2) \\ &= (f(m)\lambda_1)\lambda_2 \\ &= (f\lambda_1)(m)\lambda_2 \\ &= ((f\lambda_1)\lambda_2)(m)\end{aligned}$$

for all $m \in M$, thus

$$f(\lambda_1\lambda_2) = (f\lambda_1)\lambda_2.$$

III) Let M, $M' \in \mathrm{mod}(\Lambda)$, and suppose $h \in \mathrm{Hom}_\Lambda(M,M')$. We need to show that $h^* \in \mathrm{Hom}_\Lambda(M'^*, M^*)$. Let $f' \in M'^*$ and $\lambda \in \Lambda$. Then

$$\begin{aligned}(h^*(f'\lambda))(m) &= ((f'\lambda)h)(m) \\ &= (f'\lambda)(h(m)) \\ &= f'(h(m))\lambda \\ &= (f'h)(m)\lambda \\ &= ((f'h)\lambda)(m) \\ &= (h^*(f')\lambda)(m)\end{aligned}$$

for all $m \in M$, hence

$$h^*(f'\lambda) = h^*(f')\lambda.$$

We leave it up to the reader to check that

$$h^*(f_1' + f_2') = h^*(f_1' + f_2').$$

(ii) Suppose P is a projective Λ-module. Then there is $n \in \mathbb{N}$ such that

$$\Lambda^n \simeq P \oplus P'$$

for some projective Λ-module P', implying that

$$\mathrm{Hom}_\Lambda(\Lambda^n, \Lambda) \simeq \mathrm{Hom}_\Lambda(P \oplus P', \Lambda)$$

as Λ^{op}-modules. Moreover, by Lemma 8(ii) there are Λ^{op}-module isomorphisms

$$\mathrm{Hom}_\Lambda(P \oplus P', \Lambda) \simeq \mathrm{Hom}_\Lambda(P, \Lambda) \oplus \mathrm{Hom}_\Lambda(P', \Lambda)$$

and

$$\mathrm{Hom}_\Lambda(\Lambda^n, \Lambda) \simeq \mathrm{Hom}_\Lambda(\Lambda, \Lambda)^n.$$

Then

$$\mathrm{Hom}_\Lambda(P, \Lambda) \oplus \mathrm{Hom}_\Lambda(P', \Lambda) \simeq \mathrm{Hom}_\Lambda(\Lambda, \Lambda)^n.$$

By Lemma 9(ii),

$$\mathrm{Hom}_\Lambda(\Lambda, \Lambda)^n \simeq (\Lambda^{\mathrm{op}})^n$$

as Λ^{op}-modules, and it follows that $\mathrm{Hom}_\Lambda(P, \Lambda) = P^*$ is a direct summand of $(\Lambda^{\mathrm{op}})^n$.

□

When applying the $()^*$-functor to a minimal projective presentation

$$P_1 \xrightarrow{s} P_0 \xrightarrow{t} X \longrightarrow 0$$

of an element $X \in \mathrm{mod}(\Lambda)$, we get the exact sequence

$$0 \longrightarrow X^* \xrightarrow{t^*} P_0^* \xrightarrow{s^*} P_1^*$$

in $\mathrm{mod}(\Lambda^{\mathrm{op}})$. The following lemma states that the cokernel of s^* is finitely generated as Λ^{op}-module, and independent of choice of minimal projective presentation of X.

Lemma 41.

(i) *Let* $X \in \mathrm{mod}(\Lambda)$*, and let*

$$P_1 \xrightarrow{s} P_0 \xrightarrow{t} X \longrightarrow 0$$

be a minimal projective presentation of X*. Then the following sequence is an exact sequence in* $\mathrm{mod}(\Lambda^{\mathrm{op}})$*:*

$$0 \longrightarrow X^* \xrightarrow{t^*} P_0^* \xrightarrow{s^*} P_1^* \longrightarrow \mathrm{Cok}(s^*) \longrightarrow 0$$

(ii) *If*

$$P_1 \xrightarrow{s} P_0 \xrightarrow{t} X \longrightarrow 0$$

and

$$P_1' \xrightarrow{s'} P_0' \xrightarrow{t'} X \longrightarrow 0$$

are two minimal projective presentations of X, then

$$\mathrm{Cok}(s^*) \simeq \mathrm{Cok}(s'^*)$$

as Λ^{op}-modules.

Proof.

(i) Proposition 40(i) implies that

$$0 \longrightarrow X^* \xrightarrow{t^*} P_0^* \xrightarrow{s^*} P_1^*$$

is an exact sequence of Λ^{op}-modules. Then, by Lemma 20, we know that $\mathrm{Cok}(s^*) \in \mathrm{mod}(\Lambda^{\mathrm{op}})$.

(ii) By Lemma 38 there are isomorphisms $h_0 \in \mathrm{Hom}_\Lambda(P_0, P_0')$ and $h_1 \in \mathrm{Hom}_\Lambda(P_1, P_1')$ such that the following diagram commutes:

$$\begin{tikzcd}
P_1 \arrow[r, "s"] \arrow[d, dashed, "h_1"] & P_0 \arrow[r, "t"] \arrow[d, dashed, "h_0"] & X \arrow[r] \arrow[d, equal] & 0 \\
P_1' \arrow[r, "s'"] & P_0' \arrow[r, "t'"] & X \arrow[r] & 0
\end{tikzcd}$$

The functor $()^*$ now induces the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X^* \arrow[r, "t'^*"] \arrow[d, equal] & P_0'^* \arrow[r, "s'^*"] \arrow[d, "h_0^*"'] & P_1'^* \arrow[r] \arrow[d, "h_1^*"'] & \mathrm{Cok}(s'^*) \arrow[r] \arrow[d, dashed, "\simeq"'] & 0 \\
0 \arrow[r] & X^* \arrow[r, "t^*"] & P_0^* \arrow[r, "s^*"] & P_1^* \arrow[r] & \mathrm{Cok}(s^*) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, it follows that

$$\mathrm{Cok}(s^*) \simeq \mathrm{Cok}(s'^*)$$

as Λ^{op}-modules.

□

2.6 The transpose

In this section we will define a functor called the transpose, which will serve as a tool for our investigation throughout Chapter 3.

The results of Lemma 41 bring about a well-defined map from $\mathrm{mod}(\Lambda)$ to $\mathrm{mod}(\Lambda^{\mathrm{op}})$ up to isomorphism in $\mathrm{mod}(\Lambda^{\mathrm{op}})$.

Definition 42. Let

$$\mathrm{Tr} : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{\mathrm{op}})$$

be given by

$$\mathrm{Tr}(X) := \mathrm{Cok}(s^*),$$

where

$$\begin{tikzcd} P_1 \arrow[r, "s"] & P_0 \arrow[r, "t"] & X \arrow[r] & 0 \end{tikzcd}$$

is any minimal projective presentation of X.

We wish to investigate if Tr can be regarded also as a functor. What should this prospective functor return when applied to a morphism? To answer this question, we will apply the following lemma.

Lemma 43. *Consider the following diagram in* $\mathrm{mod}(\Lambda)$*:*

$$\begin{tikzcd} \dots \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] & P_{n-1} \arrow[r, "d_{n-1}"] & \dots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & X \arrow[r] \arrow[d, "h"] & 0 \\ \dots \arrow[r, "g_{n+1}"] & Y_n \arrow[r, "g_n"] & Y_{n-1} \arrow[r, "g_{n-1}"] & \dots \arrow[r, "g_2"] & Y_1 \arrow[r, "g_1"] & Y_0 \arrow[r, "g_0"] & Y \arrow[r] & 0 \end{tikzcd} \tag{2.26}$$

Suppose P_i is projective and $d_i d_{i+1} = 0$ for all $i \geq 0$, and suppose the lower row of the diagram is exact. Then, for all $i \geq 0$, there is $h_i \in \mathrm{Hom}_\Lambda(P_i, Y_i)$ such that the resulting diagram is commutative.

Proof. We begin with considering a small part of Diagram 2.26:

$$\begin{tikzcd} P_1 \arrow[rr, "d_1"] \arrow[dr] \arrow[dd, dashed, "h_1"] & & P_0 \arrow[r, "d_0"] \arrow[dd, dashed, "h_0"] & X \arrow[r] \arrow[dd, "h"] & 0 \\ & \mathrm{Ker}(d_0) \arrow[ur] \arrow[dd, dashed, "\mathrm{Ker}(h_0)"] & & & \\ Y_1 \arrow[rr, "g_1"] \arrow[dr] & & Y_0 \arrow[r, "g_0"] & Y \arrow[r] & 0 \\ & \mathrm{Ker}(g_0) \arrow[ur] & & & \end{tikzcd}$$

Since P_0 is projective and g_0 is an epimorphism, there is $h_0 \in \mathrm{Hom}_\Lambda(P_0, Y_0)$ such that

$$g_0 h_0 = h d_0.$$

Then there is a kernel map $\mathrm{Ker}(h_0) \in \mathrm{Hom}_\Lambda(\mathrm{Ker}(d_0), \mathrm{Ker}(g_0))$ which commutes with the rest of the diagram. Since the lower row is exact, then Y_1 is onto $\mathrm{Ker}(g_0)$, thus because P_1 is projective there exists $h_1 \in \mathrm{Hom}_\Lambda(P_1, Y_1)$ which commutes with the rest of the diagram. By continuing in this fashion, we obtain $h_i \in \mathrm{Hom}_\Lambda(P_i, Y_i)$ for all $i \geq 0$. □

Note that $h_i \in \mathrm{Hom}_\Lambda(P_i, Y_i)$ such that the resulting square right from h_i is commutative, is not necessarily unique. Suppose, for instance, that $f \in \mathrm{Hom}_\Lambda(P_0, Y_1)$ such that the composition $e_1 f$ is nonzero. Then $(h_0 + e_1 f) \in \mathrm{Hom}_\Lambda(P_0, Y_0)$ such that

$$e_o(h_0 + e_1 f) = e_0 h_0 + \underbrace{(e_0 e_1)}_{=0} f = h d_0.$$

We now return to the task of finding a procedure for how to obtain, given some $h \in \mathrm{Hom}_\Lambda(X, X')$ (where $X,\ X' \in \mathrm{mod}(\Lambda)$), a suitable morphism in either $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X), \mathrm{Tr}(X'))$ or $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X'), \mathrm{Tr}(X))$.

$$P_1 \xrightarrow{s} P_0 \xrightarrow{t} X \longrightarrow 0 \tag{2.27}$$

and

$$P_1' \xrightarrow{s'} P_0' \xrightarrow{t'} X' \longrightarrow 0 \tag{2.28}$$

be minimal projective presentations of X and X', respectively. Then Lemma 43 yields the following commutative diagram in $\mathrm{mod}(\Lambda)$:

$$\begin{tikzcd} P_1 \arrow[r, "s"] \arrow[d, "h_1"] & P_0 \arrow[r, "t"] \arrow[d, "h_0"] & X \arrow[r] \arrow[d, "h"] & 0 \\ P_1' \arrow[r, "s'"] & P_0' \arrow[r, "t'"] & X' \arrow[r] & 0 \end{tikzcd} \tag{2.29}$$

By applying $()^*$ to Diagram 2.29, we get the following commutative diagram in $\mathrm{mod}(\Lambda^{\mathrm{op}})$:

$$\begin{tikzcd} P_0'^* \arrow[r, "s'^*"] \arrow[d, "h_0^*"] & P_1'^* \arrow[r, "\hat{t}'"] \arrow[d, "h_1^*"] & \mathrm{Tr}(X') \arrow[r] \arrow[d, dashed, "(h_1^*)_{\mathrm{Cok}}"] & 0 \\ P_0^* \arrow[r, "s^*"] & P_1^* \arrow[r, "\hat{t}"] & \mathrm{Tr}(X) \arrow[r] & 0 \end{tikzcd}$$

Now h_1^* gives rise to a cokernel map $(h_1^*)_{\mathrm{Cok}} \in \mathrm{Hom}_\Lambda(\mathrm{Tr}(X'), \mathrm{Tr}(X))$, which seems like a good candidate for our $\mathrm{Tr}(h)$. However, there is a well-definedness issue which must be sorted out.

As mentioned above, the Λ-module homomorphisms h_0 and h_1 of Diagram 2.29 are not uniquely determined. If we were to choose $g_0 \in \mathrm{Hom}_\Lambda(P_0, P_0')$ and

$g_1 \in \mathrm{Hom}_\Lambda(P_1, P_1')$ instead of h_0 and h_1, then this choice would lead to a different cokernel map $(g_1^*)_{\mathrm{Cok}} \in \mathrm{Hom}_\Lambda(\mathrm{Tr}(X'), \mathrm{Tr}(X))$. We will solve this problem by regarding Tr as a functor between appropriate quotient categories of $\mathrm{mod}(\Lambda)$ and $\mathrm{mod}(\Lambda^{\mathrm{op}})$. Consider the following definition.

Definition 44.

(i) For M, $N \in \mathrm{mod}(\Lambda)$, we let

$$\mathcal{P}_\Lambda(M, N) := \{f \in \mathrm{Hom}_\Lambda(M, N) \mid f \text{ factors through a projective } \Lambda\text{-module}\},$$

and we let

$$\underline{\mathrm{Hom}}_\Lambda(M, N) := \mathrm{Hom}_\Lambda(M, N)/\mathcal{P}_\Lambda(M, N).$$

(ii) We let $\underline{\mathrm{mod}}(\Lambda)$ be the quotient category of $\mathrm{mod}(\Lambda)$ given by

$$\mathrm{Ob}(\underline{\mathrm{mod}}(\Lambda)) := \mathrm{Ob}(\mathrm{mod}(\Lambda)),$$

$$\mathrm{Hom}_{\underline{\mathrm{mod}}(\Lambda)}(M, N) := \underline{\mathrm{Hom}}_\Lambda(M, N).$$

(We have the same definition if Λ is replaced by Λ^{op}.)

We observe the following for homomorphisms which factor through projective modules:

Lemma 45. *Let M, $N \in \mathrm{mod}(\Lambda)$. If $f \in \mathcal{P}_\Lambda(M, N)$ and $P \in \mathrm{Mod}(\Lambda)$ is onto N, then f factors through P. Particularly, f factors through the projective cover $P(N)$ of N.*

Proof. Consider the following diagram in $\mathrm{Mod}(\Lambda)$:

$$\begin{tikzcd}
 & & M \arrow[dl] \arrow[dd, "f"] & \\
 & P' \arrow[dl, dashed, "\exists"'] \arrow[dr] & & \\
P \arrow[rr] & & N \arrow[r] & 0
\end{tikzcd}$$

If $f \in \mathcal{P}_\Lambda(M, N)$ then f factors through some projective Λ-module P', and because of the lifting property of P' and the fact that P is onto N, the result is obtained. $\square$

Recall that when attempting to define the transpose of a morphism, we encountered a challenge regarding well-definedness. Our problem conveniently vanishes if we instead consider the transpose to be a functor of quotient categories

$$\underline{\mathrm{mod}}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda^{\mathrm{op}}).$$

Lemma 46. *Let X, $X' \in \operatorname{mod}(\Lambda)$. Then there is a well-defined map*

$$\underline{\operatorname{Hom}}_\Lambda(X, X') \to \underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X'), \operatorname{Tr}(X))$$

which is defined as follows: Choose any representative h for the element of $\underline{\operatorname{Hom}}_\Lambda(X, X')$ to which Tr *should be applied. Let (2.27) and (2.28) be minimal projective presentations for X and X' respectively, and let $h_0 \in \operatorname{Hom}_\Lambda(P_0, P_0')$ and $h_1 \in \operatorname{Hom}_\Lambda(P_1, P_1')$ be any choice of Λ-module homomorphisms such that Diagram 2.29 commutes. Then*

$$\operatorname{Tr}(h + \mathcal{P}_\Lambda(X, X')) := (h_1^*)_{\mathrm{Cok}} + \mathcal{P}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X'), \operatorname{Tr}(X)).$$

Proof. We need to prove that the following two statements are true.

I) If $g_0 \in \operatorname{Hom}_\Lambda(P_0, P_0')$ and $g_1 \in \operatorname{Hom}_\Lambda(P_1, P_1')$ such that Diagram 2.29 commutes when h_0 and h_1 are replaced by g_0 and g_1, then

$$(h_1^*)_{\mathrm{Cok}} - (g_1^*)_{\mathrm{Cok}} \in \mathcal{P}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X'), \operatorname{Tr}(X)).$$

That is, $(h_1^*)_{\mathrm{Cok}}$ and $(g_1^*)_{\mathrm{Cok}}$ represent the same element of $\underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X'), \operatorname{Tr}(X))$.

II) If $h \in \mathcal{P}_\Lambda(X, X')$, then

$$(h_1^*)_{\mathrm{Cok}} \in \mathcal{P}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X'), \operatorname{Tr}(X)).$$

That is, one element in $\underline{\operatorname{Hom}}_\Lambda(X, X')$ maps to the same element of $\underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X'), \operatorname{Tr}(X))$ regardless of choice of representative.

I) Consider Diagram 2.30.

$$\begin{tikzcd}
P_1 \arrow[r, "s"] \arrow[d, bend right, "g_1"'] \arrow[d, "h_1"] & P_0 \arrow[r, "t"] \arrow[dl, dashed, "u"'] \arrow[d, bend right, "g_0"'] \arrow[d, "h_0"] & X \arrow[r] \arrow[d, "h"] & 0 \\
P_1' \arrow[r, "s'"] \arrow[dr] & P_0' \arrow[r, "t'"] & X' \arrow[r] & 0 \\
& \operatorname{Ker}(t') \arrow[u] & &
\end{tikzcd} \tag{2.30}$$

Since

$$ht = t'h_0 = t'g_0$$

then

$$t'(h_0 - g_0) = 0,$$

implying that $(h_0 - g_0)$ factors through $\mathrm{Ker}(t')$. Then since P_0 is projective and there is a canonical projection from P_1' onto $\mathrm{Im}(s') = \mathrm{Ker}(t')$, it follows that $(g_0 - h_0)$ factors through P_1'. That is, there is $u \in \mathrm{Hom}_\Lambda(P_0, P_1')$ such that

$$s'u = g_0 - h_0. \tag{2.31}$$

By applying the $()^*$-functor, we now get the following diagram in $\mathrm{mod}(\Lambda^{\mathrm{op}})$:

$$\begin{tikzcd}
P_0'^* \arrow[r, "s'^*"] \arrow[d, "h_0^*"'] \arrow[d, bend left, "g_0^*"] & P_1'^* \arrow[r, "\hat{t}'"] \arrow[d, "h_1^*"'] \arrow[d, bend left, "g_1^*"] \arrow[dl, "u^*"'] & \mathrm{Tr}(X') \arrow[r] \arrow[d, "(h_1^*)_{\mathrm{Cok}}"'] \arrow[d, bend left, "(g_1^*)_{\mathrm{Cok}}"] \arrow[dl, dashed, "v"'] & 0 \\
P_0^* \arrow[r, "s^*"] & P_1^* \arrow[r, "\hat{t}"] & \mathrm{Tr}(X) \arrow[r] & 0
\end{tikzcd}$$

From (2.31) we see that

$$u^* s'^* = g_0^* - h_0^*.$$

Then

$$s^* u^* s'^* = s^*(g_0^* - h_0^*) = (g_1^* - h_1^*)s'^*,$$

hence

$$(g_1^* - h_1^* - s^* u^*)s'^* = 0.$$

Then $(g_1^* - h_1^* - s^*u^*)$ factors through the cokernel of s'^*, namely $\hat{t}'$. That is, there is $v \in \mathrm{Hom}_{\mathrm{mod}(\Lambda^{\mathrm{op}})}(\mathrm{Tr}(X'), P_1^*)$ such that

$$v\hat{t}' = g_1^* - h_1^* - s^* u^*. \tag{2.32}$$

By multiplying $\hat{t}$ with (2.32) we get

$$\hat{t}v\hat{t}' = \hat{t}(g_1^* - h_1^* - s^*u^*) = ((g_1^*)_{\mathrm{Cok}} - (h_1^*)_{\mathrm{Cok}})\hat{t}',$$

and since $\hat{t}'$ is an epimorphism then

$$\hat{t}v = (g_1^*)_{\mathrm{Cok}} - (h_1^*)_{\mathrm{Cok}}.$$

That is, $((g_1^*)_{\mathrm{Cok}} - (h_1^*)_{\mathrm{Cok}})$ factors through P_1^*, hence

$$((g_1^*)_{\mathrm{Cok}} - (h_1^*)_{\mathrm{Cok}}) \in \mathcal{P}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X'), \mathrm{Tr}(X)).$$

II) Consider Diagram 2.33.

$$
\begin{tikzcd}
P_1 \arrow[r, "s"] \arrow[d, "h_1"] & P_0 \arrow[r, "t"] \arrow[d, "h_0"] \arrow[dl, dashed, "v"'] \arrow[ddl, dashed] & X \arrow[r] \arrow[d, "h"] \arrow[dl, dashed, "u"'] & 0 \\
P_1' \arrow[r, "s'"] \arrow[dr] & P_0' \arrow[r, "t'"] & X' \arrow[r] & 0 \\
& \mathrm{Ker}(t') \arrow[u] & &
\end{tikzcd}
\tag{2.33}
$$

Suppose h factors through a projective Λ-module P. Then since t' is an epimorphism, h also factors through t': There is $u \in \mathrm{Hom}_\Lambda(X, P_0)$ such that

$$t'u = h.$$

Consider the Λ-module homomorphism $(h_0 - ut)$. By multiplying with t' from the left, we get

$$t'(h_0 - ut) = t'h_0 - t'ut = t'h_0 - ht = 0,$$

hence $(h_0 - ut)$ factors through $\mathrm{Ker}(t')$. Then since P_0 is projective and there is a canonical projection from P_1' onto $\mathrm{Im}(s') = \mathrm{Ker}(t')$, it follows that $(h_0 - ut)$ also factors through P_1': There is $v \in \mathrm{Hom}_\Lambda(P_0, P_1')$ such that

$$s'v = h_0 - ut.$$

By applying $()^*$ we get

$$v^*s'^* = h_0^* - t^*u^*,$$

and the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & X'^* \arrow[r, "t'^*"] \arrow[d, "h^*"] & P_0'^* \arrow[r, "s'^*"] \arrow[d, "h_0^*"] \arrow[dl, "u^*"'] & P_1'^* \arrow[r, "\hat{t}'"] \arrow[d, "h_1^*"] \arrow[dl, "v^*"'] & \mathrm{Tr}(X') \arrow[r] \arrow[d, "(h_1^*)_{\mathrm{Cok}}"] \arrow[dl, dashed, "w"'] & 0 \\
0 \arrow[r] & X^* \arrow[r, "t^*"] & P_0^* \arrow[r, "s^*"] & P_1^* \arrow[r, "\hat{t}"] & \mathrm{Tr}(X) \arrow[r] & 0
\end{tikzcd}
$$

We now show that $(h_1^* - s^*v^*)$ factors through $\mathrm{Cok}(s'^*) = \mathrm{Tr}(X')$:

$$
\begin{aligned}
(h_1^* - s^*v^*)s'^* &= h^*s'^* - s^*v^*s'^* \\
&= h_1^*s'^* - s^*(h_0^* - t^*u^*) \\
&= h_1^*s'^* - s^*h_0^* + s^*t^*u^* \\
&= 0.
\end{aligned}
$$

Then there is $w \in \mathrm{Hom}_\Lambda(\mathrm{Tr}(X'), P_1^*)$ such that

$$w\hat{t}' = h_1^* - s^*v^*.$$

Finally, we see that

$$\hat{t}w\hat{t}' = \hat{t}(h_1^* - s^*v^*) = \hat{t}h_1^* = (h_1^*)_{\mathrm{Cok}}\hat{t}',$$

and since $\hat{t}'$ is an epimorphism, it follows that

$$(h_1^*)_{\mathrm{Cok}} = \hat{t}w.$$

Thus $(h_1^*)_{\mathrm{Cok}} \in \mathcal{P}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X'), \mathrm{Tr}(X))$.

□

Proposition 47.

(i) *The transpose constitutes a contravariant functor*

$$\mathrm{Tr} : \underline{\mathrm{mod}}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda^{\mathrm{op}}).$$

(ii) $\mathrm{Tr}^2 = 1_{\underline{\mathrm{mod}}(\Lambda)}$.

(iii) $\mathrm{Tr}(\oplus_{i=1}^n M_i) \simeq \oplus_{i=1}^n \mathrm{Tr}(M_i)$

(iv) $\mathrm{Tr}(M) = 0 \Leftrightarrow M$ *is projective.*

(v) $\mathrm{Tr}^2(M) \simeq$ *the non-projective part of* M.

Proof.

(i) We have seen that the map

$$\mathrm{Tr} : \mathrm{Ob}(\underline{\mathrm{mod}}(\Lambda)) \to \mathrm{Ob}(\underline{\mathrm{mod}}(\Lambda^{\mathrm{op}}))$$

is well-defined, and that given X, $X' \in \underline{\mathrm{mod}}(\Lambda)$ then there is a well-defined map

$$\mathrm{Tr} : \underline{\mathrm{Hom}}_\Lambda(X, X') \to \underline{\mathrm{Hom}}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X'), \mathrm{Tr}(X)).$$

However, recall from Definition **??**(v) that there are yet two assertions which must be satisfied, namely that Tr preserves composition of morphisms, and that Tr preserves the identity.

Let X, Y, $Z \in \underline{\mathrm{mod}}(\Lambda)$, and suppose $f \in \mathrm{Hom}_\Lambda(X, Y)$ and $g \in \mathrm{Hom}_\Lambda(Y, Z)$. We want to show that

$$\mathrm{Tr}(gf) = \mathrm{Tr}(f)\,\mathrm{Tr}(g).$$

Let the following commutative diagram in $\mathrm{mod}(\Lambda)$ be what we obtain from applying Lemma 43 to f and g, when the rows of the diagram represent minimal projective presentations for X, Y and Z, respectively.

$$\begin{tikzcd}
P_{X,1} \arrow[r, "s_X"] \arrow[d, "f_1"] & P_{X,0} \arrow[r, "t_X"] \arrow[d, "f_0"] & X \arrow[r] \arrow[d, "f"] & 0 \\
P_{Y,1} \arrow[r, "s_Y"] \arrow[d, "g_1"] & P_{Y,0} \arrow[r, "t_Y"] \arrow[d, "g_0"] & Y \arrow[r] \arrow[d, "g"] & 0 \\
P_{Z,1} \arrow[r, "s_Z"] & P_{Z,0} \arrow[r, "t_Z"] & Z \arrow[r] & 0
\end{tikzcd} \tag{2.34}$$

Recall from Lemma 46 that when finding the transpose of gf we may choose any $(gf)_0$ and $(gf)_1$ such that the following diagram is commutative:

$$\begin{tikzcd}
P_{X,1} \arrow[r, "s_X"] \arrow[d, "(gf)_1"] & P_{X,0} \arrow[r, "t_X"] \arrow[d, "(gf)_0"] & X \arrow[r] \arrow[d, "gf"] & 0 \\
P_{Z,1} \arrow[r, "s_Z"] & P_{Z,0} \arrow[r, "t_Z"] & Z \arrow[r] & 0
\end{tikzcd} \tag{2.35}$$

It is easy to see from Diagram 2.34 that Diagram 2.35 is commutative for the particular choices

$$(gf)_0 := g_0 f_0$$

and

$$(gf)_1 = g_1 f_1.$$

The $()^*$-functor gives rise to the following diagram in $\mathrm{mod}(\Lambda^{\mathrm{op}})$. (We may also think of Diagram 2.36 as a diagram in $\underline{\mathrm{mod}}(\Lambda^{\mathrm{op}})$, where the morphisms are any representatives for their corresponding equivalence classes.)

$$\begin{tikzcd}
P_{Z,0}^* \arrow[r, "s_Z^*"] \arrow[d, "g_0^*"] & P_{Z,1}^* \arrow[r, "\hat{t}_Z"] \arrow[d, "g_1^*"] & \mathrm{Tr}(Z) \arrow[r] \arrow[d, "\mathrm{Tr}(g)"] \arrow[dd, bend left, "\mathrm{Tr}(gf)"] & 0 \\
P_{Y,0}^* \arrow[r, "s_Y^*"] \arrow[d, "f_0^*"] & P_{Y,1}^* \arrow[r, "\hat{t}_Y"] \arrow[d, "f_1^*"] & \mathrm{Tr}(Y) \arrow[r] \arrow[d, "\mathrm{Tr}(f)"] & 0 \\
P_{X,0}^* \arrow[r, "s_X^*"] & P_{X,1}^* \arrow[r, "\hat{t}_X"] & \mathrm{Tr}(X) \arrow[r] & 0
\end{tikzcd} \tag{2.36}$$

By studying Diagram 2.36, we see that

$$\operatorname{Tr}(gf)\hat{t}_Z = \hat{t}_X(gf)_1^* = \hat{t}_X f_1^* g_1^* = \operatorname{Tr}(f)\hat{t}_Y g_1^* = \operatorname{Tr}(f)\operatorname{Tr}(g)\hat{t}_Z,$$

and since $\hat{t}_Z$ is an epimorphism it follows that

$$\operatorname{Tr}(gf) = \operatorname{Tr}(f)\operatorname{Tr}(g).$$

We finally show that

$$\operatorname{Tr}(1_X) = 1_{\operatorname{Tr}(X)}$$

for any $X \in \underline{\operatorname{mod}}(\Lambda)$. Consider Diagram 2.29. If $X' = X$, $P_0' = P_0$, $P_1' = P_1$, $s' = s$, $t' = t$ and $h = 1_X$, then by letting

$$h_0 := 1_{P_0}$$

and

$$h_1 := 1_{P_1},$$

the diagram becomes commutative. Then by applying the $()^*$-functor, we see that

$$\operatorname{Tr}(1_X) = (1_{P_1}^*)_{\operatorname{Cok}} = 1_{\operatorname{Tr}(X)}.$$

We leave it to the reader to prove the rest of the proposition.

□

We complete this section by making some comments regarding notation.

Definition 48. We will use the following notation for the transpose applied to morphisms: If $\bar{h} \in \underline{\operatorname{Hom}}_\Lambda(X, X')$, then of course $\operatorname{Tr}(\bar{h})$ is the resulting equivalence class in $\underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(Tr(X'), \operatorname{Tr}(X))$. If $h \in \operatorname{Hom}_\Lambda(X, X')$ is any representative for $\bar{h}$, then we let $\operatorname{Tr}(h) \in \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X'), \operatorname{Tr}(X))$ denote any representative for $\operatorname{Tr}(\bar{h})$ that we get when following the procedure of Lemma 46. That is, we might write

$$\operatorname{Tr}(h) = (h_1^*)_{\operatorname{Cok}} \tag{2.37}$$

for

$$\operatorname{Tr}(h + \mathcal{P}_\Lambda(X, X')) = (h_1^*)_{\operatorname{Cok}} + \mathcal{P}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X'), \operatorname{Tr}(X)).$$

Whenever we use the notation of (2.37), it should be clear that we are implicitly saying that we have chosen one representative $\operatorname{Tr}(h)$.

2.7 Auslanders defect formula

In this section we consider the exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\operatorname{mod}(\Lambda)$. Recall Lemma 23(ii), which states that $\operatorname{Hom}_R(M, N) \in \operatorname{mod}(R)$ for any M, $N \in \operatorname{mod}(\Lambda)$. Also, recall that the hom functors are left exact.

Definition 49. Let $X \in \mathrm{mod}(\Lambda)$.

(i) We define $\delta_*(X)$ by the exactness of the following sequence of R-modules:

$$0 \longrightarrow \mathrm{Hom}_\Lambda(C,X) \xrightarrow{(-\circ g)_X} \mathrm{Hom}_\Lambda(B,X) \xrightarrow{(-\circ f)_X} \mathrm{Hom}_\Lambda(A,X) \longrightarrow \delta_*(X) \longrightarrow 0 \tag{2.38}$$

(ii) We define $\delta^*(X)$ by the exactness of the following sequence of R-modules:

$$0 \longrightarrow \mathrm{Hom}_\Lambda(X,A) \xrightarrow{(f\circ -)_X} \mathrm{Hom}_\Lambda(X,B) \xrightarrow{(g\circ -)_X} \mathrm{Hom}_\Lambda(X,C) \longrightarrow \delta^*(X) \longrightarrow 0 \tag{2.39}$$

The following lemma assigns functorial structure to δ_* and δ^* defined above.

Proposition 50. *Let $X,\ X' \in \mathrm{mod}(\Lambda)$, and let $h \in \mathrm{Hom}_\Lambda(X,X')$.*

(i) *Let*

$$\delta^*(h) := \textit{The cokernel map of } (h\circ -)_A,$$

as illustrated below:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_\Lambda(C,X) \arrow[r, "(-\circ g)_X"] \arrow[d, "(h\circ -)_C"'] & \mathrm{Hom}_\Lambda(B,X) \arrow[r, "(-\circ f)_X"] \arrow[d, "(h\circ -)_B"'] & \mathrm{Hom}_\Lambda(A,X) \arrow[r] \arrow[d, "(h\circ -)_A"'] & \delta_*(X) \arrow[r] \arrow[d, dashed, "\delta^*(h)"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(C,X') \arrow[r, "(-\circ g)_{X'}"] & \mathrm{Hom}_\Lambda(B,X') \arrow[r, "(-\circ f)_{X'}"] & \mathrm{Hom}_\Lambda(A,X') \arrow[r] & \delta_*(X') \arrow[r] & 0
\end{tikzcd}$$

Then δ^* constitutes a covariant functor $\mathrm{mod}(\Lambda) \to \mathrm{mod}(R)$.

(ii) *Correspondingly, let*

$$\delta_*(h) := \textit{the cokernel map of } (-\circ h)_C,$$

as illustrated in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_\Lambda(X',A) \arrow[r, "(f\circ -)_{X'}"] \arrow[d, "(-\circ h)_A"'] & \mathrm{Hom}_\Lambda(X',B) \arrow[r, "(g\circ -)_{X'}"] \arrow[d, "(-\circ h)_B"'] & \mathrm{Hom}_\Lambda(X',C) \arrow[r] \arrow[d, "(-\circ h)_C"'] & \delta^*(X') \arrow[r] \arrow[d, dashed, "\delta_*(h)"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(X,A) \arrow[r, "(f\circ -)_X"] & \mathrm{Hom}_\Lambda(X,B) \arrow[r, "(g\circ -)_X"] & \mathrm{Hom}_\Lambda(X,C) \arrow[r] & \delta^*(X) \arrow[r] & 0
\end{tikzcd}$$

Then δ_* constitutes a contravariant functor $\mathrm{mod}(\Lambda) \to \mathrm{mod}(R)$.

Proof. By Lemma 23, $\mathrm{Hom}_\Lambda(A,X)$, $\mathrm{Hom}_\Lambda(X,C) \in \mathrm{mod}(R)$, and then by Lemma 20, so is $\delta^*(X)$ and $\delta_*(X)$. It is easy to see that the hom functors take Λ-module homomorphisms to R-module homomorphisms, and then since δ^* and δ_* applied to morphisms are defined to be cokernel maps of R-module homomorphisms, then it is clear that δ^* and δ_* take Λ-module homomorphisms to R-module homomorphisms.

We now show that δ^* and δ_* applied to morphisms have functorial structure. It is obvious that

$$\delta^*(1_X) = 1_{\delta^*(X)}$$

and that

$$\delta_*(1_X) = 1_{\delta_*(X)}.$$

For the rest of the proof, we let X, X', $X'' \in \mathrm{mod}(\Lambda)$, $h \in \mathrm{Hom}_\Lambda(X,X')$ and $h' \in \mathrm{Hom}_\Lambda(X',X'')$.

(i) We need to show that $\delta^*(h'h) = \delta^*(h')\delta^*(h)$.

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(A,X) \arrow[r, "\pi"] \arrow[d, dashed, "(h\circ -)_A"] \arrow[dd, bend right=60, "(h'h\circ -)_A"'] & \delta^*(X) \arrow[d, dashed, "\delta^*(h)"'] \arrow[dd, bend left=60, "\delta^*(h'h)"] \\
\mathrm{Hom}_\Lambda(A,X') \arrow[r, "\pi'"] \arrow[d, dashed, "(h'\circ -)_A"] & \delta^*(X') \arrow[d, dashed, "\delta^*(h')"'] \\
\mathrm{Hom}_\Lambda(A,X'') \arrow[r, "\pi''"] & \delta^*(X'')
\end{tikzcd} \tag{2.40}$$

Note that for any $u \in \mathrm{Hom}_\Lambda(A,X)$ then

$$(h'h\circ -)_A(u) = h'hu = (h'\circ -)_A(hu) = (h'\circ -)_A(h\circ -)_A(u),$$

hence $(h'h\circ -)_A = (h'\circ -)_A(h\circ -)_A$. (Note that this also follows from the fact that $\mathrm{Hom}_\Lambda(A,-)$ is a covariant functor into Ab. We now take advantage of the commutativity of the upper, lower and outer squares of Diagram 2.40:

$$\delta^*(h'h)\pi = \pi''(h'h\circ -)_A = \pi''(h'\circ -)_A(h\circ -)_A = \delta^*(h')\pi'(h\circ -)_A = \delta^*(h')\delta^*(h)\pi,$$

and since π is an epimorphism this implies that

$$\delta^*(h'h) = \delta^*(h')\delta^*(h).$$

(ii) We need to show that $\delta_*(h'h) = \delta_*(h)\delta_*(h')$.

$$
\begin{tikzcd}
\mathrm{Hom}_\Lambda(X'', C) \arrow[r, "\pi''"] \arrow[d, dashed, "(- \circ h')_C"] \arrow[dd, bend right=60, "(- \circ h'h)_C"'] & \delta_*(X'') \arrow[d, dashed, "\delta_*(h')"'] \arrow[dd, bend left=60, "\delta_*(h'h)"] \\
\mathrm{Hom}_\Lambda(X', C) \arrow[r, "\pi'"] \arrow[d, dashed, "(- \circ h)_C"] & \delta_*(X') \arrow[d, dashed, "\delta_*(h)"'] \\
\mathrm{Hom}_\Lambda(X, C) \arrow[r, "\pi"] & \delta_*(X)
\end{tikzcd}
\tag{2.41}
$$

Similar to in (i),

$$(- \circ h'h)_C = (- \circ h)_C(- \circ h')_C.$$

We now take advantage of the commutativity of the upper, lower and outer squares of Diagram 2.41:

$$\delta_*(h'h)\pi'' = \pi(-\circ h'h)_C = \pi(-\circ h)_C(-\circ h')_C = \delta_*(h)\pi'(-\circ h')_C = \delta_*(h)\delta_*(h')\pi'',$$

and since π'' is an epimorphism this implies that

$$\delta_*(h'h) = \delta_*(h)\delta_*(h').$$

$\square$

Definition 51. The functor

$$\delta^* : \mathrm{mod}(\Lambda) \to \mathrm{mod}(R)$$

is called the *covariant defect functor*, and the functor

$$\delta_* : \mathrm{mod}(\Lambda) \to \mathrm{mod}(R)$$

is called the *contravariant defect functor*.

2.8 Tensor products

In this section we will study a construction called the tensor product. Consider the following definition:

Definition 52. Let $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ and $N \in \mathrm{Mod}(\Lambda)$, and let A be an abelian group. A Λ-*balanced map* $\tau : M \times N \to A$ is a map such that

$$\begin{aligned}
\tau(m_1 + m_2, n) &= \tau(m_1, n) + \tau(m_2, n), \\
\tau(m, n_1 + n_2) &= \tau(m, n_1) + \tau(m, n_2), \\
\tau(m\lambda, n) &= \tau(m, \lambda n).
\end{aligned}$$

Definition 53. Let $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ and $N \in \mathrm{Mod}(\Lambda)$. A *tensor product* $M \otimes_\Lambda N$ of M and N over Λ, is an abelian group with a Λ-balanced map τ such that the following statement holds.

For any abelian group A and Λ-balanced map $\chi : M \times N \to A$, then there exists a unique abelian group homomorphism $\zeta : M \otimes_\Lambda N \to A$ such that $\zeta\tau = \chi$.

$$\begin{tikzcd}
M \times N \arrow[rr, "\tau"] \arrow[dr, "\forall\chi\ \Lambda\text{-balanced}"'] & & M \otimes_\Lambda N \arrow[dl, dashed, "\exists!\ \text{abelian group homomorphism } \zeta"] \\
& A &
\end{tikzcd} \tag{2.42}$$

This characteristic is called *the universal property* of τ among Λ-balanced maps. For $(m, n) \in M \times N$, we let

$$\tau((m \times n)) := m \otimes n.$$

We will employ the following results regarding tensor products from [1, Ch. 5]:

Lemma 54. *Suppose $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ and $N \in \mathrm{Mod}(\Lambda)$.*

(i) *There is a tensor product $M \otimes_\Lambda N$ which is unique up to isomorphism.*

(ii) *The elements of $M \otimes_\Lambda N$ can be written as sums*

$$\sum_{i=1}^{k} (m_i, n_i),$$

for some $k \in \mathbb{N}$, where $m_i \in M$ and $n_i \in N$ for all $1 \le i \le k$.

(iii) *We may define abelian group homomorphisms from a tensor product $M \otimes_\Lambda N$ merely in terms of what one term $m \otimes n$ of an element of $M \otimes_\Lambda N$ is mapped to, by treating $m \otimes n$ as elements of the direct product $M \times N$ and expanding linearly.*

Proof.

(i) By [1, Proposition 19.1 and 19.2, Ch. 5].

(ii) By [1, Proposition 19.4, Ch. 5]

(iii) By [1, Proposition 19.4, Ch. 5].

□

From now on the above mentioned facts are considered well-known.

Lemma 55. *Let* M, $M' \in \operatorname{mod}(\Lambda)^{\mathrm{op}}$, N, $N' \in \operatorname{Mod}(\Lambda)$, *and let* $f \in \operatorname{Hom}_\Lambda(M, M')$ *and* $g \in \operatorname{Hom}_\Lambda(N, N')$. *Then there is a unique abelian group homomorphism*

$$\begin{aligned} f \otimes g : M \otimes_\Lambda n &\to M' \otimes_\Lambda N' \\ (m \otimes n) &\mapsto f(m) \otimes g(n). \end{aligned}$$

Proof. Consider the map

$$\begin{aligned} \chi : M \times N &\to M' \otimes_\Lambda N' \\ (m, n) &\mapsto f(m) \otimes g(n). \end{aligned}$$

We first show that χ is Λ-balanced:

$$\begin{aligned} \chi(m_1 + m_2, n) &= f(m_1 + m_2) \otimes g(n) \\ &= [f(m_1) + f(m_2)] \otimes g(n) \\ &= f(m_1) \otimes g(n) + f(m_2) \otimes g(n) \\ &= \chi(m_1, n) + \chi(m_2, n), \end{aligned}$$

and by the same arguments we get

$$\chi(m, n_1 + n_2) = \chi(m, n_1) + \chi(m, n_2).$$

Last,

$$\begin{aligned} \chi(m\lambda, n) &= f(m\lambda) \otimes g(n) \\ &= f(m)\lambda \otimes g(n) \\ &= f(m) \otimes \lambda g(n) \\ &= f(m) \otimes g(\lambda n) \\ &= \chi(m, \lambda n). \end{aligned}$$

Then there exists a unique abelian group homomorphism ζ such that Diagram 2.43 commutes:

$$
\begin{tikzcd}
M \times N \arrow[rr, "\tau"] \arrow[dr, "\chi"'] & & M \otimes_\Lambda N \arrow[dl, dashed, "\exists! \zeta"] \\
& M' \otimes_\Lambda N' &
\end{tikzcd}
\tag{2.43}
$$

Since $\tau(m, n) = m \otimes n$ and $\chi(m, n) = f(m) \otimes g(n)$, then $\zeta(m \otimes n)$ must be equal to $f(m) \otimes g(n)$. This completes the proof.

□

In general, tensor products are abelian groups. Not surprisingly, in our environment the they turn out to have additional structure. Recall that $\mathrm{Mod}(\Lambda) \cup \mathrm{Mod}(\Lambda^{\mathrm{op}}) \subseteq \mathrm{Mod}(R)$, where multiplying an element of a Λ-module (or Λ^{op}-module) with $r \in R$ is defined by multiplying with $r1_\Lambda$.

Lemma 56. *Let $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$, and let $N \in \mathrm{Mod}(\Lambda)$. Then $M \otimes_\Lambda N$ is an R-module with multiplication*

$$R \times M \otimes_\Lambda N \to M \otimes_\Lambda N$$

defined in the following manner:

$$r \sum_{i=1}^{k} (m_i \otimes n_i) := \sum_{i=1}^{k} (rm_i) \otimes n_i.$$

Proof. Since $rm_i \in M$ for all $r \in R$ and $1 \leq i \leq k$, then $\sum_{i=1}^{k} (rm_i) \otimes n_i$ is obviously an element of $M \otimes_\Lambda N$. We leave it to the reader to check that the module axioms are satisfied. □

For modules over suitable rings then the action of tensoring satisfies the condition of a functor. Consider the following lemma:

Lemma 57.

(i) *For $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}}$ then*

$$M \otimes_\Lambda - : \mathrm{Mod}(\Lambda) \to \mathrm{Mod}(R)$$

is a right exact functor.

(i) *For $N \in \mathrm{Mod}(\Lambda)$ then*

$$- \otimes_\Lambda N : \mathrm{Mod}(\Lambda^{\mathrm{op}}) \to \mathrm{Mod}(R)$$

is a functor.

Proof. Consider By [3, Corollary 1.9, Ch. 1]. Replace R by Λ and S by R. Then $M \otimes_\Lambda -$ and $- \otimes_\Lambda N$ are functors

$$M \otimes_\Lambda - : \mathrm{Mod}(\Lambda) \to \mathrm{Mod}(R)$$

and

$$- \otimes_\Lambda N : \mathrm{Mod}(\Lambda^{\mathrm{op}}) \to \mathrm{Mod}(R^{\mathrm{op}}) = \mathrm{Mod}(R).$$

By [3, Theorem 2.10, Ch. 1], $M \otimes_\Lambda -$ is right exact. □

The isomorphism of the following theorem is called the Adjoint Isomorphism. It will be applied in Section 3.1.3 as a step in the first out of two main tasks in this thesis, namely to develop an isomorphism of certain modules which we have yet to study.

Theorem 58. *Let $M \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$, $N \in \mathrm{Mod}(\Lambda)$ and $L \in \mathrm{Mod}(R)$. There is an abelian group isomorphism*

$$\theta_{M,N,L} : \mathrm{Hom}_R(M \otimes_\Lambda N, L) \to \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M, L))$$

given by

$$\theta_{M,N,L}(f) := [n \mapsto f(- \otimes n)]$$

for $f \in \mathrm{Hom}_R(M \otimes_\Lambda N, L)$. That is,

$$\theta_{M,N,L}(f)(n)(m) := f(m \otimes n). \tag{2.44}$$

Moreover, $\theta_{M,N,L}$ is natural in M, N and L.

Proof. For simplicity we omit the subscripts of θ for the first part of the proof. We begin by showing that the image of θ is contained in $\mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M, L))$. We recall, by Lemma 56 and Lemma 22(ii) respectively, that the R-module structure on $M \otimes_\Lambda N$ is given by

$$r(m \otimes n) := (rm) \otimes n \tag{2.45}$$

for $r \in R$ and $m \otimes N \in M \otimes_\Lambda N$, and the Λ-module structure on $\mathrm{Hom}_R(M, L)$ is given by

$$(\lambda h)(m) := h(m\lambda) \tag{2.46}$$

for $\lambda \in \Lambda$ and $h \in \mathrm{Hom}_R(M, L)$. Let

$$f \in \mathrm{Hom}_R(M \otimes_\Lambda N, L). \tag{2.47}$$

For any $n \in N$, then given $r \in R$, m_1, $m_2 \in M$, we see that

$$\begin{aligned}
\theta(f)(n)(rm_1 + m_2) &\overset{(2.44)}{=} f((rm_1 + m_2) \otimes n) \\
&= f((rm_1) \otimes n + m_2 \otimes n) \\
&\overset{(2.45)}{=} f(r(m_1 \otimes n) + m_2 \otimes n) \\
&\overset{(2.47)}{=} rf(m_1 \otimes n) + f(m_2 \otimes n) \\
&\overset{(2.44)}{=} r\theta(f)(n)(m_1) + \theta(f)(n)(m_2),
\end{aligned}$$

thus $\theta(f)(n) \in \mathrm{Hom}_R(M, L)$ for all $n \in N$.

For $\lambda \in \Lambda$, n_1, $n_2 \in N$ and $m \in M$, then

$$\begin{aligned}
\theta(f)(\lambda n_1 + n_2)(m) &\overset{(2.44)}{=} f(m \otimes (\lambda n_1 + n_2)) \\
&= f(m \otimes \lambda n_1 + m \otimes n_2) \\
&= f(m\lambda \otimes n_1 + m \otimes n_2) \\
&\overset{(2.47)}{=} f(m\lambda \otimes n_1) + f(m \otimes n_2) \\
&\overset{(2.44)}{=} \underbrace{(\theta(f)(n_1))}_{\in \mathrm{Hom}_R(M,L)}(m\lambda) + \theta(f)(n_2)(m) \\
&\overset{(2.46)}{=} \lambda\theta(f)(n_1)(m) + \theta(f)(n_2)(m) \\
&= (\lambda\theta(f)(n_1) + \theta(f)(n_2))(m).
\end{aligned}$$

Thus

$$\theta(f)(\lambda n_1 + n_2) = \lambda\theta(f)(n_1) + \theta(f)(n_2),$$

and we have proven that $\theta(f) \in \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M, L))$.

We now show that θ is an abelian group homomorphism. Let f_1, $f_2 \in \mathrm{Hom}_R(M, L)$. Then

$$\begin{aligned}
\theta(f_1 + f_2)(n)(m) &= (f_1 + f_2)(m \otimes n) \\
&= f_1(m \otimes n) + f_2(m \otimes n) \\
&= \theta(f_1)(n)(m) + \theta(f_2)(n)(m) \\
&= (\theta(f_1) + \theta(f_2))(n)(m)
\end{aligned}$$

for any $n \in N$ and $m \in M$, thus

$$\theta(f_1 + f_2) = \theta(f_1) + \theta(f_2).$$

To prove that θ is an isomorphism of abelian groups, we will construct an inverse abelian group homomorphism θ' of θ. We proceed as follows:

I) We first construct a map $\hat{\theta}$ from $\mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M, L))$ to the set of maps $M \times N \to L$. We show that $\hat{\theta}(g)$ is a Λ-balanced map for all $g \in \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M, L))$, thus since L is an abelian group, $\hat{\theta}(g)$ gives rise to a unique abelian group homomorphism

$$\theta'(g) : M \otimes_\Lambda N \to L$$

such that $\theta'(g)\tau = \hat{\theta}(g)$:

$$\begin{tikzcd}
M \times N \arrow[rr, "\tau"] \arrow[dr, "\hat{\theta}(g)"'] & & M \otimes_\Lambda N \arrow[dl, dashed, "\theta'(g)"] \\
& L &
\end{tikzcd}$$

II) We then show that $\theta'(g) \in \mathrm{Hom}_R(M \otimes_\Lambda N, L)$.

III) Finally, we show that θ' is the inverse of θ by composing the two homomorphisms.

I) Let

$$g \in \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M, L)). \tag{2.48}$$

We define

$$\hat{\theta}(g) : M \times N \to L$$

by

$$\hat{\theta}(g)((m, n)) := g(n)(m). \tag{2.49}$$

Note that

$$g(n) \in \mathrm{Hom}_R(M, L) \tag{2.50}$$

for all $n \in N$.

For m_1, $m_2 \in M$ and $n \in N$, then

$$\begin{aligned}\hat{\theta}(g)((m_1 + m_2, n)) &\overset{(2.49)}{=} g(n)(m_1 + m_2)\\ &\overset{(2.50)}{=} g(n)(m_1) + g(n)(m_2)\\ &\overset{(2.49)}{=} \hat{\theta}(g)((m_1, n)) + \hat{\theta}(g)((m_2, n)).\end{aligned}$$

For $m \in M$ and n_1, $n_2 \in N$, then

$$\begin{aligned}\hat{\theta}(g)((m, n_1 + n_2)) &\overset{(2.49)}{=} g(n_1 + n_2)(m)\\ &\overset{(2.48)}{=} g(n_1)(m) + g(n_2)(m)\\ &\overset{(2.49)}{=} \hat{\theta}(g)((m, n_1)) + \hat{\theta}(g)((m, n_2)).\end{aligned}$$

For $m \in M$, $n \in N$ and $\lambda \in \Lambda$, then

$$\begin{aligned}\hat{\theta}(g)((m\lambda, n)) = g(n)(m\lambda) &\underset{(2.46)}{\overset{(2.50),}{=}} \lambda g(n)(m)\\ &\overset{(2.48)}{=} g(\lambda n)(m)\\ &\overset{(2.49)}{=} \hat{\theta}(g)((m, \lambda n)).\end{aligned}$$

Thus $\hat{\theta}(g)$ is Λ-balanced, and there exists an abelian group homomorphism

$$\theta'(g) : M \otimes_\Lambda N \to L$$

given by

$$\theta'(g)(m \otimes n) = \hat{\theta}((m, n)) = g(n)(m). \tag{2.51}$$

II) We know that $\theta'(g)$ is an abelian group homomorphism. To show that $\theta'(g) \in \operatorname{Hom}_R(M \otimes_\Lambda N, L)$, we must see that

$$\theta'(g)(r(m \otimes n)) = r\theta'(g)(m \otimes n)$$

for all $r \in R$:

$$\begin{aligned} \theta'(g)(r(m \otimes n)) &\overset{(2.45)}{=} \theta'(g)((rm) \otimes n) \\ &\overset{(2.49)}{=} g(n)(rm) \\ &\overset{(2.50)}{=} rg(n)(m) \\ &\overset{(2.49)}{=} r\theta'(g)(m \otimes n). \end{aligned}$$

III) We will now show that θ' is the inverse abelian group homomorphism of θ. First we verify that θ' is in fact a homomorphism of abelian groups. Let g_1, $g_2 \in \operatorname{Hom}_\Lambda(N, \operatorname{Hom}_R(M, L))$. Then

$$\begin{aligned} \theta'(g_1 + g_2)(m \otimes n) &= (g_1 + g_2)(n)(m) \\ &= g_1(n)(m) + g_2(n)(m) \\ &= \theta'(g_1)(m \otimes n) + \theta'(g_2)(m \otimes n) \\ &= (\theta'(g_1) + \theta'(g_2))(m \otimes n) \end{aligned}$$

for any $m \otimes n \in M \otimes_\Lambda N$, thus

$$\theta'(g_1 + g_2) = \theta'(g_1) + \theta'(g_2).$$

We must now show that

$$\theta'\theta = 1_{\operatorname{Hom}_R(M \otimes_\Lambda N, L)} \tag{2.52}$$

and

$$\theta\theta' = 1_{\operatorname{Hom}_\Lambda(N, \operatorname{Hom}_R(M, L))}. \tag{2.53}$$

We first show (2.52). For this we need to see that

$$\theta'\theta(f) = f$$

for all $f \in \operatorname{Hom}_R(M \otimes_\Lambda N, L)$, that is, that

$$\theta'\theta(f)(m \otimes n) = f(m \otimes n)$$

for all $m \otimes n \in M \otimes_\Lambda N$. This is easily verified:

$$\theta'(\theta(f))(m \otimes n) \overset{(2.51)}{=} \theta(f)(n)(m) \overset{(2.44)}{=} f(m \otimes n).$$

We now show (2.53). We then need to see that

$$\theta\theta'(g) = g$$

for all $g \in \operatorname{Hom}_\Lambda(N, \operatorname{Hom}_R(M, L))$, that is,

$$\theta\theta'(g)(n)(m) = g(n)(m)$$

for all $n \in N$ and $m \in M$:

$$\theta(\theta'(g))(n)(m) \overset{(2.44)}{=} \theta'(g)(m \otimes n) \overset{(2.51)}{=} g(n)(m)$$

Finally, we prove that θ is natural in M, N and L. For this part, we return to writing θ with subscripts M, N and L.

Let M and M' be to R-Λ-bimodules, and let $u : M \to M'$. Then u gives rise to the R-module homomorphism

$$[- \circ (u \otimes 1_N)]_L : \operatorname{Hom}(M' \otimes_\Lambda N, L) \to \operatorname{Hom}(M \otimes_\Lambda N, L)$$

and the Λ-module homomorphism

$$[(- \circ u)_L \circ -]_N : \operatorname{Hom}_\Lambda(N, \operatorname{Hom}_R(M', L)) \to \operatorname{Hom}_\Lambda(N, \operatorname{Hom}_R(M, L)).$$

The naturality of $\theta_{M,N,L}$ in M is by definition the commutativity of the following diagram:

$$\begin{tikzcd}
\operatorname{Hom}_R(M' \otimes_\Lambda N, L) \arrow[r, "\theta_{M',N,L}"] \arrow[d, "{[- \circ (u \otimes 1_N)]_L}"'] & \operatorname{Hom}_\Lambda(N, \operatorname{Hom}_R(M', L)) \arrow[d, "{[(- \circ u)_L \circ -]_N}"] \\
\operatorname{Hom}_R(M \otimes_\Lambda N, L) \arrow[r, "\theta_{M,N,L}"] & \operatorname{Hom}_\Lambda(N, \operatorname{Hom}_R(M, L))
\end{tikzcd}$$

Let $f \in \operatorname{Hom}_R(M' \otimes_\Lambda N, L)$. Then

$$\begin{aligned}
[(- \circ u)_L \circ -]_N \circ \theta_{M',N,L}(f) &= [(- \circ u)_L \circ -]_N(\theta_{M',N,L}(f)) \\
&= [(- \circ u)_L \circ -]_N([n \mapsto f(- \otimes n)]) \\
&= (- \circ u)([n \mapsto f(- \otimes n)]) \\
&= [n \mapsto f(u(-) \otimes n)],
\end{aligned}$$

and

$$\begin{aligned}
\theta_{M,N,L} \circ [- \circ (u \otimes 1_N)]_L(f) &= \theta_{M,N,L}([- \circ (u \otimes 1_N)]_L(f)) \\
&= \theta_{M,N,L}([f \circ (u \otimes 1_N)]_L) \\
&= [n \mapsto f \circ u \otimes 1_N(- \otimes n)] \\
&= [n \mapsto f(u(-) \otimes n)].
\end{aligned}$$

Hence $\theta_{M,N,L}$ is natural in M.

Let N, $N' \in \operatorname{Mod}(\Lambda)$, and let $v \in \operatorname{Hom}_\Lambda(N, N')$. Then v gives rise to the R-module homomorphism

$$[- \circ (1_M \otimes v)]_L : \operatorname{Hom}_R(M \otimes_\Lambda N', L) \to \operatorname{Hom}_R(M \otimes_\Lambda N', L)$$

and the Λ-module homomorphism

$$(- \circ v)_{\mathrm{Hom}_R(M,L)} : \mathrm{Hom}_\Lambda(N', \mathrm{Hom}_R(M,L)) \to \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M,L)).$$

We must show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_R(M \otimes_\Lambda N', L) \arrow[r, "\theta_{M,N',L}"] \arrow[d, "{[- \circ (1_M \otimes v)]_L}"'] & \mathrm{Hom}_\Lambda(N', \mathrm{Hom}_R(M,L)) \arrow[d, "(- \circ v)_{\mathrm{Hom}_R(M,L)}"] \\
\mathrm{Hom}_R(M \otimes_\Lambda N, L) \arrow[r, "\theta_{M,N,L}"] & \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M,L))
\end{tikzcd}$$

Let $f \in \mathrm{Hom}_R(M \otimes_\Lambda N', L)$. Then

$$\begin{aligned}
(- \circ v)_{\mathrm{Hom}_R(M,L)} \circ \theta_{M,N',L}(f) &= (- \circ v)_{\mathrm{Hom}_R(M,L)}(\theta_{M,N',L}(f)) \\
&= (- \circ v)_{\mathrm{Hom}_R(M,L)}([n' \mapsto f(- \otimes n')]) \\
&= [n \mapsto f(- \otimes v(n))],
\end{aligned}$$

and

$$\begin{aligned}
\theta_{M,N,L} \circ [- \circ (1_M \otimes v)]_L(f) &= \theta_{M,N,L}([- \circ (1_M \otimes v)]_L(f)) \\
&= \theta_{M,N,L}(f \circ (1_M \otimes v)) \\
&= [n \mapsto f \circ (1_M \otimes v)(- \otimes n)] \\
&= [n \mapsto f(- \otimes v(n))].
\end{aligned}$$

Hence $\theta_{M,N,L}$ is natural in N.

Let L, $L' \in \mathrm{Mod}(R)$, and let $w \in \mathrm{Hom}_R(L, L')$. Then w gives rise to the R-module homomorphism

$$(w \circ -)_{M \otimes_\Lambda N} : \mathrm{Hom}_R(M \otimes_\Lambda N, L) \to \mathrm{Hom}_R(M \otimes_\Lambda N, L')$$

and the l-module homomorphism

$$[(w \circ -)_M \circ -]_N : \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M,L)) \to \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M,L')).$$

We must see that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_R(M \otimes_\Lambda N, L) \arrow[r, "\theta_{M,N,L}"] \arrow[d, "(w \circ -)_{M \otimes_\Lambda N}"'] & \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M,L)) \arrow[d, "{[(w \circ -)_M \circ -]_N}"] \\
\mathrm{Hom}_R(M \otimes_\Lambda N, L') \arrow[r, "\theta_{M,N,L'}"] & \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_R(M,L'))
\end{tikzcd}$$

Let $f \in \mathrm{Hom}_R(M \otimes_\Lambda N, L)$. Then

$$\begin{aligned}
[(w \circ -)_M \circ -]_N \circ \theta_{M,N,L}(f) &= [(w \circ -)_M \circ -]_N(\theta_{M,N,L}(f)) \\
&= [(w \circ -)_M \circ -]_N([n \mapsto f(- \otimes n)]) \\
&= (w \circ -)_M \circ [n \mapsto f(- \circ n)] \\
&= [n \mapsto wf(- \otimes n)],
\end{aligned}$$

and

$$\begin{aligned}\theta_{M,N,L} \circ (w \circ -)_{M \otimes_\Lambda N}(f) &= \theta_{M,N,L}((w \circ -)_{M \otimes_\Lambda N}(f)) \\ &= \theta_{M,N,L}(wf) \\ &= [n \mapsto wf(- \otimes n)].\end{aligned}$$

Hence $\theta_{M,N,L}$ is natural in L, and the proof is completed. □

2.9 Split and almost split sequences

As mentioned in the introduction in Section 1, our primary objective is to compute almost split sequences. A formal definition is called for, but first, consider the following result for short exact sequences in $\mathrm{mod}(\Lambda)$:

Lemma 59. *For an exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathrm{mod}(\Lambda)$*, the following two statements are equivalent:*

(i) *There exists* $f' \in \mathrm{Hom}_\Lambda(B, A)$ *such that* $f'f = 1_A$*.*

(ii) *There exists* $g' \in \mathrm{Hom}_\Lambda(C, B)$ *such that* $gg' = 1_C$*.*

Proof. We will demonstrate that (i) implies (ii). The proof of the converse is similar.

Suppose $f' \in \mathrm{Hom}_\Lambda(B, A)$ such that $f'f = 1_A$. Consider the following diagram in $\mathrm{mod}(\Lambda)$:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] \arrow[dl, "f'"'] & C \arrow[r] \arrow[d, dashed, "1_C"] \arrow[dl, dashed, "g'"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

Let

$$s := 1_B - ff'.$$

Then

$$hf = f - (ff')f = f - f\underbrace{(f'f)}_{=1_A} = 0,$$

so h factors through $\mathrm{Cok}(f) = C$. That is, there is $g' \in \mathrm{Hom}_\Lambda(C, B)$ such that

$$g'g = h.$$

Then

$$(gg')g = g(g'g) = gh = g1_B - g(ff') = g - \underbrace{(gf)}_{=0}f' = 1_C g,$$

implying, since g is an epimorphism, that

$$gg' = 1_C.$$

□

We now give the definition of a split epimorphism, a split sequence and an almost split sequence.

Definition 60.

(i) Let M, $N \in \mathrm{mod}(\Lambda)$, and let $f \in \mathrm{Hom}_\Lambda(M, N)$ be an epimorphism. We say that f is a *split epimorphism* if there exists $f' \in \mathrm{Hom}_\Lambda(N, M)$ such that

$$ff' = 1_N.$$

(ii) Let δ be an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathrm{mod}(\Lambda)$.

(a) We say that δ is a *split sequence*, or that δ *splits*, if it satisfies the equivalent conditions of Lemma 59.

(b) We say that δ is an *almost split sequence* if δ does not split and the following holds: For any $Y \in \mathrm{mod}(\Lambda)$ and any $h \in \mathrm{Hom}_\Lambda(Y, C)$ which is not a split epimorphism,[5] then there is $u \in \mathrm{Hom}_\Lambda(Y, B)$ such that

$$h = gu.$$

$$\begin{tikzcd}
 & & & Y \arrow[dl, dashed, "u"'] \arrow[d, "h"] & \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

The following proposition states some facts for almost split sequences in $\mathrm{mod}(\Lambda)$.

Proposition 61.

(i) *All almost split sequences in* $\mathrm{mod}(\Lambda)$ *are of the form*

$$0 \longrightarrow D\,\mathrm{Tr}(X) \xrightarrow{f} E \xrightarrow{g} X \longrightarrow 0$$

for some $E \in \mathrm{mod}(\Lambda)$*, where* $X \in \mathrm{mod}(\Lambda)$ *is indecomposable and non-projective.*

[5] If there were $h' \in \mathrm{Hom}_\Lambda(C, X)$ such that $hh' = 1_C$, the existence of u would give a $g' := uh' \in \mathrm{Hom}_\Lambda(C, B)$ such that $gg' = 1_C$, but δ does not split.

(ii) *For any non-projective and indecomposable* $X \in \mathrm{mod}(\Lambda)$*, there is an almost split sequence*

$$0 \longrightarrow D\,\mathrm{Tr}(X) \xrightarrow{f} E \xrightarrow{g} X \longrightarrow 0$$

in $\mathrm{mod}(\Lambda)$.

Proof.

(i) If

$$0 \longrightarrow Y \xrightarrow{f} E \xrightarrow{g} X \longrightarrow 0$$

is an almost split sequence, then by [2, Proposition 1.14, Ch. 5], X is indecomposable and $Y \simeq D\,\mathrm{Tr}(X)$. If X were projective, there would be $g' \in \mathrm{Hom}_\Lambda(X, E)$ such that $gg' = 1_E$, and the sequence would split.

(ii) This follows from [2, Theorem 1.15, Ch. 5].

□

In the following, we will be working with equivalence classes of almost split sequences. We introduce the following notation:

Definition 62. Let U, $V \in \mathrm{mod}(\Lambda)$.

(i) We let

$$\Upsilon_{U,V} := \{\text{Short exact sequences starting in } U \text{ and ending in } V\}.$$

(ii) We let $\hat{\Upsilon}_{U,V} \subseteq \Upsilon_{U,V}$ be given by

$$\hat{\Upsilon}_{U,V} := \{\text{Almost split sequences starting in } U \text{ and ending in } V\}.$$

(iii) We let $\sim$ denote the following equivalence relation on $\Upsilon_{U,V}$: Two short exact sequences

$$0 \longrightarrow U \longrightarrow E \longrightarrow V \longrightarrow 0$$

and

$$0 \longrightarrow U \longrightarrow E' \longrightarrow V \longrightarrow 0$$

are *equivalent* if there exists $e \in \mathrm{Hom}_\Lambda(E, E')$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "e"] & V \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U \arrow[r] & E' \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

Note that the symmetry of this relation follows from the Five Lemma, while its transitivity and reflexivity are obvious. We find it satisfactory to omit indexing $\sim$ by U and V, but it should be clear that $\sim$ only relates short exact sequences with the same starting points and the same end points.

It can be shown that $\Upsilon_{U,V}/\sim$ exhibits abelian group structure, and that $\hat{\Upsilon}_{U,V}$ is a subgroup if one includes the equivalence class of split sequences from U to V. This topic will be revisited in the end of Section 2.10.

2.10 The artin R-algebra Γ

We now let $X \in \mathrm{mod}(\Lambda)$ be non-projective and indecomposable. Recall the factor $\underline{\mathrm{End}}_\Lambda(X)$ of $\mathrm{End}_\Lambda(X)$ from Definition 44(ii). By Proposition 24(ii), $\underline{\mathrm{End}}_\Lambda(X)$ is an artin R-algebra. In addition this algebra has some features which will be highly relevant for our further work, as a great part of Chapter 3 will consist of the development of $\underline{\mathrm{End}}_\Lambda(X)$-module isomorphisms. An abbreviating notation is called for.

Definition 63. Let
$$\Gamma := \underline{\mathrm{End}}_\Lambda(X).$$

Lemma 64. *The artin R-algebra Γ is a local ring.*

Proof. By [2, Theorem 2.2, Ch. 2], $\mathrm{End}_\Lambda(X)$ is a local ring. Then Γ is a local ring by Lemma 12. $\square$

Since Γ is a local ring, it has a unique maximal ideal. From now on we let
$$\underline{r} := \mathrm{rad}(\Gamma).$$

Recall the dual $D = \mathrm{Hom}_R(-, I)$ from Section 2.3. By Proposition 29, we know that D is a functor
$$D : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Gamma^{\mathrm{op}}),$$
and that
$$D^2(M) \simeq M$$
as Γ-modules for all $M \in \mathrm{mod}(\Gamma)$. The following lemma states that the dual preserves zero and simpleness:

Lemma 65.

(i) *If $M \in \mathrm{mod}(\Gamma)$ is nonzero, then $D(M) \in \mathrm{mod}(\Gamma^{\mathrm{op}})$ is nonzero.*

(ii) *If $M \in \mathrm{mod}(\Gamma)$ is simple, then $D(M) \in \mathrm{mod}(\Gamma^{\mathrm{op}})$ is simple.*

Proof.

(i) Let $M \in \operatorname{mod}(\Gamma)$, and suppose $m \in M$ is a nonzero element. Note that M is also an R-module, and recall that I is the injective envelope of $R/\underline{m}$ where $\underline{m}$ is the maximal ideal in R. Let $\hat{f} \in \operatorname{Hom}_R(Rm, I)$ be given by

$$\hat{f}(rm) := r + \underline{m}$$

as in the proof of Proposition 29(iii) (a). Then since

$$\iota : Rm \to M$$

is an inclusion and I is injective there is $f \in \operatorname{Hom}_R(M, I)$ such that

$$f\iota = \hat{f},$$

and

$$f(m) = f\iota(m) = \hat{f}(m) = 1_R + \underline{m} \neq 0.$$

That is, $f \in D(M)$ is nonzero.

(ii) Let $M \in \operatorname{mod}(\Gamma)$ be simple. If $M = 0$ then $D(M) = 0$, so we assume $M \neq 0$. Suppose $D(M)$ is a non-simple Γ^{op}-module. Then $D(M)$ has a nonzero, proper submodule U, and we get the following exact sequence in $\operatorname{mod}(\Gamma^{\mathrm{op}})$:

$$0 \longrightarrow U \longrightarrow D(M) \longrightarrow D(M)/U \longrightarrow 0.$$

By applying D and since $D^2(M) \simeq M$ as Γ-modules, we get

$$0 \longrightarrow D(D(M)/U) \longrightarrow M \longrightarrow D(U) \longrightarrow 0.$$

in $\operatorname{mod}(\Gamma)$. Note that $D(M)/U \neq 0$ since $U \neq D(M)$. Then since M is simple it follows that $D(D(M)/U/) \simeq D^2(M)$, and by the exactness of the above sequence we see that

$$D(U) = 0.$$

Then

$$U = 0$$

by (iii), which contradicts the assumption that U is a nonzero submodule of $D(M)$.

□

We will now give the definition of the top and the socle of a module.

Definition 66. Let S be an artinian ring, and let $M \in \operatorname{mod}(S)$. Let $\underline{r}$ denote the radical of S.

(i) The *top of M (as S-module)* is the quotient of M modulo its radical,

$$\operatorname{Top}_S(M) := M/\underline{r}M.$$

(ii) The *socle of M (as S-module)* is the sum of all simple S-submodules of M:

$$\operatorname{Soc}_S(M) := \sum\{U \mid U \text{ is a simple } S\text{-submodule of } M\}.$$

The following result will be of great importance for us, since a consequence will be that any almost split sequence ending in X is a generator for all such.

Lemma 67. *The following holds.*

(i) $\operatorname{Top}_\Gamma(\Gamma)$ *is a simple* Γ*-module.*

(ii) $\operatorname{Soc}_\Gamma(D\Gamma) \simeq D\operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$ *as* Γ*-modules.*

(iii) $\operatorname{Soc}_\Gamma(D\Gamma)$ *is a simple* Γ*-module.*

Proof.

(i) By Lemma 9(iii),
$$\operatorname{End}_\Gamma(\Gamma) \simeq \Gamma$$
as rings. Then $\operatorname{End}_\Gamma(\Gamma)$ is a local ring by Lemma 64.

Consider [2, Proposition 4.7, Ch. 1]. Since Γ is an artin R-algebra, then by Lemma 19, it is also an artinian ring. Moreover, Γ is a projective Γ-module. It follows that $\underline{r}\Gamma$ is the unique maximal submodule of Γ.

We now show that $\operatorname{Top}_\Gamma(\Gamma) = \Gamma/\underline{r}\Gamma$ is a simple Γ-module. Suppose M is a nonzero submodule of $\Gamma/\underline{r}\Gamma$. Then M is of the form
$$M = N/\underline{r}\Gamma$$
for some Γ-module N such that
$$\underline{r}\Gamma \subseteq N \subseteq \Gamma.$$
Since M is nonzero then $\underline{r}\Gamma \neq N$, so since $\underline{r}\Gamma$ is the maximal submodule of Γ, then $N = \Gamma$. Hence
$$M = \Gamma/\underline{r}\Gamma.$$

(ii) Consider the following exact sequence of Γ^{op}-modules:
$$0 \longrightarrow \underline{r} \longrightarrow \Gamma \longrightarrow \Gamma/\underline{r} \longrightarrow 0$$
By applying D we get the following exact sequence of Γ-modules:
$$0 \longrightarrow D(\Gamma/\underline{r}) \longrightarrow D(\Gamma) \longrightarrow D(\underline{r}) \longrightarrow 0$$
By [2, Proposition 3.1, Ch. 1], then $\Gamma/\underline{r}$ is a semisimple Γ^{op}-module. Thus, by Lemma 65(ii) and because D commutes with finite direct sums, the submodule $D(\Gamma/\underline{r})$ of $D\Gamma$ is a semisimple Γ-module. That is,
$$D(\Gamma/\underline{r}) \subseteq \operatorname{Soc}_\Gamma(D\Gamma).$$

By the same argument, it is clear that $D\operatorname{Soc}_\Gamma(D\Gamma)$ is a semisimple Γ^{op}-module. Moreover,

$$\operatorname{Soc}_\Gamma(D\Gamma) \subseteq D\Gamma,$$

and we get the following exact, commutative diagram in $\operatorname{mod}(\Gamma)$:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & D(\Gamma/\underline{r}) \arrow[r] \arrow[d] & D\Gamma \arrow[d, equal] & & \\
0 \arrow[r] & \operatorname{Soc}_\Gamma(D\Gamma) \arrow[r] & D\Gamma \arrow[r] & D\Gamma/\operatorname{Soc}_\Gamma(D\Gamma) \arrow[r] & 0
\end{tikzcd}$$

By applying D, we get the following commutative, exact diagram in $\operatorname{mod}(\Gamma^{\mathrm{op}})$:

$$\begin{tikzcd}
0 \arrow[r] & D(D\Gamma/\operatorname{Soc}_\Gamma(D\Gamma)) \arrow[r] & \Gamma \arrow[r, "b"] \arrow[d, equal] & D(\operatorname{Soc}_\Gamma(D\Gamma)) \arrow[r] \arrow[d, bend left, "d"] & 0 \\
0 \arrow[r, dashed] & \underline{r} \arrow[r, dashed, "a"] & \Gamma \arrow[r, "c"] & \Gamma/\underline{r} \arrow[r] \arrow[u, dashed, bend left, "e"] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

We know that $\underline{r}$ is the kernel of c, so we fill this into the diagram and get an exact bottom row. The composition $b1_\Gamma a = 0$ because $\underline{r}$ annihilates the semisimple Γ^{op}-modules, thus since $\Gamma/\underline{r}$ is the cokernel of a, there is $e \in \operatorname{Hom}_{\Gamma^{\mathrm{op}}}(\Gamma/\underline{r}, D(\operatorname{Soc}_\Gamma(D\Gamma)))$ such that

$$ec = b1_\Gamma = b.$$

Moreover, since b is an epimorphism then so is e. The composition $de \in \operatorname{End}_{\Gamma^{\mathrm{op}}}(\Gamma/\underline{r})$ is thus an isomorphism ([2, Proposition 1.4, Ch. 1]) since $l_{\Gamma^{\mathrm{op}}}(\Gamma/\underline{r}) < \infty$. It follows that e is a monomorphism in addition to being an epimorphism, that is,

$$D(\operatorname{Soc}_\Gamma(D\Gamma)) \simeq \Gamma/\underline{r} = \operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$$

as Γ^{op}-modules. Equivalently,

$$\operatorname{Soc}_\Gamma(D\Gamma) \simeq D\operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma) \tag{2.54}$$

as Γ-modules.

(iii) The result of (i) also holds if we regard Γ as a Γ^{op}-module. That is, $\operatorname{Top}_{\Gamma^{\mathrm{op}}}(\Gamma)$ is a simple Γ^{op}-module. By Lemma 65(ii) and (2.54) it follows that $\operatorname{Soc}_\Gamma(D\Gamma)$ is a simple Γ-module.

□

The following lemma is probably well-known to the reader, but we choose to include it due to its importance for the final result of this thesis, namely the algorithm presented in Section 3.2.2.

Lemma 68. *For a ring S, then a simple S-module M can be generated by any nozero element of M.*

Proof. If $m \in M\backslash\{0\}$, then m generates a nonzero submodule M' of M. Since M is simple, then $M' = M$. □

Recall from Section 2.9 the set $\Upsilon_{U,V}/\sim$ of equivalence classes of short exact sequences from U to V, and the subset $\hat{\Upsilon}_{U,V}/\sim$ of equivalence classes of almost split sequences. For the purpose of regarding $\hat{\Upsilon}_{U,V}/\sim$ as a subgroup of $\Upsilon_{U,V}/\sim$, we redefine $\hat{\Upsilon}_{U,V}/\sim$ to also contain the equivalence class of split sequences from U to V since this represents zero in $\Upsilon_{U,V}/\sim$. ([3, Corollary 7.20 and Theorem 7.21, Ch. 7].) However, for simplicity we will still refer to $\hat{\Upsilon}_{U,V}/\sim$ as the set of equivalence classes of almost exact sequences from U to V, omitting to specify each time the incorporation of a zero element.

We will now look at a convenient relation between short exact sequences and Ext functors. For a definition and basic properties of Ext, see [3, Ch. 7]. The following Proposition allows us to regard $\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$ as a Γ-module whose socle can be identified with the set of equivalence classes of almost split sequences ending in X.

Proposition 69.

(i) *Let U, $V \in \operatorname{mod}(\Lambda)$. Then $\Upsilon_{U,V}/\sim$ is an abelian group, and there is an abelian group isomorphism*

$$\operatorname{Ext}^1_\Lambda(V, U) \simeq (\Upsilon_{U,V}/\sim).$$

(ii) *Let $X \in \operatorname{mod}(\Lambda)$ be indecomposable and non-projective. We can assign to $\Upsilon_{D\operatorname{Tr}(X),X}/\sim$ a Γ-module structure such that the following holds:*

(a) $(\Upsilon_{D\operatorname{Tr}(X),X})/\sim) \in \operatorname{mod}(\Gamma)$,

(b) $\operatorname{Soc}_\Gamma(\Upsilon_{D\operatorname{Tr}(X),X}/\sim) = (\hat{\Upsilon}_{D\operatorname{Tr}(X),X}/\sim)$.

Proof.

(i) By [3, Theorem 7.21, Ch. 7].

(ii) We first observe that if $(\Upsilon_{D\operatorname{Tr}(X),X}/\sim) \in \operatorname{Mod}(\Gamma)$ then by (i) this brings about a Γ-module structure on $\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$ such that the isomorphism of (i) becomes a Γ-module isomorphism. It can be shown that

$$\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X)) \in \operatorname{mod}(R),^6$$

[6] In the proof of Proposition 83 we will see that $\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X)) = \delta_*(D\operatorname{Tr}(X))$ for a given short exact sequence δ, and by Proposition 50, $\delta_*(D\operatorname{Tr}(X)) \in \operatorname{mod}(R)$.

hence by Lemma 21 we see that $\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$, and so also $\Upsilon_{D\operatorname{Tr}(X),X}/\sim$, is finitely generated as Γ-module.

As for the rest of what was claimed, a complete demonstration will not be given in this thesis. However, the question of a Γ-module structure on a set of equivalence classes of short exact sequences will be revisited in Section 3.1.3 as part of the proof of Proposition 79. There a simplified investigation will be carried out for $\operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X))$ and $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$ for an indecomposable, non-projective $X \in \operatorname{mod}(\Lambda)$ and an arbitrary $C \in \operatorname{mod}(\Lambda)$. We will also show how to obtain the corresponding element of $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$ given an element of a subset of $\operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X))$. In the case that $C := X$ this subset turns out to be equal to $\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$ itself.

For more information, see [2, Ch. 5].

□

Chapter 3

Computing Almost Split Sequences

From now on we let X be a fixed, indecomposable and non-projective element of $\mathrm{mod}(\Lambda)$, and we fix a minimal projective presentation

$$P_1 \xrightarrow{\ s\ } P_0 \xrightarrow{\ t\ } X \longrightarrow 0$$

of X.

We are interested in generating almost split sequences ending in X. Recall Proposition 69 of Section 2.10. It turns out that if we consider $\mathrm{Ext}^1_\Lambda(X, D\,\mathrm{Tr}(X))$ to be a Γ-module by identifying it with the Γ-module $\Upsilon_{D\,\mathrm{Tr}(X),X}/\sim$, then there exists a Γ-module isomorphism

$$\breve{\omega}_X : D\Gamma \to \mathrm{Ext}^1_\Lambda(X, D\,\mathrm{Tr}(X)). \tag{3.1}$$

By Lemma 67(iii) then $\mathrm{Soc}_\Gamma(D\Gamma)$ is a simple Γ-module, and by applying the isomorphism $\breve{\omega}_X$ restricted to $\mathrm{Soc}_\Gamma(D\Gamma)$, we get the following chain of Γ-module isomorphisms:

$$\mathrm{Soc}_\Gamma(D\Gamma) \underset{\breve{\omega}_X|_{\mathrm{Soc}_\Gamma(D\Gamma)}}{\simeq} \mathrm{Soc}_\Gamma(\mathrm{Ext}^1_\Lambda(X, D\,\mathrm{Tr}(X))) \underset{\text{Prop. 69(i)}}{\simeq} \mathrm{Soc}_\Gamma(\Upsilon_{D\,\mathrm{Tr}(X),X}/\sim) \underset{\text{Prop. 69(ii)}}{\simeq} \hat{\Upsilon}_{D\,\mathrm{Tr}(X),X}/\sim .$$

That is, $\hat{\Upsilon}_{D\,\mathrm{Tr}(X),X}/\sim$ is a simple Γ-module.

Thus, by Lemma 68, any nonzero element of $\mathrm{Soc}_\Gamma(D\Gamma)$ provides, through the application of $\breve{\omega}_X$ and the isomorphism of Proposition 69(i), a generator for

$$(\hat{\Upsilon}_{D\,\mathrm{Tr}(X),X}/\sim) = \{\text{Equivalence classes of almost split sequences ending in } X\}.$$

This strongly encourages us to find $\breve{\omega}_X$. This isomorphism is obtained from a more general connection between the dual D from Section 2.3 and the defect functors δ_* and δ^* from Section 2.7, given in terms of a Γ-module isomorphism

$$\omega_{\delta,X} : D\delta^*(X) \to \delta_*(D\,\mathrm{Tr}(X))$$

whose subscript δ represents a short exact sequence in $\mathrm{mod}(\Lambda)$. We will dedicate Section 3.1 to constructing this isomorphism $\omega_{\delta,X}$. This task will include defining and composing various maps, endoved with miscellaneous structures which will be illuminated throughout the process.

Once equipped with $\omega_{\delta,X}$, we will define $\breve{\omega}_X$ in Section 3.2 to be the special case of $\omega_{\delta,X}$ when δ is the exact sequence

$$0 \longrightarrow \Omega_\Lambda(X) \xrightarrow{\iota} P(X) \xrightarrow{t} X \longrightarrow 0$$

of Definition 37(i), whence we get (3.1).

When restricting R to be a field in Section 3.2.2, the results of the prior investigation will suggest an algorithm for computing almost split sequences.

3.1 Constructing the isomorphism $\omega_{\delta,X}$

The acquisition of $\omega_{\delta,X}$ will essentially follow from the construction of two important bijections, named $\sigma_{\delta,X}$ and $\gamma_{\delta,X}$. The isomorphism $\gamma_{\delta,X}$ will be found in Section 3.1.3 by means of the Adjoint Isomorphism of Theorem 58. The isomorphism $\sigma_{\delta,X}$ will be given in terms of an algorithm in Section 3.1.2, obtained from studying a commutative diagram in $\mathrm{mod}(R)$. This diagram will in turn be found by completing exact sequences in $\mathrm{mod}(R)$ of the form

$$0 \to \mathrm{Hom}_\Lambda(X,Y) \xrightarrow{(-\circ t)_Y} \mathrm{Hom}_\Lambda(P_0,Y) \xrightarrow{(-\circ s)_Y} \mathrm{Hom}_\Lambda(P_1,Y) \tag{3.2}$$

for $Y \in \mathrm{mod}(\Lambda)$, to exact sequences

$$0 \to \mathrm{Hom}_\Lambda(X,Y) \xrightarrow{(-\circ t)_Y} \mathrm{Hom}_\Lambda(P_0,Y) \xrightarrow{(-\circ s)_Y} \mathrm{Hom}_\Lambda(P_1,Y) \xrightarrow{\phi_Y} \mathrm{Tr}(X) \otimes_\Lambda Y \to 0.$$

We thus need to show that $\mathrm{Tr}(X) \otimes_\Lambda Y$ is the cokernel of $(-\circ s)_Y$, and we are also interested in describing in detail the projection

$$\phi_Y : \mathrm{Hom}_\Lambda(P_1,Y) \to \mathrm{Tr}(X) \otimes_\Lambda Y.$$

In order to find ϕ_Y, we must first show that

$$\mathrm{Hom}_\Lambda(P,Y) \simeq \mathrm{Hom}_\Lambda(P,\Lambda) \otimes_\Lambda Y$$

as R-modules – and, of course, how this isomorphism is given – for projective P and any Y in $\mathrm{mod}(\Lambda)$. This will be done in Section 3.1.1. We will then find ϕ_Y in Section 3.1.2, by choosing an appropriate $P \in \mathcal{P}(\Lambda)$ and composing $\varphi_{P,Y}$ with a suitable R-module homomorphism.

3.1.1 A relation between homomorphism sets and tensor products

The aim of this section is to construct an R-module isomorphism

$$\varphi_{P,Y} : \mathrm{Hom}_\Lambda(P, Y) \to \mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y$$

for any $Y \in \mathrm{mod}(\Lambda)$ and any $P \in \mathcal{P}(\Lambda)$.

We recall from Proposition 40 that $\mathrm{Hom}_\Lambda(P, \Lambda) \in \mathrm{mod}(\Lambda^{\mathrm{op}})$, and then by Lemma 56 we see that $\mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y$ exhibits R-module structure as follows:

$$r \cdot f \otimes y := (rf) \otimes y.$$

We also recall from Lemma 22(iii) that

$$(rf)(p) := r(f(p))$$

defines R-module structure on $\mathrm{Hom}_\Lambda(P, Y)$.

The isomorphism $\varphi_{P,Y}$ will later be applied with some specific choices for P and Y, as one of the stages in developing the desired isomorphism

$$\omega_{\delta,X} : \delta^*(X) \to \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f).$$

It is not easy to construct a map from $\mathrm{Hom}_\Lambda(P, Y)$ to $\mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y$ directly, since although we can easily obtain an element of Y given an element of $\mathrm{Hom}_\Lambda(P, Y)$, we do not know of an obvious way of obtaining an element of $\mathrm{Hom}_\Lambda(P, \Lambda)$.

What we do have is an R-module homomorphism going in the opposite direction of what we are interested in, as stated in Lemma 70.

Lemma 70. *Let $Y \in \mathrm{mod}(\Lambda)$ and $P \in \mathcal{P}(\Lambda)$. We define*

$$\alpha_{P,Y} : \mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y \to \mathrm{Hom}_\Lambda(P, Y)$$

by

$$\alpha_{P,Y}(f \otimes y) := [p \mapsto f(p)y].$$

Then $\alpha_{P,Y}$ is an R-module homomorphism which is natural in P and in Y.

Proof. We first see that $[p \mapsto f(p)y] \in \mathrm{Hom}_\Lambda(P, Y)$ for all $f \in \mathrm{Hom}_\Lambda(P, \Lambda)$ and $y \in Y$. This follows from the underlying structure on $\mathrm{Hom}_\Lambda(P, \Lambda)$ and Y. That is,

$$f(p + \lambda p')y = f(p)y + \lambda f(p')y$$

for any p, $p' \in P$ and $\lambda \in \Lambda$.

Let $r \in R$, $f \in \mathrm{Hom}_\Lambda(P, \Lambda)$ and $y \in Y$. Then

$$\begin{aligned} r \cdot \alpha_{P,Y}(f \otimes y) &= r[p \mapsto f(p)y] \\ &= [p \mapsto r(f(p)y)] \\ &= [p \mapsto (rf(p))y] \\ &= [p \mapsto (rf)(p)y] \\ &= \alpha_{P,Y}(rf \otimes y) \\ &= \alpha_{P,Y}(r \cdot f \otimes y), \end{aligned}$$

hence α is a homomorphism of R-modules.

We now show that $\alpha_{P,Y}$ is natural in P. Let P, $P' \in \mathcal{P}(\Lambda)$, and let $h \in \mathrm{Hom}_\Lambda(P, P')$. We must show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(P', \Lambda) \otimes_\Lambda Y \arrow[r, "\alpha_{P',Y}"] \arrow[d, "(- \circ h)_\Lambda \otimes 1_Y"'] & \mathrm{Hom}_\Lambda(P', Y) \arrow[d, "(- \circ h)_Y"'] \\
\mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y \arrow[r, "\alpha_{P,Y}"] & \mathrm{Hom}_\Lambda(P, Y)
\end{tikzcd} \tag{3.3}$$

Suppose $f \otimes y \in \mathrm{Hom}_\Lambda(P', \Lambda) \otimes_\Lambda Y$. Then

$$(- \circ h)_Y \circ \alpha_{P',Y}(f \otimes y) = (- \circ h)_Y([p' \mapsto f(p')y]) = [p \mapsto f(h(p))y],$$

and

$$\alpha_{P,Y} \circ [(- \circ h)_\Lambda \otimes 1_Y](f \otimes y) = \alpha_{P,Y}(fh \otimes y) = [p \mapsto fh(p)y].$$

We have then seen that

$$(- \circ h)_Y \circ \alpha_{P',Y} = \alpha_{P,Y} \circ [(- \circ h)_\Lambda \otimes 1_Y],$$

that is, Diagram 3.3 is commutative.

To show that $\alpha_{P,Y}$ is natural in Y, we let Y, $Y' \in \mathrm{mod}(\Lambda)$ and $g \in \mathrm{Hom}_\Lambda(Y, Y')$. We need to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y \arrow[r, "\alpha_{P,Y}"] \arrow[d, "1_{\mathrm{Hom}_\Lambda(P,\Lambda)} \otimes g"'] & \mathrm{Hom}_\Lambda(P, Y) \arrow[d, "(g \circ -)_P"'] \\
\mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y' \arrow[r, "\alpha_{P,Y'}"] & \mathrm{Hom}_\Lambda(P, Y')
\end{tikzcd} \tag{3.4}$$

Let $f \otimes y \in \mathrm{Hom}_\Lambda(P, \Lambda) \otimes Y$. Then

$$(g \circ -)_P \circ \alpha_{P,Y}(f \otimes y) = (g \circ -)_P([p \mapsto f(p) \cdot y]) = [p \mapsto g(f(p) \cdot y)],$$

and

$$\alpha_{P,Y'} \circ [1_{\mathrm{Hom}_\Lambda(P,\Lambda)} \otimes g](f \otimes y) = \alpha_{P,Y'}(f \otimes g(y)) = [p \mapsto f(p) \cdot g(y)].$$

Since $f(p) \in \Lambda$ for any $p \in P$ and g is a Λ-module homomorphism, we see that

$$g(f(p) \cdot y) = f(p) \cdot g(y).$$

Then

$$(g \circ -)_P \circ \alpha_{P,Y} = \alpha_{P,Y'} \circ [1_{\mathrm{Hom}_\Lambda(P,\Lambda)} \otimes g],$$

hence Diagram 3.4 is commutative. $\square$

Note that naturality of $\alpha_{P,Y}$ in P really means that with $Y \in \mathrm{mod}(\Lambda)$ fixed then $\{\alpha_P\}_{P\in\mathcal{P}(\Lambda)}$ is a natural transformation of the contravariant functors $\mathrm{Hom}_\Lambda(-,\Lambda)\otimes_\Lambda Y$ and $\mathrm{Hom}_\Lambda(-,Y)$ from $\mathcal{P}(\Lambda)$ to $\mathrm{mod}(R)$. Analogously, naturality of $\alpha_{P,Y}$ in Y means that with $P \in \mathcal{P}(\Lambda)$ fixed then $\{\alpha_Y\}_{Y\in\mathrm{mod}(\Lambda)}$ is a natural transformation of the covariant functors $\mathrm{Hom}_\Lambda(P,\Lambda)\otimes_\Lambda -$ and $\mathrm{Hom}_\Lambda(P,-)$ from $\mathrm{mod}(\Lambda)$ to $\mathrm{mod}(R)$.

Our hope is that $\alpha_{P,Y}$ is an isomorphism of R-modules, because then its inverse will be the isomorphism that we seek. (We will see that this holds in the end of this section.)

As discussed above, we cannot see immediately what a such inverse would be. However, in the special case that $P = \Lambda$, we can take advantage of already having an element of $\mathrm{Hom}_\Lambda(\Lambda,\Lambda)$ readily at hand, namely 1_Λ. This allows us to construct a map from $\mathrm{Hom}_\Lambda(\Lambda,Y)$ to $\mathrm{Hom}_\Lambda(\Lambda,\Lambda)\otimes_\Lambda Y$, which conveniently turns out to be the inverse R-module homomorphism of $\alpha_{\Lambda,Y}$.

We will thus proceed as follows. First we construct the R-module homomorphism inverse

$$\varphi_{\Lambda,Y} : \mathrm{Hom}_\Lambda(\Lambda,Y) \to \mathrm{Hom}_\Lambda(\Lambda,\Lambda)\otimes_\Lambda Y$$

of $\alpha_{\Lambda,Y}$. We then use $\varphi_{\Lambda,Y}$ to construct an R-module homomorphism

$$\varphi_{\Lambda^n,Y} : \mathrm{Hom}_\Lambda(\Lambda^n,Y) \to \mathrm{Hom}_\Lambda(\Lambda^n,\Lambda)\otimes_\Lambda Y,$$

for any power Λ^n of Λ. Finally we take advantage of the fact that for some $n \in \mathbb{N}$ then $P \in \mathcal{P}(\Lambda)$ is isomorphic to a direct summand of Λ^n, and we construct $\varphi_{P,Y}$ with the use of $\varphi_{\Lambda^n,Y}$.

Lemma 71. *The map*

$$\varphi_{\Lambda,Y} : \mathrm{Hom}_\Lambda(\Lambda,Y) \to \mathrm{Hom}_\Lambda(\Lambda,\Lambda)\otimes_\Lambda Y$$

given by

$$\varphi_{\Lambda,Y}(g) := 1_\Lambda \otimes g(1_\Lambda)$$

is an R-module inverse of $\alpha_{\Lambda,Y}$.

Proof. It is obvious that $1_\Lambda \otimes g(1_\Lambda)$ is an element of $\mathrm{Hom}_\Lambda(\Lambda,\Lambda)\otimes_\Lambda Y$. We begin by showing that $\varphi_{\Lambda,Y}$ is a homomorphism of R-modules. Let $r \in R$ and $g \in \mathrm{Hom}_\Lambda(\Lambda,Y)$. Then

$$\begin{aligned}\varphi_{\Lambda,Y}(rg) &= 1_\Lambda \otimes (rg)(1_\Lambda)\\ &= 1_\Lambda \otimes (r1_\Lambda)(g(1_\Lambda))\\ &= (r1_\Lambda)\otimes(g(1_\Lambda))\\ &= r\cdot 1_\Lambda \otimes(g(1_\Lambda))\\ &= r\varphi_{\Lambda,Y}(g)\end{aligned}$$

We now show that

$$\varphi_{\Lambda,Y}\alpha_{\Lambda,Y} = 1_{\mathrm{Hom}_\Lambda(\Lambda,\Lambda)\otimes_\Lambda Y}$$

and

$$\alpha_{\Lambda,Y}\varphi_{\Lambda,Y} = 1_{\mathrm{Hom}_\Lambda(\Lambda,Y)}.$$

Let $f \in \mathrm{Hom}_\Lambda(\Lambda, \Lambda)$, $g \in \mathrm{Hom}_\Lambda(\Lambda, Y)$ and $y \in Y$. Then

$$\varphi_{\Lambda,Y}\alpha_{\Lambda,Y}(f \otimes y) = \varphi_{\Lambda,Y}([\lambda \mapsto f(\lambda)y]) = 1_\Lambda \otimes f(1_\Lambda)y = 1_\Lambda f(1_\Lambda) \otimes y = f \otimes y,$$

and

$$\alpha_{\Lambda,Y}\varphi_{\Lambda,Y}(g) = \alpha_{\Lambda,Y}(1_\Lambda \otimes g(1_\Lambda)) = [\lambda \mapsto 1_\Lambda(\lambda)g(1_\Lambda)] = [\lambda \mapsto \lambda g(1_\Lambda) = g(\lambda)] = g.$$

Then $\varphi_{\Lambda,Y}$ is shown to be the inverse of $\alpha_{P,Y}$. $\square$

We denote by $\varphi^n_{\Lambda,Y}$ the diagonal $n \times n$-matrix where all entries are equal to $\varphi_{\Lambda,Y}$, that is:

$$\begin{aligned} \varphi^n_{\Lambda,Y} : \mathrm{Hom}_\Lambda(\Lambda, Y)^n &\to (\mathrm{Hom}_\Lambda(\Lambda, \Lambda) \otimes_\Lambda Y)^n \\ \{f_i\}_{i=1}^n &\mapsto \{\varphi_{\Lambda,Y}(f_i)\}_{i=1}^n \end{aligned}$$

The following diagram now illustrates our three-step strategy for finding an isomorphism $\varphi_{P,Y}$ from $\mathrm{Hom}_\Lambda(P, Y)$ to $\mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y$.

$$\begin{tikzcd}
(\mathrm{Hom}_\Lambda(\Lambda, Y))^n \arrow[r, "\varphi^n_{\Lambda,Y}"] \arrow[dr, phantom, "\mathrm{I)}"] & (\mathrm{Hom}_\Lambda(\Lambda, \Lambda) \otimes_\Lambda Y)^n \arrow[d] \\
\mathrm{Hom}_\Lambda(\Lambda^n, Y) \arrow[u] \arrow[r, dashed, "\varphi_{\Lambda^n,Y}"] \arrow[dr, phantom, "\mathrm{II)}"] & \mathrm{Hom}_\Lambda(\Lambda^n, \Lambda) \otimes_\Lambda Y \arrow[d] \\
\mathrm{Hom}_\Lambda(P \oplus P', Y) \arrow[u] \arrow[r, dashed, "\varphi_{P\oplus P',Y}"] \arrow[dr, phantom, "\mathrm{III)}"] & \mathrm{Hom}_\Lambda(P \oplus P', \Lambda) \otimes_\Lambda Y \arrow[d] \\
\mathrm{Hom}_\Lambda(P, Y) \arrow[u] \arrow[r, "\varphi_{P,Y}"] & \mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y
\end{tikzcd}$$

The upper arrow of Diagram 3.1.1 represents our starting point $\varphi^n_{\Lambda,Y}$. The vertical arrows represent R-module homomorphisms which can be made available for us through the application of appropriate hom and tensor functors. We will proceed by constructing, by composition of already existing maps, the R-module homomorphisms represented by the dashed horizontal arrows of the diagram. We will start from the top and work downwards until we reach our target $\varphi_{P,Y}$, represented by the most visible arrow at the bottom of the diagram.

At this point all we can guarantee will be that $\varphi_{P,Y}$ is an R-module homomorphism. We will complete this section by demonstrating that $\varphi_{P,Y}$ is indeed the inverse of $\alpha_{P,Y}$. Hence we will have proven that $\varphi_{P,Y}$ is an isomorphism of R-modules from $\mathrm{Hom}_\Lambda(P,Y)$ to $\mathrm{Hom}_\Lambda(P,\Lambda)\otimes_\Lambda Y$, as desired.

Throughout the following process we will occasionally assume that a hom or tensor functor is a functor into $\mathrm{Mod}(R)$ without actually demonstrating that morphisms are taken to R-module homomorphisms by the functor. However, since we are in the framework of an R-algebra and R is a commutative ring, it seems plausible that this holds – and it does.

Before we can begin, we need to find a way to identify P with a direct summand of a power of Λ. We assume that Λ can be written as

$$\Lambda = \Lambda e_1 \oplus \Lambda e_2 \oplus ... \oplus \Lambda e_m$$

for some $m \in \mathbb{N}$, where the Λe_i's are all the indecomposable projective Λ-modules, and, for simplicity, that

$$\Lambda e_i \not\simeq \Lambda e_j$$

for $i,\ j \in \{1, ..., m\}$ such that $i \neq j$. Then any projective Λ-module P can be written as a direct sum of indecomposable projective Λ-modules, and by collecting equal terms, we get P expressed as a direct sum

$$P \simeq (\Lambda e_1)^{n_1} \oplus (\Lambda e_2)^{n_2} \oplus ... \oplus (\Lambda e_m)^{n_m}.$$

Let

$$n := \max_{1\leq i\leq m} n_i.$$

Then

$$P \oplus (\underbrace{(\Lambda e_1)^{n-n_1} \oplus (\Lambda e_2)^{n-n_2} \oplus ... \oplus (\Lambda e_m)^{n-n_m}}_{:=P'}) \simeq \Lambda^n.$$

We let ψ denote the above Λ-module isomorphism $P \oplus P' \to \Lambda^n$, and we let

$$\{\nu_i : \Lambda \to \Lambda^n\}_{i=1}^n$$

be the set of inclusions from Λ into Λ^n, and

$$\{\rho_i : \Lambda^n \to \Lambda\}_{i=1}^n$$

be the set of projections from Λ^n onto Λ. We can now perform the steps I) through III) illustrated by Diagram 3.1.1.

I) Let

$$\begin{aligned} \xi_1 : \mathrm{Hom}_\Lambda(\Lambda^n, Y) &\to (\mathrm{Hom}_\Lambda(\Lambda, Y)^n \\ f &\mapsto \{f\nu_i\}_{i=1}^n \end{aligned}$$

denote the Λ^{op}-module isomorphism (and thus R-module homomorphism) of Lemma 8, and let ξ_2 denote the R-module homomorphism given by

$$\begin{aligned}\xi_2 : (\mathrm{Hom}_\Lambda(\Lambda,\Lambda)\otimes_\Lambda Y)^n &\to \mathrm{Hom}_\Lambda(\Lambda^n,\Lambda)\otimes_\Lambda Y\\ \{g_i\otimes y_i\}_{i=1}^n &\mapsto \sum_{i=1}^n g_i\rho_i\otimes y_i.\end{aligned}$$

We know that since $\varphi_{\Lambda,Y}$ is a homomorphism of R-modules, then so is $\varphi_{\Lambda,Y}^n$.

$$\begin{tikzcd}
(\mathrm{Hom}_\Lambda(\Lambda,Y))^n \arrow[r, "\varphi^n_{\Lambda,Y}"] & (\mathrm{Hom}_\Lambda(\Lambda,\Lambda)\otimes_\Lambda Y)^n \arrow[d, "\xi_2"'] \\
\mathrm{Hom}_\Lambda(\Lambda^n,Y) \arrow[u, "\xi_1"] \arrow[r, dashed, "\varphi_{\Lambda^n,Y}"] & \mathrm{Hom}_\Lambda(\Lambda^n,\Lambda)\otimes_\Lambda Y
\end{tikzcd}$$

Then the composition

$$\varphi_{\Lambda^n,Y} := \xi_2\circ\varphi^n_{\Lambda,Y}\circ\xi_1$$

is a homomorphism of R-modules from $\mathrm{Hom}_\Lambda(\Lambda^n,Y)$ to $\mathrm{Hom}_\Lambda(\Lambda^n,\Lambda)\otimes_\Lambda Y$. For $h\in\mathrm{Hom}_\Lambda(\Lambda^n,Y)$, then

$$\begin{aligned}\varphi_{\Lambda^n,Y} &= \xi_2(\varphi^n_{\Lambda,Y}(\xi_1(h)))\\ &= \xi_2(\varphi^n_{\Lambda,Y}(\{h\nu_i\}_{i=1}^n))\\ &= \xi_2(\{\varphi_{\Lambda,Y}(h\nu_i)\}_{i=1}^n)\\ &= \xi_2(\{1_\Lambda\otimes h\nu_i(1_\Lambda)\}_{i=1}^n)\\ &= \sum_{i=1}^n \rho_i\otimes h\nu_i(1_\Lambda).\end{aligned}$$

II) We will now take advantage of the Λ-module isomorphism

$$\psi : P\oplus P'\to\Lambda^n.$$

Through the application of appropriate hom and tensor functors, ψ provides the R-module homomorphisms

$$(-\circ\psi^{-1})_Y : \mathrm{Hom}_\Lambda(P\oplus P',Y)\to\mathrm{Hom}_\Lambda(\Lambda^n,Y)$$

and

$$(-\circ\psi)_\Lambda\otimes 1_Y : \mathrm{Hom}_\Lambda(\Lambda^n,\Lambda)\otimes_\Lambda Y\to\mathrm{Hom}_\Lambda(P\oplus P',\Lambda)\otimes_\Lambda Y.$$

We let

$$\varphi_{P\oplus P',Y} := [(-\circ\psi)_\Lambda\otimes 1_Y]\circ\varphi_{\Lambda^n,Y}\circ[(-\circ\psi^{-1})_Y],$$

as illustrated in the following diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_\Lambda(\Lambda^n, Y) & \xrightarrow{\varphi_{\Lambda^n,Y}} & \mathrm{Hom}_\Lambda(\Lambda^n, \Lambda) \otimes_\Lambda Y \\
\big\uparrow{\scriptstyle (-\circ\psi^{-1})_Y} & & \big\downarrow{\scriptstyle (-\circ\psi)_\Lambda \otimes 1_Y} \\
\mathrm{Hom}_\Lambda(P \oplus P', Y) & \overset{\varphi_{P\oplus P',Y}}{\dashrightarrow} & \mathrm{Hom}_\Lambda(P \oplus P', \Lambda) \otimes_\Lambda Y
\end{array}$$

Then for $h \in \mathrm{Hom}_\Lambda(P \oplus P', Y)$, we get

$$\begin{aligned}
\varphi_{P\oplus P',Y}(h) &= [(-\circ\psi)_\Lambda \otimes 1_Y] \circ \varphi_{\Lambda^n,Y} \circ [(-\circ\psi^{-1})_Y](h) \\
&= [(-\circ\psi)_\Lambda \otimes 1_Y][\varphi_{\Lambda^n,Y}(h\psi^{-1})] \\
&= [(-\circ\psi)_\Lambda \otimes 1_Y]\left[\sum_{i=1}^n \rho_i \otimes h\psi^{-1}\nu_i(1_\Lambda)\right] \\
&= \sum_{i=1}^n \rho_i\psi \otimes h\psi^{-1}\nu_i(1_\Lambda).
\end{aligned}$$

III) The canonical projection

$$\pi : P \oplus P' \to P$$

of $P \oplus P'$ onto P and the inclusion

$$\mu : P \to P \oplus P'$$

of P into $P \oplus P'$ now provide R-module homomorphisms

$$(-\circ\pi)_Y : \mathrm{Hom}_\Lambda(P, Y) \to \mathrm{Hom}_\Lambda(P \oplus P', Y)$$

and

$$[(-\circ\mu)_\Lambda \otimes 1_Y] : \mathrm{Hom}_\Lambda(P \oplus P', \Lambda) \otimes_\Lambda Y \to \mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y.$$

We then get the following diagram, where we let $\varphi_{P,Y} : \mathrm{Hom}_\Lambda(P, Y) \to \mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y$ be given by

$$\varphi_{P,Y} := [(-\circ\mu)_\Lambda \otimes 1_Y] \circ \varphi_{P\oplus P',Y} \circ [(-\circ\pi)_Y].$$

$$\begin{array}{ccc}
\mathrm{Hom}_\Lambda(P \oplus P', Y) & \xrightarrow{\varphi_{P\oplus P',Y}} & \mathrm{Hom}_\Lambda(P \oplus P', \Lambda) \otimes_\Lambda Y \\
\big\uparrow{\scriptstyle (-\circ\pi)_Y} & & \big\downarrow{\scriptstyle (-\circ\mu)_\Lambda \otimes 1_Y} \\
\mathrm{Hom}_\Lambda(P, Y) & \overset{\varphi_{P,Y}}{\dashrightarrow} & \mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y
\end{array}$$

For $h \in \mathrm{Hom}_\Lambda(P, Y)$, then

$$\begin{aligned}
\varphi_{P,Y}(h) &= [(-\circ\mu)_\Lambda \otimes 1_Y] \circ \varphi_{P\oplus P',Y} \circ [(-\circ\pi)_Y](h) \\
&= [(-\circ\mu)_\Lambda \otimes 1_Y][\varphi_{P\oplus P',Y}(h\pi)] \\
&= [(-\circ\mu)_\Lambda \otimes 1_Y]\left[\sum_{i=1}^n \rho_i\psi \otimes h\pi\psi^{-1}\nu_i(1_\Lambda)\right] \\
&= \sum_{i=1}^n \rho_i\psi\mu \otimes h\pi\psi^{-1}\nu_i(1_\Lambda).
\end{aligned}$$

Proposition 72. *The map* $\varphi_{P,Y} : \mathrm{Hom}_\Lambda(P, Y) \to \mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y$ *given by*

$$\varphi_{P,Y}(h) := \sum_{i=1}^{n} \rho_i \psi\mu \otimes h\pi\psi^{-1}\nu_i(1_\Lambda)$$

for $h \in \mathrm{Hom}_\Lambda(P, Y)$*, is an isomorphism of* R*-modules which is natural in* P *and natural in* Y*.*

Proof. It is clear that $\varphi_{P,Y}$ is an R-module homomorphism, since it is constructed by composing R-module homomorphisms. We show that $\varphi_{P,Y}$ is the inverse of $\alpha_{P,Y}$ by composing the two maps.

Let $h \in \mathrm{Hom}_\Lambda(P, Y)$. Then

$$\begin{aligned}
\alpha_{P,Y}\varphi_{P,Y}(h) &= \alpha_{P,Y}\left(\sum_{i=1}^{n} \rho_i\psi\mu \otimes h\pi\psi^{-1}\nu_i(1_\Lambda)\right) \\
&= \left[p \mapsto \left(\sum_{i=1}^{n} \underbrace{\rho_i\psi\mu(p)}_{\in\Lambda} \cdot \underbrace{h\pi\psi^{-1}\nu_i}_{\Lambda\text{-hom.}}(1_\Lambda)\right)\right] \\
&= \left[p \mapsto \left(\sum_{i=1}^{n} h\pi\psi^{-1}\nu_i\rho_i\psi\mu(p)\right)\right] \\
&= \left[p \mapsto h\pi\psi^{-1} \underbrace{\left(\sum_{i=1}^{n} \nu_i\rho_i\right)}_{=1_{\Lambda^n}} \psi\mu(p)\right] \\
&= [p \mapsto h(p)] \\
&= h,
\end{aligned}$$

hence $\alpha_{P,Y}\varphi_{P,Y} = 1_{\mathrm{Hom}_\Lambda(P,Y)}$.

Let $f \otimes y \in \mathrm{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda Y$. Then

$$
\begin{aligned}
\varphi_{P,Y}\alpha_{P,Y}(f\otimes y) &= \varphi_{P,Y}([p\mapsto f(p)\cdot y])\\
&= \sum_{i=1}^{n} \rho_i\psi\mu \otimes [p\mapsto f(p)\cdot y]\left(\pi\psi^{-1}\nu_i(1_\Lambda)\right)\\
&= \sum_{i=1}^{n} \rho_i\psi\mu \otimes \underbrace{f\pi\psi^{-1}\nu_i(1_\Lambda)}_{\in\Lambda}\cdot y\\
&= \sum_{i=1}^{n} \rho_i\psi\mu \cdot f\pi\psi^{-1}\nu_i(1_\Lambda)\otimes y\\
&= \left[p\mapsto \sum_{i=1}^{n} \underbrace{\rho_i\psi\mu(p)}_{\in\Lambda}\cdot \underbrace{f\pi\psi^{-1}\nu_i}_{\Lambda\text{-hom.}}(1_\Lambda)\right]\otimes y\\
&= \left[p\mapsto \sum_{i=1}^{n} f\pi\psi^{-1}\nu_i(\rho_i\psi\mu(p))\right]\otimes y\\
&= f\pi\psi^{-1}\underbrace{\left(\sum_{i=1}^{n}\nu_i\rho_i\right)}_{=1_{\Lambda^n}}\psi\mu\otimes y\\
&= f\otimes y,
\end{aligned}
$$

so $\varphi_{P,Y}\alpha_{P,Y} = 1_{\mathrm{Hom}_\Lambda(P,\Lambda)\otimes_\Lambda Y}$.

The naturality of $\varphi_{P,Y}$ in P and Y now follows from the naturality of $\alpha_{P,Y}$ in P and Y and Lemma 2. $\square$

3.1.2 The $\sigma_{\delta,X}$-Algorithm

Recall the minimal projective presentation

$$
P_1 \xrightarrow{s} P_0 \xrightarrow{t} X \longrightarrow 0
$$

of X, and the short exact sequence δ given by

$$
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.
$$

In this section we will construct an $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-module isomorphism

$$
\sigma_{\delta,X} : \delta^*(X) \to \mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes f).
$$

For this we will use the R-module isomorphism $\varphi_{P,Y}$ developed in the previous section, with certain specific choices for P and Y. In particular, $\varphi_{P_1,A}$ will play an important role in the construction of $\sigma_{\delta,X}$.

We apply $()^*$ to the minimal projective presentation of X and get the following exact sequence in $\mathrm{mod}(\Lambda^{\mathrm{op}})$, where $\hat{t}$ denotes the canonical projection from

$\mathrm{Hom}_\Lambda(P_1, Y)$ onto $\mathrm{Tr}(X)$:

$$\mathrm{Hom}_\Lambda(P_0,\Lambda) \xrightarrow{(-\circ s)_\Lambda} \mathrm{Hom}_\Lambda(P_1,\Lambda) \xrightarrow{\hat{t}} \mathrm{Tr}(X) \longrightarrow 0 \tag{3.5}$$

Suppose $Y \in \mathrm{mod}(\Lambda)$. It can be shown that $\mathrm{Hom}_\Lambda(-, Y)$ is a functor

$$\mathrm{Hom}_\Lambda(-, Y) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(R).$$

We have seen that this functor takes objects in $\mathrm{mod}(\Lambda)$ to objects in $\mathrm{mod}(R)$ (Lemma 23)(ii)). We leave it to the reader to check that $\mathrm{Hom}_\Lambda(-, Y)$ takes Λ-module homomorphisms to R-module homomorphisms. By Lemma 57, then $-\otimes_\Lambda Y$ is a functor

$$-\otimes_\Lambda Y : \mathrm{Mod}(\Lambda^{\mathrm{op}}) \to \mathrm{Mod}(R).$$[1]

Then by applying $\mathrm{Hom}_\Lambda(-, Y)$ to the minimal projective presentation of X and $-\otimes_\Lambda Y$ to Sequence (3.5), we get the exact sequences in $\mathrm{mod}(R)$ displayed in the rows of the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_\Lambda(X,Y) \arrow[r, "(-\circ t)_Y"] & \mathrm{Hom}_\Lambda(P_0,Y) \arrow[r, "(-\circ s)_Y"] \arrow[d, "\varphi_{P_0,Y}"'] & \mathrm{Hom}_\Lambda(P_1,Y) \arrow[d, "\varphi_{P_1,Y}"'] & & \\
& & \mathrm{Hom}_\Lambda(P_0,\Lambda)\otimes_\Lambda Y \arrow[r, "(-\circ s)_\Lambda\otimes 1_Y"] & \mathrm{Hom}_\Lambda(P_1,\Lambda)\otimes_\Lambda Y \arrow[r, "\hat{t}\otimes 1_Y"] & \mathrm{Tr}(X)\otimes_\Lambda Y \arrow[r] & 0
\end{tikzcd} \tag{3.6}$$

The bottom row of Diagram 3.6 is in $\mathrm{mod}(R)$ (and not just in $\mathrm{Mod}(R)$) because of the R-module isomorphisms $\varphi_{P_0,Y}$ and $\varphi_{P_1,Y}$, as well as Lemma 20.

Definition 73. For $Y \in \mathrm{mod}(\Lambda)$, we define

$$\phi_Y : \mathrm{Hom}_\Lambda(P_1, Y) \to \mathrm{Tr}(X)\otimes_\Lambda Y$$

by

$$\phi_Y := [\hat{t}\otimes 1_Y]\circ \varphi_{P_1,Y}.$$

For $h \in \mathrm{Hom}_\Lambda(P_1, Y)$, we see that

$$\begin{aligned}
\phi_Y(h) &= [\hat{t}\otimes 1_Y](\varphi_{P_1,Y}(h)) \\
&= [\hat{t}\otimes 1_Y]\left(\sum_{i=1}^{n} \rho_i\psi\mu\otimes h\pi\psi^{-1}\nu_i(1_\Lambda)\right) \\
&= \left(\sum_{i=1}^{n} \hat{t}(\rho_i\psi\mu)\otimes h\pi\psi^{-1}\nu_i(1_\Lambda)\right).
\end{aligned}$$

[1]It can also be shown that $-\otimes_\Lambda Y$ is a functor $\mathrm{mod}(\Lambda^{\mathrm{op}} \to \mathrm{mod}(R)$, but we do not require this result for our further work.

It is evident that ϕ_Y is a homomorphism of R-modules. Moreover, ϕ_Y conveniently completes the top row of Diagram 3.6 to an exact sequence of R-modules, as advertised in the introduction of Section 3.1.

Lemma 74. *The R-module homomorphism ϕ_Y is natural in Y, and the following sequence is an exact sequence of R-modules:*

$$0 \to \mathrm{Hom}_\Lambda(X,Y) \xrightarrow{(-\circ t)_Y} \mathrm{Hom}_\Lambda(P_0,Y) \xrightarrow{(-\circ s)_Y} \mathrm{Hom}_\Lambda(P_1,Y) \xrightarrow{\phi_Y} \mathrm{Tr}(X)\otimes_\Lambda Y \to 0$$

Proof. Due to the naturality of $\varphi_{P,Y}$ in P, then Diagram 3.6 is commutative. Then the exactness of the above sequence follows directly from the definition of ϕ_Y and Lemma 10. We now show that ϕ_Y is natural in Y. It is easy to see that $\hat{t}\otimes 1_Y$ is natural in Y: Let Y, $Y' \in \mathrm{mod}(\Lambda)$, and let $h \in \mathrm{Hom}_\Lambda(Y,Y')$. Then

$$\begin{tikzcd}
\mathrm{Hom}_\Lambda(P_1,\Lambda)\otimes_\Lambda Y \arrow[r, "\hat{t}\otimes 1_Y"] \arrow[d, "1_{\mathrm{Hom}_\Lambda(P_1,\Lambda)}\otimes h"] & \mathrm{Tr}(X)\otimes_\Lambda Y \arrow[d, "1_{\mathrm{Tr}(X)}\otimes h"] \\
\mathrm{Hom}_\Lambda(P_1,\Lambda)\otimes_\Lambda Y' \arrow[r, "\hat{t}\otimes 1'_Y"] & \mathrm{Tr}(X)\otimes_\Lambda Y'
\end{tikzcd}$$

obviously commutes, since

$$[1_{\mathrm{Tr}(X)}\otimes h]\circ[\hat{t}\otimes 1_Y] = \hat{t}\otimes h = [\hat{t}\otimes 1'_Y]\circ[1_{\mathrm{Hom}_\Lambda(P_1,\Lambda)}\otimes h].$$

Recall that by Proposition 72, $\varphi_{P_1,Y}$ is also natural in Y. Then by Lemma 2(ii), the composition ϕ_Y is natural in Y. □

Starting with the exact sequence of Lemma 74, we may now construct a commutative diagram from which the desired map

$$\sigma_{\delta,X} : \delta^*(X) \to \mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes f)$$

can be obtained.

Proposition 75. *The following diagram is commutative, and the rows and columns are exact sequences of R-modules.*

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(X,A) \arrow[r, "(-\circ t)_A"] \arrow[d, "(f\circ -)_X"] & \mathrm{Hom}_\Lambda(P_0,A) \arrow[r, "(-\circ s)_A"] \arrow[d, "(f\circ -)_{P_0}"] & \mathrm{Hom}_\Lambda(P_1,A) \arrow[r, "\phi_A"] \arrow[d, "(f\circ -)_{P_1}"] & \mathrm{Tr}(X)\otimes_\Lambda A \arrow[r] \arrow[d, "1_{\mathrm{Tr}(X)}\otimes f"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(X,B) \arrow[r, "(-\circ t)_B"] \arrow[d, "(g\circ -)_X"] & \mathrm{Hom}_\Lambda(P_0,B) \arrow[r, "(-\circ s)_B"] \arrow[d, "(g\circ -)_{P_0}"] & \mathrm{Hom}_\Lambda(P_1,B) \arrow[r, "\phi_B"] \arrow[d, "(g\circ -)_{P_1}"] & \mathrm{Tr}(X)\otimes_\Lambda B \arrow[r] \arrow[d, "1_{\mathrm{Tr}(X)}\otimes g"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_\Lambda(X,C) \arrow[r, "(-\circ t)_C"] \arrow[d] & \mathrm{Hom}_\Lambda(P_0,C) \arrow[r, "(-\circ s)_C"] \arrow[d] & \mathrm{Hom}_\Lambda(P_1,C) \arrow[r, "\phi_C"] \arrow[d] & \mathrm{Tr}(X)\otimes_\Lambda C \arrow[r] \arrow[d] & 0 \\
 & \delta^*(X) \arrow[d] & 0 & 0 & 0 & \\
 & 0 & & & &
\end{tikzcd}
\tag{3.7}
$$

Proof. The exactness of the rows is stated by Lemma 74, as is the commutativity of the squares involving ϕ_A, ϕ_B and ϕ_C. The rest of the diagram commutes by associativity of composition of R-module homomorphisms (or just morphisms, in general, for a category). That is, if we compose a homomorphism with one homomorphism from the left and one from the right, then the order in which we compose does not affect the outcome. The exactness of the left column follows from the left exactness of $\mathrm{Hom}_\Lambda(X, -)$ and the definition of δ^*, while the middle columns are exact since P_0 and P_1 are projective Λ-modules. Since $\mathrm{Tr}(X) \otimes_\Lambda -$ is a right exact functor, then the right column is exact. $\square$

As mentioned above, we wish to construct a map $\sigma_{\delta,X}$ from $\delta^*(X)$ to $\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$. We will take advantage of the commutative diagram of Proposition 75. Consider the following algorithm.

The $\sigma_{\delta,X}$-Algorithm

Input: $\bar{h} \in \delta^*(X)$.

Output: $\sigma_{\delta,X}(\bar{h})$.

- Choose any preimage $h \in \mathrm{Hom}_\Lambda(X, C)$ for $\bar{h}$.
- Choose any $u \in \mathrm{Hom}_\Lambda(P_0, B)$ such that $gu = ht$.
- Find $v \in \mathrm{Hom}_\Lambda(P_1, A)$ such that $fv = us$.
- Let $\sigma_{\delta,X}(\bar{h}) := \phi_A(v)$.

end

The trail of the element $\bar{h}$ through Diagram 3.7 is illustrated below.

$$\begin{tikzcd}
& & & \phi_A(v) \\
& & v \arrow[r, "\phi_A"] & \phi_A(v) \arrow[u] \\
& u \arrow[r, "(-\circ s)_{P_0}"] & us = fv \arrow[u] & \\
h \arrow[r, "(-\circ t)_X"] & ht = gu \arrow[u] & & \\
\bar{h} \arrow[u] & & &
\end{tikzcd}$$

Proposition 76. *The $\sigma_{\delta,X}$-Algorithm presented above forms an $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-module isomorphism*

$$\sigma_{\delta,X} : \delta^*(X) \to \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$$

which is natural in δ and in X.

Proof. We first need to prove that $\sigma_{\delta,X}$ forms a well-defined map, and that its image is contained in $\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$. This requires the verification of the following five statements.

I) The choice of preimage $h \in \mathrm{Hom}_\Lambda(X, C)$ of $\bar{h}$ does not affect the outcome of the algorithm.

II) For $ht \in \mathrm{Hom}_\Lambda(P_0, C)$, there exists $u \in \mathrm{Hom}_\Lambda(P_0, B)$ such that $ht = gu$.

III) The choice of $u \in \mathrm{Hom}_\Lambda(P_0, B)$ such that $ht = gu$ does not affect the outcome of the algorithm.

IV) For $us \in \mathrm{Hom}_\Lambda(P_1, B)$, there exists unique $v \in \mathrm{Hom}_\Lambda(P_1, A)$ such that $us = fv$.

V) The outcome $\phi_A(v)$ of the algorithm is an element of $\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$.

If I) through V) are satisfied, then $\sigma_{\delta,X}$ is an $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-module homomorphism if

$$\sigma_{\delta,X}(\bar{h}\bar{e}) = \sigma_{\delta,X}(\bar{h})\bar{e} \tag{3.8}$$

for all $\bar{e} \in \underline{\mathrm{End}}_\Lambda(X)$.

We first show that I) through V) are satisfied. We find it suitable to swap the order in which the statements are proved.

II) Regard $ht \in \mathrm{Hom}_\Lambda(P_0, C)$. Since $(g \circ -)_{P_0}$ is onto $\mathrm{Hom}_\Lambda(P_0, C)$, then ht is of the form $(g \circ -)_{P_0}(u) = gu$ for some $u \in \mathrm{Hom}_\Lambda(P_0, B)$.

IV) Regard $us \in \mathrm{Hom}_\Lambda(P_1, B)$, and recall that $ht = gu$. We observe that

$$g(us) = (gu)s = ts = 0,$$

hence $us \in \mathrm{Ker}((g \circ -)_{P_1}) = \mathrm{Im}((f \circ -)_{P_1})$. That is,

$$us = (f \circ -)_{P_1}(v) = fv$$

for some $v \in \mathrm{Hom}_\Lambda(P_1, A)$. Moreover, since f is a monomorphism, such v is unique.

V) Regard $v \in \mathrm{Hom}_\Lambda(P_1, A)$, and recall that $fv = us$ for $u \in \mathrm{Hom}_\Lambda(P_0, B)$. Then

$$\begin{aligned}(1_{\mathrm{Tr}(X)} \otimes f)(\phi_A(v)) &= \phi_B \circ (f \circ -)_{P_1}(v) = \phi_B(fv) \underbrace{=}_{fv=us} \phi_B(us) \\ &= \underbrace{\phi_B \circ (- \circ s)_B}_{=0}(u),\end{aligned}$$

hence $\phi_A(v) \in \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$.

III) Suppose $gu = gu'$, and let v and v' be the elements of $\mathrm{Hom}_\Lambda(P_1, A)$ such that $us = fv$ and $u's = fv'$. We claim that $\phi_A(v) = \phi_A(v')$. We first observe that

$$u - u' \in \mathrm{Ker}((g \circ -)_{P_0}) = \mathrm{Im}((f \circ -)_{P_0},$$

so there is $w \in \mathrm{Hom}_\Lambda(P_0, A)$ such that $u - u' = fw$. Then

$$fws = (u - u')s = f(v - v'),$$

so $v - v' = ws$ since f is a monomorphism. That is,

$$v - v' \in \mathrm{Im}((- \circ s)_A) = \mathrm{Ker}(\phi_A),$$

hence $\phi_A(v - v') = 0$ which is what we wanted to show.

I) Suppose $\bar{h} = \bar{h}'$. Then $h - h' \in \mathrm{Im}((g \circ -)_X)$. That is, $h - h' = gr$ for some $r \in \mathrm{Hom}_\Lambda(X, B)$, and $(h - h')t = (gr)t$. We saw in III) that the choice of $u \in \mathrm{Hom}_\Lambda(P_0, B)$ such that $gu = (h - h')t$ does not affect the outcome, so we now choose $u = rt$. Hence $h - h'$ is mapped to

$$(- \circ s)_B(rt) = rts = 0$$

in $\mathrm{Hom}_\Lambda(P_1, B)$, since $ts = 0$. It is clear that if we continue through the diagram, then $h - h'$ will also be mapped to $0 \in \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$. That is, the choice of preimage of $\bar{h}$ in $\mathrm{Hom}_\Lambda(X, C)$ does not affect the final result.

We now show that $\sigma_{\delta,X}$ is a homomorphism of $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-modules. To check that (3.8) holds, we must first understand the $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-module structure on the domain and codomain of $\sigma_{\delta,X}$.

We leave it up to the reader to check that

$$\begin{aligned} \delta^*(X) \times \underline{\mathrm{End}}_\Lambda(X) &\to \delta^*(X) \\ (\bar{h}, \bar{e}) &\mapsto \overline{h \circ e}, \end{aligned}$$

where $h \in \mathrm{Hom}_\Lambda(X, C)$ and $e \in \mathrm{End}_\Lambda(X)$ are any representatives for $\bar{h}$ and $\bar{e}$, respectively, defines an $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-module structure on $\delta^*(X)$.

The $\underline{\mathrm{End}}_\Lambda(X)$-module structure on $\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ is provided by the Tr-functor. For $\bar{e} \in \underline{\mathrm{End}}_\Lambda(X)$ we let $\mathrm{Tr}(e)$ be a chosen representative for $\mathrm{Tr}(\bar{e})$. Then $\mathrm{Tr}(e) \otimes 1_Y$ is an R-module homomorphism

$$\mathrm{Tr}(X) \otimes_\Lambda Y \to \mathrm{Tr}(X) \otimes_\Lambda Y$$

for any $Y \in \mathrm{mod}(\Lambda)$. Consider the following commutative diagram in $\mathrm{mod}(R)$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \arrow[r, "\iota"] \arrow[d, dashed, "(\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}}"] & \mathrm{Tr}(X) \otimes_\Lambda A \arrow[r, "1_{\mathrm{Tr}(X)} \otimes f"] \arrow[d, "\mathrm{Tr}(e) \otimes 1_A"] & \mathrm{Tr}(X) \otimes_\Lambda B \arrow[r, "1_{\mathrm{Tr}(X)} \otimes g"] \arrow[d, "\mathrm{Tr}(e) \otimes 1_B"] & \mathrm{Tr}(X) \otimes_\Lambda C \arrow[r] \arrow[d, "\mathrm{Tr}(e) \otimes 1_C"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \arrow[r, "\iota"] & \mathrm{Tr}(X) \otimes_\Lambda A \arrow[r, "1_{\mathrm{Tr}(X)} \otimes f"] & \mathrm{Tr}(X) \otimes_\Lambda B \arrow[r, "1_{\mathrm{Tr}(X)} \otimes g"] & \mathrm{Tr}(X) \otimes_\Lambda C \arrow[r] & 0
\end{tikzcd}$$

(3.9)

There is a kernel R-module homomorphism

$$(\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}} : \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \to \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f),$$

satisfying

$$(\mathrm{Tr}(e) \otimes 1_A) \circ \iota = \iota \circ (\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}}.$$

Since ι is an inclusion it does not actually alter the elements that is is applied to. That is, for $q \otimes a \in \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ then $\iota(q \otimes a) = q \otimes a$, and it follows that for any $q \otimes a \in \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ then

$$\begin{aligned}(\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}}(q \otimes a) &= \iota((\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}}(q \otimes a)) \\ &= (\mathrm{Tr}(e) \otimes 1_A) \underbrace{(\iota(q \otimes a))}_{\substack{=q \otimes a \\ \in \mathrm{Tr}(X) \otimes_\Lambda A}} \\ &= (\mathrm{Tr}(e) \otimes 1_A)(\underbrace{q \otimes a}_{\in \mathrm{Tr}(X) \otimes_\Lambda A}).\end{aligned}$$

We claim that the multiplication

$$\begin{aligned}&\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \times \underline{\mathrm{End}}_\Lambda(X) \to \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \\ &\qquad (q \otimes a, \bar{e}) \mapsto (\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}}(\underbrace{q \otimes a}_{\in \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)}) = (\mathrm{Tr}(e) \otimes 1_A)(\underbrace{q \otimes a}_{\in \mathrm{Tr}(X) \otimes_\Lambda A})\end{aligned} \tag{3.10}$$

defines an $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-module structure on $\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$.

For the well-definedness of the above action, we suppose $e \in \mathcal{P}_\Lambda(X, X)$. Then $\mathrm{Tr}(e) \in \mathcal{P}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X), \mathrm{Tr}(X))$. We must show that

$$(q \otimes a)\bar{e} = (\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}} = 0 \tag{3.11}$$

for all $q \otimes a \in \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$. Since $\mathcal{P}_1^*$ is onto $\mathrm{Tr}(X)$ then by Lemma 45, $\mathrm{Tr}(e)$ factors through P_1^*: There is $u \in \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X), P_1^*)$ and $v \in \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P_1^*, \mathrm{Tr}(X))$ such that the following diagram in $\mathrm{mod}(\Lambda^{\mathrm{op}})$ is commutative:

$$\begin{tikzcd} & \mathrm{Tr}(X) \arrow[dl, "u"'] \arrow[dd, "\mathrm{Tr}(e)"] \\ P_1^* \arrow[dr, "v"'] & \\ & \mathrm{Tr}(X) \end{tikzcd}$$

Then for $Y \in \mathrm{mod}(\Lambda)$, the following is a commutative diagram in $\mathrm{mod}(R)$:

$$
\begin{tikzcd}
 & \mathrm{Tr}(X) \otimes_\Lambda Y \arrow[dl, "u \otimes 1_Y"'] \arrow[dd, "\mathrm{Tr}(e) \otimes 1_Y"] \\
P_1^* \otimes_\Lambda Y \arrow[dr, "v \otimes 1_Y"'] & \\
 & \mathrm{Tr}(X) \otimes_\Lambda Y
\end{tikzcd}
$$

Note that $P_1^* \otimes_\Lambda -$ is a covariant, exact functor

$$\mathrm{mod}(\Lambda) \to \mathrm{mod}(R),$$

since $P_1^* \in \mathrm{mod}(\Lambda^{\mathrm{op}})$ is projective. Then by filling in $P_1^* \otimes_\Lambda -$ applied to δ in the middle row of Diagram 3.9, we get the following commutative diagram in $\mathrm{mod}(R)$.

$$
\begin{tikzcd}
0 \arrow[rr] & & \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \arrow[rr, "\iota"] \arrow[dl, dashed, "(u \otimes 1_A)_{\mathrm{Ker}}"'] \arrow[dd] & & \mathrm{Tr}(X) \otimes_\Lambda A \arrow[rr] \arrow[dl, "u \otimes 1_A"'] \arrow[dd] & & \mathrm{Tr}(X) \otimes_\Lambda B \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Tr}(X) \otimes_\Lambda C \arrow[r] \arrow[dl] \arrow[dd] & 0 \\
 & 0 \arrow[rr] \arrow[dr, dashed, "(v \otimes 1_A)_{\mathrm{Ker}}"'] & & P_1^* \otimes_\Lambda A \arrow[rr] \arrow[dr, "v \otimes 1_A"'] & & P_1^* \otimes_\Lambda B \arrow[rr] \arrow[dr] & & P_1^* \otimes_\Lambda C \arrow[rr] \arrow[dr] & & 0 \\
0 \arrow[rr] & & \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \arrow[rr, "\iota"] & & \mathrm{Tr}(X) \otimes_\Lambda A \arrow[rr] & & \mathrm{Tr}(X) \otimes_\Lambda B \arrow[rr] & & \mathrm{Tr}(X) \otimes_\Lambda C \arrow[r] & 0
\end{tikzcd}
$$

We see that

$$\iota(\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}} = (\mathrm{Tr}(e) \otimes 1_A)\iota = (v \otimes 1_A) \underbrace{(u \otimes 1_A)\iota}_{=0(u \otimes 1_A)_{\mathrm{Ker}}} = 0.$$

Since ι is a monomorphism it follows that

$$(\mathrm{Tr}(e) \otimes 1_A)_{\mathrm{Ker}} = 0, \tag{3.12}$$

and (3.11) is satisfied.

We now demonstrate that (3.8) holds for all $\bar{h} \in \delta^*(X)$ and $\bar{e} \in \underline{\mathrm{End}}_\Lambda(X)$. Let $h \in \mathrm{Hom}_\Lambda(X, C)$ and $e \in \mathrm{End}_\Lambda(X)$ be representatives for $\bar{h}$ and $\bar{e}$. By Lemma 43 there is $e_0 \in \mathrm{End}_\Lambda(P_0)$ and $e_1 \in \mathrm{End}_\Lambda(P_1)$ such that the following diagram in

$\mathrm{mod}(\Lambda)$ is commutative.

$$
\begin{tikzcd}
 & P_1 \arrow[r, "s"] \arrow[d, "e_1"] & P_0 \arrow[r, "t"] \arrow[d, "e_0"] & X \arrow[r] \arrow[d, "e"] & 0 \\
 & P_1 \arrow[r, "s"] \arrow[d, dashed, "v_h"] & P_0 \arrow[r, "t"] \arrow[d, dashed, "u_h"] & X \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}
\tag{3.13}
$$

The process of the $\sigma_{\delta,X}$-Algorithm when applied to $\bar{h}$ involves finding $u_h \in \mathrm{Hom}_\Lambda(P_0, B)$ and $v_h \in \mathrm{Hom}_\Lambda(P_1, A)$ such that Diagram 3.13 commutes. If we let

$$u_{he} := u_h e_0$$

and

$$v_{he} := v_h e_1,$$

then by the commutativity of Diagram 3.13 it follows that u_{h_e} and v_{he} satisfies the requirements for the second and third step of the $\sigma_{\delta,X}$-Algorithm when applied to $\overline{he} = \bar{h}\bar{e}$. By the above discussion, it follows that

$$\sigma_{\delta,X}(\bar{h})\bar{e} = \phi_A(v_h)\bar{e}$$

and

$$\sigma_{\delta,X}(\bar{h}\bar{e}) = \phi_A(v_h e_1).$$

We claim that the following diagram in $\mathrm{mod}(R)$ is commutative:

$$
\begin{tikzcd}
\mathrm{Hom}_\Lambda(P_1, A) \arrow[r, "\varphi_{P_1,A}"] \arrow[d, "(-\circ e_1)_A"] \arrow[rr, bend left, "\phi_A"] & P_1^* \otimes_\Lambda A \arrow[r, "\hat{t}\otimes 1_A"] \arrow[d, "e_1^*\otimes 1_A"] & \mathrm{Tr}(X)\otimes_\Lambda A \arrow[r] \arrow[d, "\mathrm{Tr}(e)\otimes 1_A"] & 0 \\
\mathrm{Hom}_\Lambda(P_1, A) \arrow[r, "\varphi_{P_1,A}"] \arrow[rr, bend right, "\phi_A"'] & P_1^* \otimes_\Lambda A \arrow[r, "\hat{t}\otimes 1_A"] & \mathrm{Tr}(X)\otimes_\Lambda A \arrow[r] & 0
\end{tikzcd}
\tag{3.14}
$$

The left part of Diagram 3.14 commutes by the naturality of $\varphi_{P_1,Y}$ in P_1, as stated by Proposition 72. The commutativity of the right side of Diagram 3.14 follows directly from the construction of $\mathrm{Tr}(e)$. It follows that

$$\phi_A(v_h)\bar{e} \overset{(3.10)}{=} (\mathrm{Tr}(e)\otimes 1_A)(\phi_A(v_h)) = \phi_A(v_h e_1),$$

and $\sigma_{\delta,X}$ is shown to be an $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-module homomorphism.

To show that $\sigma_{\delta,X}$ is an isomorphism, we will apply [4, Snake Lemma 1.3.2, Ch. 1]. In order to associate our situation with that of the Snake Lemma, we find it convenient to display Diagram 3.7 in a different fashion than before. We leave out the insignificant components as well as the names of most of the maps.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathrm{Hom}_\Lambda(X,A) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(X,B) \arrow[r, "(g\circ -)_X"] \arrow[d] & \mathrm{Hom}_\Lambda(X,C) \arrow[r] \arrow[d] & \delta^*(X) \\
 & \mathrm{Hom}_\Lambda(P_0,A) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P_0,B) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P_0,C) \arrow[d] & \\
 & \mathrm{Hom}_\Lambda(P_1,A) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P_1,B) \arrow[r] \arrow[d] & \mathrm{Hom}_\Lambda(P_1,C) \arrow[d] & \\
\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes f) \arrow[r] & \mathrm{Tr}(X)\otimes_\Lambda A \arrow[r, "1_{\mathrm{Tr}(X)}\otimes f"] \arrow[d] & \mathrm{Tr}(X)\otimes_\Lambda B \arrow[r] \arrow[d] & \mathrm{Tr}(X)\otimes_\Lambda C \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{3.15}$$

Let

$$\sigma' : \mathrm{Hom}_\Lambda(X,C) \to \mathrm{Tr}(X)\otimes_\Lambda A$$

be the connecting map of the Snake Lemma. Then

$$\mathrm{Ker}(\sigma') = \mathrm{Im}((g\circ -)_X)$$

and

$$\mathrm{Im}(\sigma') = \mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes f),$$

and we get an induced isomorphism

$$\mathrm{Hom}_\Lambda(X,C)/\,\mathrm{Ker}(\sigma') \to \mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes f)$$

which is precisely the map $\sigma_{\delta,X}$, that we constructed.

□

We have given σ the subscripts δ and X. This notation is slightly inaccurate; the construction will also depend on the minimal projective presentation of X. We had, however, fixed a minimal projective presentation of X before developing the $\sigma_{\delta,X}$-algorithm. Whenever $\sigma_{\delta,X}$ is applied we implicitly assume that a minimal projective presentation of X is fixed beforehand. The same goes for the maps $\gamma_{\delta,X}$ and $\omega_{\delta,X}$ which will be constructed in the next section.

3.1.3 Achieving our first goal; finding $\omega_{\delta,X}$

Recall from Proposition 29 that the dual $D = \operatorname{Hom}_R(-, I)$ is a contravariant, exact functor from $\operatorname{mod}(R)$ to $\operatorname{mod}(R)$. In the end of this section we will finally find the isomorphism

$$\omega_{\delta,X} : D\delta^*(X) \to \delta_*(D\operatorname{Tr}(X))$$

that we are seeking. We now let δ be a fixed sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\operatorname{mod}(\Lambda)$. Will use the adjoint isomorphism θ from Theorem 58 to construct an isomorphism

$$\gamma_{\delta,X} : D\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f) \to \delta_*(D\operatorname{Tr}(X)),$$

to which we will assign additional structure to that following from θ. After this task is completed, the isomorphism $\omega_{\delta,X}$ that we wish to construct may be found immediately.

The above mentioned structure which will be assigned to $\gamma_{\delta,X}$, is that of being an $\underline{\operatorname{End}}_\Lambda(X)$-module homomorphism. (This will in turn assure, together with Proposition 76, that $\omega_{\delta,X}$ is an $\underline{\operatorname{End}}_\Lambda(X)$-module isomorphism.) For this, it will be necessary that we are familiar with in which way $D\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f)$ and $\delta_*(D\operatorname{Tr}(X))$ are endoved with $\underline{\operatorname{End}}_\Lambda(X)$-module structure. For the latter case, we need to introduce the notions of a pushout.

Definition 77. Consider the following diagram in $\operatorname{mod}(\Lambda)$:

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "v"] & N \\ L & \end{tikzcd} \tag{3.16}$$

A *pushout* of this diagram is an element $Y \in \operatorname{mod}(\Lambda)$ together with $\hat{u} \in \operatorname{Hom}_\Lambda(N, Y)$ and $\hat{v} \in \operatorname{Hom}_\Lambda(L, Y)$ such that

$$\hat{u}u = \hat{v}v,$$

and universal with this property. That is, if $Y' \in \operatorname{mod}(\Lambda)$, $u' \in \operatorname{Hom}_\Lambda(N, Y')$ and $v' \in \operatorname{Hom}_\Lambda(L, Y')$ such that

$$u'u = v'v,$$

then there exists a unique $y \in \mathrm{Hom}_\Lambda(Y, Y')$ such that

$$y\hat{u} = u'$$

and

$$y\hat{v} = v' :$$

$$\begin{tikzcd}
M \arrow[r, "u"] \arrow[d, "v"] & N \arrow[d, "\hat{u}"] \arrow[ddr, "u'"] & \\
L \arrow[r, "\hat{v}"] \arrow[drr, "v'"'] & Y \arrow[dr, dashed, "y"] & \\
 & & Y'
\end{tikzcd}$$

In this case we may also refer to Y as the *pushout of u and v*, and we call the resulting square a *pushout square*.

We will need the following facts for pushout squares:

Lemma 78.

(i) *For any M, N $L \in \mathrm{mod}(\Lambda)$, $u \in \mathrm{Hom}_\Lambda(M, N)$ and $v \in \mathrm{Hom}_\Lambda(M, L)$, then there exists a pushout of Diagram 3.16.*

(ii) *Suppose*

$$\begin{tikzcd}
M \arrow[r, "u"] \arrow[d, "v"] & N \arrow[d, "\hat{u}"] \\
L \arrow[r, "\hat{v}"] & Y
\end{tikzcd}$$

is a pushout square in $\mathrm{mod}(\Lambda)$. *Then*

(a) *$\hat{u}_{\mathrm{Cok}}$ is an isomorphism.*

(b) *If u is a monomorphism then $\hat{v}$ is a monomorphism.*

Proof. Consider [2, Proposition 5.6, Ch. 2].

(i) Take the pushout of Diagram 3.16 to be the cokernel of the Λ-module homomorphism

$$\begin{bmatrix} u \\ -v \end{bmatrix} : M \to N \oplus L.$$

(ii)

(a) Follows directly.

(b) It also follows that

$$M \xrightarrow{\begin{bmatrix} u \\ -v \end{bmatrix}} N \oplus L \xrightarrow{\begin{bmatrix} \hat{u} & \hat{v} \end{bmatrix}} Y \longrightarrow 0$$

is an exact sequence in $\operatorname{mod}(\Lambda)$. If u is a monomorphism, then so is

$$\begin{bmatrix} u \\ -v \end{bmatrix} : M \to N \oplus L,$$

which means that

$$0 \longrightarrow M \xrightarrow{\begin{bmatrix} u \\ -v \end{bmatrix}} N \oplus L \xrightarrow{\begin{bmatrix} \hat{u} & \hat{v} \end{bmatrix}} Y \longrightarrow 0$$

is an exact sequence in $\operatorname{mod}(\Lambda)$. By [2, Corollary 5.7, Ch. 2], it follows that $\hat{v}$ is a monomorphism.

□

We are now ready to define and analyze the isomorphism $\gamma_{\delta,X}$.

Proposition 79. *There is an isomorphism of* $\underline{\operatorname{End}}_\Lambda(X)$*-modules*

$$\gamma_{\delta,X} : D\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f) \to \delta_*(D\operatorname{Tr}(X)),$$

which is natural in δ *and natural in* X.

Proof. For $Y \in \operatorname{mod}(\Lambda)$, then

$$D(\operatorname{Tr}(X) \otimes_\Lambda Y) = \operatorname{Hom}_R(\operatorname{Tr}(X) \otimes_\Lambda Y, I)$$

and

$$\operatorname{Hom}_\Lambda(Y, D\operatorname{Tr}(X)) = \operatorname{Hom}_\Lambda(Y, \operatorname{Hom}_R(\operatorname{Tr}(X), I)).$$

Recall that $\operatorname{Tr}(X)$ is an R-Λ-bimodule and $I \in \operatorname{mod}(R)$. By Theorem 58, there is an abelian group isomorphism

$$\theta_{\operatorname{Tr}(X),Y,I} : D(\operatorname{Tr}(X) \otimes_\Lambda Y) \to \operatorname{Hom}_\Lambda(Y, D\operatorname{Tr}(X)) \tag{3.17}$$

which is natural in $\operatorname{Tr}(X)$ and in Y.

We now apply D to the right column of Diagram 3.7 and $\operatorname{Hom}_\Lambda(-, D\operatorname{Tr}(X))$ to δ. Then (3.17) inserted A, B and C for Y respectively, yields the following commutative diagram in Ab:

$$\begin{tikzcd}
0 \arrow[r] & D(\operatorname{Tr}(X) \otimes_\Lambda C) \arrow[r] \arrow[d, "\theta_{\operatorname{Tr}(X),C,I}", "\simeq"'] & D(\operatorname{Tr}(X) \otimes_\Lambda B) \arrow[r] \arrow[d, "\theta_{\operatorname{Tr}(X),B,I}", "\simeq"'] & D(\operatorname{Tr}(X) \otimes_\Lambda A) \arrow[r] \arrow[d, "\theta_{\operatorname{Tr}(X),A,I}", "\simeq"'] & D(\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f)) \arrow[r] \arrow[d, dashed, "\gamma_{\delta,X}", "\simeq"'] & 0 \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(C, D\operatorname{Tr}(X)) \arrow[r] & \operatorname{Hom}_\Lambda(B, D\operatorname{Tr}(X)) \arrow[r] & \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X)) \arrow[r] & \delta_*(D\operatorname{Tr}(X)) \arrow[r] & 0
\end{tikzcd} \tag{3.18}$$

Since $\theta_{\mathrm{Tr}(X),A,I}$, $\theta_{\mathrm{Tr}(X),B,I}$ and $\theta_{\mathrm{Tr}(X),C,I}$ are isomorphisms of abelian groups, then by the Five Lemma, there is an abelian group isomorphism

$$\gamma_{\delta,X} : D(\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)) \to \delta_*(D\,\mathrm{Tr}(X)) \tag{3.19}$$

such that Diagram 3.18 commutes. We need to show that $\gamma_{\delta,X}$ is also a homomorphism of $\underline{\mathrm{End}}_\Lambda(X)$-modules.

The first question that arises, is in which way $D(\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f))$ and $\delta_*(D\,\mathrm{Tr}(X))$ are $\underline{\mathrm{End}}_\Lambda(X)$-modules. Suppose $\bar{h} \in \underline{\mathrm{End}}_\Lambda(X)$. By applying the transpose functor

$$\mathrm{Tr} : \underline{\mathrm{mod}}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda^{\mathrm{op}})$$

from Section 2.6 we get $\mathrm{Tr}(\bar{h}) \in \underline{\mathrm{End}}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X))$, and in accordance with Definition 48 we let $\mathrm{Tr}(h) \in \mathrm{End}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}(X))$ denote a chosen representative for $\mathrm{Tr}(\bar{h})$. We shall see that $\mathrm{Tr}(h)$ will be involved in the action $\bar{h}$ has on elements in both $D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ and in $\delta_*(D\,\mathrm{Tr}(X))$.

We first explore the $\underline{\mathrm{End}}_\Lambda(X)$-module structure on $D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$. Recall the R-module homomorphism

$$(\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}} : \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \to \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$$

obtained from Diagram 3.9 (when replacing e by h) that we saw in the proof of Proposition 76.

The duality D now induces the R-module homomorphism

$$D((\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}}) = (-\circ(\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}})_I : D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \to D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f).$$

We claim that the multiplication

$$\underline{\mathrm{End}}_\Lambda(X) \times D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \to D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$$

defined by

$$\bar{h}\bar{z} := (- \circ (\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}})_I(\bar{z}) = \bar{z} \circ (\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}}$$

for $\bar{z} \in D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ assigns an $\underline{\mathrm{End}}_\Lambda(X)$-module structure to $D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$.

To show that this action is well-defined, we need to show that

$$\bar{h}\bar{z} = \bar{z} \circ (\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}} = 0 \tag{3.20}$$

for all $h \in \mathcal{P}_\Lambda(X, X)$ and $\bar{z} \in D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$. We saw in the proof of Proposition 76 (by (3.12)) that

$$(\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}} = 0$$

for all $h \in \mathcal{P}_\Lambda(X, X)$, so (3.20) obviously holds.

We now investigate the action of $\underline{\mathrm{End}}_\Lambda(X)$ on $D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ in more detail. Since I is injective, then for any $\bar{z} \in D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ there exists $z \in D(\mathrm{Tr}(X) \otimes_\Lambda A)$ such that

$$\bar{z} = z\iota. \tag{3.21}$$

Consider the following diagram in $\mathrm{mod}(R)$:

$$\begin{tikzcd}
\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \arrow[r, "\iota"] \arrow[d, "(\mathrm{Tr}(h)\otimes 1_A)_{\mathrm{Ker}}"] & \mathrm{Tr}(X) \otimes_\Lambda A \arrow[d, "\mathrm{Tr}(h)\otimes 1_A"] \\
\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \arrow[r, "\iota"] \arrow[d, "\bar{z}"] & \mathrm{Tr}(X) \otimes_\Lambda A \arrow[dl, dashed, "z"] \\
I &
\end{tikzcd}$$

We see that

$$\bar{h}\bar{z} = \bar{z} \circ (\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}} = z \circ (\mathrm{Tr}(h) \otimes 1_A) \circ \iota \tag{3.22}$$

for any z satisfying (3.21).

For $q \otimes a \in \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$, then

$$\begin{aligned}
(\bar{h}\bar{z})(q \otimes a) &= \bar{z}((\mathrm{Tr}(h) \otimes 1_A)_{\mathrm{Ker}}(q \otimes a)) \\
&= z(\mathrm{Tr}(h) \otimes 1_A)\iota(q \otimes a) \\
&= z(\mathrm{Tr}(h)(q) \otimes a).
\end{aligned}$$

We now check that this multiplication satisfies the associativity requirement: Let $\bar{z} \in D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ and $\bar{h}_1, \bar{h}_2 \in \underline{\mathrm{End}}_\Lambda(X)$, and suppose $z \in D(\mathrm{Tr}(X) \otimes_\Lambda A)$ satisfies (3.21) and $\mathrm{Tr}(h_1)$ and $\mathrm{Tr}(h_2)$ are representatives for $\mathrm{Tr}(\bar{h}_1)$ and $\mathrm{Tr}(\bar{h}_2)$, respectively. Then

$$\begin{aligned}
\bar{h}_1(\bar{h}_2\bar{z}) &\overset{(3.22)}{=} \bar{h}_1 \underbrace{(z \circ (\mathrm{Tr}(h_2) \otimes 1_A) \circ \iota)}_{\in D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)} \\
&\overset{(3.22)}{=} z \circ (\mathrm{Tr}(h_2) \otimes 1_A) \circ (\mathrm{Tr}(h_1) \otimes 1_A) \circ \iota \\
&= z(\mathrm{Tr}(h_2)\,\mathrm{Tr}(h_1) \otimes 1_A) \circ \iota \\
&= z(\mathrm{Tr}(h_1 h_2) \otimes 1_A) \circ \iota \\
&\overset{(3.22)}{=} (\bar{h}_1\bar{h}_2)\bar{z}.
\end{aligned}$$

We leave it up to the reader to verify that the rest of the module axioms are satisfied.

We also wish to regard $\delta_*(D\,\mathrm{Tr}(X))$ as an $\underline{\mathrm{End}}_\Lambda(X)$-module. Consider the set $\Upsilon_{D\,\mathrm{Tr}(X),C}/\sim$ of equivalence classes of short exact sequences from $D\,\mathrm{Tr}(X)$ to C. It can be shown that $\Upsilon_{D\,\mathrm{Tr}(X),C}/\sim$ is an abelian group ([3, Theorem 7.21, Ch. 7]), and we will now show how we can regard $\Upsilon_{D\,\mathrm{Tr}(X),C}/\sim$ as an $\underline{\mathrm{End}}_\Lambda(X)$-module. In the following, we will sometimes refer to the elements of $\Upsilon_{D\,\mathrm{Tr}(X),C}/\sim$ as short exact sequences, although implicitly we mean that we are regarding the equivalence class of the given sequence.

Let

$$0 \longrightarrow D\operatorname{Tr}(X) \longrightarrow E \longrightarrow C \longrightarrow 0 \tag{3.23}$$

represent an element of $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$, and let $\bar{h} \in \underline{\operatorname{End}}_\Lambda(X)$. Again we let $\operatorname{Tr}(h) \in \operatorname{End}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X))$ denote a chosen representative for $\operatorname{Tr}(\bar{h}) \in \underline{\operatorname{End}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr}(X))$. By applying the contravariant functor D to $\operatorname{Tr}(h)$, we get the Λ-module homomorphism

$$D\operatorname{Tr}(h) : D\operatorname{Tr}(X) \to D\operatorname{Tr}(X).$$

Consider the following diagram in $\operatorname{mod}(\Lambda)$:

$$\begin{tikzcd} D\operatorname{Tr}(X) \arrow[r] \arrow[d, "D\operatorname{Tr}(h)"] & E \\ D\operatorname{Tr}(X) & \end{tikzcd} \tag{3.24}$$

Let E' be the pushout of Diagram 3.24. Then in light of Lemma 78, we see that we get the following commutative diagram in $\operatorname{mod}(\Lambda)$:

$$\begin{tikzcd} 0 \arrow[r] & D\operatorname{Tr}(X) \arrow[r] \arrow[d, "D\operatorname{Tr}(h)"] & E \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D\operatorname{Tr}(X) \arrow[r] & E' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

We now define the multiplication

$$\underline{\operatorname{End}}_\Lambda(X) \times (\Upsilon_{D\operatorname{Tr}(X),C}/\sim) \to (\Upsilon_{D\operatorname{Tr}(X),C}/\sim)$$

by

$$\bar{h} \cdot (0 \to D\operatorname{Tr}(X) \to E \to C \to 0) := (0 \to D\operatorname{Tr}(X) \to E' \to C \to 0). \tag{3.25}$$

Note that the sequences of (3.25) are elements of $\Upsilon_{D\operatorname{Tr}(X),C}$ and not of $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$, but when writing these sequences we are implicitly referring to their corresponding equivalence classes in $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$. The reason why we are assigning the $\underline{\operatorname{End}}_\Lambda(X)$-module structure to $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$ instead of $\Upsilon_{D\operatorname{Tr}(X),C}$, is for the multiplication of (3.25) to be well-defined.

We now show that the associativity of multiplication with $\underline{\operatorname{End}}_\Lambda(X)$ holds. Let $\bar{h}_1, \bar{h}_2 \in \underline{\operatorname{End}}_\Lambda(X)$. Then

$$\bar{h}_1(\bar{h}_2(0 \to D\operatorname{Tr}(X) \to E \to C \to 0))$$

yields the bottom row of Diagram 3.26.

$$
\begin{tikzcd}
0 \arrow[r] & D\operatorname{Tr}(X) \arrow[r] \arrow[d, "D\operatorname{Tr}(h_2)"] & E \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D\operatorname{Tr}(X) \arrow[r] \arrow[d, "D\operatorname{Tr}(h_1)"] & E' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D\operatorname{Tr}(X) \arrow[r] & E'' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
\tag{3.26}
$$

We want to show that the bottom row of Diagram 3.26 is also what we obtain from

$$(\bar{h}_1\bar{h}_2)(0 \to D\operatorname{Tr}(X) \to E \to C \to 0) = (\overline{h_1h_2})(0 \to D\operatorname{Tr}(X) \to E \to C \to 0).$$

Then we must show that E'' is the pushout of Diagram 3.27:

$$
\begin{tikzcd}
D\operatorname{Tr}(X) \arrow[r] \arrow[d, "D\operatorname{Tr}(h_1h_2)"] & E \\
D\operatorname{Tr}(X)
\end{tikzcd}
\tag{3.27}
$$

Since D and Tr are both contravariant functors, then

$$D\operatorname{Tr}(h_1h_2) = D\operatorname{Tr}(h_1)D\operatorname{Tr}(h_2).$$

It is easy to see that E'', together with the same morphism in $\operatorname{Hom}_\Lambda(D\operatorname{Tr}(X), E'')$ and the composition of the pushout morphisms in $\operatorname{Hom}_\Lambda(E, E')$ and $\operatorname{Hom}_\Lambda(E', E'')$ from before, satisfies the first property of a pushout: That of the resulting square from Diagram 3.27 being commutative. For the universal property, we will take advantage of the fact that E' and E'' are the pushouts of

$$
\begin{tikzcd}
D\operatorname{Tr}(X) \arrow[r] \arrow[d, "D\operatorname{Tr}(h_2)"] & E \\
D\operatorname{Tr}(X)
\end{tikzcd}
\tag{3.28}
$$

and

$$
\begin{tikzcd}
D\operatorname{Tr}(X) \arrow[r] \arrow[d, "D\operatorname{Tr}(h_1)"] & E' \\
D\operatorname{Tr}(X)
\end{tikzcd}
\tag{3.29}
$$

respectively.

Then given some $E''' \in \operatorname{mod}(\Lambda)$ together with morphisms in $\operatorname{Hom}_\Lambda(E', E''')$ and $\operatorname{Hom}_\Lambda(D\operatorname{Tr}(X), E''')$ which bring about a commutative square starting from Diagram 3.27, a unique morphism in $\operatorname{Hom}_\Lambda(E'', E''')$ (with suitable properties regarding commutativity), is obtained in the following manner: Since $D\operatorname{Tr}(h_1h_2) = D\operatorname{Tr}(h_1)D\operatorname{Tr}(h_2)$ and since E' is the pushout of Diagram 3.28, there is a unique morphism in $\operatorname{Hom}_\Lambda(E', E''')$ (with suitable properties), and then since E'' is the pushout of 3.29, we get the desired morphism in $\operatorname{Hom}_\Lambda(E'', E''')$. It is easy to see that the required commutativity of the appropriate triangles is satisfied. Thus E'' is a pushout of Diagram 3.27. We will not demonstrate that the rest of the module axioms are satisfied, as it would require a thorough survey of the abelian group structure on $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$.

Recall from Proposition 69(i) that

$$\operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X)) \simeq \Upsilon_{D\operatorname{Tr}(X),C}/\sim \tag{3.30}$$

as abelian groups. Thus by regarding the elements of $\operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X))$ as equivalence classes in $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$ and by the above discussion, we get an $\underline{\operatorname{End}}_\Lambda(X)$-module structure on $\operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X))$. Moreover, we know that $\delta_*(D\operatorname{Tr}(X)) \subseteq \operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X))$ as R-modules. We claim that $\delta_*(X)$ is also an $\underline{\operatorname{End}}_\Lambda(X)$-submodule of $\operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X))$. What we will do now, is explain how an element of $\delta_*(D\operatorname{Tr}(X))$ can be identified with an element of $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$, and then show that if we multiply the resulting element of $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$ with $\bar{h} \in \underline{\operatorname{End}}_\Lambda(X)$, we get an element in $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$ which also originates from an element of $\delta_*(D\operatorname{Tr}(X))$. It would be necessary to demonstrate that the correspondence

$$\delta_*(X) \leftrightarrow (\Upsilon_{D\operatorname{Tr}(X),C}/\sim)$$

which we are about to describe is the same as including $\delta_*(D\operatorname{Tr}(X))$ into $\operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X))$ and applying the isomorphism of (3.30) for a rigorous demonstration. We will not do this here, but rather confine ourselves to a superficial motivation.

Given an element $\bar{y} \in \delta_*(D\operatorname{Tr}(X)) \subseteq \operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(X))$, then $\bar{y}$ corresponds to an equivalence class in $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$ in the following manner. Since

$$\delta_*(D\operatorname{Tr}(X)) = \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X))/\operatorname{Im}((-\circ f)_I),$$

we can choose $y \in \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X))$ such that

$$\bar{y} = y + \operatorname{Im}((-\circ f)_I).$$

Then in light of Lemma 78, we see that the pushout E of f and y yields the following commutative diagram in $\operatorname{mod}(\Lambda)$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "y"] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D\operatorname{Tr}(X) \arrow[r] & E \arrow[r] & C \arrow[r] & 0
\end{tikzcd} \tag{3.31}$$

The bottom row of this diagram is the short exact sequence which we shall identify with $\bar{y} \in \delta_*(D\operatorname{Tr}(X))$, and it can be shown that it is unique up to the equivalence $\sim$ ([3, Ch. 7]).

We now see that the resulting sequence

$$0 \longrightarrow D\operatorname{Tr}(X) \longrightarrow E' \longrightarrow C \longrightarrow 0 \tag{3.32}$$

from multiplying $\bar{h}$ with the bottom row of Diagram 3.31 "is contained in $\delta_*(D\operatorname{Tr}(X))$" in the sense that the composition $(D\operatorname{Tr}(h))y \in \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X))$ is a representative for an element

$$\overline{(D\operatorname{Tr}(h))y} \in \delta_*(D\operatorname{Tr}(X)),$$

whence (3.32) is obtained. This argues that $\delta_*(D\operatorname{Tr}(X))$ is an $\underline{\operatorname{End}}_\Lambda(X)$-submodule of $\operatorname{End}_\Lambda(C, D\operatorname{Tr}(X)) \simeq (\Upsilon_{D\operatorname{Tr}(X),C}/\sim)$, which we from now on will assume holds.

To show that $\gamma_{\delta,X}$ is an $\underline{\operatorname{End}}_\Lambda(X)$-module homomorphism, we need to look closer at how $\gamma_{\delta,X}$ behaves applied to an element $\bar{z} \in D\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f)$. Note that

$$\delta_*(D\operatorname{Tr}(X)) = \operatorname{Hom}_\Lambda(A, \operatorname{Hom}_R(\operatorname{Tr}(X), I)) / \operatorname{Im}((- \circ f)_{D\operatorname{Tr}(X)}),$$

and that $(-\circ\iota)_I$ is the canonical projection from $D(\operatorname{Tr}(X) \otimes_\Lambda A)$ onto $D\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f)$. Recall the following commutative diagram in Ab:

$$\begin{tikzcd}
D(\operatorname{Tr}(X) \otimes_\Lambda A) \arrow[r, "(- \circ \iota)_I"] \arrow[d, "\theta_{\operatorname{Tr}(X),A,I}"] & D\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f) \arrow[d, "\gamma_{\delta,X}"] \\
\operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X)) \arrow[r] & \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X)) / \operatorname{Im}((- \circ f)_{D\operatorname{Tr}(X)})
\end{tikzcd}$$

Suppose $\bar{z} \in D\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f)$. Then there is some $z \in D(\operatorname{Tr}(X) \otimes_\Lambda A)$ such that

$$\bar{z} = (- \circ \iota)_I(z) = z\iota,$$

that is, satisfying (3.21).

Since $\gamma_{\delta,X}$ is the cokernel map of $\theta_{\operatorname{Tr}(X),A,I}$, then

$$\begin{aligned}
\gamma_{\delta,X}(\bar{z}) &= \theta_{\operatorname{Tr}(X),A,I}(z) + \operatorname{Im}((- \circ f)_{D\operatorname{Tr}(X)}) \\
&= [a \mapsto z(- \otimes a)] + \operatorname{Im}((- \circ f)_{D\operatorname{Tr}(X)})
\end{aligned} \tag{3.33}$$

for any z satisfying (3.21).

Suppose $\bar{h} \in \underline{\operatorname{End}}_\Lambda(X)$. Then

$$\begin{aligned}
\gamma_{\delta,X}(\bar{h}\bar{z}) &\overset{(3.22)}{=} \gamma_{\delta,X}(z \circ (\operatorname{Tr}(h) \otimes 1_A) \circ \iota) \\
&\overset{(3.33)}{=} [a \mapsto z(\operatorname{Tr}(h)(-) \otimes a)] + \operatorname{Im}((- \circ f)_{D\operatorname{Tr}(X)}).
\end{aligned} \tag{3.34}$$

We must now show that the same element of $\Upsilon_{D\operatorname{Tr}(X),C}/\sim$ is obtained from

I) identifying $[a \mapsto z(\mathrm{Tr}(h)(-) \otimes a)] + \mathrm{Im}((- \circ f)_{D\,\mathrm{Tr}(X)})$ with an equivalence class in $\Upsilon_{D\,\mathrm{Tr}(X),C}/\sim$,

II) identifying $[a \mapsto z(- \otimes a)] + \mathrm{Im}((- \circ f)_{D\,\mathrm{Tr}(X)})$ with an equivalence class in $\Upsilon_{D\,\mathrm{Tr}(X),C}/\sim$, and then multiplying with $\bar{h}$ as in (3.25).

Note that I) yields the short exact sequence obtained from taking the pushout of f and $[a \mapsto z(\mathrm{Tr}(h)(-) \otimes a)]$, while II) yields the short exact sequence obtained from taking the pushout of f and the composition

$$D\,\mathrm{Tr}(h) \circ [a \mapsto z(- \otimes a)] = [a \mapsto D\,\mathrm{Tr}(h)(z(- \otimes a))].$$

Since

$$\begin{aligned} D\,\mathrm{Tr}(h)(z(- \otimes a)) &= (- \circ \mathrm{Tr}(h))_I(z(- \otimes a)) \\ &= z(- \otimes a) \circ \mathrm{Tr}(h) \\ &= z(\mathrm{Tr}(h)(-) \otimes a), \end{aligned}$$

we see that

$$D\,\mathrm{Tr}(h) \circ [a \mapsto z(- \otimes a)] = [a \mapsto z(\mathrm{Tr}(h)(-) \otimes a)].$$

We have then demonstrated that

$$\gamma_{\delta,X}(\bar{h}\bar{z}) = \bar{h}\gamma_{\delta,X}(\bar{z})$$

for all $\bar{z} \in D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$ and $\bar{h} \in \underline{\mathrm{End}}_\Lambda(X)$. That is, $\gamma_{\delta,X}$ is an isomorphism of $\underline{\mathrm{End}}_\Lambda(X)$-modules.

The naturality of $\gamma_{\delta,X}$ in δ and X is yet to be proven. We first show that $\gamma_{\delta,X}$ is natural in δ. Let δ and δ' be two exact sequences in $\mathrm{mod}(\Lambda)$ as displayed in Diagram 3.35, and let $(u, v, w) : \delta \to \delta'$ be a set of Λ-module homomorphisms such that the diagram commutes.

$$\begin{tikzcd} \delta : 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r] \arrow[d, "w"] & 0 \\ \delta' : 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd} \tag{3.35}$$

What we need to show is that Diagram 3.36 is commutative. The left vertical arrow represents the cokernel morphism $D(\mathrm{Tr}(X) \otimes_\Lambda -)(u)_{\mathrm{Cok}}$ induced from (u, v, w) when applying $D(\mathrm{Tr}(X) \otimes_\Lambda -)$ to δ and δ', and the right vertical arrow represents the cokernel morphism $\mathrm{Hom}_\Lambda(-, D\,\mathrm{Tr}(X))(u)_{\mathrm{Cok}}$ induced from (u, v, w) when applying $\mathrm{Hom}_\Lambda(-, D\,\mathrm{Tr}(X))$ to δ and δ'.

$$\begin{tikzcd} D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f') \arrow[r, "\gamma_{\delta',X}"] \arrow[d, "D(\mathrm{Tr}(X) \otimes_\Lambda -)(u)_{\mathrm{Cok}}"'] & \delta'_*(D\,\mathrm{Tr}(X)) \arrow[d, "\mathrm{Hom}_\Lambda(-, D\,\mathrm{Tr}(X))"'] \\ D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \arrow[r, "\gamma_{\delta,X}"] & \delta_*(D\,\mathrm{Tr}(X)) \end{tikzcd} \tag{3.36}$$

We will proceed by drawing two copies of Diagram 3.18 behind each other, the front one for δ and the back one for δ'. Consider Diagram 3.37. Here Diagram 3.36 is the one drawn with thicker edges. To simplify and save space in Diagram 3.37 we denote the contravariant functors $\mathrm{Hom}_\Lambda(-, D\,\mathrm{Tr}(X))$ and $D(\mathrm{Tr}(X) \otimes_\Lambda -)$ by F and G respectively, when applied to morphisms.

Let Diagram $\widehat{3.37}$ denote the part of Diagram 3.37 where the edges of Diagram 3.36 are left out. Since the morphisms of Diagram 3.36 are cokernel morphisms, they are constructed precisely such that the squares connecting their corresponding edges to Diagram $\widehat{3.37}$ commute. Thus, if Diagram $\widehat{3.37}$ is commutative, then so is Diagram 3.36.

$$
\begin{tikzcd}[column sep=tiny]
 & 0 \arrow[rr] & & D(\operatorname{Tr}(X)\otimes_\Lambda C') \arrow[rr, "G(g')"] \arrow[dl, "G(w)"'] \arrow[dd, "\theta_{\operatorname{Tr}(X),C',I}" near start] & & D(\operatorname{Tr}(X)\otimes_\Lambda B') \arrow[rr, "G(f')"] \arrow[dl, "G(v)"'] \arrow[dd, "\theta_{\operatorname{Tr}(X),B',I}" near start] & & D(\operatorname{Tr}(X)\otimes_\Lambda A') \arrow[rr] \arrow[dl, "G(u)"'] \arrow[dd, "\theta_{\operatorname{Tr}(X),A',I}" near start] & & D\operatorname{Ker}(1_{\operatorname{Tr}(X)}\otimes f') \arrow[rr] \arrow[dl, "G(u)_{\mathrm{Cok}}"'] \arrow[dd, "\gamma_{\delta',X}" near start] & & 0 \\
0 \arrow[rr] & & D(\operatorname{Tr}(X)\otimes_\Lambda C) \arrow[rr, "G(g)" near start] \arrow[dd, "\theta_{\operatorname{Tr}(X),C,I}"' near start] & & D(\operatorname{Tr}(X)\otimes_\Lambda B) \arrow[rr, "G(f)" near start] \arrow[dd, "\theta_{\operatorname{Tr}(X),B,I}"' near start] & & D(\operatorname{Tr}(X)\otimes_\Lambda A) \arrow[rr] \arrow[dd, "\theta_{\operatorname{Tr}(X),C,I}"' near start] & & D\operatorname{Ker}(1_{\operatorname{Tr}(X)}\otimes f) \arrow[rr] \arrow[dd, "\gamma_{\delta,X}"' near start] & & 0 & \\
 & 0 \arrow[rr] & & \operatorname{Hom}_\Lambda(C', D\operatorname{Tr}(X)) \arrow[rr, "F(g')" near start] \arrow[dl, "F(w)"'] & & \operatorname{Hom}_\Lambda(B', D\operatorname{Tr}(X)) \arrow[rr, "F(f')" near start] \arrow[dl, "F(v)"'] & & \operatorname{Hom}_\Lambda(A', D\operatorname{Tr}(X)) \arrow[rr] \arrow[dl, "F(u)"'] & & \delta'_*(D\operatorname{Tr}(X)) \arrow[rr] \arrow[dl, "F(u)_{\mathrm{Cok}}"'] & & 0 \\
0 \arrow[rr] & & \operatorname{Hom}_\Lambda(C, D\operatorname{Tr}(X)) \arrow[rr, "F(g)"] & & \operatorname{Hom}_\Lambda(B, D\operatorname{Tr}(X)) \arrow[rr, "F(f)"] & & \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X)) \arrow[rr] & & \delta_*(D\operatorname{Tr}(X)) \arrow[rr] & & 0 &
\end{tikzcd}
\tag{3.37}
$$

We now show that Diagram $\widehat{3.37}$ is commutative. Throughout this paragraph we assume that Diagram 3.37 is rotated back to normal position prior to investigation. There are four types of squares to consider: The "horizontal" ones, constituting the top and the bottom of the two cubes of the diagram, the "straight vertical" ones, constituting the front and the back side of the cubes, and the "skew vertical" ones, constituting the sides of the cubes. The commutativity of the skew vertical and the straight vertical squares follows immediately from the naturality of $\theta_{\mathrm{Tr}(X),Y,I}$ in Y. The commutativity of Diagram 3.35 implies that the horizontal squares commute, since contravariant functors switches the order in compositions of morphisms. Hence $\gamma_{\delta,X}$ is natural in δ.

We now show that $\gamma_{\delta,X}$ is natural in X. Let X, $X' \in \mathrm{mod}(\Lambda)$, and let $h \in \mathrm{Hom}_\Lambda(X, X')$. Then X and X' give raise to two different versions of Diagram 3.18. We display them both in Diagram 3.38, the one for X in front of the one for X'. We have not named all the edges in the diagram, but it should be clear from the context which morphism each one of them represents.

$$
\begin{tikzcd}[column sep=small]
& 0 \arrow[rr] & & D(\operatorname{Tr}(X') \otimes_\Lambda C) \arrow[rr] \arrow[dl] \arrow[dd, "\theta_{\operatorname{Tr}(X'),C,I}" near end] & & D(\operatorname{Tr}(X') \otimes_\Lambda B) \arrow[rr] \arrow[dl] \arrow[dd, "\theta_{\operatorname{Tr}(X'),B,I}" near end] & & D(\operatorname{Tr}(X') \otimes_\Lambda A) \arrow[rr] \arrow[dl] \arrow[dd, "\theta_{\operatorname{Tr}(X'),A,I}" near end] & & D\operatorname{Ker}(1_{\operatorname{Tr}(X')} \otimes f) \arrow[rr] \arrow[dl, thick] \arrow[dd, thick, "\gamma_{\delta,X'}" near end] & & 0 \\
0 \arrow[rr] & & D(\operatorname{Tr}(X) \otimes_\Lambda C) \arrow[rr] \arrow[dd, "\theta_{\operatorname{Tr}(X),C,I}" near start] & & D(\operatorname{Tr}(X) \otimes_\Lambda B) \arrow[rr] \arrow[dd, "\theta_{\operatorname{Tr}(X),B,I}" near start] & & D(\operatorname{Tr}(X) \otimes_\Lambda A) \arrow[rr] \arrow[dd, "\theta_{\operatorname{Tr}(X),A,I}" near start] & & D\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes f) \arrow[rr] \arrow[dd, thick, "\gamma_{\delta,X}" near start] & & 0 & \\
& 0 \arrow[rr] & & \operatorname{Hom}_\Lambda(C, D\operatorname{Tr}(X')) \arrow[rr] \arrow[dl] & & \operatorname{Hom}_\Lambda(B, D\operatorname{Tr}(X')) \arrow[rr] \arrow[dl] & & \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X')) \arrow[rr] \arrow[dl] & & \delta_*(D\operatorname{Tr}(X')) \arrow[rr] \arrow[dl, thick] & & 0 \\
0 \arrow[rr] & & \operatorname{Hom}_\Lambda(C, D\operatorname{Tr}(X)) \arrow[rr] & & \operatorname{Hom}_\Lambda(B, D\operatorname{Tr}(X)) \arrow[rr] & & \operatorname{Hom}_\Lambda(A, D\operatorname{Tr}(X)) \arrow[rr] & & \delta_*(D\operatorname{Tr}(X)) \arrow[rr] & & 0 &
\end{tikzcd}
\tag{3.38}
$$

As in the proof of the naturality of $\gamma_{\delta,X}$ in δ, we must now show that the diagram drawn with thicker edges in Diagram 3.38 is commutative. By the same arguments as above, it suffices to prove the commutativity of the part of Diagram 3.38 obtained from omitting the thicker edges. Again, we divide the squares to consider into the three types "horizontal", "straight vertical" and "skew vertical" (after rotating the diagram back to normal position). The skew vertical squares commute by the naturality of (3.17) in $\mathrm{Tr}(X)$, and the straight vertical squares commute by the naturality of (3.17) in Y. The commutativity of the bottom horizontal squares follows from the associativity of composition of morphisms.

We now show that the top horizontal squares are commutative. We will prove this for the left one; the procedure is similar for the right one. Consider the following diagram.

$$\begin{tikzcd}
\mathrm{Tr}(X)\otimes_\Lambda B \arrow[r, "1_{\mathrm{Tr}(X)}\otimes g"] \arrow[d, "\mathrm{Tr}(h)\otimes 1_B"] & \mathrm{Tr}(X)\otimes_\Lambda C \arrow[d, "\mathrm{Tr}(h)\otimes 1_C"] \\
\mathrm{Tr}(X')\otimes_\Lambda B \arrow[r, "1_{\mathrm{Tr}(X')}\otimes g"] & \mathrm{Tr}(X')\otimes_\Lambda C
\end{tikzcd}$$

It is easy to see that this diagram is commutative:

$$(\mathrm{Tr}(h)\otimes 1_C)\circ(1_{\mathrm{Tr}(X)}\otimes g) = \mathrm{Tr}(h)\otimes g = (1_{\mathrm{Tr}(X')}\otimes g)\circ(\mathrm{Tr}(h)\otimes 1_B).$$

When applying the contravariant functor D to the above diagram we obtain the top left horizontal square of Diagram 3.38, hence the latter must also be commutative.

By the above discussion the square with thicker edges of Diagram 3.38, is commutative; naturality of $\gamma_{\delta,X}$ in X has been proven. □

We have now done all the preliminary work which is necessary in order to derive the desired connection between the dual and the defect functors. Recall that by Proposition 76, there is an isomorphism of $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-modules

$$\sigma_{\delta,X}^{-1} : \mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes f) \to \delta^*(X)$$

which is natural in δ and in X. We hereby present one of the main results of this thesis.

Theorem 80. *Let*

$$\omega_{\delta,X} := \gamma_{\delta,X} D\sigma_{\delta,X}^{-1}.$$

Then $\omega_{\delta,X}$ is an isomorphism of $\underline{\mathrm{End}}_\Lambda(X)$-modules

$$\omega_{\delta,X} : D\delta^*(X) \to \delta_*(D\,\mathrm{Tr}(X)) \tag{3.39}$$

which is natural in δ and natural in X.

Proof. Since

$$\sigma_{\delta,X}^{-1} : \mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes f) \to \delta^*(X)$$

is an $\underline{\mathrm{End}}_\Lambda(X)^{\mathrm{op}}$-module isomorphism, then

$$D\sigma_{\delta,X}^{-1} : D\delta^*(X) \to D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f)$$

is an $\underline{\mathrm{End}}_\Lambda(X)$-module isomorphism. It is evident that the composition

$$D\delta^*(X) \xrightarrow{D\sigma_{\delta,X}^{-1}} D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes f) \xrightarrow{\gamma_{\delta,X}} \delta_*(D\,\mathrm{Tr}(X))$$

of two $\underline{\mathrm{End}}_\Lambda(X)$-module isomorphisms is also an isomorphism of $\underline{\mathrm{End}}_\Lambda(X)$-modules.

Since D is a functor, then naturality of $D\sigma_{\delta,X}^{-1}$ in δ and in X follows from naturality of $\sigma_{\delta,X}^{-1}$ in the same variables.

By Lemma 2(ii), a composition of natural transformations is also natural. Hence, by the above discussion and Proposition 79, the composition $\gamma_{\delta,X} D\sigma_{\delta,X}^{-1}$ is natural in δ and natural in X. □

3.2 The Almost Split Sequnece Algorithm

In this section we will show how we can employ the results of this thesis in order to compute almost split sequences for finitely generated indecomposable non-projective modules over a finite dimensional algebra over a field. As formerly announced, we will make use of the $\underline{\mathrm{End}}_\Lambda(X)$-module isomorphism $\omega_{\delta,X}$ developed in the previous section in the special case that δ is the exact sequence of Definition 37(i) given by

$$0 \longrightarrow \Omega \xrightarrow{\iota} P_0 \xrightarrow{t} X \longrightarrow 0, \tag{3.40}$$

where

$$\Omega := \Omega_\Lambda(X)$$

is a fixed kernel of P_0.

Definition 81. We let

$$\breve{\sigma}_X := \sigma_{\delta,X},$$
$$\breve{\gamma}_X := \gamma_{\delta,X}$$

and

$$\breve{\omega}_X := \omega_{\delta,X}$$

for this particular choice for δ.

Of course, whenever fixing a non-projective, indecomposable $X \in \mathrm{mod}(\Lambda)$, it will still be necessary to make a choice for P_0 and Ω in order to obtain δ. That is, δ is not uniquely determined by X. Nevertheless we find it appropriate to omit the subscrips δ from the above morphisms because P_0 and Ω are unique up to isomorphism in $\mathrm{mod}(\Lambda)$; altering these objects will not be affecting the actual structure of a morphism with δ-dependence.

3.2.1 Investigating $\breve{\omega}_X$

The aim of this brief section is to specify the domain and codomain of the $\underline{\mathrm{End}}_\Lambda(X)$-module isomorphism $\breve{\omega}_X$. For that we will need the result of the following lemma.

Lemma 82. *Let $Y \in \mathrm{mod}(\Lambda)$. Then*

$$\delta^*(Y) = \underline{\mathrm{Hom}}_\Lambda(Y, X).$$

Proof. Recall that for $Y \in \mathrm{mod}(\Lambda)$, then $\delta^*(Y)$ is defined by the exactness of the following sequence:

$$0 \longrightarrow \mathrm{Hom}_\Lambda(Y, \Omega) \xrightarrow{(\iota\,\circ\,-)_Y} \mathrm{Hom}_\Lambda(Y, P_0) \xrightarrow{(t\,\circ\,-)_Y} \mathrm{Hom}_\Lambda(Y, X) \longrightarrow \delta^*(Y) \longrightarrow 0$$

Thus $\delta^*(Y)$ is the cokernel of $(t \circ -)_Y$, that is,

$$\delta^*(Y) = \mathrm{Hom}_\Lambda(Y, X)/\,\mathrm{Im}((t \circ -)_Y). \tag{3.41}$$

We now show that

$$\mathrm{Im}(t \circ -)_Y = \mathcal{P}(Y, X).$$

If $u \in \mathrm{Im}(t \circ -)$, then u clearly factors through a projective Λ-module, namely P_0. Conversely, if u factors through some projective Λ-module P, then by Lemma 45, u will also factor through $P(X)$. Then from (3.41) we get

$$\delta^*(Y) = \mathrm{Hom}_\Lambda(Y, X)/\mathcal{P}(Y, X) = \underline{\mathrm{Hom}}_\Lambda(Y, X).$$

□

We now see that $\breve{\omega}_X$ indeed identifies the dual of $\underline{\mathrm{End}}_\Lambda(X)$ with the set of equivalence classes of short exact sequences ending in X (as $\underline{\mathrm{End}}_\Lambda(X)$-modules), as advertised in the introduction of this chapter.

Proposition 83. *The special case $\breve{\omega}_X$ of $\omega_{\delta,X}$ is an isomorphism of $\underline{\mathrm{End}}_\Lambda(X)$-modules*

$$\breve{\omega}_X : D\,\underline{\mathrm{End}}_\Lambda(X) \to \mathrm{Ext}^1_\Lambda(X, D\,\mathrm{Tr}(X)).$$

Proof. Recall from Theorem 80 that

$$\omega_{\delta,X} : D\delta^*(X) \to \delta_*(D\,\mathrm{Tr}(X)).$$

Lemma 82 implies that $\delta^*(X) = \underline{\mathrm{End}}_\Lambda(X)$, hence

$$D\delta^*(X) = D\,\underline{\mathrm{End}}_\Lambda(X). \tag{3.42}$$

Note that since $X \in \mathrm{mod}(\Lambda)$, then $\mathrm{Tr}(X) \in \mathrm{mod}(\Lambda^{\mathrm{op}})$ – hence $D\,\mathrm{Tr}(X) \in \mathrm{mod}(\Lambda)$. By applying the contravariant functor $\mathrm{Hom}_\Lambda(-, D\,\mathrm{Tr}(X))$ to δ and by [3, Theorem 7.3, Ch. 7], we get the following exact sequence:

$$0 \to \operatorname{Hom}_\Lambda(X, D\operatorname{Tr}(X)) \to \operatorname{Hom}_\Lambda(P_0, D\operatorname{Tr}(X)) \to \operatorname{Hom}_\Lambda(\Omega_\Lambda(X), D\operatorname{Tr}(X)) \cdots$$

$$\cdots \longrightarrow \operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X)) \longrightarrow \operatorname{Ext}^1_\Lambda(P_0, D\operatorname{Tr}(X)) \tag{3.43}$$

Since P_0 is projective, then $\operatorname{Ext}^1_\Lambda(P_0, D\operatorname{Tr}(X)) = 0$. We thus recognize Sequence (3.43) as Sequence (2.38) of Definition 49, where X is replaced by $D\operatorname{Tr}(X)$. That is,

$$\delta_*(D\operatorname{Tr}(X)) = \operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X)). \tag{3.44}$$

We obtain the desired result from (3.42) and (3.44). □

3.2.2 Designing the algorithm for $R = K$

Recall from Proposition 69 that identifying the elements of $\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$ with equivalence classes of short exact sequences ending in X gives rise to an $\underline{\operatorname{End}}_\Lambda(X)$-module structure on $\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$ such that this is a finitely generated $\underline{\operatorname{End}}_\Lambda(X)$-module. Furthermore, the the socle of $\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$ as $\underline{\operatorname{End}}_\Lambda(X)$-module is simple and corresponds to the set $\hat{\Upsilon}_{D\operatorname{Tr}(X),X}/\sim$ of equivalence classes of almost split sequences in $\operatorname{mod}(\Lambda)$ ending in X. This means that any nonzero element e of $\operatorname{Soc}_\Gamma(D\,\underline{\operatorname{End}}_\Lambda(X))$ can be used to generate $\hat{\Upsilon}/\sim$, since $\breve{\omega}_X(e) \in \operatorname{Soc}_\Gamma(\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$ will then be nonzero.

Our strategy is now to try to find any nonzero element of $D\,\underline{\operatorname{End}}_\Lambda(X)$, and then check that it is also contained in the socle. We do know of a nonzero element in $\underline{\operatorname{End}}_\Lambda(X)$, namely the equivalence class represented by the identity morphism from X to X; $\bar{1}_X$. How can we take advantage of this knowledge to obtain an element of $D\,\underline{\operatorname{End}}_\Lambda(X)$?

Our rescue will be the mapping of Definition 31 from Section 2.3. This can be applied once we make our final assumption; from now on we let R be a field. Then by Lemma 30, $R = K$ and the functor D is given by

$$D = \operatorname{Hom}_K(-, K) : \operatorname{mod}(K) \to \operatorname{mod}(K).$$

Lemma 84. *Consider the identity $\bar{1}_X \in \underline{\operatorname{End}}_\Lambda(X)$. Let*

$$\mathcal{B}_{\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)} := \{\breve{\sigma}_X(\bar{1}_X), w_2, ..., w_l\}$$

be a K-basis of $\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)$. Then

$$\breve{\gamma}_X(d_{\mathcal{B}_{\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)}}(\breve{\sigma}_X(\bar{1}_X))) \in \operatorname{Soc}_\Gamma(\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X)))$$

is a generator.

Proof. Let

$$\breve{\sigma}_X^{-1}(\mathcal{B}_{\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)}) := \{\bar{1}_X, \breve{\sigma}_X^{-1}(w_2), ..., \breve{\sigma}_X^{-1}(w_l)\}$$

be the K-basis of $\underline{\mathrm{End}}_\Lambda(X)$ corresponding to $\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)}$. We know that

$$\bar{1}_X \in \mathrm{Top}_{\Gamma^{\mathrm{op}}}(\underline{\mathrm{End}}_\Lambda(X))$$

is a nonzero element, hence by Lemma 67(ii) then

$$(d_{\breve{\sigma}_X^{-1}(\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)})})(\bar{1}_X) \in \mathrm{Soc}_\Gamma(D\,\underline{\mathrm{End}}_\Lambda(X))$$

is a nonzero element, thus since $\breve{\omega}_X$ is an isomorphsim of $\underline{\mathrm{End}}_\Lambda(X)$-modules,

$$\begin{aligned}\breve{\omega}_X((d_{\breve{\sigma}_X^{-1}(\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)})})(\bar{1}_X)) &= \breve{\gamma}_X(D\breve{\sigma}_X^{-1})((d_{\breve{\sigma}_X^{-1}(\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)})})(\bar{1}_X))\\ &\in \mathrm{Soc}_\Gamma(\mathrm{Ext}^1_\Lambda(X, D\,\mathrm{Tr}(X)))\end{aligned}$$

is a nonzero element. Furthermore, by Lemma 34 we know that

$$(D\breve{\sigma}_X^{-1})(d_{\breve{\sigma}_X^{-1}(\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)})}(\bar{1}_X)) = d_{\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)}}(\breve{\sigma}_X(\bar{1}_X)),$$

thus

$$\breve{\omega}_X((d_{\breve{\sigma}_X^{-1}(\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)})})(\bar{1}_X)) = \breve{\gamma}_X(d_{\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)}}(\breve{\sigma}_X(\bar{1}_X))).$$

By Lemma 68, this nonzero element is a generator for $\mathrm{Soc}_\Gamma(\mathrm{Ext}^1_\Lambda(X, D\,\mathrm{Tr}(X)))$. □

The previous lemma implies that what we are really interested in doing, is to algorithmically compute $\breve{\gamma}_X(d_{\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)}\otimes\iota)}})(\breve{\sigma}_X(\bar{1}_X))$. We observe that the $\sigma_{\delta,X}$-Algorithm of Section 3.1.2 can be drastically simplified in the special case that we are studying.

Lemma 85. *In accordance with the name change from $\sigma_{\delta,X}$ to $\breve{\sigma}_X$, we let the $\breve{\sigma}_X$-Algorithm be the resulting $\sigma_{\delta,X}$-Algorithm from letting δ be the exact sequence (3.40). Then with input $\bar{1}_X$, the $\breve{\sigma}_X$-Algorithm returns*

$$\breve{\sigma}_X(\bar{1}_X) = \phi_\Omega(w),$$

where w is the projective cover of Ω.

Proof. We begin by translating Diagram 3.7 to our current situation:

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota) \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(X,\Omega) \arrow[r, "(-\,\circ\, t)_\Omega"] \arrow[d, "(\iota\,\circ\,-)_X"] & \operatorname{Hom}_\Lambda(P_0,\Omega) \arrow[r, "(-\,\circ\, s)_\Omega"] \arrow[d, "(\iota\,\circ\,-)_{P_0}"] & \operatorname{Hom}_\Lambda(P_1,\Omega) \arrow[r, "\phi_\Omega"] \arrow[d, "(\iota\,\circ\,-)_{P_1}"] & \operatorname{Tr}(X) \otimes_\Lambda \Omega \arrow[r] \arrow[d, "1_{\operatorname{Tr}(X)} \otimes \iota"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(X,P_0) \arrow[r, "(-\,\circ\, t)_{P_0}"] \arrow[d, "(t\,\circ\,-)_X"] & \operatorname{End}_\Lambda(P_0) \arrow[r, "(-\,\circ\, s)_{P_0}"] \arrow[d, "(t\,\circ\,-)_{P_0}"] & \operatorname{Hom}_\Lambda(P_1,P_0) \arrow[r, "\phi_{P_0}"] \arrow[d, "(t\,\circ\,-)_{P_1}"] & \operatorname{Tr}(X) \otimes_\Lambda P_0 \arrow[r] \arrow[d, "1_{\operatorname{Tr}(X)} \otimes t"] & 0 \\
0 \arrow[r] & \operatorname{End}_\Lambda(X) \arrow[r, "(-\,\circ\, t)_X"] \arrow[d] & \operatorname{Hom}_\Lambda(P_0,X) \arrow[r, "(-\,\circ\, s)_X"] \arrow[d] & \operatorname{Hom}_\Lambda(P_1,X) \arrow[r, "\phi_X"] \arrow[d] & \operatorname{Tr}(X) \otimes_\Lambda X \arrow[r] \arrow[d] & 0 \\
 & \underline{\operatorname{End}}_\Lambda(X) \arrow[d] & 0 & 0 & 0 & \\
 & 0 & & & &
\end{tikzcd}
\tag{3.45}
$$

We now perform the $\breve{\sigma}_X$-Algorithm on the specific element $\bar{1}_X \in \underline{\operatorname{End}}_\Lambda(X)$. We begin by choosing the preimage $1_X \in \operatorname{End}_\Lambda(X)$. We then observe that an element $u \in \operatorname{End}_\Lambda(P_0)$ such that $tu = t$ is readily at hand; we simply choose 1_{P_0}. Next, wee must find $v \in \operatorname{Hom}_\Lambda(P_1, \Omega)$ such that $\iota v = s$. Recall from Definition 37(ii) that the projective cover w of Ω satisfies this property. At last, we evaluate ϕ_Ω in w. $\square$

The previous results give rise to the following algorithm for computing a generator for almost split sequences in $\operatorname{mod}(\Lambda)$ ending in X.

The Almost Split Sequence Algorithm

Input: $X \in \operatorname{mod}(\Lambda)$ indecomposable and non-projective. [2]

Output: A generator

$$0 \longrightarrow D\operatorname{Tr}(X) \longrightarrow E \longrightarrow X \longrightarrow 0$$

for $\hat{\Upsilon}/\sim$.

- Regard $\phi_\Omega(w) \in \operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)$ where w is the projective cover of Ω , and extend to a K-basis

$$B_{\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)} := \{\underbrace{\phi_\Omega(w)}_{:=w_1}, w_2, ..., w_l\}$$

  of $\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)$.

- Expand $B_{\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)}$ to a K-basis

$$\mathcal{B}_{\operatorname{Tr}(X) \otimes_\Lambda \Omega} := \{\phi_\Omega(w), w_2, ..., w_l, w_{l+1}, ..., w_{l+m}\}$$

  of $\operatorname{Tr}(X) \otimes_\Lambda \Omega$.

- Let $\xi : \Omega \to D\operatorname{Tr}(X)$ be the Λ-module homomorphism defined as follows. For $a \in \Omega$ then

$$\begin{aligned} \xi(a) : \operatorname{Tr}(X) &\to K \\ q &\mapsto \text{the first } K\text{-coefficient of } q \otimes a \text{ with respect to } \mathcal{B}_{\operatorname{Tr}(X) \otimes_\Lambda \Omega}. \end{aligned}$$

- Let E be the pushout of ι and ξ.

end

Theorem 86. *The Almost Split Sequence Algorithm returns a generator*

$$0 \longrightarrow D\operatorname{Tr}(X) \longrightarrow E \longrightarrow X \longrightarrow 0$$

for all almost split sequences in $\operatorname{mod}(\Lambda)$ *ending in* X*.*

Proof. By Lemma 85 then

$$\breve{\sigma}_X(\bar{1}_X) = \phi_\Omega(w),$$

so $\mathcal{B}_{\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)}$ is as in Lemma 84, implying that

$$\breve{\gamma}_X(\underbrace{d_{\mathcal{B}_{\operatorname{Ker}(1_{\operatorname{Tr}(X)} \otimes \iota)}}(\phi_\Omega(w))}_{:=\bar{z}}) := \bar{y}$$

generates $\operatorname{Soc}_\Gamma(\operatorname{Ext}^1_\Lambda(X, D\operatorname{Tr}(X))$.

[2] We also assume that a projective cover P_0 of X, a kernel Ω of P_0 and a projective cover w of Ω, are chosen beforehand as described.

As in the proof of Proposition 79, an element of $\hat{\Upsilon}/\sim$ is obtained by taking the pushout E of ι and any representative $y \in \mathrm{Hom}_\Lambda(\Omega, D\,\mathrm{Tr}(X))$ for $\bar{y}$. Since $\breve{\gamma}_X$ is the cokernel map of $\theta_{\mathrm{Tr}(X),\Omega,K}$, then if

$$y := \theta_{\mathrm{Tr}(X)\Omega,K}(z)$$

for any representative $z \in D(\mathrm{Tr}(X) \otimes_\Lambda \Omega)$ for $\bar{z}$, then y is a representative for $\breve{\gamma}_X(\bar{z}) = \bar{y}$. Let μ denote the inclusion

$$\mu : \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes \iota) \to \mathrm{Tr}(X) \otimes_\Lambda \Omega.$$

Then

$$D\mu = (- \circ \mu)_K : D(\mathrm{Tr}(X) \otimes_\Lambda \Omega) \to D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes \iota)$$

is the canonical projection from $D(\mathrm{Tr}(X) \otimes_\Lambda \Omega)$ onto its cokernel $D\,\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes \iota)$, and any $z \in D(\mathrm{Tr}(X) \otimes_\Lambda \Omega)$ such that

$$z\mu = \bar{z} \tag{3.46}$$

is a representative for $\bar{z}$. We now show that

$$z := d_{\mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}}(\phi_\Omega(w))$$

satisfies (3.46).

Suppose $q \otimes a \in \mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes \iota)$. Then applying $d_{\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes \iota)}}(\phi_\Omega(w))$ to $q \otimes a$ corresponds to expressing $q \otimes a$ in terms of $\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes \iota)}$, and then extracting the first K-coefficient. Since $\mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}$ is merely an expansion of $\mathcal{B}_{\mathrm{Ker}(1_{\mathrm{Tr}(X)} \otimes \iota)}$, then the outcome does not change if we instead include $q \otimes a$ into $\mathrm{Tr}(X) \otimes_\Lambda \Omega$ and express $q \otimes a$ in terms of $\mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}$ before extracting the first K-coefficient – which, in turn, corresponds to applying $d_{\mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}}(\phi_\Omega(w))\mu$ to $q \otimes a$.

It is then clear that we must take the pushout of ι and $\theta_{\mathrm{Tr}(X),\Omega,K}(d_{\mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}}(\phi_\Omega(w)))$ in order to obtain the desired almost split sequence. We recall from Theorem 58 that

$$\begin{aligned}
\theta_{\mathrm{Tr}(X),\Omega,K}(d_{\mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}}(\phi_\Omega(w)))(a) =& [q \mapsto d_{\mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}}(\phi_\Omega(w))(q \otimes a)] \\
=& [q \mapsto \text{the first } K\text{-coefficient of } q \otimes a \\
& \text{ with respect to } \mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}] \\
=& \xi(a)
\end{aligned}$$

for any $a \in \Omega$. Hence

$$\theta_{\mathrm{Tr}(X),\Omega,K}(d_{\mathcal{B}_{\mathrm{Tr}(X) \otimes_\Lambda \Omega}}(\phi_\Omega(w))) = \xi,$$

and this completes the proof. $\square$

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