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# Two Weight Discrete Hilbert Transforms and Systems of Reproducing Kernels

Thesis for the degree of Philosophiae Doctor

Trondheim, April 2011

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***To the memory of my brother Solomon, who lost his life on 23 February 2011 at the age of 21. Sol, I can not believe that it has been already over a month without you among us. My heart is still heavy and broken and will never again be the same. I think about you and miss you more every day. I wish heaven had a phone so I could hear your voice and laughter again. May God keep your soul in peace.***

# Abstract

The Hilbert transform has become increasingly popular over the years due to its wide ranging applications not only in mathematics, but also in many other applied areas. In a quest for more applications, studying various aspects of its two weight forms has been a subject of high interest as early as the 1970's. Of special interest is the interface of the Hilbert transform with the notions of Carleson measures and the system of reproducing kernels in spaces of analytic functions. Though these notions have proved to be of fundamental importance and ubiquitous in the development of function theoretic spaces, their properties for many significant spaces, including the model subspace of the Hardy spaces $H^2$, have not yet been well understood. The present thesis focuses on this interface and provides answers to several problems encompassing them.

The thesis consists of five chapters. The first chapter provides an up-to-date review of the relevant background literature. The remaining chapters contain results that have been published by, or intended for, international journals.

The work in chapter two covers the problems of unitarity, invertibility, boundedness, and surjective mapping properties of the two weight discrete Hilbert transforms, and a complete solution is obtained for the first one. Our solutions for the remaining problems are complete under a sparsity priori growth condition. Under such a condition, we describe bounded two weight Hilbert transforms in terms of a relatively simple $A_2$ conditions. As a consequence, computable geometric criteria have been established for invertibility of such maps. Chapter two also provides all the basic underpinnings for the materials presented in Chapter three and Chapter four, where links have been established to interpolate all our results on the weighted transforms into statements about Carleson measures and systems of reproducing kernels in certain Hilbert spaces, of which de Branges spaces and model subspaces of $H^2$, are prime examples. As an application, a connection to the Feichtinger conjecture,

which is known to be equivalent to dozens of other conjectures including the famous Kadison–Singer problem, is pointed out and verified for certain classes of spaces.

Chapter five deals again with normalized reproducing kernel Riesz bases in model subspaces of $H^2$ generated by the class of meromorphic inner functions. In this chapter, the approach to studying such bases digresses somewhat from the methods used in the preceding chapters. Here, we study the normalized kernel bases from an equality of spaces perspective. It is known that such bases can be described in terms of equality of spaces whenever the kernels are associated with points all from the real line. When the points are from the upper half-plane, it is now proved that the analogous conditions may still be sufficient while failing to be necessary.

# Preface

This thesis is submitted in partial fulfillment of the requirements for the Degree of Philosophiae (PhD) at the Norwegian University of Science and Technology (NTNU). It serves as a documentation of learning experiences that have taken place over a period of four years. The thesis work has been carried out at the Department of Mathematics from October 2006 to January 2011, with the exception of a leave of five months to take care of my then five-month-old son. Many of the results presented here are from the following papers:

[11] Y. Belov, T. Mengestie, and K. Seip, Discrete Hilbert transforms on sparse sequences, Proc. London Math. Soc., doi: 10.1112/plms/pdq053..

[12] Y. Belov, T. Mengestie, and K. Seip, Unitary discrete Hilbert transforms, J. Anal. Math., 112 (2010), 383–393.

[63] T. Mengestie, Compact Carleson measures from sparse sequences, Preprint, 2011.

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I have made a number of great friends along the way. They have helped me in one way or another in my journey as both a post-grad and a PhD student at NTNU. I extend many thanks to them all. I am especially grateful to Dr. Bedilu, Mesite, Tigist, Yemarshet, Yohannes and Wondwosen for making my stay in Trondheim pleasant; Wondimu for being a fountain of inspirations; and Bekalu for his long-time, through-thick-and-thin friendship, for his continuous encouragement, and most of all for being an authentic example of a genuine friend.

A number of colleagues and fellow graduate students have greatly contributed to good inspiring working and social environments, for which I would like to thank them all. I would like to thank all the members of the analysis group at NTNU for providing me a stimulating learning environment. The group's weakly seminars have greatly enriched my professional life in various ways.

A special thank you goes to my fellow graduate students Maru, Zerihun, and Zeytun for a number of mathematical and other discussions during coffee breaks, and for the jokes and the laughter while walking all the way from Gløshaugen to Moholt. Thanks also go to Dr. Achenef and Solomon, with whom I have had fruitful discussions through the course of this study.

Finally, I am forever indebted to my family, both near and far, for their love and never-ending support. Most important, none of this would have been possible without my beloved wife, Mignot Amare, and my joyful son, Yonas. Your patience, understanding, love, and support have been a powerful source of inspirations and strength all these years. I have felt your pain and suffering keenly especially in the final stages of the work on this thesis, during which I spent more time with my mathematics than with you.

Trondheim, January 2011

Tesfa Yigrem Mengestie

# Contents

# 1 Introduction

This thesis deals with two closely connected and recurring themes in complex and harmonic analysis; weighted discrete Hilbert transforms, and Carleson measures and systems of reproducing kernels in spaces of analytic functions. In this part, we give a brief review of the relevant background with particular emphases on the class of bounded Hilbert transforms on weighted spaces followed by its connection with Carleson measures for the shift-coinvariant subspaces of the Hardy space $H^2$.

The theory of the Hilbert transform began back in 1905 in D. Hilbert's work on a problem posed by B. Riemann concerning analytic functions which later came to be known as the Riemann–Hilbert problem (cf. [14]). Since then, it has received a lot of attention and that it has been extensively investigated in connection with a wide range of applications. Hilbert's work was originally concerned with the transform of functions defined on the circle [49], in which case the transform is given by convolutions of functions with the kernel

$$k_H(t) = \cot(t/2). \tag{1.1}$$

Many of Hilbert's earlier results were also connected to the discrete version of the transform which were latter studied further by I. Schur [46] who extended them to the continuous case, while the underlying space remained to be $L^2$ or its atomic version $\ell^2$. Usually, the transform is understood as convolutions of functions defined on the real line with the Cauchy kernel,

$$k_C(t) = (\pi t)^{-1}. \tag{1.2}$$

The transform is explicitly defined using the Cauchy principal value as

$$\widetilde{H}f(x) = \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt, \tag{1.3}$$

which makes sense almost everywhere on the real line whenever

$$\int_{\mathbb{R}} \frac{|f(t)|}{1+|t|} dt < \infty. \tag{1.4}$$

The principal value notation p.v., as always, means that a symmetric neighborhood about the pole is excluded before the limit is taken. Thus, we compute $\widetilde{H}f$ by

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t} dt = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{x-\varepsilon} \frac{f(t)}{x-t} dt + \int_{x+\varepsilon}^{\infty} \frac{f(t)}{x-t} dt \right).$$

For some applications, the class of functions for which the admissibility condition (1.4) holds remains "small" and we may require to apply the transform on functions integrable with respect to the Poisson measure on the real line. If $\pi$ denotes such measure, $d\pi(x) = (1+x^2)^{-1}dx$, then the transform of $f$ in $L^1(\pi)$ is defined by

$$\widetilde{H}_{pos} f(x) = \frac{1}{\pi} \text{ p.v.} \int_{\mathbb{R}} f(t) \left( \frac{1}{x-t} + \frac{t}{1+t^2} \right) dt, \tag{1.5}$$

where the kernel $(x-t)^{-1}$ in (1.3) is replaced by the modified kernel $(x-t)^{-1} + t(1+t^2)^{-1}$. This modification provides a wider class of functions than the class of functions for which (1.4) holds. We record our first simple example.

**Example 1.** If $f$ stands for a signal that assumes a single value at all time $t$, then condition (1.4) fails and its convolution with the Cauchy kernel diverges. But $f$ belongs to $L^1(\pi)$ and $\widetilde{H}_{pos} f$ exists.

When (1.4) holds, the two transforms $\widetilde{H}_{pos}$ and $\widetilde{H}$ are related by $\widetilde{H}_{pos} = \widetilde{H} + C$ for some absolute constant[1] $C$. It is thus essential to identify functions differing by constants in dealing with these two forms of the transforms.

When we apply the transform twice in succession to a function $f$, an interesting inverse relation occurs, namely that

$$\widetilde{H}(\widetilde{H}f) = -f$$

holds provided that the integrals defining both $f$ and $\widetilde{H}f$ converge in the underlying spaces. Thus if it exists, the inverse can be also identified as a Hilbert transform, up

[1] From now on, the same letter $C$ will denote various positive constants which may differ at different occurrences even in the same chain of inequalities. Variables indicating the dependency of $C$ will often be specified in subscripts.

to a minus sign.

Later in 1928, a fundamental result was established by Marcel Riesz [84]. It deals with functions $f$ in $L^p(\mathbb{R})$ when $1 < p < \infty$. Riesz proved that there exists a constant $C_p$ for which the inequality

$$\|\widetilde{H}f\|_{L^p(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})} \tag{1.6}$$

holds for all functions $f$ in $L^p(\mathbb{R})$. M. Riesz proved a similar result for the discrete version of the transform and also for functions defined on the circle. In fact, for each nonzero $x$, the two defining kernels in (1.1) and (1.2) are connected by the identity

$$\frac{1}{2}k_H(x) = \pi k_C(x) + \sum_{n=1}^{\infty}\Big(\frac{1}{x+2n\pi} - \frac{1}{2n\pi}\Big),$$

which may be used to transform results between the two different domains. By Pichorides's well known result [82], the best constant $C_p$ in (1.6) is given by

$$\max\big\{\tan(\pi/(2p)),\ \cot(\pi/(2p))\big\} = \big\|\widetilde{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})\big\|. \tag{1.7}$$

The same best constant holds when the operator acts on functions defined on the circle.

It may be mentioned that the Hilbert transform was a motivating example for A. Zygmund and A. Calderón [20] in their operator theoretic studies, which have profoundly influenced the development of modern harmonic analysis. Today, the Hilbert transform plays a significant role in many areas of science including mathematics, physics, and signal processing.

## 1.1 Weighted Hilbert transforms

It became of practical importance to study Hilbert transforms $\widetilde{H}$ acting on weighted spaces $L^p(\mathbb{R}, w)$ consisting of all functions $f$ satisfying

$$\|f\|_{w,p}^p = \int_{\mathbb{R}} |f(t)|^p w(t)dt < \infty.$$

The question was to characterize the weights[2] $w$ for which the norm inequality

$$\int_{\mathbb{R}} |\widetilde{H}f(x)|^p w(x)dx \le C_p \|f\|_{w,p}^p \tag{1.1.1}$$

[2] Here and in what follows, by a weight we mean, as always, a positive real valued function. At times, we may apply the name for a finite positive Borel measure.

holds for each $f$ in $L^p(\mathbb{R},w)$ and a constant $C_p$ not necessarily given by (1.7).

In 1960, Helson and Szegő [48] fully described such weights for $p=2$. The Helson–Szegő condition states that $w$ satisfies (1.1.1) if and only if it has the representation

$$w(x)=\exp\left(u(x)+\widetilde{H}v_1(x)\right) \tag{1.1.2}$$

for some $L^\infty(\mathbb{R})$ functions $u$ and $v_1$ such that $\|v_1\|_{L^\infty}<\pi/2$. Notice that the expression for the weight here involves the Hilbert transform but acting on a bounded function $v_1$.

Later, in 1971, R. Hunt, B. Muckenhoupt, and R. Wheeden [51] obtained the following remarkable and entirely different description of the weights in terms of what has become known as the Muckenhoupt's $A_p$ condition.

**Theorem 1.1.1.** *The operator $\widetilde{H}:L^p(\mathbb{R},w)\longrightarrow L^p(\mathbb{R},w)$ is bounded if and only if $w$ satisfies the Muckenhoupt's $A_p$ condition*

$$\sup_I \frac{1}{|I|}\int_I w(x)dx\left(\frac{1}{|I|}\int_I w(x)^{\frac{-1}{p-1}}dx\right)^{p-1}<\infty,$$

*where $I$ ranges over all finite intervals in $\mathbb{R}$.*

In particular for $p=2$, it implies that $A_2$ is equivalent to the Helson–Szegő condition. But to date, no direct proof has been found of this equivalence.

Let $1<p<\infty$ and $\mu$ be a positive Borel measure on the real line. We define the Hilbert transforms $\widetilde{H}_\mu$ on $L^p(\mathbb{R},\ \mu)$ by

$$\widetilde{H}_\mu f(x)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(t)}{x-t}d\mu(t) \tag{1.1.3}$$

for all $x\in\mathbb{R}\setminus\operatorname{supp}(f)$. One may then consider the question when $\widetilde{H}_\mu$ acts as a bounded linear map on the space $L^p(\mathbb{R},\mu)$, i. e., there exists an absolute constant $C_p$ such that

$$\int_{\mathbb{R}}|\widetilde{H}_\mu f|^p d\mu\le C_p\int_{\mathbb{R}}|f|^p d\mu. \tag{1.1.4}$$

Helson and Szegő again provide both a necessary and a sufficient condition when $p=2$. The condition being that $\mu$ must be absolutely continuous, $d\mu(x)=w(x)dx$ for some weight $w$ which satisfies (1.1.2). As for other ranges of $p$, the condition ensures that the measure $\mu$ has to be again absolutely continuous with the corresponding weight $w$ satisfying the same $A_p$ condition.

A thing to be noted is that the $A_p$ condition not only gives a clear and workable answer to the boundedness problem for the weighted Hilbert transform but also for several other classical operators. For instance the same Muckenhoupt $A_p$ condition is both necessary and sufficient for the weighted norm inequality (1.1.1) to hold when we replace $\widetilde{H}$ by the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(t)|dt. \tag{1.1.5}$$

Here, the supremum is taken over all finite intervals containing $x$ in $\mathbb{R}$ [66]. By further setting that $0<\alpha<n,\ \ 1<p<n/\alpha$ and $1/q = 1/p-\alpha/n$, B. Muckenhoupt and R. Wheeden [64] proved that the fractional integral operator of order $\alpha$;

$$T_\alpha f(x) = \int_{\mathbb{R}^n} |x-t|^{\alpha-n} f(t)dt, \tag{1.1.6}$$

satisfies

$$\|T_\alpha f w\|_{L^q(\mathbb{R}^n)} \le C_{(p,q)} \|fw\|_{L^p(\mathbb{R}^n)} \tag{1.1.7}$$

if and only if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)^q dx\right)^{1/q} \left(\frac{1}{|Q|}\int_Q w(x)^{\frac{-p}{p-1}} dx\right)^{\frac{p-1}{p}} < \infty, \tag{1.1.8}$$

with $Q$ ranging over all $n$ dimensional cubes.

The single weight case is now well understood for several operators including the Hilbert transform. Some applications of one weight Hilbert transforms can be found for instance in [81, 86, 88, 103] on spectral theory of stationary stochastic processes and Toeplitz operators.

## 1.2 Two weight Hilbert transforms

The problem with two weights was first raised by B. Muckenhoupt [65] in the context of more general operators. Obviously, it first attracted the attention due to the well established theory of one weight operators. The problem is to describe the pairs of weights $(v,w)$ for which $\widetilde{H}$ is bounded from $L^2(\mathbb{R},v)$ to $L^2(\mathbb{R},w)$. That is, there exists an absolute constant $C$ for which the two weights norm inequality

$$\int_{\mathbb{R}} |\widetilde{H}f(x)|^2 w(x)dx \leq C \int_{\mathbb{R}} |f(t)|^2 v(t)dt \tag{1.2.1}$$

holds for each f in $L^2(\mathbb{R},v)$ [3]. In the sequel, this will be referred to as the two weight problem. A Helson–Szegő type characterization has been again already obtained by M. Cotlar and C. Sadosky [34–36]. The condition states that for continuous pair of weights $(v,w)$, $\widetilde{H}$ is bounded if and only if there exist an analytic function $h$ in the Hardy class $H^1$ and a positive constant $C$ such that the matrix

$$\begin{pmatrix} Cw-v & Cw+v-h \\ Cw+v-\widetilde{H}h & Cw-v \end{pmatrix}$$

is positive semi definite. Thus the problem is completely solved as far as the Helson–Szegő type description is concerned[4]. The question has been to characterize the weights in terms of criteria somewhat akin to the classical $A_2$ condition for the case of single weighted transforms. One may suspect that a natural description should be one that simply requires the weights to satisfy the two weight analog

$$\sup_I \frac{1}{|I|}\int_I w(x)^{-1}dx \ \frac{1}{|I|}\int_I v(x)dx < \infty \tag{1.2.2}$$

of $A_2$. It turns out that nothing like this is sufficient for (1.2.1) to hold. This rather intriguing result was proved by F. Nazarov (cf. [68, 73]). On the other hand, given the huge degree of freedom associated with two weights in contrast with a single weight, the lack of a full $A_2$ type sufficient condition was not really unanticipated. Evidently, things look much more complicated in two weight cases. For simple operators like the Hardy operator,

$$TH_{op}f(x) = \int_0^x f(t)dt,$$

an $A_2$ type characterization has already been obtained in [19, 66]. The description is

[3] The two weight problem can be analogously stated for all $p$ in $(1,\infty)$. But here on, we will restrict ourselves mainly to the case of $p=2$.

[4] In [34, 37], they had also obtained the Helson–Szegő version of their result in $L^p$ for $p \neq 2$.

that

$$\sup_{t,\,0<t<\infty}\left(\int_t^\infty w(x)dx\right)^{\frac{1}{2}}\left(\int_0^t v(x)^{\frac{1}{2}}dx\right)^2<\infty. \tag{1.2.3}$$

On the other hand, for the classical Hardy–Littlewood function $M$, it was shown [90] that the two weights norm inequality holds if and only if

$$\int_I |M\chi_I v^{\frac{1}{2}}(x)|^2 w(x)dx \le C\int_I v^{\frac{1}{2}}(x)dx<\infty \tag{1.2.4}$$

for all characteristic functions $\chi_I$ over intervals $I$ in $\mathbb{R}$. Apart from its simplicity, the interesting aspect of this result is the solution to the boundedness problem depends only on the action of $M$ over some particular classes of functions of the form[5] $f=\chi_I v^{\frac{1}{2}}$. On the other hand, unlike the $A_2$ condition, the solution here involves the operator $M$ itself. Later, R. Wheeden [104] considered the more general case when $1<p<q<\infty$ and $0<\alpha<n$, and in which case the maximal operator is defined by

$$M_\alpha f(x)=\sup_{B:\, x\in B}\frac{1}{|B|^{1-\alpha/n}}\int_B |f(t)|dt$$

where $B$ is a ball in $\mathbb{R}^n$. He proved that the inequality

$$\left(\int_{\mathbb{R}^n}|M_\alpha f(x)|^q w(x)dx\right)^{1/q}\le C_{(p,\,q)}\left(\int_{\mathbb{R}^n}|f(x)|^p v(x)dx\right)^{1/p}$$

holds if and only if the weights $(v,w)$ satisfy

$$\sup_B\left(\int_{\mathbb{R}^n}\frac{w(x)}{\left(|B|^{1/n}+|x-x_B|\right)^{(n-\alpha)q}}dx\right)^{1/q}\left(\int_B v(x)^{-1/(p-1)}dx\right)^{\frac{p-1}{p}}<\infty, \tag{1.2.5}$$

where $x_B$ is the center of the ball $B$. As noticed in [89], inequality (1.2) holds when we replace $M_\alpha$ by the fractional integral operator $T_\alpha$ if and only if both (1.2.5) and

$$\sup_B\left(\int_{\mathbb{R}^n}w(x)dx\right)^{1/q}\left(\int_B\frac{v(x)^{\frac{-1}{p-1}}}{\left(|B|^{1/n}+|x-x_B|\right)^{(n-\alpha)(p-1)/p}}dx\right)^{\frac{p-1}{p}}<\infty$$

[5]This particular result suggested whether the two weight problem for other operators could be answered with similar conditions. The suggestion was latter refuted; for example it fails to hold for the higher dimensional Hardy operators, see [94] for counterexample.

hold. The appearance of additional terms in this and (1.2.5) deviating from the classical $A_p$ form led to the question whether similar conditions could hold for the two weight Hilbert transform. It turns out that this is indeed the case.

### 1.2.1 Improved $A_2$ type and testing conditions

Following the result of F. Nazarov, which ensures that (1.2.2) fails to imply (1.2.1) and the aforementioned modified $A_p$ form for the maximal and integral operators, recently, a new quantitative condition for the two weight problem has been found [58,71]. We will refer such a condition as an improved $A_2$ type condition. To state it, we find it convenient at this point to recast (1.2.1) in a more general form, one that permits the replacement of the weight functions $(v,w)$ by positive Borel measures $\mu$ and $\omega$ on $\mathbb{R}$, and leads to

$$\int_{\mathbb{R}} |\widetilde{H}_\mu f(x)|^2 d\omega(x) \leq C \int_{\mathbb{R}} |f(t)|^2 d\mu(t). \tag{1.2.6}$$

Note that to deal with this, we need to replace the Lebesgue measure in (1.3) by the measure $\mu$ as in (1.1.3). To see that (1.2.1) is also included in (1.2.6), one may simply set $d\omega(x) = w(x)dx, \quad d\mu(x) = v(x)dx$ and replace $f$ by $fv^{-1}$ in (1.2.6). Then (1.2.2), the natural analog of the $A_2$ conditions, takes the form

$$\sup_I \frac{\mu(I)}{|I|} \frac{\omega(I)}{|I|} < \infty, \tag{1.2.7}$$

which obviously reduces to the $A_2$ condition when the two weights are equal.

For an interval $I$ and a measure $\omega$, we define, as in [58], a variant of the Poisson integral by

$$P(I,\omega) = \int_{\mathbb{R}} \frac{|I|}{\big(|I| + \mathrm{dist}(x,I)\big)^2} d\omega(x).$$

Then the improved $A_2$ condition for two measures $\omega$ and $\mu$ states:

$$\sup_I P(I,\omega)P(I,\mu) < \infty. \tag{1.2.8}$$

It may be noted that the supremum in (1.2.8) is bigger than the supremum in (1.2.2) when we replace the weights by the corresponding positive measures. The result of F. Nazarov shows that even this strengthened $A_2$ type necessity condition is not sufficient for the two weight inequality (1.2.1). Quite recently, the necessity of this condition was also supplemented in [58] where a new and real-variable proof is obtained.

Two weight inequalities for maximal functions (as indicated in the previous subsection), maximal singular integrals and other operators with positive kernels have already been described. Those descriptions are given in terms of some obvious necessary conditions; that the operators be uniformly bounded on a restricted class of functions, namely indicators of intervals and cubes. For further details, interested readers may wish to consult the papers [56, 57, 90–93].

Suggestions then prevailed to consider additional testing conditions, as in (1.2.4), for the two weight problem which simply requires $\widetilde{H}_\mu$ and its adjoint $\widetilde{H}_\mu^*$ to be uniformly bounded on systems of characteristic functions $\chi_I$ on intervals. That is for all intervals $I$ in $\mathbb{R}$,

$$\int_{\mathbb{R}} |\widetilde{H}_\mu \chi_I(x)|^2 d\omega(x) \leq C \int_I d\mu(x) \tag{1.2.9}$$

and

$$\int_{\mathbb{R}} |\widetilde{H}_\mu^* \chi_I(x)|^2 d\mu(x) \leq C \int_I d\omega(x) \tag{1.2.10}$$

hold. Clearly, these conditions are necessary. But the converse statement does not in general follow from them alone.

In a series of papers [69–72], F. Nazarov, S. Treil, and A. Volberg have developed powerful techniques towards proving the sufficiency of these testing conditions combined with the improvement of the two weight $A_2$ condition. In their successful quest, by assuming further side conditions like doubling measure for the two weights and pivotal conditions [71], they proved that (1.2.8), (1.2.9), and (1.2.10) are indeed both necessary and sufficient for (1.2.1). Following the arguments described in those papers, quite recently, the result has been improved by M. Lacey, E. Sawyer, and I. Tuero [58] under a range of weaker side conditions which they called energy conditions. The energy conditions weaken the pivotal conditions in [71] and gives a negative answer to the question of whether the pivotal conditions were necessary.

There exists now a sizable literature on the two weight problem not only because its relation to the one weight case attracted considerable attention but also because it appears naturally in many areas for instance in perturbation theory of self-adjoint operators [73], spectral theory of Jacobi matrices, [80, 101] and Carleson measures in model subspaces of $H^2$ [69]. For further information, see the last three chapters of the monograph by A. Volberg [102].

In the next chapter, we will continue to study the discrete version of the two weight problem, paying special attention to its connection with Carleson measures and Riesz bases of reproducing kernels in spaces of meromorphic functions. Subject to an a priori sparsity condition, we will provide a solution to the problem in terms of a rather a relatively simple $A_2$ condition (cf, Theorem (2.2.1)).

## 1.3 Two weight problem and Carleson measures

We begin by recalling a few notions. Let $\mathcal{H}$ be a separable Hilbert space and $(e_n)$ a sequence of unit vectors in $\mathcal{H}$. We say $(e_n)$ is a Bessel sequence if there is a positive constant $C$ such that the inequality

$$\sum_n \left| \langle f, e_n \rangle_{\mathcal{H}} \right|^2 \leq C \|f\|^2_{\mathcal{H}}$$

holds for every $f$ in $\mathcal{H}$. The sequence $(e_n)$ is a Riesz basic sequence if there exists a positive constant $A$ such that the inequalities

$$A^{-1} \sum_n |c_n|^2 \leq \left\| \sum_n c_n e_n \right\|^2_{\mathcal{H}} \leq A \sum_n |c_n|^2 \tag{1.3.1}$$

holds for every finite sequence of scalars $(c_n)$. Equivalently, by a well-known lemma of R. Boas [15], $(e_n)$ is a Riesz basic sequence if it is a Bessel sequence for which the moment problem

$$\langle f,\ e_n \rangle_{\mathcal{H}} = a_n$$

has a solution $f$ in $\mathcal{H}$ for every square-summable sequence $(a_n)$. If, in addition, the solution is unique, we call $(e_n)$ a Riesz basis. A Riesz basis is precisely the image of an orthonormal basis under a bounded invertible operator. If, in particular, $\mathcal{H}$ is defined on some sets for which point evaluations are bounded linear functionals, then by the Riesz representation theorem, there exists a unique function $k_z$ in $\mathcal{H}$ such that

$$f(z) = \langle f,\ k_z \rangle_{\mathcal{H}}$$

for all $f$ in $\mathcal{H}$. The function $k_z(w) = \overline{k_w(z)}$ is referred to as the reproducing kernel of $\mathcal{H}$. Both the Hardy space $H^2$ and all its model subspaces $K_I^2$ are reproducing kernel Hilbert spaces with respective kernel functions

$$k_\lambda^{H^2}(z) = \frac{i}{2\pi}\frac{1}{z-\overline{\lambda}} \text{ and } k_\lambda^{K_I^2}(z) = \frac{i}{2\pi}\frac{1-I(z)\overline{I(\lambda)}}{z-\overline{\lambda}}$$

for points $z$ and $\lambda$ in the upper half-plane.

The thesis originated in an attempt to answer some questions about sequences of reproducing kernels and Carleson measures in spaces of analytic functions, more specifically in model subspaces of $H^2$. As mentioned above, such questions are closely connected with the two weight problem. Indeed, one of our main results (Theorem 2.4.1) in the next chapter gives an explicit characterization of normalized reproducing kernel Riesz bases in terms of the two weight problem. The connection with Carleson measures in model subspaces has been already established in [69].

We let $H^2$ denote the Hardy space in the upper half-plane, viewed in the usual way as a subspace of $L^2(\mathbb{R})$[6]. Given an inner function[7] $I$ in the upper half-plane, we define the model subspace $K_I^2$ as

$$K_I^2 = H^2 \ominus IH^2;$$

it is the orthogonal complement in $H^2$ of functions divisible by the inner function $I$. These spaces are, by a classical theorem of A. Beurling [13], the subspaces of $H^2$ that are invariant with respect to the backward shift. Equivalently, such subspaces can be described by

$$K_I^2 = H^2 \cap I\overline{H^2}.$$

The later description does not require the Hilbert space structure and it can be used to define the analogous subspaces in all Hardy spaces $H^p$ for all[8] $p > 0$. The spaces arise in connection with several themes and plays a significant role in operator theory. They received the name model subspaces because of their application in the Sz.-Nagy–Foias [67] model for contractions in Hilbert spaces. They are often called star-invariant or co-invariant subspaces. We refer to [27, 74–76] for more information about the model theory related to the backward shift.

We now mention a couple of examples. We will give more examples in Subsection

[6] Here we mean that every function $f$ in $H^2(\mathbb{C}_+)$ has a boundary limit function, $f_b(x) = \lim_{y\to 0^+} f(x+iy)$ almost every where on $\mathbb{R}$. The map $f \to f_b$ identifies $H^2(\mathbb{C}_+)$ by $H^2(\mathbb{R})$ which consists of functions in $L^2(\mathbb{R})$ whose Fourier transforms vanish a. e. on the negative axis.

[7] We call a bounded analytic function $I$ in $\mathbb{C}_+$ inner if $\lim_{y\to 0^+} |I(x+iy)| = 1$ for almost all $x \in \mathbb{R}$ with respect to the Lebesgue measure.

[8] $K_I^p = H^p \cap I\overline{H^p}$.

4.1.

#### Paley–Wiener spaces

For $a > 0$, the Paley–Wiener space $PW_a$ consists of entire functions of exponential type at most $a$ whose restriction to the real axis are square summable. It coincides with the space of entire functions

$$\left\{ f : f(x) = \int_{-a}^{a} g(t) e^{itx} dt,\ \ g \in L^2(-a,a) \right\};$$

the space of the Fourier image of square integrable functions supported in the interval $(-a,a)$. If we set $I(z) = e^{iaz}$, then the relation $K_I^2 = SPW_{a/2}$ identifies the Paley–Wiener spaces as model subspaces up to a unimodular factor $S(z) = e^{iaz/2}$.

#### Linear span of fractions in $L^2(\mathbb{R})$

We consider a sequence of points $z_n$ in the upper half-plane where each $z_n$ appears with multiplicities $m_n$. We assume that this sequence satisfies the Blaschke condition

$$\sum_n \frac{m_n \Im z_n}{|z_n|^2 + 1} < \infty. \tag{1.3.2}$$

Then the closed linear span in $L^2(\mathbb{R})$ of the fractions

$$\frac{1}{(z - \overline{z_n})^j},\ \ j = 1,\ 2,\ \ldots,\ m_n$$

coincides with the model subspace $K_B^2$ generated by the Blaschke product

$$B(z) = \prod_n e^{i\sigma_n} \left( \frac{z - z_n}{z - \overline{z_n}} \right)^{m_n}$$

with real sequence of points $\sigma_n$. Note that the factor $e^{i\sigma_n}$ is needed to make sure that the product is convergent. We also note that the space $K_B^2$ contains no other fractions of the form $(z - \overline{w})^{-j}$ with $w \neq z_n$ for all $n$. If the sequence $(z_n)$ fails to satisfy (1.3.2), then the span of the fractions will be the whole space $L^2(\mathbb{R})$.

### 1.3.1 Carleson Measures in $K_I^2$

A long-standing problem in the function theory of the spaces $K_I^2$ is to describe the Carleson measures, i.e., those nonnegative measure $\mu$ on the closed upper half-plane

$\overline{\mathbb{C}_+}$ for which an inequality of the form

$$\int_{\overline{\mathbb{C}_+}} |f(z)|^2 d\mu(z) \leq C\|f\|_2^2 \tag{1.3.3}$$

holds for all $f$ in $K_I^2$, either in geometric terms or more intrinsically in terms of suitable properties of the inner function $I$ [9]. This question was first posed by W. Cohn [31]. By the Closed Graph Theorem, (1.3.3) may be equivalently rephrased as boundedness of the embedding map from $K_I^2$ into $L^2(\mu)$. That is,

$$K_I^2 \subset L^2(\mu) \quad \text{and} \quad \sup_{f\in K_I^2} \frac{\|f\|_{L^2(\mu)}}{\|f\|_2} < \infty \tag{1.3.4}$$

holds for each nonzero $f$ in $K_I^2$.

In $H^2$ and more generally in $H^p$, $0 < p < \infty$, a geometric characterization of such measures was obtained by L. Carleson [26]. We state the result as follows.

**Theorem 1.3.1.** *A nonnegative measure $\mu$ on $\overline{\mathbb{C}_+}$ is a Carleson measure for $H^2$ if and only if*

$$\sup_{(x_0,l)} \frac{\mu(Q(x_0,l))}{l} < \infty \tag{1.3.5}$$

*for all squares $Q(x_0,l) = \{x+iy : x_0 < x < x_0+l,\ 0 < y < l\}$.*

It may be noted that the same condition (1.3.5) describes all the Carleson measures in $H^p$ for $0 < p < \infty$.

Clearly, every Carleson measure for $H^2$ is a Carleson measure for $K_I^2$ as well. But functions in $K_I^2$ may have nicer boundary behavior than functions in $H^2$, and therefore the class of Carleson measures will be wider for $K_I^2$. The following interesting special case has been completely understood. We say that $I$ is a one-component inner function if there exists a positive number $\varepsilon$ with $0 < \varepsilon < 1$ such that the set

$$\{z \in \mathbb{C}_+ : |I(z)| < 1-\varepsilon\} \tag{1.3.6}$$

is connected. We refer to the paper [3] for some descriptions of the class of one-component inner functions. The Carleson measures for $K_I^2$ have been completely described, first by W. Cohn [31] himself, when $I$ belongs to this class. For this case,

[9] For a function $f$ in $H^2$, we denote its $H^2$ or any of its model subspaces norm by $\|f\|_2$. Unless explicitly stated otherwise, its usage will be clear from the context.

Cohn proved that $\mu$ is a Carleson measure for $K_I^2$ if and only if (1.3.4) holds for kernel functions $k_z$ for all $z$ in the upper half-plane. The same result follows also from [2, 87] as a particular case.

Later, Cohn [30] conjectured that his result in general describes all the Carleson measures regardless of the number of components of the generating inner functions. The conjecture has been refuted by Nazarov and Volberg [69]. The underlying observation of that paper is that the problem of describing the Carleson measures for $K_I^2$ is closely linked to the two weight problem for the Hilbert transform. The link has made it possible to construct a counterexample from the latter setting.

For one-component inner functions $I$, the embedding result of Cohn can be considered as saying that the reproducing kernel thesis holds for the embedding operator from $K_I^2$ into $L^2(\mu)$. We recall that an operator in a reproducing kernel Hilbert space is said to satisfy the reproducing kernel thesis if its boundedness can be completely determined by its action on the kernel functions alone. This property holds for both boundedness and compactness of Toeplitz, Hankel [16], and the Carleson embedding operators on $H^2$.

More partial results on Carleson measures for $K_I^2$ may be found in [2, 3, 6, 8] and [32, 41, 87]. For discrete measures, the problem can be also viewed as the problem of describing Bessel sequences of normalized reproducing kernels in $K_I^2$. In Chapter three, we will study such measures in some function spaces, paying special attention to the model subspaces. As an application of the results obtained, a version of the Feichtinger conjecture in $K_I^2$ will be then considered.

### 1.3.2 Reproducing kernel Riesz bases in $K_I^2$

The study of systems of reproducing kernel Riesz bases in model subspaces has a long history. It begins with a perturbation result of Paley and Wiener [79] on systems of nonharmonic Fourier series. Paley and Wiener asked for a precise bound on $d$ ensuring that

$$\sup_n |\alpha_n - n| = d, \;\; n \in \mathbb{Z}, \; \alpha_n \in \mathbb{R},$$

imply that the system of exponentials $\left(e^{i\alpha_n t}\right)$ forms a Riesz basis in $L^2(0, 2\pi)$. They gave an affirmative answer for any $d < \pi^{-2}$. Later on, A. Ingham [52] noticed that for $d = 1/4$, the system may fail to be a Riesz basis. Their result was repeatedly revised and generalized by several authors before Kadets' [54] proved the best possible result that the exponential system forms a Riesz basis whenever $d < 1/4$. The full description of Riesz bases of exponentials was obtained later in [50] in terms of the Helson–Szegő condition.

The Fourier transform provides an isometry between $L^2(0,2\pi)$ and the model subspace $K_I^2$ generated by the inner function $I(z) = e^{2\pi iz}$. Thus the system of exponentials $\left(e^{i\alpha_n t}\right)$ in $L^2(0,2\pi)$ translates into a family of normalized reproducing kernels $S_R(\alpha_n)$ in $K_I^2$. As mentioned in Section 1.3, the subspace $K_I^2$ has the special form:

$$K_I^2 = e^{i\pi z} PW_\pi^2,$$

where $PW_\pi^2$ is the Paley–Wiener space of entire functions $f$ of exponential type not bigger than $\pi$. The problem to characterize reproducing kernel Riesz bases in model subspaces was then considered in [50], and a solution was given whenever the generating inner function $I$ and the sequences of points $(\alpha_n)$ in $\mathbb{C}_+$ satisfy the additional condition

$$\sup_n |I(\alpha_n)| < 1. \tag{1.3.7}$$

Under this condition, with $B$ denoting the Blaschke product with simple zeros $(\alpha_n)$, $S_R(\alpha_n)$ constitutes a Riesz basis in $K_I^2$ if and only if the Carleson interpolation condition [26],

$$\inf_m \prod_{n,n\neq m} \left| \frac{\alpha_m - \alpha_n}{\alpha_m - \overline{\alpha_n}} \right| > 0,$$

holds and the Toeplitz operator with symbol $I\overline{B}$ is invertible[10]. Invoking the classical Widom–Devinatz theorem [39, 105] for invertibility leads to well known and beautiful descriptions of reproducing kernel Riesz bases. Good references on this topic are [50, 74, 75].

A different approach to study Riesz bases of exponentials was developed by some authors including B. Levin (cf. [79]), and Y. Lyubarskii and K. Seip [61]. The essential role in their arguments was played by the so called generating function.

The result in [61] describes the exponential bases in terms of an $A_2$ condition involving such function. The core of their approach was to connect the problem

[10] The Toeplitz operator with symbol $\Phi \in L^\infty(\mathbb{R})$ is the map $T_\Phi : H^2 \to H^2$, $\quad T_\Phi f = P_+(\Phi f)$ where $P_+$ is the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2$.

with some mapping properties of the Hilbert transform. More precisely, they turned the problem into one about the boundedness of the discrete Hilbert transform in a weighted space of sequences.

A similar result in terms of the $A_2$ condition was also obtained in [44] for the class of de Branges spaces following a somewhat different operator theoretic approach. But the main result in that paper still requires the a priori assumption (1.3.7) to hold. The results in [44] and [61] are proved by different means and complement each other. More recently, S. Gunter [45] gave an alternative description for exponential bases in Paley–Wiener spaces. The novelty of the approach in this paper again lies on the parametrization of the generating function.

The result can be regarded as a parametrization of bases of exponentials with real frequencies by independent parameters. More partial results may be found among others in [7, 10, 30, 42].

In Chapter three, we will again study such bases in certain function spaces which includes the model subspaces. The main tool in our approach will be the two weight discrete Hilbert transform. During the course of our work, we have found it both useful and conceptually appealing to transform these problem into a study of the mapping properties of discrete Hilbert transforms. We have also learned to appreciate that the essential difficulties in dealing with the Riesz basis problem seem to appear in a more succinct form with the boundedness property of the Hilbert transform. It should be mentioned that the motivation to study the problem from this perspective first came from the works of Lyubarskii and Seip [61]. The idea was further explored in the survey made by Seip [97].

## 1.4 New necessary conditions for bounded $\widetilde{H}$

To give a flavor of the work in the subsequent chapters, we will now deduce some necessary conditions for bounded two weight discrete Hilbert transforms. For the sake of comparison, we shall first discuss an $A_2$ type condition. We begin by noting that the discrete version of the $A_2$ condition, as stated in [51, 61], reads as

$$\sup_{n,m,m\leq n} \frac{1}{(n-m+1)^2} \sum_{l=m}^{n} w_l \sum_{l=m}^{n} w_l^{-1} < \infty. \tag{1.4.1}$$

If we are now given two finite or infinite sequences of distinct points $\Gamma = (\gamma_n)$ and $\Lambda = (\lambda_j)$ in $\mathbb{C}$ and a sequence of positive numbers $v = (v_n)$, we may define the discrete Hilbert transform by

$$(a_n)_n \mapsto \left( \sum_n \frac{a_n v_n}{\lambda_j - \gamma_n} \right)_j . \tag{1.4.2}$$

To make sense of this, we assume that $\Gamma$ and $\Lambda$, viewed as subsets of $\mathbb{C}$, are disjoint. We also assume that $\Lambda$ is a subset of the set

$$(\Gamma, v)^* = \left\{ z \in \mathbb{C} : \sum_n \frac{v_n}{|z - \gamma_n|^2} < \infty \right\}$$

because we wish to define the discrete Hilbert transform in (1.4.1) for sequences $(a_n)_n$ in

$$\ell_v^2 = \left\{ (a_n)_n : \sum_n |a_n|^2 v_n < \infty \right\}.$$

We now assume the set $(\Gamma, v)^*$ be nonempty and associate another weight sequence $w = (w_j)$ with $\Lambda$, and proceed to find an $A_2$ type necessary condition for the boundedness of the operator $\widetilde{H} : \ell_v^2 \to \ell_w^2$ given by (1.4.2). To obtain a condition similar to (1.4.1), we will simply adopt those arguments described in the works of Lyubarskii and Seip [61] for single weighted discrete transforms. For each $n$, we consider two squares of the form

$$Q_1^n = [\Re\gamma_n, \Re\gamma_n + h] \times [0, h] \quad \text{and} \quad Q_2^n = [\Re\gamma_n + 2h, \Re\gamma_n + 3h] \times [0, h]$$

of length $h$ and a side lying along the real line, and a positive sequence $(a_m)$ supported on $Q_1^n$ in the sense that $a_m = 0$ if $\gamma_m \notin Q_1^n$. Then for $j$ such that $\lambda_j \in Q_2^n$, it holds that

$$\begin{aligned} \left|\widetilde{H}(a_m)\right|^2 = \left| \sum_{m:\gamma_m \in Q_1^n} \frac{a_m v_m}{\lambda_j - \gamma_m} \right|^2 &\geq \left( \sum_{m:\gamma_m \in Q_1^n} \frac{a_m v_m \Re(\lambda_j - \gamma_m)}{|\lambda_j - \gamma_m|^2} \right)^2 \\ &\geq \frac{C}{h^2} \left( \sum_{m:\gamma_m \in Q_1^n} a_m v_m \right)^2 . \end{aligned}$$

This along with the boundedness of $\widetilde{H}$ leads to

$$\frac{1}{h^2}\sum_{j:\lambda_j\in Q_2^n} w_j\left(\sum_{m:\gamma_m\in Q_1^n} a_m v_m\right)^2 \leq C\sum_{m:\gamma_m\in Q_1^n}|a_m|^2 v_m. \tag{1.4.3}$$

Setting $a_m=1$, we deduce a discrete $A_2$ type condition:

$$\sup_{n,h}\frac{1}{h^2}\sum_{m:\gamma_m\in Q_1^n} v_m \sum_{m:\lambda_m\in Q_2^n} w_m<\infty. \tag{1.4.4}$$

The condition describes the local interaction between the weights whenever $\widetilde{H}$ is bounded from $\ell^2_v$ to $\ell^2_w$. Applying $\widetilde{H}$ or its adjoint to the sequence $e^{(n)}=\left(e^{(n)}_m\right)$ in which $e^{(n)}_n=1$ and 0 otherwise leads to a global necessary condition

$$\sup_m\left\{v_m\sum_n\frac{w_n}{|\gamma_m-\lambda_n|^2},\quad w_m\sum_n\frac{v_n}{|\gamma_n-\lambda_m|^2}\right\}<\infty. \tag{1.4.5}$$

An application of the Cauchy–Schwarz inequality shows that this condition can be sufficient if the supremum is small in the sense that the sum of any of the series with respect to $m$ against the respective weight sequence $(v_m)$ or $(w_m)$ is finite. The class of transforms $\widetilde{H}$ for which this smallness holds will be described in Section 3.2.

In the subsequent chapters, we will see that (1.4.5) serves as a testing condition, which bears strong resemblance to those testing conditions introduced in [71] and [58].

We may now assume that both $\gamma_n$ and $v_n$ are indexed by the positive integers. In addition, we assume that $\gamma_n$ accumulates only at infinity in the sense that $|\gamma_n|\nearrow\infty$ when $n\to\infty$. With each positive integer $m$, we associate two other positive integers defined by

$$m_{\min}=\min\left\{l:\inf_{l>m}|\gamma_l|/|\gamma_m|\geq 2\right\}\text{ and }\ m_{\max}=\max\left\{l:\sup_{l<m}|\gamma_m|/|\gamma_l|\geq 2\right\}.$$

For instance, if $\gamma_n$ grows at least exponentially with respect to $n$, $\left(|\gamma_n|\geq\exp(n)\right)$, then $m_{\min}=m+1$ and $m_{\max}=m-1$. We note that $m_{\max}$ may not exist for at most a finite number of indices $m$. If so, we may alter those corresponding sequences $\gamma_m$. Next, we consider another sequence $\lambda_j$, which consists of points from $(\Gamma,v)^*$, such that $|\gamma_j| < |\lambda_j| < |\gamma_{j+1}|$ for each $j = 1, 2, \ldots$ and a weight sequence $w_j$ associated with it.

To obtain our next necessary condition for boundedness, we look at a sequence $a^{(m)} = (a_n^{(m)})$ in which $a_n^{(m)} = 1$ for $n \le m_{\max}$ and $0$ otherwise. We then observe that

$$\|a^{(m)}\|_{\ell_v^2}^2 = \sum_{n=1}^{m_{\max}} v_n,$$

and note that for $\lambda_j$ such that $j \ge m_{\min}$, it readily follows that

$$|\widetilde{H}a^{(m)}(\lambda_j)|^2 = \left| \sum_{n=1}^{m_{\max}} \frac{v_n}{\lambda_j - \gamma_n} \right|^2 \ge C \frac{1}{|\lambda_j|^2} \left( \sum_{n=1}^{m_{\max}} v_n \right)^2. \tag{1.4.6}$$

Taking into account the boundedness of $\widetilde{H}$, we obtain from this that

$$\sum_{n=1}^{m_{\max}} v_n \ge C \sum_{j=1}^{\infty} w_j |\widetilde{H}a^{(m)}(\lambda_j)|^2 \ge C \sum_{j=m_{\min}}^{\infty} \frac{w_j}{|\lambda_j|^2} \left( \sum_{n=1}^{m_{\max}} v_n \right)^2. \tag{1.4.7}$$

On the other hand if we choose $a^{(m)} = (a_n^{(m)})$ so that $a_n^{(m)} = 1/\overline{\gamma_n}$ for $n \ge m_{\min}$ and zero else, then

$$\|a^{(m)}\|_{\ell_v^2}^2 = \sum_{n=m_{\min}}^{\infty} \frac{v_n}{|\gamma_n|^2}$$

and for each $\lambda_j$ such that $j \le m_{\max}$, we obtain

$$|\widetilde{H}a^{(m)}(\lambda_j)|^2 = \left| \sum_{n=m_{\min}}^{\infty} \frac{v_n}{\overline{\gamma_n}(\lambda_j - \gamma_n)} \right|^2 \ge C \left( \sum_{n=m_{\min}}^{\infty} \frac{v_n}{|\gamma_n|^2} \right)^2. \tag{1.4.8}$$

Considering the boundedness of the $\widetilde{H}$ again, we find that

$$\sum_{n=m_{\min}}^{\infty} \frac{v_n}{|\gamma_n|^2} \ge C \sum_{j=1}^{\infty} w_j |\widetilde{H}a^{(m)}(\lambda_j)|^2 \ge C \sum_{j=1}^{m_{\max}} w_j \left( \sum_{n=m_{\min}}^{\infty} \frac{v_n}{|\gamma_n|^2} \right)^2. \tag{1.4.9}$$

We summarize the result of our observations in the following theorem.

**Theorem 1.4.1.** *Let the sequences $(\gamma_n, v_n)$ and $(\lambda_n, w_n)$ be constructed as above. If the operator $\widetilde{H}$ is bounded from $\ell_v^2$ to $\ell_w^2$, then*

$$\sup_{m \ge 1} v_m \sum_{n=1}^{\infty} \frac{w_n}{|\gamma_m - \lambda_n|^2} < \infty \tag{1.4.10}$$

*and*

$$\sup_{m\geq 1}\left(\sum_{l=1}^{m_{\max}} v_l \sum_{n=m_{\min}}^{\infty} \frac{w_n}{|\lambda_n|^2}+\sum_{l=1}^{m_{\max}} w_l \sum_{n=m_{\min}}^{\infty} \frac{v_n}{|\gamma_n|^2}\right)<\infty. \tag{1.4.11}$$

In the next chapter, these conditions will be studied in depth including when the target space $\ell^2_w$ is replaced by a weighted space of functions. In the special case when the sequence $\gamma_n$ grows much faster, interestingly, it turns out that such simple conditions are sufficient as well and solve the corresponding two weight problem.

## Organization of the thesis

The results of this thesis are organized into two main parts. The first part concerns the different mapping properties of the two weight discrete Hilbert transforms. This part is presented in the next chapter. The second part deals with Carleson measures and various aspects of systems of reproducing kernels in spaces of analytic functions. These are all presented in the remaining chapters.

Most of the material in Chapter 3 and Chapter 4 could be viewed as transformations of the main results from the preceding chapter into results about systems of reproducing kernels and Carleson measures in function spaces. This makes each of the chapters intertwined with its predecessor and need them to be read in sequence. The last chapter is self-contained and can be read without priori information from the preceding chapters except at few cases where we used a result from Subsection 2.4.3 in order to construct the counterexamples in Sections 5.2 and 5.3. A couple of other notions which are used in earlier chapters are restated there for the reader's convenience.

We begin all of the remaining chapters with a brief discussion of the main points to be addressed in there. The discussions could be viewed as abstracts for the main results contained in the respective chapters. As in the introduction, only a few fundamental results by other authors relevant to our work will be stated as theorems or lemmas. Others will be either simply indicated by citations or briefly mentioned without further details.

## Notation

We close this introduction with a few words on notation. Throughout the thesis, the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set in question, which may be a Hilbert space, a set of complex numbers, or a suitable index set. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$. Sometimes we will need to remove a set of points, say $S_1$ from a given set $S$. The set thus obtained will then be written $S \setminus S_1$.

We denote by $k_\lambda$ any kernel function associated with a given point $\lambda$. The space where the kernel lives will be mainly clear from the context. Given a sequence $(\lambda_n)$ of points which will frequently be viewed as a subset of $\mathbb{C}$, we then denote by $S_R(\lambda_n)$ the system of normalized reproducing kernels associated with the sequence. If $\Lambda = (\lambda_n)$, then we alternatively write $S_R(\Lambda)$ instead of $S_R(\lambda_n)$.

# 2 Two weight discrete Hilbert transforms

In this chapter we consider the weighted discrete Hilbert transforms

$$(a_n)_n \mapsto \left( \sum_n \frac{a_n v_n}{\lambda_j - \gamma_n} \right)_j \tag{2.0.1}$$

from $\ell^2_v$ to $\ell^2_w$, where $\Gamma = (\gamma_n)$ and $\Lambda = (\lambda_j)$ are disjoint sequences of points in the complex plane and $v = (v_n)$ and $w = (w_j)$ are positive weight sequences. It is shown that if such a Hilbert transform is unitary, then $\Gamma \cup \Lambda$ is a subset of a circle or a straight line, and a description of all unitary discrete Hilbert transforms is then given. Transforms of the form

$$(a_n)_n \mapsto \sum_n \frac{a_n v_n}{z - \gamma_n}$$

from $\ell^2_v$ to a weighted $L^2$ space are also studied. In the special case when $|\gamma_n|$ grows at least exponentially, bounded transforms of this kind are described in terms of a simple relative to the Muckenhoupt's $A_2$ condition. The case when $z$ is, in addition, restricted to another sequence $\Lambda$ is again studied in detail; it is shown that a bounded transform satisfying a certain admissibility condition can be split into finitely many surjective transforms, and precise geometric conditions are found for invertibility of such two weight transforms. Our method to establish these results allows a moderate weakening of the growth of $(\gamma_n)$ at least when the weight sequence $(v_n)$ is sufficiently regular. The interplay between the growth of the sequence $(\gamma_n)$ and the "smoothness" of the weight $(v_n)$ is briefly considered in the last section of the chapter.

We note that all these operator theoretic results can be interpreted as statements about systems of reproducing kernels and Carleson measures in certain Hilbert spaces of which de Branges spaces and model subspaces of $H^2$ are prime examples. This will be our main subject of study in the next two chapters.

## 2.1 Unitary discrete Hilbert transforms

This part is concerned with the unitary property of the map in (2.0.1) in the complex plane. Our discussion will be based on [12]. We begin by assuming that we are given a finite or an infinite sequence of distinct points $\Gamma = (\gamma_n)$ in $\mathbb{C}$ and a corresponding sequence of positive numbers $v = (v_n)$. We may define the weighted discrete Hilbert transform as the map

$$(a_n) \mapsto \sum_n \frac{a_n v_n}{z - \gamma_n}, \tag{2.1.1}$$

which is well defined when $(a_n)$ belongs to $\ell^2_v$ and $z$ is a point in the set $(\Gamma, v)^*$. We denote the transformation defined in (2.1.1) by $H_{(\Gamma,v)}$ and ask when there are a sequence of points $\Lambda = (\lambda_j)$ in $(\Gamma, v)^*$ and a corresponding weight sequence $w = (w_j)$ for which the map

$$(a_n)_n \mapsto \left( \sum_n \frac{a_n v_n}{\lambda_j - \gamma_n} \right)_j \tag{2.1.2}$$

is a unitary transformation[1] from $\ell^2_v$ to $\ell^2_w$. First we note that there do exist pairs of sequences $(\Gamma, v)$ and $(\Lambda, w)$ for which this holds. We may for instance set $\Gamma = \mathbb{Z},\ \Lambda = \mathbb{Z} + \frac{1}{2}$ and $w_j = v_j = 1$ for all $j$. Then as will be seen in Subsection 2.1.2, $H_{(\mathbb{Z},1)}$ constitutes a unitary map from $\ell^2_1$ to $\ell^2_1$.

To stress the dependence on the pair $(\Lambda, w)$, we will re-denote the transformation in (2.1.2) by $H_{(\Gamma,v);(\Lambda,w)}$. If $H_{(\Gamma,v);(\Lambda,w)}$ is assumed to be a unitary transformation, then both $H_{(\Gamma,v);(\Lambda,w)}$ and its adjoint map orthonormal bases into orthonormal bases in the respective spaces, from which it follows that

$$w_j = \left( \sum_n \frac{v_n}{|\lambda_j - \gamma_n|^2} \right)^{-1} \text{ and } v_n = \left( \sum_j \frac{w_j}{|\lambda_j - \gamma_n|^2} \right)^{-1}.$$

This describes the associated weight sequence $w$ in terms of the sequence $\Lambda$. Thus, it remains to describe those sequences $\Lambda$ which give rise to unitary transformations

[1] Recall that a bijective map $T : H_1 \to H_2$ between two Hilbert spaces $H_1$ and $H_2$ is a unitary transformation if $\langle Th_1, Th_2 \rangle_{H_2} = \langle h_1, h_2 \rangle_{H_1}$ for all $h_1$ and $h_2$ in $H_1$. It is an isometry as one can see by setting $h_1 = h_2$ in this formula.

$H_{(\Gamma,v);(\Lambda,w)}$.

### 2.1.1 Localization of the sequences $\Gamma$ and $\Lambda$

Our starting point is the following localization result for the sequences $\Gamma$ and $\Lambda$ generating unitary discrete Hilbert transforms.

**Theorem 2.1.1.** *If the discrete Hilbert transform*

$$H_{(\Gamma,v);(\Lambda,w)} : \ell_v^2 \to \ell_w^2$$

*is unitary, then $\Gamma \cup \Lambda$ is a subset of a circle or a straight line in $\mathbb{C}$.*

To prove the theorem, we need to recall a few concepts from projective geometry. For a four-tuple of distinct points $(z_1, z_2, z_3, z_4)$ in the extended complex plane, $\big(\mathbb{C} \cup \{\infty\}\big)$, the cross-ratio is defined by

$$\mathscr{C} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}. \tag{2.1.3}$$

Note that there exists different ways to define the cross-ratio. However, they all differ from each other by a suitable permutation of the coordinates. In general, there are six possible different values the cross-ratio can take depending on the order in which the points are listed. If any one of these ratios is real, then all of them are real. One of the fundamental properties of a cross-ratio is that it is invariant under a Möbius transformation. It means that if

$$z_k \to \frac{az_k + b}{cz_k + d}$$

with $ad - bc \neq 0$, then $\mathscr{C}$ does not change for the new quadruple image points. Such transformations map in particular circles in the Riemann sphere into circles in the Riemann sphere. As a consequence, the following classical result holds.

**Theorem 2.1.2.** *Four points $(z_1, z_2, z_3, z_4)$ of the extended complex plane lie on the same circle or a straight line if and only if their cross-ratio is real.*

The proof of this result can be found in many standard books in projective geometry or geometry of complex numbers; for instance in ( [95], P. 36). We now turn to the proof of our first local result.

### Proof of Theorem 2.1.1

In what follows, we let $e^{(n)}$ denote the vectors in the standard orthonormal basis for $\ell^2_v$. Thus $e^{(n)}$ is the sequence for which the $n$-th entry is $v_n^{-1/2}$ and all the other entries are 0.

We fix an index $m$ and observe that since $\Gamma$ is a subset of $(\Lambda, w)^*$, the function

$$G(z) = (z-\gamma_m)\sum_j \frac{w_j}{(\overline{\lambda_j}-\overline{\gamma_m})(\lambda_j - z)}$$

is well-defined for $z$ in $\Gamma$. In fact, since $H_{(\Gamma,v);(\Lambda,w)}$ is assumed to be a unitary transformation, the basis vectors $e^{(n)}$ map into an orthonormal system in $\ell^2_w$, and therefore $G$ vanishes on $\Gamma$. Thus we may write

$$G(z) = G(z) - G(\gamma_n) = (z-\gamma_n)\sum_j \frac{w_j(\lambda_j-\gamma_m)}{(\overline{\lambda_j}-\overline{\gamma_m})(\lambda_j-\gamma_n)(\lambda_j - z)},$$

where on the right-hand side we have just subtracted the respective series that define $G(z)$ and $G(\gamma_n)$. It follows that

$$\frac{G(z)}{z-\gamma_n} = \sum_j \frac{w_j(\lambda_j-\gamma_m)}{(\overline{\lambda_j}-\overline{\gamma_m})(\lambda_j-\gamma_n)(\lambda_j - z)},$$

and this function vanishes for $z$ in $\Gamma \setminus \{\gamma_n\}$. Since $H_{(\Gamma,v);(\Lambda,w)}$ is assumed to be unitary, the vectors $H_{(\Gamma,v);(\Lambda,w)}e^{(n)}$ constitute an orthonormal basis for $\ell^2_w$, and therefore the sequence

$$\left(\frac{\lambda_j-\gamma_m}{\overline{\lambda_j}-\overline{\gamma_m}}\cdot\frac{1}{\lambda_j-\gamma_n}\right)_j$$

is a multiple of the sequence $\left(1/(\overline{\lambda_j}-\overline{\gamma_n})\right)_j$. Thus the complex numbers

$$\left(\frac{\lambda_j-\gamma_m}{\lambda_j-\gamma_n}\right)^2$$

have the same argument for all $j$, and so

$$\left(\frac{(\lambda_j-\gamma_m)(\lambda_l-\gamma_n)}{(\lambda_j-\gamma_n)(\lambda_l-\gamma_m)}\right)^2 > 0$$

for $j \neq l$ and $m \neq n$. In other words, the cross ratio of the four complex numbers $\lambda_j$, $\lambda_l$, $\gamma_n$, $\gamma_m$ is real. By Theorem 2.1.2, this can only happen if the points lie on the same circle or straight line.

After having applied this argument to four arbitrary points, say $\lambda_1$, $\lambda_2$, $\gamma_1$, and $\gamma_2$, we see that in fact every point from $\Gamma \cup \Lambda$ lies on the circle or a straight line determined by the four initial points, because we may apply the same argument to any given point in $\Gamma \cup \Lambda$ along with three of the points $\lambda_1$, $\lambda_2$, $\gamma_1$, or $\gamma_2$.

### 2.1.2 The unitary transformations associated with $\Gamma$ and $v$

For a given sequence $\Gamma$ being a subset of a circle or a straight line and an associated weight sequence $v$, we wish to describe those pairs $\Lambda$ and $w$ such that $H_{(\Gamma,v);(\Lambda,w)} : \ell^2_v \to \ell^2_w$ is a unitary transformation. To begin with, we require the admissibility condition

$$\sum_n \frac{v_n}{1+|\gamma_n|^2} < \infty, \tag{2.1.4}$$

which is now a necessary and sufficient condition for $(\Gamma, v)^*$ to be nonempty; we will say that $v$ is an admissible weight sequence for $\Gamma$ whenever (2.1.4) holds.

We will assume that $\Gamma$ is a subset of the real line. The case when $\Gamma$ is a subset of a circle is completely analogous, as will be briefly commented on at the end of this section. We set

$$\varphi(z) = \sum_n v_n \left( \frac{1}{\gamma_n - z} - \frac{\gamma_n}{1+\gamma_n^2} \right) \tag{2.1.5}$$

and observe that $\varphi$ is well-defined on $(\Gamma, v)^*$ because the series in (2.1.5) converges absolutely for $z$ in $(\Gamma, v)^*$. We also note that $\varphi$ is a Herglotz function in the upper half-plane (cf. [28], Chapter 9 ). It means that $\varphi$ is analytic in $\mathbb{C}_+$, meromorphic in $\mathbb{C}$, $\overline{\varphi(z)} = \varphi(\bar{z})$ and it belongs to $\mathbb{C}_+$ whenever $z$ is in $\mathbb{C}_+$. A general Herglotz function $\psi$ in the upper half-plane can be written as

$$\psi(z) = b + cz + \int_{-\infty}^{\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t),$$

where $b$ is a real constant, $c$ a nonnegative constant, and $\mu$ a nonnegative measure on the real line such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty.$$

We will say that $\psi$ is a purely atomic Herglotz function if $c = 0$ and $\mu$ is a purely atomic measure; our function $\varphi$ is thus an example of a purely atomic Herglotz function.

Now for every real number $\alpha$, we set

$$\Lambda(\alpha) = \big\{\lambda \in (\Gamma, v)^* : \ \varphi(\lambda) = \alpha\big\}.$$

We observe that

$$\sum_n \frac{v_n(z-w)}{(w-\gamma_n)(z-\gamma_n)} = \varphi(z) - \varphi(w), \tag{2.1.6}$$

which implies that the sequences $(1/(\lambda-\gamma_n))_n$ with $\lambda$ in $\Lambda(\alpha)$ constitute an orthogonal set in $\ell^2_v$. This means that $\Lambda(\alpha)$ is at most a countable set, so that we may associate with $\Lambda(\alpha)$ a weight sequence $w(\alpha) = (w_j)$, where

$$w_j = \left(\sum_n \frac{v_n}{(\lambda_j-\gamma_n)^2}\right)^{-1} \tag{2.1.7}$$

for $\lambda_j$ in $\Lambda(\alpha)$. It is implicit in our arguments that if $H_{(\Gamma,v);(\Lambda,w)} : \ell^2_v \to \ell^2_w$ is a unitary transformation, then $\Lambda = \Lambda(\alpha)$ and $w = w(\alpha)$ for some real number $\alpha$.

We will now prove the following main theorem.

**Theorem 2.1.3.** *Let $v$ be an admissible weight sequence for $\Gamma$. If $\Gamma$ is a subset of the real line, and $\alpha$ be a real number, then the discrete Hilbert transform*

$$H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))} : \ell^2_v \to \ell^2_{w(\alpha)}$$

*is unitary if and only if $(\alpha - \varphi(z))^{-1}$ is a purely atomic Herglotz function.*

*Proof.* In this proof, we will again use the standard orthonormal basis vectors $e^{(n)}$ in $\ell^2_v$; we will denote the corresponding basis vectors in $\ell^2_{w(\alpha)}$ by $f^{(j)}$. We will use the notation $\|\cdot\|_v$ and $\|\cdot\|_w$ for the respective norms in $\ell^2_v$ and $\ell^2_w$.

It is clear that the adjoint transformation[2] to $H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}$ is again a discrete Hilbert transform. In fact, since $\Gamma$ and $\Lambda(\alpha)$ are sequences of real numbers, we have $H^*_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))} = -H_{(\Lambda(\alpha),w(\alpha));(\Gamma,v)}$, where

[2]Here and in what follows $T^*$ refers the adjoint of an operator $T$ in the Hilbert space sense, i.e the operator for which $\langle Tf, g\rangle = \langle f, T^*g\rangle$ holds.

$$H_{(\Lambda(\alpha),w(\alpha));(\Gamma,v)} : \ell^2_{w(\alpha)} \to \ell^2_v.$$

Therefore, $H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}$ is unitary if and only if both $H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}$ and $H_{(\Lambda(\alpha),w(\alpha));(\Gamma,v)}$ are isometric. Hence it suffices to check whether

$$\left(H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))})e^{(n)}\right) \text{ and } \left(H_{(\Lambda(\alpha),w(\alpha));(\Gamma,v)}f^{(j)}\right)$$

are orthonormal sequences in respectively $\ell^2_{w(\alpha)}$ and $\ell^2_v$.

The orthogonality of the vectors $H_{(\Lambda(\alpha),w(\alpha));(\Gamma,v)}f^{(j)}$ in $\ell^2_v$ has already been verified (see (2.1.6)); it is just a consequence of the definition of $\Lambda(\alpha)$. Likewise, by (2.1.7), we have automatically

$$\|H_{(\Lambda(\alpha),w(\alpha));(\Gamma,v)}f^{(j)}\|^2_v = \sum_n \frac{w_j v_n}{|\gamma_n - \lambda_j|^2} = 1.$$

So our task is to show that

$$\left(H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}e^{(n)}\right)$$

is an orthonormal sequence in $\ell^2_{w(\alpha)}$ if and only if $(\alpha - \varphi(z))^{-1}$ is a purely atomic Herglotz function.

We first assume that $(\alpha - \varphi(z))^{-1}$ is indeed a purely atomic Herglotz function. It suffices to show that there is a real constant $b$ such that

$$\frac{1}{\alpha - \varphi(z)} = b + \sum_j w_j \left(\frac{1}{\lambda_j - z} - \frac{\lambda_j}{1+\lambda_j^2}\right), \tag{2.1.8}$$

where $\lambda_j$ are the points in $\Lambda(\alpha)$ and $w_j$ are as in (2.1.7). Indeed, by symmetry, it will then follow that the numbers $\gamma_n$ are solutions to the equation

$$\sum_j w_j \left(\frac{1}{\lambda_j - z} - \frac{\lambda_j}{1+\lambda_j^2}\right) = -b,$$

so that the arguments already employed for the vectors $H_{(\Lambda(\alpha),w(\alpha));(\Gamma,v)}f^{(j)}$ apply similarly to the vectors $H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}e^{(n)}$.

We start from the representation (2.1.8), with no a priori assumption on the points $\lambda_j$ and the nonnegative numbers $w_j$ except the admissibility condition

$$\sum_j \frac{w_j}{1+\lambda_j^2} < \infty;$$

our goal is to prove that the $\lambda_j$ are in $\Lambda(\alpha)$ and that the $w_j$ are given by (2.1.7). We first observe that if we set $z = \lambda_j + iy$, then we get, by restricting to imaginary parts,

$$\frac{w_j}{y} \le \left( \sum_n \frac{y v_n}{(\lambda_j - \gamma_n)^2 + y^2} \right)^{-1},$$

whence

$$\sum_n \frac{v_n}{\left(\lambda_j - \gamma_n\right)^2} \le w_j^{-1}.$$

In other words, the points $\lambda_j$ belong to $(\Gamma, v)^*$. We now multiply each side of (2.1.8) by $z - \lambda_j$ and take the limit when $z = \lambda_j + iy$ and $y \to 0^+$; since $\lambda_j$ is in $(\Gamma, v)^*$ and $\varphi(\lambda_j) = \alpha$, this gives (2.1.7).

Suppose, on the other hand, that $(\alpha - \varphi(z))^{-1}$ is not a purely atomic Herglotz function and that the vectors $H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}e^{(n)}$ constitute an orthonormal system in $\ell^2_{w(\alpha)}$. We will show that this leads to a contradiction. To begin with, our assumption on $(\alpha - \varphi(z))^{-1}$ implies that

$$\frac{1}{\alpha - \varphi(z)} = b + \sum_j w_j \left( \frac{1}{\lambda_j - z} - \frac{\lambda_j}{1+\lambda_j^2} \right) + cz + \int_{-\infty}^{\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t), \tag{2.1.9}$$

with $\mu$ a spectral measure such that $\mu(\{\lambda_j\}) = 0$ for every $j$ and not both $c = 0$ and $\mu = 0$; the fact that the $w_j$ are given by (2.1.7) can be proved as in the first part of the proof.

We now argue in the same way as above, reversing the roles of $\Gamma$ and $\Lambda(\alpha)$. This means that we first show, by again restricting to imaginary parts, that

$$\sum_j \frac{w_j}{(\gamma_n - \lambda_j)^2} + \int_{-\infty}^{\infty} \frac{d\mu(t)}{(\gamma_n - t)^2} \le v_n^{-1}$$

for every $n$. We infer from this that both the sum and the integral on the right-hand side of (2.1.9) converge absolutely for $z = \gamma_n$. Indeed, the right-hand side of (2.1.9) vanishes for $z=\gamma_n$, and so if we put $z=\gamma_n+i\delta$ in (2.1.9), divide each side by $iy$, and let $y$ tend to 0, we get

$$v_n^{-1}=\sum_j \frac{w_j}{(\gamma_n-\lambda_j)^2}+\int_{-\infty}^{\infty}\frac{d\mu(t)}{(\gamma_n-t)^2}.$$

Since we should have $\|H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}e^{(n)}\|_{w(\alpha)}=1$, we have reached a contradiction unless $\mu=0$. On the other hand, if $\mu=0$ and $c>0$, then we also reach a contradiction because the condition for orthogonality of the vectors $H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}e^{(n)}$ becomes

$$\sum_j\left(\frac{w_j}{\gamma_m-\lambda_j}-\frac{w_j}{\gamma_n-\lambda_j}\right)=0$$

for $m\neq n$, and this is inconsistent with the right-hand side of (2.1.9) being 0 whenever $z=\gamma_n$. $\square$

A few remarks are in order. First, it should be noted that we may have $(\Gamma,v)^*\cap\mathbb{R}=\emptyset$ even if $(\Gamma,v)$ is an admissible pair. The following is an example.

**Example 2.** Pick a sequence of distinct prime numbers $p_l$ such that

$$\sum_l p_l^{-1/2}<\infty.$$

Set $\Gamma=\bigcup_l p_l^{-1}\mathbb{Z}$, and equip $\Gamma$ with the weight sequence $v$ obtained by placing a weight of magnitude $p_l^{-3/2}$ at every point of the sequence $p_l^{-1}\mathbb{Z}$.

On the other hand, if $\Gamma$ is a discrete subset of the real line, then $H_{(\Gamma,v);(\Lambda(\alpha),w(\alpha))}:\ell_v^2\to\ell_{w(\alpha)}^2$ is unitary for every $\alpha$ with one possible exception: It fails to be unitary when

$$\sum_n v_n<\infty \quad\text{and}\quad \alpha=\sum_n\frac{v_n\gamma_n}{1+\gamma_n^2}.$$

This statement follows almost immediately from Theorem 2.1.3. We get the exceptional case because the constant $c$ in the representation (2.1.9) is obtained as

$$c=\lim_{y\to\infty}\frac{1}{iy(\alpha-\varphi(iy))}.$$

If $\Gamma$ is a subset of the unit circle, then the potential (2.1.5) should be replaced by

$$\varphi(z) = \frac{i}{2} \sum_n v_n \frac{\gamma_n + z}{\gamma_n - z}; \tag{2.1.10}$$

the analysis goes through in the same way, and we obtain a statement completely analogous to Theorem 2.1.3. Note, however, that for discrete sets $\Gamma$ on the unit circle, there will be no exceptional value for $\alpha$ because there is no linear term '$cz$' in the general representation of a Herglotz function. Indeed, a Herglotz function $\psi$ in the unit disk is of the form

$$\psi(z) = b + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),$$

where $b$ is a real constant and $\mu$ a nonnegative measure on the circle.

Finally, as will be seen in Section 4.5, the unitary transformations obtained from Theorem 2.1.3 (and its counterpart for the unit circle) correspond precisely to Clark's orthonormal bases [29], and the exceptional case here is also like the exceptional case in Clark's result. From this point of view, Theorem 2.1.3 is essentially a reformulation of Clark's theorem.

## 2.2 Bounded discrete Hilbert transforms

When we now turn to questions about boundedness, surjectivity, and invertibility, results of the same generality as in the previous section seem at present out of reach. The results to be presented below are complete only when the discrete Hilbert transforms are defined on particularly sparse sequences. We will nevertheless present the problems in the most general setting, as we believe they merit further investigations, and we will (next chapter) emphasize the connection with topics such as Carleson measures and Riesz bases of normalized reproducing kernels in Hilbert spaces of analytic functions; this will lead us to the most intriguing general question, namely whether or not the Feichtinger conjecture holds true for systems of reproducing kernel Bessel sequences in such spaces. This and the next two sections are based on [11].

We now consider the transform $H_{(\Gamma,v)}$ defined by (2.1.1) and ask if we may describe those nonnegative measures $\mu$ on $(\Gamma, v)^*$ such that $H_{(\Gamma,v)}$ acts as a bounded map from $\ell_v^2$ to $L^2\big((\Gamma,v)^*,\mu\big)$. This question is another version of the long-standing problem of finding criteria akin to the Muckenhoupt $A_2$ condition for boundedness of two weight Hilbert transforms.

We now also assume that both $\Gamma = (\gamma_n)$ and the weight sequence $v = (v_n)$ are indexed by the positive integers. The main result of this section is a solution to the boundedness problem when $\Gamma$ is exponentially or super-exponentially "sparse", i.e., when we have

$$\inf_{n\geq 1} |\gamma_{n+1}|/|\gamma_n| > 1. \tag{2.2.1}$$

In this case, $(\Gamma,v)^*$ is nonempty and in fact equal to $\mathbb{C}\setminus\Gamma$ if and only if

$$\sum_{n=1}^{\infty} \frac{v_n}{1+|\gamma_n|^2} < \infty. \tag{2.2.2}$$

When we consider the boundedness problem for such sparse sequences $\Gamma$, it is quite natural to partition $\mathbb{C}$ in the following way. Set $\Omega_1 = \big\{z\in\mathbb{C}: \ |z| < \big(|\gamma_1|+|\gamma_2|\big)/2\big\}$ and then

$$\Omega_n = \big\{z\in\mathbb{C}: \ \big(|\gamma_{n-1}|+|\gamma_n|\big)/2 \leq |z| < \big(|\gamma_n|+|\gamma_{n+1}|\big)/2\big\}$$

for $n\geq 2$.

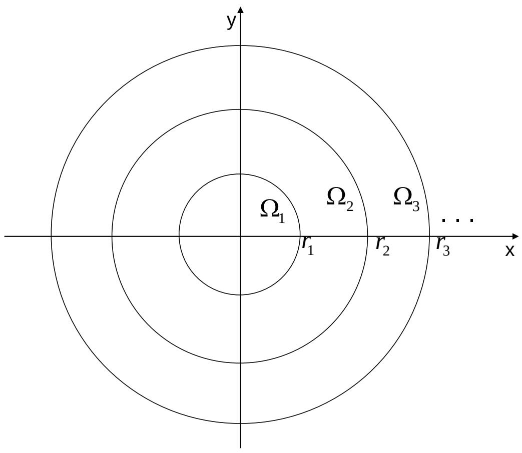


Figure 2.1: $r_n = \big(|\gamma_n|+|\gamma_{n+1}|\big)/2$.

Our solution to the boundedness problem reads as follows.

**Theorem 2.2.1.** *Suppose that the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that $v$ is an admissible weight sequence for $\Gamma$. If $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma)=0$, then the map $H_{(\Gamma,v)}$ is bounded* *from $\ell^2_v$ to $L^2(\mathbb{C},\mu)$ if and only if*

$$\sup_{n\geq 1}\int_{\Omega_n}\frac{v_n d\mu(z)}{|z-\gamma_n|^2}<\infty \tag{2.2.3}$$

*and*

$$\sup_{n\geq 1}\left(\sum_{l=1}^{n} v_l \sum_{m=n+1}^{\infty}\int_{\Omega_m}\frac{d\mu(z)}{|z|^2}+\sum_{m=1}^{n}\mu\left(\Omega_m\right)\sum_{l=n+1}^{\infty}\frac{v_l}{|\gamma_l|^2}\right)<\infty. \tag{2.2.4}$$

It should be noted that neither (2.2.3) nor (2.2.4) alone is in general sufficient for the boundedness of $H_{(\Gamma,v)}$. In other words, the two conditions are independent of each other in the sense that no one implies the other. We will give simple examples illustrating this in Subsection 2.2.2 . It should also be noted that the condition is symmetric in the two measures

$$\sum_{n=1}^{\infty} v_n\delta_{\gamma_n}$$

and $\mu$. This is natural since the theorem also gives a necessary and sufficient condition for the adjoint transformation

$$f\mapsto\left(\int_{\mathbb{C}}\frac{f(z)d\mu(z)}{\bar{z}-\overline{\gamma_n}}\right)_n$$

to be bounded from $L^2(\mathbb{C},\mu)$ to $\ell^2_v$. The condition (2.2.4) can be understood as a simple relative to the classical Muckenhoupt's $A_2$ condition.

Besides its simplicity, the main virtue of Theorem 2.2.1 is its role as a tool in our study of surjectivity and invertibility of discrete Hilbert transforms in the subsequent sections.

### 2.2.1 Proof of Theorem 2.2.1

In what follows, we will use the notation

$$V_1=1,\quad V_n=\sum_{j=1}^{n-1}v_j,\qquad\text{and}\qquad P_n=\sum_{j=n+1}^{\infty}\frac{v_j}{|\gamma_j|^2}. \tag{2.2.5}$$

Note that with these notations, condition (2.2.4) can be replaced by a dyadic version

$$\sup_{n\geq 1}\left(V_n\sum_{V_n\leq V_m\leq 2V_n}\int_{\Omega_m}\frac{d\mu(z)}{|z|^2}+P_n\sum_{P_n\leq P_m\leq 2P_n}\mu(\Omega_m)\right)<\infty. \tag{2.2.6}$$

**Proof of the necessity of the conditions in Theorem 2.2.1**

The necessity of the conditions has been already established in Section 1.4 when $H_{(\Gamma,v)}$ acts from $\ell^2_v$ to another weighted sequence space $\ell^2_w$. We shall now redo the arguments replacing the target space by $L^2(\mathbb{C},\mu)$. We observe first that the necessity of (2.2.3) is obvious: Just apply $H_{(\Gamma,v)}$ to the sequence $e^{(n)}=(e^{(n)}_m)$ with $e^{(n)}_n=1$ and $e^{(n)}_m=0$ for $m\neq n$.

To show that (2.2.4) is also a necessary condition, we begin by looking at the sequence $c^{(n)}=(c^{(n)}_m)$ so that $c^{(n)}_m=1$ for $m<n$ and $c^{(n)}_m=0$ otherwise. We observe that $\|c^{(n)}\|^2_v=V_n$ and note that for $z$ in $\Omega_l$ and $l\geq n$ we have

$$|H_{(\Gamma,v)}c^{(n)}(z)|^2=\left|\sum_{m=1}^{n-1}\frac{v_m}{z-\gamma_m}\right|^2\gtrsim\frac{V_n^2}{|z|^2}. \tag{2.2.7}$$

Taking into account the boundedness of $H_{(\Gamma,v)}$, we deduce from this that

$$V_n\gtrsim\int_{\mathbb{C}}|H_{(\Gamma,v)}c^{(n)}(z)|^2d\mu(z)=\sum_{k=1}^{\infty}\int_{\Omega_k}\left|\sum_{m=1}^{n-1}\frac{v_m}{z-\gamma_m}\right|^2d\mu(z)\gtrsim V_n^2\sum_{m\geq n}\int_{\Omega_m}\frac{d\mu(z)}{|z|^2}.$$

On the other hand, if we set $a^{(n)}=(a^{(n)}_m)$ so that $a^{(n)}_m=1/\overline{\gamma_m}$ for $m>n$ and $a^{(n)}_m=0$ otherwise, then $\|a^{(n)}\|^2_v=P_n$. We note that for $z$ in $\Omega_l$ and $l\leq n$ we have

$$|H_{(\Gamma,v)}a^{(n)}(z)|^2=\left|\sum_{m=n+1}^{\infty}\frac{v_m}{\overline{\gamma_m}(z-\gamma_m)}\right|^2\gtrsim P_n^2.$$

Thus

$$P_n\gtrsim\int_{\mathbb{C}}|H_{(\Gamma,v)}a^{(n)}(z)|^2d\mu(z)\gtrsim P_n^2\sum_{m\leq n}\mu(\Omega_m).$$

**Proof of the sufficiency of the conditions in Theorem 2.2.1**

Let $a=(a_n)$ be an arbitrary sequence in $\ell^2_v$. We make first the following estimate:

$$\int_{\Omega_n} |H_{(\Gamma,v)}a(z)|^2 d\mu(z) \le 3\int_{\Omega_n}\left(\left|\sum_{m=1}^{n-1}\frac{a_m v_m}{z-\gamma_m}\right|^2+\frac{|a_n|^2v_n^2}{|z-\gamma_n|^2}+\left|\sum_{m=n+1}^{\infty}\frac{a_m v_m}{z-\gamma_m}\right|^2\right)d\mu(z)$$
$$\lesssim \int_{\Omega_n}\left(|z|^{-2}\left(\sum_{m=1}^{n-1}|a_m|v_m\right)^2+\left(\sum_{m=n+1}^{\infty}\frac{|a_m|v_m}{|\gamma_m|}\right)^2\right)d\mu(z)+|a_n|^2v_n;$$

here we used the Cauchy–Schwarz inequality, (2.2.1) and (2.2.3). Hence it remains for us to show that

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}|a_m|v_m\right)^2\int_{\Omega_n}|z|^{-2}d\mu(z)\lesssim\sum_{j=1}^{\infty}|a_j|^2v_j \tag{2.2.8}$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{m=n+1}^{\infty}\frac{|a_m|v_m}{|\gamma_m|}\right)^2\mu(\Omega_n)\lesssim\sum_{j=1}^{\infty}|a_j|^2v_j. \tag{2.2.9}$$

We consider first (2.2.8). To simplify the writing, we set

$$\tau_n=\left(\int_{\Omega_n}|z|^{-2}d\mu(z)\right)^{\frac{1}{2}}.$$

By duality, we have

$$\left(\sum_{n=1}^{\infty}\tau_n^2\left(\sum_{m=1}^{n-1}|a_m|v_m\right)^2\right)^{\frac{1}{2}}=\sup_{\|(c_n)\|_{\ell^2}=1}\sum_{n=1}^{\infty}|c_n|\tau_n\sum_{m=1}^{n-1}|a_m|v_m.$$

Since

$$\sum_{n=1}^{\infty}|c_n|\tau_n\sum_{m=1}^{n-1}|a_m|v_m=\sum_{m=1}^{\infty}|a_m|v_m\sum_{n=m+1}^{\infty}|c_n|\tau_n,$$

it suffices to show that the $\ell^2$-norm of

$$\alpha_m=v_m^{\frac{1}{2}}\sum_{n=m+1}^{\infty}|c_n|\tau_n$$

is bounded by a constant times the $\ell^2$-norm of $(c_n)$. To this end, we note that the Cauchy–Schwarz inequality gives

$$|\alpha_m|^2 \le v_m \sum_{n=m+1}^{\infty} |c_n|^2 V_n^{-\frac{1}{2}} \sum_{j=m+1}^{\infty} \tau_j^2 V_j^{\frac{1}{2}}.$$

By (2.2.4), we see that

$$\sum_{j:2^l V_m < V_j \le 2^{l+1} V_m} \tau_j^2 V_j^{\frac{1}{2}} \lesssim \frac{1}{2^{\frac{l}{2}} V_{m+1}^{\frac{1}{2}}}$$

for $l \ge 0$. Summing these inequalities, we get

$$\sum_{j=m+1}^{\infty} \tau_j^2 V_j^{\frac{1}{2}} \lesssim \frac{1}{V_{m+1}^{\frac{1}{2}}}.$$

Hence

$$|\alpha_m|^2 \lesssim \frac{v_m}{V_{m+1}^{\frac{1}{2}}} \sum_{n=m+1}^{\infty} |c_n|^2 V_n^{-\frac{1}{2}}.$$

This gives us

$$\begin{aligned} \sum_{m=1}^{\infty} |\alpha_m|^2 &\lesssim \sum_{m=1}^{\infty} \frac{v_m}{V_{m+1}^{\frac{1}{2}}} \sum_{n=m+1}^{\infty} |c_n|^2 V_n^{-\frac{1}{2}} \\ &= \sum_{n=1}^{\infty} |c_n|^2 V_n^{-\frac{1}{2}} \sum_{m=1}^{n-1} \frac{v_m}{V_{m+1}^{\frac{1}{2}}} \end{aligned}$$

when we change the order of summation, and so (2.2.8) follows because

$$V_n^{-\frac{1}{2}} \sum_{m=1}^{n-1} \frac{v_m}{V_{m+1}^{\frac{1}{2}}} \le V_n^{-\frac{1}{2}} \int_0^{V_n} x^{-\frac{1}{2}} dx = 2. \tag{2.2.10}$$

We next consider (2.2.9). We note to begin with that the Cauchy–Schwarz inequality gives

$$\left( \sum_{m=n+1}^{\infty} \frac{|a_m| v_m}{|\gamma_m|} \right)^2 \le \sum_{m=n+1}^{\infty} |a_m|^2 v_m P_{m-1}^{\frac{1}{2}} \sum_{j=n+1}^{\infty} \frac{v_j}{P_{j-1}^{\frac{1}{2}} |\gamma_j|^2}.$$

Since

$$\sum_{j=n+1}^{\infty} \frac{v_j}{P_{j-1}^{\frac{1}{2}} |\gamma_j|^2} \le \int_0^{P_n} x^{-\frac{1}{2}} dx \le 2P_n^{\frac{1}{2}},$$

it follows that

$$\sum_{n=1}^{\infty} \mu(\Omega_n) \left( \sum_{m=n+1}^{\infty} \frac{|a_m| v_m}{|\gamma_m|} \right)^2 \lesssim \sum_{n=1}^{\infty} \mu(\Omega_n) P_n^{\frac{1}{2}} \sum_{m=n+1}^{\infty} |a_m|^2 v_m P_{m-1}^{\frac{1}{2}},$$

which becomes

$$\sum_{n=1}^{\infty} \mu(\Omega_n) \Big( \sum_{m=n+1}^{\infty} \frac{|a_m| v_m}{|\gamma_m|} \Big)^2 \lesssim \sum_{m=1}^{\infty} |a_m|^2 v_m P_{m-1}^{\frac{1}{2}} \sum_{n=1}^{m-1} \mu(\Omega_n) P_n^{\frac{1}{2}}$$

when we change the order of summation. From (2.2.4) it follows that

$$\begin{aligned} \sum_{n=1}^{m-1} \mu(\Omega_n) P_n^{\frac{1}{2}} &\lesssim \sum_{l=0}^{\infty} \sum_{n: 2^l P_{m-1} \le P_n \le 2^{l+1} P_{m-1}} \mu(\Omega_n) P_n^{\frac{1}{2}} \\ &\lesssim \frac{1}{P_{m-1}^{\frac{1}{2}}} \sum_{l=0}^{\infty} \frac{1}{2^{\frac{l}{2}}} \lesssim \frac{1}{P_{m-1}^{\frac{1}{2}}}, \end{aligned}$$

and we get (2.2.9).

## Special cases

Condition (2.2.3) of Theorem 2.2.1 is a condition on the local behavior of $\mu$, while condition (2.2.4) deals with its global behavior. Combining the two conditions, we see that (2.2.3) may be replaced by a stronger global necessary condition:

$$\sup_{n \ge 1} \int_{\mathbb{C}} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} < \infty. \tag{2.2.11}$$

This is in fact immediate because

$$\begin{aligned} \int_{\mathbb{C}} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} &= \sum_{m=1}^{\infty} \int_{\Omega_m} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \simeq \int_{\Omega_n} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \\ &+ \frac{v_n}{|\gamma_n|^2} \sum_{m=1}^{n-1} \mu(\Omega_m) + v_n \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2}. \end{aligned} \tag{2.2.12}$$

One could also arrive at (2.2.11) by simply applying $H_{(\Gamma,v)}$ to the sequence $e^{(n)} = (e_m^{(n)})_m$ where $e_n^{(n)} = v_n^{-\frac{1}{2}}$ and 0 for $n \neq m$.

We note that the sparsity assumption (2.2.1) plays no role in establishing (2.2.11) (and hence (2.2.3)).

We single out two cases in which (2.2.4) is automatically fulfilled once either this condition or the original one (2.2.3) holds.

**Corollary 2.2.2.** *Suppose the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that the numbers $v_n$ grow at least exponentially and that the numbers $v_n/|\gamma_n|^2$ decay at least exponentially with $n$. If $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma)=0$, then the operator $H_{(\Gamma,v)}$ is bounded from $\ell^2_v$ to $L^2(\mathbb{C},\mu)$ if and only if*

$$\sup_{n\geq 1}\int_{\Omega_n}\frac{v_n d\mu(z)}{|z-\gamma_n|^2}<\infty. \tag{2.2.13}$$

To see this it is enough to verify condition (2.2.4), namely that

$$\sup_{n\geq 1}\sum_{m=1}^{n}\mu\left(\Omega_m\right)\sum_{l=n+1}^{\infty}\frac{v_l}{|\gamma_l|^2}\simeq\sup_{n\geq 1}\frac{v_{n+1}}{|\gamma_{n+1}|^2}\sum_{m=1}^{n}\int_{\Omega_m}d\mu(z).$$

The right-hand quantity is bounded (up to a constant multiple) by

$$\sup_{n\geq 1}\frac{v_{n+1}}{|\gamma_{n+1}|^2}\sum_{m=1}^{n}\frac{|\gamma_{m+1}|^2}{v_{m+1}}\int_{\Omega_m}\frac{v_m}{|z|^2}d\mu(z)<\infty;$$

here we used the exponential growth of the numbers $|\gamma_n|^2/v_n$ to compare $|\gamma_{m+1}|^2/v_{m+1}$ with $v_m/|z|^2$ for each $z$ in $\Omega_m$.

**Corollary 2.2.3.** *Suppose the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that $(v_n)$ is summable. If $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma)=0$, then the operator $H_{(\Gamma,v)}$ is bounded from $\ell^2_v$ to $L^2(\mathbb{C},\mu)$ if and only if*

$$\sup_{n\geq 1}\int_{\mathbb{C}}\frac{v_n d\mu(z)}{|z-\gamma_n|^2}<\infty. \tag{2.2.14}$$

This corollary also follows immediately from Theorem 2.2.1 by simply looking at the splitting in (2.2.12).

Condition (2.2.4) may become simpler if additional assumption are made on the data $(\gamma_n,v_n)$. If, for instance, we assume that there exists a positive constant $C$ such that

$$P_m \le C \frac{V_{m+1}}{|\gamma_{m+1}|^2} \tag{2.2.15}$$

for every $m > 1$, then it implies that

$$v_n \le C V_n \tag{2.2.16}$$

holds for $n \ge 1$, and hence $V_n$ grows at most exponentially. We will see in the lemma below that (2.2.15) is equivalent to (2.2.16) and exponential decay of $V_n/|\gamma_n|^2$ along sufficiently sparse arithmetic progressions.

**Lemma 2.2.4.** *Assume (2.2.15) holds. Then there exists a positive integer $N$ such that*

$$\frac{V_{n+N}}{|\gamma_{n+N}|^2} \le \frac{1}{2} \frac{V_n}{|\gamma_n|^2} \tag{2.2.17}$$

*holds for every positive integer $n$.*

*Proof.* We set $n_1 = 1$ and define $n_l$ inductively for $l = 1, 2, \dots$ by letting $n_l$ be the smallest index $n$ such that $V_{n_l}/V_{n_{l-1}} \ge 2$. Since $V_n$ grows at most exponentially, there is a constant $K$ such that also $V_{n_l}/V_{n_{l-1}} \le K$ for all $l > 1$. We first note that the estimate

$$\sum_{j=1}^{\infty} \frac{V_{n_{l+j}}}{|\gamma_{n_{l+j}}|^2} \le M \frac{V_{n_l}}{|\gamma_{n_l}|^2}$$

for some positive constant $M$ is an immediate consequence of (2.2.15). We then observe that if a sequence of positive numbers $c_l$ satisfies

$$\sum_{j=1}^{\infty} c_{l+j} \le M c_l,$$

then in particular $c_{l+j} \le M c_{l+m}$ for $j > m$, and therefore

$$c_{l+j} \le \frac{M}{L} \sum_{m=1}^{L} c_{l+m} \le \frac{M^2}{L} c_l$$

for $j > L$. It follows that if $n_{l-1} \le n < n_l$ and $n_{l+j} \le n + N < n_{l+j}$, then

$$\frac{V_{n+N}}{|\gamma_{n+N}|^2} \le \min\left(\frac{K^j}{c^N}, \frac{K^2 M^2}{j-1}\right) \frac{V_n}{|\gamma_n|^2},$$

where $c = \inf_n |\gamma_{n+1}|/|\gamma_n| > 1$. Since, independently of $j$,

$$\min\left(\frac{K^j}{c^N}, \frac{K^2M^2}{j-1}\right) \leq \frac{1}{2}$$

for sufficiently large $N$, the result follows. □

Our claim that (2.2.16) and (2.2.17) together represent a reformulation of (2.2.15) is now immediate because the implication in the other direction is trivial. In what follows we will use the following consequence of (2.2.16) and (2.2.17):

$$\frac{V_{n+j}}{|\gamma_{n+j}|^2} \lesssim 2^{-j/N} \frac{V_n}{|\gamma_n|^2}. \tag{2.2.18}$$

**Corollary 2.2.5.** *Suppose that the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that $v$ is an admissible weight sequence for $\Gamma$ for which (2.2.15) holds. If $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma) = 0$, then the map $H_{(\Gamma,v)}$ is bounded from $\ell^2_v$ to $L^2(\mathbb{C},\mu)$ if and only if (2.2.3) and*

$$\sup_{n\geq 1} V_n \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} < \infty \tag{2.2.19}$$

*hold.*

The corollary follows from Theorem 2.2.1 and the above lemma because

$$\begin{aligned}
\sup_{n\geq 1} P_n \sum_{m=1}^{n-1} \mu(\Omega_m) \quad &\lesssim \quad \sup_{n\geq 1} P_n \sum_{m=1}^{n-1} |\gamma_{m+1}|^2 \int_{\Omega_m} |z|^{-2} d\mu(z) \\
&\lesssim \quad \sup_{n\geq 1} \frac{V_{n+1}}{|\gamma_{n+1}|^2} \sum_{m=1}^{n-1} \frac{|\gamma_{m+1}|^2}{V_{m-1}} < \infty,
\end{aligned}$$

for the last inequality we in particular used $V_{n+1} \leq (1+C)^3 V_{n-2}$ with $C$ the absolute constant in (2.2.15).

### 2.2.2 Bessel sequences

We now switch to discrete Hilbert transforms and require thus that $\mu$ be a purely atomic measure. In other words, we are interested in the case when there are a sequence of points $\Lambda = (\lambda_j)$ in $(\Gamma, v)^*$ and a corresponding weight sequence

$w = (w_j)$ such that the discrete Hilbert transform $H_{(\Gamma,v);(\Lambda;w)}$ is bounded from $\ell^2_v$ to $\ell^2_w$. As will be explained in Section 3.2, this means that we will be dealing with Bessel sequences of normalized reproducing kernels for certain Hilbert spaces of analytic functions.

We now record the following consequence of the Open Mapping Theorem [85, p. 73].

**Lemma 2.2.6.** *Suppose $T$ is a bounded linear transformation from a Hilbert space $\mathscr{H}_1$ to another Hilbert space $\mathscr{H}_2$. Then $T$ is surjective if and only if the adjoint transformation $T^*$ is bounded from below.*

If we let $T$ be the map $f \mapsto (\langle f, f_j\rangle_{\mathscr{H}})$ from $\mathscr{H}$ to $\ell^2$, then we find

$$T^*(c_j) = \sum_j c_j f_j.$$

Thus it follows from Lemma 2.2.6 that $(f_j)$ is a Riesz basic sequence if and only if it is a Bessel sequence for which the moment problem $\langle f, f_j\rangle_{\mathscr{H}} = a_j$ has a solution $f$ in $\mathscr{H}$ for every square-summable sequence $(a_j)$. We may also set $T = H_{(\Gamma,v);(\Lambda,w)}$ and find that Lemma 2.2.6 gives the necessary condition

$$w_j \simeq \left(\sum_{n=1}^{\infty} \frac{v_n}{|\lambda_j - \gamma_n|^2}\right)^{-1} \tag{2.2.20}$$

for surjectivity of the transformation $H_{(\Gamma,v);(\Lambda,w)}$. To see this, observe that the lemma implies

$$\sum_{m=1}^{\infty} \left|\sum_{n=1}^{\infty} \frac{a_n w_n}{\gamma_m - \overline{\lambda_n}}\right|^2 v_m \simeq \sum_{n=1}^{\infty} |a_n|^2 w_n \tag{2.2.21}$$

for each $\ell^2_w$-summable sequence $(a_n)$. The desired conclusion follows once up on setting $a_n = 1$, for $n = j$ and 0 otherwise in (2.2.21).

When $\Gamma$ and $v$ are given and $\Lambda$ is a sequence in $(\Gamma, v)^*$, we will say that the sequence given by (2.2.20) is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$. In the next chapter, we will indeed see that the sequence $w_j^{-\frac{1}{2}}$ constitutes the norm of the reproducing kernels $k_{\lambda_j}$ for certain Hilbert spaces of analytic functions.

We want to disentangle condition (2.2.4). To this end, we split any given sequence

$\Lambda$ into three disjoint sequences:

$$\Lambda^{(0)} = \left\{ \lambda \in \Lambda : \text{ if } \lambda \text{ is in } \Omega_n, \text{ then } \frac{v_n}{|\lambda - \gamma_n|^2} \geq \max\left( \frac{V_n}{|\lambda|^2}, P_n \right) \right\}.$$

$$\Lambda^{(V)} = \left\{ \lambda \in \Lambda : \text{ if } \lambda \text{ is in } \Omega_n, \text{ then } \frac{V_n}{|\lambda|^2} > \max\left( \frac{v_n}{|\lambda - \gamma_n|^2}, P_n \right) \right\}.$$

$$\Lambda^{(P)} = \left\{ \lambda \in \Lambda : \text{ if } \lambda \text{ is in } \Omega_n, \text{ then } P_n > \max\left( \frac{v_n}{|\lambda - \gamma_n|^2}, \frac{V_n}{|\lambda|^2} \right) \right\}.$$

We say that a sequence $\Lambda$ is $V$-lacunary if

$$\sup_n \# \left( \Lambda \cap \bigcup_{m:\ 2^n \leq V_m \leq 2^{n+1}} \Omega_m \right) < \infty$$

and $P$-lacunary if

$$\sup_n \# \left( \Lambda \cap \bigcup_{m:\ 2^{-n-1} \leq P_m \leq 2^{-n}} \Omega_m \right) < \infty.$$

We then have the following interesting reformulation of Theorem 2.2.1.

**Theorem 2.2.7.** *Suppose the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that $v$ is an admissible weight sequence for $\Gamma$. Let $\Lambda$ be a sequence in $(\Gamma, v)^*$, and let $w$ be the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$. Then $H_{(\Gamma,v);(\Lambda,w)}$ is a bounded transformation if and only if $\sup_n \#(\Lambda \cap \Omega_n) < \infty$, $\Lambda^{(V)}$ is a $V$-lacunary sequence, $\Lambda^{(P)}$ is a $P$-lacunary sequence, and*

$$\sup_{n \geq 1} \left( V_n \sum_{m \geq n} \sum_{\lambda \in \Lambda^{(0)} \cap \Omega_m} \frac{|\lambda - \gamma_m|^2}{v_m |\lambda|^2} + P_n \sum_{m \leq n} \sum_{\lambda \in \Lambda^{(0)} \cap \Omega_m} \frac{|\lambda - \gamma_m|^2}{v_m} \right) < \infty. \tag{2.2.22}$$

For each point $\lambda_j$ in $\Lambda^{(0)}$, the Bessel weight sequence can be estimated by

$$w_j^{-1} \simeq \sum_{n=1}^{\infty} \frac{v_n}{|\lambda_j - \gamma_n|^2} \simeq \frac{v_j}{|\lambda_j - \gamma_j|^2} \tag{2.2.23}$$

which follows from the sparsity condition (2.2.1), and it implicitly appears in condition (2.2.22). Qualitatively, this result for Bessel sequences is rather unexpected. The sequence splits naturally into three sequences: one sequence $\Lambda^{(0)}$ being near the points $(\gamma_n)$ with a geometric condition on its distortion from $(\gamma_n)$ and other two sequences $\Lambda^{(V)}$ and $\Lambda^{(P)}$ being "exponentially more sparse" than the sequence $(\gamma_n)$ and with no further restriction on their locations. This splitting into a "super-thin" sequence $\Lambda^{(V)} \bigcup \Lambda^{(P)}$ and a "distorted" sequence $\Lambda^{(0)}$ represents a phenomenon not previously recorded, as far as we know.

Corollary 2.2.2 and Corollary 2.2.3, when restricted to the case of Bessel sequences, describe two situations in which the "super-thin" part does not appear, for different reasons: Corollary 2.2.2 covers the case when $V_n$ grows exponentially and $P_n$ decays exponentially with $n$; $\Lambda^{(V)}$ and $\Lambda^{(P)}$ can then both be "absorbed" in $\Lambda^{(0)}$. Corollary 2.2.3 covers the case when $V_n$ is uniformly bounded so that $\Lambda^{(V)}$ can only be a finite sequence; the sequence $\Lambda^{(P)}$ can again be "absorbed" in $\Lambda^{(0)}$.

We conclude that the most interesting situation occurs when either $v_n/|\gamma_n|^2 = o(P_n)$ or $v_n = o(V_n)$ and $V_n \to \infty$ as $n \to \infty$. These two cases will be studied in depth in Section 2.4.

We finish this section by constructing the examples promised in Section 2.2, which show that neither condition (2.2.3) nor (2.2.4) is sufficient for the boundedness of the Hilbert transform $H_{(\Gamma,v)}$. We can make our constructions following the corresponding conditions in Theorem 2.2.7.

**Example 3.** For each $n$, set $\gamma_n = 2^n$ and the associated weight sequence $v_n$ equals 1. We construct a sequence $\Lambda = (\lambda_j)$ by picking a single point $\lambda_j = (j\log(j+1))^{-1}\gamma_j$ from each annulus $\Omega_j$. If we now set $w_j = 1$ for each $j$, then $\Lambda$ fails to satisfy (2.2.22). On the other hand if we pick the sequence $\lambda_{n,j} = \gamma_n + \gamma_n/j, \quad j = 1,2,\ldots$ then

$$\Lambda_1 = \left(\bigcup_{j=1}^{\infty} \lambda_{n,j}\right)_n$$

easily meets condition (2.2.22) while there exists no uniform bound on the number of its points found on each annulus $\Omega_n$.

## 2.3 Surjective discrete Hilbert transforms

Our next general question is the following: If $w = (w_j)$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$ and $H_{(\Gamma,v);(\Lambda,w)}$ is a bounded transformation, is it possible to split $\Lambda$ into a finite union of subsequences $\Lambda'$ such that, with $w'$ denoting the subsequence of $w$ corresponding to $\Lambda'$, each of the transformations $H_{(\Gamma,v);(\Lambda',w')}$ is surjective? As it will be explained in the next chapter, this question would have a positive answer should the well known Feichtinger conjecture hold true. The following result gives a positive answer to this question when (2.2.1) holds.

**Theorem 2.3.1.** *Suppose the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that $v$ is an admissible weight sequence for $\Gamma$. If $\Lambda$ is a sequence in $\mathbb{C} \setminus \Gamma$, $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$, and the transformation $H_{(\Gamma,v);(\Lambda,w)}$ is bounded, then $\Lambda$ admits a splitting into a finite union of subsequences such that, for each subsequence $\Lambda'$ and corresponding subsequence $w'$ of $w$, the transformation $H_{(\Gamma,v);(\Lambda',w')}$ is surjective.*

The geometry of the sequence $\Lambda$ which generates a bounded map $H_{(\Gamma,v);(\Lambda,w)}$ plays an essential role to make the splitting through the required properties.

### 2.3.1 Proof of Theorem 2.3.1

By Theorem 2.2.7 the points of $\Lambda$ splits naturally into three subsequences. This splitting is our starting point for proving the theorem. Indeed, we use the same splitting as in the theorem and treat the three sequences $\Lambda^{(0)}$, $\Lambda^{(V)}$, and $\Lambda^{(P)}$ separately. We also use Lemma 2.2.6, i.e., we make a splitting so that, for each subsequence $\Lambda'$ with associated weight sequence, the adjoint transformation $H_{(\Lambda',w');(\Gamma,v)}$ is bounded below. From now on, we will use the notations

$$W_n = \sum_{m=1}^{n-1} w_m \quad \text{and} \quad Q_n = \sum_{m=n+1}^{\infty} \frac{w_m}{|\lambda_m|^2}. \tag{2.3.1}$$

### 2.3.2 The splitting of $\Lambda^{(0)}$

We may assume that there is at most one point $\lambda_n$ in $\Lambda^{(0)}$ from each annulus $\Omega_n$; we denote the corresponding weights by $w_n$. Let $\Lambda' = (\lambda_{n_j})$ be a subsequence of $\Lambda^{(0)}$

with corresponding weight sequence $w' = (w_{n_j})$, and let $a = (a_{n_j})$ be an arbitrary $\ell^2_{w'}$-sequence. Since

$$|\xi - \eta|^2 \geq |\xi|^2 - 2|\xi||\eta| + |\eta|^2 \geq \frac{1}{2}|\xi|^2 - |\eta|^2$$

for arbitrary complex numbers $\xi$ and $\eta$, we have

$$|H_{(\Lambda',w');(\Gamma,v)}a(\gamma_{n_j})|^2 = \left|\sum_{l=1}^{\infty} \frac{a_{n_l} w_{n_l}}{\overline{\gamma_{n_j} - \lambda_{n_l}}}\right|^2$$

$$\geq \frac{1}{2}\frac{|a_{n_j}|^2 w_{n_j}^2}{|\lambda_{n_j} - \gamma_{n_j}|^2} - 2\left|\sum_{l=1}^{j-1} \frac{a_{n_l} w_{n_l}}{\overline{\gamma_{n_j} - \lambda_{n_l}}}\right|^2 - 2\left|\sum_{l=j+1}^{\infty} \frac{a_{n_l} w_{n_l}}{\overline{\gamma_{n_j} - \lambda_{n_l}}}\right|^2.$$

On the other hand, the sparsity condition (2.2.1) gives that

$$w_{n_j} \simeq \frac{|\lambda_{n_j} - \gamma_{n_j}|^2}{v_{n_j}} \tag{2.3.2}$$

for each point $\lambda_{n_j} \in \Lambda^{(0)}$. Therefore, by the definition of $\Lambda^{(0)}$, there is a positive constant $c$ such that

$$\|H_{(\Lambda',w');(\Gamma,v)}a\|_v^2 \geq c\|a\|_{w'}^2 - 2\sum_{j=1}^{\infty}\left(\left|\sum_{l=1}^{j-1} \frac{a_{n_l} w_{n_l}}{\overline{\gamma_{n_j} - \lambda_{n_l}}}\right|^2 + \left|\sum_{l=j+1}^{\infty} \frac{a_{n_l} w_{n_l}}{\overline{\gamma_{n_j} - \lambda_{n_l}}}\right|^2\right) v_{n_j}.$$

Hence it remains for us to show that, for a given $\varepsilon > 0$, we may obtain

$$\sum_{j=1}^{\infty}\left(\sum_{l=1}^{j-1} |a_{n_l}| w_{n_l}\right)^2 \frac{v_{n_j}}{|\lambda_{n_j}|^2} \leq \varepsilon \sum_{j=1}^{\infty} |a_{n_j}|^2 w_{n_j} \tag{2.3.3}$$

and

$$\sum_{j=1}^{\infty}\left(\sum_{l=j+1}^{\infty} \frac{|a_{n_l}| w_{n_l}}{|\lambda_{n_l}|}\right)^2 v_{n_j} \leq \varepsilon \sum_{j=1}^{\infty} |a_{n_j}|^2 w_{n_j} \tag{2.3.4}$$

for every subsequence $\Lambda'$ in a finite splitting of $\Lambda^{(0)}$.

We proceed as in the proof of Theorem 2.2.1. Thus we set $\tau_j = v_{n_j}^{\frac{1}{2}}/|\lambda_{n_j}|$ and consider first (2.3.3). By duality,

$$\left(\sum_{j=1}^{\infty} \tau_j^2 \left(\sum_{l=1}^{j-1} |a_{n_l}| w_{n_l}\right)^2\right)^{\frac{1}{2}} = \sup_{\|(c_j)\|_{\ell^2}=1} \sum_{j=1}^{\infty} |c_j| \tau_j \sum_{l=1}^{j-1} |a_{n_l}| w_{n_l}.$$

Since

$$\sum_{j=1}^{\infty}|c_j|\tau_j\sum_{l=1}^{j-1}|a_{n_l}|w_{n_l}=\sum_{l=1}^{\infty}|a_{n_l}|w_{n_l}\sum_{j=l+1}^{\infty}|c_j|\tau_j,$$

it suffices to show that the $\ell^2$-norm of

$$\alpha_l=w_{n_l}^{\frac{1}{2}}\sum_{j=l+1}^{\infty}|c_j|\tau_j$$

can be made smaller than $\varepsilon$ times the $\ell^2$-norm of $(c_j)$. To this end, we note that the Cauchy–Schwarz inequality gives

$$|\alpha_l|^2\le w_{n_l}\sum_{j=l+1}^{\infty}|c_j|^2W_{n_j}^{-\frac{1}{2}}\sum_{m=l+1}^{\infty}\tau_m^2W_{n_m}^{\frac{1}{2}}.$$

Using (2.2.4), we get

$$\sum_{m=l+1}^{\infty}\tau_m^2W_{n_m}^{\frac{1}{2}}\lesssim\frac{1}{W_{n_{l+1}}^{\frac{1}{2}}}.$$

Hence

$$|\alpha_l|^2\lesssim\frac{w_{n_l}}{W_{n_{l+1}}^{\frac{1}{2}}}\sum_{j=l+1}^{\infty}|c_j|^2W_{n_j}^{-\frac{1}{2}}.$$

This gives us

$$\sum_{l=1}^{\infty}|\alpha_l|^2\lesssim\sum_{j=1}^{\infty}|c_j|^2W_{n_j}^{-\frac{1}{2}}\sum_{l=1}^{j-1}\frac{w_{n_l}}{W_{n_{l+1}}^{\frac{1}{2}}},$$

and so (2.3.3) would follow if we could obtain

$$\sum_{l=1}^{j-1}\frac{w_{n_l}}{W_{n_{l+1}}^{\frac{1}{2}}}\le c\varepsilon W_{n_j}^{\frac{1}{2}}\tag{2.3.5}$$

for an absolute constant $c$.

Having singled out this goal, we proceed to consider (2.3.4). We note to begin with that the Cauchy–Schwarz inequality gives

$$\left(\sum_{l=j+1}^{\infty}\frac{|a_{n_l}|w_{n_l}}{|\lambda_{n_j}|}\right)^2\le\sum_{l=j+1}^{\infty}|a_{n_l}|^2w_{n_l}Q_{n_{l-1}}^{\frac{1}{2}}\sum_{m=j+1}^{\infty}\frac{w_{n_m}}{Q_{n_{m-1}}^{\frac{1}{2}}|\lambda_{n_m}|^2}.$$

Now our goal will be to obtain

$$\sum_{m=j+1}^{\infty} \frac{w_{n_m}}{Q_{n_{m-1}}^{\frac{1}{2}} |\lambda_{n_m}|^2} \leq c\varepsilon Q_{n_j}^{\frac{1}{2}}. \tag{2.3.6}$$

Indeed, this would imply

$$\begin{aligned}\sum_{j=1}^{\infty} \left( \sum_{l=j+1}^{\infty} \frac{|a_{n_l}| w_{n_l}}{|\lambda_{n_l}|} \right)^2 v_{n_j} &\lesssim \varepsilon \sum_{j=1}^{\infty} v_{n_j} Q_{n_j}^{\frac{1}{2}} \sum_{l=j+1}^{\infty} |a_{n_l}|^2 w_{n_l} Q_{n_{l-1}}^{\frac{1}{2}} \\ &= \varepsilon \sum_{l=1}^{\infty} |a_{n_l}|^2 w_{n_l} Q_{n_{l-1}}^{\frac{1}{2}} \sum_{j=1}^{l-1} v_{n_j} Q_{n_j}^{\frac{1}{2}}.\end{aligned}$$

By (2.2.4), we have

$$\sum_{j=1}^{l-1} v_{n_j} Q_{n_j}^{\frac{1}{2}} \lesssim \frac{1}{Q_{n_{l-1}}^{\frac{1}{2}}},$$

and so it will suffice to have (2.3.6).

In order to obtain the two estimates (2.3.5) and (2.3.6) for every subsequence in our finite splitting of $\Lambda^{(0)}$, we make a splitting according to the following algorithm:

(1) Let $\delta$ be a small positive number to be chosen later. Select those $n$ for which $w_n > \delta W_n$. If we choose $\Lambda'$ to consist of every $N$-th $\lambda_n$ in the corresponding subsequence of $\Lambda^{(0)}$, then we get

$$\sum_{l=1}^{j-1} \frac{w_{n_l}}{W_{n_{l+1}}^{\frac{1}{2}}} \leq \frac{2}{\delta N} W_{n_j}^{\frac{1}{2}}$$

by again comparing the sum to the integral of the function $x^{-\frac{1}{2}}$ over the interval from $0$ to $W_{n_j}$. Thus we achieve our goal if we choose $N$ to be of the order of magnitude $1/(\delta\varepsilon)$.

(2) Return to those points $\lambda_{n_j}$ not selected in (1). For these we have $w_{n_j} \leq \delta W_{n_j}$. Group these points into blocks of points with consecutive indices such that for each block

$$\delta \leq \sum_j w_{n_j} W_{n_j}^{-1} < 2\delta.$$

Construct new subsequences by picking every $N$-th block from this sequence of blocks. Then some elementary estimates, again using comparisons with an integral, lead to the following inequality:

$$\sum_{l=1}^{j-1} \frac{w_{n_l}}{W_{n_{l+1}}^{\frac{1}{2}}} \leq \frac{16\delta}{1-(1-2\delta)^N} W_{n_j}^{\frac{1}{2}},$$

where we sum over the new subsequence. Thus it would suffice if we choose $N$ to be roughly $1/\delta$ and $\delta$ to be a suitable constant times $\varepsilon$.

(3) Take one of the subsequences selected in (1) or (2) and consider the subsequence of this subsequence, say $\Lambda' = (\lambda_{n_j})$, along which $w_{n_j}|\lambda_{n_j}|^{-2} > \delta Q_{n_j}$. If we select a new subsequence by picking every $N$-th $\lambda_{n_j}$ in the sequence $\Lambda'$, then the sum in (2.3.6) becomes smaller than $2/(\delta N)Q_{n_j}$ by the same argument as in (1). Again our goal is achieved if we choose $N$ to be of the order of magnitude $1/(\delta\varepsilon)$.

(4) Take again one of the subsequences selected in (1) or (2) and consider those subsequences of these for which we have $w_{n_j}|\lambda_{n_j}|^{-2} \leq \delta Q_{n_j}$. Group the points in these subsequences into blocks of points with consecutive indices such that for each block

$$\delta \leq \sum_j w_{n_j}|\lambda_{n_j}|^{-2} Q_{n_j}^{-1} < 2\delta.$$

Now construct new subsequences by picking every $N$-th block from this sequence of blocks. Then as in point (2) we get

$$\sum_{m=j+1}^{\infty} \frac{w_{n_m}}{Q_{n_{m-1}}^{\frac{1}{2}} |\lambda_{n_m}|^2} \leq \frac{16\delta}{1-(1-2\delta)^N} Q_{n_j}^{\frac{1}{2}}.$$

(Here the summation is again over the new subsequence.) We observe once more that it would suffice if we choose $N$ to be roughly $1/\delta$ and $\delta$ to be a suitable constant times $\varepsilon$.

### 2.3.3 The splitting of $\Lambda^{(V)}$

The splitting of $\Lambda^{(V)}$ is almost identical to that of $\Lambda^{(0)}$. We will now use the estimate

$$|H_{(\Lambda',w');(\Gamma,v)}a(\gamma_n)|^2 \geq \frac{1}{2}\frac{|a_{n_j}|^2 w_{n_j}^2}{|\overline{\lambda_{n_j}} - \overline{\gamma_n}|^2} - 2\left|\sum_{l=1}^{j-1} \frac{a_{n_l} w_{n_l}}{\overline{\lambda_{n_j}} - \overline{\gamma_n}}\right|^2 - 2\left|\sum_{l=j+1}^{\infty} \frac{a_{n_l} w_{n_l}}{\overline{\gamma_n} - \overline{\lambda_{n_l}}}\right|^2.$$

The reason we write '$\gamma_n$' instead of '$\gamma_{n_j}$' is that we need to sum over several annuli $\Omega_n$ in order to estimate the norm of $\|a\|_w$. Indeed, we may assume that $\lambda_{n_j}$ belongs to a union of annuli $\Omega_n$, denoted by $\Delta_j$, such that

$$\sum_{\gamma_n \in \Delta_j} \frac{v_n}{|\lambda_{n_j} - \gamma_n|^2} \geq \frac{1}{10}\frac{V_{n_j}}{|\lambda_j|^2},$$

with the sets $\Delta_j$ being pairwise disjoint. Therefore, by the definition of $\Lambda^{(V)}$, there is a constant $c$ such that

$$\sum_{\gamma_n \in \Delta_j} |a_{n_j}|^2 w_{n_j}^2 \frac{v_n}{|\lambda_{n_j} - \gamma_n|^2} \geq c|a_{n_j}|^2 w_{n_j}.$$

Hence we obtain

$$\|H_{(\Lambda',w');(\Gamma,v)}a\|_v^2 \geq c\|a\|_{w'}^2 - 2\sum_{j=1}^{\infty}\sum_{\gamma_n \in \Delta_j}\left(\left|\sum_{l=1}^{j-1} \frac{a_{n_l} w_{n_l}}{\overline{\lambda_{n_j}} - \overline{\gamma_n}}\right|^2 + \left|\sum_{l=j+1}^{\infty} \frac{a_{n_l} w_{n_l}}{\overline{\lambda_{n_j}} - \overline{\gamma_n}}\right|^2\right) v_{n_j}.$$

The splitting is then done in essentially the same way as above, repeating the reasoning based on the estimate (2.3.2).

### 2.3.4 The splitting of $\Lambda^{(P)}$

We use once more (2.3.3), but this time we may assume that $\lambda_{n_j}$ belongs to a union of annuli $\Omega_n$, again denoted by $\Delta_j$, such that

$$\sum_{\gamma_n \in \Delta_j} \frac{v_n}{|\lambda_{n_j} - \gamma_n|^2} \geq \frac{1}{10} P_{n_j},$$

with the sets $\Delta_j$ being pairwise disjoint. Therefore, by the definition of $\Lambda^{(P)}$, there is a constant $c$ such that

$$\sum_{\gamma_n \in \Delta_j} |a_{n_j}|^2 w_{n_j}^2 \frac{v_n}{|\lambda_{n_j} - \gamma_n|^2} \geq c|a_{n_j}|^2 w_{n_j}.$$

Hence we obtain

$$\|H_{(\Lambda',w');(\Gamma,v)}a\|_v^2 \geq c\|a\|_{w'}^2 - 2\sum_{j=1}^{\infty}\sum_{\gamma_n\in\Delta_j}\left(\left|\sum_{l=1}^{j-1}\frac{a_{n_l}w_{n_l}}{\overline{\lambda_{n_j}}-\overline{\gamma_n}}\right|^2+\left|\sum_{l=j+1}^{\infty}\frac{a_{n_l}w_{n_l}}{\overline{\lambda_{n_j}}-\overline{\gamma_n}}\right|^2\right)v_{n_j},$$

and proceed as outlined in the previous paragraph.

## 2.4 Invertible discrete Hilbert transforms

We proceed now to our next main result, which is a general statement about invertible discrete Hilbert transforms. The observation that leads to this result, is that the inverse transformation, if it exists, can be identified effectively as another discrete Hilbert transform.

To make a precise statement, we introduce the following terminology. We say that a sequence $\Lambda$ of distinct points in $(\Gamma,v)^*$ is a uniqueness sequence for $H_{(\Gamma,v)}$ if there is no nonzero vector $a$ in $\ell_v^2$ such that $H_{(\Gamma,v)}a$ vanishes on $\Lambda$; we say that $\Lambda$ is an exact uniqueness sequence for $H_{(\Gamma,v)}$ if it is a uniqueness sequence for $H_{(\Gamma,v)}$, but fails to be so on the removal of any one of the points in $\Lambda$. If $\Lambda$ is an exact uniqueness sequence for $H_{(\Gamma,v)}$, then we say that a nontrivial function $G$ defined on $(\Gamma,v)^*$ is a generating function for $\Lambda$ if $G$ vanishes on $\Lambda$ but, for every $\lambda_j$ in $\Lambda$, there is a nonzero vector $a^{(j)}$ in $\ell_v^2$ such that

$$G(z) = (z-\lambda_j)H_{(\Gamma,v)}a^{(j)}(z)$$

for every $z$ in $(\Gamma,v)^*$. It is clear that if a generating function exists, then it is unique up to multiplication by a nonzero constant.

We note that if $\Lambda$ is an exact uniqueness sequence for $H_{(\Gamma,v)}$, then there exists a unique element $e=(e_n)$ in $\ell_v^2$ such that $H_{(\Gamma,v);(\Lambda,w)}e=(1,0,0,...)$. We set $\nu=(\nu_n)$ and $\varpi=(\varpi_j)$, where

$$\nu_n = v_n|\lambda_1-\gamma_n|^2|e_n|^2, \tag{2.4.1}$$

$\varpi_1 = w_1^{-1}$, and

$$\varpi_j = w_j^{-1}|\lambda_j-\lambda_1|^{-2}\left|\sum_{n=1}^{\infty}\frac{e_nv_n}{(\lambda_j-\gamma_n)^2}\right|^{-2}, \tag{2.4.2}$$

presuming the series appearing in the latter expression converges absolutely. We will see that, plainly, we have absolute convergence of this series whenever $\Lambda$ admits a generating function.

Our next result reads as follows.

**Theorem 2.4.1.** *Suppose that every exact uniqueness sequence for $H_{(\Gamma,v)}$ admits a generating function. Let $\Lambda$ be a sequence in $(\Gamma,v)^*$, and let $w$ be the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma,v)$. Then $H_{(\Gamma,v);(\Lambda,w)}$ is an invertible transformation if and only if*

*(1) $\Lambda$ is an exact uniqueness sequence for $H_{(\Gamma,v)}$ and*

*(2) the transformations $H_{(\Gamma,v);(\Lambda,w)}$ and $H_{(\Lambda,\varpi);(\Gamma,v)}$ are bounded.*

Note that when we write '$H_{(\Lambda,\varpi);(\Gamma,v)}$ is bounded', it is implicitly understood that $\Gamma \subset (\Lambda,\varpi)^*$.

We may observe that if $\gamma_n \to \infty$ when $n \to \infty$, then the function

$$\Phi(z) = (z-\lambda_1)\sum_{n=1}^{\infty}\frac{e_n v_n}{z-\gamma_n}, \tag{2.4.3}$$

and its reciprocal $\Psi = 1/\Phi$ are meromorphic functions in $\mathbb{C}$, and $\Phi$ is then the generating function for $\Lambda$. We may then rewrite the expressions for $v$ and $\varpi$ as

$$v_n = \frac{v_n}{|\Psi'(\gamma_n)|^2} \quad \text{and} \quad \varpi_j = w_j^{-1}|\Phi'(\lambda_j)|^{-2}. \tag{2.4.4}$$

Combining Theorem 2.4.1 with Theorem 2.2.1, we will obtain computable and geometric invertibility criteria when $\Gamma$ is a sparse sequence as defined by (2.2.1).

### 2.4.1 Proof of Theorem 2.4.1

It is clear that if the mapping $H_{(\Gamma,v);(\Lambda,w)}$ is invertible, then $\Lambda$ is an exact uniqueness sequence for $H_{(\Gamma,v)}$, which in turn implies that there is a unique element $e=(e_n)$ in $\ell^2_v$ such that $H_{(\Gamma,v)}e$ vanishes on $\Gamma\setminus\{\lambda_1\}$ and takes the value 1 at $\lambda_1$. Then

$$G(z) = (z-\lambda_1)\sum_{n=1}^{\infty}\frac{e_n v_n}{z-\gamma_n}$$

is a generating function for $\Lambda$. Since by assumption $G(\lambda_j)=0$ for $j>1$, we may write

$$G(z) = G(z) - G(\lambda_j) = (z-\lambda_j)\sum_{n=1}^{\infty}\frac{e_n v_n(\gamma_n-\lambda_1)}{(\gamma_n-\lambda_j)(z-\gamma_n)},$$

where on the right-hand side we have just subtracted the respective series that define $G(z)$ and $G(\lambda_j)$. Since $G$ is a generating function for $\Lambda$, it follows that

$$\sum_{n=1}^{\infty} \frac{|e_n|^2|\gamma_n-\lambda_1|^2 v_n}{|\gamma_n-\lambda_j|^2} < \infty.$$

In particular, the sequence

$$e^{(j)} = \left( e_n \frac{\gamma_n-\lambda_1}{\gamma_n-\lambda_j} \left( \sum_{m=1}^{\infty} \frac{e_m v_m(\lambda_1-\gamma_m)}{(\lambda_j-\gamma_m)^2} \right)^{-1} \right)_n$$

will be the unique vector in $\ell^2_v$ such that $H_{(\Gamma,v)}e^{(j)}(\lambda_l)$ is 0 when $l \neq j$ and 1 for $l = j$.

To simplify the writing, we set

$$\alpha_j = \left( \sum_{m=1}^{\infty} \frac{e_m v_m(\lambda_1-\gamma_m)}{(\lambda_j-\gamma_m)^2} \right)^{-1};$$

thus if $b = (b_1, b_2, ..., b_l, 0, 0, ...)$ is a sequence with only finitely many nonzero entries, then the sequence

$$a = \left( e_n(\gamma_n-\lambda_1) \sum_{j=1}^{l} \frac{b_j\alpha_j}{\gamma_n-\lambda_j} \right)_n \tag{2.4.5}$$

will be the unique vector in $\ell^2_v$ such that $H_{(\Gamma,v);(\Lambda,w)}a = b$. This means that we have identified a linear transformation, defined on a dense subset of $\ell^2_w$, that must be the inverse transformation to $H_{(\Gamma,v);(\Lambda,w)}$, should it exist. Hence, under the assumption that $\Lambda$ is an exact uniqueness sequence for $H_{(\Gamma,v)}$, a necessary and sufficient condition for invertibility of $H_{(\Gamma,v);(\Lambda,w)}$ is that the linear transformation defined by (2.4.5) extends to a bounded transformation on $\ell^2_w$. An equivalent condition is that the transformation $H_{(\Lambda,\varpi);(\Gamma,\nu)}$ be bounded, where

$$\nu_n = v_n|\lambda_1-\gamma_n|^2|e_n|^2$$

and

$$\varpi_j = w_j^{-1} \left| \sum_{n=1}^{\infty} \frac{e_n v_n(\lambda_1-\gamma_n)}{(\lambda_j-\gamma_n)^2} \right|^{-2} = w_j^{-1}|\lambda_j-\lambda_1|^{-2} \left| \sum_{n=1}^{\infty} \frac{e_n v_n}{(\lambda_j-\gamma_n)^2} \right|^{-2}.$$

In the final step, we used the definition of the sequence $(e_n)$.

An interesting feature of our results for sparse sequences is that invertibility implies that $\Lambda$ is a perturbation of $\Gamma$, in a sense to be made precise. As a consequence, we will see that there may exist bounded transformations $H_{(\Gamma,v);(\Lambda,w)}$ such that no infinite subsequence $\Lambda'$ of $\Lambda$ is also a subsequence of another sequence $\Lambda''$ for which the associated Hilbert transform is invertible.

### 2.4.2 Localization of $\Lambda$ when $\Gamma$ is a sparse sequence

We will for the rest of this section consider two interesting special cases. The main point of this subsection will be that, although $\Lambda$ may possibly have a nontrivial splitting into three sequences $\Lambda^{(0)}$, $\Lambda^{(V)}$, $\Lambda^{(P)}$ (cf. the discussion in Subsection 2.2.2), the invertibility of $H_{(\Gamma,v);(\Lambda,w)}$ forces the sequences $\Lambda^{(V)}$ and $\Lambda^{(P)}$ to be trivial, in a sense to be made precise.

We assume as before that $\Gamma=(\gamma_n)$ is indexed by the positive integers, and that the sequence is sparse in the sense that (2.2.1) holds. We retain the notation

$$V_n=\sum_{m=1}^{n-1} v_m \quad \text{and} \quad P_n=\sum_{m=n+1}^{\infty} \frac{v_m}{|\gamma_m|^2}$$

from the previous section. In the discussion below, the sets

$$D_n(v;M)=\left\{\lambda\in\Omega_n : \frac{Mv_n}{|\lambda-\gamma_n|^2}\geq \max\left(\frac{V_n}{|\lambda|^2},P_n\right)\right\},$$

defined for every admissible weight sequence $v$ and positive number $M$, will play an essential role. If $M$ is fixed and either $v_n=o(V_n)$ or $v_n/|\gamma_n|^2=o(P_n)$ when $n\to\infty$, then these sets are essentially disks centered at $\gamma_n$ with radii that are $o(|\gamma_n|)$ when $n\to\infty$. In such situations, the splitting of a sequence $\Lambda$ into the three sequences $\Lambda^{(0)}$, $\Lambda^{(V)}$, $\Lambda^{(P)}$ may be nontrivial, in the sense that

$$\Lambda\setminus\bigcup_n D_n(v;M)$$

may be an infinite sequence for every positive $M$.

We will assume that $\Lambda=(\lambda_n)$ is a sequence disjoint from $\Gamma$, indexed by a sequence of integers $(n_0,n_0+1,n_0+2,\ldots)$ and ordered such that the moduli $|\lambda_n|$ increase with $n$. For convenience, we assume that $\lambda_{n_0}\neq 0$. The choice of $n_0$ is made such that $\Lambda$ is "aligned" with $\Gamma$. More precisely, we will say that $\Lambda$ is a $v$-perturbation of $\Gamma$ if $n_0$ can be chosen such that, for a sufficiently large $M$, $\lambda_n$ is in $D_n(v;M)$ for all but possibly a finite number of indices $n$. If $\Lambda$ is a $v$-perturbation of $\Gamma$, it will be implicitly understood that $n_0$ is chosen so that the two sequences are "aligned" in this way.

A $v$-perturbation $\Lambda$ of $\Gamma$ will be said to be, respectively

- an exact $v$-perturbation of $\Gamma$ if $n_0 = 1$;
- a $v$-perturbation of $\Gamma$ of deficiency $n_0 - 1$ if $n_0 > 1$;
- a $v$-perturbation of $\Gamma$ of excess $1 - n_0$ if $n_0 < 1$.

The main results of this subsection are the following two local theorems.

**Theorem 2.4.2.** *Suppose $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$ and that $v_n = o(V_n)$ when $n \to \infty$. If, in addition, the transformation $H_{(\Gamma,v);(\Lambda,w)}$ is invertible, then $\Lambda$ is either an exact $v$-perturbation of $\Gamma$ or a $v$-perturbation of deficiency* $1$.

**Theorem 2.4.3.** *Suppose $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$ and that $v_n/|\gamma_n|^2 = o(P_n)$ when $n \to \infty$. If, in addition, the transformation $H_{(\Gamma,v);(\Lambda,w)}$ is invertible, then $\Lambda$ is either an exact $v$-perturbation of $\Gamma$ or a $v$-perturbation of $\Gamma$ of excess* $1$.

Note the contrast between these results and Theorem 2.2.7; $\Lambda$ has no nontrivial $V$-lacunary or $P$-lacunary subsequences when $H_{(\Gamma,v);(\Lambda,w)}$ is an invertible transformation. We will see in the next subsection that, quite remarkably, all the three cases—exactness, deficiency 1, and excess 1—may occur.

### Proof of Theorem 2.4.2 and Theorem 2.4.3

The proof of the two theorems require several steps. In order to structure the proof, we formulate each of the main steps as separate lemmas. Each lemma is in fact of independent interest. We begin with a simple estimate, to be used repeatedly in what follows. It concerns the quantity

$$\rho_n = \prod_{m=\max(1,n_0)}^{n} \frac{|\gamma_m|^2}{|\lambda_m|^2},$$

which will appear prominently in our conditions for invertibility. We use again the notation introduced in (2.3.1), i.e., we set

$$W_n = \sum_{m=n_0}^{n-1} w_m \quad \text{and} \quad Q_n = \sum_{m=n+1}^{\infty} \frac{w_m}{|\lambda_m|^2}.$$

**Lemma 2.4.4.** *If $\Lambda$ is a v-perturbation of $\Gamma$ and $|\gamma_n| \simeq |\lambda_n|$, then we have both*

$$\left|\log \frac{\rho_m}{\rho_n}\right|^2 \lesssim (V_{m+1} - V_{n+1})(Q_n - Q_m) \tag{2.4.6}$$

*and*

$$\left|\log \frac{\rho_m}{\rho_n}\right|^2 \lesssim (W_{m+1} - W_{n+1})(P_n - P_m) \tag{2.4.7}$$

*when $m > n$. If, in addition, either $v_n = o(V_n)$ or $v_n/|\gamma_n|^2 = o(P_n)$ when $n \to \infty$, then $\log \rho_n = o(n)$ when $n \to \infty$.*

*Proof.* Since $|\gamma_n| \simeq |\lambda_n|$, we have

$$\begin{aligned} \left|\log \frac{\rho_m}{\rho_n}\right| = 2\left|\sum_{l=n+1}^{m} \log \frac{|\gamma_l|}{|\lambda_l|}\right| \quad &\leq \quad 2\sum_{l=n+1}^{m} \log\left(1 + \left|1 - \frac{|\gamma_l|}{|\lambda_l|}\right|\right) \\ &\lesssim \sum_{l=n+1}^{m} \left|1 - \frac{|\gamma_l|}{|\lambda_l|}\right|. \end{aligned} \tag{2.4.8}$$

Hence, by the Cauchy–Schwarz inequality, we get

$$\left|\log \frac{\rho_m}{\rho_n}\right|^2 \lesssim \sum_{l=n+1}^{m} v_l \sum_{j=n+1}^{m} \frac{|\gamma_j - \lambda_j|^2}{v_j |\lambda_j|^2},$$

which is the desired estimate (2.4.6) since $w_j \simeq |\lambda_j - \gamma_j|^2 / v_j$. Another application of the Cauchy–Schwarz inequality to (2.4.8) gives

$$\left|\log \frac{\rho_m}{\rho_n}\right|^2 \lesssim \sum_{l=n+1}^{m} \frac{|\gamma_l - \lambda_l|^2}{v_l} \sum_{j=n+1}^{m} \frac{v_j}{|\gamma_j|^2},$$

which is the second estimate (2.4.7).

Finally, starting again from (2.4.8) and using the Cauchy–Schwarz inequality a third time, we get

$$|\log \rho_n|^2 \lesssim n \sum_{l=\max(1,n_0)}^{n} \frac{|\gamma_l-\lambda_l|^2}{|\lambda_l|^2} \lesssim n \sum_{l=\max(1,n_0)}^{n} \min\left(\frac{v_l}{V_l}, \frac{v_l}{|\gamma_l|^2 P_l}\right),$$

where in the last step we used that $\Lambda$ is a $v$-perturbation of $\Gamma$. This relation gives the last statement in the lemma, namely that $\log \rho_n = o(n)$ when either $v_n = o(V_n)$ or $v_n/|\gamma_n|^2 = o(P_n)$ as $n \to \infty$. □

We next prove the following lemma, which is really a corollary to Theorem 2.2.1. It also shows why the discs $D_n(v;M)$ appear naturally in our study of invertible discrete Hilbert transforms.

**Lemma 2.4.5.** *Suppose that either $v_n = o(V_n)$ or $v_n/|\gamma_n|^2 = o(P_n)$ when $n \to \infty$. If, in addition, $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma) = 0$ and the map $H_{(\Gamma,v)}$ is both bounded and bounded below from $\ell^2_v$ to $L^2(\mathbb{C},\mu)$, then there exist positive numbers $M$ and $\delta$ such that*

$$\int_{D_n(v;M)} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \geq \delta$$

*for all but finitely many indices $n$.*

*Proof.* Applying the assumption about boundedness below to any sequence with only one nonzero entry, we find that there is a positive number $\sigma$ independent of $n$ such that

$$\int_{\mathbb{C}} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \geq \sigma$$

for every $n$. On the other hand, since $|\gamma_n|$ grows at least exponentially and $H_{(\Gamma,v)}$ is bounded from $\ell^2_v$ to $L^2(\mathbb{C},\mu)$, we have

$$\sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \lesssim v_n \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} \lesssim \min\left(\frac{v_n}{V_n}, \frac{v_n}{|\gamma_n|^2 P_n}\right)$$

and

$$\sum_{m=1}^{n-1} \int_{\Omega_m} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \lesssim \frac{v_n}{|\gamma_n|^2} \sum_{m=1}^{n-1} \int_{\Omega_m} d\mu(z) \lesssim \min\left(\frac{v_n}{V_n}, \frac{v_n}{|\gamma_n|^2 P_n}\right),$$

which by assumption tend to 0 when $n \to \infty$. We also have

$$\int_{\Omega_n \setminus D_n(v;M)} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \leq \frac{1}{M} \int_{\Omega_n} \max\left(\frac{V_n}{|\lambda|^2}, P_n\right) d\mu(z) \lesssim \frac{1}{M},$$

again using the condition for boundedness of the map $H_{(\Gamma,v)} : \ell^2_v \to L^2(\mathbb{C},\mu)$. The result follows with $\delta = \sigma/2$ if we choose a sufficiently large $M$. $\square$

The preceding lemma shows that if the transformation $H_{(\Gamma,v);(\Lambda,w)}$ is invertible, then $\Lambda$ must contain a subsequence that is a $v$-perturbation of $\Gamma$. The next two lemmas show that $\Lambda$ itself must be a $v$-perturbation of $\Gamma$.

**Lemma 2.4.6.** *Suppose that $v_n = o(V_n)$ when $n \to \infty$. If, in addition, $\Lambda$ is an exact $v$-perturbation of $\Gamma$, then $\Lambda$ is a uniqueness sequence for $H_{(\Gamma,v)}$.*

*Proof.* We argue by contradiction. So suppose there is a nonzero vector $a = (a_n)$ in $\ell^2_v$ such that $H_{(\Gamma,v)}a$ vanishes on $\Lambda$. This means that there is a nonzero entire function $J(z)$ such that

$$\sum_{n=1}^{\infty} \frac{a_n v_n}{z-\gamma_n} = J(z) \prod_{m=1}^{\infty} \frac{1-z/\lambda_m}{1-z/\gamma_m}$$

for every $z$ in $\mathbb{C} \setminus \Gamma$. Applying Cauchy–Schwarz on the left hand-side we obtain

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{a_n v_n}{z-\gamma_n} \right|^2 \quad &\leq \quad \|(a_m)\|_v^2 \left( \frac{V_n}{|z|^2} + \frac{v_n}{|z-\gamma_n|^2} + \frac{v_{n+1}}{|\gamma_{n+1}|^2} \right) \\ &\simeq \|(a_m)\|_v^2 \left( \frac{V_n}{|z|^2} + \frac{v_n}{|z-\gamma_n|^2} \right). \end{aligned} \tag{2.4.9}$$

Note that in the estimation we used the fact that $v_n$ grows at most sub-exponentially because of the assumption and hence $v_m/|\gamma_m|^2$ decays exponentially. If we now choose $M$ sufficiently large, then we have

$$\frac{V_n}{|z|^2} \gtrsim |J(z)|^2 \rho_n$$

for $z$ in $\Omega_n \setminus D_n(v;M)$. Since $v_n = o(V_n)$ when $n \to \infty$, the left-hand side is bounded by $e^{-\delta n}$ for some positive $\delta$, while, by Lemma 2.4.4, $\rho_n = e^{o(n)}$

when $n\to\infty$. Thus the maximum of $|J(z)|$ in $\Omega_n\setminus D_n(v;M)$ tends to 0 when $n\to\infty$, which is a contradiction unless $J(z)\equiv 0$. □

**Lemma 2.4.7.** *Suppose that $v_n/|\gamma_n|^2 = o(P_n)$ when $n\to\infty$. If, in addition, $\Lambda$ is a $v$-perturbation of $\Gamma$ of excess $1$, then $\Lambda$ is a uniqueness sequence for $H_{(\Gamma,v)}$.*

*Proof.* We argue again by contradiction and assume that there is a nonzero vector $a=(a_n)$ in $\ell^2_v$ such that $H_{(\Gamma,v)}a$ vanishes on $\Lambda$. In this case, it follows that there is a nonzero entire function $J(z)$ such that

$$\sum_{n=1}^{\infty}\frac{a_nv_n}{z-\gamma_n} = J(z)(z-\lambda_0)\prod_{m=1}^{\infty}\frac{1-z/\lambda_m}{1-z/\gamma_m}$$

for every $z$ in $\mathbb{C}\setminus\Gamma$. As in the preceding proof, after applying Cauchy–Schwarz on the left-hand side, if we choose $M$ sufficiently large, we then have

$$P_n \gtrsim |J(z)|^2|z|^2\rho_n$$

for $z$ in $\Omega_n\setminus D_n(v;M)$. Since $v_n/|\gamma_n|^2 = o(P_n)$ when $n\to\infty$, we have that $P_n/|z|^2$ is bounded by $e^{-\delta n}$ for some positive number $\delta$, while, by Lemma 2.4.4, $\rho_n = e^{o(n)}$ when $n\to\infty$. Thus the maximum of $|J(z)|$ in $\Omega_n\setminus D_n(v;M)$ tends to 0 when $n\to\infty$, which is a contradiction unless $J(z)\equiv 0$. □

We finally prove two lemmas that, together with the previous three lemmas, give the precise restrictions stated in Theorem 2.4.2 and Theorem 2.4.3 and complete their proofs.

**Lemma 2.4.8.** *Suppose that $v_n = o(V_n)$ when $n\to\infty$. If, in addition, $\Lambda$ is a $v$-perturbation of $\Gamma$ of deficiency $2$, then $\Lambda$ is not a uniqueness sequence for $H_{(\Gamma,v)}$.*

*Proof.* We may write

$$\frac{c}{(z-\gamma_1)(z-\gamma_2)}\prod_{m=3}^{\infty}\frac{1-z/\lambda_m}{1-z/\gamma_n} = \sum_{n=1}^{\infty}\frac{a_nv_n}{z-\gamma_n}+h(z),$$

where $h$ is an entire function and

$$|a_n|^2 v_n^2 \simeq \frac{|\gamma_n - \lambda_n|^2}{|\gamma_n|^4} \rho_n.$$

Since $\Lambda$ is a $v$-perturbation, we therefore get

$$\sum_{n=1}^{\infty} |a_n|^2 v_n \lesssim \sum_{n=1}^{\infty} \frac{\rho_n}{|\gamma_n|^2 V_n} < \infty,$$

where in the final step we used that the ratio $\rho_n/V_n$ grows at most sub-exponentially. We then get

$$|h(z)|^2 \lesssim \frac{\rho_n}{|z|^4} + \frac{V_n}{|z|^2}$$

when $z$ is in $D_n(v;M)$ with $M$ sufficiently large. Using again that both $\rho_n$ and $V_n$ grow at most sub-exponentially, we have that $h(z) \to 0$ when $z \to \infty$, which means that $h \equiv 0$. $\square$

**Lemma 2.4.9.** *Suppose that $v_n/|\gamma_n|^2 = o(P_n)$ when $n \to \infty$. If, in addition, $\Lambda$ is a $v$-perturbation of $\Gamma$ of deficiency 1, then $\Lambda$ is not a uniqueness sequence for $H_{(\Gamma,v)}$.*

*Proof.* In this case, we may write

$$\frac{c}{z-\gamma_1} \prod_{m=2}^{\infty} \frac{1 - z/\lambda_m}{1 - z/\gamma_m} = \sum_{n=1}^{\infty} \frac{a_n v_n}{z - \gamma_n} + h(z),$$

where $h$ is an entire function and

$$|a_n|^2 v_n^2 \simeq \frac{|\gamma_n - \lambda_n|^2}{|\gamma_n|^2} \rho_n.$$

Since $\Lambda$ is a $v$-perturbation, we get

$$\sum_{n=1}^{\infty} |a_n|^2 v_n \lesssim \sum_{n=1}^{\infty} \frac{\rho_n}{|\gamma_n|^2 P_n} < \infty,$$

where we now used that the ratio $\rho_n/P_n$ grows at most sub-exponentially. It follows that

$$|h(z)|^2 \lesssim \frac{\rho_n}{|z|^2} + P_n$$

when $z$ is in $D_n(v;M)$ with $M$ sufficiently large. We conclude that $h(z) \to 0$

when $z \to \infty$, which means that $h \equiv 0$. □

### 2.4.3 Geometric criteria for invertibility of $H_{(\Gamma,v);(\Lambda,w)}$

After the preliminary results of the previous subsection, we may now state our geometric conditions for invertibility of $H_{(\Gamma,v);(\Lambda,w)}$ when $\Gamma$ is a sparse sequence. We begin with the case when $v_n = o(V_n)$ as $n \to \infty$.

**Theorem 2.4.10.** *Suppose $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$ and that $V_n \to \infty$ and $v_n = o(V_n)$ when $n \to \infty$. Then the transformation $H_{(\Gamma,v);(\Lambda,w)}$ is invertible if and only if*

$$\sup_{n\geq 1} V_n Q_n < \infty \tag{2.4.10}$$

*and one of the following two conditions holds:*

*(1) $\Lambda$ is an exact $v$-perturbation of $\Gamma$ and there are positive constants $C$ and $\delta$ such that*

$$\frac{\rho_m}{\rho_n} \leq C \left(\frac{V_m}{V_n}\right)^{1-\delta} \tag{2.4.11}$$

*whenever $m > n$.*

*(2) $\Lambda$ is a $v$-perturbation of $\Gamma$ of deficiency 1 and there are positive constants $C$ and $\delta$ such that*

$$\frac{\rho_m}{\rho_n} \geq C \left(\frac{V_m}{V_n}\right)^{1+\delta} \tag{2.4.12}$$

*whenever $m > n$.*

It is quite remarkable that the essential quantitative conditions for invertibility, found in (1) and (2), only depend on the moduli of the complex numbers $\gamma_n/\lambda_n$, and beautifully interconnected with the weight sequence $(v_n)$. As will be explained in the next chapter, the result gives a geometric characterization of Riesz bases of normalized reproducing kernels in some spaces of meromorphic functions of which the de Branges spaces are leading examples.

We note that in the case when

$$\sum_{n=1}^{\infty} v_n < \infty,$$

the result is much simpler and less delicate. Then, as can be seen from the proof of part (1) of Theorem 2.4.10, the following consequence holds.

**Corollary 2.4.11.** *Suppose $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$ and the sequence $(v_n)$ is summable. Then the transformation $H_{(\Gamma,v);(\Lambda;w)}$ is invertible if and only if $\Lambda$ is an exact $v$-perturbation of $\Gamma$ and*

$$\sup_{n\geq 1} Q_n < \infty.$$

In the case when $v_n/|\gamma_n|^2 = o(P_n)$, we have the following counterpart to Theorem 2.4.10.

**Theorem 2.4.12.** *Suppose $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$ and that $v_n/|\gamma_n|^2 = o(P_n)$ when $n \to \infty$. Then the transformation $H_{(\Gamma,v);(\Lambda,w)}$ is invertible if and only if*

$$\sup_{n\geq 1} W_n P_n < \infty$$

*and one of the following two conditions holds:*

*(1) $\Lambda$ is an exact $v$-perturbation of $\Gamma$ and there are positive constants $C$ and $\delta$ such that*

$$\frac{\rho_m}{\rho_n} \geq C \left(\frac{P_m}{P_n}\right)^{1-\delta} \tag{2.4.13}$$

*whenever $m > n$.*

*(2) $\Lambda$ is a $v$-perturbation of $\Gamma$ of excess $1$ and there are positive constants $C$ and $\delta$ such that*

$$\frac{\rho_m}{\rho_n} \leq C \left(\frac{P_m}{P_n}\right)^{1+\delta} \tag{2.4.14}$$

*whenever $m > n$.*

There is a slight lack of symmetry between the two theorems; while it may happen that $\sup_n V_n < \infty$, we will always have that $P_n \to 0$. Therefore, no precaution is needed concerning the decay of $P_n$.

### 2.4.4 Kadets'-$1/4$ type stability results

We will now show how the above two results can be used to obtain results similar to Kadets'-$1/4$ theorem for complex exponentials [54]. We note that the $L^p$ version of the Kadets'-$1/4$ theorem can be found in [61] where a complete description of the complete interpolating sequences for the Paley–Wiener space is obtained.

**Corollary 2.4.13.** *Suppose that both $v_n = o(V_n)$ and $V_n \to \infty$ when $n \to \infty$, and write $\Gamma = (\gamma_n)$ and $\Lambda = (\lambda_n)$, with both sequences indexed by the positive integers. Moreover, assume that there exists a positive constant $C$ such that*

$$\frac{|\gamma_n - \lambda_n|}{|\gamma_n|} \leq C \frac{v_n}{V_n} \tag{2.4.15}$$

*for every positive integer $n$.*

*(1) If, in addition, there is a real constant $c < 1/2$ such that*

$$\frac{|\gamma_n|}{|\lambda_n|} - 1 \leq c \frac{v_n}{V_n}$$

*for all sufficiently large $n$, then $H_{(\Gamma,v);(\Lambda,w)}$ is an invertible transformation.*

*(2) If, on the other hand, there is a positive constant $c > 1/2$ such that*

$$\frac{|\gamma_n|}{|\lambda_n|} - 1 \geq c \frac{v_n}{V_n}$$

*for all sufficiently large $n$, then $H_{(\Gamma,v);(\Lambda^{(1)},w^{(1)})}$ is an invertible transformation, where $\Lambda^{(1)} = (\lambda_2, \lambda_3, \ldots)$ and $w^{(1)} = (w_2, w_3, \ldots)$.*

It follows from (2.4.15) that $H_{(\Gamma,v);(\Lambda,w)}$ is a bounded transformation, while the respective conditions in (1) and (2) imply that the inverse transformations are bounded, subject to the proviso that, when (2) holds, one point be removed from $\Lambda$. This rather puzzling result can be seen as an analogue of the Kadets'-$1/4$ theorem for complex exponentials. We note that if we have the precise relation

$$\frac{|\gamma_n|}{|\lambda_n|} - 1 = \frac{1}{2} \frac{v_n}{V_n}, \tag{2.4.16}$$

then neither $H_{(\Gamma,v);(\Lambda,w)}$ nor $H_{(\Gamma,v);(\Lambda^{(1)},w^{(1)})}$ is an invertible transformation (see the next example below). Another curious point is that if we replace the condition that $V_n \to \infty$ by the assumption that $\sup_n V_n < \infty$, then (2.4.15) automatically implies that $H_{(\Gamma,v);(\Lambda,w)}$ is an invertible transformation.

To arrive at the results stated in the corollary, we note that if

$$\frac{|\gamma_n - \lambda_n|}{|\lambda_n|} \lesssim \frac{v_n}{V_n},$$

then

$$Q_n = \sum_{m=n+1}^{\infty} \frac{w_m}{|\lambda_m|^2} \lesssim \sum_{m=n+1}^{\infty} \frac{v_m}{V_{m+1}^2} \leq \frac{1}{V_{n+1}},$$

where in the last step we compared the sum with the integral of $1/x^2$ from $V_{n+1}$ to $\infty$. We also have, assuming $|\gamma_n|/|\lambda_n| - 1 \leq c v_n/V_n$, that

$$\log\frac{\rho_m}{\rho_n} \leq 2c\left(1+o(1)\right) \sum_{j=n+1}^{m} \frac{v_l}{V_l} = 2c\big(1+o(1)\big)\log\frac{V_m}{V_n} \tag{2.4.17}$$

when $m > n$ and $n \to \infty$. In view of Theorem 2.4.10, this gives part (1) of the corollary; part (2) follows by the same argument, with the inequality in (2.4.17) reversed.

We now construct an example to show that when (2.4.4) holds the invertibility of the operators in the corollary may fail.

**Example 4.** For each $n$, we set $\gamma_n = 2^n$ and $v_n = 1$. Then if we consider a sequence of real points $\Lambda = (\lambda_n)$ where

$$\lambda_n = \frac{(n-1)2^{n+1}}{2n-1},$$

then $\Lambda$ satisfies (2.4.15). But from a simple computation, it follows that neither (2.4.11) nor (2.4.12) holds.

We have the following statement, in complete analogy with Corollary 2.4.13 and with the same proof:

**Corollary 2.4.14.** *Suppose that $v_n/|\gamma_n|^2 = o(P_n)$ when $n \to \infty$ and that*

$$\sup_{n\geq 1} W_n P_n < \infty.$$

*(1) If, in addition, $\Lambda$ is an exact $v$-perturbation of $\Gamma$ and there is a real constant $c < 1/2$ such that*

$$\frac{|\lambda_n|}{|\gamma_n|} - 1 \leq c \frac{v_n}{|\gamma_n|^2 P_n}$$

*for all sufficiently large $n$, then $H_{(\Gamma,v);(\Lambda,w)}$ is an invertible transformation.*

*(2) If, on the other hand, $\Lambda$ is a $v$-perturbation of $\Gamma$ of excess 1 and there is a positive constant $c > 1/2$ such that*

$$\frac{|\lambda_n|}{|\gamma_n|} - 1 \geq c \frac{v_n}{|\gamma_n|^2 P_n}$$

*for all sufficiently large $n$, then $H_{(\Gamma,v);(\Lambda,w)}$ is an invertible transformation.*

In the next two subsections, we will present the proof of Theorem 2.4.10; the proof of Theorem 2.4.12 is completely analogous and will therefore be omitted.

### 2.4.5 Proof of Theorem 2.4.10

In addition to the results of Subsection 2.4.2, we will need the following simple facts.

**Lemma 2.4.15.** *Let $c = (c_n)$ be a sequence of positive numbers.*

*(i) If there is a constant $C$ such that*

$$\sum_{m=1}^{n-1} c_m \leq C c_n$$

*for $n > 1$, then there is a positive constant $\delta$ such that $c_m/c_n \geq C 2^{\delta(m-n)}$ whenever $m > n$.*

*(ii) If there is a constant $C$ such that*

$$\sum_{m=n+1}^{\infty} c_m \leq C c_n$$

*for every positive integer $n$, then there is a positive constant $\delta$ such that $c_m/c_n \leq C 2^{-\delta(m-n)}$ whenever $m > n$.*

*Proof.* We consider (i). The assumption implies that

$$Nc_{n-1} \leq N \sum_{m=1}^{n-1} c_m \leq C \sum_{m=n}^{n+N-1} c_m \leq C^2 c_{n+N}.$$

which means that if we choose $N > 2C^2$, then $c_{n+j(N+1)} \geq 2^j c_n$. The result follows if we choose $\delta = 1/(N+2)$.

To prove (ii), we again apply the assumption which implies

$$Nc_{n-1} \leq N \sum_{m=n-1}^{\infty} c_m \leq C \sum_{m=n-2+N+1}^{n-2+2N} c_m \leq C^2 C_{n-2+N}$$

and the rest can be performed in a similar way as in $(i)$. □

### Proof of the necessity of the conditions in Theorem 2.4.10

We turn to the proof of the necessity of the conditions in Theorem 2.4.10. Thus we begin by assuming that $H_{(\Gamma,v);(\Lambda,w)}$ is an invertible transformation. Since this means that, in particular, $H_{(\Gamma,v);(\Lambda,w)}$ is a bounded transformation, we must have

$$\sup_{n\geq 1} V_n Q_n < \infty.$$

Also, in view of Theorem 2.4.2, we already know that $\Lambda$ is either an exact $v$-perturbation of $\Gamma$ or a $v$-perturbation of $\Gamma$ of deficiency 1. Thus it remains only to establish the necessity of the conditions in parts (1) and (2), under the respective assumptions of exactness and deficiency 1.

We treat the two cases separately:

#### (1) $\Lambda$ *is an exact* $v$*-perturbation of* $\Gamma$.

Since $v_n = o(V_n)$, the weight sequence $w = (w_n)$ defined by (2.2.20) satisfies

$$w_n \simeq \frac{|\gamma_n - \lambda_n|^2}{v_n}. \tag{2.4.18}$$

As a consequence, we now obtain simple estimates for the weight sequences $\nu = (\nu_n)$ and $\varpi = (\varpi_j)$ appearing in Theorem 2.4.1.

We begin by noting that if $\Lambda$ is a $v$-perturbation of $\Gamma$ and an exact uniqueness sequence for $H_{(\Gamma,v)}$, then there is a constant $c$ such that

$$\sum_{n=1}^{\infty} \frac{e_n v_n}{z-\gamma_n} = \frac{c}{z-\gamma_1} \prod_{m=2}^{\infty} \frac{1-z/\lambda_m}{1-z/\gamma_n} \tag{2.4.19}$$

for every $z$ in $\mathbb{C} \setminus \Gamma$, where again $e=(e_n)$ is the vector such that $H_{(\Gamma,v);(\Lambda,w)}e = (1,0,0,\ldots)$. Indeed, the expression on the left-hand side can have zeros only at the points $\lambda_m$ for $m>1$, since $\Lambda$ is assumed to be an exact uniqueness sequence for $H_{(\Gamma,v)}$. From (2.4.19) we obtain

$$|e_n|^2 v_n^2 \simeq \frac{|\lambda_n-\gamma_n|^2}{|\lambda_n|^2}\rho_n,$$

and, therefore, using (2.4.1) and (2.4.18), we obtain

$$v_n \simeq w_n \rho_n. \tag{2.4.20}$$

On the other hand, differentiating (2.4.19) at $z=\lambda_n$, we get

$$\left| \sum_{l=1}^{\infty} \frac{e_l v_l}{(\lambda_n-\gamma_l)^2} \right| \simeq \frac{|\gamma_n|}{|\lambda_n|^2|\lambda_n-\gamma_n|} \prod_{m=1}^{n-1} \frac{|\gamma_m|}{|\lambda_m|}.$$

Thus using (2.4.2) and again (2.4.18), we obtain

$$\varpi_n \simeq v_n \rho_n^{-1}. \tag{2.4.21}$$

To simplify the writing, we set

$$V_n^{(\rho,0)} = \sum_{m=1}^{n-1} v_m \rho_m^{-1} \quad \text{and} \quad P_n^{(\rho,0)} = \sum_{m=n+1}^{\infty} v_m |\lambda_m|^{-2} \rho_m{-1}$$

as well as

$$W_n^{(\rho,0)} = \sum_{m=1}^{n-1} w_n \rho_n \quad \text{and} \quad Q_n^{(\rho,0)} = \sum_{m=n+1}^{\infty} w_n \rho_n |\lambda_n|^{-2}.$$

By Theorem 2.4.1 and Theorem 2.2.1, we must have

$$\sup_{n\geq 1} V_n^{(\rho,0)} Q_n^{(\rho,0)} < \infty;$$

we will now show that the estimate in part (1) is a consequence of this condition.

We set $n_1=2$ and define $n_j$ inductively by requiring $V_{n_{j+1}-1}/V_{n_j} < 2 \leq V_{n_{j+1}}/V_{n_j}$. By (2.4.6) of Lemma 2.4.4 and the uniform boundedness of $V_n Q_n$, it follows that there are constants $c$ and $C$ such that $c < \rho_n/\rho_m \leq C$ when $n$ and $m$ both lie in the interval $[n_j, n_{j+1}]$. Hence we have

$$V_{n_j}^{(\rho,0)} \simeq \sum_{l=1}^{j} V_{n_l}\rho_{n_l}^{-1}. \tag{2.4.22}$$

Now if

$$Q_{n_j} - Q_{n_{j+1}} \geq \frac{\varepsilon}{V_{n_{j+1}}}, \tag{2.4.23}$$

then our condition $\sup_n V_n^{(\rho,0)} Q_n^{(\rho,0)} < \infty$ and (2.4.22) imply that there exists a constant $C$ such that

$$\sum_{l=1}^{j} V_{n_l}\rho_{n_l}^{-1} \leq C V_{n_{j+1}}\rho_{n_{j+1}}^{-1}. \tag{2.4.24}$$

If, on the other hand, we have

$$Q_{n_j} - Q_{n_{j+1}} < \frac{\varepsilon}{V_{n_{j+1}}},$$

then an application of (2.4.2) of Lemma 2.4.4 gives $\rho_{n_{j+1}}/\rho_{n_j} \leq 5/4$ if $\varepsilon$ is sufficiently small. Hence we have

$$\frac{V_{n_{j+1}}\rho_{n_j}}{V_{n_j}\rho_{n_{j+1}}} \geq \frac{8}{5},$$

which means that $V_{n_j}\rho_{n_j}^{-1}$ increases exponentially on any set of consecutive integers $j$ for which (2.4.23) fails. Combining (2.4.24) with the latter estimate, we therefore get that

$$\sum_{l=1}^{j} V_{n_l}\rho_{n_l}^{-1} \leq \left(\frac{5}{8}C + \frac{8}{3}\right) V_{n_{j+1}}\rho_{n_{j+1}}^{-1}$$

when (2.4.23) fails and $\varepsilon$ is sufficiently small. Thus (2.4.24) holds for every index $j$ if the constant $C$ is suitably adjusted. Hence, by part (i) of Lemma 2.4.15, there exists a constant $C$ such that

$$\frac{\rho_{n_{j+l}}}{\rho_{n_j}} \leq C\frac{V_{n_{j+l}}}{V_{n_j}} 2^{-\delta l} \leq C\left(\frac{V_{n_{j+l}}}{V_{n_j}}\right)^{1-\delta/2},$$

where in the last step we used that $V_{n_{j+1}}/V_{n_j} \leq 4$ for sufficiently large $j$. We are done since it suffices to establish (2.4.11) for $n = n_j$ and $m = n_{j+l}$.

**(2) $\Lambda$ *is a* $v$*-perturbation of* $\Gamma$ *of deficiency* $1$**

As in the previous case, we begin by finding estimates for the weight sequences $v = (v_n)$ and $\varpi = (\varpi_j)$ appearing in Theorem 2.4.1. If $\Lambda$ is a $v$-perturbation of $\Gamma$ of deficiency $1$ and an exact uniqueness sequence for $H_{(\Gamma,v)}$, then there is a constant $c$ such that

$$\sum_{n=1}^{\infty} \frac{e_n v_n}{z-\gamma_n} = \frac{c}{(z-\gamma_1)(z-\gamma_2)} \prod_{m=3}^{\infty} \frac{1-z/\lambda_m}{1-z/\gamma_n}$$

for every $z$ in $\mathbb{C}\setminus\Gamma$, where again $e=(e_n)$ is the vector such that $H_{(\Gamma,v);(\Lambda,w)}e = (1,0,0,\ldots)$. Arguing in the same way as in the preceding case, we obtain from this relation the estimates

$$v_n \simeq w_n \rho_n |\gamma_n|^{-2} \tag{2.4.25}$$

and

$$\varpi_n \simeq v_n \rho_n^{-1} |\gamma_n|^2. \tag{2.4.26}$$

We now set

$$V_n^{(\rho,1)} = \sum_{m=1}^{n-1} v_n |\gamma_n|^2 \rho_n^{-1} \quad \text{and} \quad P_n^{(\rho,1)} = \sum_{m=n+1}^{\infty} v_n \rho_n$$

as well as

$$W_n^{(\rho,1)} = \sum_{m=1}^{n-1} w_n |\gamma_n|^{-2} \rho_n \quad \text{and} \quad Q_n^{(\rho,1)} = \sum_{m=n+1}^{\infty} w_n |\gamma_n|^{-4} \rho_n.$$

By Theorem 2.4.1 and Theorem 2.2.1, we must have

$$\sup_{n\geq 1} W_n^{(\rho,1)} P_n^{(\rho,1)} < \infty$$

; we will now show that also the estimate in part (2) is a consequence of this condition.

We let the sequence $(n_j)_j$ be as above and find that

$$P_{n_j}^{(\rho,1)} \simeq \sum_{l=j+1}^{\infty} V_{n_l} \rho_{n_l}^{-1} \tag{2.4.27}$$

whenever $j \geq 1$. Now if

$$Q_{n_{j+1}} - Q_{n_j} \geq \frac{\varepsilon}{V_{n_{j+1}}}, \tag{2.4.28}$$

then it follows from the condition $\sup_n W_n^{(\rho,1)} P_n^{(\rho,1)} < \infty$ and (2.4.27) that

$$\sum_{l=j+1}^{\infty} V_{n_l} \rho_{n_l}^{-1} \lesssim V_{n_j} \rho_{n_j}^{-1}. \tag{2.4.29}$$

As in the preceding case, we find that, if $\varepsilon$ is sufficiently small, then $V_{n_j}\rho_{n_j}^{-1}$ increases exponentially on any set of consecutive integers $j$ for which (2.4.28) fails. The relation (2.4.27) implies that no such set is infinite; thus there is an infinite sequence of indices $n_j$ for which (2.4.29) holds, and there must in fact be a uniform bound on the number of points found in any set of consecutive integers $j$ for which (2.4.28) fails. We may infer from this argument that in fact (2.4.29) holds for every index $n_j \geq 1$. Finally, we invoke part (ii) of Lemma 2.4.15, which implies that there is a constant $C$ such that

$$\frac{\rho_{n_{j+l}}}{\rho_{n_j}} \geq C \frac{V_{n_{j+l}}}{V_{n_j}} 2^{\delta l} \geq C \left( \frac{V_{n_{j+l}}}{V_{n_j}} \right)^{1+\delta},$$

and we are done since it suffices to establish (2.4.12) for $n = n_j$ and $m = n_{j+l}$.

### Proof of the sufficiency of the conditions in Theorem 2.4.10

We begin by noting that the condition

$$\sup_{n \geq 1} V_n Q_n < \infty$$

implies that $H_{(\Gamma,v);(\Lambda,w)}$ is a bounded transformation. Indeed, (2.2.3) in Theorem 2.2.1 holds trivially when

$$\mu = \sum_{n=1}^{\infty} w_n \delta_{\lambda_n}.$$

We also have

$$W_n \lesssim \frac{|\gamma_n|^2}{V_n} \quad \text{and} \quad P_n \lesssim \frac{v_n}{|\gamma_n|^2}$$

by the assumptions that $v_n = o(V_n)$ and $\sup_n V_n Q_n < \infty$. Therefore, Theorem 2.2.1 allows us to conclude that $H_{(\Gamma,v);(\Lambda,w)}$ is a bounded transformation.

We will now use Theorem 2.4.1 and show that the respective conditions in part

(1) and part (2) in Theorem 2.4.10 imply those in Theorem 2.4.1. The sequence $(n_j)_j$ will be the same as in the previous subsection.

### (1) $\Lambda$ *is an exact* $v$*-perturbation of* $\Gamma$

We already know from Lemma 2.4.6 that if $\Lambda$ is an exact $v$-perturbation of $\Gamma$, then $\Lambda$ is a uniqueness sequence for $H_{(\Gamma,v)}$. To check that $\Lambda$ is in fact an exact uniqueness sequence for $H_{(\Gamma,v)}$, we note that we may write

$$\frac{c}{z-\gamma_1}\prod_{m=2}^{\infty}\frac{1-z/\lambda_m}{1-z/\gamma_n}=\sum_{n=1}^{\infty}\frac{a_n v_n}{z-\gamma_n}+h(z),$$

where $h$ is an entire function and

$$|a_n|^2 v_n \simeq \frac{w_n}{|\gamma_n|^2}\rho_n.$$

By the assumption that $\sup_n V_n Q_n < \infty$, we have

$$\sum_{n=1}^{\infty}|a_n|^2 v_n \lesssim \sum_{j=1}^{\infty}\frac{\rho_{n_j}}{V_{n_j}},$$

which, in view of (2.4.11), implies that $(a_n)$ is in $\ell^2_v$. In particular, we then have

$$|h(z)|^2 \lesssim \frac{\rho_n}{|z|^2}+\frac{V_n}{|z|^2}$$

when $z$ is in $D_n(v;M)$ with $M$ sufficiently large. Thus $h(z)\to 0$ when $z\to\infty$ which means that $h\equiv 0$.

It remains only to verify that $H_{(\Lambda,\varpi);(\Gamma,v)}$ is a bounded transformation. By Theorem 2.2.1, we need to show that we have both

$$\sup_{n\geq 1} W_n^{(\rho,0)}P_n^{(\rho,0)}<\infty \quad\text{and}\quad \sup_{n\geq 1} V_n^{(\rho,0)}Q_n^{(\rho,0)}<\infty.$$

To this end, we note that since $\rho_n$ can only grow sub-exponentially, we have

$$\sup_{n\geq 1} W_n^{(\rho,0)}P_n^{(\rho,0)}<\infty$$

by the same argument that gave $\sup_n W_n P_n < \infty$. Since $\sup_n V_n Q_n < \infty$, we have

$$V_n^{(\rho,0)}Q_n^{(\rho,0)} \lesssim \sum_{n_j<n}\frac{V_{n_j}}{\rho_{n_j}}\frac{\rho_n}{V_n};$$

here the right-hand side is uniformly bounded whenever (2.4.11) holds.

#### (2) $\Lambda$ *is a* $v$*-perturbation of* $\Gamma$ *of deficiency* $1$

In view of Lemma 2.4.8, we will have that $\Lambda$ is an exact uniqueness sequence for $H_{(\Gamma,v)}$ if we can show that there is no nonzero $a$ in $\ell^2_v$ such that $H_{(\Gamma,v)}a$ vanishes on $\Lambda$. To show this, we assume to the contrary that such a sequence $a$ exists. Then there is a constant $c$ such that

$$\sum_{n=1}^{\infty} \frac{a_n v_n}{z-\gamma_n} = \frac{c}{z-\gamma_1} \prod_{m=2}^{\infty} \frac{1-z/\lambda_m}{1-z/\gamma_n}. \tag{2.4.30}$$

By estimating each side of (2.4.30) for $z$ in $D_n(v;M)$ with $M$ sufficiently large, we get

$$V_n \sum_{m=1}^{\infty} |a_m|^2 v_m \gtrsim \rho_n.$$

But this is a contradiction, because (2.4.12) implies that $\rho_n/V_n$ is an increasing sequence.

It remains only to verify that $H_{(\Lambda,\varpi);(\Gamma,v)}$ is a bounded transformation. To this end, we note that

$$\sup_{n\geq 1} V_n^{(\rho,1)} Q_n^{(\rho,1)} < \infty$$

holds trivially because $1/\rho_n$ can only grow sub-exponentially, while

$$W_n^{(\rho,1)} P_n^{(\rho,1)} \lesssim \sum_{n_j<n} \frac{P_{n_j}}{\rho_{n_j}} \frac{\rho_n}{P_n},$$

which is uniformly bounded when (2.4.12) holds.

## 2.5 Interplay between the growth of $\Gamma$ and “smoothness” of $v$

As remarked earlier, our methods used in the previous sections allow for a moderate weakening of the growth condition (2.2.1), at least when the sequence $(v_n)$ is sufficiently regular. In this section we will consider an example of such interplay between the growth of the sequence $\Gamma = (\gamma_n)$ and the “smoothness” of $v = (v_n)$. We first note that for any given admissible pair sequence $(\gamma_n, v_n)$, if $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma)=0$ and the operator $H_{(\Gamma,v)}$ is bounded from $\ell^2_v$ to $L^2(\mathbb{C},\mu)$, then the estimate

$$\beta_n^2=\int_{\mathbb{C}}\frac{1}{|z-\gamma_n|^2}d\mu(z)\lesssim\frac{1}{v_n} \tag{2.5.1}$$

holds for each $n$. Thus condition (2.5.1) ( and hence (2.2.3)) remains in general necessary independent of the growth of $(\gamma_n)$. When the sequence $(\beta_n)$ has small growth in the sense that it belongs to $\ell^2_v$, the condition is sufficient as well. The class of transforms for which this holds will be described in Section 3.2.3.

The goal is now to weaken the sparseness condition (2.2.1) on $(\gamma_n)$ and compensate it by instead requiring some sort of regularity from the weight sequence $(v_n)$. In what follows we will replace (2.2.1) by the weaker condition

$$|\gamma_{n+j}-\gamma_n|\geq c\frac{|j||\gamma_n|}{n^\alpha}, \tag{2.5.2}$$

whenever $|j|\leq n^\alpha,\ 0<\alpha<1$ and a positive constant $c$ independent of the positive integer $n$. For convenience, we also set $\gamma_0=0$. We observe that $|\gamma_n|=\exp(n^{1-\alpha})$ is an example of a sequence $\gamma_n$ of "minimal" growth satisfying (2.5.2). Note that if we allowed $\alpha=0$, then we would be back to the previous situation since in this case $|\gamma_n|$ grows at least exponentially. On the other hand, if we allowed $\alpha\geq 1$, then the minimal growth of $|\gamma_n|$ would be; power for $\alpha=1$ and logarithmic for $\alpha>1$, and for such growth our general methods do not apply.
We further assume that the weight sequence $v_n$ satisfies a regularity condition[3]:

$$n^{2\alpha}v_{2n+j}\lesssim V_n, \tag{2.5.3}$$

for $|j|\leq n^\alpha$. We observe that $v_n$ can have at most a power growth when $\alpha\leq\frac{1}{2}$. The growth of $(v_n)$ needs also to be "smooth". For instance if we set $v_n=n$ whenever $n=2^m$ and 1 otherwise, then $v$ does not satisfy condition (2.5.3). To see this, take $j=0$, and observe that

$$n^{2\alpha}v_{2n}=2^{2\alpha m}v_{2^{m+1}}=2^{(2\alpha+1)m+1}\simeq 2^{(2\alpha+1)m}V_{2^m}$$

which fails to satisfy (2.5.3) when $m\to\infty$.

With these a priori assumptions at hand, we ask as before the questions about

[3]Clearly, this condition is not optimal. A thorough investigation is awaiting.

boundedness, invertibility and surjective decomposition of the operators $H_{(\Gamma,v)}$. In this section, we will be only dealing with the boundedness problem. All the other questions about surjective decomposition and invertibility properties of $H_{(\Gamma,v)}$ can be dealt with in a similar manner.

**Theorem 2.5.1.** *Suppose that the sequence $\Gamma$ satisfies the sparseness condition (2.5.2) and $v$ satisfies the regularity condition (2.5.3). If $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma)=0$, then the map $H_{(\Gamma,v)}$ is bounded from $\ell^2_v$ to $L^2(\mathbb{C},\mu)$ if and only if*

$$\sup_{n\geq 1}\int_{\Omega_n}\frac{v_n d\mu(z)}{|z-\gamma_n|^2}<\infty \tag{2.5.4}$$

*and*

$$\sup_{n\geq 1}\left(V_n\sum_{m=n}\int_{\Omega_m}\frac{d\mu(z)}{|z|^2}+P_n\sum_{m=1}^{n}\mu(\Omega_m)\right)<\infty. \tag{2.5.5}$$

The theorem draws a similar conclusion as Theorem 2.2.1 from a weaker hypothesis on the growth of the sequence $(\gamma_n)$, but with an additional restriction on the variation of the sequence $(v_n)$. Obviously, the result falls short of addressing all possible interplays between $\Gamma$ and the weight sequence $w$. A comprehensive smoothness condition for the weight sequences remains yet to be found or assumed.

## Proof of Theorem 2.5.1

To prove the necessity of condition (2.5.5), we argue as in the proof of the necessity of Theorem 2.2.1, i.e. we look at the sequence $c^{(n)}=(c_m^{(n)})$ so that $c_m^{(n)}=1$ for $m<2n$ and $c_m^{(n)}=0$ otherwise. We observe that $\|c^{(n)}\|_v^2=V_{2n}$ and note that for $z$ in $\Omega_l$ and $l\geq 2n$ we have

$$|H_{(\Gamma,v)}c^{(n)}(z)|^2=\left|\sum_{m=1}^{2n-1}\frac{v_m}{z-\gamma_m}\right|^2\gtrsim\frac{V_{2n}^2}{|z|^2}. \tag{2.5.6}$$

Taking into account the boundedness of $H_{(\Gamma,v)}$, we deduce from this that

$$\begin{aligned}V_{2n}\gtrsim\int_{\mathbb{C}}|H_{(\Gamma,v)}c^{(n)}(z)|^2d\mu(z) &= \sum_{k=1}^{\infty}\int_{\Omega_k}\left|\sum_{m=1}^{2n-1}\frac{v_m}{z-\gamma_m}\right|^2d\mu(z)\\ &\gtrsim V_{2n}^2\sum_{m\geq 2n}\int_{\Omega_m}\frac{d\mu(z)}{|z|^2}.\end{aligned}$$

The necessity of the remaining part of (2.5.5) can be handled in the same way.

We now turn to the proof of the sufficiency. We let $(a_n)$ be a sequence in $\ell_v^2$ and make, as before, the following estimate:

$$\begin{aligned}\int_{\Omega_n} |H_{(\Gamma,v)}a(z)|^2 d\mu(z) \;\lesssim\; & \int_{\Omega_n} \left| \sum_{m<n-n^\alpha} \frac{a_m v_m}{z-\gamma_m} \right|^2 d\mu(z) + \int_{\Omega_n} \left| \sum_{|j|\le n^\alpha} \frac{a_{n+j} v_{n+j}}{z-\gamma_{n+j}} \right|^2 d\mu(z) \\ & + \int_{\Omega_n} \left| \sum_{m>n+n^\alpha} \frac{a_m v_m}{z-\gamma_m} \right|^2 d\mu(z), \end{aligned} \tag{2.5.7}$$

which follows from the Cauchy–Schwarz inequality. Using the growth condition (2.5.2), we further split the second integral on the right-hand side of (2.5.7) into

$$\int_{\Omega_n} \frac{|a_n|^2 v_n^2}{|z-\gamma_n|^2} d\mu(z) + \int_{\Omega_n} \frac{n^{2\alpha}}{|z|^2} \left( \sum_{j:j\neq 0,\ |j|\le n^\alpha} \frac{|a_{n+j}| v_{n+j}}{|j|} \right)^2 d\mu(z). \tag{2.5.8}$$

Taking the sum with respect to $n$, we observe that the sum involving the first and the third integrals on the right-hand side of (2.5.7) can be handled following the same arguments used to establish (2.2.8) and (2.2.9). The sum over $n$ of the first term in (2.5.8) is bounded by a constant times $\|(a_n)\|_v^2$ as follows by (2.2.3). The remaining task is to show that

$$\sum_{n=1}^{\infty} \int_{\Omega_n} \frac{n^{2\alpha}}{|z|^2} \left( \sum_{j,|j|\le n^\alpha} \frac{|a_{n+j}| v_{n+j}}{|j|+1} \right)^2 d\mu(z) \lesssim \|(a_n)\|_v^2 \tag{2.5.9}$$

and

$$\sum_{n=1}^{\infty} \int_{\Omega_n} \left( \sum_{m>n+n^\alpha} \frac{|a_m| v_m}{|z-\gamma_m|} \right)^2 d\mu(z) \lesssim \|(a_n)\|_v^2. \tag{2.5.10}$$

We first consider (2.5.9). Applying Cauchy–Schwarz and the regularity condition (2.5.3), we obtain

$$\begin{aligned} & \int_{\Omega_n} \frac{n^{2\alpha}}{|z|^2} \left( \sum_{j,\ |j|\le n^\alpha} \frac{|a_{n+j}| v_{n+j}}{|j|+1} \right)^2 d\mu(z) \\ & \lesssim V_{[n/2]} \sum_{j,\ |j|\le n^\alpha} |a_{n+j}|^2 v_{n+j} \int_{\Omega_n} \frac{d\mu(z)}{|z|^2} \end{aligned}$$

where $[n/2]$ refers to the greatest integer not bigger than the number $n/2$. For notational convenience, setting

$$\tau_n^2 = \int_{\Omega_n} |z|^{-2} d\mu(z)$$

as before, we note that

$$\sum_{n=j}^{\infty} \tau_n^2 \lesssim \frac{1}{V_j} \tag{2.5.11}$$

by our assumption (2.5.5). It follows that the double sum

$$\sum_{n=1}^{\infty} \tau_n^2 V_{[n/2]} \sum_{j,\ |j|\leq n^\alpha} |a_{n+j}|^2 v_{n+j}$$

is bounded a constant times

$$\sum_{l=0}^{\infty} \sup_{2^l \leq n \leq 2^{l+1}} \left( V_{[n/2]} \sum_{j,\ |j|\leq n^\alpha} |a_{n+j}|^2 v_{n+j} \right) \sum_{n=2^l}^{2^{l+1}} \tau_n^2$$

$$\lesssim \sum_{l=0}^{\infty} \sup_{2^l \leq n \leq 2^{l+1}} \sum_{j,\ |j|\leq n^\alpha} |a_{n+j}|^2 v_{n+j} \lesssim \|(a_n)\|_v^2,$$

where for the later estimate we used (2.5.11).

# 3 Carleson measures and systems of reproducing kernels

In this chapter we will translate our problems and main results from the preceding chapter into statements about systems of reproducing kernels and Carleson measures in certain Hilbert space of meromorphic functions, with particular emphasis on applications to de Branges spaces and model subspaces of the Hardy space $H^2$. Descriptions of Carleson measures and Riesz bases of normalized reproducing kernels for some of these spaces follow from those results. In particular, a connection to the Feichtinger conjecture is pointed out, and we verify that for certain classes of Hilbert spaces. While dealing with Carleson measures, the reproducing kernel thesis is ubiquitous. It is proved that some of our solutions to the Carleson measure problem may be explicitly interpreted as the statement that this thesis holds. Compactness and Schatten class membership of the embedding maps induced by Carleson measures are also considered. This and the next chapter are based on the papers [11, 12, 63].

## 3.1 A class of Hilbert spaces

We begin by recalling a few definitions that will be used quite often in the sequel. Let $\mathscr{H}$ be a separable Hilbert space which consists of complex-valued functions defined on some set $\Omega$ in $\mathbb{C}$. We will say that a sequence $\Lambda$ of distinct points in $\Omega$ is a uniqueness sequence if no nonzero function in $\mathscr{H}$ vanishes on $\Lambda$; we say that $\Lambda$ is an exact uniqueness sequence for $\mathscr{H}$ if it is a uniqueness sequence for $\mathscr{H}$, but fails to be so on the removal of any one of the points in $\Lambda$. If $\Lambda$ is an exact uniqueness sequence for $\mathscr{H}$, then we say that a nontrivial function $G$ defined on $\Omega$ is a generating function for $\Lambda$ if $G$ vanishes on $\Lambda$ but, for every $\lambda_j$ in $\Lambda$, there is a nonzero function $g_j$ in $\mathscr{H}$ such that

$$G(z) = (z-\lambda_j)g_j(z) \tag{3.1.1}$$

for every $z$ in $\Omega$. It is clear that if a generating function exists, it is unique up to multiplication by a nonzero constant. If not, our assumption implies that there exists a sequence of functions $g_j$ in $\mathscr{H}$ such that $g_j(\lambda_m)$ equals 0 when $m \neq j$ and 1 for $m = j$. If there exists another sequence of functions $h_j$ which satisfy (3.1.1), then we observe that

$$g_j - \frac{1}{h_j(\lambda_j)} h_j \in \mathscr{H}$$

and vanishes on $\Lambda$ and contradicts its uniqueness property.

We will assume that $\mathscr{H}$ satisfies the following three axioms:

(Ax1) $\mathscr{H}$ has a reproducing kernel $k_\lambda$ at every point $\lambda$ in $\Omega$, i.e., the point evaluation functional $k_\lambda : f \to f(\lambda)$ is continuous in $\mathscr{H}$ for every $\lambda$ in $\Omega$.

(Ax2) Every exact uniqueness sequence for $\mathscr{H}$ admits a generating function.

(Ax3) There exists a sequence of distinct points $\Gamma = (\gamma_n)$ in $\Omega$ such that the sequence of normalized reproducing kernels $S_R(\gamma_n)$ constitutes a Riesz basis for $\mathscr{H}$. In addition, there is at least one point $z$ in $\Omega \setminus \Gamma$ for which $k_z \neq 0$.

The second axiom (Ax2) may be viewed as a weak statement about the possibility of dividing out zeros. To see this, we may observe that (Ax2) holds trivially if $\mathscr{H}$ has the property that whenever $f(\lambda) = 0$ for some $f$ in $\mathscr{H}$ and $\lambda$ in $\Omega$, we have that $f(z)/(z-\lambda)$ also belongs to $\mathscr{H}$. Indeed, if $\Lambda$ is an exact uniqueness sequence, then there exists a unique function $g_j$ in $\mathscr{H}$ such that $g_j(\lambda_l) = 1$ for $l = j$ and 0 otherwise. We fix an index $n_0$ and set $G(z) = (z-\lambda_{n_0})g_{n_0}(z)$. It follows from the the hypothesis that

$$f_n(z) = G(z)/(z-\lambda_n) = g_{n_0}(z) + (\lambda_n - \lambda_{n_0})g_{n_0}(z)/(z-\lambda_{n_0})$$

also belongs to $\mathscr{H}$.

On the other hand, (Ax2) and (Ax3) lead to a representation of functions in $\mathscr{H}$ (see below) which shows that if $\lambda$ is a point in $\Omega \setminus \overline{\Gamma}$ such that $k_\lambda \neq 0$, then $f(z)/(z-\lambda)$ is in $\mathscr{H}$ whenever $f$ is in $\mathscr{H}$ and $f(\lambda) = 0$. In general, however, this division property need not hold at the accumulation points of $\Gamma$ when we only assume (Ax2).

A prime example of such spaces is the Paley–Wiener space $PW_\pi$. For this space,

$$k_\lambda(z) = \frac{\sin \pi(z-\overline{\lambda})}{\pi(z-\overline{\lambda})}.$$

Axiom $(Ax3)$ is satisfied with an orthonormal basis of reproducing kernels associated with the sequence of integers

$$\left(\frac{\sin \pi(z-n)}{\pi(z-n)}\right)_{n\in\mathbb{Z}},$$

leading to what is known as the cardinal series or the Shannon sampling theorem. We will give more examples of such spaces in the next chapter.

The Riesz basis $S_R(\gamma_n)$ has a biorthogonal basis, which we will call $(g_n)$. By axiom (Ax2), we may write $G(z) = c_n(z-\gamma_n)g_n(z)$ for some nonzero constant $c_n$. We use the suggestive notation $G'(\gamma_n)$ for the value of $G(z)/(z-\gamma_n)$ at $\gamma_n$. We have $G'(\gamma_n) \neq 0$ because otherwise $G(z)/(z-\gamma_n)$ would be identically zero, which can only happen if all functions in $\mathscr{H}$ vanish at every point in $\Omega \setminus \Gamma$; this would contradict the last part of $(Ax3)$. By the uniqueness of the biorthogonal sequence $(g_n)$, we now have

$$g_n(z) = \frac{G(z)}{G'(\gamma_n)(z-\gamma_n)}$$

for every $n$. The function $G$, which is unique up to a multiplicative constant, is the generating function for $\Gamma$. We may assume that $G$ does not vanish at any point $\lambda$ in $\Omega \setminus \overline{\Gamma}$, because then $G(z)/(z-\lambda)$ would be a vector in $\mathscr{H}$ vanishing at every point in $\Gamma$. Hence $G(z)/(z-\lambda)$ would be identically zero, which again would be in contradiction with the second part of $(Ax3)$.

The sequence $g_n$ is also a Riesz basis for $\mathscr{H}$ (cf. [106], p. 29), and therefore every vector $h$ in $\mathscr{H}$ can be written as

$$h(z) = \sum_n h(\gamma_n)\frac{G(z)}{G'(\gamma_n)(z-\gamma_n)}, \tag{3.1.2}$$

where the sum converges with respect to the norm of $\mathscr{H}$ and

$$\|h\|^2_{\mathscr{H}} \simeq \sum_n \frac{|h(\gamma_n)|^2}{\|k_{\gamma_n}\|^2_{\mathscr{H}}} < \infty.$$

Since point evaluation at every point $z$ is a bounded linear functional, (3.1.2) also converges pointwise in $\Omega \setminus \Gamma$. Note that by (3.1.2) we have

$$h(z) = \sum_n \frac{h(\gamma_n)}{\|k_{\gamma_n}\|_{\mathscr{H}}} \cdot \frac{\|k_{\gamma_n}\|_{\mathscr{H}} G(z)}{G'(\gamma_n)(z-\gamma_n)},$$

and by the assumption that $h \mapsto \big(h(\gamma_n)/\|k_{\gamma_n}\|_{\mathscr{H}}\big)$ is a bijective map from $\mathscr{H}$ to $\ell^2$, it follows that

$$\sum_n \frac{\|k_{\gamma_n}\|^2_{\mathscr{H}}}{|G'(\gamma_n)|^2 |z-\gamma_n|^2} < \infty \tag{3.1.3}$$

whenever $z$ is in $\Omega \setminus \Gamma$. We set

$$v_n = \frac{\|k_{\gamma_n}\|^2_{\mathscr{H}}}{|G'(\gamma_n)|^2}$$

and observe that by the last part of axiom (Ax3), there is at least one such $z$ in $\Omega \setminus \Gamma$. Therefore, (3.1.3) implies that

$$\sum_n \frac{v_n}{1+|\gamma_n|^2} < \infty. \tag{3.1.4}$$

We may now change our viewpoint: Given a sequence of distinct complex numbers $\Gamma = (\gamma_n)$ and a weight sequence $v = (v_n)$ that satisfy the admissibility condition (3.1.4), we introduce the space $\mathscr{H}(\Gamma, v)$ consisting of all functions

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n v_n}{z-\gamma_n}$$

for which

$$\|f\|^2_{\mathscr{H}(\Gamma,v)} = \sum_{n=1}^{\infty} |a_n|^2 v_n < \infty,$$

assuming that the set $(\Gamma, v)^*$ is nonempty. Thus we obtain the value of a function $f$ in $\mathscr{H}(\Gamma, v)$ at a point $z$ in $(\Gamma, v)^*$ by computing a discrete Hilbert transform. We note that the inner product of functions $f$ and $g$ generated by $\ell^2_v$-summable sequences $(a_n)$ and $(b_n)$ is

$$\langle f, g\rangle_{\mathscr{H}(\Gamma,v)} = \sum_{n=1}^{\infty} a_n \overline{b_n} v_n.$$

From the preceding discussion, we observe that the class of Hilbert space $\mathscr{H}$ introduced above is isometric to some space of meromorphic functions in $\Omega$. We summarize all these observations as follows.

**Proposition 3.1.1.** *Let $\mathscr{H}$ and $\Omega$ be as above, and $\Gamma = (\gamma_n)$ consisting of distinct points in $\Omega$ such that $S_R(\gamma_n)$ is a Riesz basis in $\mathscr{H}$. Then there exist a generating function $G$ in $Hol(\Omega)$[1] and a positive weight sequence $(v_n)$ such that*

$$f \in \mathscr{H} \Leftrightarrow f(z) = G(z) \sum_{n=1}^{\infty} \frac{a_n v_n}{z - \gamma_n}, \tag{3.1.5}$$

*and $\|f\|_{\mathscr{H}}^2 \simeq \sum_{n=1}^{\infty} |a_n|^2 v_n$ for some $\ell_v^2$- summable sequence $(a_n)$.*

In other words, the space $\mathscr{H}(\Gamma, v)$ introduced above coincides with $\mathscr{H}/G$, and both spaces consist of functions in $\Omega$ with simple poles contained in the sequence $\Gamma$. From now on, $\mathscr{H}(\Gamma, v)$ will be our natural object to study.

## 3.2 Carleson measures in $\mathscr{H}(\Gamma, v)$

Carleson measures have proved to be objects of fundamental importance in the study of function spaces since they were introduced in the late 1950's by L. Carleson [26] for studying the problem of interpolation by bounded analytic functions. They play an important role harmonic analysis, complex analysis and partial differential equations. In this section we will discuss these objects in the spaces $\mathscr{H}(\Gamma, v)$.

We say that a nonnegative measure $\mu$ on $(\Gamma, v)^*$ is a Carleson measure for $\mathscr{H}(\Gamma, v)$ if the inequality

$$\int_{(\Gamma, v)^*} |f(z)|^2 d\mu(z) \lesssim \|f\|_{\mathscr{H}(\Gamma, v)}^2$$

holds for every $f$ in $\mathscr{H}(\Gamma, v)$. It is now immediate that $\mu$ is a Carleson measure for $\mathscr{H}(\Gamma, v)$ if and only if the map $H_{(\Gamma, v)}$ is bounded from $\ell_v^2$ to $L^2\big((\Gamma, v)^*, \mu\big)$. If $\Gamma$ satisfies the sparseness condition (2.2.1), then Theorem 2.2.1 describes all such measures for $\mathscr{H}(\Gamma, v)$. Translating Theorem 2.2.7 to this setting gives all Bessel sequences of normalized reproducing kernels in $\mathscr{H}(\Gamma, v)$. We also note that the discrete version of Corollary 2.2.2 ensures the existence of a uniform bound on the number of points from $\Lambda$ found in each shell $\Omega_n$ is both a necessary and sufficient

[1] $Hol(\Omega)$ denotes the class of holomorphic functions on $\Omega$.

condition for $S_R(\Lambda)$ to be a Bessel sequence.

Since the sparseness condition is the main tool in the development of this result it is very unlikely that the result describes all the Carleson measures in $\mathscr{H}(\Gamma, v)$. In fact, the necessity of the analogous conditions to (2.2.3) and (2.2.4) for the general case has already been established in Theorem 1.4.1. The priori sparsity assumption plays a crucial role in the proof of the converse statement. But we believe that even these partial results give interesting information about the general problem.

### 3.2.1 Reproducing kernel thesis property in $\mathscr{H}(\Gamma, v)$

It is not always easy to determine whether a given operator on a function space possesses important properties, such as boundedness, compactness and Schatten class membership. For reproducing kernel Hilbert spaces, one fruitful approach has been to employ a small class of test functions, namely the reproducing kernels, such that the operator's properties may be determined by its action on these functions alone. In general, there exists no reason why this should be true. But many important results from harmonic analysis may be interpreted as examples of this phenomenon, for example the Carleson measure theorem and Cohn's [31] embedding result on model subspaces generated by one-component inner functions fall into this.

On the other hand, as pointed in Section 1.3, Cohn's embedding conjecture [30] for all model subspaces which was later refuted by Nazarov and Volberg [69] serves as an example that the property does not hold in general.

A natural problem for us is now whether our Carleson measure results on the spaces $\mathscr{H}(\Gamma, v)$ could be interpreted as another example of this property. Alternatively stated, we are interested in whether conditions (2.2.3) and (2.2.4) can be established by applying sequences of reproducing kernel test functions from $\mathscr{H}(\Gamma, v)$. We are able to establish this whenever the weight sequence $(v_n)$ possess some regularity conditions.

We first note that the reproducing kernel of $\mathscr{H}(\Gamma, v)$ at a point $z$ in $(\Gamma, v)^*$ is explicitly given by

$$k_z(\zeta) = \sum_{n=1}^{\infty} \frac{v_n}{(\overline{z} - \overline{\gamma_n})(\zeta - \gamma_n)}; \tag{3.2.1}$$

this is a direct consequence of the definition of $\mathscr{H}(\Gamma, v)$. Indeed, if $g_\lambda$ from $\mathscr{H}(\Gamma, v)$,

$$g_\lambda(z) = \sum_{n=1}^{\infty} \frac{b_n^\lambda v_n}{z-\gamma_n},$$

stands for the kernel function at the point $\lambda$, then for any

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n v_n}{z-\gamma_n}$$

in $\mathscr{H}(\Gamma, v)$, we have

$$\langle f, g_\lambda \rangle_{\mathscr{H}(\Gamma,v)} = \sum_{n=1}^{\infty} a_n \overline{b_n^\lambda} = f(\lambda) = \sum_{n=1}^{\infty} \frac{a_n v_n}{\lambda - \gamma_n}.$$

This means that

$$\sum_{n=1}^{\infty} a_n v_n \Big( \overline{b_n^\lambda} - \frac{1}{\lambda - \gamma_n} \Big) = 0$$

for all sequence $a_n \in \ell_v^2$. This happens only if

$$b_n^\lambda = \frac{1}{\overline{\lambda} - \overline{\gamma_n}}.$$

We could also directly observe the explicit expression for the kernel from the general fact that every kernel function has the series expansion

$$k_z = \sum_n \overline{e_n(z)} e_n$$

for any orthonormal basis $(e_n)$ of the given space. In particular setting

$$e_n(z) = v_n^{\frac{1}{2}}/(z-\gamma_n),$$

an orthonormal basis in $\mathscr{H}(\Gamma, v)$, immediately gives (3.2.1) as required.

When $\Gamma$ satisfies the sparsity condition (2.2.1), the norm of the reproducing kernels at each point $\lambda \in \Omega_m$, $m > 1$ can be estimated by

$$\begin{aligned} \|k_\lambda\|^2_{\mathscr{H}(\Gamma,v)} &= \sum_{n=1}^{\infty} \frac{v_n}{|\lambda-\gamma_n|^2} \simeq \frac{V_m}{|\lambda|^2} + \frac{v_m}{|\gamma_m - \lambda|^2} + P_m \\ &\simeq \max\big\{ V_m |\lambda|^{-2},\ v_m |\gamma_m - \lambda|^{-2},\ P_m \big\}. \end{aligned} \tag{3.2.2}$$

Furthermore, for any point $z$, we may write

$$|k_\lambda(z)|^2 = \left| \frac{v_m}{\overline{(\lambda-\gamma_m)}(z-\gamma_m)} + \sum_{n=1}^{m-1} \frac{v_n}{\overline{(\lambda-\gamma_n)}(z-\gamma_n)} + \sum_{n=m+1}^{\infty} \frac{v_n}{\overline{(\lambda-\gamma_n)}(z-\gamma_n)} \right|^2,$$

and try to compare the three terms appearing here depending on the position of $\lambda$ relative to $\gamma_n$. We consider the case when $v_n$ grows at least exponentially and $v_n|\gamma_n|^{-2}$ decreases exponentially with respect to $n$. We then pick a sequence of points $(\lambda_m)$ such that $\lambda_m$ in $\Omega_m$ is chosen sufficiently close to $\gamma_m$ in such a way that

$$|k_{\lambda_m}(z)|^2 \gtrsim \frac{v_m^2|z-\gamma_m|^{-2}}{|\lambda_m-\gamma_m|^2} \tag{3.2.3}$$

uniformly holds for $z \in \Omega_m$. Such a choice is possible since $|\lambda_m - \gamma_m|$ can be made as small as we wish while $|z-\gamma_m|$ is bounded by

$$\max\{|\gamma_m-\gamma_{m-1}|/2, |\gamma_m-\gamma_{m+1}|/2\}.$$

If $\mu$ is a Carleson measure for $\mathscr{H}(\Gamma,v)$, then an appeal to (3.2.2) and (3.2.3) leads to

$$v_m|\gamma_m-\lambda_m|^{-2} \gtrsim \int_{\mathbb{C}} |k_{\lambda_m}(z)|^2 d\mu(z) \gtrsim \int_{\Omega_m} \frac{v_m^2|z-\gamma_m|^{-2}}{|\lambda_m-\gamma_m|^2} d\mu(z)$$

from which condition (2.2.3) and (2.2.11) follow. We now record this observation into the following corollary.

**Corollary 3.2.1.** *Suppose the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that the numbers $v_n$ grow at least exponentially and that the numbers $v_n/|\gamma_n|^2$ decay at least exponentially with $n$. If $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma)=0$, then the following are equivalent.*

*(i) The operator $H_{(\Gamma,v)}$ is bounded from $\ell_v^2$ to $L^2(\mathbb{C},\mu)$.*

*(ii) $\mu$ is a Carleson measure for $\mathscr{H}(\Gamma,v)$.*

*(iii)*

$$\sup_{n\geq 1} \int_{\Omega_n} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} < \infty. \tag{3.2.4}$$

*(iv)*

$$\sup_{\lambda\in(\Gamma,v)^*} \|k_\lambda\|_{\mathscr{H}(\Gamma,v)}^{-2} \int_{\mathbb{C}} |k_\lambda(z)|^2 d\mu(z) < \infty.$$

### 3.2.2 Vanishing Carleson measures in $\mathscr{H}(\Gamma, v)$

Among all bounded linear operators on Hilbert spaces, the compact ones have many properties similar to those of finite rank transformations. In fact in such spaces, every compact operator is a norm limit of a sequence of finite rank operators.

Our main objective in this part is to identify those Carleson measures $\mu$ for which the embedding maps $I_\mu$ from $\mathscr{H}(\Gamma, v)$ into $L^2(\mathbb{C}, \mu)$ are compact. Whenever $\mu$ induces such an embedding, we call it a vanishing or compact Carleson measure for $\mathscr{H}(\Gamma, v)$. For the Hardy spaces $H^p$, such measures have been characterized by a simple geometric condition, namely that;

$$\lim_{l \to 0} \frac{\mu(Q(x_0, l))}{l} = 0 \tag{3.2.5}$$

for each squares of the form $Q(x_0, l) = \{x + iy \in \mathbb{C} : x_0 < x < x_0 + l,\ 0 < y < l\}$. This description can be equivalently stated in terms of reproducing kernels on $H^2$ as

$$\lim_{|\lambda| \to \infty} \int_{\mathbb{C}} \frac{\Im \lambda \, d\mu(z)}{|z - \overline{\lambda}|^2} = 0.$$

In the closed unit disc $\overline{\mathbb{D}}$, the corresponding measures were explicitly studied by Power [83] and characterized by a similar geometric condition[2].

Vanishing Carleson measures appear naturally in the study of compact composition operators in various function spaces. As far as their characterization is concerned, there exists a general "folk theorem": once the Carleson measures are described by a certain "big oh" condition, vanishing Carleson measures are then characterized by the corresponding "little oh" counterparts. From this perspective, the natural candidates to characterize the vanishing Carleson measures in $\mathscr{H}(\Gamma, v)$ would be

$$\int_{\Omega_n} \frac{d\mu(z)}{|z - \gamma_n|^2} = o(v_n^{-1}) \tag{3.2.6}$$

$$\sum_{m=1}^{n} \mu(\Omega_m) = o(P_n^{-1}) \tag{3.2.7}$$

[2]The corresponding result on the disc could be read by simple change of variables. That is $\eta \in \mathbb{D} \Leftrightarrow z = z(\eta) = \frac{-i(\eta+1)}{\eta-1} \in \mathbb{C}_+$, and then $dz = \frac{2i}{(\eta-1)^2} d\eta$.

and
$$\sum_{m=n}^{\infty}\int_{\Omega_m}\frac{d\mu(z)}{|z|^2}=o(V_n^{-1}) \tag{3.2.8}$$
as $n\to\infty$. It turns out that these are indeed the right conditions in the space $\mathscr{H}(\Gamma,v)$. We note that since our space is reflexive, $\mu$ induces a compact embedding if and only if each weakly convergent sequence in $\mathscr{H}(\Gamma,v)$ converges in norm in $L^2(\mathbb{C},\mu)$. The necessity of the above conditions can be easily verified. We may first choose a sequence of test functions
$$q_n(z)=\frac{\sqrt{v_n}}{z-\gamma_n}.$$
The sequence converges weakly to zero in $\mathscr{H}(\Gamma,v)$. This is a particular case of a much more general statement which says that any orthonormal sequence in a Hilbert space converges weakly to zero[3]. This along with compactness of $\mu$ yields
$$0=\lim_{n\to\infty}\int_{\mathbb{C}}|q_n(z)|^2d\mu(z)$$
from which the first condition (3.2.6) follows. To prove the necessity of the remaining conditions, we recall a few general facts. It is well known that a weakly convergent sequence is uniformly norm bounded. In general, the converse statement does not hold. But under additional assumption, the following particular case of Nordgren's [77] result holds.

**Lemma 3.2.2.** *Let $(f_n)$ be a sequence of functions in $\mathscr{H}(\Gamma,v)$. Then $(f_n)$ converges weakly to zero (weekly null) if and only if it converges pointwise to zero and*
$$\sup_n\|f_n\|_{\mathscr{H}(\Gamma,v)}<\infty. \tag{3.2.9}$$

Next we consider a sequence of unit norm functions defined by
$$g_n(z)=\frac{1}{\sqrt{P_n}}\sum_{m=n+1}^{\infty}\frac{v_m}{\gamma_m(z-\gamma_m)}.$$

If $z$ belongs to the shell $\Omega_N$, then $|g_n(z)|\simeq P_n^{\frac{1}{2}}$ whenever $n>N$ and converges pointwise to zero as $n\to\infty$. Thus by the above lemma, the sequence $g_n$ is weakly null. Taking into account compactness of $\mu$, we have

[3] This is an immediate consequence of Bessel property of orthonormal sequences.

$$0 = \lim_{n\to\infty} \int_{\mathbb{C}} |g_n(z)|^2 d\mu(z) \geq \lim_{n\to\infty} P_n \sum_{m=1}^{n} \mu(\Omega_m)$$

from which (3.2.7) follows. On the other hand, If $\sup_n V_n < \infty$, then (3.2.8) trivially holds for each Carleson measure $\mu$. We shall thus consider the case when $V_n \to \infty$ as $n \to \infty$. In this case, we may consider another sequence of unit norm test functions

$$h_n(z) = \frac{1}{V_n^{\frac{1}{2}}} \sum_{m=1}^{n-1} \frac{v_m}{z - \gamma_m}.$$

It can be easily verified that $h_n$ converges pointwise to zero, and by Lemma 3.2.2 it constitutes a weakly null sequence. If $\mu$ induces a compact embedding, we then have

$$0 = \lim_{n\to\infty} \int_{\mathbb{C}} |h_n(z)|^2 d\mu(z) \gtrsim \lim_{n\to\infty} V_n \sum_{k=n}^{\infty} \int_{\Omega_k} \frac{d\mu(z)}{|z|^2},$$

which gives the remaining assertion in (3.2.7).

**Theorem 3.2.3.** *Suppose that the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that $v$ is an admissible weight sequence for $\Gamma$. A nonnegative measure $\mu$ on $\mathbb{C}$ with $\mu(\Gamma) = 0$ is a compact Carleson measure for $\mathscr{H}(\Gamma, v)$ if and only if*

$$\lim_{n\to\infty} \int_{\Omega_n} \frac{v_n d\mu(z)}{|z - \gamma_n|^2} = 0 \tag{3.2.10}$$

*and*

$$\lim_{n\to\infty} \left( V_n \sum_{m=n}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} + P_n \sum_{m=1}^{n} \mu(\Omega_m) \right) = 0. \tag{3.2.11}$$

*Proof.* The "only if part" was already established in the previous paragraphs. Assume conversely that the conditions (3.2.10) and (3.2.11) hold, and consider a weakly null sequence

$$f_n(z) = \sum_{m=1}^{\infty} \frac{a_m^n v_m}{z - \gamma_m}$$

in $\mathscr{H}(\Gamma, v)$. Then an appeal to the classical Riesz representation theorem gives that for each sequence $(b_m)$ in $\ell_v^2$, we have

$$\sum_{m=1}^{\infty} a_m^n v_m \overline{b_m} \longrightarrow 0$$

whenever $n \to \infty$. Taking in particular $b^{(l)} = \left(b_m^{(l)}\right) = 1$ for $m = l$ and $b_m^{(l)} = 0$ otherwise implies

$$\lim_{n\to\infty} |a_m^n| v_m = 0 \tag{3.2.12}$$

for each $m$. We may first make the following splitting:

$$\begin{aligned}\sum_{l=1}^{\infty} \int_{\Omega_l} |f_n(z)|^2 d\mu(z) \;&\lesssim\; \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{1}{|z|^2} \left( \sum_{m=1}^{l-1} |a_m^n| v_m \right)^2 d\mu(z) \\ &+\; \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{|a_l^n|^2 v_l^2}{|z-\gamma_l|^2} d\mu(z) + \sum_{l=1}^{\infty} \mu(\Omega_l) \left( \sum_{m=l+1}^{\infty} \frac{|a_m^n| v_m}{|\gamma_m|} \right)^2,\end{aligned}$$

which follows from Cauchy–Schwarz and the growth condition (2.2.1). It suffices to show that each of the three right-hand sums converges to zero when $n \to \infty$.

We first show that

$$\lim_{n\to\infty} \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{|a_l^n|^2 v_l^2}{|z-\gamma_l|^2} d\mu(z) = 0. \tag{3.2.13}$$

From (3.2.10), for each small $\varepsilon > 0$, there exists $N$ for which

$$\int_{\Omega_l} \frac{v_l}{|z-\gamma_l|^2} d\mu(z) < \varepsilon$$

when $l > N$. It follows that

$$\begin{aligned}\sum_{l=1}^{\infty} \int_{\Omega_l} \frac{|a_l^n|^2 v_l^2}{|z-\gamma_l|^2} d\mu(z) \;&\lesssim\; \sum_{l=1}^{N} |a_l^n|^2 v_l \int_{\Omega_l} \frac{v_l}{|z-\gamma_l|^2} d\mu(z) + \varepsilon \sum_{l=N+1}^{\infty} |a_l^n|^2 v_l \\ &\lesssim\; \sum_{l=1}^{N} |a_l^n|^2 v_l + \varepsilon;\end{aligned}$$

here we used (3.2.9) and (3.2.10). Taking the limit $n \to \infty$ in (3.2.14) and invoking (3.2.12) leads to the desired conclusion (3.2.13).

We next prove that

$$\lim_{n\to\infty} \sum_{l=1}^{\infty} \int_{\Omega_l} \frac{1}{|z|^2} \left( \sum_{m=1}^{l-1} |a_m^n| v_m \right)^2 d\mu(z) = 0. \tag{3.2.14}$$

Here we only need to modify the arguments used to establish (2.2.8) in the previous chapter. We keep the notation $\tau_l$ from that chapter. By duality, we have

$$\begin{aligned}\left(\sum_{l=1}^{\infty}\tau_l^2\left(\sum_{m=1}^{l-1}|a_m^n|v_m\right)^2\right)^{\frac{1}{2}} &= \sup_{\|c_l\|_\ell^2=1}\sum_{l=1}^{\infty}\tau_l|c_l|\sum_{m=1}^{l-1}|a_m^n|v_m\\ &\leq \sup_{\|c_l\|_\ell^2=1}\sum_{m=1}^{\infty}|a_m^n|v_m\sum_{l=m+1}^{\infty}\tau_l|c_l|.\end{aligned}$$

The Cauchy–Schwarz inequality applied to the last sum gives

$$\left(\sum_{l=m+1}^{\infty}\tau_l|c_l|\right)^2 \leq \sum_{l=m+1}^{\infty}\tau_l^2V_l^{\frac{1}{2}}\sum_{j=m+1}|c_j|^2V_j^{-\frac{1}{2}}. \tag{3.2.15}$$

By (3.2.11), we observe that for each $\varepsilon > 0$, there exists $N_1$ for which

$$\sum_{l:2^kV_m<V_l\leq 2^{k+1}V_m}\tau_l^2V_l^{\frac{1}{2}} \lesssim \frac{\varepsilon}{2^{k/2}V_{m+1}^{\frac{1}{2}}}$$

for $k \geq 0$ and $m \geq N_1$. Summing these inequalities for $m \geq N_1$, we get

$$\sum_{l=m+1}^{\infty}\tau_l^2V_l^{\frac{1}{2}} \lesssim \frac{\varepsilon}{V_{m+1}^{\frac{1}{2}}}. \tag{3.2.16}$$

Combining (3.2.15) with (3.2.16), we find that

$$\begin{aligned}\sum_{m=1}^{\infty}v_m\left(\sum_{l=m+1}^{\infty}\tau_l|c_l|\right)^2 &= \sum_{m=1}^{N_1}v_m\left(\sum_{l=m+1}^{\infty}\tau_l|c_l|\right)^2+\sum_{m=N_1+1}^{\infty}v_m\left(\sum_{l=m+1}^{\infty}\tau_l|c_l|\right)^2\\ &\lesssim \sum_{m=1}^{N_1}\frac{v_m}{V_{m+1}}\sum_{j=m+1}|c_j|^2V_j^{-\frac{1}{2}}+\varepsilon\sum_{m=N_1+1}^{\infty}\frac{v_m}{V_{m+1}}\sum_{j=m+1}^{\infty}|c_j|^2V_j^{-\frac{1}{2}}\\ &\lesssim \underbrace{\sum_{m=1}^{N_1}\frac{v_m}{V_{m+1}}\sum_{j=m+1}|c_j|^2V_j^{-\frac{1}{2}}}_{=C}+\varepsilon,\end{aligned}$$

where in the last inequality we used (2.2.10). To obtain (3.2.14), we see that

$$\begin{aligned}\sum_{m=1}^{\infty}|a_m^n|^2 v_m \sum_{m=1}^{\infty}\left(\sum_{l=m+1}^{\infty}\tau_l|c_l|\right)^2 &\lesssim C\sum_{m=1}^{N_1}|a_m^n|^2 v_m+\varepsilon\sum_{m=N_1}^{\infty}|a_m^n|^2 v_m\\ &\lesssim \sum_{m=1}^{N_1}|a_m^n|^2 v_m \longrightarrow 0\end{aligned}$$

as $n\to\infty$ which follows from (3.2.12).

It remains to prove that

$$\lim_{n\to\infty}\sum_{l=1}^{\infty}\mu(\Omega_l)\left(\sum_{m=l+1}^{\infty}\frac{|a_m^n|v_m}{|\gamma_m|}\right)^2=0. \tag{3.2.17}$$

We note to begin with that the Cauchy–Schwarz inequality gives

$$\sum_{l=1}^{\infty}\mu(\Omega_l)\left(\sum_{m=l+1}^{\infty}\frac{|a_m^n|v_m}{|\gamma_m|}\right)^2\le\sum_{m=l+1}^{\infty}|a_m^n|^2 v_m P_{m-1}^{\frac{1}{2}}\sum_{j=l+1}^{\infty}\frac{v_j}{P_{j-1}^{\frac{1}{2}}|\gamma_j|^2}.$$

Since

$$\sum_{j=l+1}^{\infty}\frac{v_j}{P_{j-1}^{\frac{1}{2}}|\gamma_j|^2}\le\int_0^{P_l}x^{-\frac{1}{2}}dx\le 2P_l^{\frac{1}{2}},$$

it follows that

$$\sum_{l=1}^{\infty}\mu(\Omega_l)\left(\sum_{m=l+1}^{\infty}\frac{|a_m^n|v_m}{|\gamma_m|}\right)^2\lesssim\sum_{l=1}^{\infty}\mu(\Omega_l)P_l^{\frac{1}{2}}\sum_{m=l+1}^{\infty}|a_m^n|^2 v_m P_{m-1}^{\frac{1}{2}},$$

which becomes

$$\sum_{l=1}^{\infty}\mu(\Omega_l)\left(\sum_{m=l+1}^{\infty}\frac{|a_m^n|v_m}{|\gamma_m|}\right)^2\lesssim\sum_{m=1}^{\infty}|a_m^n|^2 v_m P_{m-1}^{\frac{1}{2}}\sum_{l=1}^{m-1}\mu(\Omega_l)P_l^{\frac{1}{2}}$$

when we change the order of summation. By (3.2.11), for each $\varepsilon>0$, there exists again an $N_2$ for which for $m\ge N_2$ it follows that

$$\sum_{l:2^k P_{m-1}\le P_l\le 2^{k+1}P_{m-1}}\mu(\Omega_l)P_l^{\frac{1}{2}}\lesssim\frac{\varepsilon}{P_{m-1}^{\frac{1}{2}}2^{k/2}}.$$

Summing these inequalities with respect to $k$ gives

$$\sum_{l=1}^{m-1} \mu(\Omega_l) P_l^{\frac{1}{2}} \lesssim \frac{\varepsilon}{P_{m-1}^{\frac{1}{2}}},$$

and we get

$$\begin{aligned} \sum_{m=1}^{\infty} |a_m^n|^2 v_m P_{m-1}^{\frac{1}{2}} \sum_{l=1}^{m-1} \mu(\Omega_l) P_l^{\frac{1}{2}} &\lesssim \sum_{m=1}^{N_2} |a_m^n|^2 v_m + \varepsilon \sum_{m=N_2+1}^{\infty} |a_m^n|^2 v_m \\ &\lesssim \sum_{m=1}^{N_2} |a_m^n|^2 v_m \longrightarrow 0 \end{aligned}$$

as $n \to \infty$ which again follows from (3.2.12). □

Our next result provides a necessary condition for all compact operators acting on $\mathscr{H}(\Gamma, v)$ in terms of the reproducing kernels when the weight sequence $v_n$ is not summable. The question whether the converse statement holds remains open.

**Proposition 3.2.4.** *Suppose the sparseness condition* (2.2.1) *holds and that* $v_n$ *is not summable. If* $T$ *is any compact operator from* $\mathscr{H}(\Gamma, v)$ *to a normed space* $\mathcal{H}$, *then*

$$\lim_{|\lambda| \to \infty} \left\| T k_\lambda / \|k_\lambda\|_{\mathscr{H}(\Gamma, v)} \right\|_{\mathcal{H}} = 0. \tag{3.2.18}$$

*Proof.* We need to show that $k_\lambda / \|k_\lambda\|_{\mathscr{H}(\Gamma, v)}$ converges weakly to zero in$\mathscr{H}(\Gamma, v)$ as $|\lambda| \to \infty$.
We shall verify Lemma 3.2.2. Assume $\lambda \in \Omega_m$ and $z$ belongs to the shell $\Omega_n$ for some $n < m$. Then we estimate:

$$\frac{|k_\lambda(z)|}{\|k_\lambda\|_{\mathscr{H}(\Gamma, v)}} \lesssim \min\left\{ \frac{|\lambda - \gamma_m|}{v_m^{\frac{1}{2}}}, \frac{|\lambda|}{V_m^{\frac{1}{2}}}, \frac{1}{P_m^{\frac{1}{2}}} \right\} \left( \frac{V_n}{|\lambda||z|} + \frac{v_n |\lambda|^{-1}}{|z - \gamma_n|} + \frac{1}{|\lambda|} \sum_{k=1}^{m-1} \frac{v_k}{|\gamma_k|} + \frac{v_m |\gamma_m|^{-1}}{|\lambda - \gamma_m|} + P_m \right).$$

Expanding out the product, we obtain

$$\frac{|k_\lambda(z)|}{\|k_\lambda\|_{\mathscr{H}(\Gamma, v)}} \lesssim \frac{V_n}{V_m^{\frac{1}{2}} |z|} + \frac{v_n}{|z - \gamma_n| V_m^{\frac{1}{2}}} + \frac{1}{V_m^{\frac{1}{2}}} \sum_{k=1}^{m-1} \frac{v_k}{|\gamma_k|} + \frac{v_m^{\frac{1}{2}}}{|\gamma_m|} + P_m^{\frac{1}{2}} \longrightarrow 0$$

as $m \to \infty (|\lambda| \to \infty)$. □

**Corollary 3.2.5.** *Suppose the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that the numbers $v_n$ grow at least exponentially and that the numbers $v_n/|\gamma_n|^2$ decay at least exponentially with $n$. If $\mu$ is a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma) = 0$, then the following are equivalent.*

*(i)* $\mu$ *is a compact Carleson measure for* $\mathscr{H}(\Gamma, v)$.

*(ii)*
$$\lim_{n\to\infty} \int_{\Omega_n} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} = 0.$$

*(iii)*
$$\lim_{|\lambda|\to\infty} \int_{\mathbb{C}} \|k_\lambda\|^{-2}_{\mathscr{H}(\Gamma,v)} \big|k_\lambda(z)\big|^2 d\mu(z) = 0.$$

The corollary follow from the above proposition and the arguments used to establish the thesis property in Corollary 3.2.1.

Because of the Open Mapping Theorem, $H_{(\Gamma,v);(\Lambda,w)}$ can not be both surjective and compact. But $(w_j)$ could be still of the form in (2.2.20) under compactness. The point is now whether the super-thin phenomenon associated to Bessel sequence of normalized reproducing kernels in $\mathscr{H}(\Gamma, v)$, observed in Theorem 2.2.7, still happens when $H_{(\Gamma,v);(\Lambda,w)}$ is a compact operator. By Theorem 3.2.3, more precisely its discrete version, it follows that no such phenomena occurs in this case.

### 3.2.3 Schatten class membership

Another important class of operators is the trace ideals or the Schatten class. It constitutes a special class of compact operators. Let $T$ be a compact operator between two separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then there exist orthonormal bases $(e_n)$ and $(\sigma_n)$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and a null sequence of nonnegative scalars $(s_n(T))$ such that

$$Tx = \sum_n s_n(T)\langle x, e_n\rangle_{\mathcal{H}_1} \sigma_n$$

for each $x \in \mathcal{H}_1$. The sequence $(s_n(T))$ constitutes the singular values (s-numbers) of $T$,

$$s_n(T) = \inf\big\{\|T-K\|_{\mathcal{H}_1\to\mathcal{H}_2},\ \text{rank K} \leq n-1\big\},$$

which coincides with the eigenvalues of the positive operator $(T^*T)^{1/2} = |T|$ on $\mathcal{H}_1$.[4] For $p > 0$, the Schatten class $S_p(\mathcal{H}_1, \mathcal{H}_2)$ consists of all such operators $T$ for which the singular values $s_n(T)$ forms a sequence in $\ell^p$. If $1 \leq p < \infty$, then $S_p(\mathcal{H}_1, \mathcal{H}_2)$ is a Banach space equipped with the norm

$$\|T\|_{S_p} = \left(\sum_n |s_n(T)|^p\right)^{1/p} < \infty.$$

In particular, $S_2(\mathcal{H}_1, \mathcal{H}_2)$ and $S_1(\mathcal{H}_1, \mathcal{H}_2)$ represent the two most important classes which are often referred to as Hilbert–Schmidt and trace class (nuclear) operators respectively. If $T$ belongs to the former class, then its norm can be equivalently computed as

$$\|T\|_{S_2} = \left(\sum_n \|Te_n\|^2_{\mathcal{H}_2}\right)^{\frac{1}{2}} \tag{3.2.19}$$

with any orthonormal basis $(e_n)$ of $\mathcal{H}_1$.

We refer to the monographs [43] and [107] for the basic facts about the Schatten classes.

If $T \in S_1(\mathcal{H}_1, \mathcal{H}_1)$, we may define its trace as

$$\operatorname{tr}(T) = \sum_n \langle Te_n, e_n \rangle_{\mathcal{H}_1} \tag{3.2.20}$$

for any orthonormal basis $(e_n)$ of $\mathcal{H}_1$. Note that the series converges absolutely and is independent of the choice of the orthonormal basis (cf. [107], p. 19). In particular, if $T$ is positive, we further have

$$\operatorname{tr}(T) = \|T\|_{S_1}.$$

A natural question of interest is to ask when a compact Carleson measure $\mu$ induces a Schatten class embedding map $I_\mu$ from $\mathscr{H}(\Gamma, v)$ into $L^2(\mathbb{C}, \mu)$. Our answer essentially depends on how fast the sequence of the integrals

$$\int_{\mathbb{C}} \frac{v_n d\mu(z)}{|z - \gamma_n|^2}$$

decays. For $p = 2$, the next apparently well-known result, gives the precise quan-

[4] We note that $s_n(T)$ can be defined for any bounded operator $T$. But $s_n(T) \to 0$ if and only if $T$ is compact.

tification as a particular case.

**Theorem 3.2.6.** *Let $I_\mu$ be a bounded embedding map from a reproducing kernel Hilbert space $\mathcal{H}$ into $L^2(\Omega,\mu)$. Then $I_\mu$ belongs to $S_2(\mathcal{H},L^2(\Omega,\mu))$ if and only if*

$$\|I_\mu\|_{S_2}^2 = \int_\Omega \|k_z\|_{\mathcal{H}}^2 \, d\mu(z) < \infty.$$

This classical result classifies the Hilbert–Schmidt membership of $I_\mu$ in terms of its actions on the reproducing kernels alone.

*Proof.* For completeness, we include a short proof of the theorem. We may compute the series in (3.2.19) using any orthonormal basis $(e_n)$ in $\mathcal{H}$. That is

$$\begin{aligned}\|I_\mu\|_{S_2}^2 &= \sum_{n=1}^{\infty} \langle I_\mu e_n, I_\mu e_n\rangle_{L^2(\Omega,\mu)} \\ &= \lim_{m\to\infty} \sum_{n=1}^{m} \int_\Omega |e_n(z)|^2 d\mu(z) \\ &= \int_\Omega \sum_{n=1}^{\infty} |e_n(z)|^2 d\mu(z) = \int_\Omega \|k_z\|_{\mathcal{H}}^2 \, d\mu(z);\end{aligned}$$

here we used the Lebesgue's monotone convergence theorem to interchange the sum and the integral signs. □

An immediate consequence of this result is that if the sequences $\Gamma$ and $v$ constitute an admissible pair and $\Gamma$ satisfies the sparseness assumption (2.2.1), then for any nonnegative measure $\mu$ on $\mathbb{C}$ with $\mu(\Gamma)=0$, we have that $I_\mu \in S_2\big(\mathscr{H}(\Gamma,v), L^2(\mathbb{C},\mu)\big)$ if and only if

$$\int_{\mathbb{C}} \|k_z\|_{\mathscr{H}(\Gamma,v)}^2 d\mu(z) \simeq \sum_{m=1}^{\infty} \int_{\Omega_m} \left( \frac{V_m}{|z|^2} + \frac{v_m}{|z-\gamma_m|^2} + P_m \right) d\mu(z) < \infty.$$

This is equivalent to saying that

$$\sum_{m=1}^{\infty} \int_{\Omega_m} \frac{v_m d\mu(z)}{|z-\gamma_m|^2} < \infty \tag{3.2.21}$$

$$\sum_{m=1}^{\infty} V_m \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} < \infty \tag{3.2.22}$$

and

$$\sum_{m=1}^{\infty} P_m \mu(\Omega_m) < \infty. \tag{3.2.23}$$

In the case when the target space $L^2(\mathbb{C}, \mu)$ has a reproducing kernel, we obtain a similar description of Hilbert–Schmidt embedding maps from the following general result.

**Theorem 3.2.7.** *Let $T$ be a bounded operator from a Hilbert space $\mathcal{H}$ into a reproducing kernel Hilbert subspace of $L^2(\Omega, \mu)$. Then $T$ belongs to $S_2(\mathcal{H}, L^2(\Omega, \mu))$ if and only if*

$$\|T\|_{S_2}^2 = \int_\Omega \|T^* k_z\|_{\mathcal{H}}^2 \, d\mu(z) < \infty.$$

*Proof.* Let $(e_n)$ be any orthonormal basis in $\mathcal{H}$. We wish to show that the series in (3.2.19) converges with the norm of the sequences computed in $L^2(\Omega, \mu)$. We have that

$$\begin{aligned} \|T\|_{S_2}^2 = \sum_{n=1}^{\infty} \|Te_n\|_{L^2(\Omega,\ \mu)}^2 &= \lim_{m\to\infty} \sum_{n=1}^{m} \|Te_n\|_{L^2(\Omega,\ \mu)}^2 \\ &= \int_\Omega \sum_{n=1}^{\infty} |Te_n(z)|^2 \, d\mu(z) \end{aligned}$$

which follows by Lebesgue's monotone convergence theorem. By the reproducing property of the kernels, we obtain

$$\sum_{n=1}^{\infty} |Te_n(z)|^2 = \sum_{n=1}^{\infty} |\langle Te_n, k_z \rangle|^2 = \sum_{n=1}^{\infty} |\langle e_n, T^* k_z \rangle|^2 = \|T^* k_z\|_{\mathcal{H}}^2,$$

where the last equality is due to Parseval's identity. □

When we now turn to the discrete Hilbert transform $\widetilde{H}$ considered in Section 1.4, we have that $\widetilde{H} \in S_2(\ell_v^2, \ell_w^2)$ if and only if

$$\sum_j \sum_n \frac{w_j v_n}{|\lambda_j - \gamma_n|^2} < \infty. \tag{3.2.24}$$

If the sequence $(\gamma_n)$ satisfy the growth condition (2.2.1), then (3.2.24) simplifies to

$$\sum_j \frac{w_j v_j}{|\lambda_j - \gamma_j|^2} < \infty$$

and

$$\sum_j w_j \left( \frac{V_j}{|\lambda_j|^2} + P_j \right) < \infty.$$

To prove our next main results, we need the following general result ( [107], p. 20–21).

**Lemma 3.2.8.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $T : \mathcal{H}_1 \mapsto \mathcal{H}_2$ be a compact operator. Then for each $p > 0$, the following are equivalent.*

*(i)* $T \in S_p(\mathcal{H}_1, \mathcal{H}_2)$.

*(ii)* $T^* \in S_p(\mathcal{H}_2, \mathcal{H}_1)$.

*(iii)* $|T| \in S_p(\mathcal{H}_1, \mathcal{H}_2)$.

*(iv)* $|T|^p = \left(T^*T\right)^{p/2} \in S_1(\mathcal{H}_1)$.

*(v)* $T^*T \in S_{p/2}(\mathcal{H}_1)$.

If any one of the above holds, we also have that

$$\|T\|_{S_p} = \|T^*\|_{S_p} = \big\||T|\big\|_{S_p} = \big\||T|^p\big\|_{S_1}^{1/p} = \big\|T^*T\big\|_{S_{\frac{p}{2}}}^{\frac{1}{2}}.$$

Our next result which provides a sufficient condition for Schatten p-class membership, involves the series

$$\sum_{n=1}^{\infty} \left( \int_{\Omega_n} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \right)^{\frac{p}{2}} \tag{3.2.25}$$

for all exponents $p$.

**Theorem 3.2.9.** *Suppose that the sequences $\Gamma$ and $v$ constitute an admissible pair and $\Gamma$ satisfies the sparsity condition (2.2.1). Let $\mu$ be a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma) = 0$. Then $I_\mu \in S_p\big(\mathscr{H}(\Gamma, v), L^2(\mathbb{C}, \mu)\big)$ if*

*(i)* $0 < p \leq 2$, *and the series in (3.2.25),*

$$\sum_{n=1}^{\infty} \left( v_n \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} \right)^{\frac{p}{2}} \quad \text{and} \quad \sum_{n=1}^{\infty} \left( \frac{v_n}{|\gamma_n|^2} \sum_{m=1}^{n-1} \mu(\Omega_m) \right)^{\frac{p}{2}}$$

*are finite.*

*(ii)* $p \geq 2$, *and the series in* (3.2.25),

$$\sum_{n=1}^{\infty}\left(V_n \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2}\right)^{\frac{p}{2}} \quad \textit{and} \quad \sum_{n=1}^{\infty}\left(P_n \sum_{m=1}^{n} \mu(\Omega_m)\right)^{\frac{p}{2}}$$

*are finite.*

*Proof.* By Lemma (3.2.8), $I_\mu$ belongs to $S_p\big(\mathscr{H}(\Gamma, v), L^2(\mathbb{C}, \mu)\big)$ if and only if $(I_\mu^* I_\mu)^{p/2}$ belongs to the trace class for $\mathscr{H}(\Gamma, v)$. We first consider when $0 < p \leq 2$. Applying the trace formula with the sequence $e_n(z) = v_n^{\frac{1}{2}}/(z - \gamma_n)$ we have that

$$\sum_{n=1}^{\infty}\langle (I_\mu^* I_\mu)^{\frac{p}{2}} e_n, e_n\rangle \leq \sum_{n=1}^{\infty}\langle I_\mu^* I_\mu e_n, e_n\rangle^{\frac{p}{2}} = \sum_{n=1}^{\infty}\left(\int_{\mathbb{C}} \frac{v_n d\mu(z)}{|z-\gamma_n|^2}\right)^{\frac{p}{2}}. \tag{3.2.26}$$

By the sparsity assumption, we have that the right-hand sum in (3.2.26) is comparable to

$$\sum_{n=1}^{\infty}\left(\int_{\Omega_n} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} + v_n \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} + \frac{v_n}{|\gamma_n|^2} \sum_{m=1}^{n-1} \mu(\Omega_m\right)^{\frac{p}{2}}$$

from which $(i)$ follows. Note that the inequality in (3.2.26) is due to a general result in (cf. [107], p. 24).

To prove (ii), we only need to check the conditions for $p = 2$ and $p = \infty$. The estimates for the remaining exponents $p$ will follow by complex interpolation between the spaces $S_2(\mathscr{H}(\Gamma, v), L^2(\mathbb{C}, \mu))$ and $S_\infty(\mathscr{H}(\Gamma, v), L^2(\mathbb{C}, \mu))$. When $p = 2$, the first series in (ii) is exactly condition (3.2.21). The remaining estimates in (3.2.22) and (3.2.23) can be easily deduced from the second and third series in (ii). On the other hand, when $p = \infty$, the conditions in (ii), simplify to those conditions in Theorem 2.2.1. $\square$

We now assume that the weight sequence $v_n$ enjoys some smoothness in the sense that

$$V_n \simeq v_n \quad \text{and} \quad P_n \simeq \frac{v_{n+1}}{|\gamma_{n+1}|^2}. \tag{3.2.27}$$

**Corollary 3.2.10.** *Suppose that the sequences* $\Gamma$ *and* $v$ *are an admissible pair and satisfy conditions* (2.2.1) *and* (3.2.27). *Let* $\mu$ *be a nonnegative measure* *on $\mathbb{C}$ with $\mu(\Gamma) = 0$ and $p \geq 2$. Then $I_\mu \in S_p\big(\mathscr{H}(\Gamma, v), L^2(\mathbb{C}, \mu)\big)$ if and only if*

$$\sum_{n=1}^{\infty} \left( \int_{\Omega_n} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \right)^{\frac{p}{2}} < \infty \tag{3.2.28}$$

$$\sum_{n=1}^{\infty} \left( v_n \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} \right)^{\frac{p}{2}} < \infty \tag{3.2.29}$$

*and*

$$\sum_{n=1}^{\infty} \left( \frac{v_n}{|\gamma_n|^2} \sum_{m=1}^{n-1} \mu(\Omega_m) \right)^{\frac{p}{2}} < \infty. \tag{3.2.30}$$

*Proof.* The sufficiency of the conditions follows by the theorem above. We may note that the smoothness assumption for $p = 2$ is not really needed since

$$\sum_{m=1}^{\infty} P_m \mu(\Omega_m) = \sum_{m=1}^{\infty} \mu(\Omega_m) \sum_{n=m+1} \frac{v_n}{|\gamma_n|^2} = \sum_{n=1}^{\infty} \frac{v_n}{|\gamma_n|^2} \sum_{m=1}^{n-1} \mu(\Omega_m)$$

which coincides with (3.2.30), and (3.2.29) follows from

$$\begin{aligned} \sum_{m=1}^{\infty} V_m \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} &= \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} v_n \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} \\ &= \sum_{n=1}^{\infty} v_n \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2}. \end{aligned}$$

On the other hand, when $p \geq 2$, the inequality in (3.2.26) gets reversed from which the necessity of the conditions follows. $\square$

We remark that the conditions in the corollary fail to imply the boundedness condition (2.2.4) for $p > 2$ if we remove the smoothness assumption (3.2.27). A simple example that illustrates this is the following.

**Example 5.** Set $v_n = 1$ for each $n$ and construct a Carleson measure $\mu$ for $\mathscr{H}(\Gamma, v)$ for which

$$t_n = \sum_{m=n+1}^{\infty} \int_{\Omega_m} \frac{d\mu(z)}{|z|^2} \simeq \frac{1}{\left(n \log(n+1)^2\right)^{\frac{2}{p}}}.$$

Then it is easily seen that

$$\sup_n n\, t_n = \infty$$

when $p > 2$.

**Corollary 3.2.11.** *Suppose the sequence $\Gamma$ satisfies the sparseness condition (2.2.1) and that $v_n \in \ell^1$. Let $\mu$ be a nonnegative measure on $\mathbb{C}$ with $\mu(\Gamma) = 0$ and $p \geq 2$. Then $I_\mu \in S_p\big(\mathscr{H}(\Gamma,v), L^2(\mathbb{C},\mu)\big)$ if and only if*

$$\sum_{n=1}^{\infty} \left( \int_{\mathbb{C}} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \right)^{\frac{p}{2}} < \infty. \tag{3.2.31}$$

In the case when the sequence $v_n$ is summable, the bounded embedding maps are identified by Corollary 2.2.3. On the other hand, for $p = 2$, Theorem 3.2.6 implies

$$\int_{\mathbb{C}} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} \leq \sum_{n=1}^{\infty} \int_{\Omega_n} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} < \infty. \tag{3.2.32}$$

The condition for other exponents $p$ follows by interpolation and hence the sufficiency follows. When $p \geq 2$, the inequality in (3.2.26) gets reversed from which the necessity of the condition also follows.

## 3.3 Reproducing kernel Riesz bases in $\mathscr{H}(\Gamma,v)$

Given a sequence $\Lambda = (\lambda_j)$ in $(\Gamma,v)^*$, we associate with it the corresponding sequence of normalized reproducing kernels $S_R(\Lambda)$. We observe that if $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma,v)$, then the transformation $H_{(\Gamma,v);(\Lambda,w)}$ is invertible if and only if the system $S_R(\Lambda)$ is a Riesz basis for $\mathscr{H}(\Gamma,v)$. If $\mathscr{H}(\Gamma,v)$ is obtained from a space $\mathscr{H}$ satisfying (Ax1), (Ax2), (Ax3), as described in one of the previous sections, then Theorem 2.4.1 applies. In the special case when $\gamma_n \to \infty$ as $n \to \infty$, we may write the meromorphic function defined in (2.4.3) as $\Phi = F/G$, with $G$ again denoting the generating function for $\Gamma$ and $F$ an entire function with a simple zero at each point $\lambda_j$. Then the expressions appearing in (2.4.4) can be restated as

$$\nu_n = \frac{v_n |F(\gamma_n)|^2}{|G'(\gamma_n)|^2} \quad \text{and} \quad \varpi_j = \frac{|G(\lambda_j)|^2}{w_j |F'(\lambda_j)|^2},$$

which expresses the weights in a natural way in terms of the generating functions. Then Theorem 2.4.1 translates into the following statement in $\mathscr{H}(\Gamma,v)$.

**Theorem 3.3.1.** *Let $\Lambda$ be a sequence in $(\Gamma,v)^*$, and let $w$ be the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma,v)$. Then the following statements are equivalent.*

*(i) The sequence $S_R(\Lambda)$ is a Riesz basis for $\mathscr{H}(\Gamma,v)$;*

*(ii) $\Lambda$ is an exact uniqueness sequence for $\mathscr{H}(\Gamma,v)$ and the transformations $H_{(\Gamma,v);(\Lambda,w)}$ and $H_{(\Lambda,\varpi);(\Gamma,v)}$ are bounded.*

The theorem reduces the Riesz bases problem into one about boundedness of two weighted discrete Hilbert transforms. Thus the essential difficulties in the problem seem to appear in a particularly succinct form in this formulation. In other words the specific challenges to the given space should be limited to the study of its Carleson measures or more generally to the two weight problem for the Hilbert transform. Though the latter property is yet to be understood well, the Helson–Szegő type condition has already been established [34, 36] and a weaker version of the basis problem will neatly follow from this link.

### 3.3.1 Reproducing kernel Riesz bases from sparse sequences

We now turn to the case when the sequence $(\gamma_n)$ satisfies the sparseness condition (2.2.1). As before we let $G$ denote the generating functions for $\Gamma$, $F$ an entire function with simple zeros at each $\lambda_j$ in $\Lambda \subset (\Gamma,v)^*$ and

$$\Phi(z) = \frac{F(z)}{G(z)}.$$

In addition, we introduce the following notations:

$$h_n = \frac{|G(\lambda_n)|^2 \|k_{\lambda_n}\|^2_{\mathscr{H}(\Gamma,v)}}{|F'(\lambda_n)|^2}, \quad H_n = \sum_{m=1}^{n-1} \sum_{\lambda_j \in \Omega_m} h_j$$

and

$$W_n = \sum_{m=n+1}^{\infty} \sum_{\lambda_j \in \Omega_m} \frac{h_j}{|\lambda_j|^2}.$$

Then our next translation of the results from the previous chapter reads as follows.

**Corollary 3.3.2.** *Suppose the sequence $\Gamma$ satisfies the sparseness condition* (2.2.1) *and that $v$ is an admissible weight sequence for $\Gamma$. Let $\Lambda = (\lambda_n)$ be a* *sequence in* $(\Gamma, v)^*$ *and* $w$ *be its weight sequence with respect to* $(\Gamma, v)$*. Then* $S_R(\Lambda)$ *is a Riesz basis in* $\mathscr{H}(\Gamma, v)$ *if and only if it is complete and minimal,* $\sup_n \#(\Lambda \cap \Omega_n) < \infty$*,*

$$\sup_{n\geq 1} \sup_{\lambda_j \in \Omega_n} \frac{\|k_{\lambda_j}\|_{\mathscr{H}(\Gamma,v)}}{v_n^{\frac{1}{2}} |\gamma_n - \lambda_j|} \frac{|F(\gamma_n)G(\lambda_j)|}{|F'(\lambda_j)G'(\gamma_n)|} < \infty, \tag{3.3.1}$$

*and*

$$\sup_{n\geq 1} \left( H_n \sum_{m=n}^{\infty} \frac{|F(\gamma_m)|^2}{v_m |G'(\gamma_m)|^2 |\gamma_m|^2} + W_n \sum_{m=1}^{n} \frac{|F(\gamma_m)|^2}{v_m |G'(\gamma_m)|^2} \right) < \infty. \tag{3.3.2}$$

This result could be read in the following way: condition (3.3.1) is a separation condition whenever $S_R(\lambda_n)$ constitutes a Riesz basis in $\mathscr{H}(\Gamma, v)$. Indeed, if two points $\lambda_k$ and $\lambda_l$ from $\Lambda$ are close enough, then the numbers $|F'(\lambda_k)|$ and $|F'(\lambda_l)|$ gets smaller and contradicts (3.3.1). The other condition (3.3.2) gives a sort of "balance" on the distribution of the sequences $(\lambda_n)$, and plays a role as a "replacement" for the $A_2$ condition.

**Proof of Corollary 3.3.2.** The result is a direct consequence of Theorems 2.2.1 and 3.3.1. We shall give here an alternative proof for the necessity. We assume that $(k_\lambda / \|k_\lambda\|_{\mathscr{H}(\Gamma,v)})_{\lambda\in\Lambda}$ constitutes a Riesz basis. Then the Bessel property ensures that there exists a uniform bound on the number of points from $\Lambda$ found in each annulus $\Omega_n$. For each square summable sequence $(a_n)$, the interpolation problem

$$f(\lambda) = a_\lambda \|k_\lambda\|_{\mathscr{H}(\Gamma,v)}$$

has also a unique solution $f$ in $\mathscr{H}(\Gamma, v)$. We solve the problem by means of the Lagrange-type formula

$$f(z) = \Phi(z) \sum_{\lambda\in\Lambda} a_\lambda \frac{\|k_\lambda\|_{\mathscr{H}(\Gamma,v)}}{\Phi'(\lambda)(z-\lambda)}, \tag{3.3.3}$$

which makes sense at least for finite sequence $(a_\lambda)$. The fact that the series convergence in $\mathscr{H}(\Gamma, v)$ for infinite sequence can be verified by duality. On the other hand, there exists an $\ell_v^2$ sequence $(c_m)$ such that $f$ has the expansion

$$f(z) = \sum_{m=1}^{\infty} \frac{c_m v_m}{z - \gamma_m}$$

for which we have $\|f\|^2_{\mathscr{H}(\Gamma,v)} \simeq \sum_{m=1}^{\infty} |c_m|^2 v_m$. Because of minimality and completeness, we may compute the sequence $c_m$ via (3.3.3). That is

$$\begin{aligned} c_m v_m = \lim_{z\to\gamma_m}(z-\gamma_m)f(z) &= \sum_{\lambda\in\Lambda} a_\lambda \frac{\|k_{\lambda_n}\|_{\mathscr{H}(\Gamma,v)}}{\Phi'(\lambda)(\gamma_m-\lambda)} \lim_{z\to\gamma_m}(z-\gamma_m)\Phi(z) \\ &= \frac{F(\gamma_m)}{G'(\gamma_m)} \sum_{\lambda\in\Lambda} a_\lambda \frac{\|k_\lambda\|_{\mathscr{H}(\Gamma,v)}/\Phi'(\lambda)}{\gamma_m-\lambda} \\ &= \frac{F(\gamma_m)}{G'(\gamma_m)} \sum_{\lambda\in\Lambda} a_\lambda \frac{\sqrt{h_\lambda}}{\gamma_m-\lambda}. \end{aligned} \tag{3.3.4}$$

Now a similar argument made to prove Theorem 2.2.1 shows that the inequality

$$\sum_{m=1}^{\infty} |c_m|^2 v_m = \sum_{m=1}^{\infty} \frac{|F(\gamma_m)|^2}{v_m |G'(\gamma_m)|^2} \left| \sum_{\lambda\in\Lambda} a_\lambda \frac{\sqrt{h_\lambda}}{\gamma_m-\lambda} \right|^2 \lesssim \sum_{\lambda\in\Lambda} |a_\lambda|^2 \simeq \|f\|^2_{\mathscr{H}(\Gamma,v)}$$

holds for all sequences $(a_\lambda) \in \ell^2$ only if (3.3.1) and (3.3.2) hold.

To be able to apply Corollary 3.3.2, we need to have a full description of those complete and minimal sequences $S_R(\Lambda)$. Our next result states as follows.

**Theorem 3.3.3.** *Let $(\Gamma,v)$ be an admissible pair and $\Lambda \subset (\Gamma,v)^*$. Then $S_R(\Lambda)$ is complete and minimal in $\mathscr{H}(\Gamma,v)$ if and only if*

$$\limsup_{y\to\infty} \frac{|F(iy)|}{|G(iy)|y} = 0, \tag{3.3.5}$$

*and at least one of the following two conditions hold:*

$$\sum_{n=1}^{\infty} \frac{|F(\gamma_n)|^2}{v_n|G'(\gamma_n)|^2} = \infty, \tag{3.3.6}$$

$$\limsup_{y\to\infty} \frac{|F(iy)|}{|G(iy)|} > 0. \tag{3.3.7}$$

*Proof.* To prove the theorem we argue as follows. From exactness of the system we observe that $\Phi(z)/(z-\lambda_1)$ belongs to $\mathscr{H}(\Gamma,v)$ and so (3.3.5) holds. On the other hand, writing the partial fraction decomposition for $\Phi$,

$$\Phi(z) = \sum_{m=1}^{\infty} \frac{a_m v_m}{z-\gamma_m} + g(z)$$

with $g$ an entire function, we observe that either

$$\limsup_{y\to\infty}|g(iy)|>0$$

in which case (3.3.7) holds or $g=0$ in which case

$$a_m=\lim_{z\to\gamma_m}\frac{1}{v_m}(z-\gamma_m)\Phi(z)=\frac{F(\gamma_m)}{v_mG'(\gamma_m)}$$

is not square-summable. If not, $\Phi$ will live in $\mathscr{H}(\Gamma,v)$. Assuming the conditions of the theorem, we note that $\Phi\notin\mathscr{H}(\Gamma,v)$ by (3.3.6) and (3.3.7). On the other hand, (3.3.5) implies $\Phi/(z-\lambda_1)\in\mathscr{H}(\Gamma,v)$, which leads to the desired conclusion. □

We note that in the special case when the points $(\gamma_n)$ satisfy (2.2.1), we get a more geometrical sufficient condition when $\Lambda$ has a subsequence $\Lambda'$ satisfying the conditions in Lemma 2.4.6 or 2.4.7.

We will now clarify a point considered in the previous chapter, namely the relation between "super-thin" sequences and Riesz bases of normalized reproducing kernels in $\mathscr{H}(\Gamma,v)$. We begin by noting that if in addition the weight sequence $v$ has the property that

$$v_n=o(V_n)\quad\text{or}\quad v_n/|\gamma_n|^2=o(P_n)\tag{3.3.8}$$

when $n\to\infty$, then Theorem 2.4.10 and Theorem 2.4.12 give interesting geometric criteria for normalized reproducing kernel Riesz bases in $\mathscr{H}(\Gamma,v)$. The translation into this discrete setting of Theorem 2.2.7 is surprisingly subtle: The sequence $\Lambda$ splits naturally into three subsequences, one that should be viewed as a perturbation of $\Gamma$ and then two sequences satisfying only certain "extreme" sparseness conditions. Translating Lemma 2.4.5 to this setting along with (3.3.8) shows that, the points that generate normalized kernel Riesz bases are all from the discs $D_n(v;M)$. In other words, all Riesz bases appear as perturbations of the canonical basis associated with the sequence $(\gamma_n)$, and if $\Lambda$ is an $\Lambda^{(V)}$-lacunary or $\Lambda^{(P)}$-lacunary sequence with infinitely many points outside every set

$$\bigcup_{n=1}^{\infty}D_n(v;M),$$

then $\Lambda$ is not a subsequence of any $(\alpha_n)$ such that $S_R(\alpha_n)$ is a Riesz basis for $\mathscr{H}(\Gamma, v)$.

## 3.4 The Feichtinger conjecture

The Feichtinger conjecture claims that every bounded frame in a separable Hilbert space can be expressed as a finite union of Riesz basic sequences. In an interesting series of papers [21–24], it has been revealed that the conjecture is equivalent to a number of other long-standing problems including the Kadison–Singer problem first formulated by R. Kadison and I. Singer in [55].

The Kadison–Singer problem, which grew out of mathematical physics and quantum mechanics, was first stated in 1959 [5]. The attention around the problem slowed down especially from the mid 1960's until 1981 when J. Anderson [4] introduced the idea of paving and showed that the problem is equivalent to what is now known as the Paving conjecture. The paving idea generated a lot of interest and many authors including J. Bourgain and L. Tzafriri have published several papers on this topic. By 1991, ideas on paving had run out and the momentum around the problem again went down.

Another breakthrough came in 2006 when P. Casazza and J. Tremain [21] showed that the problem is equivalent to several unsolved problems in different areas of research in both pure and applied mathematics. We refer to the papers [21, 24] for all these historical accounts and the different reformulations on the various aspects of the problem.

Though a significant amount of effort has been invested in trying to solve these conjectures, the general problem remains yet to be solved[6]. When we return to the Feichtinger conjecture, we may refer to the recent paper [25] for a weaker version where it is proved that it suffices to make the decomposition into a finite union of frame sequences. We recall that a sequence of vectors $(f_n)$ is a frame sequence in a Hilbert space $H$ if it constitutes a frame for its closed linear span.

[5]Implicitly, it has been already contained in the 1937 P. M. Dirac's famous book on foundations of quantum mechanics.

[6]In September 2006, workshop on the Kadison–Singer problem was held at the AIM institute in Palo Alto, organized by P. Casazza, R. Kadison, and D. Larson. Part of the goal of the workshop was to initiate people to work together on the different version of the problem and to keep the subject alive until it gets resolved.

We are here interested in the version of the conjecture that involves unit norm Bessel functions which we state as:

**The Feichtinger conjecture:** *Every Bessel sequence of unit vectors in a separable Hilbert space can be expressed as a finite union of Riesz basic sequences.*

An interesting approach to the Feichtinger conjecture is to restrict attention to normalized reproducing kernels for so-called model subspaces of $H^2$. This special case does not appear to be much easier than the general one, owing to the profound richness of structure and variety of the class of model subspaces. The lack of general results on the geometry of Bessel sequences (which is a particular case of the Carleson measure problem) and Riesz bases is an obvious challenge when we address the Feichtinger conjecture in this setting. Bessel sequences from kernel functions are well understood for many classical spaces of functions for instance Hardy, Bergman and Fock spaces, and the validity of the conjecture follows from various known results about sampling and interpolation in these spaces. Model subspaces therefore constitute a natural object of study as far as our version of the Feichtinger conjecture is concerned. This view will be justified more in the next subsection.

It was recently shown by A. Baranov and K. Dyakonov [5] that the Feichtinger conjecture holds true for Bessel sequences of normalized reproducing kernels for $K_I^2$ when either $I$ is a one-component inner function or the points $\lambda_n$ satisfy

$$\sup_n |I(\lambda_n)| < 1. \tag{3.4.1}$$

In the latter case, the complete description of Riesz basic sequences from [50] plays an essential role in their argument[7]. A. Baranov and K. Dyakonov used their result for the case when (3.4.1) holds to treat the general case of one-component inner functions. Their approach was to split the half-plane into two regions, one in which $|I(z)|$ is bounded away from 1 and another in which a perturbation argument for Clark bases applies. In Subsection 2.2.2 we have already observed a situation where no splitting of this kind can be made. Indeed, we encountered examples of Bessel

[7]The geometry of normalized reproducing kernel Riesz basic sequences in $K_I^2$ is well understood when the associated sequence of points satisfy condition (3.4.1). The general case is also briefly considered in [50]. But no workable or explicit solution is obtained. For further information, we refer to [50] or the monograph by Seip [98], where a complete analysis can be found.

sequences of normalized reproducing kernels which cannot be associated with a perturbation of any Riesz basis. Our examples show that the methods of [5] can not be extended beyond the case of one-component inner functions. In the next subsection we will identify a collection of more model subspaces for which the problem can be completely understood. Our result complements the findings of A. Baranov and K. Dyakonov [5].

### 3.4.1 The Feichtinger conjecture in $\mathscr{H}(\Gamma, v)$

We now turn to the special case of normalized reproducing kernels for $\mathscr{H}(\Gamma, v)$. Given a sequence $\Lambda = (\lambda_j)$ in $(\Gamma, v)^*$, we associate with it the corresponding sequence of normalized reproducing kernels $S_R(\lambda_j)$ in $\mathscr{H}(\Gamma, v)$. We observe that if $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$, then the transformation $H_{(\Gamma,v);(\Lambda,w)}$ is bounded if and only if the system $S_R(\Lambda)$ is a Bessel sequence in $\mathscr{H}(\Gamma, v)$. Moreover, this transformation is both bounded and surjective if and only if the system $S_R(\Lambda)$ is a Riesz basic sequence in $\mathscr{H}(\Gamma, v)$. If, in addition, $\Gamma$ satisfies the sparseness condition (2.2.1), then it follows from Theorem 2.3.1 that the conjecture holds true for Bessel sequences of normalized reproducing kernels in $\mathscr{H}(\Gamma, v)$, and this result applies for all classes of spaces considered in the next chapter. This special case of the conjecture pertaining to discrete Hilbert transforms appears as an interesting setting in which the ramifications of the general Feichtinger conjecture could be explored.

Recently, S. Lata and V. Paulsen [59] obtained two more equivalences of the Feichtinger conjecture that involve only reproducing kernel Hilbert spaces, specifically for every space contractively[8] contained in the Hardy space $H^2$. The main point of [59] is that not only it suffices to verify the conjecture in such spaces but it interestingly reduces the question about general Bessel sequences to special class of functions which have more structure in our disposal. More specifically, they proved that the conjecture holds true if one can partition each Bessel sequence of normalized kernel functions in each contractively contained subspaces of $H^2$ into finitely many Riesz basic sequences.

[8] A subspace $\mathcal{H}_S$ of the Hardy space $H^2$ is contractively contained in $H^2$ if the inclusion map from $\mathcal{H}_S$ to $H^2$ is a contraction, i.e. $\|f\|_{H^2} \leq \|f\|_{\mathcal{H}_S}$ for every $f \in \mathcal{H}_S$.

# 4 Examples of spaces $\mathscr{H}(\Gamma, v)$

The prime examples of Hilbert spaces belonging to the general class described in Section 3.1 are found among so-called de Branges spaces and model subspaces of $H^2$. In this chapter we discuss how these fundamental spaces fit into our class of Hilbert space $\mathscr{H}(\Gamma, v)$. An interesting aspect of our approach is that it allows us to pay an implicit revisit to the characterization of the orthogonal bases of reproducing kernels introduced by L. de Branges and D. Clark. If a Hilbert space of complex-valued functions defined on a subset of $\mathbb{C}$ satisfies a few basic axioms and has more than one orthogonal bases of reproducing kernels, then it is shown that these bases are all of Clark's type. In other words, there are no other orthogonal bases of reproducing kernels than those already introduced and studied by L. de Branges [38] and D. Clark [29].

In the last part of the chapter we will give a negative answer to a question of A. Baranov about the relation between the growth of the phase function of $I$ at real points generating a Bessel sequence of normalized reproducing kernels in $K_I^2$.

## 4.1 de Branges spaces

To begin with, we note that de Branges spaces may be defined in terms of axioms that are very similar to those introduced above. Indeed, a Hilbert space $\mathscr{H}$ of entire functions which contains a nonzero element is called a de Branges space if it satisfies the following three axioms:

(H1) $\mathscr{H}$ has a reproducing kernel $k_\lambda$ at every point $\lambda$ in $\mathbb{C}$, i.e., the point evaluation functional $k_\lambda : f \to f(\lambda)$ is continuous in $\mathscr{H}$ for every $\lambda$ in $\mathbb{C}$.

(H2) If $f$ is in $\mathscr{H}$ and $f(\lambda) = 0$ for some point $\lambda$ in $\mathbb{C}$, then $f(z)(z-\bar{\lambda})/(z-\lambda)$ is in $\mathscr{H}$ and has the same norm as $f$.

(H3) The function $\overline{f(\bar{z})}$ belongs to $\mathscr{H}$ whenever $f$ belongs to $\mathscr{H}$, and it has the same norm as $f$.

The general reference for de Branges spaces is the book [38]. The leading example of a de Branges space is again the Paley–Wiener space $PW_\pi$.

A space $\mathscr{H}$ that satisfies (H1), (H2), (H3), will in particular satisfy (Ax1), (Ax2), (Ax3) with $\Omega = \mathbb{C}$. Indeed, we observe that then (H1) and (Ax1) coincide, and it is also plain that (H2) implies (Ax2). Indeed, if $\Lambda$ is an exact uniqueness set, then there exists a unique function $g_j$ such that $g_j(\lambda_m) = 1$ for $m = j$ and 0 otherwise. We fix $n_0$ and observe that

$$\frac{z-\overline{\lambda_n}}{z-\lambda_n} g_{n_0}(z) = g_{n_0}(z) + \frac{(\lambda_n - \overline{\lambda_n}) g_{n_0}(z)}{z-\lambda_n}.$$

Then by $(H2)$, it follows that $g_{n_0}(z)/(z-\lambda_n)$ belongs to $\mathscr{H}$. Thus, $G(z) = (z-\lambda_{n_0}) g_{n_0}(z)$ constitutes a generating function for $\Lambda$ because

$$f_n(z) = G(z)/(z-\lambda_n) = g_{n_0}(z) + (\lambda_n - \lambda_{n_0}) g_{n_0}(z)/(z-\lambda_{n_0})$$

also belongs to $\mathscr{H}$. We observe that if we choose $\lambda$ nonreal, by axiom $(H2)$ there exists a nonzero function $f$ in the space for which $\lambda$ is not included in its zero set. Then $k_\lambda$ is nonzero and the last part of axiom (Ax3) follows. One of the basic results in de Branges's theory is that a space that satisfies (H1), (H2), (H3), will have an orthogonal basis consisting of reproducing kernels $k_{\gamma_n}$ with $\Gamma = (\gamma_n)$ being a sequence of real points. Thus, in particular, (H1), (H2), (H3) imply that our third general axiom (Ax3) holds. We will recall this fundamental result below. In the case of the Paley–Wiener space, we have an orthogonal basis of reproducing kernels associated with the sequence of integers.

Another way of defining de Branges spaces is as follows. We say that an entire function $E$ belongs to the Hermite–Biehler (HB) class if it has no real zeros and satisfies

$$|E(z)| > |E(\bar{z})|, \quad z \in \mathbb{C}_+.$$

Each such function $E$ generates a space $H(E)$ consisting of all entire functions $f$ such that both $f/E$ and $f^*/E$ belong to the Hardy space $H^2$ where $f^*(z) = \overline{f(\bar{z})}$.

If we equip $H(E)$ with the standard inner product

$$\langle f,g\rangle_{H(E)} = \int_{-\infty}^{\infty} \frac{f(x)\overline{g(x)}}{|E(x)|^2}dx,$$

then it becomes a reproducing kernel Hilbert space with kernel function

$$k_\lambda(z) = \frac{i}{2\pi}\frac{E(z)\overline{E(\lambda)} - E(z)^*\overline{E(\lambda)^*}}{z-\overline{\lambda}} \tag{4.1.1}$$

at each point $\lambda$ in $\mathbb{C}$. In particular when $\lambda$ is real we have

$$\|k_\lambda\|^2_{H(E)} = \frac{1}{\pi}\varphi'(\lambda)|E(\lambda)|^2 \tag{4.1.2}$$

where $\varphi$ refers to the phase function of $E$, i.e. a continuous function in $\mathbb{R}$ such that $E(t)e^{i\varphi(t)}$ is real for each $t$. The point of interest to us is that $H(E)$ is in addition a de Branges space, and the following basic result of de Branges gives that every de Branges space can be obtained in this way via a function $E$ in the Hermite–Biehler class (cf. [38], p. 57). We arrive at the Paley–Wiener space by setting $E(z) = e^{-i\pi z}$.

**Theorem 4.1.1.** *A Hilbert space $\mathscr{H}$ of entire functions which contains a non zero element, and satisfies the axioms (H1), (H2), and (H3) is equal isometrically to some space $H(E)$.*

We shall now state one of the fundamental results in de Branges spaces concerning the existence of orthogonal bases of reproducing kernels associated to sequence of points on the real line (cf. [38], p. 55).

**Theorem 4.1.2.** *Let E be an HB class function with an associated phase function $\varphi$ such that $\varphi(\gamma_n) = \alpha + n\pi, n \in \mathbb{Z}, \alpha, \gamma_n \in \mathbb{R}$. If $e^{i\alpha}E - e^{-i\alpha}E^* \notin H(E)$, then $S_R(\gamma_n)$ constitutes an orthonormal basis for $H(E)$, and also the property $e^{i\alpha}E - e^{-i\alpha}E^* \in H(E)$ holds for at most one $\alpha$ modulo $\pi$.*

It follows from the preceding remarks that all the results from the previous chapter and sections apply to de Branges spaces with orthogonal bases of reproducing kernels located at a sequence of nonzero real points $\gamma_n$ such that

$$\inf_n |\gamma_{n+1}|/|\gamma_n| > 1.$$

## 4.2 Model subspaces of $H^2$

We now turn to the model subspaces of the Hardy space $H^2$. The elements of $K_I^2$ (originally defined in $\mathbb{C}_+$) have meromorphic extensions into $\mathbb{C}$ if the function $I$ has such an extension[1]. In this case, we have the relation $I = E^*/E$ and the function $E$ in the $HB$ class is unique up to an entire function with no zeros on both the upper and the lower half-planes and real valued on the real line.

Clearly, the map $f \mapsto f/E$ is unitary from $H(E)$ to $K_I^2$. Thus de Branges spaces can be viewed as a subclass of the collection of all model subspaces of $H^2$.

It is a well established fact that all model subspaces satisfy axiom $(Ax1)$ from Section 3.1. We now prove that every model subspace satisfies also axiom (Ax2). This is obvious if we consider $K_I^2$ as a space of functions on the upper half-plane, but for our purposes it is essential that we also include those points on the real line at which point evaluation makes sense. We will need the fact that the reproducing kernel for $K_I^2$ at some point $\zeta$ in the upper half-plane is

$$k_\zeta(z) = \frac{i}{2\pi} \cdot \frac{1-\overline{I(\zeta)}I(z)}{z-\overline{\zeta}}.$$

This formula extends to each point on the real line at which every function in $K_I^2$ has a nontangential limit whose modulus is bounded by a constant times the $H^2$ norm of the function. This immediately holds if for instance $I$ is a meromorphic inner functions. For a general $I$, a paper of P. Ahern and D. Clark [1] gives that these are exactly the points $\zeta$ at which $I$ has an angular derivative, i.e., at which both $I$ and $I'$ have non-tangential limits and $|I(\zeta)| = 1$. In other words, for a real point $\zeta$ :

$$k_\zeta \in K_I^2 \Leftrightarrow |I'(\zeta)| = a + \sum_n \frac{2\Im z_n}{|\zeta - z_n|^2} + \int_{\mathbb{R}} \frac{d\psi(t)}{(t-\zeta)^2} < \infty, \qquad (4.2.1)$$

where $(z_n)$ constitutes the zeros of the Blaschke factor in the factorization

$$I(z) = \gamma \exp(iaz) B(z) I_\psi(z).$$

The same conclusion also follows from a more general result due to W. Cohn [33]:

[1] An inner function is meromorphic if it accumulates at infinity. Each such function $I$ is described by an $HB$ class function $E$ such that $I = E^*/E$. Details can be read in [47] where a proof is given.

for $1 < p < \infty$, the kernel function $k_\zeta$ belongs to the Hardy space $H^p$ if and only if (4.2.1) holds when 2 is replaced by $p$.

The singular inner function $I_\psi$ is defined by

$$I_\psi(z) = \exp\left(i\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)d\psi(t)\right).$$

In this case we always have

$$\|k_\zeta\|^2_{K_I^2} = \frac{|I'(\zeta)|}{2\pi}. \tag{4.2.2}$$

**Lemma 4.2.1.** *The Hilbert space $K_I^2$, viewed as a space of functions on the set*

$$\Omega = \{z = x+iy :\ y \geq 0 \textit{ and } f \mapsto f(z) \textit{ is bounded}\},$$

*satisfies axiom (Ax2) of Section 3.1.*

To make the proof more transparent, we single out the main technical ingredient as a separate lemma.

**Lemma 4.2.2.** *If $x_0$ is a point on the real line at which the point evaluation functional for $K_I^2$ is bounded, then*

$$\lim_{y\to 0}\|k_{x_0+iy} - k_{x_0}\|_{H^2} = 0.$$

*Proof.* Assuming $I(x_0) = 1$, we may write

$$\frac{2\pi}{i}(k_{x_0+iy}(t) - k_{x_0}(t)) = \frac{1-\overline{I(x_0+iy)}I(t)}{t-\overline{(x_0+iy)}} - \frac{1-I(t)}{t-x_0}.$$

The right-hand difference can be further rearranged into

$$\frac{(1-\overline{I(x_0+iy)})I(t)}{t-\overline{(x_0+iy)}} - \frac{(1-I(t))iyt}{(t-x_0)(t-\overline{(x_0+iy)})}.$$

Here the first term has $H^2$ norm bounded by a constant times $y^{\frac{1}{2}}$ in view of the theorem of Ahern and Clark [1], while the $H^2$ norm of the second term tends to 0 when $y \to 0$, by Lebesgue's dominated convergence theorem. □

*Proof of Lemma 4.2.1.* Let $\Lambda$ be an exact uniqueness set for $K_I^2$ consisting of points in $\Omega$. We will let $g_j$ denote the unique function in $K_I^2$ such that $g_j(\lambda_l)$ equals 0 when $l \neq j$ and 1 for $l = j$. We can choose an arbitrary point in $\Lambda$, say $\lambda_1$, and choose $G(z) = (z-\lambda_1)g_1(z)$ as our candidate for a generating function. It is plain that if $\lambda_j$ is a point in the open half-plane, then

$$g_j(z) = \frac{G(z)}{G'(\lambda_j)(z-\lambda_j)}.$$

The difficulty occurs if $\lambda_j$ is a point on the real line. In this case, if we replace $\lambda_j$ by $\lambda_j + i\varepsilon$, then the modified sequence $\Lambda^{(\varepsilon)}$ will still be an exact uniqueness sequence for $K_I^2$ with $\varepsilon$ sufficiently small. In fact, by Lemma 4.2.2, the function $g_1$ vanishing on $\Lambda^{(\varepsilon)} \setminus \{\lambda_1\}$ will vary continuously with $\varepsilon$. Thus the corresponding generating function $G_\varepsilon(z)$ will tend to $G(z)$ for every point in the upper half-plane when $\varepsilon \to 0$. On the other hand, another application of Lemma 4.2.2 gives that

$$\frac{G_\varepsilon(z)}{G_\varepsilon'(\lambda_j+i\varepsilon)(z-\lambda_j+i\varepsilon)} \to g_j(z)$$

in $K_I^2$ when $\varepsilon \to 0$. Lemma 4.2.2 also gives that $G_\varepsilon'(\lambda_j+i\varepsilon)$ converges to a finite number, say $1/\alpha$, and we may therefore conclude that

$$g_j(z) = \alpha\frac{G(z)}{z-\lambda_j}.$$

□

As for axiom (Ax3), it remains an open problem, posed by N. Nikol‘skiĭ ( cf. [76], p. 210), to decide whether every model subspace $K_I^2$ has a Riesz basis of normalized reproducing kernels. Thus it is not known whether the class of spaces introduced in Section 3.1 includes all model subspaces. However, there exists an interesting class of model subspaces that actually possess orthogonal bases of reproducing kernels associated with sequences of real points. Such bases, to be discussed briefly below, are called Clark bases [29]. We also note that if the inner function $I$ happens to be an interpolating Blaschke product, then it is immediate that $K_I^2$ has a Riesz basis of normalized reproducing kernels associated with the sequence of zeros of $I$.

The spaces $K_I^2$ that possess Clark bases, correspond precisely to those spaces

$\mathscr{H}(\Gamma,\nu)$ for which $\Gamma$ is a real sequence. To get from $\mathscr{H}(\Gamma,\nu)$ to the corresponding space $K_I^2$, we construct the Herglotz function

$$\varphi(z) = \sum_{n=1}^{\infty} \nu_n \left( \frac{1}{\gamma_n - z} - \frac{\gamma_n}{1+\gamma_n^2} \right). \tag{4.2.3}$$

Then

$$I(z) = \frac{\varphi(z) - i}{\varphi(z) + i} \tag{4.2.4}$$

will be an inner function in the upper half-plane with

$$\Gamma = \{t \in \mathbb{R} : I(t) = 1\}, \quad \text{and} \quad |I'(\gamma_n)| = 2/\nu_n.$$

Then the map $f \mapsto (1-I)f$ will be a unitary map from $\mathscr{H}(\Gamma,\nu)$ to $K_I^2$; it is implicit in this construction that in fact every function in $K_I^2$ has a non-tangential limit at each point $\gamma_n$ and also that the corresponding point evaluation functional is bounded at $\gamma_n$. Note that in this case

$$\mu = \sum_{n=1}^{\infty} \nu_n \delta_{\gamma_n},$$

where $\delta_t$ denotes the Dirac measure at the point $t$ is the Clark measure for the function $I$.

Similarly, if $\Gamma$ is the zero sequence of an interpolating Blaschke product $B$ in the upper half-plane, then we may set $\nu_n \simeq \Im \gamma_n$ and $\overline{\Gamma} = (\overline{\gamma_n})$. Then the map $f \mapsto 2\sqrt{\pi} f$ will be a unitary map from $\mathscr{H}(\overline{\Gamma},\nu)$ to $K_B^2$.

Along with the above question, N. Nikol‘skiĭ ( cf. [76], p. 210) has also raised the question to decide the class of reproducing kernel Riesz basic sequences in $K_I^2$ which could be extended to a reproducing kernel Riesz basis into the whole space. From the results in the previous chapter, we observe that not all infinite subsequences can be extended to a Riesz basis into the whole space $K_I^2$.

We conclude that our general discussion applies to model subspaces $K_I^2$ that possess Clark bases or when $I$ is an interpolating Blaschke product in the upper half-plane.

Since positive results on Carleson measures are scarce, we mention without proof the following observation: A suitable adaption of Theorem 2.2.1 gives a description of any Carleson measure $\mu$ restricted to a cone

$$\Gamma_{x_0} = \left\{z = x + iy : \ |z - x_0| < Cy\right\};$$

here $x_0$ is an arbitrary real point and $C$ a positive constant. To arrive at this result, one may represent the space by means of its Clark basis or more generally as an $L^2$ space with respect to a Clark measure [29], and act similarly as in Subsection 2.2.1.

By the observation made at the end of Section 3.3, the problem of describing all Riesz bases of normalized reproducing kernels for $K_I^2$ is part of the problem of deciding when discrete Hilbert transforms $H_{(\Gamma,v);(\Lambda,w)}$ are bounded. The most far-reaching result known about such bases is that found in [50] dealing with the case when (3.4.1) holds. The general result in [50] for this particular case leads to a description of all Riesz bases of normalized reproducing kernels for the Paley–Wiener space and also for a wider class of de Branges spaces known as weighted Paley–Wiener spaces [62]. As pointed out in Subsection 1.3.2, one of the main points of [50] is that when (3.4.1) holds, one can transform the problem into a question about invertibility of Toeplitz operators and then apply the Devinatz–Widom theorem. Another approach, closer in spirit to the present work, can be found in [61], where the Riesz basis problem is explicitly related to a boundedness problem for Hilbert transforms.

## 4.3 Fock-type spaces

It may be noted that our work gives a full description of the Carleson measures and the Riesz bases of normalized reproducing kernels for certain Fock-type spaces studied recently by A. Borichev and Y. Lyubarskii [17]. The spaces $\mathcal{F}_\varphi$ considered by these authors consist of all entire functions $f$ such that

$$\|f\|_\varphi^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(|z|)} dm(z) < \infty,$$

where $\varphi$ is a positive, increasing, and unbounded function on $[0, \infty)$ and $m$ denotes Lebesgue area measure on $\mathbb{C}$. The main point of [17] is that if $\varphi$ grows “at most as fast” as $[\log(1+r)]^2$, then the corresponding space $\mathcal{F}_\varphi$ has a Riesz basis of reproducing kernels and, conversely, if the growth of $\varphi$ is “faster” than $[\log(1+r)]^2$, then no such basis exists. It is proved that when $\varphi(r) = [\log(1+r)]^2$, we can choose such a basis associated with a sequence $\Gamma = (\gamma_n)$ satisfying $|\gamma_n| = e^{n/2}$; if $\varphi(r) = [\log(r+1)]^\alpha$ with $1 < \alpha < 2$, then the growth of $|\gamma_n|$ will be super-exponential.

We note that the study of such bases for Fock-type spaces began with the results of Seip [96] which shows that the classical Fock space, $\varphi(x) = x^2$, contains no basis of reproducing kernels. When $\varphi$ grows faster than this, similar result was obtained in [18]. In the case when $\varphi(x) = x - \frac{3}{2}\log x$, the absence of such basis was established in [53]. On the other hand, when the growth of $\varphi$ behaves like the logarithmic function, $\varphi(x) \simeq \log x$, then the space $\mathcal{F}_\varphi$ becomes finite dimensional and obviously contains such bases. Thus the result of A. Borichev and Y. Lyubarskii is meant to address the remaining gap when $\varphi$ grows more slowly than $x^2$ but more rapidly than $c\log x$, in which case $\mathcal{F}_\varphi$ has infinite dimension.

In view of the discussions made in the previous chapter and Section 4.1, the results of Borichev and Lyubarskii clarify when a Fock-type space equals a de Branges space, i.e., the two spaces consist of the same entire functions and have equivalent norms. It would be of interest to find a direct proof of this equality; transforming the area integral in Fock-space into a line integral in de Branges space. Indeed, we conjecture that for each $f$ in $\mathcal{F}_\varphi$,

$$\|f\|_\varphi^2 \simeq \int_0^\infty \left| \sum_{n=0}^\infty a_n x^{n+1/2} e^{-\varphi(x)} \right|^2 dx =: I_f \tag{4.3.1}$$

holds. We will verify $I_f \lesssim \|f\|_\varphi^2$ in what follows. Thus the problem is, in fact, to show that the other estimate $I_f \gtrsim \|f\|_\varphi^2$.

Since

$$\lim_{r\to\infty} \exp(n\log r - \varphi(r)) = 0$$

for all $n$, the spaces $\mathcal{F}_\varphi$ contain all the polynomials, and in particular $(z^n), n \geq 0$ constitutes an orthogonal basis for $\mathcal{F}_\varphi$. Each $f$ in $\mathcal{F}_\varphi$ has a series expansion

$$f(z) = \sum_{n=0}^\infty a_n z^n$$

and its norm can be estimated as

$$\begin{aligned}
\|f\|_\varphi^2 &= \int_{\mathbb{C}} \left| \sum_{n=0}^{\infty} a_n z^n \right|^2 e^{-2\varphi(|z|)} dm(z) \\
&= \int_{\mathbb{C}} \left| \sum_{n=0}^{\infty} a_n z^n \right|^2 e^{-2\log(1+|z|)^\alpha} dm(z) \\
&\simeq \sum_{n=0}^{\infty} |a_n|^2 \int_0^\infty e^{(2n+1)\log r - 2\varphi(r)} dr \\
&\simeq \sum_{n=0}^{\infty} |a_n|^2 \exp\left(2(\alpha-1)\Big(\frac{n+1}{\alpha}\Big)^{\alpha/(\alpha-1)}\right). \qquad (4.3.2)
\end{aligned}$$

To simplify the writing, we set

$$\eta_n = (\alpha-1)\Big(\frac{n+1}{\alpha}\Big)^{\alpha/(\alpha-1)}$$

and observe that the sequence of functions

$$\exp\big((2n+1)\log r - 2\varphi(r)\big)$$

attain the extremum values at the points $\eta_n$. Applying the substitution $t = \log r$, we write

$$\begin{aligned}
I_f &\simeq \int_{-\infty}^{\infty} \left| \sum_{n=0}^{\infty} a_n e^{(n+1)t - t^\alpha} \right|^2 dt \\
&\leq \int_{-\infty}^{\infty} \left( \sum_{n=0}^{\infty} |a_n| e^{\eta_n} \, e^{-\left(\eta_n + t^\alpha - (n+1)t\right)} \right)^2 dt. \qquad (4.3.3)
\end{aligned}$$

We set $J_m = [\frac{m}{2} - \frac{1}{4},\ \frac{m}{2} + \frac{1}{4}]$ and observe that $t \simeq m/2$ whenever $t$ belongs to $J_m$. We estimate the integral when $t \geq -1/4$. The remaining piece can be essentially handled in the same manner. Applying the Cauchy–Schwarz inequality, we have that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \int_{J_m} \left( \sum_{n=0}^{m} |a_n| e^{\eta_n} \, e^{-\left(\eta_n + t^\alpha - (n+1)t\right)} \right)^2 dt \\
&\quad \lesssim \sum_{m=0}^{\infty} \sum_{n=0}^{m} |a_n|^2 e^{2\eta_n} \, e^{-\left(\eta_n + (m/2)^\alpha - \frac{(n+1)m}{2}\right)} \qquad (4.3.4)
\end{aligned}$$

since

$$\sum_{n=0}^{\infty} e^{-\left(\eta_n+(m/2)^{\alpha}-\frac{(n+1)m}{2}\right)}$$

is uniformly bounded by an absolute constant. By interchanging the order of summation, we find that the left-hand double sum in (4.3.4) is also bounded by $\|f\|_{\varphi}^2$.

Similarly, by Cauchy–Schwarz, it follows that

$$\begin{aligned}\sum_{m=0}^{\infty}\int_{J_m}\left(\sum_{n=m+1}^{\infty}|a_n|e^{\eta_n}\, e^{-\left(\eta_n+t^{\alpha}-(n+1)t\right)}\right)^2 dt\\ \lesssim \sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}|a_n|^2 e^{2\eta_n}\, e^{-\left(\eta_n+(m/2)^{\alpha}-\frac{(n+1)m}{2}\right)}\\ = \sum_{n=0}^{\infty}|a_n|^2 e^{2\eta_n}\sum_{m=0}^{n} e^{-\left(\eta_n+(m/2)^{\alpha}-\frac{(n+1)m}{2}\right)}\\ \lesssim \|f\|_{\varphi}^2.\end{aligned} \tag{4.3.5}$$

We will in what follows make some computations to simplify further our results for the space $\mathcal{F}_{\varphi}$ for all radial weight functions $\varphi(z)=(\log(1+|z|))^{\alpha}$ with $1<\alpha\leq 2$.

**Case 1:** $\varphi(r)=\big(\log(1+r)\big)^2$

For this case, it has been proved that the normalized reproducing kernels associated with the sequence $(\gamma_n)=(\exp(n/2+i\theta_n))$ constitutes a Riesz basis for each real sequence $(\theta_n)$. We arrive at the de Branges space when we in particular set $\theta_n=0$ for each $n$. To proceed further, we need the following lemma from [17].

**Lemma 4.3.1.** *Let $\Gamma=(\gamma_n)$. Then the following holds.*

*(i) For each point $z$ in $\mathbb{C}$, the estimate*

$$\|k_z\|_{\varphi}^2 \simeq \frac{e^{2\varphi(|z|)}}{1+|z|^2} \tag{4.3.6}$$

*holds.*

*(ii) The product*

$$G(z)=\prod_{\gamma_n\in\Gamma}\left(1-\frac{z}{\gamma_n}\right)$$

*converges uniformly on compact sets in $\mathbb{C}$ and satisfies the estimate*

$$|G(z)| \simeq \frac{e^{\varphi(|z|)}\operatorname{dist}(z,\Gamma)}{|z|^{3/2}} \tag{4.3.7}$$

*for each $z \in \mathbb{C}$.*

As in (4.3.7), we notice that the estimate

$$|G'(\gamma_n)| \simeq |\gamma_n|^{-3/2} e^{\varphi(|\gamma_n|)} \simeq \exp\left((n^2-3n)/4\right) \tag{4.3.8}$$

also holds. Setting the corresponding weight sequence

$$v_n = \frac{\|k_{\gamma_n}\|_\varphi^2}{|G'(\gamma_n)|^2+1} \simeq |\gamma_n|, \tag{4.3.9}$$

we find that Corollary 2.2.2 immediately gives the Carleson measures for $\mathcal{F}_\varphi$, namely that; a nonnegative measure $\mu$ on $\mathbb{C}$ with $\mu(\Gamma) = 0$ is a Carleson measure for $\mathcal{F}_\varphi$ if and only if

$$\sup_{n\geq 1} \int_{\Omega_n} \frac{|\gamma_n| d\mu(z)}{|z-\gamma_n|^2} < \infty.$$

Because of the sparseness condition on both sequences $(\gamma_n)$ and $(v_n)$, the norm of the kernel functions in $\mathscr{H}(\Gamma, v)$ can be easily estimated. That is if $\lambda_j$ belongs to $\Omega_j$, then

$$\begin{aligned} \|k_{\lambda_j}\|^2_{\mathscr{H}(\Gamma,v)} = \sum_{n=1}^{\infty} \frac{v_n}{|\lambda_j-\gamma_n|^2} &\simeq \frac{v_{j-1}}{|\lambda_j|^2} + \frac{v_j}{\operatorname{dist}^2(\lambda_j,\Gamma)} + \frac{v_{j+1}}{|\gamma_{j+1}|^2} \\ &\simeq \frac{|\gamma_j|}{\operatorname{dist}^2(\lambda_j,\Gamma)}. \end{aligned} \tag{4.3.10}$$

From this, (4.3.7), (4.3.8), and (4.3.9), we observe that the basis conditions (3.3.1) simplifies to

$$\sup_{n\geq 1} \sup_{\lambda_j \in \Omega_n} \frac{|F(\gamma_n)|}{|\gamma_n - \lambda_j||F'(\lambda_j)|} < \infty, \tag{4.3.11}$$

while (3.3.2) becomes

$$\sup_{n\geq 1} \left( H_n \sum_{m=n}^{\infty} \frac{|F(\gamma_m)|^2}{\exp\left(m^2/2\right)} + W_n \sum_{m=1}^{n} \frac{|F(\gamma_m)|^2}{\exp\left((m^2-2m)/2\right)} \right) < \infty.$$

**Case 2:** $\varphi(r) = \big(\log(1+r)\big)^{1+\delta}, \quad 0<\delta<1$

We denote $\alpha_n = \log \|z^n\|_\varphi^2, n \geq 0$, and $r_0 = 0$ and $r_n = \exp\big((\alpha_{n+1} - \alpha_{n-1})/4\big)$ for $n \geq 1$. Then one of the main results from [17] ensures that the sequence of normalized reproducing kernels associated with the points $(r_n e^{i\theta_n})$ forms a Riesz basis for each real sequence $(\theta_n)$ again. As before setting $\theta_n = 0$ for each $n$ leads to the de Branges spaces. The reproducing kernel of $\mathcal{F}_\varphi$ at a point $\lambda$ is

$$k_\lambda(\zeta) = \sum_{n=0}^{\infty} \overline{\lambda}^n \frac{\zeta^n}{\|z^n\|_\varphi^2} = \sum_{n=0}^{\infty} \overline{\lambda}^n \zeta^n e^{-\alpha_n}$$

and hence

$$\|k_\lambda\|_{\mathcal{F}_\varphi}^2 = \sum_{n=0}^{\infty} |\lambda|^{2n} e^{-\alpha_n}. \tag{4.3.12}$$

To compute the series, we need to describe the growth of the sequence $\alpha_m$. We have

$$\begin{aligned} \|z^m\|_{\mathcal{F}_\varphi}^2 &= \int_{\mathbb{C}} |z|^{2m} \exp\big(-2\varphi(|z|)\big) dm(z) \\ &= 2\pi \int_0^\infty r^{2m} \exp\big(-2(\log^+ r)^{1+\delta}\big) r dr \\ &\simeq \int_0^\infty \exp\big((2m+2)t - 2t^{1+\delta}\big) dt. \end{aligned}$$

It suffices to describe the asymptotic behavior of the last integral when $m \to \infty$. Invoking Saddle point approximation, we obtain

$$\int_0^\infty \exp\big((2m+2)t - 2t^{1+\delta}\big) dt \simeq m^{\frac{1-\delta}{\delta}} \exp\Big(2\delta(1+\delta)^{-\frac{\delta+1}{\delta}} (m+1)^{\frac{\delta+1}{\delta}}\Big),$$

which shows that the sequence $(\alpha_m)$ has a polynomial growth faster than second degree. From this along with (4.3.12) we observe that for each $\gamma_j = r_j e^{i\theta_j}$ we have

$$\|k_{\gamma_j}\|_{\mathcal{F}_\varphi}^2 \simeq \exp\Big(\frac{\alpha_{j+1} - \alpha_{j-1}}{2}\Big) = |\gamma_j|^2. \tag{4.3.13}$$

We need to estimate the weight sequence $v_m$. For each point $z \in \mathbb{C} \cap \Omega_m$, we first compute

$$|G(z)| \simeq \frac{|z|^{m-1} \operatorname{dist}(z, \Gamma)}{|\gamma_m|} \prod_{n=1}^{m-1} |\gamma_n|^{-1},$$

and so

$$\begin{aligned}\log|G(z)| &\simeq (m-1)\log|z| - \log|\gamma_m| + \log\operatorname{dist}(z,\Gamma) - \sum_{n=1}^{m-1}(\alpha_{n+1}-\alpha_{n-1})/4 \\ &\simeq (m-1)\log|z| - \log|\gamma_m| + \log\operatorname{dist}(z,\Gamma) - m\log|\gamma_{m-1}|.\end{aligned}$$

This implies that

$$|G(z)| \simeq \frac{|z|^{m-1}\operatorname{dist}(z,\Gamma)}{|\gamma_m||\gamma_{m-1}|^m}, \tag{4.3.14}$$

and in particular, for points in $\Gamma$ we find

$$|G'(\gamma_m)| \simeq \frac{|\gamma_m|^{m-2}}{|\gamma_{m-1}|^m} \tag{4.3.15}$$

from which we get the weight sequence

$$v_m \simeq \frac{\|k_{\gamma_m}\|^2_{\mathcal{F}_\varphi}}{|G'(\gamma_m)|^2+1} \simeq \frac{|\gamma_{m-1}|^{2m}}{|\gamma_m|^{2m-6}}. \tag{4.3.16}$$

Since it again suffices to consider the asymptotic behavior of the $\alpha_m's$ when $m$ goes to infinity, we observe that $v_m$ has a "super-exponential" decay. Thus the Carleson measures for $\mathcal{F}_\varphi$ follows from Corollary 2.2.3. That is, a non-negative measure $\mu$ on $\mathbb{C}$ with $\mu(\Gamma)=0$ is a Carleson measure for $\mathcal{F}_\varphi$ if and only if

$$\sup_{n\geq 1}\int_{\mathbb{C}} \frac{v_n d\mu(z)}{|z-\gamma_n|^2} < \infty. \tag{4.3.17}$$

A simple computation gives the estimate

$$\|k_{\lambda_j}\|^2_{\mathscr{H}(\Gamma,v)} \simeq \frac{v_j}{\operatorname{dist}^2(\lambda_j,\Gamma)}.$$

This is deducible along the lines of (4.3.10). It remains to apply this together with the estimates (4.3.14), (4.3.16) and (4.3.15), and observe that condition (3.3.1) simplifies to

$$\sup_{n\geq 1}\sup_{\lambda_j\in\Omega_n} \frac{|\lambda_j\gamma_n^{-1}|^{n-1}|F(\gamma_n)|}{|\gamma_n-\lambda_j||F'(\lambda_j)|} < \infty.$$

On the other hand, applying the relation $v_m|G'(\gamma_m)|^2 \simeq \|k_{\gamma_m}\|^2_{\mathcal{F}_\varphi}$, condition

(3.3.2) for this special setting also reduces to

$$\sup_{n=1}^{\infty}\left(H_n\sum_{m\geq n}\frac{|F(\gamma_m)|^2}{|\gamma_m|^4}+W_n\sum_{m=1}^{n}\frac{|F(\gamma_m)|^2}{|\gamma_m|^2}\right)<\infty.$$

## 4.4 Orthogonal bases of reproducing kernels

We wish to describe those spaces $\mathscr{H}$ which admit orthogonal bases of reproducing kernels. We note that this family of spaces is part of the much larger family of spaces $\mathscr{H}$ that admits Riesz bases of normalized reproducing kernels. Since each space of the latter kind can be equipped with an equivalent norm such that one of the Riesz bases becomes an orthonormal basis (cf. [106], p. 33), the question of interest is when a space $\mathscr{H}$ has more than one orthogonal basis of reproducing kernels. We note that if $\Lambda=(\lambda_j)$ is a sequence in $(\Gamma,v)^*$ associated with a weight sequence $w=(w_j)$, where

$$w_j=\|k_{\lambda_j}\|^{-2}_{\mathscr{H}(\Gamma,v)}=\left(\sum_n\frac{v_n}{|\lambda_j-\gamma_n|^2}\right)^{-1},$$

then $S_R(\Lambda)$ is an orthonormal basis for $\mathscr{H}(\Gamma,v)$ if and only if $H_{(\Gamma,v):(\Lambda,w)}:\ell^2_v\to\ell^2_w$ is a unitary transformation. Thus from the two Subsections 2.1.1 and 2.1.2 we conclude:

If the space $\mathscr{H}(\Gamma,v)$ has an orthogonal bases of reproducing kernels, then $\Gamma$ is a subset of a straight line or a circle. Moreover, when $\Gamma$ is a subset of the real line, the orthogonal bases of reproducing kernels for $\mathscr{H}(\Gamma,v)$ are obtained from the unitary transformations described by Theorem 2.1.3; an analogous result holds when $\Gamma$ is a subset of the unit circle.

## 4.5 Relation to Clark's Bases

We are now finally prepared to point out the correspondence between our description of unitary discrete Hilbert transforms and the orthogonal bases of reproducing kernels studied by de Branges [38] and Clark [29]. We restrict to Clark's bases; the only difference between the two cases is that Clark considered the case of the unit circle while de Branges worked on the real line with, in our terminology, $|\gamma_n|\to\infty$. Said differently, de Branges studies with the class of meromorphic inner functions while Clark treated the general case. The result of de Branges has already been placed in context in Section 4.1.

Suppose $\varphi$ is of the form (2.1.10) with $\Gamma = (\gamma_n)$ a sequence of distinct points on the unit circle. Then the function

$$I(z) = \frac{\varphi(z) - i}{\varphi(z) + i}$$

is an inner function in the open unit disk $\mathbb{D}$. We associate with $I$ the model subspace $K_I^2$ of the Hardy space $H^2$ of the unit disk. Since $1/(1 - \overline{\zeta} z)$ is the reproducing kernel for $H^2$ at a point $\zeta$ in $\mathbb{D}$, the reproducing kernel for $K_I^2$ at the same point $\zeta$ is

$$k_\zeta(z) = \frac{1 - \overline{I(\zeta)} I(z)}{1 - \overline{\zeta} z}.$$

This formula extends to each point on the unit circle at which every function in $K_I^2$ has a radial limit whose modulus is bounded by a constant times the $H^2$ norm of the function.

A computation shows that

$$i \frac{1 + I(z)}{1 - I(z)} = \varphi(z)$$

which according to Clark's theorem means that the reproducing kernels

$$k_{\gamma_n}(z) = \frac{1 - I(z)}{1 - \overline{\gamma_n} z}$$

constitute an orthogonal basis for $K_I^2$. In fact, Clark's theorem says that if $\beta$ is a point on the unit circle and the spectral measure of the Herglotz function

$$\varphi_\beta(z) = i \frac{\beta + I(z)}{\beta - I(z)}$$

is purely atomic, then the reproducing kernels associated with the spectrum of $\varphi_\beta$ also constitute an orthogonal basis for $K_I^2$. The spectral measures generated in this way correspond precisely to the spectral measures of the functions

$$\frac{1}{\alpha - \varphi(z)}$$

with $\alpha$ any real number.

Having observed this correspondence, we conclude that a Hilbert space $\mathscr{H}$ of the type considered in the previous chapter can have more than one orthogonal basis of reproducing kernels only if $\mathscr{H}$ is, up to trivial modifications, a model space $K_I^2$ either in the unit disk or in the upper half-plane. In other words, there are no other orthogonal bases of reproducing kernels than those already introduced and studied by L. de Branges [38] and D. Clark [29].

An additional wonder, which can be seen from Clark's theorem or indeed by a straightforward computation, is that the norm in $\mathscr{H}$ can always be computed as an $L^2$ integral over a circle or a straight line.

## 4.6 Baranov's Separation Problem

A classical theorem of Plancherel–Pólya (cf. [60]) states that for real sequences $(x_j)$ such that

$$\inf_{m\neq j} |x_j - x_m| > 0,$$

the inequality

$$\sum_j |f(x_j)|^2 \lesssim \|f\|^2_{L^2(\mathbb{R})} \tag{4.6.1}$$

holds for all entire functions $f$ of exponential type say $\omega$ whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. As explained in Section 1.3, the inner function which generates the Paley–Wiener space here is $I(z) = \exp(i\omega z)$. The above separation condition can be equivalently stated as

$$\inf_{m\neq j} |\psi(x_j) - \psi(x_m)| > 0$$

where $\psi(t) = \omega t$ for each real $t$ is the continuous branch of the argument of $I$, i. e. $I(t) = \exp(i\psi(t))$ for each $t \in \mathbb{R}$.

In what follows we discuss the analogue of Plancherel–Pólya's result in the class of model spaces generated by meromorphic inner functions. Let $I$ be a meromorphic inner function and $\varphi$ be a continuous branch of its argument. It holds that

$$2\pi\|k_t\|^2 = |I'(t)| = \varphi'(t)$$

which follows from (4.2.2). We may then likewise consider a sequence of real points $(t_n)$ satisfying the separation condition

$$\inf_n \Big(\varphi(t_{n+1}) - \varphi(t_n)\Big) > 0. \tag{4.6.2}$$

For one-component inner functions $I$, it is known that the Plancherel–Pólya type inequality

$$\sum_n \frac{|f(t_n)|^2}{\varphi'(t_n)} \lesssim \|f\|_{K_I^2}^2 \tag{4.6.3}$$

holds for all functions $f$ in $K_I^2$ whenever the points $(t_n)$ satisfy (4.6.2). This is no longer true if the inner function is not one-component. Counterexamples can be found in [10]. One can also construct other examples by simply extending any sequence of points that gives rise to a $V$-lacunary sequence in Theorem 2.2.7.

We stress that the problems of deciding the Carleson measures and reproducing kernel Riesz bases in model subspaces rely on the geometry of the generating inner functions. A good example in this regard is the case of Cohn's embedding theorem [31] in conjunction with the result of F. Nazarov and A. Volberg [69], which asserts that uniform embedding of all the reproducing kernels may not characterize the Carleson measures in model subspaces if the generating inner function is not one-component. Additional example valid only for one-component case can be a perturbation result of W. Cohn [30] with respect to small changes in the argument of the generating inner function. Counterexamples for this when the inner function has more components can be found in [10].

Inspired by the preceding connection between separation and the Bessel property, A. Baranov posed the question whether condition (4.6.2) is necessary for (4.6.3): *there exists $M$ such that for any $J = [a,b]$ with $\varphi(b) - \varphi(a) = 1$*

$$\#\{n : t_n \in J\} \leq M.$$

The question has again a positive answer for one-component inner functions. The main objective in this section is prove that the answer in general is negative. Indeed, a slight modification of our general approach in Section 2.2 to construct Bessel sequences will lead to a suitable counterexample. This provides one more example of the fact that the Carleson measure problem changes quite substantially when we move from one-component to infinitely many component inner functions.

In the remaining part of this section we present an example that gives a negative answer to Baranov's question. Since it introduces no additional complications, we may first state the problem in a more general form as follows.

**Question.** *Suppose that $\Gamma$ and $\Lambda$ are disjoint sequences of real numbers and that $\gamma_n \nearrow \infty$. If $w$ is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$ and $H_{(\Gamma,v);(\Lambda,w)}$ is a bounded transformation, then is it true that there is a uniform bound on the number of $\lambda_j$ such that $|\gamma_n| \leq |\lambda_j| \leq |\gamma_{n+1}|$?*

### 4.6.1 An example answering Baranov's question

We will now modify our construction to obtain an example that gives a negative answer to Baranov's question.

We assume that $(t_n)$ is a sequence of positive numbers such that

$$\inf_{n \geq 1} \frac{t_{n+1}}{t_n} > 1.$$

In addition, we will assume that, for each positive integer $n$, we have the following cluster of $n$ points:

$$\gamma_{n,l} = t_n + l - 1, \quad 1 \leq l \leq n.$$

We denote this finite sequence by $\Gamma_n$ and set

$$\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n.$$

We will consider the simplest case when the corresponding weight sequence $v$ is identically $1$, i.e., $v_{n,l} = 1$ for every point $\gamma_{n,l}$ in $\Gamma$.

It may be noted that if we want to describe the measures $\mu$ for which $H_{(\Gamma,1)}$ is bounded from $\ell^2$ to $L^2(\mathbb{C} \setminus \Gamma, \mu)$, then it suffices to consider the behavior of $\mu$ in the Carleson squares

$$S_n = \{z = x + iy : \ |x - \gamma_{n,1}| \leq 2n, \ 0 \leq y \leq 4n\}.$$

Indeed, outside these squares, each cluster $\Lambda_n$ has basically the same effect as if a single point were located at, say, $\lambda_{n,1}$ with weight $n$. This means that Theorem 2.2.1 applies to describe the behavior of $\mu$ outside the squares $S_n$. In fact, by this observation, one may obtain a complete solution to the boundedness problem for these particular sequences $\Gamma$ and $v$. We omit this description here and confine the discussion to a suitable example solving Baranov's problem.

The preceding notes indicate that the sequence $\Lambda$ should be placed inside the union of the squares $S_n$. We set

$$\lambda_{n,s} = \gamma_{n,1} - 2^s, \quad 0 \le s \le \log_2 n$$

and then $\Lambda_n = (\lambda_{n,s})_s$ with $s$ running from 0 to $[\log_2 n]$ (the integer part of $\log_2 n$), and

$$\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n.$$

We observe that

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{l=1}^{n} \frac{1}{|\gamma_{m,l} - \lambda_{n,s}|^2} &\simeq \sum_{l=1}^{n} \frac{1}{|\gamma_{n,l} - \lambda_{n,s}|^2} + \sum_{m=1}^{n-1} \frac{n}{|\lambda_{n,s}|^2} + \sum_{m=n+1}^{\infty}\sum_{l=1}^{n} \frac{1}{|\gamma_{m,l}|^2} \\ &\simeq |\lambda_{n,s} - \gamma_{n,1}|^{-1} = 2^{-s},\end{aligned}$$

from which we have

$$w_{n,s} \simeq 2^s.$$

These numbers constitute the sequence $w$, which is the Bessel weight sequence for $\Lambda$ with respect to $(\Gamma, v)$. We now state our result which appears in [11].

**Theorem 4.6.1.** *If the sequences $\Gamma$, $\Lambda$ and $w$ are constructed as above, then $H_{(\Gamma,1);(\Lambda,w)}$ is a bounded transformation.*

The interesting point, giving a negative answer to Baranov's question, is that there are more than $\log_2 n$ points from $\Lambda$ between the neighboring clusters $\Lambda_{n-1}$ and $\Lambda_n$.

*Proof of the theorem.* Let $a = (a_{m,l})$ be an arbitrary $\ell^2$-sequence associated with $\Gamma$ and set

$$H_{(\Gamma,1):(\Lambda,w)} a(\lambda) = \sum_{m=1}^{\infty}\sum_{l=1}^{m} \frac{a_{m,l}}{\lambda - \gamma_{m,l}}$$

for each point $\lambda$ in $(\Gamma, v)^*$. An application of the Cauchy–Schwarz inequality gives

$$\begin{aligned}\sum_{s=0}^{[\log_2 n]} |H_{(\Gamma,1):(\Lambda,w)} a(\lambda_{n,s})|^2 w_{n,s} &\lesssim \sum_{s=0}^{[\log_2 n]} \left( \sum_{l=1}^{n} \frac{|a_{n,l}|}{|\lambda_{n,s} - \gamma_{n,l}|} \right)^2 w_{n,s} + \\ &\quad \sum_{s=0}^{[\log_2 n]} \left( \left( \sum_{m=1}^{n-1}\sum_{l=1}^{m} \frac{|a_{m,l}|}{|\lambda_{n,s}|} \right)^2 + \left( \sum_{m=n+1}^{\infty}\sum_{l=1}^{m} \frac{|a_{m,l}|}{|\gamma_{n,l}|} \right)^2 \right) w_{n,s} \\ &\lesssim \frac{n^3}{t_n^2} \|a\|_{\ell^2}^2 + \sum_{s=0}^{[\log_2 n]} 2^s \left( \sum_{l=1}^{n} \frac{|a_{n,l}|}{|\lambda_{n,s} - \gamma_{n,l}|} \right)^2.\end{aligned}$$

The summation over $n$ of the first term on the right-hand side causes no problem because $t_n$ grows at least exponentially with respect to $n$. We therefore concentrate on the second term

$$A_n = \sum_{s=0}^{[\log_2 n]} 2^s \left( \sum_{l=1}^{n} \frac{|a_{n,l}|}{2^s + l - 1} \right)^2.$$

The Cauchy–Schwarz inequality again gives

$$\begin{aligned}
\left( \sum_{l=1}^{n} \frac{|a_{n,l}|}{2^s + l - 1} \right)^2 &\leq \sum_{j=1}^{n} \frac{j^{-\frac{1}{2}}}{2^s + j - 1} \sum_{l=1}^{n} \frac{l^{\frac{1}{2}} |a_{n,l}|^2}{2^s + l - 1} \\
&= \left( \sum_{j=1}^{2^s} \frac{j^{-\frac{1}{2}}}{2^s + j - 1} + \sum_{j=2^s+1}^{n} \frac{j^{-\frac{1}{2}}}{2^s + j - 1} \right) \sum_{l=1}^{n} \frac{l^{\frac{1}{2}} |a_{n,l}|^2}{2^s + l - 1}.
\end{aligned}$$

The sum of the two sums on the right-hand side is bounded by a constant times $2^{-\frac{s}{2}}$, and so it follows that

$$A_n \leq \sum_{s=0}^{[\log_2 n]} 2^{\frac{s}{2}} \sum_{l=1}^{n} \frac{l^{\frac{1}{2}} |a_{n,l}|^2}{2^s + l - 1}.$$

Changing the order of summation and using that

$$\sum_{s=0}^{[\log_2 n]} \frac{2^{\frac{s}{2}}}{2^s + l - 1} \lesssim l^{-1/2},$$

we finally obtain the desired estimate:

$$A_n \lesssim \sum_{j=1}^{n} |a_{n,j}|^2.$$

$\square$

# 5 Reproducing kernel Riesz bases from equality of spaces

Following the arguments used in the work of J. Ortega–Cerdà and K. Seip [61], A. Baranov [7] described reproducing kernel Riesz bases associated to real points for model subspaces in terms of equality of spaces. In this chapter we will study the natural analogue of his result with the real points being replaced by sequences of points located in the upper half-plane. We show that the analogous conditions are indeed sufficient but not in general necessary. We will also discuss invertibility of Toeplitz operators from this equality of spaces perspective. Roughly speaking, the work in this part may be viewed as a remark on the interrelationship among three objects; invertible Toeplitz operators, equality of spaces and Riesz bases of reproducing kernels in model subspaces generated by the class of meromorphic inner functions. It is shown that none of these can be described in terms of the others in a sense to be made precise.

## 5.1 Equality of spaces

We say that a sequence $(f_j)$ in a Hilbert space $\mathcal{H}$ is a frame if there exists a positive constant $C$ such that the inequalities

$$C^{-1}\|f\|_{\mathcal{H}}^2 \leq \sum_j |\langle f, f_j\rangle_{\mathcal{H}}|^2 \leq C\|f\|_{\mathcal{H}}^2 \tag{5.1.1}$$

hold for functions $f$ in $\mathcal{H}$. While the lower inequality ensures completeness with $\ell^2$ norm control over the coefficients of a frame system, i.e. each $f$ in $\mathcal{H}$ can be approximated by a finite combination

$$\sum_j c_j f_j \quad \text{with } \|(c_j)\|_{\ell^2} \lesssim \|f\|_{\mathcal{H}},$$

the upper inequality encompasses frames as a special class of Bessel sequences.

A frame constitutes a Riesz basis if and only if it ceases to be a frame after the removal of any one of its elements. It may be noted that K. Seip [99] constructed frames of exponentials in $L^2(-\pi,\pi)$ that contains no Riesz basis subsequence.

The approach we intend to follow here begins with a problem of R. Duffin and A. Schaeffer [40] to describe real sequences which generate Fourier frames in $L^2(-\pi,\pi)$. In an interesting paper [78], J. Ortega–Cerdà and K. Seip have solved the problem by equivalently describing the sampling sequences in the Paley–Wiener space $PW_\pi$, in terms of equality of two spaces[1]. Their result reads:

**Theorem 5.1.1.** *A separated real sequence*[2] $(t_n)$ *is sampling*[3] *for* $PW_\pi$ *if and only if there exist entire functions* $E$ *and* $F$ *in the HB class such that*

*(i)* $H(E) = PW_\pi$ *and*

*(ii)* $(t_n)$ *constitutes the zero sequence of* $EF + E^*F^*$.

Following their approach, A. Baranov [7] was able to prove the following two more general results in model subspaces generated by the class of meromorphic inner functions.

**Theorem 5.1.2.** *Let* $E$ *be an HB class function,* $I = E^*/E$ *and* $(t_n)$ *be a sequence of real points for which* $S_R(t_n)$ *constitutes a frame for* $K_I^2$. *Then there exist entire functions* $E_1$ *in the HB class and* $E_2$ *either in the HB class or a constant such that*

*(i)* $H(E) = H(E_1)$,

*(ii) the sequence* $(t_n)$ *constitutes a zero sequence for the function* $E_1E_2 - E_1^*E_2^*$ *and*

*(iii)* $1 - I_1I_2 \notin L^2(\mathbb{R})$ *with* $I_1 = E_1^*/E_1$ *and* $I_2 = E_2^*/E_2$.

These conditions are about the lower inequality in (5.1.1), and require the Bessel property to be sufficient as well. In particular, if $E_2$ is a constant, then the next stronger result holds which also reveals that the overcompleteness of a frame comes from the existence of a second entire function $E_2$ in the $HB$ class.

[1] Here and in what follows by equality of two spaces we mean equality as a set equipped with equivalent norms. We denote by $H_1 = H_2$ if the spaces $H_1$ and $H_2$ satisfy such a relation.

[2] Here we mean that the points are separated in the Euclidean distance.

[3] We recall that $\Lambda$ is a sampling sequence for a reproducing kernel Hilbert space $\mathscr{H}$ if $S_R(\Lambda)$ constitutes a frame for $\mathscr{H}$.

**Theorem 5.1.3.** *Let $E$ be an HB class function, $I = E^*/E$ and $(t_n)$ be a sequence of real points. Then $S_R(t_n)$ is a Riesz basis in $K_I^2$ if and only if there exists an HB class function $E_1$ such that*

*(i) $H(E) = H(E_1)$ and*

*(ii) the sequence $(t_n)$ is the zero set of the function $I_1 - 1$ and $I_1 - 1 \notin L^2(\mathbb{R})$ where $I_1 = E_1^*/E_1$.*

Baranov's result provides a new approach to study reproducing kernel Riesz bases, bypassing the usual appeal to either invertible properties of Toeplitz operators or an $A_2$ condition involving generator functions. The above results are all dealing with when the points associated with the kernel functions are real. In particular if $\psi$ denotes the increasing branch of the argument of $I_1$, then the points $t_n$ satisfy the relation

$$\psi(t_n) = \alpha + 2\pi n,\ n \in \mathbb{Z}$$

for some $\alpha \in [0, 2\pi)$. A natural question is then whether an analog of Theorem 5.1.3, with $I_1 - 1$ replaced by a meromorphic inner function, holds when we associate the kernel functions with a sequence of points located in the upper half-plane. It turns out that such analogues are indeed sufficient but not in general necessary. This seems rather natural since the condition equality of spaces is so strong. We now prove the following.

**Theorem 5.1.4.** *Let $E$ be an HB class function, $I = E^*/E$ and $(\lambda_n) \subset \mathbb{C}_+$. Then $S_R(\lambda_n)$ is a Riesz basis in $K_I^2$ if there exists an interpolating Blaschke product $B = E_1^*/E_1$, $E_1$ an HB class function such that*

*(i) $H(E) = H(E_1)$ and*

*(ii) the sequence $(\lambda_n)$ constitutes the zero set of $B$.*

We will construct counterexamples in Section 5.2 to show that (i) is not in general necessary. On the other hand, by Theorem 1 in [50], and since the Carleson interpolation condition implies the Blaschke condition, (ii) is always necessary. From Theorems 5.1.3 and 5.1.4, we conclude that larger perturbations of reproducing kernel Riesz bases is admissible with points from the upper half-plane than along the real line.

To be able to apply these results, we need to have a characterization of those entire functions $E_1$ for which the spaces $H(E)$ and $H(E_1)$ are equal. An obvious sufficient condition is of course when $|E_1(z)| \simeq |E(z)|$ holds for each $z \in \mathbb{C}_+ \cup \mathbb{R}$. The converse statement here is not true. This problem was considered in [7] and a solution has been given in terms of increasing branches of the arguments of $E_1$ and $E$. For weighted Paley–Wiener spaces, the same problem was treated in [62].

In general, checking equality of two spaces for given functions $E$ and $E_1$ is practically quite hard. One reason, as stated in the next lemma from [7], is that it can be equivalently reformulated in terms of another longstanding open problem; Carleson measures in model subspaces. To state the lemma, we need to recall the Smirnov class functions $\mathscr{N}^+$. An analytic function $f$ is said to be in $\mathscr{N}^+$ if the representation $f = g/h$ holds for some $H^\infty$ functions $g$ and $h$ with $h$ outer as well. Note that since $h$ is outer, the ratio is well defined.

**Lemma 5.1.5.** *Let $E$ and $E_1$ be HB class functions, and $I = E^*/E$, $I_1 = E_1^*/E_1$ and $w = E/E_1$. Then $H(E) = H(E_1)$ if and only if*

*(i)* $w,\ w^{-1} \in \mathscr{N}^+ \cap L^2(\pi)$

*(ii)* $\mu = w^2 dm$ *and* $\mu_1 = w^{-2} dm$ *are Carleson measures for the space* $K_I^2$ *and* $K_{I_1}^2$ *respectively where* $dm$ *and* $\pi$ *respectively stand to the Lebesgue and Poisson measures on the real line.*

The lemma again complements the fact that the Riesz basis problem is a special case of the Carleson measure or the two weight problems for the Hilbert transform (cf. Theorem (2.4.1)). We state one more extension of a theorem from [7]. The result is an immediate consequence of Theorem 5.1.4. We keep the notation $\widetilde{H}_{pos}$ from (1.5) for the Hilbert transform when it acts on functions integrable with respect to the Poisson measure on the real line. That is for $g$ in $L^1(\pi)$;

$$\widetilde{H}_{pos} g(x) = p.v. \frac{1}{\pi} \int_{\mathbb{R}} \left( \frac{1}{x-t} + \frac{t}{t^2+1} \right) g(t) dt.$$

**Corollary 5.1.6.** *Let $I$ be a meromorphic inner function with an increasing branch of argument $\varphi$ and $(\lambda_n) \subset \mathbb{C}_+$. Then $S_R(\lambda_n)$ is a Riesz basis in $K_I^2$ if there exists a meromorphic inner function $I_1$ with an increasing branch of argument $\varphi_1$ such that* *(i)* $\varphi - \varphi_1 \in L^1(\pi)$ *and* $\widetilde{H}_{pos}(\varphi - \varphi_1) \in L^\infty(\mathbb{R})$

*(ii) the sequence* $(\lambda_n)$ *constitutes the zero set of* $I_1$.

This result was proved in [7] for the case when the sequence $\lambda_n$ consists only of real points. The proof was based on Theorem 5.1.3 and another general result (Theorem 3.2) from [7]. The corollary will follow from a similar proof. We only have to use this time Theorem 5.1.4 in place of Theorem 5.1.3.

## Proof of Theorem 5.1.4

We may first note that $B$ being an interpolating Blaschke product in the hypothesis, which is known from Theorem 1.1 in [50], makes our proof easy. We will use arguments similar to those used by Baranov in [7]. We should only argue using normalized reproducing kernel Riesz bases associated with sequence of points from $\mathbb{C}_+$ instead of de Branges basis. We include a proof for the sake of completeness. For convenience, denote by $K_z$, $k_z$, $K^1_z$ and $k^1_z$ the reproducing kernels of the spaces $H(E)$, $K^2_I$, $H(E_1)$ and $K^2_B$ respectively at the point $z$. If $(\lambda_n)$ is the zero set of an interpolating Blaschke product $B$, then the family of normalized reproducing kernels associated to $(\lambda_n)$ constitutes a Riesz basis in $K^2_B$. This result is due to Shapiro and Shields [100]. In view of the unitary isomorphism $f \mapsto E_1 f$ from $K^2_B$ onto $H(E_1)$, which in particular maps reproducing kernels onto reproducing kernels, this holds true if and only if the system

$$\left\{ \frac{E_1 k^1_{\lambda_n}}{\|k^1_{\lambda_n}\|_2} \right\} = \left\{ \frac{K^1_{\lambda_n}}{\|k^1_{\lambda_n}\|_2 \overline{E_1(\lambda_n)}} \right\} = \left\{ \frac{K^1_{\lambda_n}}{\|K^1_{\lambda_n}\|_{H(E_1)}} \right\}$$

constitutes a Riesz basis for $H(E_1)$ where the equalities are due to the kernels relation

$$K^1_z(w) = E_1(w)\overline{E_1(z)} \frac{i}{2\pi} \left( \frac{1 - \overline{I_1(z)} I_1(w)}{w - \bar{z}} \right) = \overline{E_1(z)} E_1(w) k^1_z(w) \qquad (5.1.2)$$

for points $z$ and $w$ in the upper half-plane. Equivalently, it means that the interpolation problem

$$f(\lambda_n) = a_n$$

has a unique solution $f$ in $H(E_1)$ whenever the admissibility condition

$$\sum_{n=1}^{\infty} |a_n|^2 \|K^1_{\lambda_n}\|^{-2}_{H(E_1)} < \infty$$

holds. By duality and the hypothesis we have that

$$\begin{aligned} \|K^1_z\|_{H(E_1)} &= \sup_{\substack{g\in H(E_1)\\ \|g\|_{H(E_1)}=1}} \left|\langle g, K^1_z\rangle\right|_{H(E_1)} = \sup_{\substack{g\in H(E_1)\\ \|g\|_{H(E_1)}=1}} |g(z)| \\ &\simeq \sup_{\substack{g\in H(E)\\ \|g\|_{H(E)}=1}} |g(z)| = \|K_z\|_{H(E)} \end{aligned} \tag{5.1.3}$$

for each point $z$ in $\mathbb{C}_+$ and in particular for the $\lambda_n's$. It follows that for each sequence $c_n$ satisfying

$$\sum_{n=1}^{\infty} |c_n|^2 \|K_{\lambda_n}\|^{-2}_{H(E)} \simeq \sum_{n=1}^{\infty} |c_n|^2 \|K^1_{\lambda_n}\|^{-2}_{H(E_1)} < \infty,$$

there exists a unique function $f$ in $H(E) = H(E_1)$ such that $f(\lambda_n) = c_n$. This proves that $(\lambda_n)$ is a complete interpolating sequence for $H(E)$ and so is for $K^2_I$.

## 5.2 Equality of spaces fails to be necessary

In this section we are concerned with constructing counterexamples which would lead to the conclusion that the natural analog of A. Baranov's result (Theorem 5.1.3) fails to hold in general. We will exhibit two different examples using one and infinitely many component inner functions. This, in addition, is meant to stress the fact that results valid in model subspaces generated by the class of one-component inner functions may in general fail when the generating inner function has infinitely many components.

**Example 6.** The condition fails to be necessary even for the case of the classical Paley–Wiener space. This can be easily seen from Baranov's example [7] where he explained that equality of spaces may also fail to follow from invertibility of Toeplitz operators. We present the example here to make the exposition self-contained. Let $E(z) = \exp(-\pi i z)$ and

$$E_1(z) = \lim_{R\to\infty} \prod_{|\lambda|<R} \left(1 - \frac{z}{\lambda_n}\right)$$

with a sequence $(\lambda_n)$;

$$\lambda_n = \begin{cases} n+i, & n \leq 0 \\ n+\delta+i, & n > 0 \end{cases}$$

where $0 < \delta < 1/4$. Then $I(z) = E^*(z)/E(z) = \exp(2\pi i z)$ and $B = E_1^*/E_1$. By Kadets'-1/4 theorem, the system of exponentials $(e^{i\lambda_n t})$ constitutes a Riesz basis in $L^2(0,2\pi)$. We claim that $H(E) \neq H(E_1)$. Were it not, then setting $\varphi$ and $\varphi_1$ respectively as increasing branches of the arguments of the inner functions $I$ and $B$, we have

$$\|k_t\|^2_{H(E)} \simeq |E(t)|^2\varphi'(t) \simeq |E_1(t)|^2\varphi_1'(t) \simeq \|k_t\|^2_{H(E_1)}$$

for each real point $t$. It is rather a simple estimate that for all such points $\varphi'(t) \simeq \varphi_1'(t) \simeq 1$ and hence

$$\left|\frac{E(t)}{E_1(t)}\right| \simeq |t|^\delta \to \infty$$

when $|t| \to \infty$ and results again in a contradiction.

We now turn to the case of infinitely many component inner functions. We may first note that each entire function $E$ in $HB$ class admits the factorization $E(z) = S(z)P(z)$ with $S$ an entire function which assumes real values on the real line and can have only real zeros, and

$$P(z) = \alpha e^{-aiz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\overline{z_n}}\right) e^{z\Re(1/z_n)} \tag{5.2.1}$$

where $a \geq 0$, $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, and the sequence $z_n$ in $\mathbb{C}_+$ satisfies the Blaschke condition. If $I$ is a meromorphic inner function identified by such $E$, then for each $z$ in $\mathbb{C}_+$, we have

$$I(z) = \frac{E^*}{E}(z) = \frac{\overline{\alpha}}{\alpha} e^{2aiz} \prod_{n=1}^{\infty} \frac{1 - z/z_n}{1 - z/\overline{z_n}}$$

which is always independent of the parameter $S$. In other words, the inner function $I = E^*/E = P^*/P$ acquires all of its structure only from the product factor $P$. This simple fact will be used effectively to construct our next example.

**Example 7.** We consider a model subspace $K_B^2$ with $B$ a Blaschke product with simple zeros at the points $z_n = \gamma_n + i$, indexed by the positive integers and $\gamma_n$ satisfying the growth condition (2.2.1), That is

$$\inf_n \gamma_{n+1}/\gamma_n > 1. \tag{5.2.2}$$

The system $k_{z_n}$, $n = 1, 2, \ldots,$ constitutes a Riesz basis in $K_B^2$. Another way of phrasing this property is to say that the map

$$(a_n) \mapsto \sum_{n=1}^{\infty} \frac{a_n}{z - \overline{z_n}}$$

is a Hilbert space isomorphism from $\ell^2$ onto $K_B^2$. An application of this makes use of another immediate consequence, namely that the norm equivalence

$$\|f\|_2^2 \simeq \sum_{n=1}^{\infty} |\langle f, k_{z_n}\rangle|^2 = \sum_{n=1}^{\infty} |f(z_n)|^2 \tag{5.2.3}$$

holds for functions $f$ in $K_B^2$. If $Z = (z_n)$ and $\lambda$ is a point in the upper half-plane, then

$$\|k_\lambda\|_2^2 \simeq \max\left\{n|\lambda|^{-2}, \operatorname{dist}^{-2}(\lambda, Z)\right\} \tag{5.2.4}$$

for some positive integer $n$. The reason such a simple estimate holds for $\|k_\lambda\|_2$ is the “minimal” interaction between the zeros of B implied by our a priori growth condition (5.2.2): Geometrically, this almost lack of interaction is reflected in the (essential) lack of intersection between the disks

$$D_n = \left\{z \in \mathbb{C}_+ : |z - z_n| \lesssim |z_n|/\sqrt{n}\right\}.$$

We shall now proceed to construct our example. Consider the sequence

$$(\lambda_n) = \left(\gamma_n\left(1 + \frac{1}{n\log(n+1)}\right) + i\right) \subset \bigcup_{n=1}^{\infty} D_n \tag{5.2.5}$$

where each $\lambda_n$ belongs to the respective $D_n$. By Theorem 2.4.10 ((2.4.10) and (2.4.11) holds) we observe that such a sequence gives rise to a reproducing kernel Riesz basis in $K_B^2$. Setting

$$E(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{z_m}\right), \quad E_1(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{\lambda_m}\right),$$

$$I = E_1^*/E_1 \quad \text{and} \quad \rho_n = \prod_{k=1}^{n} \frac{|z_k|}{|\lambda_k|},$$

we claim that $\|K^1_{\lambda_n}\|_{H(E_1)} \not\simeq \|K_{\lambda_n}\|_{H(E)}$. Were it not, then

$$\|K^1_{\lambda_n}\|^2_{H(E_1)} \quad = \quad |E_1(\lambda_n)|^2 \|k^1_{\lambda_n}\|^2_{K^2_I} \simeq \frac{\Im\lambda_n |\lambda_n|^{2n-2}}{\prod_{k=1}^{n} |\lambda_k|^2} \tag{5.2.6}$$

and applying (5.2.4), we also have

$$\|K_{\lambda_n}\|^2_{H(E)} = |E(\lambda_n)|^2 \|k_{\lambda_n}\|^2_{K^2_I} \simeq \frac{|\lambda_n|^{2n-2}}{\prod_{k=1}^{n} |z_k|^2}. \tag{5.2.7}$$

Invoking (5.1.3) would imply that

$$1 = \Im\lambda_n \simeq \frac{1}{\rho_n} \to \infty$$

when $n \to \infty$ and yields a contradiction.

## 5.3 Invertibility of Toeplitz operators

As pointed out earlier, when condition (1.3.7) holds, the essential part of the Riesz basis condition involves the invertibility of certain Toeplitz operators. We now consider the inverse question, namely whether the Toeplitz operator $T_{I\overline{B_\Lambda}}$ is necessarily invertible whenever $S_R(\Lambda)$ constitutes a Riesz basis in $K^2_I$ where $\overline{B_\Lambda}$ here refers to the Blaschke product with zero set $\Lambda$. It turns out that the answer to this question is in general negative. For one-component inner functions $I$, this was already noticed in [7]. The answer remains negative when the generating inner function possesses infinitely many components. To see this, one can use the space $K^2_B$ introduced in the above example and observe that the zero set of the Blaschke product $B_\Lambda$;

$$\Lambda = \Big(\gamma_n(1+1/n^2) + i/\log(n+1)\Big), \tag{5.3.1}$$

generates a reproducing kernel Riesz basis in $K^2_B$ (cf. Theorem 2.4.10) while the Toeplitz operator with symbol $B\overline{B_\Lambda}$ fails to be invertible.

Invertibility of the Toeplitz operator is not a necessity for equality of spaces either. We refer to remark 6.5 (1) in [7] for a counterexample which first appeared in [62]. Conversely, the example in Subsection 4.2 clarifies that invertibility again fails to imply equality of spaces. To see this, first observe that condition (1.3.7) holds for $\Lambda$ and $I = E^*/E$. Thus $(e^{i\lambda t})_{\lambda\in\Lambda}$ is a Riesz basis and implies invertibility of the Toeplitz operator with symbol $I\overline{B_\Lambda}$. To this effect, the basis property implies neither equality of spaces nor invertibility of the Toeplitz operator though it easily follows from the former (cf. Theorem 5.1.4).

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