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Quivers and admissible relations of tensor products and trivial extensions

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Abstract

We show how to find quotients of path algebras isomorphic to the tensor product

$$\Lambda \otimes_k \Gamma,$$

the triangular matrix algebra

$$\begin{pmatrix} \Gamma & 0 \\ M & \Lambda \end{pmatrix}$$

and the trivial extension

$$\Lambda \ltimes N,$$

where Λ and Γ are quotients of path algebras, M a Λ–Γ-bimodule, and N a Λ-bimodule.

We also solve the following problem: Given a quotient of a path algebra

$$kQ/I,$$

where k is a field, Q a quiver and $I \subseteq kQ$ an ideal satisfying

$$J_Q^t \subseteq I \subseteq J_Q \qquad \text{for some } t \geq 2$$

(where J_Q denotes the ideal generated by the arrows in Q); find an isomorphic quotient of a path algebra

$$kQ'/I'$$

where the ideal I' satisfies

$$J_{Q'}^{t'} \subseteq I' \subseteq J_{Q'}^2 \qquad \text{for some } t' \geq 2.$$

Notation

Notation	*Meaning*	*Defined on page*
General:		
k	a field	
$\mathbb{N}$	the set of natural numbers: $\{1, 2, 3, \ldots\}$	
$\mathbb{N}_0$	the set of non-negative integers: $\{0, 1, 2, \ldots\}$	
In an algebra Λ, with a subset $X \subseteq \Lambda$:		
$\langle X \rangle$	the ideal in Λ generated by X	
In a quotient algebra Λ/I:		
$[x]$	the equivalence class of x (for $x \in \Lambda$)	
For a quiver Q:		
Q_0	the set of vertices in Q	
Q_1	the set of arrows in Q	
Q_n	the set of paths of length n in Q	13
Q_*	the set of non-trivial paths in Q	13
Q_+	the set of non-trivial paths in Q	13
$Q_?$	the set of vertices and arrows in Q	13
J_Q	the ideal in kQ generated by the arrows	13
$Q \circlearrowleft_{(A,h,t)}$	augmented quiver	68
$Q \circlearrowleft_{B}$	augmented quiver	68
For a path q in a quiver:		
$\mathfrak{h}(q)$	the head of q	13
$\mathfrak{t}(q)$	the tail of q	13
$l(q)$	the length of q	13
For an element λ of a path algebra kQ:		
$\mathrm{terms}(\lambda)$	set of terms in λ	13
$\mathrm{terms}_X(\lambda)$	terms in λ contained in $\langle X \rangle$ (for $X \subseteq kQ$)	89
$\mathrm{coefficient}(q, \lambda)$	coefficient of q in λ (for a path $q \in Q_*$)	13
$\mathrm{minlength}(\lambda)$	minimal length of path occuring in λ	13
$\mathfrak{h}(\lambda)$	the head of λ (only defined if λ is uniform)	14
$\mathfrak{t}(\lambda)$	the tail of λ (only defined if λ is uniform)	14
For a set of relations $\rho \subseteq kQ$ in a path algebra:		
$\overset{\rho}{\sim}$	equivalence modulo ρ	14
For two quivers Q and R:		
$Q \times R$	product quiver	24
$Q \cup R$	union of quivers	20
$R \xrightarrow{(A,h,t)} Q$	augmented union of quivers	62
$R \xrightarrow{B} Q$	augmented union of quivers	62
For a product quiver $Q \times R$:		
π_1, π_2	projection maps	25
inc_1^v	inclusion map at a vertex $v \in R_0$	25, 35
inc_2^u	inclusion map at a vertex $u \in Q_0$	25, 35
$\mathrm{inc}_1(X)$	set of all inclusions of a subset $X \subseteq kQ$	36
$\mathrm{inc}_2(Y)$	set of all inclusions of a subset $Y \subseteq kR$	36
$\kappa(Q, R)$	set of commutativity relations	29

Notation	*Meaning*	*Defined on page*
For an augmented union of quivers $R \xrightarrow{B} Q$ with B a uniform basis of a bimodule M:		
$\mu(Q, R, B)$	relation set induced by module structure	63
$\overrightarrow{m}$	element of $k(R \xrightarrow{B} Q)$ corresponding to m (for $m \in M$)	63
For an augmented quiver $Q \circlearrowleft_B$ with B a uniform basis of a bimodule M:		
$\nu(Q, B)$	relation set induced by module structure	70
$\xi(Q, B)$	relation set for products of B-arrows	70
$\overrightarrow{m}$	element of $k(Q \circlearrowleft_B)$ corresponding to m (for $m \in M$)	68
For a uniform element m of a bimodule:		
$\mathfrak{h}(m)$	the head of m	50
$\mathfrak{t}(m)$	the tail of m	50
For an arrow α (or arrows $\alpha_1, \ldots, \alpha_n$) and an element s (or elements $s_1, \ldots, s_n$):		
$\mathrm{subst}_{(\alpha,s)}$	substitution map	88
$\mathrm{tr}_{\{(\alpha_1,s_1),\ldots,(\alpha_n,s_n)\}}$	translation map	98

Name conventions

Name(s)	*Use*
Λ, Γ	k-algebras
λ, γ	elements of k-algebras
Q, R, S	quivers
u, v	vertices of quivers
α, β	arrows of quivers (in general)
α, β, γ, δ, ε, ζ	arrows of quivers (in concrete examples)
q, r, p	paths in quivers
$\mathfrak{r}$, $\mathfrak{s}$	relations
ρ, σ, τ	sets of relations
M	module
m	module element
b	module basis element (in general)
a, **b**, **c**, **d**, **e**, **f**	module basis elements (in concrete examples)
f	function
ϕ, ψ	k-algebra homomorphisms (or k-vector space homomorphisms)

When two quivers Q and R are involved, the conventions for names of vertices, arrows, paths, relations and relation sets are as follows:

$$\begin{array}{llll} u \in Q_0, & \alpha \in Q_1, & q \in Q_*, & \mathfrak{r} \in \rho \subseteq kQ; \\ v \in R_0, & \beta \in R_1, & r \in R_*, & \mathfrak{s} \in \sigma \subseteq kR. \end{array}$$

A variable name with a bar over denotes some helping variable related to the unbared variable of the same name. For example, in order to define an algebra homomorphism ϕ from a path algebra quotient $kQ/\langle\rho\rangle$, we will often first define a homomorphism $\overline{\phi}$ from kQ.

Preface

This thesis represents the work of my final year as master student in mathematics at Norges teknisk-naturvitenskapelige universitet (NTNU).

There are several persons I want to thank at this point.

First of all, I would like to thank my advisor Øyvind Solberg, who suggested the topic of my thesis, and who has given me much useful feedback on my work and answered lots of questions.

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Finally, thanks to Kristin – for the illustrations on pages 12, 22, 58, 84 and 108, for reading and criticizing the whole thesis, and for everything else.

Øystein Skartsæterhagen
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Contents

Introduction

Purpose and motivation

The purpose of this thesis is to describe concrete methods for finding the quivers and relations of certain algebras. The algebras we consider are made by combining one or more quotients of path algebras, and in some cases a bimodule over the algebras. When bimodules are involved, we assume these to be given by representations over quivers (Chapter 3 describes how to relate bimodule structure to representations). Thus, all the ingredients for forming the new algebra are given in a concrete way.

In effect, we describe algorithms that take as input one or more quivers and sets of relations, and possibly a representation over a quiver, and produces as output a new quiver and a set of relations.

The motivation for doing this is a desire to make computer programs that can find quivers and relations for given algebras. The intention is that the methods presented in this thesis should be precise enough that a computer program can be based quite directly on them, but they are not tied to any particular programming language or computer algebra system. Thus, our descriptions are mostly given in terms of simple operations which can be implemented in a straightforward way in a computer program.

There is one notable exception to this: In Chapter 5, we assume that we are able to check whether a given element of a path algebra lies in a given ideal (this is needed in line 9 of Algorithm 1 on page 91), which is not a trivial operation. In a computer program, this can be done using Gröbner bases.

It should also be noted that the relation sets we produce may be much larger than necessary, as can be seen in examples 5.5 and 5.6. In practice it will therefore be preferable to combine our methods with an algorithm for reducing a given generating set of an ideal to one which is minimal.

During the writing of this thesis, the author has begun implementing the methods described herein in the package QPA (Quivers and Path Algebras)[1] for the computer algebra system GAP[2]. At the time of this writing, the method for tensor products of algebras (Chapter 2) is implemented, and the method for turning an algebra with "preadmissible" relation set into one with admissible relation set (Chapter 5) is partially implemented.

[1] `http://www.math.ntnu.no/~oyvinso/QPA/`
[2] `http://www.gap-system.org/`

Overview of the thesis

Chapter 1 contains general definitions and results which will be used throughout the thesis.

In Chapter 2, we consider the problem of finding a quotient of a path algebra isomorphic to the tensor product

$$kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle$$

of two quotients of path algebras. We show that the quiver for this algebra is the *product quiver* $Q \times R$ given by

$$(Q \times R)_0 = Q_0 \times R_0,$$
$$(Q \times R)_1 = (Q_0 \times R_1) \cup (Q_1 \times R_0);$$

and that its set of relations is

$$\mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q, R),$$

where $\mathrm{inc}_1(\rho)$ and $\mathrm{inc}_2(\sigma)$ consist of the relations in ρ and σ transferred into the path algebra $k(Q \times R)$ over the product quiver, while $\kappa(Q, R)$ consists of relations making all squares of the form

$$\begin{array}{ccc} \mathfrak{t}(\alpha) \times \mathfrak{t}(\beta) & \xrightarrow{\mathfrak{t}(\alpha)\times\beta} & \mathfrak{t}(\alpha) \times \mathfrak{h}(\beta) \\ {\scriptstyle \alpha\times\mathfrak{t}(\beta)}\downarrow & & \downarrow{\scriptstyle \alpha\times\mathfrak{h}(\beta)} \\ \mathfrak{h}(\alpha) \times \mathfrak{t}(\beta) & \xrightarrow[\mathfrak{h}(\alpha)\times\beta]{} & \mathfrak{h}(\alpha) \times \mathfrak{h}(\beta) \end{array} \qquad \text{for arrows } \alpha \in Q_1 \text{ and } \beta \in R_1$$

in the product quiver $Q \times R$ commute. The main result of this chapter can thus be stated as

$$kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle \cong k(Q \times R)/\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q, R)\rangle\,.$$

In Chapter 3, we show how bimodules over quotients of path algebras can be described by representations over quivers, using the result of Chapter 2. The connection to tensor products is the fact that a Λ–Γ-bimodule M can be viewed as a left $\Lambda \otimes_k \Gamma^{\mathrm{op}}$-module. Thus, a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule M can be described by a representation over the product quiver $Q \times R^{\mathrm{op}}$ respecting the relations

$$\mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma^{\mathrm{op}}) \cup \kappa(Q, R^{\mathrm{op}}).$$

The first two sections of Chapter 3 give the necessary background for the use of bimodules in the following chapter. The rest of the chapter contains some additional results which are interesting in their own right, but will not be used later.

In Chapter 4, we consider two closely related problems: Finding quotients of path algebras isomorphic to the lower triangular matrix algebra

$$\begin{pmatrix} kR/\langle\sigma\rangle & 0 \\ M & kQ/\langle\rho\rangle \end{pmatrix}$$

and the trivial extension

$$kQ/\langle\rho\rangle \ltimes N,$$

where M is a $kQ/\langle\sigma\rangle$–$kR/\langle\rho\rangle$-bimodule, and N is a $kQ/\langle\rho\rangle$-bimodule. In both cases, we use the very simple strategy of adding one arrow

$$\overrightarrow{b}$$

for each basis element b of the module to the original quiver(s). We then create relations to make a path of the form

$$\alpha\,\overrightarrow{b} \quad \text{or} \quad \overrightarrow{b}\,\beta$$

be the same as the linear combination of arrows corresponding to the product αb or $b\beta$. The problem with this strategy is that the relation set we produce is usually not admissible: If I is the ideal generated by the relations, we do not necessarily have

$$J^t \subseteq I \subseteq J^2 \qquad \text{for some } t,$$

but only

$$J^t \subseteq I \subseteq J \qquad \text{for some } t;$$

where J is the ideal generated by the arrows. This motivates the next chapter, which gives a solution to this problem.

In Chapter 5, we consider algebras

$$kQ/\langle\rho\rangle$$

where the relation set ρ is not admissible but only satisfies a weaker condition which we call *preadmissibility*:

$$J^t \subseteq \langle\rho\rangle \subseteq J \qquad \text{for some } t.$$

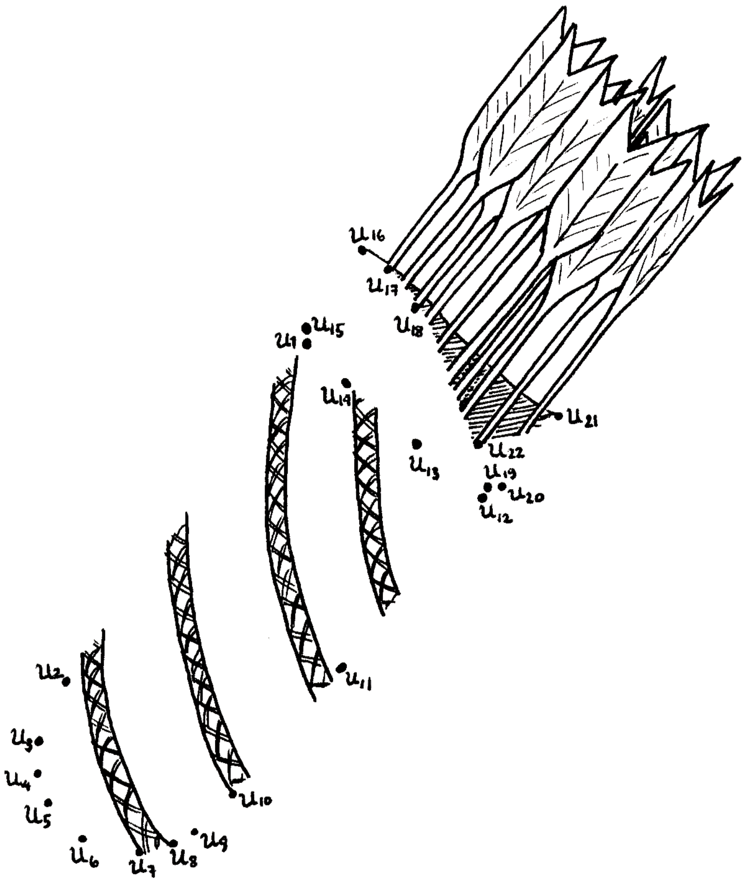
We show how we can transform the quiver and relations of this algebra to get an isomorphic algebra

$$kQ'/\langle\rho'\rangle$$

where ρ' is admissible.

Prerequisites

It is assumed that the reader has some knowledge of quivers and path algebras, for example from the NTNU course MA3203 Ring theory. The reader should also be familiar with tensor products; at NTNU, this is covered by the course MA3204 Homological algebra.



Exercise: Make a quiver by drawing arrows

$$\alpha_i : u_i \longrightarrow u_{i+1}$$

for $i \in \{1, \ldots, 21\}$.

Chapter 1

Quivers and path algebras

In this chapter, we will establish the terminology to be used throughout the thesis, recall some facts we need, and show a few basic results.

1.1 Definitions and conventions

We use k to denote some fixed field.

For a quiver Q, we have the sets Q_0 and Q_1 of vertices and arrows, respectively. We define Q_n for any $n > 1$ to be the set of all paths of length n. We further define the sets

$$
\begin{aligned}
Q_* &= \bigcup_{n \in \mathbb{N}_0} Q_n && \text{(all paths in } Q\text{)},\\
Q_+ &= \bigcup_{n \in \mathbb{N}} Q_n && \text{(all non-trivial paths in } Q\text{)},\\
Q_? &= Q_0 \cup Q_1 && \text{(vertices and arrows in } Q\text{)}.
\end{aligned}
$$

We define J_Q to be the ideal in kQ generated by the arrows; that is,

$$J_Q = \langle Q_1 \rangle .$$

We denote the endpoints of a path q by $\mathfrak{h}(q)$ for the head end, and $\mathfrak{t}(q)$ for the tail. Thus, an arrow $\alpha \colon u \to v$ has $\mathfrak{h}(\alpha) = v$ and $\mathfrak{t}(\alpha) = u$. The length of a path q is denoted $l(q)$.

Definition. Let $\lambda \in kQ$ be an element of a path algebra. We define $\operatorname{terms}(\lambda)$ to be the set of terms when λ is written as a linear combination of paths; that is,

$$\operatorname{terms}\left(\sum_{i \in I} x_i q_i\right) = \{\, x_i q_i \mid i \in I \,\},$$

where the x_i are non-zero elements of k and the q_i are distinct elements of Q_*.

For a path $q \in Q_*$, we define coefficient(q, λ) to be the coefficient of q in λ. We define minlength(λ) to be the minimal length of a path occuring in λ; that is,

$$\text{minlength}(\lambda) = \min\{\, l(q) \mid q \in Q_* \text{ with coefficient}(q, \lambda) \neq 0 \,\}. \qquad \blacksquare$$

Definition. Let $\lambda \in kQ$ be an element of a path algebra. Given vertices u and v in Q, we say that λ is (u, v)**-uniform** if

$$u\lambda v = \lambda.$$

The element λ is **uniform** if it is (u, v)-uniform for some pair of vertices (u, v). Given an arrow α, we say that λ is α**-uniform** if it is $(\mathfrak{h}(\alpha), \mathfrak{t}(\alpha))$-uniform.

For a (u, v)-uniform element $\lambda \in kQ$, we define $\mathfrak{h}(\lambda) = u$ and $\mathfrak{t}(\lambda) = v$. ■

♪ Let Q be a quiver, and u and v vertices in Q. An element of the path algebra kQ is (u, v)-uniform if and only if it is a linear combination of paths from v to u.■

The algebras we will be concerned with are quotients of path algebras – that is, algebras of the form kQ/I for some quiver Q and (two-sided) ideal $I \subseteq kQ$. When describing such an ideal, we will always use a finite generating set consisting of uniform elements. We call such a set a **set of relations** (or **relation set**), and each of its elements a **relation**.

Definition. For a path algebra kQ, a set $\rho \subseteq kQ$ of relations is **admissible** if

$$J_Q^t \subseteq \langle \rho \rangle \subseteq J_Q^2$$

for some $t \geq 2$, and **preadmissible** if

$$J_Q^t \subseteq \langle \rho \rangle \subseteq J_Q$$

for some $t \geq 2$.

A number t such that

$$J_Q^t \subseteq \langle \rho \rangle$$

is called a **path length bound** for the quotient algebra $kQ/\langle \rho \rangle$. ■

Except in Chapter 4 (where we perform certain operations that destroy admissibility) and Chapter 5 (where we repair the damage done in Chapter 4), all our relation sets will be admissible, and we will just say "relation set" when we mean "admissible relation set".

Definition. Let Q be a quiver and $\rho \in kQ$ a set of relations. We define the equivalence relation $\overset{\rho}{\sim}$ on kQ by

$$\lambda_1 \overset{\rho}{\sim} \lambda_2 \iff \lambda_1 - \lambda_2 \in \langle \rho \rangle\,.$$

We say that λ_1 and λ_2 are **equivalent modulo** ρ if $\lambda_1 \overset{\rho}{\sim} \lambda_2$. ■

1.2 Modules and representations

When discussing modules over a path algebra quotient $kQ/\langle\rho\rangle$, we will use representations over the quiver Q respecting the relations ρ to describe the modules. We have the following correspondence between modules and representations (see [1], pages 56–57):

For a $kQ/\langle\rho\rangle$-module M, the corresponding representation over Q is (V, f), where the vector space V_u at a vertex $u \in Q_0$ is defined by

$$V_u = uM, \tag{1.1}$$

and the linear map

$$f_\alpha \colon V_{\mathfrak{t}(\alpha)} \to V_{\mathfrak{h}(\alpha)}$$

for an arrow $\alpha \in Q_1$ is defined by

$$f_\alpha(m) = \alpha m \qquad \text{for } m \in V_{\mathfrak{t}(\alpha)}. \tag{1.2}$$

For a representation (V, f) over Q respecting ρ, the corresponding $kQ/\langle\rho\rangle$-module M is

$$M = \bigoplus_{u \in Q_0} V_u \tag{1.3}$$

as k-vector space. For an element $m \in M$ and a vertex $u \in Q_0$, denote by m_u the component of m belonging to the vector space V_u. Then the scalar multiplication of M is induced by the following rules for vertices and arrows: Let $m \in M$ be a module element, $u \in Q_0$ a vertex and $\alpha \in Q_1$ an arrow. The products um and αm are given componentwise by

$$(um)_v = \begin{cases} m_v & \text{if } v = u, \\ 0 & \text{otherwise;} \end{cases} \tag{1.4}$$

$$(\alpha m)_v = \begin{cases} f_\alpha(m_{\mathfrak{t}(\alpha)}) & \text{if } v = \mathfrak{h}(\alpha), \\ 0 & \text{otherwise;} \end{cases} \tag{1.5}$$

for each vertex $v \in Q_0$.

This correspondence is an equivalence between the category of finite dimensional $kQ/\langle\rho\rangle$-modules and the category of representations over Q respecting ρ ([1], Proposition 1.7). We will view this correspondence as an identification; thus, for us, a module and the corresponding representation is the same thing.

When dealing with a module over a quotient of a path algebra, we will often need a k-basis for the module. Furthermore, we do not want an arbitrary basis, but one that behaves nicely with respect to multiplication with paths. In terms of representations, we want each basis element to be contained in one of the vector spaces of the representation. We will call such elements *left uniform*, by analogy with the concept of uniform elements of a path algebra.

Definition. Let $kQ/\langle\rho\rangle$ be a quotient of a path algebra, M a $kQ/\langle\rho\rangle$-module and m an element of M. For a vertex $u \in Q_0$, we say that m is **left u-uniform** if

$$um = m.$$

The element m is **left uniform** if it is left u-uniform for some vertex u. ■

Definition. Let $kQ/\langle\rho\rangle$ be a quotient of a path algebra, M a $kQ/\langle\rho\rangle$-module and B a k-basis for M. Then B is a **left uniform basis** if every element of B is left uniform. ■

The following example shows a left uniform basis, and additionally introduces a notation we will use to name the elements of such a basis for a given representation.

Example 1.1. Let Q be the quiver

$$Q\colon \quad u_1 \xrightarrow{\alpha} u_2 \xrightarrow{\beta} u_3$$

Let (V, f) the following representation over Q:

$$(V, f)\colon \quad k \xrightarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} k^2 \xrightarrow{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} k^2$$

Let M be the kQ-module corresponding to (V, f). We can view M as the direct sum

$$M = V_{u_1} \oplus V_{u_2} \oplus V_{u_3} = k \oplus k^2 \oplus k^2.$$

Choose a basis

$$B = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$$

for M, where

$$\mathbf{a} = (1, 0, 0)$$

$$\mathbf{b} = \left(0, \begin{pmatrix}1\\0\end{pmatrix}, 0\right) \qquad \mathbf{d} = \left(0, 0, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$\mathbf{c} = \left(0, \begin{pmatrix}0\\1\end{pmatrix}, 0\right) \qquad \mathbf{e} = \left(0, 0, \begin{pmatrix}0\\1\end{pmatrix}\right)$$

Then B is a left uniform basis for M. The basis element $\mathbf{a}$ is left u_1-uniform, the elements $\mathbf{b}$ and $\mathbf{c}$ are left u_2-uniform, and the elements $\mathbf{d}$ and $\mathbf{e}$ are left u_3-uniform. We also see that $\{a\}$ is a basis for V_{u_1}, $\{b, c\}$ is a basis for V_{u_2}, and $\{d, e\}$ is a basis for V_{u_3}. In other words, the set of u-uniform elements in B is a basis for V_u, for any vertex $u \in Q_0$.

When we have a representation and want to give names to left uniform basis elements of the module in this manner, we will use notation like the following as shorthand for the above:

$$B\colon \quad \mathbf{a} \longrightarrow \begin{pmatrix}\mathbf{b}\\\mathbf{c}\end{pmatrix} \longrightarrow \begin{pmatrix}\mathbf{d}\\\mathbf{e}\end{pmatrix}$$

■

1.3 Describing algebra homomorphisms

We will often need to construct algebra homomorphisms. We usually prefer to describe the homomorphisms by what they do on some smaller set than the whole domain, such as a basis or a generating set (for path algebras, the set $Q_?$ of vertices and arrows is a fine generating set). The following two lemmata give criteria for when such descriptions induce algebra homomorphisms.

Lemma 1.1. *Let Λ and Γ be k-algebras, and let B be a basis for Λ. If*

$$\phi\colon \Lambda \to \Gamma$$

is a k-module homomorphism satisfying

1. $\phi(1_\Lambda) = 1_\Gamma$, *and*
2. $\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$ *for any b_1 and b_2 in B,*

then ϕ is a k-algebra homomorphism.

Proof. The only thing that needs to be checked is that ϕ preserves multiplication on *any* elements, not just basis elements. Let

$$\sum_i x_i b_i \quad \text{and} \quad \sum_j y_j b_j$$

be two elements of Λ, written as k-linear combinations of basis elements (the x_i and y_j are elements of k, the b_i and b_j elements of B). Then we have

$$\begin{aligned}
\phi\Big(\sum_i x_i b_i\Big) \cdot \phi\Big(\sum_j y_j b_j\Big) &= \Big(\sum_i x_i \cdot \phi(b_i)\Big) \cdot \Big(\sum_j y_j \cdot \phi(b_j)\Big) \\
&= \sum_i \sum_j x_i y_j \cdot \phi(b_i)\phi(b_j) \\
&\overset{!}{=} \sum_i \sum_j x_i y_j \cdot \phi(b_i b_j) \\
&= \phi\Big(\sum_i \sum_j x_i y_j b_i b_j\Big) \\
&= \phi\bigg(\Big(\sum_i x_i b_i\Big) \cdot \Big(\sum_j y_j b_j\Big)\bigg),
\end{aligned}$$

where the marked equality follows from criterion 2 of the Lemma, and the other equalities follow from k-linearity of ϕ or reorganization of sums. $\square$

Lemma 1.2. *Let Q be a quiver and Γ a k-algebra. Let*

$$f : Q_? \to \Gamma$$

be a function defined on the vertices and arrows of Q. Assume that f satisfies the following conditions:

1. $\sum_{u\in Q_0} f(u) = 1_\Gamma$;
2. $qr = 0$ *implies* $f(q)\cdot f(r) = 0$ *for* q *and* r *in* $Q_?$;
3. $f(u) = f(u)^2$ *for* $u \in Q_0$;
4. $f(\mathfrak{h}(\alpha))\cdot f(\alpha) = f(\alpha) = f(\alpha)\cdot f(\mathfrak{t}(\alpha))$ *for* $\alpha \in Q_1$.

Then f *extends uniquely to a* k*-algebra homomorphism*

$$\phi\colon kQ \to \Gamma.$$

Proof. We first define ϕ as a vector space homomorphism, by defining its actions on basis elements (that is, paths):

$$\begin{aligned} \phi(u) &= f(u) && \text{for a vertex } u \in Q_0,\\ \phi(\alpha_1\cdots\alpha_n) &= f(\alpha_1)\cdots f(\alpha_n) && \text{for a path } \alpha_1\cdots\alpha_n \in Q_+. \end{aligned}$$

Now we can use Lemma 1.1 to show that ϕ in fact is an algebra homomorphism. The first condition of Lemma 1.1 (identity is preserved) follows directly from condition 1 of this lemma:

$$\phi(1_{kQ}) = \phi\Big(\sum_{u\in Q_0} u\Big) = \sum_{u\in Q_0} f(u) = 1_\Gamma.$$

For the second condition (preservation of multiplication on basis elements), consider two paths q and r. The case $qr = 0$ is taken care of by condition 2, so we may assume $qr \neq 0$. Now we have several possibilities:

- Both q and r are vertices: Then $q = r$, and we have

  $$\phi(qr) = \phi(q) = f(q) \overset{(3)}{=} f(q)^2 = \phi(q)^2 = \phi(q)\cdot\phi(r)$$

  by condition 3.

- The path q is a vertex, and r is a non-trivial path of length $n > 0$: Then $r = \alpha_1\cdots\alpha_n$ for some arrows α_i with $q = \mathfrak{h}(\alpha_1)$, and we have

  $$\begin{aligned}\phi(qr) &= \phi(\alpha_1\cdots\alpha_n) = f(\alpha_1)\cdots f(\alpha_n) \overset{(4)}{=} f(q)\cdot(f(\alpha_1)\cdots f(\alpha_n))\\ &= \phi(q)\cdot\phi(\alpha_1\cdots\alpha_n) = \phi(q)\cdot\phi(r)\end{aligned}$$

  by condition 4. The opposite case where q is a non-trivial path and r a vertex is completely analogous.

- Both q and r are non-trivial paths: Then $q = \alpha_1\cdots\alpha_n$ and $r = \beta_1\cdots\beta_m$ for some arrows α_i and β_i, and we have

  $$\begin{aligned}\phi(qr) &= \phi(\alpha_1\cdots\alpha_n\cdot\beta_1\cdots\beta_m) = f(\alpha_1)\cdots f(\alpha_n)\cdot f(\beta_1)\cdots f(\beta_m)\\ &= \phi(\alpha_1\cdots\alpha_n)\cdot\phi(\beta_1\cdots\beta_m) = \phi(q)\cdot\phi(r).\end{aligned}$$

□

1.4 Some algebra constructions

The main theme of this thesis is to investigate certain constructions that take one or two algebras as arguments and produce a new algebra. For each construction we consider, it turns out that if the original algebras are quotients of path algebras, then the new algebra is isomorphic to some quotient of a path algebra. Our goal is to show how the quiver and a set of relations for the new algebra can be computed based on the quivers and relations of the original ones.

In this section, we consider two algebra constructions where it is very easy to find the quiver and relations for the new algebra: the opposite algebra of an algebra and the direct product of two algebras. This should give a taste of the kind of problems we are going to solve in the following chapters. Additionally, the definitions and results of this section will be useful to us many times throughout the thesis.

First consider the opposite algebra of a path algebra. To turn the order of multiplications around, what we need to do to the quiver is simply to reverse the direction of every arrow (the vertices stay the same). We therefore make the following definition.

Definition. Let Q be a quiver. We define the **opposite quiver** of Q, denoted Q^{op}, by

$$\begin{aligned} Q_0^{\mathrm{op}} &= Q_0, \\ Q_1^{\mathrm{op}} &= \{\, \alpha^{\mathrm{op}} \colon \mathfrak{h}(\alpha) \to \mathfrak{t}(\alpha) \mid \alpha \in Q_1 \,\}. \end{aligned}$$

For any path $q \in Q_*$, we define the **opposite path** $q^{\mathrm{op}} \in Q_*^{\mathrm{op}}$ by

$$\begin{aligned} q^{\mathrm{op}} &= q && \text{if } q \text{ is a vertex,} \\ q^{\mathrm{op}} &= \alpha_n^{\mathrm{op}} \cdots \alpha_1^{\mathrm{op}} && \text{if } q = \alpha_1 \cdots \alpha_n \text{ for arrows } \alpha_1, \ldots, \alpha_n. \end{aligned}$$ ■

We observe that the path algebra over the opposite quiver Q^{op} is isomorphic to the opposite algebra of the path algebra over Q:

$$k(Q^{\mathrm{op}}) \cong (kQ)^{\mathrm{op}}. \tag{1.6}$$

We view this isomorphism as an identification, and write just kQ^{op} without parentheses for this algebra. We extend the map

$$^{\mathrm{op}} \colon Q_* \to Q_*^{\mathrm{op}}$$

by linearity to a vector space homomorphism

$$^{\mathrm{op}} \colon kQ \to kQ^{\mathrm{op}}.$$

Applying this map to an element $\lambda \in kQ$ gives the element in $k(Q^{\mathrm{op}})$ which corresponds (under the isomorphism of Equation (1.6)) to the opposite element of λ in $(kQ)^{\mathrm{op}}$.

For a path algebra quotient $kQ/\langle\rho\rangle$, we get the opposite algebra by taking the opposite quiver and the opposite of each relation:

$$(kQ/\langle\rho\rangle)^{\mathrm{op}} \cong kQ^{\mathrm{op}}/\langle\rho^{\mathrm{op}}\rangle\,.$$

We view this isomorphism, too, as an identification.

Now consider the direct product

$$kQ \times kR$$

of two path algebras kQ and kR. This algebra contains everything from kQ and everything from kR, so it seems reasonable that its quiver should contain everything from Q and everything from R. The following definition defines a name and a notation for such a quiver.

Definition. Let Q and R be quivers. We define the **union** $Q \cup R$ of Q and R to be the quiver obtained by taking the unions of the vertex and arrow sets of Q and R:

$$\begin{aligned}(Q \cup R)_0 &= Q_0 \cup R_0\\ (Q \cup R)_1 &= Q_1 \cup R_1.\end{aligned} \qquad \blacksquare$$

When saying that taking the union $Q \cup R$ of the quivers should produce a proper quiver for the product $kQ \times kR$ of path algebras, we silently assumed that the quivers Q and R did not have any vertices or arrows in common. If they did, those vertices or arrows would appear only once in the union $Q \cup R$, making the path algebra $k(Q \cup R)$ smaller than the product algebra $kQ \times kR$. To make such assumptions (which we will use several times) precise, we give the following definition.

Definition. We say that the quivers Q and R are **disjoint** if their vertex sets are disjoint; that is, if

$$Q_0 \cap R_0 = \emptyset. \qquad \blacksquare$$

So far, we have guessed that the product $kQ \times kR$ is isomorphic to the path algebra over the union of the quivers:

$$kQ \times kR \cong k(Q \cup R).$$

If there are relations involved – that is, if we have an algebra of the form $kQ/\langle\rho\rangle \times kR/\langle\sigma\rangle$ – we could guess that what we need to do is to take the union of these as well:

$$kQ/\langle\rho\rangle \times kR/\langle\sigma\rangle \cong k(Q \cup R)/\langle\rho \cup \sigma\rangle\,.$$

We will now show that these guesses are true.

Proposition 1.3. *Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras (where Q and R are disjoint quivers, and ρ and σ are sets of relations). Then*

$$kQ/\langle\rho\rangle \times kR/\langle\sigma\rangle \cong k(Q \cup R)/\langle\rho \cup \sigma\rangle$$

as k-algebras, with isomorphisms induced by the correspondences

$$([q], 0) \leftrightarrow [q] \qquad \textit{for a path } q \in Q_*$$
$$(0, [r]) \leftrightarrow [r] \qquad \textit{for a path } r \in R_*.$$

Proof. Define the algebra homomorphism

$$\overline{\phi}\colon k(Q \cup R) \to kQ/\langle\rho\rangle \times kR/\langle\sigma\rangle$$

by the actions

$$\overline{\phi}(q) = ([q], 0) \qquad \text{for } q \in Q_?,$$
$$\overline{\phi}(r) = (0, [r]) \qquad \text{for } r \in R_?$$

on vertices and arrows (the conditions of Lemma 1.2 are satisfied, so this induces an algebra homomorphism).

We see that $\langle\rho \cup \sigma\rangle \subseteq \ker\overline{\phi}$, since

$$\overline{\phi}(\mathfrak{r}) = ([\mathfrak{r}], 0) = (0, 0) \qquad \text{for } \mathfrak{r} \in \rho,$$
$$\overline{\phi}(\mathfrak{s}) = (0, [\mathfrak{s}]) = (0, 0) \qquad \text{for } \mathfrak{s} \in \sigma.$$

We want to show that $\ker\overline{\phi} \subseteq \langle\rho \cup \sigma\rangle$ as well. Let $\lambda \in k(Q \cup R)$ be any element. Then we can write λ as $\lambda_Q + \lambda_R$ for some elements $\lambda_Q \in kQ$ and $\lambda_R \in kR$. If $\lambda \in \ker\overline{\phi}$, then

$$0 = \overline{\phi}(\lambda) = ([\lambda_Q], [\lambda_R]),$$

so

$$\lambda_Q \in \langle\rho\rangle \qquad \text{and} \qquad \lambda_R \in \langle\sigma\rangle\,.$$

This means that

$$\lambda \in \langle\rho\rangle + \langle\sigma\rangle = \langle\rho \cup \sigma\rangle\,.$$

We thus have $\ker\overline{\phi} = \langle\rho \cup \sigma\rangle$, and $\overline{\phi}$ induces an algebra monomorphism

$$\phi\colon k(Q \cup R)/\langle\rho \cup \sigma\rangle \to kQ/\langle\rho\rangle \times kR/\langle\sigma\rangle$$

with

$$\phi([q]) = ([q], 0) \qquad \text{for } q \in Q_*,$$
$$\phi([r]) = (0, [r]) \qquad \text{for } r \in R_*.$$

We further see that ϕ is onto: For any element $([\lambda_Q], [\lambda_R])$ of $kQ/\langle\rho\rangle \times kR/\langle\sigma\rangle$, we have

$$([\lambda_Q], [\lambda_R]) = \phi([\lambda_Q + \lambda_R]).$$

Thus, ϕ is an algebra isomorphism. □

Chapter 2

Tensor product of algebras

Given two k-algebras Λ and Γ, their tensor product $\Lambda \otimes_k \Gamma$ over k is again a k-algebra, with multiplication induced by the rule

$$(\lambda_1 \otimes \gamma_1)(\lambda_2 \otimes \gamma_2) = \lambda_1\lambda_2 \otimes \gamma_1\gamma_2$$

for simple tensors (where λ_1 and λ_2 are elements of Λ, and γ_1 and γ_2 elements of Γ). We are interested in the case where Λ and Γ are quotients of path algebras. Then their tensor product is isomorphic to a quotient of a path algebra. We will show how to find the quiver and relations of this algebra based on the quivers and relations of Λ and Γ.

Some bibliographical notes regarding the contents of this chapter: The name "product quiver" and the notation $Q \times R$ (see definition on page 24) are taken from [2]. Our Proposition 2.2 is the same as Proposition 3 in [2], but that paper does not include the more general result which we state as Proposition 2.4. The result of Proposition 2.4 can be found in [3] as Lemma 1.3, but with quite different notation.

2.1 The quiver product

Let Q and R be quivers. Assuming that $kQ \otimes_k kR$ actually is a quotient of a path algebra, what should its quiver look like?

We know how to describe $kQ \otimes_k kR$ as a k-vector space: It has a basis consisting of all elements $q \otimes r$ where q and r are paths in Q and R, respectively.

Of these elements, we observe that each element $u \otimes v$ which is made by combining two vertices $u \in Q_0$ and $v \in R_0$ has the following property:

$$\begin{aligned}
(u \otimes v)(q \otimes r) &= \begin{cases} q \otimes r & \text{if } \mathfrak{h}(q) = u \text{ and } \mathfrak{h}(r) = v, \\ 0 & \text{otherwise;} \end{cases} \\
(q \otimes r)(u \otimes v) &= \begin{cases} q \otimes r & \text{if } \mathfrak{t}(q) = u \text{ and } \mathfrak{t}(r) = v, \\ 0 & \text{otherwise;} \end{cases}
\end{aligned}$$

for any paths $q \in Q_*$ and $r \in R_*$. This looks like the way vertices in a quiver behave under multiplication, so it seems reasonable that the quiver for the tensor product will have one vertex for every pair (u, v) of vertices from each of the two original quivers.

If $\alpha\colon u \to u'$ is an arrow in Q and v a vertex in R, the element $\alpha \otimes v$ acts like a path from $u \otimes v$ to $u' \otimes v$, in the sense that it is unchanged under multiplication by these on the appropriate side:

$$(\alpha \otimes v)(u \otimes v) = \alpha \otimes v = (u' \otimes v)(\alpha \otimes v).$$

Furthermore, since such an element cannot be written as the product of two other basis elements, it should correspond to an arrow in the tensor product's quiver. The same applies to elements of the form $u \otimes \beta$ where u is a vertex and β an arrow. We thus create an arrow for each pair of arrow/vertex or vertex/arrow from the original quivers.

Now any basis element of the tensor product can be decomposed as a product of those we have already considered. Take for example an element

$$(\alpha_2\alpha_1) \otimes (\beta_2\beta_1),$$

where

$$u_1 \overset{\alpha_1}{\to} u_2 \overset{\alpha_2}{\to} u_3 \qquad \text{and} \qquad v_1 \overset{\beta_1}{\to} v_2 \overset{\beta_2}{\to} v_3$$

are paths in the respective quivers; it can be written as

$$(\alpha_2\alpha_1) \otimes (\beta_2\beta_1) = (\alpha_2 \otimes v_3)(\alpha_1 \otimes v_3)(u_1 \otimes \beta_2)(u_1 \otimes \beta_1)\,. \tag{2.1}$$

In the quiver we indicated above, this corresponds to a path of length four.

Thus, we see that every basis element of the tensor product is represented by some path in our quiver. But there may be several paths corresponding to the same basis element. Consider again the decomposition in Equation (2.1). The same element could also be written as the product

$$(\alpha_2\alpha_1) \otimes (\beta_2\beta_1) = (\alpha_2 \otimes v_3)(u_2 \otimes \beta_2)(\alpha_1 \otimes v_2)(u_1 \otimes \beta_1)\,.$$

So the quiver we suggest will not only produce all basis elements of the tensor product – it may produce some of them several times, as different paths.

This is not desirable behaviour, but since it can be solved by introducing some relations in the path algebra, we shall postpone worrying about it until the next section. For the moment, we will be happy with what we have found and make the following definition based on our observations.

Definition. The **product quiver** of two quivers Q and R, denoted $Q \times R$, has vertex set

$$(Q \times R)_0 = Q_0 \times R_0$$

and arrow set

$$(Q \times R)_1 = (Q_1 \times R_0) \cup (Q_0 \times R_1)\,,$$

with heads and tails for arrows given by

$$\begin{aligned} \mathfrak{h}(\alpha \times v) &= \mathfrak{h}(\alpha) \times v, & \mathfrak{t}(\alpha \times v) &= \mathfrak{t}(\alpha) \times v & &\text{for } \alpha \in Q_1 \text{ and } v \in R_0; \\ \mathfrak{h}(u \times \beta) &= u \times \mathfrak{h}(\beta), & \mathfrak{t}(u \times \beta) &= u \times \mathfrak{t}(\beta) & &\text{for } u \in Q_0 \text{ and } \beta \in R_1. \end{aligned} \qquad \blacksquare$$

We can visualize the product quiver of Q and R as a rectangular grid of vertices with arrows filled in, such that each column is a copy of Q and each row a copy of R.

A simple example should make the construction clear.

Example 2.1. Let Q and R be the quivers

$$Q\colon\ u_1 \xrightarrow{\alpha} u_2\ ,$$

$$R\colon\ v_1 \xrightarrow{\beta} v_2 \xrightarrow{\gamma} v_3\ .$$

Their product $Q \times R$ is

$$Q \times R\colon \quad \begin{array}{ccccc} u_1 \times v_1 & \xrightarrow{u_1 \times \beta} & u_1 \times v_2 & \xrightarrow{u_1 \times \gamma} & u_1 \times v_3 \\ \downarrow{\scriptstyle \alpha \times v_1} & & \downarrow{\scriptstyle \alpha \times v_2} & & \downarrow{\scriptstyle \alpha \times v_3} \\ u_2 \times v_1 & \xrightarrow{u_2 \times \beta} & u_2 \times v_2 & \xrightarrow{u_2 \times \gamma} & u_2 \times v_3 \end{array} \qquad \blacksquare$$

We will now define some functions for moving paths into and out of a product quiver. For a product quiver $Q \times R$, we define the two projection maps

$$\pi_1\colon (Q \times R)_* \to Q_*, \qquad \pi_2\colon (Q \times R)_* \to R_*$$

by

$$\pi_1\Big(\prod_i (q_i \times r_i)\Big) = \prod_i q_i\ , \qquad \pi_2\Big(\prod_i (q_i \times r_i)\Big) = \prod_i r_i\ ;$$

where each $q_i \times r_i$ is either a vertex or arrow in $Q \times R$. For a vertex v in R, we define the inclusion map of Q into $Q \times R$ at v,

$$\mathrm{inc}_1^v\colon Q_* \to (Q \times R)_*,$$

by

$$\begin{aligned} \mathrm{inc}_1^v(u) &= u \times v & &\text{for a vertex } u \text{ in } Q \\ \mathrm{inc}_1^v\left(\prod_i \alpha_i\right) &= \prod_i (\alpha_i \times v) & &\text{for arrows } \alpha_i \text{ in } Q. \end{aligned}$$

The inclusion map

$$\mathrm{inc}_2^u\colon R_* \to (Q \times R)_*$$

for a vertex u in Q is defined similarly.

♪ The projection and inclusion maps preserve products of paths. That is, if we have paths

$$\begin{aligned}
&p_1 \text{ and } p_2 \text{ in } (Q \times R)_* \text{ with } p_1 p_2 \neq 0,\\
&q_1 \text{ and } q_2 \text{ in } Q_* \text{ with } q_1 q_2 \neq 0,\\
&r_1 \text{ and } r_2 \text{ in } R_* \text{ with } r_1 r_2 \neq 0;
\end{aligned}$$

then

$$\begin{aligned}
\pi_1(p_1 p_2) &= \pi_1(p_1) \cdot \pi_1(p_2) & \qquad \mathrm{inc}_1^v(q_1 q_2) &= \mathrm{inc}_1^v(q_1) \cdot \mathrm{inc}_1^v(q_2)\\
\pi_2(p_1 p_2) &= \pi_2(p_1) \cdot \pi_2(p_2) & \qquad \mathrm{inc}_2^u(r_1 r_2) &= \mathrm{inc}_2^u(r_1) \cdot \mathrm{inc}_2^u(r_2)
\end{aligned}$$

(for all vertices $v \in R_0$ and $u \in Q_0$). ■

Example 2.2. We illustrate the inclusion and projection maps with the product quiver

$$Q \times R\colon \qquad \begin{array}{ccccc}
u_1 \times v_1 & \xrightarrow{u_1 \times \beta} & u_1 \times v_2 & \xrightarrow{u_1 \times \gamma} & u_1 \times v_3\\
\big\downarrow{\scriptstyle \alpha \times v_1} & & \big\downarrow{\scriptstyle \alpha \times v_2} & & \big\downarrow{\scriptstyle \alpha \times v_3}\\
u_2 \times v_1 & \xrightarrow{u_2 \times \beta} & u_2 \times v_2 & \xrightarrow{u_2 \times \gamma} & u_2 \times v_3
\end{array}$$

from Example 2.1.

The three possible inclusions of the arrow $\alpha \in Q_1$ are

$$\mathrm{inc}_1^{v_1}(\alpha) = \alpha \times v_1, \qquad \mathrm{inc}_1^{v_2}(\alpha) = \alpha \times v_2, \qquad \mathrm{inc}_1^{v_3}(\alpha) = \alpha \times v_3.$$

The two possible inclusions of the arrow $\beta \in R_1$ are

$$\mathrm{inc}_2^{u_1}(\beta) = u_1 \times \beta, \qquad \mathrm{inc}_2^{u_2}(\beta) = u_2 \times \beta.$$

Similarly, there are two inclusions of the path $\gamma\beta \in R_*$:

$$\mathrm{inc}_2^{u_1}(\gamma\beta) = (u_1 \times \gamma)(u_1 \times \beta), \qquad \mathrm{inc}_2^{u_2}(\gamma\beta) = (u_2 \times \gamma)(u_2 \times \beta).$$

The paths

$$\alpha \times v_1, \qquad (u_1 \times \gamma)(u_1 \times \beta) \qquad \text{and} \qquad (u_2 \times \gamma)(\alpha \times v_2)(u_1 \times \beta)$$

in $Q \times R$ have projections

$$\begin{aligned}
\pi_1(\alpha \times v_1) &= \alpha & \qquad \pi_2(\alpha \times v_1) &= v_1\\
\pi_1((u_1 \times \gamma)(u_1 \times \beta)) &= u_1 & \qquad \pi_2((u_1 \times \gamma)(u_1 \times \beta)) &= \gamma\beta\\
\pi_1((u_2 \times \gamma)(\alpha \times v_2)(u_1 \times \beta)) &= \alpha & \qquad \pi_2((u_2 \times \gamma)(\alpha \times v_2)(u_1 \times \beta)) &= \gamma\beta
\end{aligned}$$ ■

Let us now introduce some convenient notation. Note that when we write a path in a product quiver as a product of arrows, much of the information is redundant. Take for example the path

$$(\alpha \times v_3)(u_1 \times \gamma)$$

in the product quiver of Example 2.1. Once we know that the path should be the product of some inclusion of α and some inclusion of γ (in that order), the only possibility is to include α at v_3 (the head of γ) and γ at u_1 (the tail of α). So we would prefer to dispense with all the unnecessary information and write this path simply as

$$\alpha\gamma.$$

To make such notation unambiguous, we must assume that the quivers Q and R are disjoint; otherwise a single expression could be interpreted both as a path in the product quiver and as a path in one of the factors. Thus we will, for the rest of this chapter, assume that the operands in any quiver product are disjoint.[1]

First define the products pq, qp, pr and rp of a path p in $Q \times R$ with a path from one of the factors (q in Q and r in R) by

$$\begin{aligned} qp &= \mathrm{inc}_1^{\pi_2(\mathfrak{h}(p))}(q) \cdot p \\ pq &= p \cdot \mathrm{inc}_1^{\pi_2(\mathfrak{t}(p))}(q) \\ rp &= \mathrm{inc}_2^{\pi_1(\mathfrak{h}(p))}(r) \cdot p \\ pr &= p \cdot \mathrm{inc}_2^{\pi_1(\mathfrak{t}(p))}(r). \end{aligned}$$

Next define the products qr and rq of one path from each of the factors (q in Q and r in R) by

$$\begin{aligned} qr &= \mathrm{inc}_1^{\mathfrak{h}(r)}(q) \cdot \mathrm{inc}_2^{\mathfrak{t}(q)}(r) \\ rq &= \mathrm{inc}_2^{\mathfrak{h}(q)}(r) \cdot \mathrm{inc}_1^{\mathfrak{t}(r)}(q). \end{aligned}$$

The general rule behind all these definitions is that we include any path which is not already in the product quiver at the unique vertex where it makes sense to include it in order for the product to be nonzero, if at all possible.

Observe that using this notation, any path in $Q \times R$ can be written as a product of arrows and vertices from Q and R. Furthermore, any path with nontrivial projection in both coordinates can be written as a product of arrows from Q and R.

Example 2.3. We apply the new notation to the product quiver

$$Q \times R\colon \quad \begin{array}{ccccc} u_1 \times v_1 & \xrightarrow{u_1 \times \beta} & u_1 \times v_2 & \xrightarrow{u_1 \times \gamma} & u_1 \times v_3 \\ \downarrow{\scriptstyle \alpha \times v_1} & & \downarrow{\scriptstyle \alpha \times v_2} & & \downarrow{\scriptstyle \alpha \times v_3} \\ u_2 \times v_1 & \xrightarrow{u_2 \times \beta} & u_2 \times v_2 & \xrightarrow{u_2 \times \gamma} & u_2 \times v_3 \end{array}$$

from Example 2.1.

[1] We will not use the assumption of disjoint quivers for anything besides simplifying the notation, so all our results hold for non-disjoint quivers as well.

We can write the vertices as

$$u_1v_1 = u_1 \times v_1 \qquad u_2v_1 = u_2 \times v_1$$
$$u_1v_2 = u_1 \times v_2 \qquad u_2v_2 = u_2 \times v_2$$
$$u_1v_3 = u_1 \times v_3 \qquad u_2v_3 = u_2 \times v_3$$

and the arrows as

$$\alpha v_1 = \alpha \times v_1 \qquad u_1\beta = u_1 \times \beta \qquad u_1\gamma = u_1 \times \gamma$$
$$\alpha v_2 = \alpha \times v_2 \qquad u_2\beta = u_2 \times \beta \qquad u_2\gamma = u_2 \times \gamma$$
$$\alpha v_3 = \alpha \times v_3$$

For the longer paths we have, for example,

$$\begin{aligned}
\beta\alpha &= (u_2 \times \beta)(\alpha \times v_1)\\
\alpha\beta &= (\alpha \times v_2)(u_1 \times \beta)\\
u_1\gamma\beta &= (u_1 \times \gamma)(u_1 \times \beta)\\
u_2\gamma\beta &= (u_2 \times \gamma)(u_2 \times \beta)\\
\alpha\gamma\beta &= (\alpha \times v_3)(u_1 \times \gamma)(u_1 \times \beta)\\
\gamma\alpha\beta &= (u_2 \times \gamma)(\alpha \times v_2)(u_1 \times \beta)\\
\gamma\beta\alpha &= (u_2 \times \gamma)(u_2 \times \beta)(\alpha \times v_1)
\end{aligned}$$

■

For any quiver Q, we have the path length function $l\colon Q_* \to \mathbb{N}_0$ given by

$$l(q) = \begin{cases} 0 & \text{if } q \text{ is a vertex} \\ n & \text{if } q \text{ is the product of } n \text{ arrows.} \end{cases}$$

For the product quiver $Q \times R$, define the component path length functions $l_1 = l \circ \pi_1$ and $l_2 = l \circ \pi_2$. Then define the decomposed path length function $l_*\colon (Q \times R)_* \to \mathbb{N}_0 \times \mathbb{N}_0$ by

$$l_*(p) = (l_1(p), l_2(p)).$$

Note that for $p \in (Q \times R)_*$, we have $l(p) = l_1(p) + l_2(p)$.

Example 2.4. Consider the paths

$$\begin{aligned}
u_1v_1 &= u_1 \times v_1,\\
\alpha\beta &= (\alpha \times v_2)(u_1 \times \beta),\\
u_1\gamma\beta &= (u_1 \times \gamma)(u_1 \times \beta),\\
\alpha\gamma\beta &= (\alpha \times v_3)(u_1 \times \gamma)(u_1 \times \beta)
\end{aligned}$$

in the product quiver $Q \times R$ of Example 2.1. We have

$$\begin{array}{rrrr}
l(u_1v_1) = 0 & l_1(u_1v_1) = 0 & l_2(u_1v_1) = 0 & l_*(u_1v_1) = (0,0)\\
l(\alpha\beta) = 2 & l_1(\alpha\beta) = 1 & l_2(\alpha\beta) = 1 & l_*(\alpha\beta) = (1,1)\\
l(u_1\gamma\beta) = 2 & l_1(u_1\gamma\beta) = 0 & l_2(u_1\gamma\beta) = 2 & l_*(u_1\gamma\beta) = (0,2)\\
l(\alpha\gamma\beta) = 3 & l_1(\alpha\gamma\beta) = 1 & l_2(\alpha\gamma\beta) = 2 & l_*(\alpha\gamma\beta) = (1,2)
\end{array}$$

■

2.2 Tensor product of path algebras

We will now show that taking the quiver product actually produces the appropriate quiver for the tensor product of two path algebras. For the moment we consider only actual path algebras; we shall generalize to quotients of path algebras in the next section.

Let us first tackle the problem that bothered us earlier: The product quiver does not represent tensor product basis elements in a unique way. Consider two arrows $\alpha\colon u_1 \to u_2$ in Q and $\beta\colon v_1 \to v_2$ in R. Then a portion of the product quiver $Q \times R$ looks like

$$\begin{array}{ccc} u_1 \times v_1 & \xrightarrow{u_1\times\beta} & u_1 \times v_2 \\ {\scriptstyle \alpha\times v_1}\downarrow & & \downarrow{\scriptstyle \alpha\times v_2} \\ u_2 \times v_1 & \xrightarrow{u_2\times\beta} & u_2 \times v_2. \end{array}$$

Here we have two distinct paths

$$\alpha\beta = (\alpha \times v_2)(u_1 \times \beta) \qquad \text{and} \qquad \beta\alpha = (u_2 \times \beta)(\alpha \times v_1)$$

in the product quiver, but the corresponding elements in the tensor product are the same:

$$(\alpha \otimes v_2)(u_1 \otimes \beta) = \alpha \otimes \beta = (u_2 \otimes \beta)(\alpha \otimes v_1).$$

In other words, we would want such a square in the product quiver to commute. So let us make it commute!

The obvious way to do this is to take all differences

$$\alpha\beta - \beta\alpha$$

(for arrows α in Q and β in R) as relations. We define $\kappa(Q, R)$ to be the set consisting of these relations for the product quiver $Q \times R$.

Definition. Let Q and R be quivers. The set $\kappa(Q, R)$ of **commutativity relations** in the product quiver $Q \times R$ is

$$\kappa(Q, R) = \{\, \alpha\beta - \beta\alpha \mid \alpha \in Q_1, \beta \in R_1 \,\}. \qquad \blacksquare$$

We shall now show that the commutativity relations $\kappa(Q, R)$ are exactly the relations we need to turn the path algebra $k(Q \times R)$ over the product quiver into an algebra isomorphic to the tensor product algebra $kQ \otimes_k kR$. This result is given in Proposition 2.2. A technical detail of the proof has been separated out as Lemma 2.1.

Lemma 2.1. *Let Q and R be quivers. For any path p in $Q \times R$,*

$$p \overset{\kappa(Q,R)}{\sim} \pi_1(p) \cdot \pi_2(p).$$

Proof. This result just states that we can reorder the arrows of a path in $Q \times R$ to get all the path's Q-arrows to the left of its R-arrows, without changing the path modulo $\kappa(Q, R)$. Intuitively, this is obvious: $\kappa(Q, R)$ lets us reverse the order of any pair of arrows which are not from the same quiver, so we can let each Q-arrow "jump over" any R-arrow to its left it until we have a path where all Q-arrows occur to the left of all R-arrows.

Let us now prove the result in a more rigorous way. We shall use induction on the length $l_*(p) = (l_1(p), l_2(p))$. Order $\mathbb{N}_0 \times \mathbb{N}_0$ by

$$(m, n) \leq (m', n') \quad \text{if } m \leq m' \text{ and } n \leq n'.$$

For a path p whose projection onto one of the components is a vertex, p is *equal* to $\pi_1(p) \cdot \pi_2(p)$. This constitutes the base step of the induction.

Now for the inductive step. Let p be a path with nontrivial projection in both components, that is, $l_1(p) > 0$ and $l_2(p) > 0$. Assume the result to be true for all paths p' with $l_*(p') < l_*(p)$ (with the ordering of $\mathbb{N}_0 \times \mathbb{N}_0$ given above). Write the projections of p as

$$\pi_1(p) = \alpha_n \cdots \alpha_1, \qquad \pi_2(p) = \beta_m \cdots \beta_1;$$

where the α_i are arrows in Q and the β_j are arrows in R. To avoid problems with empty products later, let[2] $\alpha_0 = \mathfrak{t}(\alpha_1)$ and $\beta_0 = \mathfrak{t}(\beta_1)$. Then we have

$$\pi_1(p) = \alpha_n \cdots \alpha_0, \qquad \pi_2(p) = \beta_m \cdots \beta_0.$$

If we write p as a product of arrows, the leftmost arrow must be the appropriate inclusion of either α_n or β_m. Let us investigate these two cases.

First, assume that $p = \alpha_n p'$, where

$$\pi_1(p') = \alpha_{n-1} \cdots \alpha_0, \qquad \pi_2(p') = \beta_m \cdots \beta_0.$$

Then we have

$$\begin{aligned} p &= \alpha_n p' \\ &\overset{\kappa(Q,R)}{\sim} \alpha_n \cdot \pi_1(p') \cdot \pi_2(p') \\ &= \alpha_n(\alpha_{n-1} \cdots \alpha_0)(\beta_m \cdots \beta_0) \\ &= \pi_1(p) \cdot \pi_2(p), \end{aligned}$$

where the equivalence follows from the induction assumption.

Next, assume $p = \beta_m p'$, where

$$\pi_1(p') = \alpha_n \cdots \alpha_0, \qquad \pi_2(p') = \beta_{m-1} \cdots \beta_0.$$

[2]Note that we here temporarily break our name conventions and give vertices Greek names.

Then

$$\begin{aligned}
p &= \beta_m p' \\
&\overset{\kappa(Q,R)}{\sim} \beta_m \cdot \pi_1(p') \cdot \pi_2(p') \\
&= \beta_m(\alpha_n \cdots \alpha_0)(\beta_{m-1} \cdots \beta_0) \\
&= (\beta_m \alpha_n)(\alpha_{n-1} \cdots \alpha_0)(\beta_{m-1} \cdots \beta_0) \\
&\overset{\kappa(Q,R)}{\sim} (\alpha_n \beta_m)(\alpha_{n-1} \cdots \alpha_0)(\beta_{m-1} \cdots \beta_0) \\
&= \alpha_n(\beta_m \alpha_{n-1} \cdots \alpha_0)(\beta_{m-1} \cdots \beta_0) \\
&\overset{\kappa(Q,R)}{\sim} \alpha_n(\alpha_{n-1} \cdots \alpha_0 \beta_m)(\beta_{m-1} \cdots \beta_0) \\
&= \pi_1(p) \cdot \pi_2(p).
\end{aligned}$$

□

We are now ready to state and prove the result we want for tensor products of path algebras.

Proposition 2.2. ([2], Proposition 3) *For quivers Q and R, we have*

$$kQ \otimes_k kR \cong k(Q \times R)/\langle \kappa(Q,R) \rangle$$

as k-algebras, with isomorphisms induced by

$$\begin{aligned}
q \otimes r &\mapsto [qr] && \textit{for } q \in Q_* \textit{ and } r \in R_*, \\
\pi_1(p) \otimes \pi_2(p) &\leftarrow [p] && \textit{for } p \in (Q \times R)_*.
\end{aligned}$$

Proof. We will show that the maps described in the result are well-defined algebra homomorphisms, and then that they are the inverses of each other.

Define a k-module homomorphism $\overline{\phi}\colon k(Q \times R) \to kQ \otimes_k kR$ by

$$\overline{\phi}(p) = \pi_1(p) \otimes \pi_2(p)$$

for any path p in $Q \times R$, extending to $k(Q \times R)$ by linearity. We show that $\overline{\phi}$ is a k-algebra homomorphism by checking that the conditions of Lemma 1.1 are satisfied. We first show that $\overline{\phi}$ preserves the identity element:

$$\begin{aligned}
\overline{\phi}(1_{k(Q\times R)}) &= \overline{\phi}\Bigg(\sum_{\substack{u \in Q_0 \\ v \in R_0}} u \times v\Bigg) = \sum_{\substack{u \in Q_0 \\ v \in R_0}} \overline{\phi}(u \times v) = \sum_{\substack{u \in Q_0 \\ v \in R_0}} u \otimes v \\
&= \Bigg(\sum_{u \in Q_0} u\Bigg) \otimes \Bigg(\sum_{v \in R_0} v\Bigg) = 1_{kQ} \otimes 1_{kR} = 1_{kQ \otimes kR}.
\end{aligned}$$

Second, we must show that $\overline{\phi}$ preserves products of basis elements. Let p and p' be paths in $Q \times R$. If $\mathfrak{t}(p) = \mathfrak{h}(p')$, the product pp' is a path, and we have

$$\begin{aligned}
\overline{\phi}(pp') &= \pi_1(pp') \otimes \pi_2(pp') = (\pi_1(p) \cdot \pi_1(p')) \otimes (\pi_2(p) \cdot \pi_2(p')) \\
&= (\pi_1(p) \otimes \pi_2(p)) \cdot (\pi_1(p') \otimes \pi_2(p')) = \overline{\phi}(p) \cdot \overline{\phi}(p').
\end{aligned}$$

If $\mathfrak{t}(p) \neq \mathfrak{h}(p')$, then we must have

$$\mathfrak{t}(\pi_1(p)) \neq \mathfrak{h}(\pi_1(p')) \quad \text{or} \quad \mathfrak{t}(\pi_2(p)) \neq \mathfrak{h}(\pi_2(p')),$$

since

$$\mathfrak{h}(p) = \mathfrak{h}(\pi_1(p)) \times \mathfrak{h}(\pi_2(p)) \quad \text{and} \quad \mathfrak{t}(p) = \mathfrak{t}(\pi_1(p)) \times \mathfrak{t}(\pi_2(p)).$$

This means that at least one of

$$\pi_1(p) \cdot \pi_1(p') \quad \text{and} \quad \pi_2(p) \cdot \pi_2(p')$$

is zero, so

$$(\pi_1(p) \cdot \pi_1(p')) \otimes (\pi_2(p) \cdot \pi_2(p')) = 0.$$

We thus have

$$\begin{aligned}\overline{\phi}(p) \cdot \overline{\phi}(p') &= (\pi_1(p) \otimes \pi_2(p)) \cdot (\pi_1(p') \otimes \pi_2(p')) \\ &= (\pi_1(p) \cdot \pi_1(p')) \otimes (\pi_2(p) \cdot \pi_2(p')) = 0 = \overline{\phi}(0) = \overline{\phi}(pp').\end{aligned}$$

This completes the proof of $\overline{\phi}$ being an algebra homomorphism.

We have $\kappa(Q, R) \subseteq \ker \overline{\phi}$, since

$$\overline{\phi}(\alpha\beta - \beta\alpha) = \overline{\phi}(\alpha\beta) - \overline{\phi}(\beta\alpha) = \alpha \otimes \beta - \alpha \otimes \beta = 0$$

for any element $\alpha\beta - \beta\alpha$ of $\kappa(Q, R)$. Thus there is an induced algebra homomorphism

$$\phi\colon k(Q \times R)/\langle\kappa(Q, R)\rangle \to kQ \otimes_k kR,$$

with

$$\phi([p]) = \overline{\phi}(p) = \pi_1(p) \otimes \pi_2(p)$$

for any path p in $Q \times R$.

Let us now define the algebra homomorphism we need in the opposite direction. Define the k-module homomorphism

$$\psi\colon kQ \otimes_k kR \to k(Q \times R)/\langle\kappa(Q, R)\rangle$$

by

$$\psi(q \otimes r) = [qr],$$

on basis elements (q and r paths in Q and R, respectively), extending to $kQ \otimes_k kR$ by linearity. We again use Lemma 1.1 to check that we have an algebra homomorphism. We first show that the identity is preserved:

$$\begin{aligned}\psi(1_{kQ \otimes kR}) = \psi(1_{kQ} \otimes 1_{kR}) &= \psi\left(\left(\sum_{u \in Q_0} u\right) \otimes \left(\sum_{v \in R_0} v\right)\right) \\ &= \psi\left(\sum_{\substack{u \in Q_0 \\ v \in R_0}} u \otimes v\right) = \sum_{\substack{u \in Q_0 \\ v \in R_0}} \psi(u \otimes v) = \sum_{\substack{u \in Q_0 \\ v \in R_0}} [uv] = \left[\sum_{\substack{u \in Q_0 \\ v \in R_0}} uv\right] \\ &= [1_{k(Q \times R)}] = 1_{k(Q \times R)/\langle\kappa(Q,R)\rangle}.\end{aligned}$$

Second, we show that ψ preserves multiplication of basis elements. For basis elements $q \otimes r$ and $q' \otimes r'$, we have

$$\begin{aligned}\psi((q \otimes r)(q' \otimes r')) &= \psi(qq' \otimes rr') = [qq'rr'] \stackrel{!}{=} [qrq'r'] = [qr][q'r'] \\ &= \psi(q \otimes r) \cdot \psi(q' \otimes r'),\end{aligned}$$

where the marked equality follows from Lemma 2.1.

With both maps defined, the only thing that remains is to show that their compositions are the respective identities. It is sufficient to show this on a generating set of each algebra. We have

$$\phi\psi(q \otimes r) = \phi([qr]) = \pi_1(qr) \otimes \pi_2(qr) = q \otimes r$$

for a basis element $q \otimes r$ (with $q \in Q_*$ and $r \in R_*$) in $kQ \otimes_k kR$, and

$$\begin{aligned}\psi\phi([u \times v]) &= \psi(u \otimes v) = [uv] = [u \times v] && \text{(for } u \in Q_0 \text{ and } v \in R_0\text{)},\\ \psi\phi([\alpha \times v]) &= \psi(\alpha \otimes v) = [\alpha v] = [\alpha \times v] && \text{(for } \alpha \in Q_1 \text{ and } v \in R_0\text{)},\\ \psi\phi([u \times \beta]) &= \psi(u \otimes \beta) = [u\beta] = [u \times \beta] && \text{(for } u \in Q_0 \text{ and } \beta \in R_1\text{)}\end{aligned}$$

for cosets of vertices and arrows in $k(Q \times R)/\langle\kappa(Q, R)\rangle$ (these generate the algebra since their representatives generate $k(Q \times R)$). □

We will now try this result out in an example.

Example 2.5. We continue using the quivers

$$Q\colon\ u_1 \xrightarrow{\alpha} u_2$$

$$R\colon\ v_1 \xrightarrow{\beta} v_2 \xrightarrow{\gamma} v_3$$

from Example 2.1. We use Proposition 2.2 to find a quiver and set of relations for the tensor product algebra

$$kQ \otimes_k kR.$$

We remember from Example 2.1 that the product quiver of Q and R is

$$Q \times R\colon\quad \begin{array}{ccccc} u_1 \times v_1 & \xrightarrow{u_1 \times \beta} & u_1 \times v_2 & \xrightarrow{u_1 \times \gamma} & u_1 \times v_3 \\ \downarrow{\scriptstyle \alpha \times v_1} & & \downarrow{\scriptstyle \alpha \times v_2} & & \downarrow{\scriptstyle \alpha \times v_3} \\ u_2 \times v_1 & \xrightarrow{u_2 \times \beta} & u_2 \times v_2 & \xrightarrow{u_2 \times \gamma} & u_2 \times v_3 \end{array}$$

The set of commutativity relations in $Q \times R$ is

$$\kappa(Q, R) = \{\beta\alpha - \alpha\beta, \gamma\alpha - \alpha\gamma\}.$$

By Proposition 2.2, we have

$$kQ \otimes_k kR \cong k(Q \times R)/\langle\kappa(Q, R)\rangle\,,$$

with isomorphisms

$$\phi\colon kQ \otimes_k kR \to k(Q \times R)/\langle \kappa(Q,R) \rangle\,,$$
$$\psi\colon k(Q \times R)/\langle \kappa(Q,R) \rangle \to kQ \otimes_k kR$$

induced by

$$\phi(q \otimes r) = [qr] \qquad \text{for } q \in Q_* \text{ and } r \in R_*,$$
$$\psi([p]) = \pi_1(p) \otimes \pi_2(p) \qquad \text{for } p \in (Q \times R)_*.$$

Let us look at how elements of the tensor product $kQ \otimes_k kR$ are represented in the path algebra quotient $k(Q \times R)/\langle \kappa(Q,R) \rangle$. Assume that k is the field of rationals. Let λ and γ be the elements

$$\lambda = 2\alpha + 3u_2,$$
$$\gamma = 5\gamma\beta + \beta$$

in kQ and kR, respectively. We will find the element of $k(Q \times R)/\langle \kappa(Q,R) \rangle$ which corresponds to $\lambda \otimes \gamma$. We have

$$\begin{aligned}\phi(\lambda \otimes \gamma) &= \phi\big(10(\alpha \otimes \gamma\beta) + 2(\alpha \otimes \beta) + 15(u_2 \otimes \gamma\beta) + 3(u_2 \otimes \beta)\big) \\ &= 10 \cdot \phi(\alpha \otimes \gamma\beta) + 2 \cdot \phi(\alpha \otimes \beta) + 15 \cdot \phi(u_2 \otimes \gamma\beta) + 3 \cdot \phi(u_2 \otimes \beta) \\ &= [10\alpha\gamma\beta + 2\alpha\beta + 15u_2\gamma\beta + 3u_2\beta].\end{aligned}$$

For the opposite direction, consider the element

$$[2\gamma\beta\alpha + \gamma\alpha\beta] \in k(Q \times R)/\langle \kappa(Q,R) \rangle\,.$$

We will see how this can be interpreted as an element of $kQ \otimes_k kR$. We have

$$\begin{aligned}\psi([2\gamma\beta\alpha + \gamma\alpha\beta]) &= \psi(2[\gamma\beta\alpha] + [\gamma\alpha\beta]) \\ &= 2 \cdot \psi([\gamma\beta\alpha]) + \psi([\gamma\alpha\beta]) \\ &= 2(\pi_1(\gamma\beta\alpha) \otimes \pi_2(\gamma\beta\alpha)) + \pi_1(\gamma\alpha\beta) \otimes \pi_2(\gamma\alpha\beta) \\ &= 2(\alpha \otimes \gamma\beta) + \alpha \otimes \gamma\beta \\ &= 3(\alpha \otimes \gamma\beta).\end{aligned}$$

■

2.3 Quotients

Armed with the above nice result for path algebras, let us now turn to the more general case of quotients of path algebras. We will show that the tensor product

$$kQ/\langle \rho \rangle \otimes_k kR/\langle \sigma \rangle$$

is isomorphic to a quotient of the path algebra

$$k(Q \times R),$$

where Q and R are quivers, and $\rho \subseteq kQ$ and $\sigma \subseteq kR$ are sets of relations.

We first show a small diagrammatical result which we will need in our proof.

Lemma 2.3. *Let T be a ring and*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & A_2 & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle j} & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{i} & B & \xrightarrow{p} & C_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} & & \\
0 & \longrightarrow & Z & \longrightarrow & C_2 & \xrightarrow{p'} & D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

a commutative diagram of T-modules with exact rows and columns. Then the sequence

$$0 \longrightarrow \operatorname{im} i + \operatorname{im} j \overset{\iota}{\hookrightarrow} B \xrightarrow{p'q=q'p} D \longrightarrow 0$$

is exact.

Proof. We immediately have that $p'q$ is an epimorphism (it is the composition of two epimorphisms) and ι a monomorphism (it is an inclusion). The composition

$$(p'q)\iota = (q'p)\iota$$

is zero because $p(b) = 0$ for $b \in \operatorname{im} i$ and $q(b) = 0$ for $b \in \operatorname{im} j$.

Now what remains is to show that $\ker p'q \subseteq \operatorname{im} \iota$. This can be done by the following diagram chase. Start with an element $b \in \ker p'q$. Chase it down, then left, then up, to get an element $a_1 \in A_1$ with $qi(a_1) = q(b)$. Chase $b - i(a_1)$ up to an element $a_2 \in A_2$. Then

$$i(a_1) + j(a_2) = i(a_1) + (b - i(a_1)) = b,$$

so $b \in \operatorname{im} i + \operatorname{im} j = \operatorname{im} \iota$. $\square$

Now we will show how the tensor product $kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle$ can be described as a quotient of the path algebra $k(Q \times R)$. In order to define the ideal we want to divide $k(Q \times R)$ by, we need some way of transfering elements of ρ and σ to $k(Q \times R)$. We extend the product quiver inclusion maps

$$\begin{aligned} \mathrm{inc}_1^v &: Q_* \to (Q \times R)_* && \text{for } v \in R_0, \\ \mathrm{inc}_2^u &: R_* \to (Q \times R)_* && \text{for } u \in Q_0 \end{aligned}$$

by linearity, reusing the names, to get k-module homomorphisms

$$\begin{aligned} \mathrm{inc}_1^v &: kQ \to k(Q \times R) && \text{for } v \in R_0, \\ \mathrm{inc}_2^u &: kR \to k(Q \times R) && \text{for } u \in Q_0. \end{aligned}$$

For each relation $\mathfrak{r} \in \rho$, there are $|R_0|$ possible inclusions $\mathrm{inc}_1^v(\mathfrak{r})$ of $\mathfrak{r}$ into $k(Q \times R)$, corresponding to elements $\mathfrak{r} \otimes v$ of the tensor product. But all these are zero in $kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle$, so we can include them all in the set of relations we divide $k(Q \times R)$ by. Similarly, this set of relations should contain all $|Q_0|$ inclusions $\mathrm{inc}_2^u(\mathfrak{s})$ of each relation $\mathfrak{s}$ in σ.

Given subsets $X \subseteq kQ$ and $Y \subseteq kR$, we define the notation

$$\begin{aligned} \mathrm{inc}_1(X) &= \{\, \mathrm{inc}_1^v(x) \mid x \in X, v \in R_0 \,\} \\ \mathrm{inc}_2(Y) &= \{\, \mathrm{inc}_2^u(x) \mid y \in Y, u \in Q_0 \,\} \end{aligned}$$

for the sets consisting of all inclusions of elements of X and Y, respectively, into the product quiver $Q \times R$. Using this notation, the relation sets

$$\rho \subseteq kQ \qquad \text{and} \qquad \sigma \subseteq kR$$

are transferred to

$$\mathrm{inc}_1(\rho) \subseteq k(Q \times R) \qquad \text{and} \qquad \mathrm{inc}_2(\sigma) \subseteq k(Q \times R)$$

in the path algebra over the product quiver.

Furthermore, we must, as for the non-quotient case, divide out by commutativity relations.

With this in mind, we now state and prove the main result of this chapter.

Proposition 2.4. ([3], Lemma 1.3) *Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras. Then*

$$kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle \cong k(Q \times R)/\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q, R)\rangle$$

as k-algebras, with isomorphisms induced by

$$\begin{aligned} [q] \otimes [r] &\mapsto [qr] && \text{for } q \in Q_* \text{ and } r \in R_*, \\ [\pi_1(p)] \otimes [\pi_2(p)] &\leftarrow\!\shortmid [p] && \text{for } p \in (Q \times R)_*. \end{aligned}$$

Proof. We will show the result by first expressing

$$kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle$$

as a quotient of $kQ \otimes_k kR$, then translating to a quotient of $k(Q \times R)$ by using Proposition 2.2.

We have exact sequences

$$0 \longrightarrow \langle\rho\rangle \overset{\iota_1}{\hookrightarrow} kQ \twoheadrightarrow kQ/\langle\rho\rangle \longrightarrow 0$$

and

$$0 \longrightarrow \langle\sigma\rangle \overset{\iota_2}{\hookrightarrow} kR \twoheadrightarrow kR/\langle\sigma\rangle \longrightarrow 0\ .$$

By combining these we can form the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \langle\rho\rangle \otimes_k \langle\sigma\rangle & \longrightarrow & \langle\rho\rangle \otimes_k kR & \longrightarrow & \langle\rho\rangle \otimes_k kR/\langle\sigma\rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \iota_1 \otimes \mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & kQ \otimes_k \langle\sigma\rangle & \xrightarrow{\mathrm{id}\otimes\iota_2} & kQ \otimes_k kR & \longrightarrow & kQ \otimes_k kR/\langle\sigma\rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & kQ/\langle\rho\rangle \otimes_k \langle\sigma\rangle & \longrightarrow & kQ/\langle\rho\rangle \otimes_k kR & \longrightarrow & kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where each map is the tensor product of a map from one of the above sequences with the appropriate identity map. The diagram commutes because

$$(\mathrm{id} \otimes g) \circ (f \otimes \mathrm{id}) = f \otimes g = (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g)$$

for any maps f and g, and identity maps id. Since all k-modules are flat (k is a field), the rows and columns are exact.

By Lemma 2.3 (and by identifying $\langle\rho\rangle \otimes_k kR$ with $\mathrm{im}(\iota_1 \otimes \mathrm{id}_{kR})$ and $kQ \otimes_k \langle\sigma\rangle$ with $\mathrm{im}(\mathrm{id}_{kQ} \otimes \iota_2)$), we get an exact sequence

$$0 \longrightarrow \big(\langle\rho\rangle \otimes_k kR\big) + \big(kQ \otimes_k \langle\sigma\rangle\big) \hookrightarrow kQ \otimes_k kR \xrightarrow{\vartheta} \frac{kQ}{\langle\rho\rangle} \otimes_k \frac{kR}{\langle\sigma\rangle} \longrightarrow 0$$

of k-vector spaces and k-linear maps, where ϑ is given by

$$\vartheta(q \otimes r) = [q] \otimes [r]$$

for paths $q \in Q_*$ and $r \in R_*$. Furthermore, $kQ \otimes_k kR$ and $kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle$ are k-algebras, and ϑ is a k-algebra homomorphism (this is easy to see, since the maps $kQ \twoheadrightarrow kQ/\langle\rho\rangle$ and $kR \twoheadrightarrow kR/\langle\sigma\rangle$ are k-algebra homomorphisms).

Since $kQ \otimes_k kR \cong k(Q \times R)/\langle\kappa(Q, R)\rangle$ (by Proposition 2.2), this effectively gives a k-algebra epimorphism

$$k(Q \times R)/\langle\kappa(Q, R)\rangle \to kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle\,.$$

To express $kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle$ as a quotient of $k(Q \times R)$, we want to find the kernel of this epimorphism. The following diagram illustrates the situation, with ϕ and ψ the isomorphisms from Proposition 2.2.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \big(\langle\rho\rangle \otimes_k kR\big) + \big(kQ \otimes_k \langle\sigma\rangle\big) & \hookrightarrow & kQ \otimes_k kR & \xrightarrow{\vartheta} & \frac{kQ}{\langle\rho\rangle} \otimes_k \frac{kR}{\langle\sigma\rangle} & \longrightarrow & 0 \\
 & & & & {\scriptstyle \phi}\uparrow \cong \downarrow{\scriptstyle \psi} & & \| & & \\
0 & \longrightarrow & \ker(\vartheta\phi) & \hookrightarrow & \frac{k(Q \times R)}{\langle\kappa(Q,R)\rangle} & \xrightarrow{\vartheta\phi} & \frac{kQ}{\langle\rho\rangle} \otimes_k \frac{kR}{\langle\sigma\rangle} & \longrightarrow & 0
\end{array}$$

It is clear that

$$\ker(\vartheta\phi) = \psi(\ker\vartheta) = \psi\big((\langle\rho\rangle \otimes_k kR) + (kQ \otimes_k \langle\sigma\rangle)\big).$$

By applying ψ to a generating set of $\langle\rho\rangle \otimes_k kR + kQ \otimes_k \langle\sigma\rangle$ (as ideal in $kQ \otimes_k kR$) we will get a generating set of $\ker(\vartheta\phi)$ (as ideal in $k(Q \times R)/\langle\kappa(Q,R)\rangle$). Let us therefore look at generating sets for $\langle\rho\rangle \otimes_k kR$ and $kQ \otimes_k \langle\sigma\rangle$.

For subsets $A \subseteq kQ$ and $B \subseteq kR$, denote by $A \otimes B$ the subset

$$\{\, a \otimes b \mid a \in A,\ b \in B \,\}$$

of $kQ \otimes_k kR$. Using this notation, we see that the ideals $\langle\rho\rangle \otimes_k kR$ and $kQ \otimes_k \langle\sigma\rangle$ are generated by the sets

$$\rho \otimes R_0 \qquad \text{and} \qquad Q_0 \otimes \sigma,$$

respectively.

Let $x \otimes v$ be an element of $\rho \otimes R_0$, with $x \in \rho$ and $v \in R_0$. Write x as $\sum_i a_i q_i$ for some $a_i \in k$ and $q_i \in Q_*$. Then

$$\begin{aligned}
\psi(x \otimes v) &= \sum_i a_i \cdot \psi(q_i \otimes v) = \sum_i a_i [q_i v] \\
&= \left[\sum_i a_i\,(q_i v)\right] = \left[\sum_i a_i \operatorname{inc}_1^v(q_i)\right] = \left[\operatorname{inc}_1^v\left(\sum_i a_i q_i\right)\right] \\
&= \big[\operatorname{inc}_1^v(x)\big].
\end{aligned}$$

Thus $\psi(\rho \otimes R_0)$ consists of all elements in $k(Q \times R)/\langle\kappa(Q,R)\rangle$ of the form $[\operatorname{inc}_1^v(x)]$ for $x \in \rho$ and $v \in R_0$. These elements are the cosets modulo $\langle\kappa(Q,R)\rangle$ of the elements of $\operatorname{inc}_1(\rho)$, so they generate the ideal

$$\frac{\langle \operatorname{inc}_1(\rho)\rangle + \langle\kappa(Q,R)\rangle}{\langle\kappa(Q,R)\rangle}$$

of $k(Q \times R)/\langle\kappa(Q,R)\rangle$.

Similarly,

$$\psi(Q_0 \otimes \sigma) = \left\{\ \big[\operatorname{inc}_2^u(y)\big]\ \middle|\ u \in Q_0,\ y \in \sigma\ \right\},$$

which is a generating set for

$$\frac{\langle \operatorname{inc}_2(\sigma)\rangle + \langle\kappa(Q,R)\rangle}{\langle\kappa(Q,R)\rangle}.$$

Thus we have

$$\begin{aligned}
\psi\big(\,\langle\rho\rangle \otimes_k kR + kQ \otimes_k \langle\sigma\rangle\,\big) &= \psi\big(\,\langle\rho \otimes R_0\rangle + \langle Q_0 \otimes \sigma\rangle\,\big) \\
&= \big\langle \psi(\rho \otimes R_0)\big\rangle + \big\langle \psi(Q_0 \otimes \sigma)\big\rangle \\
&= \frac{\langle \operatorname{inc}_1(\rho)\rangle + \langle\kappa(Q,R)\rangle}{\langle\kappa(Q,R)\rangle} + \frac{\langle \operatorname{inc}_2(\sigma)\rangle + \langle\kappa(Q,R)\rangle}{\langle\kappa(Q,R)\rangle}
\end{aligned}$$

$$= \frac{\langle \mathrm{inc}_1(\rho)\rangle + \langle \mathrm{inc}_2(\sigma)\rangle + \langle \kappa(Q,R)\rangle}{\langle \kappa(Q,R)\rangle}$$
$$= \frac{\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q,R)\rangle}{\langle \kappa(Q,R)\rangle}.$$

This means that

$$\ker(\vartheta\phi) = \frac{\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q,R)\rangle}{\langle \kappa(Q,R)\rangle},$$

so $\vartheta\phi$ induces an isomorphism

$$\theta\colon \frac{k(Q\times R)/\langle \kappa(Q,R)\rangle}{\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q,R)\rangle / \langle \kappa(Q,R)\rangle} \xrightarrow{\cong} kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle$$

given by

$$\theta([[x]]) = \vartheta\phi([x])$$

for an element $x \in k(Q\times R)$. The action of θ on generators (cosets of paths) is

$$\theta([[p]]) = \vartheta\phi([p]) = \vartheta(\pi_1(p) \otimes \pi_2(p)) = [\pi_1(p)] \otimes [\pi_2(p)]$$

for a path $p \in (Q\times R)_*$. The inverse θ^{-1} is given on generators by

$$\theta^{-1}([q] \otimes [r]) = [[qr]]$$

for paths $q \in Q_*$ and $r \in R_*$, since $\pi_1(qr) = q$ and $\pi_2(qr) = r$.

By the Third Isomorphism Theorem (for algebras),

$$\frac{k(Q\times R)/\langle \kappa(Q,R)\rangle}{\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q,R)\rangle / \langle \kappa(Q,R)\rangle} \cong \frac{k(Q\times R)}{\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q,R)\rangle}$$

by the correspondence

$$[[x]] \leftrightarrow [x].$$

So we have

$$kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle \cong \frac{k(Q\times R)/\langle \kappa(Q,R)\rangle}{\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q,R)\rangle / \langle \kappa(Q,R)\rangle},$$

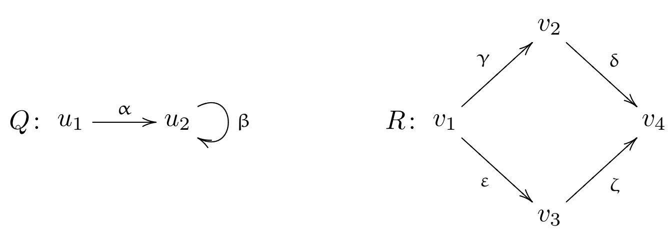
with isomorphisms induced by

$$[q] \otimes [r] \mapsto [qr] \qquad \text{for } q \in Q_* \text{ and } r \in R_*,$$
$$[\pi_1(p)] \otimes [\pi_2(p)] \leftarrow\!\shortmid [p] \qquad \text{for } p \in (Q\times R)_*.$$

□

We will now use this proposition in an example.

Example 2.6. Let Q and R be the quivers



Let $\rho \subseteq kQ$ and $\sigma \subseteq kR$ be the following sets of relations:

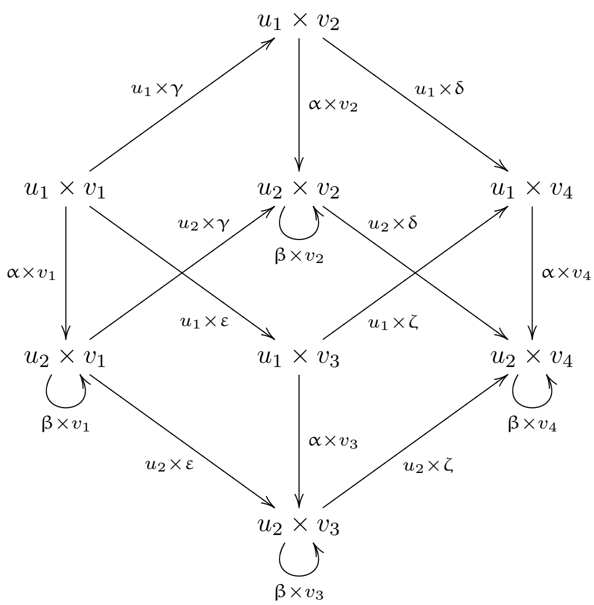
$$\begin{aligned}\rho &= \{\beta^2\},\\ \sigma &= \{\delta\gamma - \zeta\varepsilon\}.\end{aligned}$$

We will use Proposition 2.4 to find a quotient of a path algebra which is isomorphic to the tensor product

$$kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle\,.$$

The product quiver $Q \times R$ is

$Q \times R$:



The relations we get by including ρ and σ into $Q \times R$ are

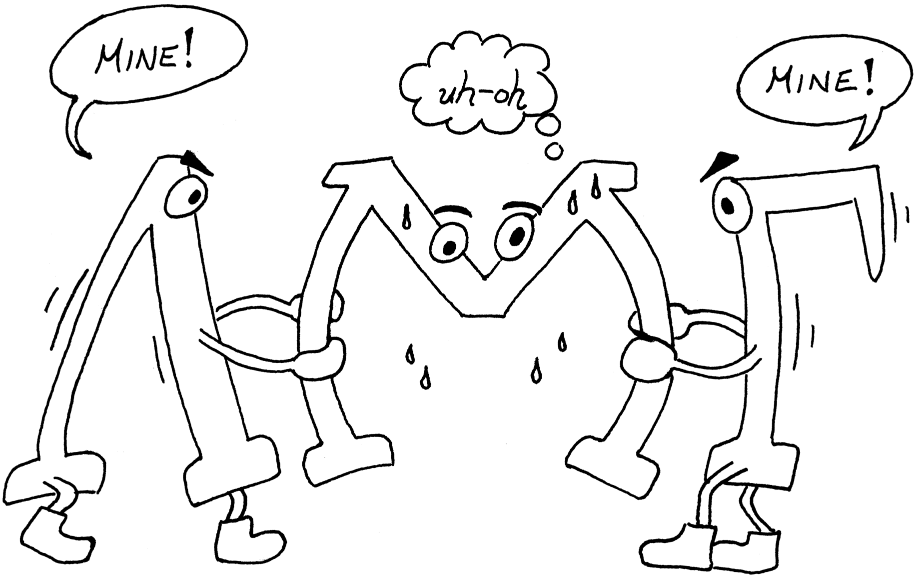
$$\begin{aligned}\mathrm{inc}_1(\rho) &= \{\beta^2 v_1, \beta^2 v_2, \beta^2 v_3, \beta^2 v_4\},\\ \mathrm{inc}_2(\sigma) &= \{\delta\gamma u_1 - \zeta\varepsilon u_1, \delta\gamma u_2 - \zeta\varepsilon u_2\}.\end{aligned}$$

The set $\kappa(Q, R)$ of commutativity relations is

$$\begin{aligned}\kappa(Q,R) = \{&\alpha\gamma - \gamma\alpha, \alpha\delta - \delta\alpha, \alpha\varepsilon - \varepsilon\alpha, \alpha\zeta - \zeta\alpha,\\ &\beta\gamma - \gamma\beta, \beta\delta - \delta\beta, \beta\varepsilon - \varepsilon\beta, \beta\zeta - \zeta\beta\}.\end{aligned}$$

By Proposition 2.4, we have

$$kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle \cong k(Q \times R)/\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma) \cup \kappa(Q, R)\rangle\,. \qquad \blacksquare$$


MINE!
uh-oh
MINE!

Chapter 3

Bimodules

In this chapter, we show how bimodules over quotients of path algebras can be described by representations over quivers. The first two sections of this chapter give bimodule versions of the definitions and results of Section 1.2. Then Section 3.3 shows how we can turn a representation for a bimodule ${}_{\Lambda}M_{\Gamma}$ into representations for the one-sided modules ${}_{\Lambda}M$ and M_{Γ}. Finally, Section 3.4 shows how to find a representation for the Λ–Λ-bimodule structure of an algebra Λ.

3.1 Representations for bimodules

If Λ and Γ are k-algebras, then a Λ–Γ-bimodule is the same as a left $\Lambda \otimes_k \Gamma^{\mathrm{op}}$-module. If M is such a module, then the correspondence between these two module structures is given by

$$\lambda * m * \gamma = (\lambda \otimes \gamma^{\mathrm{op}}) \circledast m \tag{3.1}$$

for elements $\lambda \in \Lambda$, $\gamma \in \Gamma$ and $m \in M$, where $*$ denotes the multiplication of M as Λ–Γ-bimodule, and $\circledast$ the multiplication of M as $\Lambda \otimes_k \Gamma^{\mathrm{op}}$-module.

For (quotients of) path algebras, this means that a bimodule can be described by a representation over the product quiver. Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras. Then, by the above correspondence and Proposition 2.4, a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule can be described by a representation over $Q \times R^{\mathrm{op}}$ respecting the relation set

$$\mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma^{\mathrm{op}}) \cup \kappa(Q, R^{\mathrm{op}}).$$

In this section, we will find out how we can turn such a representation into a bimodule, and vice versa. We first consider an example of a representation over a product quiver, to see how the induced bimodule structure behaves.

Example 3.1. Let Q and R be the quivers

$$Q\colon\ u_1 \xrightarrow{\alpha} u_2\ ,$$

$$R\colon\ v_1 \xrightarrow{\beta} v_2 \xrightarrow{\gamma} v_3\ .$$

By Proposition 2.2, we have

$$kQ \otimes_k kR^{\mathrm{op}} \cong k(Q \times R^{\mathrm{op}})/\langle \kappa(Q, R^{\mathrm{op}})\rangle \tag{3.2}$$

with isomorphisms induced by

$$\begin{aligned} q \otimes r^{\mathrm{op}} &\mapsto [qr^{\mathrm{op}}] && \text{for } q \in Q_* \text{ and } r \in R_*, \\ \pi_1(p) \otimes \pi_2(p) &\leftarrow\!\shortmid [p] && \text{for } p \in (Q \times R^{\mathrm{op}})_*. \end{aligned}$$

Here $Q \times R^{\mathrm{op}}$ is the product quiver

$$Q \times R^{\mathrm{op}}\colon \quad \begin{array}{ccccc} u_1 \times v_1 & \xleftarrow{u_1 \times \beta^{\mathrm{op}}} & u_1 \times v_2 & \xleftarrow{u_1 \times \gamma^{\mathrm{op}}} & u_1 \times v_3 \\ \downarrow{\scriptstyle \alpha \times v_1} & & \downarrow{\scriptstyle \alpha \times v_2} & & \downarrow{\scriptstyle \alpha \times v_3} \\ u_2 \times v_1 & \xleftarrow{u_2 \times \beta^{\mathrm{op}}} & u_2 \times v_2 & \xleftarrow{u_2 \times \gamma^{\mathrm{op}}} & u_2 \times v_3 \end{array}$$

and $\kappa(Q, R^{\mathrm{op}})$ is the commutativity relation set

$$\kappa(Q, R^{\mathrm{op}}) = \{\alpha\beta^{\mathrm{op}} - \beta^{\mathrm{op}}\alpha,\ \alpha\gamma^{\mathrm{op}} - \gamma^{\mathrm{op}}\alpha\}.$$

Let (V, f) be the following representation over $Q \times R^{\mathrm{op}}$:

$$(V, f)\colon \quad \begin{array}{ccccc} k^2 & \xleftarrow{\binom{1}{1}} & k & \xleftarrow{1} & k \\ {\scriptstyle (1\ \ -1)}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ k & \longleftarrow & 0 & \longleftarrow & k \end{array}$$

Since both squares in the representation commute, the representation respects the relation set $\kappa(Q, R^{\mathrm{op}})$. Let M be the corresponding $k(Q \times R^{\mathrm{op}})/\langle \kappa(Q, R^{\mathrm{op}})\rangle$-module.

We will see how this induces a kQ–kR-bimodule structure on M, by first turning M into a left $kQ \otimes_k kR^{\mathrm{op}}$-module by the isomorphism of Equation (3.2), and then turning it into a kQ–kR-bimodule by Equation (3.1).

To distinguish between the different module structures, we use the symbols $*$ and $\circledast$ for the scalar multiplications of M as left $k(Q \times R^{\mathrm{op}})/\langle \kappa(Q, R^{\mathrm{op}})\rangle$-module and left $kQ \otimes_k kR^{\mathrm{op}}$-module, respectively. We thus write:

$$\begin{aligned} \chi * m \qquad & \text{for } \chi \in k(Q \times R^{\mathrm{op}})/\langle \kappa(Q, R^{\mathrm{op}})\rangle \text{ and } m \in M, \\ (\lambda \otimes \gamma^{\mathrm{op}}) \circledast m \qquad & \text{for } \lambda \in kQ,\ \gamma \in kR \text{ and } m \in M. \end{aligned}$$

By the correspondence of Equation 3.1, we have

$$\lambda m \gamma = (\lambda \otimes \gamma^{\mathrm{op}}) \circledast m$$

for an element $m \in M$, and algebra elements $\lambda \in kQ$ and $\gamma \in kR$. We can express left multiplication with a path $q \in Q_*$ by

$$\begin{aligned} qm = q \cdot m \cdot 1_{kR} &= (q \otimes 1_{kR^{\mathrm{op}}}) \circledast m \\ &= (q \otimes (v_1 + v_2 + v_3)) \circledast m \\ &= (q \otimes v_1) \circledast m + (q \otimes v_2) \circledast m + (q \otimes v_3) \circledast m \\ &= (qv_1) * m + (qv_2) * m + (qv_3) * m. \end{aligned}$$

Similarly, right multiplication with a path $r \in R_*$ is expressed by

$$\begin{aligned} mr &= 1_{kQ} \cdot m \cdot r = (1_{kQ} \otimes r^{\mathrm{op}}) \circledast m \\ &= ((u_1 + u_2) \otimes r^{\mathrm{op}}) \circledast m \\ &= (u_1 \otimes r^{\mathrm{op}}) \circledast m + (u_2 \otimes r^{\mathrm{op}}) \circledast m \\ &= (u_1 r^{\mathrm{op}}) * m + (u_2 r^{\mathrm{op}}) * m. \end{aligned}$$

Let B be the left uniform basis for M (as $k(Q \times R^{\mathrm{op}})$-module) given by the following diagram:

$$B\colon \quad \begin{array}{ccccc} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} & \longleftarrow & \mathbf{c} & \longleftarrow & \mathbf{d} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{e} & \longleftarrow & 0 & \longleftarrow & \mathbf{f} \end{array}$$

We compute all products of the basis element $\mathbf{d}$ with paths of Q on the left and paths of R on the right:

$$\begin{aligned} u_1\mathbf{d} &= (u_1v_1) * \mathbf{d} + (u_1v_2) * \mathbf{d} + (u_1v_3) * \mathbf{d} = 0 + 0 + \mathbf{d} = \mathbf{d} \\ u_2\mathbf{d} &= (u_2v_1) * \mathbf{d} + (u_2v_2) * \mathbf{d} + (u_2v_3) * \mathbf{d} = 0 + 0 + 0 = 0 \\ \alpha\mathbf{d} &= (\alpha v_1) * \mathbf{d} + (\alpha v_2) * \mathbf{d} + (\alpha v_3) * \mathbf{d} = 0 + 0 + \mathbf{f} = \mathbf{f} \\ \mathbf{d}v_1 &= (u_1v_1) * \mathbf{d} + (u_2v_1) * \mathbf{d} = 0 + 0 = 0 \\ \mathbf{d}v_2 &= (u_1v_2) * \mathbf{d} + (u_2v_2) * \mathbf{d} = 0 + 0 = 0 \\ \mathbf{d}v_3 &= (u_1v_3) * \mathbf{d} + (u_2v_3) * \mathbf{d} = \mathbf{d} + 0 = \mathbf{d} \\ \mathbf{d}\beta &= (u_1\beta^{\mathrm{op}}) * \mathbf{d} + (u_2\beta^{\mathrm{op}}) * \mathbf{d} = 0 + 0 = 0 \\ \mathbf{d}\gamma &= (u_1\gamma^{\mathrm{op}}) * \mathbf{d} + (u_2\gamma^{\mathrm{op}}) * \mathbf{d} = \mathbf{c} + 0 = \mathbf{c} \\ \mathbf{d}(\gamma\beta) &= (u_1\beta^{\mathrm{op}}\gamma^{\mathrm{op}}) * \mathbf{d} + (u_2\beta^{\mathrm{op}}\gamma^{\mathrm{op}}) * \mathbf{d} = (\mathbf{a} + \mathbf{b}) + 0 = \mathbf{a} + \mathbf{b} \end{aligned}$$

There are a few observation to make here. The basis element $\mathbf{d}$ lies in the vector space $V_{u_1 \times v_3}$ of the representation. We see that in order for a product $q\mathbf{d}$ to be nonzero, the path q must have u_1 as its tail, and in order for a product $\mathbf{d}r$ to be nonzero, the path r must have v_3 as its head. Furthermore, the vertices u_1 and v_3 act as a left and right identity, respectively, on d. When it comes to multiplication by arrows, we see that left multiplication of $\mathbf{d}$ by α is given by the map $f_{\alpha \times v_3}$ of the representation, while right multiplication of $\mathbf{d}$ by γ is given by the map $f_{u_1 \times \gamma^{\mathrm{op}}}$ of the representation. ■

The moral of the above example is that when we represent a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule by a representation over $Q \times R^{\mathrm{op}}$, the linear maps in the representation corresponding to arrows of the form

$$\alpha \times v \qquad \text{for } \alpha \in Q_1 \text{ and } v \in R_0$$

describe left multiplication by α, while the maps corresponding to arrows of the form

$$u \times \beta^{\mathrm{op}} \qquad \text{for } u \in Q_0 \text{ and } \beta \in R_1$$

describe right multiplication by β.

We will now aim to give a precise description of the correspondence between bimodules and representations over product quivers. We will first describe how bimodules correspond to *modules* over product quivers. Then it will be easy to see (although somewhat tedious to describe precisely) how they correspond to representations.

The following proposition gives the relationship between left multiplication by paths of the product quiver and simultaneous left and right multiplication by paths in a bimodule.

Proposition 3.1. *Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras. Then any $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule is also a left module over*

$$k(Q \times R^{\mathrm{op}})/\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma^{\mathrm{op}}) \cup \kappa(Q, R^{\mathrm{op}})\rangle\,,$$

and vice versa. Let M be such a module, and denote its scalar multiplication as bimodule by $*$ and its scalar multiplication as left module by $\circledast$. Then the correspondence between these two module structures is given by

$$q * m * r = [qr^{\mathrm{op}}] \circledast m$$

for a module element $m \in M$ and paths $q \in Q_*$ and $r \in R_*$.

Proof. We know that $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodules correspond to left modules over

$$kQ/\langle\rho\rangle \otimes_k kR^{\mathrm{op}}/\langle\sigma^{\mathrm{op}}\rangle$$

by Equation (3.1), and these again correspond to left modules over

$$k(Q \times R^{\mathrm{op}})/\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma^{\mathrm{op}}) \cup \kappa(Q, R^{\mathrm{op}})\rangle$$

by the isomorphism of Proposition 2.4. Following these two correspondences, we get

$$q * m * r = (q \otimes r^{\mathrm{op}}) \bullet m = [qr^{\mathrm{op}}] \circledast m$$

for a module element $m \in M$ and paths $q \in Q_*$ and $r \in R_*$, where $\bullet$ denotes the scalar multiplication of M as left module over

$$kQ/\langle\rho\rangle \otimes_k kR^{\mathrm{op}}/\langle\sigma^{\mathrm{op}}\rangle\,. \qquad \square$$

When using the correspondence of the above proposition to convert a left module over the product quiver into a bimodule, it is convenient to know the result of one-sided multiplication of a module element by a path. The following proposition tells us how to find this.

Proposition 3.2. *Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras, and let M be a left module over*

$$k(Q \times R^{\mathrm{op}})/\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma^{\mathrm{op}}) \cup \kappa(Q, R^{\mathrm{op}})\rangle\,,$$

with scalar multiplication denoted by $\circledast$. *Then, in the induced* $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-*bimodule structure on* M, *we have*

$$qm = \sum_{v \in R_0} (qv) \circledast m \qquad \text{for a path } q \in Q_* \text{ and an element } m \in M,$$
$$mr = \sum_{u \in Q_0} (ur^{\mathrm{op}}) \circledast m \qquad \text{for a path } r \in R_* \text{ and an element } m \in M.$$

Proof. Using Proposition 3.1, we compute

$$\begin{aligned} qm = q \cdot m \cdot 1_{kR/\langle\sigma\rangle} &= q \cdot m \cdot \Big(\sum_{v \in R_0} [v] \Big) \\ &= \sum_{v \in R_0} qmv = \sum_{v \in R_0} (qv) \circledast m \end{aligned}$$

for a module element $m \in M$ and a path $q \in Q_*$. An analogous computation gives

$$mr = \sum_{u \in Q_0} (ur^{\mathrm{op}}) \circledast m$$

for a module element $m \in M$ and a path $r \in R_*$. □

Now we can create an explicit description of the correspondence between bimodules and product quiver representations by using the two above propositions, together with the usual correspondence between modules and representations for the algebra

$$k(Q \times R^{\mathrm{op}})/\langle \mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma^{\mathrm{op}}) \cup \kappa(Q, R^{\mathrm{op}}) \rangle \,.$$

The following two propositions do precisely this. First we show how to go from a bimodule to a representation.

Proposition 3.3. *Let* $kQ/\langle\rho\rangle$ *and* $kR/\langle\sigma\rangle$ *be quotients of path algebras, and let* M *be a* $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$*-bimodule. Then the representation over* $Q \times R^{\mathrm{op}}$ *corresponding to* M *is* (V, f), *where the vector spaces are*

$$V_{u \times v} = uMv$$

for each pair of vertices $u \in Q_0$ *and* $v \in R_0$, *and the maps are given by*

$$\begin{aligned} f_{\alpha \times v} \colon V_{\mathfrak{t}(\alpha) \times v} &\to V_{\mathfrak{h}(\alpha) \times v} \\ x &\mapsto \alpha x \end{aligned}$$

for an arrow $\alpha \in Q_1$ *and a vertex* $v \in R_0$, *and*

$$\begin{aligned} f_{u \times \beta^{\mathrm{op}}} \colon V_{u \times \mathfrak{h}(\beta)} &\to V_{u \times \mathfrak{t}(\beta)} \\ x &\mapsto x\beta \end{aligned}$$

for a vertex $u \in Q_0$ *and an arrow* $\beta \in R_1$.

Proof. We use Proposition 3.1 and the correspondence between modules and representations described in Section 1.2.

To distinguish the module structures on M, we denote the scalar multiplication of M as left $k(Q \times R^{\mathrm{op}})$-module by $*$.

For a pair of vertices $u \times v \in Q_0 \times R_0$, we get that the vector space $V_{u\times v}$ at the vertex $u \times v$ of the product quiver is

$$V_{u\times v} = (u \times v) * M = uMv,$$

by first using Equation (1.1), and then Proposition 3.1.

We will now describe the linear map

$$f_{\alpha\times v} \colon V_{\mathfrak{t}(\alpha)\times v} \to V_{\mathfrak{h}(\alpha)\times v}$$

for an arrow $\alpha \times v$ of the product quiver, where α is an arrow in Q and v is a vertex in R. For an element $m \in V_{\mathfrak{t}(\alpha)\times v}$, we have

$$f_{\alpha\times v}(m) = (\alpha \times v) * m = \alpha m v = \alpha m,$$

by first using Equation (1.2), then Proposition 3.1, and finally the fact that $mv = m$, since m is an element of $V_{\mathfrak{t}(\alpha)\times v}$.

Finding the linear map $f_{u\times\beta^{\mathrm{op}}}$ for a vertex $u \in Q_0$ and an arrow $\beta \in R_1$ can be done in a similar way. ☐

Now we will show how to go from a representation to a bimodule.

Proposition 3.4. *Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras, and let (V, f) be a representation over $Q \times R^{\mathrm{op}}$ respecting the relations*

$$\mathrm{inc}_1(\rho) \cup \mathrm{inc}_2(\sigma^{\mathrm{op}}) \cup \kappa(Q, R^{\mathrm{op}}).$$

Then the corresponding $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule M is

$$M = \bigoplus_{\substack{u\in Q_0\\ v\in R_0}} V_{u\times v}$$

as k-vector space. For an element $m \in M$ and a pair of vertices $u \times v \in Q_0 \times R_0$, denote by $m_{u\times v}$ the component of m belonging to the vector space $V_{u\times v}$.

The scalar multiplications of elements of m by algebra elements on the left and right are defined as follows. Let m be an element of M. For vertices $u' \in Q_0$ and $v' \in R_0$, the products $u'm$ and mv' are given componentwise by

$$(u'm)_{u\times v} = \begin{cases} m_{u\times v} & \text{if } u = u' \\ 0 & \text{otherwise} \end{cases}$$

$$(mv')_{u\times v} = \begin{cases} m_{u\times v} & \text{if } v = v' \\ 0 & \text{otherwise} \end{cases}$$

for every pair of vertices $u \times v \in Q_0 \times R_0$. For arrows $\alpha \in Q_1$ and $\beta \in R_1$, the products αm and $m\beta$ are given componentwise by

$$(\alpha m)_{u\times v} = \begin{cases} f_{\alpha\times v}(m_{\mathfrak{t}(\alpha)\times v}) & \textit{if } u = \mathfrak{h}(\alpha) \\ 0 & \textit{otherwise} \end{cases}$$

$$(m\beta)_{u\times v} = \begin{cases} f_{u\times \beta^{\mathrm{op}}}(m_{u\times \mathfrak{h}(\beta)}) & \textit{if } v = \mathfrak{t}(\beta) \\ 0 & \textit{otherwise} \end{cases}$$

for every pair of vertices $u \times v \in Q_0 \times R_0$. For arbitrary algebra elements $\lambda \in kQ/\langle\rho\rangle$ and $\gamma \in kR/\langle\sigma\rangle$, the products λm and $m\gamma$ are defined by extending from the above definitions for vertices and arrows using the module axioms.

Proof. We use Proposition 3.2 and the correspondence between modules and representations described in Section 1.2.

To distinguish the module structures on M, we denote the scalar multiplication of M as left $k(Q \times R^{\mathrm{op}})$-module by $*$.

Equation (1.3) gives the description of M as vector space:

$$M = \bigoplus_{\substack{u\in Q_0 \\ v\in R_0}} V_{u\times v}.$$

We need to find the result of multiplying a module element with a vertex or arrow on either side.

We first compute the product $u'm$ of a vertex $u' \in Q_0$ and a module element $m \in M$. By Proposition 3.2, we have

$$u'm = \sum_{v\in R_0} (u' \times v) * m.$$

For a pair of vertices $u \times v \in Q_0 \times R_0$, the $(u \times v)$-component of $u'm$ is

$$\begin{aligned}(u'm)_{u\times v} &= \Big(\sum_{v'\in R_0} (u' \times v') * m\Big)_{u\times v} = \sum_{v'\in R_0} \big((u' \times v') * m\big)_{u\times v} \\ &= \begin{cases} m_{u\times v} & \text{if } u = u', \\ 0 & \text{otherwise;} \end{cases}\end{aligned}$$

where the last equation follows from using Equation (1.4) on $(u' \times v') * m$:

$$\big((u' \times v') * m\big)_{u\times v} = \begin{cases} m_{u\times v} & \text{if } u \times v = u' \times v', \\ 0 & \text{otherwise.} \end{cases}$$

The computation for the product mv' of a module element $m \in M$ and a vertex $v' \in R_0$ is similar.

We now compute the product αm of an arrow $\alpha \in Q_1$ and a module element $m \in M$. By Proposition 3.2, we have

$$\alpha m = \sum_{v\in R_0} (\alpha \times v) * m.$$

For a pair of vertices $u \times v \in Q_0 \times R_0$, the $(u \times v)$-component of αm is

$$(\alpha m)_{u\times v} = \Big(\sum_{v'\in R_0} (\alpha \times v') * m\Big)_{u\times v} = \sum_{v'\in R_0} \big((\alpha \times v') * m\big)_{u\times v}$$
$$= \begin{cases} f_{\alpha\times v}(m_{\mathfrak{t}(\alpha)\times v}) & \text{if } u = \mathfrak{h}(\alpha), \\ 0 & \text{otherwise;} \end{cases}$$

where the last equation follows from using Equation (1.5) on $(\alpha \times v') * m$:

$$\big((\alpha \times v') * m\big)_{u\times v} = \begin{cases} f_{\alpha\times v'}(m_{\mathfrak{t}(\alpha)\times v}) & \text{if } u \times v = \mathfrak{h}(\alpha) \times v', \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we compute the product $m\beta$ of a module element $m \in M$ and an arrow $\beta \in R_1$. This is similar to the above computation for αm, except that we need to change to β^{op} when translating to the left module structure. By Proposition 3.2, we have

$$m\beta = \sum_{u\in Q_0} (u \times \beta^{\mathrm{op}}) * m.$$

For a pair of vertices $u \times v \in Q_0 \times R_0$, the $(u \times v)$-component of $m\beta$ is

$$(m\beta)_{u\times v} = \Big(\sum_{u'\in Q_0} (u' \times \beta^{\mathrm{op}}) * m\Big)_{u\times v} = \sum_{u'\in Q_0} \big((u' \times \beta^{\mathrm{op}}) * m\big)_{u\times v}$$
$$= \begin{cases} f_{u\times\beta^{\mathrm{op}}}(m_{u\times\mathfrak{h}(\beta)}) & \text{if } u = \mathfrak{t}(\beta), \\ 0 & \text{otherwise;} \end{cases}$$

where the last equation follows from using Equation (1.5) on $(u' \times \beta^{\mathrm{op}}) * m$:

$$\big((u' \times \beta^{\mathrm{op}}) * m\big)_{u\times v} = \begin{cases} f_{u'\times\beta^{\mathrm{op}}}(m_{u\times\mathfrak{t}(\beta^{\mathrm{op}})}) & \text{if } u \times v = u' \times \mathfrak{h}(\beta^{\mathrm{op}}), \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Now that we have established our desired correspondence, we will identify bimodules over quotients of path algebras with representations over the product quiver respecting the necessary relations, just as we identify left modules over a quotient of a path algebra with representations over its quiver.

3.2 Uniform elements and bases

For one-sided modules over path algebras, we have the notions of left uniform elements and left uniform bases. We will now define similar concepts for bimodules. Since these involve multiplication on both sides, we use the term "uniform" instead of "left uniform". We also extend the notion of "heads" and "tails" to uniform bimodule elements.

Definition. Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras, M a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule and m an element of M. Given vertices $u \in Q_0$ and $v \in R_0$, we say that m is **(u, v)-uniform** if

$$umv = m.$$

The element m is **uniform** if it is (u, v)-uniform for some pair of vertices (u, v).

If m is (u, v)-uniform, we define the **head** and **tail** of m by $\mathfrak{h}(m) = u$ and $\mathfrak{t}(m) = v$, respectively. ■

♩ An element m of a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule M is (u, v)-uniform if and only if it is left $(u \times v)$-uniform when M is viewed as a left $k(Q \times R^{\mathrm{op}})$-module. ■

♩ If kQ is a path algebra and $I \subseteq kQ$ an ideal, then we have two different definitions of uniformity for elements of I. We can either use the definition on page 14, viewing the elements of I as elements of the path algebra, or we can use the definition above, viewing I as a kQ-bimodule. Fortunately, these two definitions coincide. ■

Definition. Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras, M a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule and B a k-basis for M. Then B is a **uniform basis** if every element of B is uniform. ■

♩ If B is a basis for a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule M, then B is uniform if and only if it is a left uniform basis for M as left $k(Q \times R^{\mathrm{op}})$-module. ■

We have already (in Example 1.1) introduced a diagrammatical notation for describing a left uniform basis. We can use the same notation to describe a uniform basis, as shown in the following example.

Example 3.2. Consider the kQ–kR-bimodule M from Example 3.1, given by the representation

$$\begin{array}{ccccc} k^2 & \xleftarrow{\binom{1}{1}} & k & \xleftarrow{1} & k \\ {\scriptstyle (1\ \ -1)}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ k & \longleftarrow & 0 & \longleftarrow & k \end{array}$$

over the product quiver

$$Q \times R^{\mathrm{op}}\colon \quad \begin{array}{ccccc} u_1 \times v_1 & \xleftarrow{u_1 \times \beta^{\mathrm{op}}} & u_1 \times v_2 & \xleftarrow{u_1 \times \gamma^{\mathrm{op}}} & u_1 \times v_3 \\ \downarrow{\scriptstyle \alpha \times v_1} & & \downarrow{\scriptstyle \alpha \times v_2} & & \downarrow{\scriptstyle \alpha \times v_3} \\ u_2 \times v_1 & \xleftarrow{u_2 \times \beta^{\mathrm{op}}} & u_2 \times v_2 & \xleftarrow{u_2 \times \gamma^{\mathrm{op}}} & u_2 \times v_3 \end{array}$$

In Example 3.1, we had a left uniform basis B for M as left $k(Q \times R)$-module. By the note above, B is then a uniform basis for M as kQ–kR-bimodule. The basis B was given by the following diagram:

$$B\colon \quad \begin{array}{ccccc} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} & \longleftarrow & \mathbf{c} & \longleftarrow & \mathbf{d} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{e} & \longleftarrow & 0 & \longleftarrow & \mathbf{f} \end{array}$$

We will from now on use such diagrams to describe uniform bases of bimodules.

In a diagram like this, the (u, v)-uniform elements of the basis are placed at vertex $u \times v$, for each pair of vertices $u \in Q_0$ and $v \in R_0$. In our basis B, we see that $\mathbf{a}$ and $\mathbf{b}$ are (u_1, v_1)-uniform, $\mathbf{c}$ is (u_1, v_2)-uniform, and so on. Thus, the heads and tails of the basis elements are:

$$\begin{aligned} \mathfrak{h}(\mathbf{a}) &= u_1 & \mathfrak{t}(\mathbf{a}) &= v_1 \\ \mathfrak{h}(\mathbf{b}) &= u_1 & \mathfrak{t}(\mathbf{b}) &= v_1 \\ \mathfrak{h}(\mathbf{c}) &= u_1 & \mathfrak{t}(\mathbf{c}) &= v_2 \\ \mathfrak{h}(\mathbf{d}) &= u_1 & \mathfrak{t}(\mathbf{d}) &= v_3 \\ \mathfrak{h}(\mathbf{e}) &= u_2 & \mathfrak{t}(\mathbf{e}) &= v_1 \\ \mathfrak{h}(\mathbf{f}) &= u_2 & \mathfrak{t}(\mathbf{f}) &= v_3 \end{aligned}$$

■

The fact that we define heads and tails of uniform bimodule elements indicates that we want to think of these elements as being similar to paths. The motivation behind this becomes clear in Chapter 4, where we will make arrows that represent uniform basis elements of a bimodule. Then we will use the head and tail of each basis element as head and tail of the arrow we construct to represent it.

3.3 Forgetting module structure

Given a kQ–kR-bimodule M, we can forget one of its module structures to obtain a left kQ-module or a right kR-module. We shall now show how to find representations over Q and R^{op} for these module structures, given a representation for M over $Q \times R^{\text{op}}$.

It turns out that there is a very simple method for doing this: If we view the product quiver $Q \times R^{\text{op}}$ as a rectangular grid where each column is a copy of Q and each row a copy of R^{op}, we can find the representation for M over Q or R^{op} simply by summing along the rows or columns. The following proposition makes this precise.

Proposition 3.5. *Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras. Let M be a $kQ/\langle\rho\rangle \otimes_k kR/\langle\sigma\rangle$-module given by a representation $(V^\times, f^\times)$ over $Q \times R^{\text{op}}$. Define the representation (V^Q, f^Q) over Q where the vector spaces are*

$$V_u^Q = \bigoplus_{v \in R_0} V_{u \times v}^\times$$

for vertices $u \in Q_0$, and the maps are

$$\begin{aligned} f_\alpha^Q \colon V_{\mathfrak{t}(\alpha)}^Q &\to V_{\mathfrak{h}(\alpha)}^Q \\ (x_v)_{v \in R_0} &\mapsto (f_{\alpha \times v}^\times(x_v))_{v \in R_0} \end{aligned}$$

for arrows $\alpha \in Q_1$. Similarly, define the representation $(V^{R^{\text{op}}}, f^{R^{\text{op}}})$ over R^{op} where the vector spaces are

$$V_u^{R^{\text{op}}} = \bigoplus_{u \in Q_0} V_{u \times v}^\times$$

for each vertex $u \in R_0$, and the maps are

$$f^{R^{\mathrm{op}}}_{\beta^{\mathrm{op}}} : V^{R^{\mathrm{op}}}_{\mathfrak{h}(\beta)} \to V^{R^{\mathrm{op}}}_{\mathfrak{t}(\beta)}$$
$$(x_u)_{u\in Q_0} \mapsto (f^{\times}_{u\times\beta^{\mathrm{op}}}(x_u))_{u\in Q_0}$$

for each arrow $\beta \in R_1$.

Then (V^Q, f^Q) is a representation for M as left kQ-module, and $(V^{R^{\mathrm{op}}}, f^{R^{\mathrm{op}}})$ is a representation for M as right kR-module.

Proof. We show that (V^Q, f^Q) is a representation for M as left kQ-module. That $(V^{R^{\mathrm{op}}}, f^{R^{\mathrm{op}}})$ is a representation for M as right kR-module can be shown in a similar way.

Let $\overline{M}$ be the left kQ-module given by the representation (V^Q, f^Q). We will show that
$$\overline{M} \cong M$$
as left kQ-modules. Then it will follow that $\overline{M}$ is also a $kQ/\langle\rho\rangle$-module, and isomorphic to M as such.

For an element m of M and vertices $u \in Q_0$ and $v \in R_0$, we denote the component of m belonging to the vector space $V^{\times}_{u\times v}$ by $m_{u\times v}$. For an element m of $\overline{M}$ and a vertex $u \in Q_0$, we denote the component of m belonging to the vector space V^Q_u by m_u. For an element $x \in V^Q_u$, we denote the component of x belonging to the vector space $V^{\times}_{u\times v}$ by x_v.

As vector spaces, $\overline{M}$ and M are clearly isomorphic, since

$$\overline{M} = \bigoplus_{u\in Q_0} V^Q_u = \bigoplus_{u\in Q_0} \Big(\bigoplus_{v\in R_0} V^{\times}_{u\times v} \Big) \cong \bigoplus_{\substack{u\in Q_0 \\ v\in R_0}} V^{\times}_{u\times v} = M.$$

We have a vector space isomorphism

$$\phi\colon \overline{M} \to M$$

given componentwise by
$$\phi(m)_{u\times v} = (m_u)_v$$
for an element $m \in \overline{M}$ and vertices $u \in Q_0$ and $v \in R_0$.

Now we only need to check that ϕ is a kQ-module homomorphism. Since we already know that it is a vector space homomorphism, it is enough to show that it preserves multiplication by vertices and arrows.

Let m be an element of $\overline{M}$. Given a vertex $u' \in Q_0$, we have (by Proposition 3.4 and Equation (1.4))

$$(u' \cdot \phi(m))_{u\times v} \overset{3.4}{=} \begin{cases} \phi(m)_{u\times v} = (m_u)_v & \text{if } u = u', \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\phi(u'm)_{u\times v} = ((u'm)_u)_v \overset{(1.4)}{=} \begin{cases} (m_u)_v & \text{if } u = u', \\ 0 & \text{otherwise;} \end{cases}$$

for any pair of vertices $u \in Q_0$ and $v \in R_0$, and thus

$$u' \cdot \phi(m) = \phi(u'm).$$

Given an arrow $\alpha \in Q_1$, we have (by Proposition 3.4 and Equation (1.5))

$$(\alpha \cdot \phi(m))_{u\times v} \overset{3.4}{=} \begin{cases} f^{\times}_{\alpha\times v}(\phi(m)_{\mathfrak{t}(\alpha)\times v}) = f^{\times}_{\alpha\times v}((m_{\mathfrak{t}(\alpha)})_v) & \text{if } u = \mathfrak{h}(\alpha), \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\begin{aligned} \phi(\alpha m)_{u\times v} &= ((\alpha m)_u)_v \\ &\overset{(1.5)}{=} \begin{cases} (f^{Q}_{\alpha}(m_{\mathfrak{t}(\alpha)}))_v = f^{\times}_{\alpha\times v}((m_{\mathfrak{t}(\alpha)})_v) & \text{if } u = \mathfrak{h}(\alpha), \\ 0 & \text{otherwise;} \end{cases} \end{aligned}$$

for any pair of vertices $u \in Q_0$ and $v \in R_0$, and thus

$$\alpha \cdot \phi(m) = \phi(\alpha m). \qquad \square$$

Example 3.3. Let us apply the above proposition to the bimodule M from Example 3.1. We had the representation

$$(V, f)\colon \quad \begin{array}{ccccc} k^2 & \xleftarrow{\binom{1}{1}} & k & \xleftarrow{1} & k \\ {\scriptstyle (1\ \ -1)}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ k & \longleftarrow & 0 & \longleftarrow & k \end{array}$$

over $Q \times R^{\mathrm{op}}$ for M as kQ–kR-bimodule, and a uniform basis B for M given by

$$B\colon \quad \begin{array}{ccccc} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} & \longleftarrow & \mathbf{c} & \longleftarrow & \mathbf{d} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{e} & \longleftarrow & 0 & \longleftarrow & \mathbf{f} \end{array}$$

By Proposition 3.5, M as left kQ-module is given by the representation

$$\begin{array}{c} k^4 \\ \Big\downarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ k^2 \end{array}$$

over Q. This can be viewed as the sum of the columns in (V, f). From the uniform basis B, we derive a left uniform basis

$$\begin{array}{c} \begin{pmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\\ \mathbf{d}\end{pmatrix} \\ \Big\downarrow \\ \begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix} \end{array}$$

for M as left kQ-module.

Similarly, M as right kR-module is given by the representation

$$k^3 \xleftarrow{\begin{pmatrix}1\\1\\0\end{pmatrix}} k \xleftarrow{\begin{pmatrix}1 & 0\end{pmatrix}} k^2 \ ,$$

over R^{op}. This can be viewed as the sum of the rows in (V, f). From the uniform basis B, we derive a left uniform basis

$$\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{e}\end{pmatrix} \longleftarrow \mathbf{c} \longleftarrow \begin{pmatrix}\mathbf{d}\\ \mathbf{f}\end{pmatrix}$$

for M as left kR^{op}-module. ∎

3.4 The enveloping algebra

If Λ is a k-algebra, then the **enveloping algebra** of Λ is

$$\Lambda^{\mathrm{e}} = \Lambda \otimes_k \Lambda^{\mathrm{op}}.$$

Thus a Λ-bimodule is the same as a left Λ^{e}-module. In particular, we can view Λ as a left Λ^{e}-module. For a path algebra $kQ/\langle\rho\rangle$, this means that the $kQ/\langle\rho\rangle$-bimodule structure of $kQ/\langle\rho\rangle$ can be described by a representation over the product quiver $Q \times Q^{\mathrm{op}}$. We will now look at how we can use Proposition 3.3 to find this representation.

Consider a path algebra $\Lambda = kQ$. Let (V, f) be the representation over $Q \times Q^{\mathrm{op}}$ for Λ as Λ-bimodule that we get from Proposition 3.3. The vector space at vertex $u \times v$ (for u and v in Q_0) is

$$V_{u\times v} = u\Lambda v.$$

This is the set of all (u, v)-uniform elements, and has a basis consisting of all paths from v to u. The linear map

$$f_{\alpha\times v} \colon V_{\mathfrak{t}(\alpha)\times v} \to V_{\mathfrak{h}(\alpha)\times v}$$

for an arrow $\alpha \in Q_1$ and a vertex $v \in Q_0$ is defined by left multiplication with α. On our basis, consisting of paths, this just means concatenating with α on the left. Similarly, the map

$$f_{u\times\beta^{\mathrm{op}}}\colon V_{u\times\mathfrak{h}(\beta)} \to V_{u\times\mathfrak{t}(\beta)}$$

for an arrow $\beta \in Q_1$ and a vertex $u \in Q_0$ is given on basis elements by concatenating with β on the right.

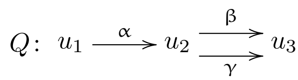
If Λ is a quotient of a path algebra, we can find a representation for Λ as Λ-bimodule in a similar way, except that the basis for the vector space

$$u\Lambda v$$

does not necessarily include representatives of all paths from v to u, only a linearly independent subset.

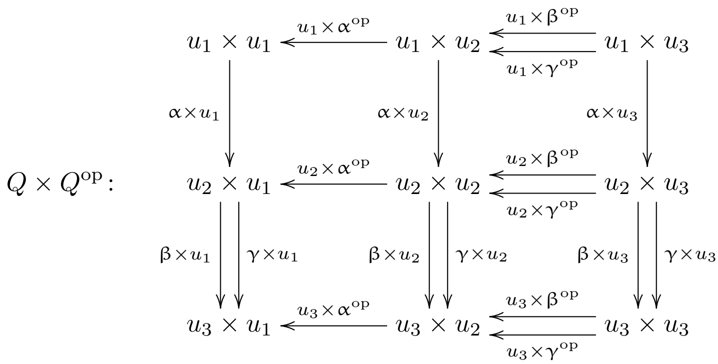
The following example shows the process of finding a representation for the bimodule structure of a path algebra.

Example 3.4. Let Q be the quiver

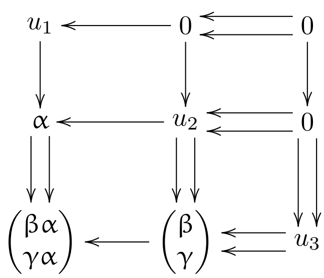


We will find a representation (V, f) over $Q \times Q^{\mathrm{op}}$ for kQ as kQ-bimodule.

The product quiver $Q \times Q^{\mathrm{op}}$ is



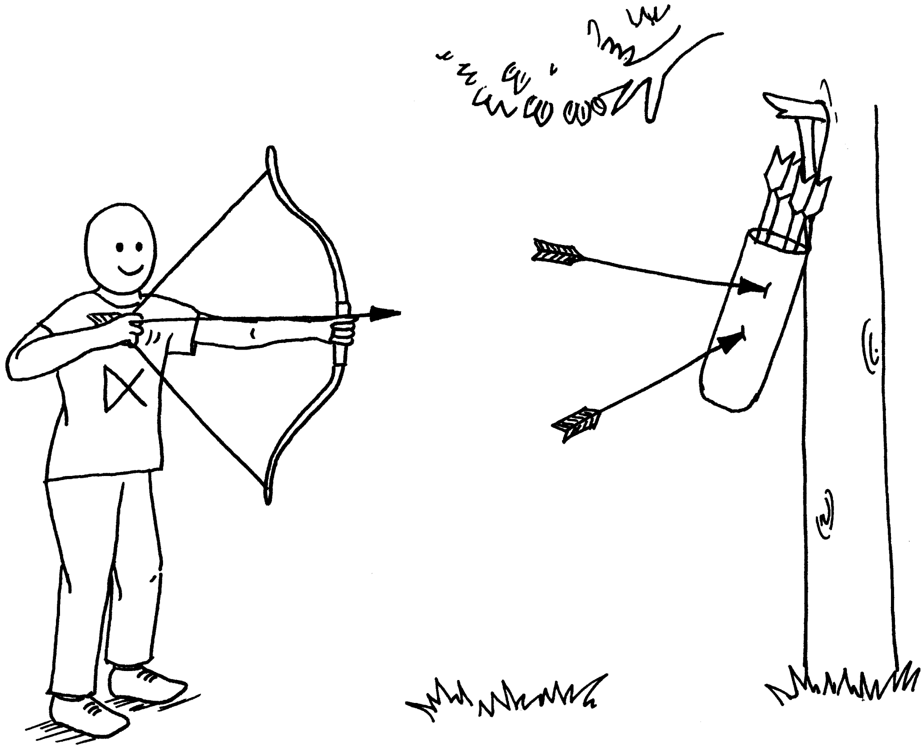
By taking the set of all paths from v to u as basis for the vector space $V_{u\times v}$ of the representation, we get the following uniform basis:



Since the maps of the representation are given, on basis elements, by left or right concatenation with the appropriate arrow, we see that the representation is

$$(V,f):\quad \begin{array}{ccccc} k & \longleftarrow & 0 & \leftleftarrows & 0 \\ {\scriptstyle 1}\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ k & \xleftarrow{\;1\;} & k & \leftleftarrows & 0 \\ {\scriptstyle \binom{1}{0}}\downdownarrows{\scriptstyle \binom{0}{1}} & & {\scriptstyle \binom{1}{0}}\downdownarrows{\scriptstyle \binom{0}{1}} & & \downdownarrows \\ k^2 & \xleftarrow[\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)]{} & k^2 & \overset{\binom{1}{0}}{\underset{\binom{0}{1}}{\leftleftarrows}} & k \end{array}$$

∎



Chapter 4

Triangular matrix algebras and trivial extensions

In this chapter, we will look at two different constructions involving one or two algebras and a bimodule.

Given two k-algebras Λ and Γ, and a Λ–Γ-bimodule M, we have the lower triangular matrix algebra

$$\begin{pmatrix} \Gamma & 0 \\ M & \Lambda \end{pmatrix} = \left\{ \begin{pmatrix} \gamma & 0 \\ m & \lambda \end{pmatrix} \middle| \lambda \in \Lambda, \gamma \in \Gamma, m \in M \right\},$$

where the addition and multiplication are defined as usual matrix addition and matrix multiplication, respectively.

A similar construction, called *trivial extension*, involves a k-algebra Λ and a Λ-bimodule M. The trivial extension of Λ by M, denoted $\Lambda \ltimes M$, is a new k-algebra. As a vector space, it is the direct sum of Λ and M. The multiplication is given by

$$(\lambda_1, m_1)(\lambda_2, m_2) = (\lambda_1\lambda_2, \lambda_1 m_2 + m_1\lambda_2).$$

for elements (λ_1, m_1) and (λ_2, m_2) of $\Lambda \ltimes M$.

Trivial extensions can be viewed as a generalization of triangular matrix algebras, since any triangular matrix algebra is isomorphic to a trivial extension (this is Proposition 4.7):

$$\begin{pmatrix} \Gamma & 0 \\ M & \Lambda \end{pmatrix} \cong (\Lambda \times \Gamma) \ltimes M.$$

When the algebras involved are quotients of path algebras, a triangular matrix algebra or trivial extension is isomorphic to a quotient of a path algebra. The goal of this chapter is to describe how to find the quivers and relations of these algebras.

The methods we will use produce quivers which may contain superfluous arrows, and relations that make these arrows equal to other paths. Thus, the relation sets are not necessarily admissible. In the next chapter, we will look at how we can fix this problem.

4.1 Triangular matrix algebras

Consider two quotients of path algebras $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$, and a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule M. We shall look at how to compute the triangular matrix algebra

$$\begin{pmatrix} kR/\langle\sigma\rangle & 0 \\ M & kQ/\langle\rho\rangle \end{pmatrix}$$

as a quotient of a path algebra. We will for convenience assume that the quivers Q and R are disjoint.

Let us first discuss the idea of the method we will be using. To avoid unnecessary complications, we assume that the sets ρ and σ of relations are empty, so that our algebra is just

$$\begin{pmatrix} kR & 0 \\ M & kQ \end{pmatrix}$$

with M a kQ–kR-bimodule. When we come to the more formal description later, the addition of relations will not cause any problems.

First consider the case $M = 0$. Then the matrix algebra is isomorphic to the direct product of kQ and kR, which we know (from Proposition 1.3) is isomorphic to the path algebra over the union $Q \cup R$ of the quivers:

$$\begin{pmatrix} kR & 0 \\ 0 & kQ \end{pmatrix} \cong kQ \times kR \cong k(Q \cup R).$$

Under this isomorphism, paths of $Q \cup R$ correspond to elements of the matrix algebra as follows:

$$q \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \qquad \text{for } q \in Q_*,$$

$$r \leftrightarrow \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{for } r \in R_*.$$

Now, if $M \neq 0$, we could start with the quiver $Q \cup R$ (and the above correspondence between paths and matrices), and then add some paths to represent the module elements. Let B be a uniform k-basis for M. Then each element b of B is uniform, and thus has a head $\mathfrak{h}(b) \in Q_0$ and a tail $\mathfrak{t}(b) \in R_0$. We have

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{h}(b) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} \mathfrak{t}(b) & 0 \\ 0 & 0 \end{pmatrix},$$

so the element

$$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

could be represented by a path from $\mathfrak{t}(b)$ to $\mathfrak{h}(b)$. Let us create an arrow

$$\overrightarrow{b} : \mathfrak{t}(b) \to \mathfrak{h}(b)$$

for each basis element $b \in B$, and add these to the quiver $Q \cup R$. We denote the resulting quiver by $R \xrightarrow{B} Q$. We use the following correspondence between the new arrows and elements of the matrix algebra:

$$\overrightarrow{b} \leftrightarrow \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \qquad\qquad \text{for } b \in B.$$

Now every element of the matrix algebra is represented by some linear combination of paths in the quiver $R \xrightarrow{B} Q$. But this quiver contains too much! Consider a basis element $b \in B$ and an arrow $\alpha \in Q_1$, with

$$\mathfrak{t}(\alpha) = \mathfrak{h}(b).$$

Then αb is some element of M, and thus it can be written as a linear combination of basis elements from B, say

$$\alpha b = \sum_i x_i b_i$$

for basis elements $b_i \in B$ and coefficients $x_i \in k$. In the matrix algebra, we have

$$\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \sum_i x_i b_i & 0 \end{pmatrix} = \sum_i x_i \begin{pmatrix} 0 & 0 \\ b_i & 0 \end{pmatrix}.$$

When we look at the corresponding elements of $k(R \xrightarrow{B} Q)$, we see that

$$\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \quad \text{corresponds to} \quad \alpha \overrightarrow{b},$$

which is a path of length two, while

$$\sum_i x_i \begin{pmatrix} 0 & 0 \\ b_i & 0 \end{pmatrix} \quad \text{corresponds to} \quad \sum_i x_i \overrightarrow{b_i},$$

which is a linear combination of arrows. To get an isomorphism between the matrix algebra and the new algebra we are producing, we need these to be the same. So we introduce the relation

$$\alpha \overrightarrow{b} - \sum_i x_i \overrightarrow{b_i},$$

together with similar relations for all other products

$$\alpha \overrightarrow{b} \qquad \text{and} \qquad \overrightarrow{b} \beta$$

of basis elements $b \in B$ and arrows $\alpha \in Q_1$ and $\beta \in R_1$. We call the set of these relations

$$\mu(Q, R, B).$$

Using the quiver and relations we have created now, we will get

$$\begin{pmatrix} kR & 0 \\ M & kQ \end{pmatrix} \cong k(R \xrightarrow{B} Q)/\langle \mu(Q, R, B) \rangle\,.$$

This is stated (in a slightly more general version) later in this section as Proposition 4.1, and we will prove it in Section 4.3.

Now that we have an idea of what we are going to do, let us write more precise definitions of the quiver

$$R \xrightarrow{B} Q$$

and the set of relations

$$\mu(Q, R, B)$$

in terms of the quivers Q and R and the uniform bimodule basis B.

Note that for creating the new quiver, we do not use the fact that B is a basis for a bimodule – we only use it as a set with a head and a tail for each element. The definition therefore only assumes a set together with head and tail functions.

Definition. Let Q and R be disjoint quivers and A a finite set, and let h and t be functions

$$h\colon A \to Q_0,$$
$$t\colon A \to R_0.$$

We define the **augmented union** of R and Q with (A, h, t), denoted

$$R \xrightarrow{(A,h,t)} Q,$$

by

$$(R \xrightarrow{(A,h,t)} Q)_0 = Q_0 \cup R_0,$$
$$(R \xrightarrow{(A,h,t)} Q)_1 = Q_1 \cup R_1 \cup \overrightarrow{A};$$

where $\overrightarrow{A}$ is a set consisting of one arrow

$$\overrightarrow{a} : t(a) \to h(a)$$

for each element a of A.

When it is clear from the context what the functions h and t are, we will omit them from the notation and write just

$$R \xrightarrow{A} Q$$

for

$$R \xrightarrow{(A,h,t)} Q.$$ ■

Example 4.1. Let Q and R be the quivers

$$Q\colon\ u_1 \xrightarrow{\alpha} u_2,$$

$$R\colon\ v_1 \xrightarrow{\beta} v_2 \xrightarrow{\gamma} v_3.$$

Let A be the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, and define the functions

$$h\colon A \to Q_0,$$
$$t\colon A \to R_0$$

by

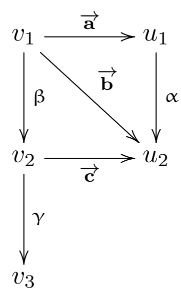
$$h(\mathbf{a}) = u_1, \qquad t(\mathbf{a}) = v_1,$$
$$h(\mathbf{b}) = u_2, \qquad t(\mathbf{b}) = v_1,$$
$$h(\mathbf{c}) = u_2, \qquad t(\mathbf{c}) = v_2.$$

Then we have

$$\overrightarrow{A} = \{\ \overrightarrow{\mathbf{a}} : v_1 \to u_1,\ \ \overrightarrow{\mathbf{b}} : v_1 \to u_2,\ \ \overrightarrow{\mathbf{c}} : v_2 \to u_2\ \},$$

and the quiver $R \xrightarrow{(A,h,t)} Q$ is

$R \xrightarrow{(A,h,t)} Q$:



■

If Q and R are disjoint quivers, M a kQ–kR-bimodule and B a uniform basis for M, then we have an augmented union

$$R \xrightarrow{B} Q,$$

where the head and tail functions are understood to be given by the usual $\mathfrak{h}$ and $\mathfrak{t}$. We then define a k-vector space homomorphism

$$\overrightarrow{} : M \to k(R \xrightarrow{B} Q)$$

by linear extension of the function from B to $\overrightarrow{B}$ mapping b to $\overrightarrow{b}$.

In the definition below, we use the $\overrightarrow{}$ homomorphism to express the relation set $\mu(Q, R, B)$ for a triangular matrix algebra.

Definition. Let Q and R be disjoint quivers, M a kQ–kR-bimodule, and B a uniform basis for M. Define the set $\mu(Q, R, B)$ of relations in $k(Q \xrightarrow{B} R)$ by

$$\begin{aligned} \mu(Q, R, B) = \{\ \alpha\,\overrightarrow{b} - \overrightarrow{\alpha b}\ \mid \alpha \in Q_1 \text{ and } b \in B \text{ with } \mathfrak{t}(\alpha) = \mathfrak{h}(b)\ \} \\ \cup \{\ \overrightarrow{b}\,\beta - \overrightarrow{b\beta}\ \mid b \in B \text{ and } \beta \in R_1 \text{ with } \mathfrak{t}(b) = \mathfrak{h}(\beta)\ \} \end{aligned}$$

■

We are now ready to formulate our result for expressing triangular matrix algebras as quotients of path algebras.

Proposition 4.1. *Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras, with Q and R disjoint quivers. Let M be a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule and B a uniform basis for M. Then*

$$\begin{pmatrix} kR/\langle\sigma\rangle & 0 \\ M & kQ/\langle\rho\rangle \end{pmatrix} \cong k(R \xrightarrow{B} Q)/\langle\rho \cup \sigma \cup \mu(Q,R,B)\rangle$$

as k-algebras, with isomorphisms induced by

$$\begin{pmatrix} 0 & 0 \\ 0 & [q] \end{pmatrix} \leftrightarrow [q] \qquad \text{for a path } q \in Q_*,$$
$$\begin{pmatrix} [r] & 0 \\ 0 & 0 \end{pmatrix} \leftrightarrow [r] \qquad \text{for a path } r \in R_*,$$
$$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \leftrightarrow \left[\overrightarrow{b}\right] \qquad \text{for a basis element } b \in B.$$ ■

♩ The relation set $\rho\cup\sigma\cup\mu(Q,R,B)$ in Proposition 4.1 is not necessarily admissible, even if ρ and σ are admissible in kQ and kR. The problem is that elements of $k(R \xrightarrow{B} Q)$ of the form $\overrightarrow{\alpha b}$ or $\overrightarrow{b\beta}$, used in $\mu(Q,R,B)$, are linear combinations of arrows. If any of these are nonzero, we will not get

$$\langle\rho \cup \sigma \cup \mu(Q,R,B)\rangle \subseteq J^2_{R \xrightarrow{B} Q}.$$

We do, however, always have

$$J^t_{R \xrightarrow{B} Q} \subseteq \langle\rho \cup \sigma \cup \mu(Q,R,B)\rangle \subseteq J_{R \xrightarrow{B} Q}$$

for some t (provided that ρ and σ are admissible), so the relation set is preadmissible. We show this in Proposition 4.2 later in this section.

In Chapter 5, we will see how to turn a quotient of a path algebra with preadmissible relation set into one with admissible relation set. We will then be able to create a new quiver S with admissible relation set $\tau \subseteq kS$, such that

$$\begin{pmatrix} kR/\langle\sigma\rangle & 0 \\ M & kQ/\langle\rho\rangle \end{pmatrix} \cong k(R \xrightarrow{B} Q)/\langle\rho \cup \sigma \cup \mu(Q,R,B)\rangle \cong kS/\langle\tau\rangle\,.$$ ■

We will prove Proposition 4.1 later (in Section 4.3), after proving the corresponding result for trivial extensions in Proposition 4.5. Right now we will only look at how the procedure works on a simple example.

Example 4.2. Let Q and R be the quivers

$$Q\colon\ u_1 \xrightarrow{\alpha} u_2\ ,$$

$$R\colon\ v_1 \xrightarrow{\beta} v_2 \xrightarrow{\gamma} v_3\ .$$

Then the product quiver $Q \times R^{\mathrm{op}}$ is

$$Q \times R^{\mathrm{op}}\colon \quad \begin{array}{ccccc} u_1 \times v_1 & \xleftarrow{u_1\times\beta^{\mathrm{op}}} & u_1 \times v_2 & \xleftarrow{u_1\times\gamma^{\mathrm{op}}} & u_1 \times v_3 \\ \downarrow{\scriptstyle \alpha\times v_1} & & \downarrow{\scriptstyle \alpha\times v_2} & & \downarrow{\scriptstyle \alpha\times v_3} \\ u_2 \times v_1 & \xleftarrow{u_2\times\beta^{\mathrm{op}}} & u_2 \times v_2 & \xleftarrow{u_2\times\gamma^{\mathrm{op}}} & u_2 \times v_3 \end{array}$$

Let M be the kQ–kR-bimodule given by the following representation over $Q \times R^{\mathrm{op}}$ (observe that the representation respects the relations $\kappa(Q, R^{\mathrm{op}})$, since both squares commute):

$$\begin{array}{ccccc} k^2 & \xleftarrow{\binom{1}{1}} & k & \xleftarrow{1} & k \\ {\scriptstyle (1\ \ -1)}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ k & \longleftarrow & 0 & \longleftarrow & k \end{array}$$

We will find a quotient of a path algebra which is isomorphic to the triangular matrix algebra
$$\begin{pmatrix} kR & 0 \\ M & kQ \end{pmatrix}$$
by using the method of Proposition 4.1.

Let B be the following uniform basis for M:

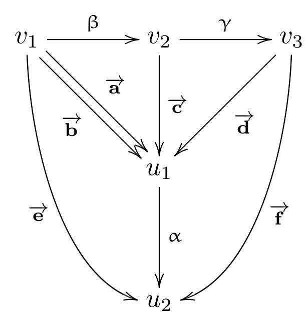
$$B\colon \quad \begin{array}{ccccc} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} & \longleftarrow & \mathbf{c} & \longleftarrow & \mathbf{d} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{e} & \longleftarrow & 0 & \longleftarrow & \mathbf{f} \end{array}$$

Following the definition of the augmented union $R \xrightarrow{B} Q$, we get new arrows

$$\overrightarrow{B} = \{\, \overrightarrow{\mathbf{a}}\colon v_1 \to u_1,\ \overrightarrow{\mathbf{b}}\colon v_1 \to u_1,\ \overrightarrow{\mathbf{c}}\colon v_2 \to u_1, \\ \overrightarrow{\mathbf{d}}\colon v_3 \to u_1,\ \overrightarrow{\mathbf{e}}\colon v_1 \to u_2,\ \overrightarrow{\mathbf{f}}\colon v_3 \to u_2 \,\},$$

and our quiver is thus

$R \xrightarrow{B} Q$:



We will now compute the relation set $\mu(Q, R, B)$. We get the following relations by taking every combination of an arrow $\alpha \in Q_1$ and a basis element $b \in B$ with $\mathfrak{t}(\alpha) = \mathfrak{h}(b)$:

$$\begin{aligned} &\alpha\overrightarrow{\mathbf{a}} - \overrightarrow{\alpha\mathbf{a}} = \alpha\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{e}}, \\ &\alpha\overrightarrow{\mathbf{b}} - \overrightarrow{\alpha\mathbf{b}} = \alpha\overrightarrow{\mathbf{b}} - \overrightarrow{(-\mathbf{e})} = \alpha\overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{e}}, \\ &\alpha\overrightarrow{\mathbf{c}} - \overrightarrow{\alpha\mathbf{c}} = \alpha\overrightarrow{\mathbf{c}} - \overrightarrow{0} = \alpha\overrightarrow{\mathbf{c}}, \\ &\alpha\overrightarrow{\mathbf{d}} - \overrightarrow{\alpha\mathbf{d}} = \alpha\overrightarrow{\mathbf{d}} - \overrightarrow{\mathbf{f}}. \end{aligned}$$

We get the following relations by taking every combination of a basis element $b \in B$ and an arrow $\beta \in R_1$ with $\mathfrak{t}(b) = \mathfrak{h}(\beta)$:

$$\begin{aligned} &\overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{c}\beta} = \overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{a}+\mathbf{b}} = \overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}}, \\ &\overrightarrow{\mathbf{d}}\gamma - \overrightarrow{\mathbf{d}\gamma} = \overrightarrow{\mathbf{d}}\gamma - \overrightarrow{\mathbf{c}}, \\ &\overrightarrow{\mathbf{f}}\gamma - \overrightarrow{\mathbf{f}\gamma} = \overrightarrow{\mathbf{f}}\gamma - \overrightarrow{0} = \overrightarrow{\mathbf{f}}\gamma. \end{aligned}$$

We thus have the relation set

$$\begin{aligned} \mu(Q, R, B) = \{\, &\alpha\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{e}},\ \alpha\overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{e}},\ \alpha\overrightarrow{\mathbf{c}},\ \alpha\overrightarrow{\mathbf{d}} - \overrightarrow{\mathbf{f}}, \\ &\overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}},\ \overrightarrow{\mathbf{d}}\gamma - \overrightarrow{\mathbf{c}},\ \overrightarrow{\mathbf{f}}\gamma \,\} \end{aligned}$$

Now Proposition 4.1 tells us that

$$\begin{pmatrix} kR & 0 \\ M & kQ \end{pmatrix} \cong k(R \xrightarrow{B} Q)/\langle \mu(Q, R, B) \rangle \,.$$

This is, however, not completely satisfactory, since the relation set $\mu(Q, R, B)$ is not admissible. In Example 5.5, we will continue this example and find a path algebra quotient with admissible relation set which is isomorphic to

$$k(R \xrightarrow{B} Q)/\langle \mu(Q, R, B) \rangle \,. \qquad \blacksquare$$

We now show that the relation set produced by Proposition 4.1 is preadmissible.

Proposition 4.2. *Let $kQ/\langle\rho\rangle$ and $kR/\langle\sigma\rangle$ be quotients of path algebras, with Q and R disjoint quivers, and ρ and σ admissible sets of relations. Let M be a $kQ/\langle\rho\rangle$–$kR/\langle\sigma\rangle$-bimodule and B a uniform basis for M. Then the relation set*

$$\rho \cup \sigma \cup \mu(Q, R, B) \subseteq k(R \xrightarrow{B} Q)$$

is preadmissible.

Proof. We must show that

$$J^t_{R \xrightarrow{B} Q} \subseteq \langle \rho \cup \sigma \cup \mu(Q, R, B) \rangle \subseteq J_{R \xrightarrow{B} Q}$$

for some t.

We immediately have

$$\langle \rho \cup \sigma \cup \mu(Q, R, B) \rangle \subseteq J_{R \xrightarrow{B} Q}.$$

To get the other inclusion, choose t_1 and t_2 such that

$$J_Q^{t_1} \subseteq \langle \rho \rangle \qquad \text{and} \qquad J_R^{t_2} \subseteq \langle \sigma \rangle \, .$$

We show that

$$J^{t_1+t_2}_{R \xrightarrow{B} Q} \subseteq \langle \rho \cup \sigma \cup \mu(Q, R, B) \rangle \, .$$

Let p be a path of length $t_1 + t_2$ in $R \xrightarrow{B} Q$. If p does not contain any arrow of $\overrightarrow{B}$, then it is completely contained in either Q or R, and thus lies in either $\langle \rho \rangle$ or $\langle \sigma \rangle$. Otherwise, we have

$$p = q \, \overrightarrow{b} \, r$$

for some basis element $b \in B$ and paths $q \in Q_*$ and $r \in R_*$. Since the length of p is $t_1 + t_2$, we must have either

$$l(q) \geq t_1 \qquad \text{or} \qquad l(r) \geq t_2.$$

Thus, we either have q in $\langle \rho \rangle$ or r in $\langle \sigma \rangle$. In all cases, we get

$$p \in \langle \rho \cup \sigma \rangle \, ,$$

so we have

$$J^{t_1+t_2}_{R \xrightarrow{B} Q} \subseteq \langle \rho \cup \sigma \cup \mu(Q, R, B) \rangle \, .$$

□

4.2 Trivial extensions

In this section, we will describe how a trivial extension

$$kQ/\langle \rho \rangle \ltimes M,$$

where M is a $kQ/\langle \rho \rangle$-bimodule, can be computed as a quotient of a path algebra.

The idea is similar to our procedure for triangular matrix algebras in the previous section. Let B be a uniform basis for M. We construct a new quiver which consists of Q and one new arrow for each basis element of M, denoted

$$Q \circlearrowleft {\scriptstyle B}.$$

We define a relation set $\nu(Q, B)$ (analogous to the relation set $\mu(Q, R, B)$ in the previous section) consisting of relations of the forms

$$\alpha \, \overrightarrow{b} - \overrightarrow{\alpha b} \qquad \text{and} \qquad \overrightarrow{b} \, \beta - \overrightarrow{b \beta}$$

for arrows α and β in Q, and basis elements $b \in B$. These relations capture the way elements of the path algebra act on module basis elements (this is described more precisely in Lemma 4.3).

Since

$$(0, m) \cdot (0, m') = 0$$

in $kQ/\langle\rho\rangle \ltimes M$ for any elements m and m' of M, we need additional relations to kill products of the arrows that represent basis elements of M. We therefore define a relation set $\xi(Q, B)$ consisting of all possible products of two such arrows.

The result we will show (in Proposition 4.5) is that

$$kQ/\langle\rho\rangle \ltimes M \cong k(Q \circlearrowleft {\scriptstyle B})/\langle\rho \cup \nu(Q, B) \cup \xi(Q, B)\rangle\,.$$

We first state a precise definition of the notation

$$Q \circlearrowleft {\scriptstyle B}$$

which we will use for the quiver of a trivial extension.

Definition. Let Q be a quiver, A a finite set, and let h and t be functions

$$h\colon A \to Q_0,$$
$$t\colon A \to Q_0.$$

We define the **augmented quiver** $Q \circlearrowleft {\scriptstyle (A,h,t)}$ of Q with (A, h, t) by

$$(Q \circlearrowleft {\scriptstyle (A,h,t)})_0 = Q_0,$$
$$(Q \circlearrowleft {\scriptstyle (A,h,t)})_1 = Q_1 \cup \overrightarrow{A};$$

where $\overrightarrow{A}$ is a set consisting of one arrow

$$\overrightarrow{a} : t(a) \to h(a)$$

for each element a of A.

When it is clear from the context what the functions h and t are, we will omit them from the notation and write just

$$Q \circlearrowleft {\scriptstyle A}$$

for

$$Q \circlearrowleft {\scriptstyle (A,h,t)}.$$

■

If Q is a quiver, M a kQ-bimodule and B a uniform basis for M, then we have an augmented quiver

$$Q \circlearrowleft {\scriptstyle B},$$

where the head and tail functions are understood to be given by the usual $\mathfrak{h}$ and $\mathfrak{t}$. We then define a k-vector space homomorphism

$$\overrightarrow{}\colon M \to k(Q \circlearrowleft {\scriptstyle B})$$

by linear extension of the function from B to $\overrightarrow{B}$ mapping b to $\overrightarrow{b}$. This is analogous to what we did for augmented unions of quivers with bimodules in the previous section. The similarities between augmented unions of quivers and augmented quivers are further explored in the following note.

♪ The definition of the augmented quiver above is essentially the same as the definition of an augmented union of quivers on page 62, except that it is based on a quiver on the form

$$\bullet \circlearrowright$$

instead of one on the form

$$\bullet \longrightarrow \bullet\,.$$

We have chosen to make two separate definitions since we only need these two special cases. We could however have used the following more general definition instead.

Let $\mathscr{Q}$ be a quiver. Assign a quiver $Q^{(u)}$ to each vertex u in $\mathscr{Q}$, and a finite set A_α and functions

$$h_\alpha \colon A_\alpha \to Q_0^{(\mathfrak{h}(\alpha))},$$
$$t_\alpha \colon A_\alpha \to Q_0^{(\mathfrak{t}(\alpha))}$$

to each arrow α in $\mathscr{Q}$. Then we define, for each arrow α in $\mathscr{Q}$, the set $\overrightarrow{A_\alpha}$ consisting of an arrow

$$\overrightarrow{a} \colon t_\alpha(a) \to h_\alpha(a)$$

for each element a of A_α. We define the *combined quiver* $\mathscr{Q}^{(Q,A,h,t)}$ by

$$(\mathscr{Q}^{(Q,A,h,t)})_0 = \bigcup_{u \in \mathscr{Q}_0} Q_0^{(u)},$$
$$(\mathscr{Q}^{(Q,A,h,t)})_1 = \Big(\bigcup_{u \in \mathscr{Q}_0} Q_1^{(u)} \Big) \cup \Big(\bigcup_{\alpha \in \mathscr{Q}_1} \overrightarrow{A_\alpha} \Big).$$

We can further define the following notation for the combined quiver $\mathscr{Q}^{(Q,A,h,t)}$: Draw the quiver $\mathscr{Q}$, with the quiver $Q^{(u)}$ placed at vertex u for each $u \in \mathscr{Q}_0$, and $(A_\alpha, h_\alpha, t_\alpha)$ placed by the arrow α for each $\alpha \in \mathscr{Q}_1$.

By taking $\mathscr{Q}$ to be

$$\bullet \circlearrowright \qquad \text{or} \qquad \bullet \longrightarrow \bullet\,,$$

we get a combined quiver which is the same as an augmented quiver or augmented union of quivers, and the notation is exactly the same as the one we originally defined for these. ■

Having defined the quiver we want to use, let us now define the sets $\nu(Q, B)$ and $\xi(Q, B)$ of relations.

Definition. Let Q be a quiver, M a kQ-bimodule and B a uniform basis for M. Define the relation sets $\nu(Q,B)$ and $\xi(Q,B)$ in $k(Q \circlearrowleft B)$ by

$$\begin{aligned}\nu(Q,B) &= \left\{ \alpha\overrightarrow{b} - \overrightarrow{\alpha b} \;\middle|\; \alpha \in Q_1 \text{ and } b \in B \text{ with } \mathfrak{t}(\alpha) = \mathfrak{h}(b) \right\} \\ &\quad \cup \left\{ \overrightarrow{b}\alpha - \overrightarrow{b\alpha} \;\middle|\; b \in B \text{ and } \alpha \in Q_1 \text{ with } \mathfrak{t}(b) = \mathfrak{h}(\alpha) \right\}, \\ \xi(Q,B) &= \left\{ \overrightarrow{b_1}\,\overrightarrow{b_2} \;\middle|\; b_1 \text{ and } b_2 \text{ in } B \text{ with } \mathfrak{t}(b_1) = \mathfrak{h}(b_2) \right\}.\end{aligned}$$

■

Now we have the ingredients we need to describe trivial extensions as quotients of path algebras. Before we prove the main result, we will show some useful properties of the relation sets $\nu(Q,B)$ and $\xi(Q,B)$. We give these as two lemmata.

Our first lemma says that modulo the relation set $\nu(Q,B)$, the new arrows $\overrightarrow{b}$ in the quiver $Q \circlearrowleft B$ behave just like the corresponding basis elements b of M with respect to multiplication by elements of the algebra kQ.

Lemma 4.3. *Let kQ be a path algebra, M a kQ-bimodule, and B a uniform basis for M. Then*

$$\lambda\overrightarrow{m}\gamma \overset{\nu(Q,B)}{\sim} \overrightarrow{\lambda m\gamma}$$

for any elements λ and γ of kQ, and element m of M.

Proof. Throughout this proof we will write just ν for $\nu(Q,B)$.

It is sufficient to show that

$$\lambda\overrightarrow{m} \overset{\nu}{\sim} \overrightarrow{\lambda m} \quad \text{and} \quad \overrightarrow{m}\gamma \overset{\nu}{\sim} \overrightarrow{m\gamma}.$$

We show $\lambda\overrightarrow{m} \overset{\nu}{\sim} \overrightarrow{\lambda m}$; the proof for $\overrightarrow{m}\gamma \overset{\nu}{\sim} \overrightarrow{m\gamma}$ is analogous.

We build up the result in several steps by first showing some special cases:

1. $u\overrightarrow{b} \overset{\nu}{\sim} \overrightarrow{ub}$ for a vertex $u \in Q_0$ and a basis element $b \in B$: If $u = \mathfrak{h}(\overrightarrow{b})$, then

$$u\overrightarrow{b} = \overrightarrow{b} = \overrightarrow{ub};$$

otherwise,

$$u\overrightarrow{b} = 0 = \overrightarrow{0} = \overrightarrow{ub}.$$

2. $\alpha\overrightarrow{b} \overset{\nu}{\sim} \overrightarrow{\alpha b}$ for an arrow $\alpha \in Q_1$ and a basis element $b \in B$: If $\mathfrak{t}(\alpha) = \mathfrak{h}(\overrightarrow{b})$, then

$$\alpha\overrightarrow{b} - \overrightarrow{\alpha b} \in \nu;$$

otherwise,

$$\alpha\overrightarrow{b} = 0 = \overrightarrow{0} = \overrightarrow{\alpha b}.$$

3. $q\overrightarrow{m} \overset{\nu}{\sim} \overrightarrow{qm}$ for a vertex or arrow $q \in Q_?$ and an element $m \in M$: Write

$$m = \sum_i c_i b_i,$$

where the b_i are basis elements in B and the c_i coefficients from k. Then we have

$$\begin{aligned} q\overrightarrow{m} &= q \cdot \overrightarrow{\sum c_i b_i} = q \sum c_i \overrightarrow{b_i} = \sum c_i \cdot q\,\overrightarrow{b_i} \overset{\nu}{\sim} \sum c_i \cdot \overrightarrow{qb_i} \\ &= \overrightarrow{\sum c_i \cdot qb_i} = \overrightarrow{q \sum c_i b_i} = \overrightarrow{qm}, \end{aligned}$$

where the equivalence follows from steps 1 and 2.

4. $q\overrightarrow{m} \overset{\nu}{\sim} \overrightarrow{qm}$ for any path $q \in Q_*$ and an element $m \in M$: Paths of length 0 and 1 are taken care of by step 3, so assume $l(q) > 1$. Write

$$q = \alpha_1 \cdots \alpha_n,$$

where each α_i is an arrow in Q. Then, by applying step 3 repeatedly, we have

$$\begin{aligned} q\overrightarrow{m} &= \alpha_1 \cdots \alpha_{n-1}\alpha_n \overrightarrow{m} \\ &\overset{\nu}{\sim} \alpha_1 \cdots \alpha_{n-1} \overrightarrow{\alpha_n m} \\ &\overset{\nu}{\sim} \cdots \overset{\nu}{\sim} \overrightarrow{\alpha_1 \cdots \alpha_n \cdot m} = \overrightarrow{qm}. \end{aligned}$$

Now we are ready to show that $\lambda\overrightarrow{m} \overset{\nu}{\sim} \overrightarrow{\lambda m}$ for an element $\lambda \in kQ$ and an element $m \in M$. Write

$$\lambda = \sum_i c_i q_i,$$

where the q_i are paths in Q and the c_i coefficients from k. Then we have, by using step 4,

$$\begin{aligned} \lambda\overrightarrow{m} &= \left(\sum c_i q_i\right)\overrightarrow{m} = \sum c_i q_i \overrightarrow{m} \overset{\nu}{\sim} \sum c_i \overrightarrow{q_i m} \\ &= \overrightarrow{\sum c_i q_i m} = \overrightarrow{\left(\sum c_i q_i\right) m} = \overrightarrow{\lambda m}. \end{aligned}$$ □

The next lemma essentially shows that when we already have the relations $\nu(Q,B)$, adding the relations $\xi(Q,B)$ is sufficient for killing every element of $k(Q \circlearrowleft B)$ which corresponds to a product of two elements of M. Such an element must be a linear combination of paths containing two arrows from $\overrightarrow{B}$; we show that all these paths are killed by the relations $\nu(Q,B) \cup \xi(Q,B)$.

Lemma 4.4. *Let kQ be a path algebra, M a kQ-bimodule, and B a uniform basis for M. Then any path in $k(Q \circlearrowleft B)$ which contains two or more arrows of $\overrightarrow{B}$ lies in the ideal*

$$\langle \nu(Q,B) \cup \xi(Q,B) \rangle .$$

Proof. It is enough to show that any path

$$\overrightarrow{b_1}\, q\, \overrightarrow{b_2} \qquad \text{for } b_1 \text{ and } b_2 \text{ in } B, \text{ and } q \in Q_*$$

lies in $\langle \nu(Q,B) \cup \xi(Q,B) \rangle$. Let

$$b_1 q = \sum_i c_i b_i \qquad \text{(with each } c_i \text{ in } k \text{ and each } b_i \text{ in } B\text{)}$$

be the expansion of the module element $b_1 q$ as a linear combination of basis elements. Then we have, by Lemma 4.3,

$$\overrightarrow{b_1}\, q\, \overrightarrow{b_2} \overset{\nu(Q,B)}{\sim} \overrightarrow{b_1 q}\, \overrightarrow{b_2} = \sum_i c_i \overrightarrow{b_i}\, \overrightarrow{b_2} \in \langle \xi(Q,B) \rangle\,.$$

□

We now state and prove the main result of this section.

Proposition 4.5. *Let $kQ/\langle\rho\rangle$ be a quotient of a path algebra, M a $kQ/\langle\rho\rangle$-bimodule, and B a uniform basis for M. Then*

$$kQ/\langle\rho\rangle \ltimes M \cong k(Q \circlearrowleft B)/\langle \rho \cup \nu(Q,B) \cup \xi(Q,B) \rangle$$

as k-algebras, with isomorphisms induced by

$$([q],0) \leftrightarrow [q] \qquad \textit{for any path } q \textit{ in } Q,$$
$$(0,b) \leftrightarrow \left[\overrightarrow{b}\right] \qquad \textit{for any basis element } b \textit{ in } B.$$

Proof. Define a k-algebra homomorphism

$$\overline{\phi} \colon k(Q \circlearrowleft B) \to kQ/\langle\rho\rangle \ltimes M$$

by the following actions on vertices and arrows (remember that $(Q \circlearrowleft B)_1 = Q_1 \cup \overrightarrow{B}$):

$$u \mapsto ([u],0) \qquad \text{for a vertex } u \in (Q \circlearrowleft B)_0 = Q_0,$$
$$\alpha \mapsto ([\alpha],0) \qquad \text{for an arrow } \alpha \in Q_1,$$
$$\overrightarrow{b} \mapsto (0,b) \qquad \text{for an arrow } \overrightarrow{b} \in \overrightarrow{B}.$$

This gives a well-defined algebra homomorphism, by Lemma 1.2 (it is straightforward to check that the conditions of the lemma are satisfied).

We observe that $\overline{\phi}$ is an epimorphism: The set

$$\{\, ([q],0) \mid q \in Q_* \,\} \cup \{\, (0,b) \mid b \in B \,\}$$

is contained in the image of $\overline{\phi}$ since

$$\overline{\phi}(q) = ([q],0) \qquad \text{for } q \in Q_* \text{ and}$$
$$\overline{\phi}(\overrightarrow{b}) = (0,b) \qquad \text{for } b \in B,$$

and this set generates $kQ/\langle\rho\rangle \ltimes M$ as vector space.

We want to show that the relation sets ρ, $\nu(Q,B)$ and $\xi(Q,B)$ are mapped to zero by $\overline{\phi}$. We have

$$\overline{\phi}(\mathfrak{r}) = ([\mathfrak{r}],0) = (0,0)$$

for any $\mathfrak{r} \in \rho$. To see that $\nu(Q,B)$ is mapped to zero, consider an arrow $\alpha \in Q_1$ and a basis element $b \in B$ with $\mathfrak{t}(\alpha) = \mathfrak{h}(b)$. Then αb is an element of M and thus some linear combination $\sum_i c_i b_i$ of basis elements $b_i \in B$ with coefficients $c_i \in k$. We compute

$$\begin{aligned}\overline{\phi}(\alpha \overrightarrow{b}) &= \overline{\phi}(\alpha) \cdot \overline{\phi}(\overrightarrow{b}) = ([\alpha],0) \cdot (0,b) = (0, \alpha b) \\ &= \Big(0, \sum_i c_i b_i\Big) = \sum_i c_i \cdot (0, b_i) = \sum_i c_i \cdot \overline{\phi}(\overrightarrow{b_i}) \\ &= \overline{\phi}\Big(\sum_i c_i \overrightarrow{b_i}\Big) = \overline{\phi}\Big(\overrightarrow{\sum_i c_i b_i}\Big) = \overline{\phi}(\overrightarrow{\alpha b})\end{aligned}$$

to get

$$\overline{\phi}(\alpha \overrightarrow{b} - \overrightarrow{\alpha b}) = 0.$$

A similar argument holds for elements of $\nu(Q,B)$ which are of the form $\overrightarrow{b}\beta - \overrightarrow{b\beta}$. For elements $\overrightarrow{b_1}\,\overrightarrow{b_2}$ of $\xi(Q,B)$, we have

$$\overline{\phi}(\overrightarrow{b_1}\,\overrightarrow{b_2}) = \overline{\phi}(\overrightarrow{b_1}) \cdot \overline{\phi}(\overrightarrow{b_2}) = (0,[b_1]) \cdot (0,[b_2]) = (0,0).$$

This means that

$$\rho \cup \nu(Q,B) \cup \xi(Q,B) \subseteq \ker \overline{\phi}\,,$$

and thus

$$\langle \rho \cup \nu(Q,B) \cup \xi(Q,B) \rangle \subseteq \ker \overline{\phi}\,.$$

So $\overline{\phi}$ induces an algebra homomorphism

$$\phi \colon k(Q \circlearrowleft B)/\langle \rho \cup \nu(Q,B) \cup \xi(Q,B) \rangle \to kQ/\langle \rho \rangle \ltimes M,$$

with

$$\begin{aligned}\phi([q]) &= ([q],0) && \text{for a path } q \in Q_*, \\ \phi\Big(\big[\overrightarrow{b}\big]\Big) &= (0,b) && \text{for a basis element } b \in B.\end{aligned}$$

Now we only need to show that

$$\ker \overline{\phi} \subseteq \langle \rho \cup \nu(Q,B) \cup \xi(Q,B) \rangle$$

to establish that ϕ is an isomorphism.

Let $\lambda \in k(Q \circlearrowleft B)$ be any element. We can write this element as

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2,$$

where λ_0 is a linear combination of paths containing no arrows from $\overrightarrow{B}$ (that is, $\lambda_0 \in kQ$), λ_1 is a combination of paths containing exactly one arrow from $\overrightarrow{B}$, and λ_2 is a combination of paths containing two or more arrows from $\overrightarrow{B}$.

Let us look at what the elements λ_0, λ_1 and λ_2 are mapped to by $\overline{\phi}$. Since $\lambda_0 \in kQ$, we have

$$\overline{\phi}(\lambda_0) = ([\lambda_0], 0).$$

We can write λ_1 as

$$\lambda_1 = \sum_i c_i \cdot q_i \overrightarrow{b_i} r_i$$

for some coefficients $c_i \in k$, paths q_i and r_i in Q_* and basis elements $b_i \in B$. Let

$$\lambda_1' = \sum_i c_i \cdot q_i b_i r_i$$

be the corresponding element of M; then

$$\begin{aligned}\overline{\phi}(\lambda_1) &= \sum_i c_i \cdot \phi(q_i) \cdot \phi(\overrightarrow{b_i}) \cdot \phi(r_i) = \sum_i c_i \cdot ([q_i], 0) \cdot (0, b_i) \cdot ([r_i], 0) \\ &= \sum_i c_i \cdot (0, q_i b_i r_i) = \Big(0, \sum_i c_i \cdot q_i b_i r_i\Big) = (0, \lambda_1').\end{aligned}$$

We see that $\overline{\phi}(\lambda_2) = (0, 0)$, since

$$\begin{aligned}\overline{\phi}(\overrightarrow{b_1}\, q\, \overrightarrow{b_2}) &= \overline{\phi}(\overrightarrow{b_1}) \cdot \overline{\phi}(q) \cdot \overline{\phi}(\overrightarrow{b_2}) \\ &= (0, b_1) \cdot ([q], 0) \cdot (0, b_2) = (0, b_1 q) \cdot (0, b_2) = (0, 0)\end{aligned}$$

for any basis elements b_1 and b_2 in B, and any path $q \in Q_*$. We thus have

$$\begin{aligned}\overline{\phi}(\lambda) &= \overline{\phi}(\lambda_0) + \overline{\phi}(\lambda_1) + \overline{\phi}(\lambda_2) = ([\lambda_0], 0) + (0, \lambda_1') + (0, 0) \\ &= ([\lambda_0], \lambda_1').\end{aligned}$$

Assume that $\overline{\phi}(\lambda) = 0$. Then $\lambda_0 \in \langle \rho \rangle$ and $\lambda_1' = 0$. By Lemma 4.3, we have

$$\lambda_1 \overset{\nu(Q,B)}{\sim} \overrightarrow{\lambda_1'} = \overrightarrow{0} = 0,$$

and thus $\lambda_1 \in \langle \nu(Q, B) \rangle$. By Lemma 4.4, we have $\lambda_2 \in \langle \nu(Q, B) \cup \xi(Q, B) \rangle$. This means that

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 \in \langle \rho \cup \nu(Q, B) \cup \xi(Q, B) \rangle,$$

and we have thus shown that

$$\ker \overline{\phi} \subseteq \langle \rho \cup \nu(Q, B) \cup \xi(Q, B) \rangle. \qquad \square$$

♩ Just like in the case of triangular matrix algebras (see the note on page 64), the relation set we produce for a trivial extension is preadmissible (we show this in Proposition 4.6 later in this section), but not necessarily admissible. In Chapter 5, we will see how we can produce an isomorphic quotient of a path algebra where the relation set is admissible. ■

Let us look at an example of a trivial extension to see how Proposition 4.5 works in practice.

Example 4.3. Let Q be the quiver

$$Q\colon\ u_1 \xrightarrow{\alpha} u_2 \xrightarrow{\beta} u_3\ .$$

Then the product quiver $Q \times Q^{\mathrm{op}}$ is

$$Q \times Q^{\mathrm{op}}\colon \qquad \begin{array}{ccccc} u_1 \times u_1 & \xleftarrow{u_1\times\alpha^{\mathrm{op}}} & u_1 \times u_2 & \xleftarrow{u_1\times\beta^{\mathrm{op}}} & u_1 \times u_3 \\ {\scriptstyle \alpha\times u_1}\downarrow & & {\scriptstyle \alpha\times u_2}\downarrow & & {\scriptstyle \alpha\times u_3}\downarrow \\ u_2 \times u_1 & \xleftarrow{u_2\times\alpha^{\mathrm{op}}} & u_2 \times u_2 & \xleftarrow{u_2\times\beta^{\mathrm{op}}} & u_2 \times u_3 \\ {\scriptstyle \beta\times u_1}\downarrow & & {\scriptstyle \beta\times u_2}\downarrow & & {\scriptstyle \beta\times u_3}\downarrow \\ u_3 \times u_1 & \xleftarrow{u_3\times\alpha^{\mathrm{op}}} & u_3 \times u_2 & \xleftarrow{u_3\times\beta^{\mathrm{op}}} & u_3 \times u_3 \end{array}$$

Let M be the kQ-bimodule given by the following representation over $Q \times Q^{\mathrm{op}}$:

$$M\colon \qquad \begin{array}{ccccc} 0 & \longleftarrow & k & \xleftarrow{1} & k \\ \downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \longleftarrow & k & \xleftarrow[1]{} & k \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

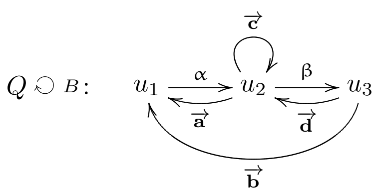
Let B the following uniform basis for M:

$$B\colon \qquad \begin{array}{ccccc} 0 & \longleftarrow & \mathbf{a} & \longleftarrow & \mathbf{b} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & \mathbf{c} & \longleftarrow & \mathbf{d} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

We use Proposition 4.5 to find a quotient of a path algebra which is isomorphic to the trivial extension

$$kQ \ltimes M.$$

The quiver $Q \circlearrowleft B$ is



The relation sets $\nu(Q,B)$ and $\xi(Q,B)$ are

$$\nu(Q,B) = \{\, \alpha\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{c}},\ \alpha\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{d}},\ \beta\overrightarrow{\mathbf{c}},\ \beta\overrightarrow{\mathbf{d}},\ \overrightarrow{\mathbf{a}}\alpha,\ \overrightarrow{\mathbf{c}}\alpha,\ \overrightarrow{\mathbf{b}}\beta - \overrightarrow{\mathbf{a}},\ \overrightarrow{\mathbf{d}}\beta - \overrightarrow{\mathbf{c}}\,\},$$
$$\xi(Q,B) = \{\, \overrightarrow{\mathbf{a}}\overrightarrow{\mathbf{c}},\ \overrightarrow{\mathbf{a}}\overrightarrow{\mathbf{d}},\ \overrightarrow{\mathbf{c}}^2,\ \overrightarrow{\mathbf{c}}\overrightarrow{\mathbf{d}}\,\}.$$

By Proposition 4.5, we have

$$kQ \ltimes M \cong \frac{k(Q \circlearrowleft B)}{\langle \nu(Q,B) \cup \xi(Q,B)\rangle},$$

with isomorphisms

$$\phi\colon kQ \ltimes M \to \frac{k(Q \circlearrowleft B)}{\langle \nu(Q,B) \cup \xi(Q,B)\rangle}$$

and

$$\psi\colon \frac{k(Q \circlearrowleft B)}{\langle \nu(Q,B) \cup \xi(Q,B)\rangle} \to kQ \ltimes M$$

given by

$$\begin{aligned} \phi(\lambda, m) &= [\lambda + \overrightarrow{m}] && \text{for } (\lambda, m) \in kQ \ltimes M, \\ \psi([q]) &= (q, 0) && \text{for a path } q \in Q_*, \\ \psi([\overrightarrow{b}]) &= (0, b) && \text{for a basis element } b \in B. \end{aligned}$$

Now we have a quotient of a path algebra which is isomorphic to the trivial extension $kQ \ltimes M$, but its relation set is not admissible. We will continue this example in Example 5.6, where we will find a quiver R and admissible relation set $\sigma \subseteq kR$ such that

$$kQ \ltimes M \cong \frac{k(Q \circlearrowleft B)}{\langle \nu(Q,B) \cup \xi(Q,B)\rangle} \cong kR/\langle\sigma\rangle\,.$$ ■

We now show that the relation set produced by Proposition 4.5 is preadmissible.

Proposition 4.6. *Let $kQ/\langle\rho\rangle$ be a quotient of a path algebra (with ρ an admissible relation set), M a $kQ/\langle\rho\rangle$-bimodule, and B a uniform basis for M. Then the relation set*

$$\rho \cup \nu(Q,B) \cup \xi(Q,B) \subseteq k(Q \circlearrowleft B)$$

is preadmissible.

Proof. We immediately have

$$\langle \rho \cup \nu(Q,B) \cup \xi(Q,B)\rangle \subseteq J_{Q \circlearrowleft B}.$$

Choose t such that

$$J_Q^t \subseteq \langle\rho\rangle\,.$$

We show that

$$J^{2t}_{Q \circlearrowleft B} \subseteq \langle \rho \cup \nu(Q,B) \cup \xi(Q,B) \rangle \, .$$

Let p be a path in $Q \circlearrowleft B$ of length $2t$. We have one of the following three cases, depending on how many arrows from $\overrightarrow{B}$ the path p contains:

1. If p does not contain any arrow from $\overrightarrow{B}$, then $p \in Q_*$, and we get $p \in \langle \rho \rangle$ since the length of p is greater than t.

2. If p contains exactly one arrow from $\overrightarrow{B}$, then

   $$p = q \, \overrightarrow{b} \, r$$

   for some basis element $b \in B$, and some paths q and r in Q. Since p has length $2t$, either q or r must have length at least t and thus lie in $\langle \rho \rangle$. So we get $p \in \langle \rho \rangle$.

3. If p contains two or more arrows from $\overrightarrow{B}$, then $p \in \langle \nu(Q,B) \cup \xi(Q,B) \rangle$ by Lemma 4.4.

In all cases, we get

$$p \in \langle \rho \cup \nu(Q,B) \cup \xi(Q,B) \rangle \, ,$$

so we have

$$J^{2t}_{Q \circlearrowleft B} \subseteq \langle \rho \cup \nu(Q,B) \cup \xi(Q,B) \rangle \, . \qquad \square$$

4.3 Return of the triangular matrix algebras

We will now return to the result we left unproved in Section 4.1, namely Proposition 4.1. We will see that a lower triangular matrix algebra is isomorphic to a trivial extension, and that the path algebra quotient we want for the triangular matrix algebra is equal to the one we produce for the trivial extension using Proposition 4.5.

We first describe how a lower triangular matrix algebra can be converted to a trivial extension.

Proposition 4.7. *Let Λ and Γ be k-algebras, and M a Λ–Γ-bimodule. Then M can be viewed as a $(\Lambda \times \Gamma)$-bimodule by*

$$(\lambda, \gamma)m = \lambda m,$$
$$m(\lambda, \gamma) = m\gamma$$

for elements $(\lambda, \gamma) \in \Lambda \times \Gamma$ and $m \in M$; and we have

$$\begin{pmatrix} \Gamma & 0 \\ M & \Lambda \end{pmatrix} \cong (\Lambda \times \Gamma) \ltimes M$$

as k-algebras, with isomorphisms given by

$$\begin{pmatrix} \gamma & 0 \\ m & \lambda \end{pmatrix} \leftrightarrow ((\lambda, \gamma), m)$$

for $\lambda \in \Lambda$, $\gamma \in \Gamma$ and $m \in M$.

♪ This result is given (without proof) on page 79 in [1].

Proof. It is easy to check that the multiplication defined above gives a $(\Lambda \times \Gamma)$-bimodule structure on M.

As k-vector spaces,

$$\begin{pmatrix} \Gamma & 0 \\ M & \Lambda \end{pmatrix} \qquad \text{and} \qquad (\Lambda \times \Gamma) \ltimes M$$

are clearly isomorphic, since they are both isomorphic to the direct sum $\Lambda \oplus \Gamma \oplus M$. The obvious vector space isomorphism is the function

$$\phi\colon \begin{pmatrix} \Gamma & 0 \\ M & \Lambda \end{pmatrix} \to (\Lambda \times \Gamma) \ltimes M$$

defined by

$$\phi \begin{pmatrix} \gamma & 0 \\ m & \lambda \end{pmatrix} = ((\lambda, \gamma), m).$$

Now we only need that ϕ is in fact a k-algebra homomorphism. We see this by checking that it preserves multiplication and the identity element:

$$\begin{aligned} \phi\left(\begin{pmatrix} \gamma_1 & 0 \\ m_1 & \lambda_1 \end{pmatrix} \cdot \begin{pmatrix} \gamma_2 & 0 \\ m_2 & \lambda_2 \end{pmatrix}\right) &= \phi \begin{pmatrix} \gamma_1\gamma_2 & 0 \\ m_1\gamma_2 + \lambda_1 m_2 & \lambda_1\lambda_2 \end{pmatrix} \\ &= ((\lambda_1\lambda_2, \gamma_1\gamma_2), m_1\gamma_2 + \lambda_1 m_2) \\ &= ((\lambda_1, \gamma_1), m_1) \cdot ((\lambda_2, \gamma_2), m_2) \\ &= \phi \begin{pmatrix} \gamma_1 & 0 \\ m_1 & \lambda_1 \end{pmatrix} \cdot \phi \begin{pmatrix} \gamma_2 & 0 \\ m_2 & \lambda_2 \end{pmatrix}, \\ \phi(1_{\left(\begin{smallmatrix} \Gamma & 0 \\ M & \Lambda \end{smallmatrix}\right)}) = \phi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= ((1,1),0) = 1_{(\Lambda \times \Gamma) \ltimes M}. \end{aligned}$$

□

In a similar way, we can express an augmented union of quivers as an augmented quiver.

Proposition 4.8. *Let Q and R be disjoint quivers, A a finite set, and h and t functions*

$$h\colon A \to Q_0,$$
$$t\colon A \to R_0.$$

Let h' and t' be the functions defined by extending the codomains of h and t to $(Q \cup R)_0$:

$$h' \colon A \to (Q \cup R)_0, \qquad h'(a) = h(a) \quad \text{for } a \in A;$$
$$t' \colon A \to (Q \cup R)_0, \qquad t'(a) = t(a) \quad \text{for } a \in A.$$

Then the augmented union of R and Q by (A, h, t) is the same as the augmented quiver of $Q \cup R$ by (A, h', t'):

$$R \xrightarrow{(A,h,t)} Q = (Q \cup R) \circlearrowleft (A, h', t').$$

Proof. This result follows immediately from the definitions of augmented union of quivers and augmented quiver, since the set $\overrightarrow{A}$ of new arrows is the same in both cases. □

We will now describe the relationship between a path algebra quotient on the form

$$\frac{k(R \xrightarrow{B} Q)}{\langle \rho \cup \sigma \cup \mu(Q, R, B) \rangle}$$

and one on the form

$$\frac{k\big((Q \cup R) \circlearrowleft B\big)}{\langle \rho \cup \sigma \cup \nu(Q \cup R, B) \cup \xi(Q \cup R, B) \rangle}.$$

The first of these is what we want to be isomorphic to the triangular matrix algebra

$$\begin{pmatrix} kR/\langle \sigma \rangle & 0 \\ M & kQ/\langle \rho \rangle \end{pmatrix},$$

and the second we know (by Proposition 4.5) to be isomorphic to the trivial extension

$$k(Q \cup R) \ltimes M.$$

The following proposition shows how the quivers and relation sets of these algebras are related.

Proposition 4.9. *Let Q and R be disjoint quivers, let M be a kQ–kR-bimodule, and let B be a uniform basis for M.*

Then M has an induced $k(Q \cup R)$-bimodule structure with scalar multiplication (which we denote by $*$) induced by

$$q * m = qm \qquad r * m = 0$$
$$m * q = 0 \qquad m * r = mr$$

for a module element $m \in M$ and paths $q \in Q_*$ and $r \in R_*$. The basis B is still uniform with respect to this module structure, and we have the equality

$$R \xrightarrow{B} Q = (Q \cup R) \circlearrowleft B$$

of quivers. The relation sets $\nu(Q \cup R, B)$ *and* $\xi(Q \cup R, B)$ *are given by*

$$\begin{aligned} \nu(Q \cup R, B) &= \mu(Q, R, B), \\ \xi(Q \cup R, B) &= \emptyset. \end{aligned}$$

Proof. It is easy to see that the multiplication $*$ described above gives a $k(Q \cup R)$-bimodule structure on M.

For any (u, v)-uniform element $m \in M$ (as kQ–kR-bimodule), we have

$$u * m * v = umv = m,$$

so m is still (u, v)-uniform when we view M as $k(Q \cup R)$-bimodule. Therefore, the basis B is uniform for M as $k(Q \cup R)$-bimodule, with the endpoints

$$\mathfrak{h}(b) \qquad \text{and} \qquad \mathfrak{t}(b)$$

of each basis element $b \in B$ the same as in the original bimodule structure. Then, by Proposition 4.8, we have

$$R \xrightarrow{B} Q = (Q \cup R) \circlearrowleft B.$$

Since $\mathfrak{h}(b) \in Q_0$ and $\mathfrak{t}(b) \in R_0$ for every basis element $b \in B$, and the quivers Q and R are disjoint, we have

$$\begin{aligned} \nu(Q \cup R, B) &= \left\{ \alpha \overrightarrow{b} - \overrightarrow{\alpha b} \;\middle|\; \alpha \in (Q \cup R)_1 \text{ and } b \in B \text{ with } \mathfrak{t}(\alpha) = \mathfrak{h}(b) \right\} \\ &\qquad \cup \left\{ \overrightarrow{b}\alpha - \overrightarrow{b\alpha} \;\middle|\; b \in B \text{ and } \alpha \in (Q \cup R)_1 \text{ with } \mathfrak{t}(b) = \mathfrak{h}(\alpha) \right\} \\ &= \left\{ \alpha \overrightarrow{b} - \overrightarrow{\alpha b} \;\middle|\; \alpha \in Q_1 \text{ and } b \in B \text{ with } \mathfrak{t}(\alpha) = \mathfrak{h}(b) \right\} \\ &\qquad \cup \left\{ \overrightarrow{b}\alpha - \overrightarrow{b\alpha} \;\middle|\; b \in B \text{ and } \alpha \in R_1 \text{ with } \mathfrak{t}(b) = \mathfrak{h}(\alpha) \right\} \\ &= \mu(Q, R, B). \end{aligned}$$

We see that

$$\mathfrak{t}(b_1) \neq \mathfrak{h}(b_2)$$

for any basis elements b_1 and b_2 in B, since $\mathfrak{t}(b_1)$ is a vertex in R, $\mathfrak{h}(b_2)$ is a vertex in Q, and the quivers Q and R are disjoint. Thus we have

$$\begin{aligned} \xi(Q, B) &= \left\{ \overrightarrow{b_1}\,\overrightarrow{b_2} \;\middle|\; b_1 \text{ and } b_2 \text{ in } B \text{ with } \mathfrak{t}(b_1) = \mathfrak{h}(b_2) \right\} \\ &= \emptyset. \end{aligned}$$

□

With the above results we have almost proved Proposition 4.1; what remains is only to put the pieces together.

Proof (of **Proposition 4.1***).* The following diagram shows how we get the isomorphism we want (the dotted arrow in the left column) by combining four propositions:

$$\begin{array}{ccc}
\begin{pmatrix} kR/\langle\sigma\rangle & 0 \\ M & kQ/\langle\rho\rangle \end{pmatrix} & \xleftrightarrow[\text{(Prop. 4.7)}]{\cong} & (kQ/\langle\rho\rangle \times kR/\langle\sigma\rangle) \ltimes M \\
 & & \updownarrow {\scriptstyle \text{(Prop. 1.3)}\ \cong} \\
\Big\updownarrow \cong & & k(Q\cup R)/\langle\rho\cup\sigma\rangle \ltimes M \\
 & & \updownarrow {\scriptstyle \text{(Prop. 4.5)}\ \cong} \\
\dfrac{k(R \xrightarrow{B} Q)}{\langle\rho\cup\sigma\cup\mu(Q,R,B)\rangle} & \xleftrightarrow[\text{(Prop. 4.9)}]{=} & \dfrac{k(Q\cup R) \circlearrowleft B}{\langle\rho\cup\sigma\cup\nu(Q\cup R,B)\cup\xi(Q\cup R,B)\rangle}
\end{array}$$

We get a $k(Q \cup R)$-bimodule structure on M in two ways: through propositions 4.7 and 1.3, and through Proposition 4.9. We need these bimodule structures to be the same for the above diagram to make sense. If we denote the scalar multiplication of M as $(kQ/\langle\rho\rangle \times kR/\langle\sigma\rangle)$-module by $\circledast$ and the scalar multiplication of M as $k(Q \cup R)$-module by $*$, then propositions 4.7 and 1.3 give

$$\begin{aligned}
[q] * m &= ([q], 0) \circledast m = [q]m \\
[r] * m &= (0, [r]) \circledast m = 0 \cdot m = 0 \\
m * [q] &= m \circledast ([q], 0) = m \cdot 0 = 0 \\
m * [r] &= m \circledast (0, [r]) = m[r]
\end{aligned}$$

for a module element $m \in M$ and paths $q \in Q_*$ and $r \in R_*$. This is the same as the $k(Q \cup R)$-module structure we get on M from Proposition 4.9.

Now we have established

$$\begin{pmatrix} kR/\langle\sigma\rangle & 0 \\ M & kQ/\langle\rho\rangle \end{pmatrix} \cong k(R \xrightarrow{B} Q)/\langle\rho \cup \sigma \cup \mu(Q,R,B)\rangle\,,$$

and we only need to check that our isomorphisms act the way we want on elements; namely,

$$\begin{pmatrix} 0 & 0 \\ 0 & [q] \end{pmatrix} \leftrightarrow [q] \qquad \text{for a path } q \in Q_*,$$

$$\begin{pmatrix} [r] & 0 \\ 0 & 0 \end{pmatrix} \leftrightarrow [r] \qquad \text{for a path } r \in R_*,$$

$$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \leftrightarrow \left[\overrightarrow{b}\right] \qquad \text{for a basis element } b \in B.$$

If we chase an element of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & [q] \end{pmatrix} \in \begin{pmatrix} kR/\langle\sigma\rangle & 0 \\ M & kQ/\langle\rho\rangle \end{pmatrix}$$

or

$$[q] \in k(R \xrightarrow{B} Q)/\langle\rho \cup \sigma \cup \mu(Q,R,B)\rangle$$

around our diagram, we get

$$\begin{array}{ccc} \begin{pmatrix} 0 & 0 \\ 0 & [q] \end{pmatrix} & \longleftrightarrow & (([q],0),0) \\ & & \updownarrow \\ & & ([q],0) \\ & & \updownarrow \\ [q] & \longleftrightarrow & [q] \end{array}$$

A completely similar chase gives

$$\begin{pmatrix} [r] & 0 \\ 0 & 0 \end{pmatrix} \leftrightarrow [r].$$

If we chase an element of the form

$$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \in \begin{pmatrix} kR/\langle\sigma\rangle & 0 \\ M & kQ/\langle\rho\rangle \end{pmatrix}$$

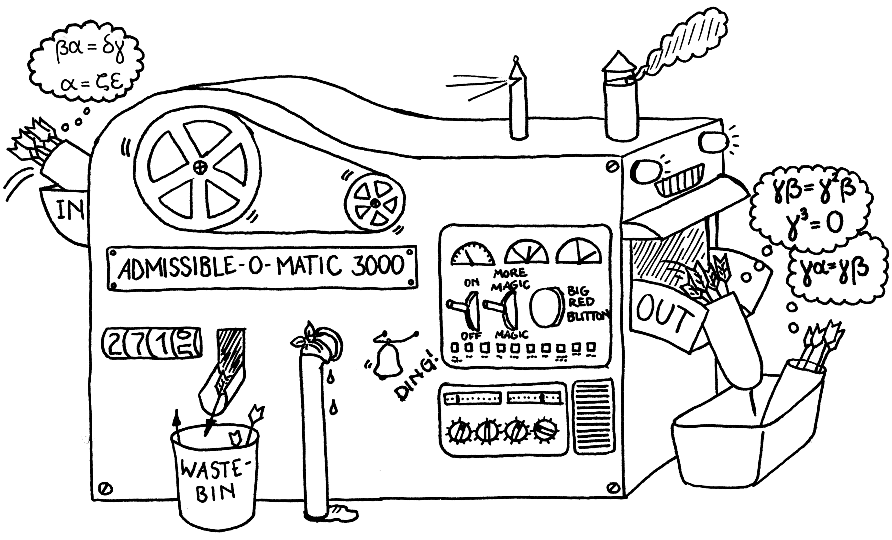
or

$$\left[\overrightarrow{b}\right] \in k(R \xrightarrow{B} Q)/\langle \rho \cup \sigma \cup \mu(Q,R,B)\rangle \,,$$

we get

$$\begin{array}{ccc} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} & \longleftrightarrow & ((0,0),b) \\ & & \updownarrow \\ & & (0,b) \\ & & \updownarrow \\ \left[\overrightarrow{b}\right] & \longleftrightarrow & \left[\overrightarrow{b}\right] \end{array}$$

□



$\beta\alpha = \delta\gamma$
$\alpha = \zeta\varepsilon$
IN
ADMISSIBLE-O-MATIC 3000
ON
OFF
MORE MAGIC
MAGIC
BIG RED BUTTON
DING!
WASTE-BIN
OUT
$\gamma\beta = \gamma^2\beta$
$\gamma^3 = 0$
$\gamma\alpha = \gamma\beta$

Chapter 5

From preadmissible to admissible relation sets

In Chapter 4, we constructed path algebra quotients $kQ/\langle\rho\rangle$ where the relation set ρ did not satisfy the usual admissibility condition

$$J_Q^t \subseteq \langle\rho\rangle \subseteq J_Q^2 \qquad \text{for some } t \geq 2, \tag{5.1}$$

but only the weaker condition

$$J_Q^t \subseteq \langle\rho\rangle \subseteq J_Q \qquad \text{for some } t \geq 2. \tag{5.2}$$

That is, the relations may contain paths of length 1.

Recall that we call a relation set ρ satisfying equation (5.1) *admissible*, and one satisfying equation (5.2) *preadmissible*. We usually want our relation sets to be admissible. In this chapter, we will look at how to turn a path algebra quotient $kQ/\langle\rho\rangle$ with ρ preadmissible into an isomorphic algebra $kQ'/\langle\rho'\rangle$ with ρ' admissible.

5.1 Idea

Let kQ be a path algebra, and $\rho \subseteq kQ$ a preadmissible set of relations. Our basic idea is that if we have a relation

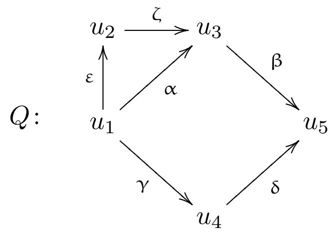
$$\alpha - \sum_{i \in I} c_i q_i \in \rho$$

for some arrow α, paths q_i and coefficients $c_i \in k$, then

$$[\alpha] = \left[\sum_{i \in I} c_i q_i\right]$$

in $kQ/\langle\rho\rangle$. We can thus remove α from the quiver without losing anything – the element that was represented by α in $kQ/\langle\rho\rangle$ is still represented by $\sum c_i q_i$. Before discussing why this idea does not work in general, we consider a simple example where it works.

Example 5.1. Let Q be the quiver

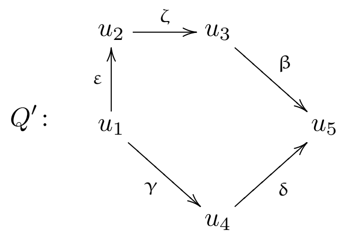


and let

$$\rho = \{\beta\alpha - \delta\gamma,\ \alpha - \zeta\varepsilon\}$$

be a set of relations in kQ. Then ρ is preadmissible but not admissible. The offending relation is $\alpha - \zeta\varepsilon$, so we try to remove the arrow α, replacing it by $\zeta\varepsilon$.

We then obtain the new quiver



and relation set

$$\rho' = \{\beta\zeta\varepsilon - \delta\gamma\}.$$

The relation we used for removing α, namely $\alpha - \zeta\varepsilon$, becomes zero when we replace α by $\zeta\varepsilon$. Therefore, this relation disappears in ρ'.

The new relation set ρ' is admissible, and we see that $kQ/\langle\rho\rangle \cong kQ'/\langle\rho'\rangle$ by the following correspondence between basis elements of the two algebras:

$$\begin{array}{rcll} kQ/\langle\rho\rangle & & kQ'/\langle\rho'\rangle & \\ [u_i] & \leftrightarrow & [u_i] & \text{for } i \in \{1,\dots,5\} \\ [\zeta\varepsilon] = [\alpha] & \leftrightarrow & [\zeta\varepsilon] & \\ [q] & \leftrightarrow & [q] & \text{for } q \in \{\beta,\gamma,\delta,\varepsilon,\zeta,\beta\zeta\} \\ [\beta\zeta\varepsilon] = [\beta\alpha] = [\delta\gamma] & \leftrightarrow & [\delta\gamma] = [\beta\zeta\varepsilon] & \end{array}$$

■

By generalization from this example, we get the following procedure for turning a path algebra quotient $kQ/\langle\rho\rangle$ with preadmissible relation set into an isomorphic algebra $kQ'/\langle\rho'\rangle$ with admissible relation set: As long as there are relations containing paths of length one left, we choose one such relation, say

$$\alpha - \sum_{i \in I} c_i q_i,$$

where α is an arrow in Q, the q_i are paths in Q and the c_i coefficients from k, such that α does not occur in any of the q_i. We remove this relation from ρ, and remove the arrow α from Q, replacing all references to α in other relations by $\sum c_i q_i$.

In certain situations, in particular when the quiver Q does not contain oriented cycles, this is sufficient. In the general situation, however, we may run into problems.

The problem is that there may exist relations on the form

$$\alpha - \sum_{i \in I} c_i q_i,$$

but only ones where the arrow α is contained in one or more of the paths q_i. In this case, it does not make sense to remove α from the quiver and replace it by $\sum c_i q_i$ in all the relations, because some relations would then still refer to the now non-existing arrow α.

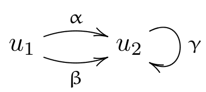
To solve this problem, we perform the substitution

$$\alpha \mapsto \sum_{i \in I} c_i q_i$$

repeatedly on the expression $\sum c_i q_i$, which we want to use as a replacement for α. By doing this, we will eventually get all occurences of α to disappear (modulo ρ). The following example illustrates the process.

Example 5.2. Let Q be the quiver

Q:



and let

$$\rho = \{\alpha - \gamma\alpha - \gamma\beta,\ \gamma^2\alpha,\ \gamma^3\}$$

be a set of relations in kQ. The relation keeping ρ from being admissible is

$$\alpha - \gamma\alpha - \gamma\beta.$$

We compute

$$\begin{aligned}\alpha &\overset{\rho}{\sim} \gamma\alpha + \gamma\beta \\ &\overset{\rho}{\sim} \gamma(\gamma\alpha + \gamma\beta) + \gamma\beta = \gamma^2\alpha + \gamma^2\beta + \gamma\beta \\ &\overset{\rho}{\sim} \gamma^2\beta + \gamma\beta.\end{aligned}$$

We first performed the substitution $\alpha \mapsto \gamma\alpha + \gamma\beta$ twice. Then we observed that in the new expression, $\gamma^2\alpha + \gamma^2\beta + \gamma\beta$, the term containing α was an element of $\langle\rho\rangle$, so we removed that term. The final expression is equivalent to α without containing α, and we can create a new quiver with relations by removing α from the quiver and substituting

$$\alpha \mapsto \gamma^2\beta + \gamma\beta$$

in the relations.

The new quiver is

$$Q'\colon \quad u_1 \xrightarrow[\beta]{} u_2 \circlearrowright \gamma\ ,$$

and the new relation set

$$\rho' = \{\gamma\beta - \gamma^2\beta - \gamma^3\beta,\ \gamma^4\beta + \gamma^3\beta,\ \gamma^3\}.$$

Note that here, unlike the case in Example 5.1, the relation we used for removing α does *not* disappear in the new relation set.

(Many of the terms in ρ' are clearly redundant; we can simplify it to

$$\rho'' = \{\gamma\beta - \gamma^2\beta,\ \gamma^3\},$$

which generates the same ideal.) ■

5.2 Removing an arrow

We now turn from specific examples to the general case. We will create an algorithm which follows the idea of Example 5.2.

To describe substitutions such as those performed in Example 5.2, we define a class of algebra homomorphisms, indexed by the arrow to be removed and its replacement.

Definition. Let Q be a quiver, α an arrow in Q, and s an α-uniform element of kQ. Then the **substitution map**

$$\mathrm{subst}_{(\alpha,s)}\colon kQ \to kQ$$

is an algebra homomorphism defined on vertices and arrows by

$$\mathrm{subst}_{(\alpha,s)}(q) = \begin{cases} q & \text{if } q \neq \alpha \\ s & \text{if } q = \alpha \end{cases}$$

for $q \in Q_?$. The conditions of Lemma 1.2 are satisfied, so this gives a well-defined algebra homomorphism. ■

♩ If the arrow α does not occur in the substitution value s, we will view the substitution map $\mathrm{subst}_{(\alpha,s)}$ as a homomorphism from kQ to kQ', where Q' is the quiver made by removing α from Q. ■

Example 5.3. First consider Example 5.1. The substitution induced from the relation

$$\alpha - \zeta\varepsilon,$$

which we used to get rid of α, is described by the substitution map

$$\mathrm{subst}_{(\alpha,\zeta\varepsilon)}\colon kQ \to kQ'.$$

For example, we have

$$\mathrm{subst}_{(\alpha,\zeta\varepsilon)}(\beta\alpha - \delta\gamma) = \beta\zeta\varepsilon - \delta\gamma.$$

Furthermore, the map

$$\phi\colon kQ/\langle\rho\rangle \to kQ'/\langle\rho'\rangle$$

given by

$$\phi([\lambda]) = [\mathrm{subst}_{(\alpha,\zeta\varepsilon)}(\lambda)]$$

is an isomorphism.

Now consider Example 5.2. The substitutions we used there are described by the substitution maps

$$\mathrm{subst}_{(\alpha,\gamma\alpha+\gamma\beta)}\colon kQ \to kQ$$

and

$$\mathrm{subst}_{(\alpha,\gamma^2\beta+\gamma\beta)}\colon kQ \to kQ'.$$

■

We will use substitution maps in the context of a quotient algebra $kQ/\langle\rho\rangle$ where the relation set ρ is preadmissible. We want our substitution maps to respect the relations ρ, in the sense that for any element λ of kQ, the element $\mathrm{subst}_{(\alpha,s)}(\lambda)$ is a representative of the same equivalence class as λ. The following lemma shows that this is true, given that α and s are representatives of the same equivalence class.

Lemma 5.1. *Let Q, Q', α and s be as in the definition above, and let ρ be a preadmissible set of relations in kQ, with*

$$\alpha \overset{\rho}{\sim} s.$$

Then $\mathrm{subst}_{(\alpha,s)}$ *takes any element of kQ to one that is equivalent modulo ρ. That is,*

$$\mathrm{subst}_{(\alpha,s)}(\lambda) \overset{\rho}{\sim} \lambda \qquad \text{for any } \lambda \in kQ.$$

Proof. We immediately have

$$\mathrm{subst}_{(\alpha,s)}(q) \overset{\rho}{\sim} q$$

for any vertex or arrow $q \in Q_?$. Since the equivalence relation $\overset{\rho}{\sim}$ respects multiplication and addition, this extends to

$$\mathrm{subst}_{(\alpha,s)}(\lambda) \overset{\rho}{\sim} \lambda$$

for any element $\lambda \in kQ$. □

We introduce some new notation which will be used in Algorithm 1.

Definition. Let $\lambda \in kQ$ be an element and $X \subseteq kQ$ a subset of a path algebra kQ. We define

$$\mathrm{terms}_X(\lambda) = \mathrm{terms}(\lambda) \cap \langle X\rangle$$

to be the set of terms in λ which are contained in the ideal generated by X. ■

♪ If α is an arrow, then

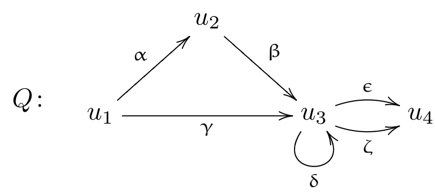
$$\text{terms}_{\{\alpha\}}(\lambda)$$

is the set of terms in λ which contain α. ■

Algorithm 1, Eliminate arrow$(\alpha, \mathfrak{r}_\alpha, Q, \rho)$, describes how we can turn a path algebra quotient with preadmissible relation set into one where the relation set is closer to being admissible. We will later (see Algorithm 2, page 99) use this repeatedly to reach a path algebra quotient with admissible relation set.

We demonstrate how the algorithm works by walking through it in an example.

Example 5.4. Let Q be the quiver



and let

$$\rho = \{\, \gamma - \delta\gamma - \beta\alpha,\ \delta^3,\ \epsilon - \epsilon\delta - \zeta + \zeta\delta^2,\ \zeta\gamma - \epsilon\beta\alpha \,\}$$

be a set of relations in kQ. We see that ρ is preadmissible, but not admissible.

We have three possible arrows to remove. We can either use the relation

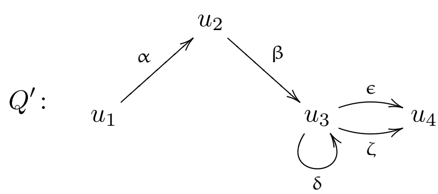
$$\gamma - \delta\gamma - \beta\alpha$$

to remove γ, or use the relation

$$\epsilon - \epsilon\delta - \zeta + \zeta\delta^2$$

to remove either ϵ or ζ. We choose the first of these possibilities. Thus, we will perform Eliminate arrow$(\gamma, \gamma - \delta\gamma - \beta\alpha, Q, \rho)$.

The first two lines of the algorithm set Q' to be the quiver



Then we get

$$c_\gamma = \text{coefficient}(\gamma, \gamma - \delta\gamma - \beta\alpha) = 1,$$

and

$$s_0 = \gamma - (\gamma - \delta\gamma - \beta\alpha) = \delta\gamma + \beta\alpha.$$

We initialize the loop counter i to 0. In the test of the **while** loop, we get

$$\text{terms}_{\{\gamma\}}(s_0) = \{\delta\gamma\} \neq \emptyset,$$

Algorithm 1 ELIMINATE ARROW$(\alpha, \mathfrak{r}_\alpha, Q, \rho)$

Input: A quiver Q, a preadmissible set of relations $\rho \subseteq kQ$, an arrow $\alpha \in Q_1$, and a relation $\mathfrak{r}_\alpha \in \rho$ with coefficient$(\alpha, \mathfrak{r}_\alpha) \neq 0$.

Output: A quiver Q' (which is equal to Q with the arrow α removed), a preadmissible set of relations $\rho' \subseteq kQ'$ and an element $s \in kQ'$, such that

$$kQ/\langle\rho\rangle \cong kQ'/\langle\rho'\rangle$$

as k-algebras, with isomorphisms given by

$$\begin{aligned} [\lambda] &\mapsto [\mathrm{subst}_{(\alpha,s)}(\lambda)] && \text{for } \lambda \in kQ, \\ [\lambda] &\leftarrow\!\shortmid [\lambda] && \text{for } \lambda \in kQ'. \end{aligned}$$

```
Q'_0 := Q_0
Q'_1 := Q_1 − {α}
c_α := coefficient(α, r_α)
s_0 := α − c_α^{−1} r_α      # Solve r_α = c_α(α − s_0) for s_0.
i := 0
while terms_{α}(s_i) ≠ ∅ :
      i := i + 1
      s̄_i := subst_(α,s_0)(s_{i−1})
      T_i := terms(s̄_i) − terms_ρ(s̄_i)
      s_i := ∑_{t∈T_i} t
n := i
s := s_n
ρ' := { subst_(α,s)(r) | r ∈ ρ }
return (Q', ρ', s)
```

so we enter the loop. The first iteration increments i to 1 and produces the values

$$\begin{aligned}
\overline{s_1} &= \text{subst}_{(\gamma,s_0)}(s_0) = \delta(\delta\gamma + \beta\alpha) + \beta\alpha = \delta^2\gamma + \delta\beta\alpha + \beta\alpha, \\
T_1 &= \text{terms}(\overline{s_1}) - \text{terms}_\rho(\overline{s_1}) = \{\, \delta^2\gamma,\ \delta\beta\alpha,\ \beta\alpha \,\} - \emptyset = \{\, \delta^2\gamma,\ \delta\beta\alpha,\ \beta\alpha \,\}, \\
s_1 &= \sum_{t \in T_1} t = \delta^2\gamma + \delta\beta\alpha + \beta\alpha.
\end{aligned}$$

Now we get

$$\text{terms}_{\{\gamma\}}(s_1) = \{\delta^2\gamma\} \neq \emptyset,$$

in the loop test, so we continue with a second iteration. The loop counter i is incremented to 2, and we get the values

$$\begin{aligned}
\overline{s_2} &= \text{subst}_{(\gamma,s_1)}(s_1) = \delta^2(\delta^2\gamma + \delta\beta\alpha + \beta\alpha) + \delta\beta\alpha + \beta\alpha \\
&= \delta^4\gamma + \delta^3\beta\alpha + \delta^2\beta\alpha + \delta\beta\alpha + \beta\alpha, \\
T_2 &= \text{terms}(\overline{s_2}) - \text{terms}_\rho(\overline{s_2}) = \{\, \delta^4\gamma,\ \delta^3\beta\alpha,\ \delta^2\beta\alpha,\ \delta\beta\alpha,\ \beta\alpha \,\} - \{\, \delta^4\gamma,\ \delta^3\beta\alpha \,\} \\
&= \{\, \delta^2\beta\alpha,\ \delta\beta\alpha,\ \beta\alpha \,\}, \\
s_2 &= \sum_{t \in T_2} t = \delta^2\beta\alpha + \delta\beta\alpha + \beta\alpha.
\end{aligned}$$

Now we have

$$\text{terms}_{\{\gamma\}}(s_2) = \emptyset,$$

so the loop stops here. The remaining lines set

$$\begin{aligned}
n &= 2, \\
s &= s_2 = \delta^2\beta\alpha + \delta\beta\alpha + \beta\alpha, \\
\rho' &= \{\, \text{subst}_{(\gamma,s)}(\mathfrak{r}) \mid \mathfrak{r} \in \rho \,\} \\
&= \{\, -\delta^3\beta\alpha,\ \delta^3,\ \epsilon - \epsilon\delta - \zeta + \zeta\delta^2,\ \zeta\delta^2\beta\alpha + \zeta\delta\beta\alpha + \zeta\beta\alpha - \epsilon\beta\alpha \,\}.
\end{aligned}$$

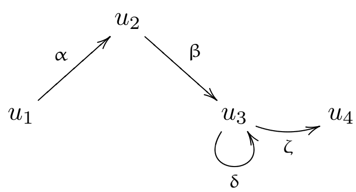
We observe that in our new algebra $kQ'/\langle\rho'\rangle$, the relation set is still not admissible, but now there is only one relation which contains paths of length one, namely,

$$\epsilon - \epsilon\delta - \zeta + \zeta\delta^2.$$

We apply our algorithm again to the new algebra. Now we can remove either ϵ or ζ; we choose ϵ. Thus, we perform ELIMINATE ARROW$(\epsilon, \epsilon - \epsilon\delta - \zeta + \zeta\delta^2, Q', \rho')$ to get a new quiver Q'' and relation set ρ''.

The first lines of the algorithm give us the new quiver

Q'':



and the initial substitution value

$$s_0 = \epsilon\delta + \zeta - \zeta\delta^2.$$

The first iteration of the **while** loop gives

$$\begin{aligned}
\overline{s_1} &= \mathrm{subst}_{(\epsilon,s_0)}(s_0) = \epsilon\delta^2 + \zeta\delta - \zeta\delta^3 + \zeta - \zeta\delta^2,\\
T_1 &= \mathrm{terms}(\overline{s_1}) - \mathrm{terms}_{\rho'}(\overline{s_1}) = \{\, \epsilon\delta^2,\ \zeta\delta,\ \zeta,\ -\zeta\delta^2 \,\},\\
s_1 &= \sum_{t\in T_1} t = \epsilon\delta^2 + \zeta\delta + \zeta - \zeta\delta^2.
\end{aligned}$$

The second iteration gives

$$\begin{aligned}
\overline{s_2} &= \mathrm{subst}_{(\epsilon,s_1)}(s_1) = \epsilon\delta^4 + \zeta\delta^3 - \zeta\delta^4 + \zeta\delta + \zeta,\\
T_2 &= \mathrm{terms}(\overline{s_2}) - \mathrm{terms}_{\rho'}(\overline{s_2}) = \{\, \zeta\delta,\ \zeta \,\},\\
s_2 &= \sum_{t\in T_2} t = \zeta\delta + \zeta.
\end{aligned}$$

We have

$$\mathrm{terms}_{\{\epsilon\}}(s_2) = \emptyset,$$

so the iteration stops here. The last part of the algorithm gives

$$\begin{aligned}
n &= 2,\\
s &= s_2 = \zeta\delta + \zeta,\\
\rho'' &= \{\, \mathrm{subst}_{(\epsilon,s)}(\mathfrak{r}) \mid \mathfrak{r} \in \rho' \,\} = \{\, -\delta^3\beta\alpha,\ \delta^3,\ 0,\ \zeta\delta^2\beta\alpha \,\}.
\end{aligned}$$

Now we have (provided the algorithm is correct)

$$kQ/\langle\rho\rangle \cong kQ'/\langle\rho'\rangle \cong kQ''/\langle\rho''\rangle\,,$$

and the relation set ρ'' is admissible. ■

Let us now show that the ELIMINATE ARROW algorithm works as intended. We will first (Proposition 5.2) show that it terminates, and then (Lemma 5.3, Lemma 5.4 and Proposition 5.5) that it produces a correct result.

Proposition 5.2. ELIMINATE ARROW$(\alpha, \mathfrak{r}_\alpha, Q, \rho)$ *terminates after at most* $t-2$ *iterations of the* ***while*** *loop, where* t *is a path length bound for* $kQ/\langle\rho\rangle$ *(that is,* $J_Q^t \subseteq \langle\rho\rangle$*).*

Proof. The main observation to make is that any path which occurs in s_i and contains α has length at least $i+2$. We shall show this soon, but let us first see why it is sufficient for proving the proposition.

If the **while** loop runs through iteration number $t-2$, then s_{t-2} is produced. Since any path in s_{t-2} which contains α has length at least t, such a path must lie in $\langle\rho\rangle$. But s_{t-2} is defined to be the sum of the terms of $\overline{s_{t-2}}$ which are *not* in $\langle\rho\rangle$, so there is no such path. Therefore,

$$\text{terms}_{\{\alpha\}}(s_{t-2}) = \emptyset,$$

and the loop does not continue.

Let us now prove the result we claimed above. For any element $\lambda \in kQ$, denote by $m(\lambda)$ the minimal length of a path in λ containing α, or ∞ if there is no such path; that is,

$$m(\lambda) = \min\big(\{\, l(q) \mid q \in \langle\alpha\rangle \text{ with coefficient}(q,\lambda) \neq 0 \,\} \cup \{\infty\}\big).$$

Then our claim is that

$$m(s_i) \geq i + 2$$

for every s_i produced by the algorithm. We will show this by induction.

For the base case of the induction, we need to show that $m(s_0) \geq 2$. The definition of s_0 ensures that

$$\text{coefficient}(\alpha, s_0) = 0,$$

and thus any path in s_0 containing α must have length at least 2.

For the inductive step, we begin by showing that

$$m(\text{subst}_{(\alpha,s_0)}(\lambda)) \geq m(\lambda) + 1$$

for any $\lambda \in kQ$. Any path q in

$$\text{subst}_{(\alpha,s_0)}(\lambda)$$

must be equal to some path q' in λ, with every occurence of α replaced by some path in s_0. Assume q to be a path of minimal length containing α.[1] Then q' must contain α as well, and we have

$$q' = q_0 \prod_{j=1}^{m} (\alpha q_j)$$

for some positive integer m, and paths $q_0, \ldots, q_m$ which do not contain α. We get

$$q = q_0 \prod_{j=1}^{m} (r_j q_j)$$

for some paths $r_1, \ldots, r_m$ in s_0. Since ρ is preadmissible, we have

$$\text{minlength}(s_0) \geq 1,$$

[1] If no path of $\text{subst}_{(\alpha,s_0)}(\lambda)$ contains α, then

$$m(\text{subst}_{(\alpha,s_0)}(\lambda)) = \infty \geq m(\lambda) + 1,$$

and we are done.

and thus

$$l(r_j) \geq 1$$

for every r_j. This means that

$$l(q) \geq l(q').$$

But since we assume that q contains α, some r_j must contain α and thus have length at least 2 (since $m(s_0) \geq 2$). Thus

$$l(q) \geq l(q') + 1.$$

So we have

$$m(\mathrm{subst}_{(\alpha,s_0)}(\lambda)) = l(q) \geq l(q') + 1 \geq m(\lambda) + 1.$$

Now the inductive step follows easily: Assuming

$$m(s_i) \geq i + 2,$$

we get

$$m(s_{i+1}) \geq m(\overline{s_{i+1}}) = m(\mathrm{subst}_{(\alpha,s_0)}(s_i)) \geq m(s_i) + 1 = i + 3. \qquad \square$$

Now that we know our algorithm terminates, we will show its correctness. We first show some technical details in the following two lemmata.

Lemma 5.3. *In* ELIMINATE ARROW$(\alpha, \mathfrak{r}_\alpha, Q, \rho)$, *the substitution value s is equivalent to α modulo ρ; that is,*

$$[\alpha] = [s] \qquad \text{in } kQ/\langle\rho\rangle.$$

Proof. We have

$$\alpha - s_0 = c_\alpha^{-1}\mathfrak{r}_\alpha \in \langle\rho\rangle\,,$$

so $\alpha \overset{\rho}{\sim} s_0$.

For any $i \in \{1, \dots n\}$, we have

$$s_{i-1} \overset{\rho}{\sim} \mathrm{subst}_{(\alpha,s_0)}(s_{i-1}) = \overline{s_i} \overset{\rho}{\sim} s_i,$$

where the first equivalence follows from Lemma 5.1, and the last from the fact that s_i is obtained from $\overline{s_i}$ by removing terms that lie in $\langle\rho\rangle$.

By repeated application of the above, we have

$$\alpha \overset{\rho}{\sim} s_0 \overset{\rho}{\sim} s_1 \overset{\rho}{\sim} \cdots \overset{\rho}{\sim} s_n = s. \qquad \square$$

Lemma 5.4. *In* ELIMINATE ARROW$(\alpha, \mathfrak{r}_\alpha, Q, \rho)$,

$$\langle\rho'\rangle = \langle\rho\rangle \cap kQ'.$$

Proof. We have

$$\langle \rho' \rangle \subseteq kQ' \subseteq kQ.$$

We will first show $\langle \rho' \rangle \subseteq \langle \rho \rangle$, then $\langle \rho \rangle \cap kQ' \subseteq \langle \rho' \rangle$.

Any element of ρ' is of the form

$$\mathrm{subst}_{(\alpha,s)}(\mathfrak{r})$$

for some $\mathfrak{r} \in \rho$. Since $\alpha \overset{\rho}{\sim} s$ (Lemma 5.3), we have by Lemma 5.1 that

$$\mathrm{subst}_{(\alpha,s)}(\mathfrak{r}) - \mathfrak{r} \in \langle \rho \rangle \,,$$

and thus

$$\mathrm{subst}_{(\alpha,s)}(\mathfrak{r}) \in \langle \rho \rangle \,.$$

This means that $\rho' \subseteq \langle \rho \rangle$. Thus

$$\langle \rho' \rangle \subseteq \langle \rho \rangle \,,$$

where both ideals are in kQ. But since $kQ' \subseteq kQ$, the ideal generated by ρ' in kQ' is contained in the ideal generated by ρ' in kQ.

Now we will show $\langle \rho \rangle \cap kQ' \subseteq \langle \rho' \rangle$. Let

$$\lambda \in \langle \rho \rangle \cap kQ'.$$

Then $\mathrm{subst}_{(\alpha,s)}(\lambda) \in \langle \rho' \rangle$ since $\lambda \in \langle \rho \rangle$, and $\mathrm{subst}_{(\alpha,s)}(\lambda) = \lambda$ since $\lambda \subseteq kQ'$. Thus

$$\lambda \in \langle \rho' \rangle \,. \qquad \square$$

Proposition 5.5. *The algorithm* Eliminate arrow$(\alpha, \mathfrak{r}_\alpha, Q, \rho)$ *produces a correct result. More precisely,*

$$kQ/\langle \rho \rangle \cong kQ'/\langle \rho' \rangle$$

as k-algebras, with the maps

$$[\lambda] \mapsto [\mathrm{subst}_{(\alpha,s)}(\lambda)] \qquad \text{for } \lambda \in kQ,$$
$$[\lambda] \leftarrow [\lambda] \qquad \text{for } \lambda \in kQ'$$

being isomorphisms; and ρ' is a preadmissible set of relations.

Proof. Let

$$\pi \colon kQ \to kQ/\langle \rho \rangle \,,$$
$$\pi' \colon kQ' \to kQ'/\langle \rho' \rangle$$

be the natural projections.

Define the algebra homomorphism $\overline{\phi} \colon kQ \to kQ'/\langle \rho' \rangle$ by

$$\overline{\phi} = \pi' \circ \mathrm{subst}_{(\alpha,s)} \colon kQ \xrightarrow{\mathrm{subst}_{(\alpha,s)}} kQ' \xrightarrow{\pi'} kQ'/\langle \rho' \rangle \,.$$

Since, by the definition of ρ',

$$\mathrm{subst}_{(\alpha,s)}(\rho) = \rho',$$

we have $\overline{\phi}(\rho) = 0$. Thus $\overline{\phi}$ induces an algebra homomorphism

$$\phi\colon kQ/\langle\rho\rangle \to kQ'/\langle\rho'\rangle\,,$$

with

$$\phi([\lambda]) = [\mathrm{subst}_{(\alpha,s)}(\lambda)] \qquad \text{for } \lambda \in kQ.$$

For the opposite direction, define the algebra homomorphism $\overline{\psi}\colon kQ' \to kQ/\langle\rho\rangle$ by

$$\overline{\psi} = \pi \circ \mathrm{inc}\colon kQ' \hookrightarrow kQ \xrightarrow{\pi} kQ/\langle\rho\rangle\,.$$

Since $\rho' \subseteq \langle\rho\rangle$ (by Lemma 5.4), we have $\overline{\psi}(\rho') = 0$, and thus $\overline{\psi}$ induces an algebra homomorphism

$$\psi\colon kQ'/\langle\rho'\rangle \to kQ/\langle\rho\rangle\,,$$

with

$$\psi([\lambda]) = [\lambda] \qquad \text{for any } \lambda \in kQ'.$$

Now it is easy to see that ϕ and ψ are inverses of each other:

$$\begin{aligned} \phi\psi([\lambda]) &= \phi([\lambda]) = [\mathrm{subst}_{(\alpha,s)}(\lambda)] \stackrel{!}{=} [\lambda] && \text{for } \lambda \in kQ', \\ \psi\phi([\lambda]) &= \psi([\mathrm{subst}_{(\alpha,s)}(\lambda)]) \stackrel{*}{=} \psi([\lambda]) = [\lambda] && \text{for } \lambda \in kQ. \end{aligned}$$

The equality marked "!" follows from the fact that

$$\mathrm{subst}_{(\alpha,s)}(\lambda) = \lambda$$

for any $\lambda \in kQ'$, since Q' does not contain α. The equality marked "*" follows from Lemma 5.1 and Lemma 5.3.

We have thus established the desired isomorphisms. All that is left now is to check that ρ' is preadmissible.

Since ρ is preadmissible, we have

$$J_Q^t \subseteq \langle\rho\rangle \subseteq J_Q$$

for some t. Taking intersections with kQ' gives

$$J_Q^t \cap kQ' \subseteq \langle\rho\rangle \cap kQ' \subseteq J_Q \cap kQ'.$$

Since

$$J_{Q'}^i = J_Q^i \cap kQ'$$

for any i, and $\langle\rho\rangle \cap kQ' = \langle\rho'\rangle$ (by Lemma 5.4), this means that

$$J_{Q'}^t \subseteq \langle\rho'\rangle \subseteq J_{Q'},$$

so ρ' is preadmissible. ☐

5.3 Repeat until admissible

To go all the way from a path algebra quotient with preadmissible relation set to one with admissible relation set, we may need to perform the ELIMINATE ARROW algorithm many times. In order to describe the resulting isomorphism from the original algebra to the new one, we define a generalization of the substitution map.

Definition. Let Q and Q' be quivers with

$$\begin{aligned} Q'_0 &= Q_0 \\ Q'_1 &= Q_1 - \{\alpha_1, \ldots, \alpha_n\}, \end{aligned}$$

for some arrows $\alpha_1, \ldots, \alpha_n$ in Q, and let $s_1, \ldots, s_n$ be elements of kQ' such that s_i is α_i-uniform for each i. Then the **translation map**

$$\mathrm{tr}_{\{(\alpha_1,s_1),\ldots,(\alpha_n,s_n)\}} \colon kQ \to kQ'$$

is an algebra homomorphism defined on vertices and arrows by

$$\mathrm{tr}_{\{(\alpha_1,s_1),\ldots,(\alpha_n,s_n)\}}(q) = \begin{cases} s_i & \text{if } q = \alpha_i \text{ (for some } i \in \{1,\ldots,n\}), \\ q & \text{otherwise,} \end{cases}$$

for $q \in Q_?$. The conditions of Lemma 1.2 are satisfied, so this gives a well-defined algebra homomorphism. ■

♪ A translation map is like several substitution maps performed simultaneously. It is clear that

$$\mathrm{tr}_{\{(\alpha_1,s_1),\ldots,(\alpha_n,s_n)\}} = \mathrm{subst}_{(\alpha_1,s_1)} \circ \cdots \circ \mathrm{subst}_{(\alpha_n,s_n)}$$

whenever the translation map is defined. Furthermore, this function is independent of the order of the (α_i, s_i) pairs. ■

Algorithm 2, PREADMISSIBLE TO ADMISSIBLE(Q, ρ), describes the complete process for turning a path algebra quotient with preadmissible relation set into one with admissible relation set. Note that all the interesting work here is done by the ELIMINATE ARROW algorithm; PREADMISSIBLE TO ADMISSIBLE simply applies that algorithm repeatedly (in the **while** loop) until the resulting relation set is admissible, then (in the **for** loop) combines all the generated substitutions to make one translation map.

It is clear that the algorithm terminates, since one arrow is removed in each iteration of the **while** loop.

We will now show that the algorithm is correct. Certain technical details regarding the translation map are proved separately in the following two lemmata, before the main result is given in Proposition 5.8.

Lemma 5.6. *In* PREADMISSIBLE TO ADMISSIBLE(Q, ρ)*, each substitution value* s_i *(for* $i \in \{1, \ldots, n\}$*) is an* α_i*-uniform element of* kQ' *(this means that the translation map* $\mathrm{tr}_{\{(\alpha_1,s_1),\ldots,(\alpha_n,s_n)\}}$ *is defined).*

Algorithm 2 PREADMISSIBLE TO ADMISSIBLE(Q, ρ)

Input: A quiver Q and a preadmissible set of relations $\rho \subseteq kQ$.
Output: A quiver Q', an admissible set of relations ρ' and a set

$$\{(\alpha_1, s_1), \ldots, (\alpha_1, s_n)\}$$

with each α_i an arrow in Q and each s_i an element of kQ; such that

$$kQ/\langle\rho\rangle \cong kQ'/\langle\rho'\rangle$$

as k-algebras, with isomorphisms given by

$$\begin{aligned} [\lambda] &\mapsto [\mathrm{tr}_{\{(\alpha_1,s_1),\ldots,(\alpha_n,s_n)\}}(\lambda)] && \text{for } \lambda \in kQ, \\ [\lambda] &\leftarrow\!\shortmid [\lambda] && \text{for } \lambda \in kQ'. \end{aligned}$$

$Q^{(0)} := Q$
$\rho_0 := \rho$
$i := 0$
while there is some relation $\mathfrak{r} \in \rho_i$ with minlength$(\mathfrak{r}) = 1$:
    $i := i + 1$
    Choose a relation $\mathfrak{r}_i \in \rho_{i-1}$ with minlength$(\mathfrak{r}_i) = 1$
    Choose an arrow $\alpha_i \in Q_1^{(i-1)}$ with coefficient$(\alpha_i, \mathfrak{r}_i) \neq 0$.
    $(Q^{(i)}, \rho_i, \overline{s_i}) :=$ ELIMINATE ARROW$(\alpha_i, \mathfrak{r}_i, Q^{(i-1)}, \rho_{i-1})$
$n := i$
$s_n := \overline{s_n}$
for $j \in (n-1, n-2, \ldots, 1)$:
    $s_j := \mathrm{tr}_{\{(\alpha_{j+1},s_{j+1}),\ldots,(\alpha_n,s_n)\}}(\overline{s_j})$
$Q' := Q^{(n)}$
$\rho' := \rho_n$
return $(Q', \rho', \{(\alpha_1, s_1), \ldots, (\alpha_n, s_n)\})$

Proof. First note that every $\overline{s_i}$ is an α_i-uniform element of $kQ^{(i)}$.

We will show the desired result by induction on i from n to 1. The case $i = n$ is clear, since $s_n = \overline{s_n}$, which is an α_n-uniform element of $kQ^{(n)} = kQ'$.

For the inductive step, assume that s_j is an α_j-uniform element of kQ' for every $j \in \{i+1, \dots, n\}$. This ensures that the translation map

$$\mathrm{tr}_{\{(\alpha_{j+1}, s_{j+1}), \dots, (\alpha_n, s_n)\}}$$

in line 12 of the algorithm is defined. Since $\overline{s_i}$ lies in $kQ^{(i)}$, it does not contain any of the arrows $\alpha_1, \dots, \alpha_i$. After applying the translation map, any occurences of the arrows $\alpha_{i+1}, \dots, \alpha_n$ are replaced by elements of kQ'. Thus, s_i is an element of kQ'. Furthermore, applying a translation map does not destroy uniformity, so s_i is α_i-uniform. □

Lemma 5.7. *In* Preadmissible to admissible(Q, ρ),

$$\mathrm{subst}_{(\alpha_n, \overline{s_n})} \circ \cdots \circ \mathrm{subst}_{(\alpha_1, \overline{s_1})} = \mathrm{tr}_{\{(\alpha_1, s_1), \dots, (\alpha_n, s_n)\}} .$$

Proof. We use induction on i from n to 1, showing

$$\mathrm{subst}_{(\alpha_n, \overline{s_n})} \circ \cdots \circ \mathrm{subst}_{(\alpha_i, \overline{s_i})} = \mathrm{tr}_{\{(\alpha_i, s_i), \dots, (\alpha_n, s_n)\}}$$

for every i. For the base case ($i = n$), we have

$$\mathrm{subst}_{(\alpha_n, \overline{s_n})} = \mathrm{subst}_{(\alpha_n, s_n)} = \mathrm{tr}_{\{(\alpha_n, s_n)\}} .$$

For the inductive step, assume

$$\mathrm{subst}_{(\alpha_n, \overline{s_n})} \circ \cdots \circ \mathrm{subst}_{(\alpha_{i+1}, \overline{s_{i+1}})} = \mathrm{tr}_{\{(\alpha_{i+1}, s_{i+1}), \dots, (\alpha_n, s_n)\}} .$$

To show that

$$\mathrm{subst}_{(\alpha_n, \overline{s_n})} \circ \cdots \circ \mathrm{subst}_{(\alpha_i, \overline{s_i})} = \mathrm{tr}_{\{(\alpha_i, s_i), \dots, (\alpha_n, s_n)\}},$$

it is enough to show that the two maps give the same result on any arrow and vertex of Q, since they are algebra homomorphisms. For α_i, we have

$$\begin{aligned}\big(\mathrm{subst}_{(\alpha_n, \overline{s_n})} \circ \cdots \circ \mathrm{subst}_{(\alpha_i, \overline{s_i})}\big)(\alpha_i) &= \big(\mathrm{tr}_{\{(\alpha_{i+1}, s_{i+1}), \dots, (\alpha_n, s_n)\}} \circ \mathrm{subst}_{(\alpha_i, \overline{s_i})}\big)(\alpha_i) \\ &= \mathrm{tr}_{\{(\alpha_{i+1}, s_{i+1}), \dots, (\alpha_n, s_n)\}}(\overline{s_i}) \\ &= s_i \\ &= \mathrm{tr}_{\{(\alpha_i, s_i), \dots, (\alpha_n, s_n)\}}(\alpha_i).\end{aligned}$$

For any other arrow or vertex $q \in Q_? - \{\alpha_i\}$, we have

$$\begin{aligned}\big(\mathrm{subst}_{(\alpha_n, \overline{s_n})} \circ \cdots \circ \mathrm{subst}_{(\alpha_i, \overline{s_i})}\big)(q) &= \big(\mathrm{subst}_{(\alpha_n, \overline{s_n})} \circ \cdots \circ \mathrm{subst}_{(\alpha_{i+1}, \overline{s_{i+1}})}\big)(q) \\ &= \mathrm{tr}_{\{(\alpha_{i+1}, s_{i+1}), \dots, (\alpha_n, s_n)\}}(q) \\ &= \mathrm{tr}_{\{(\alpha_i, s_i), \dots, (\alpha_n, s_n)\}}(q),\end{aligned}$$

since the substitution of s_i for α_i does not affect q. □

Proposition 5.8. *The algorithm* Preadmissible to admissible(Q, ρ) *produces a correct result.*

Proof. We need to show two things:

1. That $kQ/\langle\rho\rangle \cong kQ'/\langle\rho'\rangle$ by the maps

$$[\lambda] \mapsto [\mathrm{tr}_{\{(\alpha_1,s_1),\dots,(\alpha_n,s_n)\}}] \qquad \text{for } \lambda \in kQ,$$
$$[\lambda] \leftarrow\!\shortmid [\lambda] \qquad \text{for } \lambda \in kQ'.$$

2. That ρ' is admissible.

From Proposition 5.5, we have

$$kQ/\langle\rho\rangle = kQ^{(0)}/\langle\rho_0\rangle \cong kQ^{(1)}/\langle\rho_1\rangle \cong \cdots \cong kQ^{(n)}/\langle\rho_n\rangle = kQ'/\langle\rho'\rangle\,.$$

We compose the isomorphisms we get from Proposition 5.5 to obtain the isomorphisms we want. From right to left, all the maps are $[\lambda] \mapsto [\lambda]$, so their composition is also $[\lambda] \mapsto [\lambda]$. From left to right, the composition of isomorphisms is

$$\mathrm{subst}_{(\alpha_n,\overline{s_n})} \circ \cdots \circ \mathrm{subst}_{(\alpha_1,\overline{s_1})},$$

which by Lemma 5.7 is the same as

$$\mathrm{tr}_{\{(\alpha_1,s_1),\dots,(\alpha_n,s_n)\}}\,.$$

The Eliminate arrow algorithm ensures that each relation set ρ_i is preadmissible, so $\rho' = \rho_n$ is preadmissible. Since every relation $\mathfrak{r} \in \rho'$ has

$$\mathrm{minlength}(\mathfrak{r}) > 1$$

by the condition of the **while** loop, we also have

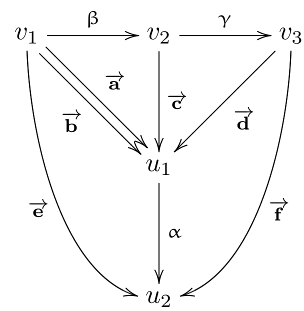
$$\langle\rho'\rangle \subseteq J^2_{Q'}.$$

□

We will now apply our algorithm to two examples from the previous chapter.

Example 5.5. This example is a continuation of Example 4.2. We have the quiver

$R \xrightarrow{B} Q$:



and preadmissible relation set

$$\mu(Q,R,B) = \{\, \alpha\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{e}},\ \alpha\overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{e}},\ \alpha\overrightarrow{\mathbf{c}},\ \alpha\overrightarrow{\mathbf{d}} - \overrightarrow{\mathbf{f}},\ \overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}},\ \overrightarrow{\mathbf{d}}\gamma - \overrightarrow{\mathbf{c}},\ \overrightarrow{\mathbf{f}}\gamma \,\}$$

for the triangular matrix algebra

$$\begin{pmatrix} kR & 0 \\ M & kQ \end{pmatrix}.$$

We will apply PREADMISSIBLE TO ADMISSIBLE$(R \xrightarrow{B} Q, \mu(Q,R,B))$ to get a new quiver S and an admissible relation set τ such that

$$\begin{pmatrix} kR & 0 \\ M & kQ \end{pmatrix} \cong k(R \xrightarrow{B} Q)/\langle \mu(Q,R,B) \rangle \cong kS/\langle \tau \rangle\,.$$

We start with $\rho_0 = \mu(Q,R,B)$. The **while** loop produces the following values (note that there is an arbitrary choice involved for each $\mathfrak{r}_i$ except $\mathfrak{r}_4$, and that we do not include relations which are zero).

First iteration:

$$\mathfrak{r}_1 = \alpha\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{e}}$$
$$\alpha_1 = \overrightarrow{\mathbf{e}}$$
$$\overline{s_1} = \alpha\overrightarrow{\mathbf{a}}$$
$$\rho_1 = \{\alpha\overrightarrow{\mathbf{b}} + \alpha\overrightarrow{\mathbf{a}},\ \alpha\overrightarrow{\mathbf{c}},\ \alpha\overrightarrow{\mathbf{d}} - \overrightarrow{\mathbf{f}},\ \overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}},\ \overrightarrow{\mathbf{d}}\gamma - \overrightarrow{\mathbf{c}},\ \overrightarrow{\mathbf{f}}\gamma\}$$

Second iteration:

$$\mathfrak{r}_2 = \alpha\overrightarrow{\mathbf{d}} - \overrightarrow{\mathbf{f}}$$
$$\alpha_2 = \overrightarrow{\mathbf{f}}$$
$$\overline{s_2} = \alpha\overrightarrow{\mathbf{d}}$$
$$\rho_2 = \{\alpha\overrightarrow{\mathbf{b}} + \alpha\overrightarrow{\mathbf{a}},\ \alpha\overrightarrow{\mathbf{c}},\ \overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}},\ \overrightarrow{\mathbf{d}}\gamma - \overrightarrow{\mathbf{c}},\ \alpha\overrightarrow{\mathbf{d}}\gamma\}$$

Third iteration:

$$\mathfrak{r}_3 = \overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}}$$
$$\alpha_3 = \overrightarrow{\mathbf{a}}$$
$$\overline{s_3} = \overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{b}}$$
$$\rho_3 = \{\alpha\overrightarrow{\mathbf{c}}\beta,\ \alpha\overrightarrow{\mathbf{c}},\ \overrightarrow{\mathbf{d}}\gamma - \overrightarrow{\mathbf{c}},\ \alpha\overrightarrow{\mathbf{d}}\gamma\}$$

Fourth iteration:

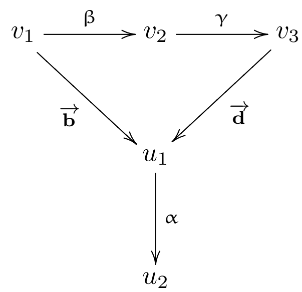
$$\begin{aligned}
\mathfrak{r}_4 &= \overrightarrow{\mathbf{d}}\gamma - \overrightarrow{\mathbf{c}} \\
\alpha_4 &= \overrightarrow{\mathbf{c}} \\
\overline{s_4} &= \overrightarrow{\mathbf{d}}\gamma \\
\rho_4 &= \{\alpha\overrightarrow{\mathbf{d}}\gamma\beta,\ \alpha\overrightarrow{\mathbf{d}}\gamma\}
\end{aligned}$$

Since there is no relation $\mathfrak{r}$ in ρ_4 with minlength$(\mathfrak{r}) = 1$, the **while** loop terminates here, and we get $n = 4$. Line 10 and the **for** loop produce the following values:

$$\begin{aligned}
s_4 &= \overline{s_4} = \overrightarrow{\mathbf{d}}\gamma \\
s_3 &= \operatorname{tr}_{\{(\alpha_4,s_4)\}}(\overline{s_3}) = \operatorname{tr}_{\{(\overrightarrow{\mathbf{c}},\overrightarrow{\mathbf{d}}\gamma)\}}(\overrightarrow{\mathbf{c}}\beta - \overrightarrow{\mathbf{b}}) = \overrightarrow{\mathbf{d}}\gamma\beta - \overrightarrow{\mathbf{b}} \\
s_2 &= \operatorname{tr}_{\{(\alpha_3,s_3),(\alpha_4,s_4)\}}(\overline{s_2}) = \operatorname{tr}_{\{(\overrightarrow{\mathbf{a}},\overrightarrow{\mathbf{d}}\gamma\beta-\overrightarrow{\mathbf{b}}),(\overrightarrow{\mathbf{c}},\overrightarrow{\mathbf{d}}\gamma)\}}(\alpha\overrightarrow{\mathbf{d}}) = \alpha\overrightarrow{\mathbf{d}} \\
s_1 &= \operatorname{tr}_{\{(\alpha_2,s_2),(\alpha_3,s_3),(\alpha_4,s_4)\}}(\overline{s_1}) = \operatorname{tr}_{\{(\overrightarrow{\mathbf{f}},\alpha\overrightarrow{\mathbf{d}}),(\overrightarrow{\mathbf{a}},\overrightarrow{\mathbf{d}}\gamma\beta-\overrightarrow{\mathbf{b}}),(\overrightarrow{\mathbf{c}},\overrightarrow{\mathbf{d}}\gamma)\}}(\alpha\overrightarrow{\mathbf{a}}) \\
&= \alpha\overrightarrow{\mathbf{d}}\gamma\beta - \alpha\overrightarrow{\mathbf{b}}
\end{aligned}$$

The resulting quiver S is $R \xrightarrow{B} Q$ with the arrows $\overrightarrow{\mathbf{e}}$, $\overrightarrow{\mathbf{f}}$, $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{c}}$ removed; that is,

S:



The resulting relation set is

$$\rho_4 = \{\alpha\overrightarrow{\mathbf{d}}\gamma\beta,\ \alpha\overrightarrow{\mathbf{d}}\gamma\}.$$

It is clear that the first relation here is superfluous, so we let

$$\tau = \{\alpha\overrightarrow{\mathbf{d}}\gamma\}.$$

We then have

$$k(R \xrightarrow{B} Q)/\langle\mu(Q,R,B)\rangle \cong kS/\langle\tau\rangle\,,$$

with isomorphisms given by

$$\begin{aligned}
&[\lambda] \mapsto \left[\operatorname{tr}_{\{(\overrightarrow{\mathbf{e}},\alpha\overrightarrow{\mathbf{d}}\gamma\beta-\alpha\overrightarrow{\mathbf{b}}),(\overrightarrow{\mathbf{f}},\alpha\overrightarrow{\mathbf{d}}),(\overrightarrow{\mathbf{a}},\overrightarrow{\mathbf{d}}\gamma\beta-\overrightarrow{\mathbf{b}}),(\overrightarrow{\mathbf{c}},\overrightarrow{\mathbf{d}}\gamma)\}}(\lambda)\right] && \text{for } \lambda \in k(R \xrightarrow{B} Q), \\
&[\lambda] \leftarrow\!\shortmid [\lambda] && \text{for } \lambda \in kS.
\end{aligned}$$

Combining with the isomorphism from Example 4.2, we get

$$\begin{pmatrix} kQ & 0 \\ M & kR \end{pmatrix} \cong kS/\langle\tau\rangle \,.$$

Let us create isomorphisms

$$\phi\colon \begin{pmatrix} kQ & 0 \\ M & kR \end{pmatrix} \to kS/\langle\tau\rangle$$

and

$$\psi\colon kS/\langle\tau\rangle \to \begin{pmatrix} kQ & 0 \\ M & kR \end{pmatrix}$$

by composing the isomorphisms we have. We can describe ϕ by its actions on basis elements:

$$\begin{aligned}
\phi\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} &= [q] && \text{for } q \in Q_*, \\
\phi\begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix} &= [r] && \text{for } r \in R_*, \\
\phi\begin{pmatrix} 0 & 0 \\ \mathbf{a} & 0 \end{pmatrix} &= \left[\overrightarrow{\mathbf{d}}\gamma\beta - \overrightarrow{\mathbf{b}}\right], \\
\phi\begin{pmatrix} 0 & 0 \\ \mathbf{b} & 0 \end{pmatrix} &= \left[\overrightarrow{\mathbf{b}}\right], \\
\phi\begin{pmatrix} 0 & 0 \\ \mathbf{c} & 0 \end{pmatrix} &= \left[\overrightarrow{\mathbf{d}}\gamma\right], \\
\phi\begin{pmatrix} 0 & 0 \\ \mathbf{d} & 0 \end{pmatrix} &= \left[\overrightarrow{\mathbf{d}}\right], \\
\phi\begin{pmatrix} 0 & 0 \\ \mathbf{e} & 0 \end{pmatrix} &= \left[\alpha\overrightarrow{\mathbf{d}}\gamma\beta - \alpha\overrightarrow{\mathbf{b}}\right], \\
\phi\begin{pmatrix} 0 & 0 \\ \mathbf{f} & 0 \end{pmatrix} &= \left[\alpha\overrightarrow{\mathbf{d}}\right].
\end{aligned}$$

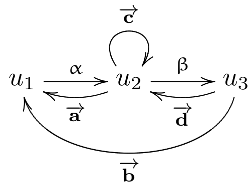
We describe ψ by its actions on equivalence classes of vertices and arrows:

$$\begin{aligned}
\psi([q]) &= \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} && \text{for } q \in Q_?, \\
\psi([r]) &= \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix} && \text{for } r \in R_?, \\
\psi([\overrightarrow{\mathbf{b}}]) &= \begin{pmatrix} 0 & 0 \\ \mathbf{b} & 0 \end{pmatrix}, \\
\psi([\overrightarrow{\mathbf{d}}]) &= \begin{pmatrix} 0 & 0 \\ \mathbf{d} & 0 \end{pmatrix}.
\end{aligned}$$

■

Example 5.6. This example is a continuation of Example 4.3. We have the quiver

$Q \circlearrowleft_B$:



and preadmissible relation set

$$\begin{aligned}\nu(Q,B)\cup\xi(Q,B) = \{&\alpha\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{c}},\ \alpha\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{d}},\ \beta\overrightarrow{\mathbf{c}},\ \beta\overrightarrow{\mathbf{d}},\\ &\overrightarrow{\mathbf{a}}\alpha,\ \overrightarrow{\mathbf{c}}\alpha,\ \overrightarrow{\mathbf{b}}\beta - \overrightarrow{\mathbf{a}},\ \overrightarrow{\mathbf{d}}\beta - \overrightarrow{\mathbf{c}},\\ &\overrightarrow{\mathbf{a}}\overrightarrow{\mathbf{c}},\ \overrightarrow{\mathbf{a}}\overrightarrow{\mathbf{d}},\ \overrightarrow{\mathbf{c}}^2,\ \overrightarrow{\mathbf{c}}\overrightarrow{\mathbf{d}}\}\end{aligned}$$

for the trivial extension

$$kQ \ltimes M.$$

We will apply PREADMISSIBLE TO ADMISSIBLE($Q \circlearrowleft_B, \nu(Q,B)\cup\xi(Q,B)$) to get a new quiver R and an admissible relation set σ such that

$$kQ \ltimes M \cong \frac{k(Q \circlearrowleft_B)}{\langle \nu(Q,B)\cup\xi(Q,B)\rangle} \cong kR/\langle\sigma\rangle\,.$$

We start with $\rho_0 = \nu(Q,B)\cup\xi(Q,B)$. The **while** loop produces the following values (as in the previous example, we do not write zero relations):

First iteration:

$$\begin{aligned}\mathfrak{r}_1 &= \alpha\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{c}}\\ \alpha_1 &= \overrightarrow{\mathbf{c}}\\ \overline{s_1} &= \alpha\overrightarrow{\mathbf{a}}\\ \rho_1 &= \{\alpha\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{d}},\ \beta\alpha\overrightarrow{\mathbf{a}},\ \beta\overrightarrow{\mathbf{d}},\ \overrightarrow{\mathbf{a}}\alpha,\ \alpha\overrightarrow{\mathbf{a}}\alpha,\ \overrightarrow{\mathbf{b}}\beta - \overrightarrow{\mathbf{a}},\ \overrightarrow{\mathbf{d}}\beta - \alpha\overrightarrow{\mathbf{a}},\\ &\qquad \overrightarrow{\mathbf{a}}\alpha\overrightarrow{\mathbf{a}},\ \overrightarrow{\mathbf{a}}\overrightarrow{\mathbf{d}},\ \alpha\overrightarrow{\mathbf{a}}\alpha\overrightarrow{\mathbf{a}},\ \alpha\overrightarrow{\mathbf{a}}\overrightarrow{\mathbf{d}}\}\end{aligned}$$

Second iteration:

$$\begin{aligned}\mathfrak{r}_2 &= \alpha\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{d}}\\ \alpha_2 &= \overrightarrow{\mathbf{d}}\\ \overline{s_2} &= \alpha\overrightarrow{\mathbf{b}}\\ \rho_2 &= \{\beta\alpha\overrightarrow{\mathbf{a}},\ \beta\alpha\overrightarrow{\mathbf{b}},\ \overrightarrow{\mathbf{a}}\alpha,\ \alpha\overrightarrow{\mathbf{a}}\alpha,\ \overrightarrow{\mathbf{b}}\beta - \overrightarrow{\mathbf{a}},\ \alpha\overrightarrow{\mathbf{b}}\beta - \alpha\overrightarrow{\mathbf{a}},\\ &\qquad \overrightarrow{\mathbf{a}}\alpha\overrightarrow{\mathbf{a}},\ \overrightarrow{\mathbf{a}}\alpha\overrightarrow{\mathbf{b}},\ \alpha\overrightarrow{\mathbf{a}}\alpha\overrightarrow{\mathbf{a}},\ \alpha\overrightarrow{\mathbf{a}}\alpha\overrightarrow{\mathbf{b}}\}\end{aligned}$$

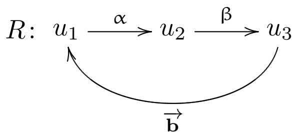
Third iteration:

$$\begin{aligned}
\mathfrak{r}_3 &= \overrightarrow{\mathbf{b}}\beta - \overrightarrow{\mathbf{a}} \\
\alpha_3 &= \overrightarrow{\mathbf{a}} \\
\overline{s_3} &= \overrightarrow{\mathbf{b}}\beta \\
\rho_3 &= \{\beta\alpha\overrightarrow{\mathbf{b}}\beta,\ \beta\alpha\overrightarrow{\mathbf{b}},\ \overrightarrow{\mathbf{b}}\beta\alpha,\ \alpha\overrightarrow{\mathbf{b}}\beta\alpha, \\
&\qquad \overrightarrow{\mathbf{b}}\beta\alpha\overrightarrow{\mathbf{b}}\beta,\ \overrightarrow{\mathbf{b}}\beta\alpha\overrightarrow{\mathbf{b}},\ \alpha\overrightarrow{\mathbf{b}}\beta\alpha\overrightarrow{\mathbf{b}}\beta,\ \alpha\overrightarrow{\mathbf{b}}\beta\alpha\overrightarrow{\mathbf{b}}\}
\end{aligned}$$

Since there is no relation $\mathfrak{r}$ in ρ_3 with minlength$(\mathfrak{r}) = 1$, the **while** loop terminates here, and we get $n = 3$. Line 10 and the **for** loop produce the following values:

$$\begin{aligned}
s_3 &= \overline{s_3} = \overrightarrow{\mathbf{b}}\beta \\
s_2 &= \operatorname{tr}_{\{(\alpha_3, s_3)\}}(\overline{s_2}) = \operatorname{tr}_{\{(\overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{b}}\beta)\}}(\alpha\overrightarrow{\mathbf{b}}) = \alpha\overrightarrow{\mathbf{b}} \\
s_1 &= \operatorname{tr}_{\{(\alpha_2, s_2), (\alpha_3, s_3)\}}(\overline{s_1}) = \operatorname{tr}_{\{(\overrightarrow{\mathbf{d}}, \alpha\overrightarrow{\mathbf{b}}), (\overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{b}}\beta)\}}(\alpha\overrightarrow{\mathbf{a}}) = \alpha\overrightarrow{\mathbf{b}}\beta
\end{aligned}$$

The resulting quiver R is $Q \circlearrowleft B$ with the arrows $\overrightarrow{\mathbf{c}}$, $\overrightarrow{\mathbf{d}}$ and $\overrightarrow{\mathbf{a}}$ removed; that is,



The resulting relation set is

$$\begin{aligned}
\rho_3 = \{&\beta\alpha\overrightarrow{\mathbf{b}}\beta,\ \beta\alpha\overrightarrow{\mathbf{b}},\ \overrightarrow{\mathbf{b}}\beta\alpha,\ \alpha\overrightarrow{\mathbf{b}}\beta\alpha,\ \overrightarrow{\mathbf{b}}\beta\alpha\overrightarrow{\mathbf{b}}\beta, \\
&\overrightarrow{\mathbf{b}}\beta\alpha\overrightarrow{\mathbf{b}},\ \alpha\overrightarrow{\mathbf{b}}\beta\alpha\overrightarrow{\mathbf{b}}\beta,\ \alpha\overrightarrow{\mathbf{b}}\beta\alpha\overrightarrow{\mathbf{b}}\}.
\end{aligned}$$

It is clear that we only need the relations $\beta\alpha\overrightarrow{\mathbf{b}}$ and $\overrightarrow{\mathbf{b}}\beta\alpha$ to generate $\langle\rho_3\rangle$, so we let

$$\sigma = \{\beta\alpha\overrightarrow{\mathbf{b}},\ \overrightarrow{\mathbf{b}}\beta\alpha\}.$$

We then have

$$\frac{k(Q \circlearrowleft B)}{\langle \nu(Q,B) \cup \xi(Q,B) \rangle} \cong kR/\langle\sigma\rangle\,,$$

with isomorphisms given by

$$\begin{aligned}
[\lambda] &\mapsto \left[\operatorname{tr}_{\{(\overrightarrow{\mathbf{c}}, \alpha\overrightarrow{\mathbf{b}}\beta), (\overrightarrow{\mathbf{d}}, \alpha\overrightarrow{\mathbf{b}}), (\overrightarrow{\mathbf{a}}, \overrightarrow{\mathbf{b}}\beta)\}}(\lambda)\right] && \text{for } \lambda \in k(Q \circlearrowleft B), \\
[\lambda] &\hookleftarrow [\lambda] && \text{for } \lambda \in kR.
\end{aligned}$$

Combining with the isomorphism from Example 4.3, we get

$$kQ \ltimes M \cong kR/\langle\sigma\rangle\,.$$

Let us create isomorphisms

$$\phi\colon kQ \ltimes M \to kR/\langle\sigma\rangle\,,$$
$$\psi\colon kR/\langle\sigma\rangle \to kQ \ltimes M$$

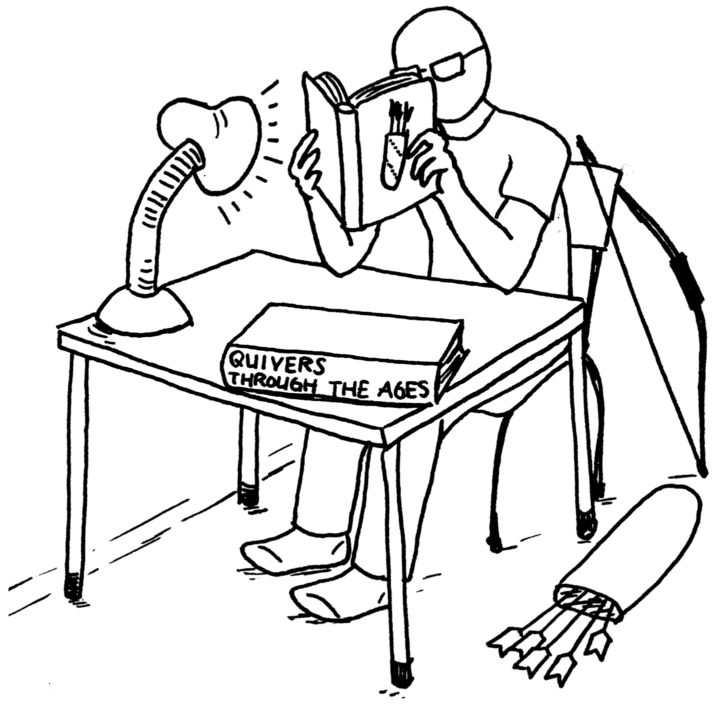
by composing the isomorphisms we have. We can describe ϕ by its actions on basis elements:

$$\begin{aligned}
\phi(q,0) &= [q] && \text{for } q \in Q_*,\\
\phi(0,\mathbf{a}) &= \left[\overrightarrow{\mathbf{b}}\,\beta\right],\\
\phi(0,\mathbf{b}) &= \left[\overrightarrow{\mathbf{b}}\right],\\
\phi(0,\mathbf{c}) &= \left[\alpha\,\overrightarrow{\mathbf{b}}\,\beta\right],\\
\phi(0,\mathbf{d}) &= \left[\alpha\,\overrightarrow{\mathbf{b}}\right].
\end{aligned}$$

We describe ψ by its actions on equivalence classes of vertices and arrows:

$$\begin{aligned}
\psi\big([q]\big) &= (q,0) && \text{for } q \in Q_?,\\
\psi\big(\left[\overrightarrow{\mathbf{b}}\right]\big) &= (0,\mathbf{b}).
\end{aligned}$$

■


QUIVERS
THROUGH THE AGES

Bibliography

[1] M. Auslander, I. Reiten, and S.O. Smalø. *Representation Theory of Artin Algebras*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[2] Martin Herschend. Tensor products on quiver representations. *Journal of Pure and Applied Algebra*, 212(2):452 – 469, 2008.

[3] Zbigniew Leszczyński. On the representation type of tensor product of algebras. *Fundamenta Mathematicae*, 144:143–161, 1994.

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A page number in **boldface** indicates a reference to a definition. A page number in *italics* indicates a reference to a result.