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Subcategory Classifications in Tensor Triangulated Categories

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PROBLEM DESCRIPTION

Look at classifications of thick triangulated subcategories in triangulated categories through the work of Paul Balmer on tensor triangulated categories.

ABSTRACT

It is known that the thick tensor-ideal subcategories in a tensor triangulated category can be classified via its prime ideal spectrum.

We use this to provide new proofs of two well-known classifications theorems: that of the thick tensor-closed triangulated subcategories of the stable category of modules over a finite group algebra, and that of the thick triangulated subcategories of the derived category of perfect complexes over a commutative Noetherian ring.

PREFACE

This thesis represents the work of my final year as a student at the Natural Science with Teacher Education programme at NTNU. It was written under supervision of Petter Bergh in the field of category theory.

First and foremost I would like to thank Petter Bergh for his guidance and razor-sharp intuition that helped me get back on track in moments of bewilderment. I would also like to thank the International Student Festival in Trondheim for distracting me all through January and February, and thus pushing me to literally move in at Matteland afterwards, my beloved home for the last couple of months. Thanks to Torkil, who accidentally was the only guy in Matteland who knew about the derived category, and thus had to suffer through many a mumbling, confusing question. Also thanks to him for looking over this thesis. Thanks to my big happy family at Matteland, for the delicious meals at Hangaren, the spontaneous umbrella-dances at the reading hall, the coffee-breaks, the late night sessions, the joy and the tears. Thanks to all the people joining me in my nocturnal adventures at Matteland, and to all the people in the world with a skewed sleeping pattern. Thanks to my other friends for enduring my disappearance in March. It will be quite interesting to see if I have any left when I step out into daylight tomorrow.

CONTENTS

CHAPTER 1

INTRODUCTION

Verdier [32] and Puppe [11] discovered independently in the 60's that certain, seemingly very different, classes of categories shared some remarkable properties. They all had particular "triangles", linked together in a very specific structure. Since then, the concept of triangulated categories has gradually invaded a long list of mathematical disciplines, from representation theory and algebraic geometry to commutative algebra and algebraic topology, and is still seizing new land. Quite powerful tools are also available, once a category is proven to be triangulated. However, to really bring out the big guns, it is often proclaimed that one needs more structure.

One such, quite modest, augmentation is to assume a *tensor product*, here just a symmetric monoidal structure which behaves nicely with respect to the triangulation. This is a mild condition, and many triangulated categories do in fact come with such a structure. Several authors have employed this to boost up their toolbox. Among them are Hovey, Palmieri and Strickland [18], May [23] and Garkusha [14]. However, a common feature of all these is that they also assume further structure. Perhaps the only elaborate work on tensor triangulated categories in their pure generality is that of Balmer, first in [2] and later followed up with [3], and this is the context in which this thesis is written; it is largely based on [2].

One of the main merits of Balmer's article is a classification of the thick tensor-closed triangulated subcategories of a tensor triangulated category (our Theorem 3.7), placing it in the tradition of thick subcategory classifications: Ever since Devinatz, Hopkins and Smith [10] succeeded in the stable homotopy category of spectra, a major obsession for many a category theorist has been to find ways to describe the thick triangulated subcategories of a triangulated category.

Balmer [2] dubs the thick tensor-closed triangulated subcategories of a tensor triangulated category $\mathscr{K}$ *thick* $\otimes$*-ideals*, and defines *prime* and *radical* such ideals in the obvious ways. He then introduces the *prime ideal spectrum* $\operatorname{Spc}\mathscr{K}$, the collection of primes endowed with a topology, and a corresponding *support*, defined on any $a \in \mathscr{K}$ to be the closed subset

$$\operatorname{supp} a := \{\mathscr{P} \in \operatorname{Spc} \mathscr{K} \mid a \notin \mathscr{P}\}$$

Finally, he calls any topological space linked to $\mathscr{K}$ in a similar way a *support data*. In this language, Balmer manages to give a bijection between the radical thick $\otimes$-ideals of $\mathscr{K}$ and the subsets of $\operatorname{Spc} \mathscr{K}$ that are unions of supports.

The problem with Balmer's classification, however, is that it really does not make us much wiser at first glance; the entities at the other end of the bijection seem as hard to compute as the subcategories themselves. But another result stirs hope: if we have another classification on $\mathscr{K}$ by another "nice" support data, then Balmer [2, Theorem 5.2] states that this support data must be isomorphic to $\{\operatorname{Spc} \mathscr{K}, \operatorname{supp}\}$.

It is therefore tempting to ask if one sometimes might be able to prove the mentioned isomorphism independently of the classification, and thereby arriving at the latter via a translation of Balmer's classification. This could provide a general path for proving classification theorems, and perhaps allow us to find sensible classifications even in categories where we do not yet have any. The first of these concerns is answered here. The answer is 'yes', at least if we require some extra structure on $\mathscr{K}$ and its support data (Theorem 3.13). Theorem 3.13, though not a very strong result, is the pinnacle of this thesis, as it is its main original contribution.

Crucially, Theorem 3.13 turns out to be applicable on this thesis' two main examples of tensor triangulated categories: the stable category of modules over a finite group algebra, and the derived category of perfect complexes over a commutative Noetherian ring. This enables us to provide new proofs of the celebrated classification theorems of Benson-Carlson-Rickard [7] and Hopkins-Neeman [17, 25].

In Chapter 2 we give a brief introduction to triangulated categories, and introduce Balmer's machinery of tensor triangulated categories. Section 3.1 settles Balmer's classification. In the following Section 3.2 we provide conditions for a support data on $\mathscr{K}$ to be isomorphic to $\{\operatorname{Spc} \mathscr{K}, \operatorname{supp}\}$, enabling a corresponding translation of Balmer's classification. Chapter 4 and Chapter 5 are devoted to the two examples.

In this thesis we assume the reader has a basic knowledge of category theory, commutative algebra and homological algebra.

CHAPTER 2

TENSOR TRIANGULATED CATEGORIES

As we do not assume any prior contact with triangulated categories, we here give a brief presentation of the axioms as well as some very basic properties which we will need. After that, we move on to introduce $\otimes$-triangulated categories in the sense of Balmer [2], and look at the prime ideal spectrum of such a category, with its corresponding support.

2.1 Triangulated categories

A *triangulated category* is an additive category together with a so-called translation functor and a specific *triangulated structure* defined on it. To get to the precise definition of a triangulated category, we first need to introduce some basic language:

A *category with translation* $(\mathscr{K}, \Sigma)$ is a category $\mathscr{K}$ together with an auto-equivalence

$$\Sigma : \mathscr{K} \longrightarrow \mathscr{K}$$

called the *translation functor* (also labeled the *shift* or *suspension* functor). The translation functor is assumed to be additive if our category is.

In a category with translation a *triangle* is a sequence of objects and morphisms on the form

$$a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{w} \Sigma a$$

and a *morphism of triangles* is a triple (f, g, h) of morphisms such that the following diagram commutes:

$$\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "f"] & b \arrow[r, "v"] \arrow[d, "g"] & c \arrow[r, "w"] \arrow[d, "h"] & \Sigma a \arrow[d, "\Sigma f"] \\
a' \arrow[r, "u'"] & b' \arrow[r, "v'"] & c' \arrow[r, "w'"] & \Sigma a'
\end{tikzcd}$$

Such a morphism is called an *isomorphism of triangles* if f, g and h are isomorphisms.

Definition 2.1 (Triangulated category)**.** A triangulated category is an additive category with translation $(\mathscr{K}, \Sigma)$ endowed with a collection of triangles, called *distinguished triangles* (d.t. for short), satisfying the following four axioms:

TR1

a) A triangle isomorphic to a d.t. is again a d.t.

b) $a \xrightarrow{1} a \longrightarrow 0 \longrightarrow \Sigma a$ is a d.t. for every $a \in \mathscr{K}$

c) For any morphism $u : a \to b$ there is a d.t. $a \xrightarrow{u} b \longrightarrow c \longrightarrow \Sigma a$

TR2 (Rotation). $a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{w} \Sigma a$ is a d.t. if and only if

$b \xrightarrow{v} c \xrightarrow{w} \Sigma a \xrightarrow{-\Sigma u} \Sigma b$ is.

TR3 Any diagram of two d.t.'s

$$\begin{tikzcd}
a \arrow[r, "u"] \arrow[d, "f"] & b \arrow[r, "v"] \arrow[d, "g"] & c \arrow[r, "w"] & \Sigma a \arrow[d, "\Sigma f"] \\
a' \arrow[r, "u'"] & b' \arrow[r, "v'"] & c' \arrow[r, "w'"] & \Sigma a'
\end{tikzcd}$$

where the first square commute can be completed to a morphism of triangles.

TR4 (The octahedral axiom). Given three d.t.'s

$$a \xrightarrow{u} b \longrightarrow c' \longrightarrow \Sigma a$$

$$b \xrightarrow{v} c \longrightarrow a' \longrightarrow \Sigma b$$

$$a \xrightarrow{v \circ u} c \longrightarrow b' \longrightarrow \Sigma a$$

there exists a d.t.

$$c' \longrightarrow b' \longrightarrow a' \longrightarrow \Sigma c'$$

such that the following diagram commutes

$$\begin{tikzcd}
a \arrow[rr, bend left, "v\circ u"] \arrow[dr, "u"'] & & c \arrow[rr, bend left] \arrow[dr] & & a' \arrow[rr, dashed, bend left] \arrow[dr] & & \Sigma c' \\
& b \arrow[ur, "v"'] \arrow[dr] & & b' \arrow[ur, dashed] \arrow[dr] & & \Sigma b \arrow[ur] & \\
& & c' \arrow[ur, dashed] \arrow[rr, bend right] & & \Sigma a \arrow[ur] & &
\end{tikzcd}$$

Remark 2.2. It can be shown that TR3 is actually superfluous, as it can be derived from the other axioms (see [23, Lemma 2.2]).

In an additive category we have the notions of additive functors and additive subcategories (full subcategories closed under finite coproducts). For these to be called triangulated, however, we need further properties:

Definition 2.3 (Triangle functor)**.** A *triangle functor* is an additive functor

$$F : \mathscr{K} \longrightarrow \mathscr{L}$$

between two triangulated categories, together with a natural isomorphism

$$\phi : F\Sigma \longrightarrow \Sigma F$$

such that for any d.t. $a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{w} \Sigma a$ in $\mathscr{K}$, the triangle

$$Fa \xrightarrow{Fu} Fb \xrightarrow{Fv} Fc \xrightarrow{\phi_a \circ Fw} \Sigma Fa$$

is a d.t. in $\mathscr{L}$.

A triangle functor is also called an *exact* functor.

Definition 2.4 (Triangulated subcategory)**.** A *triangulated subcategory* of a triangulated category $\mathscr{K}$ is an additive subcategory $\mathscr{L} \subset \mathscr{K}$ that is closed under isomorphisms and translation, and has the property that whenever two of the objects a, b, c in a d.t. $a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{w} \Sigma a$ belong to $\mathscr{L}$, then so does the third.

Remark 2.5. Note that, by the rotation axiom (TR2) and the condition that the subcategory must be closed under translation, the last requirement is equivalent to demand that, for instance, $b, c \in \mathscr{L} \Rightarrow a \in \mathscr{L}$.

One type of triangulated subcategories is of particular interest when it comes to subcategory-classifications, and is also in focus in this thesis: the *thick* subcategories.

Definition 2.6. A *thick subcategory* of an additive category is a subcategory $\mathscr{S}$ with the property that whenever $a \simeq b \oplus c$ and $a \in \mathscr{S}$, then also $b, c \in \mathscr{S}$.

Basic properties

We derive some easy consequences of the axioms. All this, and more, can be found in Neemans book [27]. Another good introduction is [16].

Proposition 2.7 (Composition of morphisms)**.** *In any distinguished triangle*

$$a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{w} \Sigma a$$

the compositions $v \circ u$ and $w \circ v$ are zero.

Proof. By TR1, $a \xrightarrow{1} a \longrightarrow 0 \longrightarrow \Sigma a$ is a d.t. The diagram

$$\begin{tikzcd} a \arrow[r, "1"] \arrow[d, "1"] & a \arrow[r] \arrow[d, "u"] & 0 \arrow[r] & \Sigma a \arrow[d, "1"] \\ a \arrow[r, "u"] & b \arrow[r, "v"] & c \arrow[r, "w"] & \Sigma a \end{tikzcd}$$

can then by TR3 be completed to a morphism of triangles with a morphism from 0 to c. The commutativity of the resulting diagram then gives that the composition $v \circ u$ must be zero. Via the rotation axiom (TR2) we also get $w \circ v = 0$. (Rotate and use the same argument). □

Proposition 2.8 (Long exact sequences)**.** *Applying* $\mathrm{Hom}(d, -) := \mathrm{Hom}_{\mathscr{K}}(d, -)$ *on a d.t.* $a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{w} \Sigma a$ *gives a long exact sequence of abelian groups:*

$$\longrightarrow \mathrm{Hom}(d, \Sigma^i a) \longrightarrow \mathrm{Hom}(d, \Sigma^i b) \longrightarrow \mathrm{Hom}(d, \Sigma^i c) \longrightarrow \mathrm{Hom}(d, \Sigma^{i+1} a) \longrightarrow$$

Proof. From the rotation axiom it suffices to show that

$$\mathrm{Hom}(d, a) \xrightarrow{u_*} \mathrm{Hom}(d, b) \xrightarrow{v_*} \mathrm{Hom}(d, c)$$

is exact. Clearly $\mathrm{Im}\, u_* \subset \mathrm{Ker}\, v_*$, as $v \circ u = 0 \Rightarrow v_* \circ u_* = 0$. To see the other inclusion, look at $f \in \mathrm{Ker}\, v_*$ and the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & d \arrow[r, "1"] & d \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\ \Sigma^{-1} c \arrow[r, "-\Sigma^{-1} w"] & a \arrow[r, "u"] & b \arrow[r, "v"] & c \end{tikzcd}$$

The rows are d.t.'s by TR1 and TR2, and the right square commutes by the choice of f. TR2 and TR3 then give us a morphism $g : d \to a$ making the diagram commute. In particular $u \circ g = f$, so $f \in \mathrm{Im}\, u_*$. □

A similar argument proves the corresponding result for $\mathrm{Hom}_{\mathscr{K}}(-, d)$.

Proposition 2.9 (Triangulated 5-Lemma)**.** *If we have a morphism of d.t.'s*

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"] & b \arrow[r] \arrow[d, "g"] & c \arrow[r] \arrow[d, "h"] & \Sigma a \arrow[d, "\Sigma f"] \\ a' \arrow[r] & b' \arrow[r] & c' \arrow[r] & \Sigma a' \end{tikzcd}$$

with f and g isomorphisms, then so is also h.

Proof. Apply the functor $\mathrm{Hom}_{\mathscr{K}}(c', -)$ on the two d.t.'s. This gives us, by Proposition 2.8, the following commutative diagram with exact rows:

$$\begin{tikzcd} \mathrm{Hom}(c', a) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}(c', b) \arrow[r] \arrow[d, "g_*"] & \mathrm{Hom}(c', c) \arrow[r] \arrow[d, "h_*"] & \mathrm{Hom}(c, \Sigma a) \arrow[r] \arrow[d, "\Sigma f_*"] & \mathrm{Hom}(c, \Sigma b) \arrow[d, "\Sigma g_*"] \\ \mathrm{Hom}(c', a') \arrow[r] & \mathrm{Hom}(c', b') \arrow[r] & \mathrm{Hom}(c', c') \arrow[r] & \mathrm{Hom}(c, \Sigma a') \arrow[r] & \mathrm{Hom}(c, \Sigma b') \end{tikzcd}$$

With f and g isomorphisms we see that all the downward morphisms are isomorphisms, except possibly h_*. But by the 5-lemma for modules we get that h_* then must be an isomorphism too.

Now, by the surjectivity of h_*, there must be a morphism $t \in \mathrm{Hom}(c', c)$ which maps to the identity on c', i.e. $h \circ t = \mathrm{id}_{c'}$. So h has a left inverse. The same argument with the contravariant Hom-functor gives us a right inverse. So h is indeed an isomorphism. $\square$

Proposition 2.10. *If* $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$ *and* $a' \longrightarrow b' \longrightarrow c' \longrightarrow \Sigma a'$ *are d.t.'s, then so is also the coproduct* $a \oplus a' \longrightarrow b \oplus b' \longrightarrow c \oplus c' \longrightarrow \Sigma(a \oplus a')$ *with morphisms inherited componentwise (the last via the natural isomorphism $\phi : \Sigma a \oplus \Sigma a' \to \Sigma(a \oplus a')$ given by the additivity of Σ).*

Proof. By TR1 there is a $d \in \mathscr{K}$ such that the upper row of the following diagram is a d.t.

$$\begin{tikzcd} a \oplus a' \arrow[r] & b \oplus b' \arrow[r] & d \arrow[r] & \Sigma(a \oplus a') \\ a \arrow[r] \arrow[u, "\binom{1}{0}"] & b \arrow[r] \arrow[u, "\binom{1}{0}"] & c \arrow[r] & \Sigma a \arrow[u, "\Sigma\binom{1}{0}"] \end{tikzcd}$$

TR3 provides us with a morphism $f : c \to d$ causing the diagram to commute. Similarly, we get a morphism $f' : c' \to d$ for the other d.t. Now, add up the two morphisms of d.t.'s (via ϕ) and look at the following resulting diagram

$$\begin{tikzcd} a \oplus a' \arrow[r] \arrow[d, equal] & b \oplus b' \arrow[r] \arrow[d, equal] & d \arrow[r] & \Sigma(a \oplus a') \arrow[d, equal] \\ a \oplus a' \arrow[r] & b \oplus b' \arrow[r] & c \oplus c' \arrow[r] \arrow[u, "(f\ f')"] & \Sigma(a \oplus a') \end{tikzcd}$$

which clearly commutes. We want to show that the morphism $(f\ f')$ is an isomorphism. Since we do not yet know that the bottom row is a d.t., we can not use the triangulated 5-lemma directly to prove this. Observe, however, that in the proof of Proposition 2.9 the only time we used the fact that the rows were d.t.'s was when assuring that the Hom-functor gave rise to long exact sequences. But this fact still holds in our bottom row since $\mathrm{Hom}(d, a \oplus a') \simeq \mathrm{Hom}(d, a) \oplus \mathrm{Hom}(d, a')$ and coproducts of long exact sequences must again be exact.

So, indeed, the 5-lemma holds in this case too. Thus the two rows are isomorphic, implying that the bottom row is also a d.t. (by TR1). □

Remark 2.11. This proof can be generalized to arbitrary coproducts, and also dualized to give the analogue result for products (see [27, Proposition 1.2.1]).

2.2 Tensor triangulated categories

In this thesis we follow the definition of Balmer [2] of a *tensor triangulated category*, which is simply a triangulated category equipped with a symmetric monoidal structure with unity that is exact in each variable. More precisely, the definition is:

Definition 2.12 (Tensor triangulated category)**.** A *tensor triangulated category* is a triangulated category with a *tensor product* $\otimes : \mathscr{K} \times \mathscr{K} \to \mathscr{K}$ such that, for any $a, b, c \in \mathscr{K}$, the following hold:

i) $a \otimes b \simeq b \otimes a$. (Symmetric)

ii) $(a \otimes b) \otimes c \simeq a \otimes (b \otimes c)$. (Monoidal)

iii) There is an object $\mathbb{1} \in \mathscr{K}$ such that $\mathbb{1} \otimes a \simeq a$ for all a. (Neutral element)

iv) $- \otimes a$ and $a \otimes -$ are triangle functors. (Exact)

Remark 2.13. Beware that the analogy to tensor products in module categories is not as one probably would expect. Firstly, we do not assume any universal property as we do with modules. Also, one would perhaps assume that *the homotopy category* $\mathrm{K}(R)$ of chain complexes of modules over a commutative ring R modulo the null-homotopic chain maps (which arguably is the "simplest" triangulated category derived from $\mathrm{mod}\, R$) would be $\otimes$-triangulated with tensor product $\otimes_R$. But this is not the case (condition **iv)** fails). However, in many $\otimes$-triangulated categories our tensor product actually descends from the module category tensor product. This, we will see, is the case in this thesis' two main examples, the stable module category over a finite group algebra and the derived category of perfect complexes over a commutative Noetherian ring.

Observe that the tensor product must commute with finite coproducts:

Proposition 2.14. *Let $\mathscr{K}$ be a $\otimes$-triangulated category. Then, for $a, b, c \in \mathscr{K}$, we have*

$$(a \oplus b) \otimes c \simeq (a \otimes c) \oplus (b \otimes c)$$

Proof. Start by taking the coproduct of the following triangles

$$a \xrightarrow{1} a \longrightarrow 0 \longrightarrow \Sigma a$$

$$0 \longrightarrow b \xrightarrow{1} b \longrightarrow 0$$

which are d.t.'s by TR1 and TR2. The resulting triangle

$$a \xrightarrow{\binom{1}{0}} a \oplus b \xrightarrow{(0\ 1)} b \xrightarrow{0} \Sigma a$$

is a d.t. by Proposition 2.10. Now, from the exactness of $\otimes$, the top row of the following diagram is a d.t.:

$$\begin{tikzcd}
a \otimes c \arrow[r, "\binom{1}{0}\otimes 1"] \arrow[d, equal] & (a \oplus b) \otimes c \arrow[r, "(0\ 1)\otimes 1"] \arrow[d, dashed, "f"] & b \otimes c \arrow[r, "0"] \arrow[d, equal] & \Sigma(a \otimes c) \arrow[d, equal] \\
a \otimes c \arrow[r, "\binom{1\otimes 1}{0}"] & (a \otimes c) \oplus (b \otimes c) \arrow[r, "(0\ 1\otimes 1)"] & b \otimes c \arrow[r, "0"] & \Sigma(a \otimes c)
\end{tikzcd}$$

The bottom row is a d.t. by the same coproduct argument as above, replacing a with $a \otimes c$ and b with $b \otimes c$. This diagram can be completed to a morphism of triangles by TR3 (via rotation). The induced morphism

$$f : (a \oplus b) \otimes c \longrightarrow (a \otimes c) \oplus (b \otimes c)$$

must now be an isomorphism by the triangulated 5-lemma (Proposition 2.9). □

Central to this thesis' discussion is the notion of *tensor ideals*. Tensor ideals are somewhat in analogy to ideals in rings, viewing coproduct as addition and tensor product as multiplication. (Although note that our system can not be considered a ring in this way, even up to isomorphism, as we do not have inverses with respect to coproduct.) The tensor ideals of interest are the thick ones:

Definition 2.15 (Thick tensor ideal)**.** A *thick tensor ideal* is a thick triangulated subcategory $\mathscr{J}$ of a $\otimes$-triangulated category $\mathscr{K}$ that obeys the following criterion:

$$a \in \mathscr{K} \text{ and } b \in \mathscr{J} \Rightarrow a \otimes b \in \mathscr{J}$$

Note that, by the symmetric property of $\otimes$, this will also imply $b \otimes a \in \mathscr{J}$. Also, check that intersections of thick $\otimes$-ideals are again thick $\otimes$-ideals.

For a collection $\mathscr{S} \subset \mathscr{K}$ we adopt the notation $\langle \mathscr{S} \rangle$ for the smallest thick $\otimes$-ideal containing $\mathscr{S}$.

Prime ideals and Zariski topology

From now on let $\mathscr{K}$ always be an *essentially small* $\otimes$-triangulated category. Recall that $\mathscr{K}$ is essentially small if the collection of isomorphism classes in $\mathrm{Ob}(\mathscr{K})$ is a set.

In algebraic geometry the set of prime ideals of a commutative ring is repeatedly turned into a topological space via the *Zariski topology*. Here, we follow the approach of Balmer [2] to transfer this idea to $\otimes$-triangulated categories with the aim of using it as a tool in the study of the thick $\otimes$-ideals. We introduce primes in the most obvious way, though pay attention to the slightly unfamiliar definition of the topology.

Definition 2.16 (Prime)**.** A *prime* of $\mathscr{K}$ is a proper thick $\otimes$-ideal $\mathscr{P} \subsetneq \mathscr{K}$ such that

$$a \otimes b \in \mathscr{P} \Rightarrow a \in \mathscr{P} \text{ or } b \in \mathscr{P}$$

Definition 2.17 (Spectrum)**.** The *spectrum* of $\mathscr{K}$, denoted $\operatorname{Spc}\mathscr{K}$, is the set of all primes of $\mathscr{K}$.

Remark 2.18. Note that $\operatorname{Spc}\mathscr{K}$ is indeed a set, as $\mathscr{K}$ is assumed to be essentially small and triangulated subcategories are closed under isomorphisms.

We now define the announced topology:

Definition 2.19 (Zariski topology)**.** The *Zariski topology on* $\operatorname{Spc}\mathscr{K}$ is the topology given by defining the closed subsets to be the sets

$$\mathrm{Z}(\mathscr{S}) = \{\mathscr{P} \in \operatorname{Spc}\mathscr{K} \mid \mathscr{S} \cap \mathscr{P} = \emptyset\}$$

for any collection of objects $\mathscr{S} \subset \mathscr{K}$.

That is, we can construct all our closed subsets by starting with a collection of objects in $\mathscr{K}$ and collect all the primes that do not intersect with it. The open subsets of our topology are, by definition, the complements of the closed. Denote them by:

$$\mathrm{U}(\mathscr{S}) = \operatorname{Spc}\mathscr{K} \setminus Z(\mathscr{S}) = \{\mathscr{P} \in \operatorname{Spc}\mathscr{K} \mid \mathscr{S} \cap \mathscr{P} \neq \emptyset\}$$

(i.e. all the primes that intersect with our chosen collection).

The support

The closed subsets corresponding to the individual objects in $\mathscr{K}$ will play an important role in the task of classifying thick $\otimes$-ideals of $\mathscr{K}$ later in this thesis, and, given their shared properties with different entities dubbed supports found in nature, they are named accordingly by Balmer [2]:

Definition 2.20. The *support* of $a \in \mathscr{K}$ is defined to be

$$\operatorname{supp} a := Z(\{a\}) = \{\mathscr{P} \in \operatorname{Spc}\mathscr{K} \mid a \notin \mathscr{P}\}$$

Observe that, since triangulated subcategories are closed under isomorphisms

$$a \simeq b \Rightarrow \operatorname{supp} a = \operatorname{supp} b$$

For a collection of objects $\mathscr{S} \subset \mathscr{K}$ we define *the support of* $\mathscr{S}$ to be the union of the supports of its objects:

Definition 2.21. For $\mathscr{S} \subset \mathscr{K}$, define

$$\operatorname{supp} \mathscr{S} := \bigcup_{a \in \mathscr{S}} \operatorname{supp} a$$

Note that this is just the primes not encapsulating $\mathscr{S}$:

$$\operatorname{supp} \mathscr{S} = \{\mathscr{P} \in \operatorname{Spc} \mathscr{K} \mid \mathscr{S} \not\subset \mathscr{P}\}$$

Observe also that $\operatorname{supp} a$ for $a \in \mathscr{K}$ is a closed subset, but $\operatorname{supp} \mathscr{S}$ for a collection $\mathscr{S} \subset \mathscr{K}$ is not necessarily closed.

The reason why we are interested in supports is that they satisfy some very handy properties:

Proposition 2.22. *The following hold on* $a, b, c \in \mathscr{K}$*:*

i) $\operatorname{supp} 0 = \emptyset$ *and* $\operatorname{supp} \mathbb{1} = \operatorname{Spc} \mathscr{K}$

ii) $\operatorname{supp}(a \oplus b) = \operatorname{supp} a \cup \operatorname{supp} b$

iii) $\operatorname{supp}(a \otimes b) = \operatorname{supp} a \cap \operatorname{supp} b$

iv) $\operatorname{supp} \Sigma a = \operatorname{supp} a$

v) $\operatorname{supp}(a) \subset \operatorname{supp} b \cup \operatorname{supp} c$ *for a d.t.* $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$.

Proof.

i) $\operatorname{supp} 0 = \emptyset$ because all additive subcategories, and thus primes, must contain 0. $\operatorname{supp} \mathbb{1} = \operatorname{Spc} \mathscr{K}$ as no proper $\otimes$-ideal can contain $\mathbb{1}$.

ii) If $\mathscr{P} \in \operatorname{supp}(a \oplus b)$, we have $a \oplus b \notin \mathscr{P}$ and a, b cannot both be in $\mathscr{P}$. Hence either $\mathscr{P} \in \operatorname{supp} a$ or $\mathscr{P} \in \operatorname{supp} b$.

Conversely, if $\mathscr{P} \notin \operatorname{supp}(a \oplus b)$ we get $a \oplus b \in \mathscr{P}$, implying $a, b \in \mathscr{P}$ by the thickness of $\mathscr{P}$. But that gives $\mathscr{P} \notin \operatorname{supp} a \cup \operatorname{supp} b$.

iii) If $\mathscr{P} \in \operatorname{supp}(a \otimes b)$, we have $a \otimes b \notin \mathscr{P}$, ensuring that none of a, b could be in $\mathscr{P}$ (because it is a $\otimes$-ideal). Thus $\mathscr{P} \in \operatorname{supp} a \cap \operatorname{supp} b$.

On the other hand, if $\mathscr{P} \notin \operatorname{supp}(a \otimes b)$, we get $a \otimes b \in \mathscr{P}$ and, since $\mathscr{P}$ is prime, at least one of a, b must be in $\mathscr{P}$. Hence, for instance, $\mathscr{P} \notin \operatorname{supp} a$.

iv) Triangulated subcategories, and therefore also primes, are closed under translation.

v) Given a prime $\mathscr{P}$, it is impossible to have $b, c \in \mathscr{P}$ and $a \notin \mathscr{P}$ (since $\mathscr{P}$ is triangulated). Thus if $\mathscr{P} \in \operatorname{supp} a$, we can not both have $\mathscr{P} \notin \operatorname{supp} b$ and $\mathscr{P} \notin \operatorname{supp} c$, hence the result. □

Remark 2.23. By **iii)** and the fact that $\mathrm{Z}(\mathscr{S}) = \bigcap_{a \in \mathscr{S}} \operatorname{supp} a$ for a collection of objects $\mathscr{S}$, we see that $\{\operatorname{supp} a \mid a \in \mathscr{K}\}$ forms a basis for the closed subsets of $\operatorname{Spc} \mathscr{K}$.

Support data

The properties of Proposition 2.22 are indeed so nice that they are given a name:

Definition 2.24. A *support data* on a $\otimes$-triangulated category $\mathscr{K}$ is a pair (X, σ) of a topological space X and an assignment σ associating to each $a \in \mathscr{K}$ a closed subset $\sigma(a) \subset X$ satisfying the rules of Proposition 2.22 (with X in the place of $\operatorname{Spc} \mathscr{K}$).

Moreover, a *morphism* of support data on $\mathscr{K}$

$$f : (X, \sigma) \longrightarrow (Y, \tau)$$

is a continuous function $f : X \to Y$ with $\sigma(a) = f^{-1}(\tau(a))$ for all $a \in \mathscr{K}$. Such a morphism is called an *isomorphism* if f is also a homeomorphism.

We see that $(\operatorname{Spc} \mathscr{K}, \operatorname{supp})$ holds a very special place among support data, in the sense that all other maps into it:

Theorem 2.25. *Given a support data (X, σ) on $\mathscr{K}$, there is a morphism of support data*

$$f : (X, \sigma) \longrightarrow (\operatorname{Spc} \mathscr{K}, \operatorname{supp})$$

given by $f(x) = \{a \in \mathscr{K} \mid x \notin \sigma(a)\}$

Proof. It is straightforward to check that $f(x)$ is a prime, for instance it is thick by

$$x \notin \sigma(a_1 \oplus a_2) = \sigma(a_1) \cup \sigma(a_2) \Rightarrow x \notin \sigma(a_1), \sigma(a_2)$$

Furthermore, the equivalence $x \in \sigma(a) \Leftrightarrow a \notin f(x) \Leftrightarrow f(x) \in \operatorname{supp} a$ gives us $f^{-1}(\operatorname{supp} a) = \sigma(a)$. This also assures the continuity of f, since $\{\operatorname{supp} a \mid a \in \mathscr{K}\}$ is a basis for the closed subsets of $\operatorname{Spc} \mathscr{K}$ by Remark 2.23. □

The above morphism is actually unique (see [2, Theorem 3.2]).

CHAPTER 3

CLASSIFICATION OF THICK $\otimes$-IDEALS

One of the main goals of category theorists in their quest for understanding the bigger structures of a category is to decipher its subcategories. Hence it is of great interest to find ways to describe these. Often one would look at a particular type of subcategories, in the case of triangulated categories recurringly the thick triangulated ones. A number of such classifications has already been accomplished. The first landmark classification was that of Devinatz, Hopkins and Smith [10] of the thick subcategories of the *stable homotopy category of spectra*, whose idea soon spread to other categories, notably the two examples we treat in this thesis.

In the context of $\otimes$-triangulated categories, Balmer [2, Theorem 4.10] found, using the support defined in the previous section, a classification of the "radical" thick $\otimes$-ideals, a result we present in detail here. Unfortunately, Balmer's classification does not seem to be very helpful at first glance, and the author does not employ this theorem further in his thesis. Here, however, we pick up this thread, and find that under certain circumstances, i.e. subjecting our category to a number of restrictions, Balmer's classification can be translated into something more meaningful. This turns out to be successful in our two examples, and thus providing a new way of arriving at those (already known) classifications.

3.1 Balmer's classification

Balmer's classification is established through the introduction of two elementary notions.

The radical

In complete analogy to commutative ring theory, define the *radical* of a thick ⊗-ideal to be

Definition 3.1 (Radical)**.** The *radical* $\sqrt{\mathscr{J}}$ of a tick ⊗-ideal $\mathscr{J} \subset \mathscr{K}$ is

$$\sqrt{\mathscr{J}} = \{a \in \mathscr{K} \mid \exists n \geq 1, a^{\otimes n} \in \mathscr{J}\}$$

If $\sqrt{\mathscr{J}} = \mathscr{J}$, we say that $\mathscr{J}$ is *radical.*

As usual, we get the following theorem:

Theorem 3.2. *The radical of a thick ⊗-ideal $\mathscr{J} \subset \mathscr{K}$ is equal to the intersection of all the primes in* $\mathrm{Spc}(\mathscr{K})$ *containing $\mathscr{J}$:*

$$\sqrt{\mathscr{J}} = \bigcap_{\mathscr{J} \subset \mathscr{P}} \mathscr{P}$$

To prove this we will need to introduce a technical lemma, asserting that for a pair $(\mathscr{S}, \mathscr{J})$ of a ⊗-*multiplicative* collection $\mathscr{S}$ (defined to be a collection of objects closed under tensor multiplication containing $\mathbb{1}$) and a thick ⊗-ideal $\mathscr{J}$ with $\mathscr{J} \cap \mathscr{S} = \emptyset$, we can always construct a prime encapsulating $\mathscr{J}$ that still does not intersect with $\mathscr{S}$:

Lemma 3.3. *Let $\mathscr{J} \subset \mathscr{K}$ be a thick ⊗-ideal, and $\mathscr{S} \subset \mathscr{K}$ a ⊗-multiplicative collection of objects disjoint from $\mathscr{J}$. Then there exists a prime $\mathscr{P} \in \mathrm{Spc}\,\mathscr{K}$ with $\mathscr{J} \subset \mathscr{P}$ and $\mathscr{P} \cap \mathscr{S} = \emptyset$.*

Proof. We will show this by using Zorn's Lemma on the set $\mathbf{F}$ of the isomorphism classes of the thick ⊗-ideals $\mathscr{A}$ satisfying the wanted properties of the prime, together with the condition $a \otimes c \in \mathscr{A} \Rightarrow a \in \mathscr{A}$ for $c \in \mathscr{S}$ and $a \in \mathscr{K}$. We will see that an element maximal with respect to inclusion in $\mathbf{F}$ is prime.

Observe that $\mathbf{F}$ is non-empty, as $\mathscr{A}_0 := \{a \in \mathscr{K} \mid \exists c \in \mathscr{S}, a \otimes c \in \mathscr{J}\}$ is easily seen to be an element: First, $\mathscr{A}_0$ clearly satisfies the required properties to be in $\mathbf{F}$. And moreover, it is, for instance, thick by seeing that if $a \oplus b \in \mathscr{A}_0$, there must be a $c \in \mathscr{S}$ such that $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \in \mathscr{J}$, which by the thickness of $\mathscr{J}$ would imply $a, b \in \mathscr{A}_0$.

Let $\mathscr{P}$ be a an element in $\mathbf{F}$ maximal with respect to inclusion. Such an element exists by Zorn's Lemma and the fact that, for any chain $\mathscr{B}_1 \subset \mathscr{B}_2 \subset \cdots$ in $\mathbf{F}$, the union $\bigcup_i \mathscr{B}_i \in \mathbf{F}$ is an upper boundary. We aim to show that $\mathscr{P}$ must be prime.

To settle this, suppose $a \otimes b \in \mathscr{P}$ with $b \notin \mathscr{P}$, and look at the thick ⊗-ideal

$$\mathscr{A}_1 := \{d \in \mathscr{K} \mid a \otimes d \in \mathscr{P}\}$$

We know that $\mathscr{A}_1$ can not be in $\mathbf{F}$, because $b \in \mathscr{A}_1$ implies $\mathscr{P} \subsetneq \mathscr{A}_1$ and $\mathscr{P}$ is maximal. We see easily that $\mathscr{A}_1$ satisfies all the requirements to be in $\mathbf{F}$, except possibly $\mathscr{A}_1 \cap \mathscr{S} = \emptyset$. This condition must therefore fail. Thus we have a $d \in \mathscr{A}_1 \cap \mathscr{S}$, that is, a $d \in \mathscr{S}$ with $a \otimes d \in \mathscr{P}$. By the last condition imposed on the elements of $\mathbf{F}$ this gives $a \in \mathscr{P}$, proving that $\mathscr{P}$ is prime. □

We can now prove Theorem 3.2:

Proof. If $a \in \sqrt{\mathscr{J}}$, we have $a^{\otimes n} \in \mathscr{J}$ for some n. For each prime $\mathscr{P}$ containing $\mathscr{J}$ we must therefore, by the property of primes, have $a \in \mathscr{P}$. Hence $a \in \bigcap_{\mathscr{J} \subset \mathscr{P}} \mathscr{P}$.

Now, suppose $a \notin \sqrt{\mathscr{J}}$. Then the $\otimes$-multiplicative collection

$$\mathscr{S} = \{a^{\otimes n} \mid n \geq 1\} \cup \{\mathbb{1}\}$$

does not intersect with $\mathscr{J}$. By Lemma 3.3 we then have that there is a prime containing $\mathscr{J}$ that is disjoint from $\mathscr{S}$, and thus $a \notin \bigcap_{\mathscr{J} \subset \mathscr{P}} \mathscr{P}$. □

Note that this gives us that, by being an intersection of such, the radical of a thick $\otimes$-ideal is again a thick $\otimes$-ideal.

The subcategory supported on a set of primes

Definition 3.4. *The subcategory supported on* a set of primes $Y \subset \operatorname{Spc} \mathscr{K}$ is the full subcategory $\mathscr{K}_Y$ of $\mathscr{K}$ with objects given by

$$\mathscr{K}_Y := \{a \in \mathscr{K} \mid \operatorname{supp} a \subset Y\}$$

It is then easy to see the following

Lemma 3.5. *Given $Y \subset \operatorname{Spc} \mathscr{K}$, $\mathscr{K}_Y$ is the intersection of the primes not in Y:*

$$\mathscr{K}_Y = \bigcap_{\mathscr{P} \notin Y} \mathscr{P}$$

Proof. Let $a \in \mathscr{K}_Y$. If there is a prime $\mathscr{P} \notin Y$ that does not contain a, then $\mathscr{P} \in \operatorname{supp} a \subset Y$, a contradiction on $\mathscr{P}$. Hence $a \in \bigcap_{\mathscr{P} \notin Y} \mathscr{P}$. It is also clear that the support of any object in $\bigcap_{\mathscr{P} \notin Y} \mathscr{P}$ must be a set of primes that is in Y. □

We show that $\mathscr{K}_Y$ is a thick $\otimes$-ideal:

Lemma 3.6. *For $Y \subset \operatorname{Spc} \mathscr{K}$, $\mathscr{K}_Y$ is a thick $\otimes$-ideal.*

Proof. This follows straightforward from Proposition 2.22:

Thick: Let $c \simeq a \oplus b$ be in $\mathscr{K}_Y$. Then $\operatorname{supp} c = \operatorname{supp}(a \oplus b) = \operatorname{supp} a \cup \operatorname{supp} b \subset Y$. In particular, $\operatorname{supp} a \subset Y$ and $\operatorname{supp} b \subset Y$, so $a, b \in \mathscr{K}_Y$.

$\otimes$-ideal: Let $a \in \mathscr{K}_Y$ and $b \in \mathscr{K}$. We then get $\operatorname{supp}(a \otimes b) = \operatorname{supp} a \cap \operatorname{supp} b \subset \operatorname{supp} a \subset Y$. Thus $a \otimes b \in \mathscr{K}_Y$.

Triangulated: Let $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$ be a d.t. with $b, c \in \mathscr{K}_Y$. Then $\operatorname{supp} a \subset \operatorname{supp} b \cup \operatorname{supp} c \subset Y$, hence we get $a \in \mathscr{K}_Y$. □

The classification theorem

With this settled we can prove Balmer's classification theorem:

Theorem 3.7 (Balmer)**.** *Let* $\mathbf{S}$ *be the set of all the subsets* $Y \subset \mathrm{Spc}(\mathscr{K})$ *that can be written as the support of a collection of objects (i.e.* $Y = \mathrm{supp}\,\mathscr{M}$ *for some* $\mathscr{M} \subset \mathscr{K}$*), and let* $\mathbf{R}$ *be the set of all radical thick* $\otimes$*-ideals. Then there is an order-preserving bijection* $F : \mathbf{S} \to \mathbf{R}$ *given by*

$$F : Y \longmapsto \mathscr{K}_Y$$

with inverse

$$F^{-1} : \mathscr{J} \longmapsto \mathrm{supp}\,\mathscr{J}$$

Proof. We first check that both assignments are well-defined: Clearly $\mathrm{supp}\,\mathscr{J} \in \mathbf{S}$. Moreover, $\mathscr{K}_Y$ is radical because by Proposition 2.22

$$\mathrm{supp}\,a^{\otimes n} = \mathrm{supp}\,a \cap ... \cap \mathrm{supp}\,a = \mathrm{supp}\,a$$

which gives $a^{\otimes n} \in \mathscr{K}_Y \Rightarrow a \in \mathscr{K}_Y$. The two assignments are also clearly order-preserving. We now show that they are bijections by proving that $F^{-1}F$ and FF^{-1} are equal to the respective identities:

($\mathrm{supp}\,\mathscr{K}_Y = Y$). By Lemma 3.5 we know that $\mathscr{K}_Y$ is the intersection of all the primes not in Y. With $Y = \mathrm{supp}\,\mathscr{M}$ these primes must be exactly the ones encapsulating $\mathscr{M}$. This gives us $\mathscr{M} \subset \mathscr{K}_Y$, which again implies $Y = \mathrm{supp}\,\mathscr{M} \subset \mathrm{supp}\,\mathscr{K}_Y$. For the other inclusion note that if a prime $\mathscr{P}$ is not in $\mathrm{supp}\,\mathscr{M}$, it must contain $\mathscr{M}$ and therefore also $\mathscr{K}_Y$ (the intersection of all such primes). Hence $\mathscr{P}$ is not in $\mathrm{supp}\,\mathscr{K}_Y$ either.

($\mathscr{K}_{\mathrm{supp}\,\mathscr{J}} = \mathscr{J}$). As above, $\mathscr{K}_{\mathrm{supp}\,\mathscr{J}}$ is the intersection of all primes containing $\mathscr{J}$. With $\mathscr{J}$ radical this is just $\mathscr{J}$ by Theorem 3.2. □

Recall that a topological space X is called *Noetherian* if it satisfies the *descending chain condition* on closed subsets, that is, for any chain

$$Y_1 \supset Y_2 \supset Y_3 \supset \cdots$$

of closed subsets in X there is an R such that $Y_r = Y_{r+1}$ for all $r \geq R$.

We then see that if $\mathrm{Spc}\,\mathscr{K}$ is Noetherian, to demand that $Y = \mathrm{supp}\,\mathscr{S}$ for a collection of objects $\mathscr{S} \subset \mathscr{K}$ is equivalent to require Y to be *specialization closed*, i.e. a union of closed subsets. This because $\mathrm{Spc}\,\mathscr{K}$ Noetherian would imply that any closed subset $\mathrm{Z}(\mathscr{S}) = \bigcap_{a \in \mathscr{S}} \mathrm{supp}\,a$ could be rewritten as a finite intersection $\bigcap_{i=1}^{n} \mathrm{supp}\,a_i = \mathrm{supp}(a_1 \otimes \cdots \otimes a_n)$. Thus with $\mathrm{Spc}\,\mathscr{K}$ Noetherian the above theorem is a bijection between the specialization closed subsets of $\mathrm{Spc}\,\mathscr{K}$ and the radical thick $\otimes$-ideals of $\mathscr{K}$.

3.2 Conditions for isomorphism of support data

The classification theorem just given is not necessarily very beneficial, as the pair $(\operatorname{Spc}\mathscr{K}, \operatorname{supp})$ appears to be as difficult to compute as the $\otimes$-ideals themselves. However, under some extra conditions $(\operatorname{Spc}\mathscr{K}, \operatorname{supp})$ can be proven to be isomorphic to another, more tangible, support data on $\mathscr{K}$, providing us with a translation of Theorem 3.7 into something much more sensible.

In this section we formulate a set of constraints on $\mathscr{K}$ and its support data (X, σ), which indeed will prove sufficient to turn the morphism in Theorem 2.25 into an isomorphism of support data. These conditions turn out to be fulfilled in the two main examples presented in this thesis, the stable category of modules over a finite group algebra and the derived category of perfect complexes of modules over a commutative Noetherian ring (both essentially small $\otimes$-triangulated categories). This will enable us to translate Theorem 3.7 into the renowned classification theorems of Benson-Carlson-Rickard [7] and Hopkins-Neeman [17, 25].

This section, together with the proofs of the corresponding classification theorems, is the main part of this thesis, as it is the only that claims some sort of originality. This is also where our approach differs from Balmer's. Balmer [2] proves the isomorphism of the support data *given* the classification theorems; here we prove the isomorphism independently, allowing the isomorphism to deliver the classification.

The conditions we are soon to impose on $\mathscr{K}$ and its support data will turn out to be quite elaborate. Although they are enough to make (X, σ) isomorphic to $(\operatorname{Spc}\mathscr{K}, \operatorname{supp})$, it is not clear whether they are all needed. It might well be that a simpler set of assumptions will suffice, and given more time it would surely have been interesting to do further research on what conditions are in fact necessary for the isomorphism to hold.

The conditions on $\mathscr{K}$

The first imposed condition on $\mathscr{K}$ is that it should be contained in a "bigger" $\otimes$-triangulated category $\mathscr{L}$, which allows for arbitrary coproducts (with which the tensor product still commutes). Moreover, $\mathscr{K}$ is required to hold a special place in this category. To describe this we will need to introduce two basic notions:

Definition 3.8. A *compact* object in a category $\mathscr{L}$ is an object $a \in \mathscr{L}$ such that for any set $\{b_i \mid i \in I\}$ of objects in $\mathscr{L}$ whose coproduct exists, the canonical map

$$\bigoplus_{i\in I} \operatorname{Hom}_{\mathscr{L}}(a, b_i) \longrightarrow \operatorname{Hom}_{\mathscr{L}}(a, \bigoplus_{i\in I} b_i)$$

is an isomorphism.

Note that this is equivalent to saying that all maps $a \to \bigoplus_{i\in I} b_i$ factors through $\bigoplus_{i\in I'} b_i$ for a finite subset $I' \subset I$.

Definition 3.9. A *localizing* subcategory of a triangulated category $\mathscr{L}$ is a thick triangulated subcategory of $\mathscr{L}$ that is closed with respect to formation of arbitrary coproducts.

For a collection $\mathscr{S} \subset \mathscr{L}$ we denote by $\operatorname{loc}\langle\mathscr{S}\rangle$ the smallest localizing subcategory containing $\mathscr{S}$.

The restriction we now impose on $\{\mathscr{K}, \mathscr{L}\}$ is that $\mathscr{L} = \operatorname{loc}\langle\mathscr{K}\rangle$ and that $\mathscr{K}$ represents exactly the compact objects of $\mathscr{L}$.

Remark 3.10. This is the same as saying that $\mathscr{L}$ is *compactly generated* with $\mathscr{K}$ its compact objects (see for instance [28, Definition 2.5]).

Now, in this setting, the highly regarded *Brown Representability Theorem* holds (see [26, Theorem 4.1]), which as a corollary asserts that every triangle functor $F : \mathscr{L} \to \mathscr{L}$ which preserves coproducts has a right adjoint. In particular, the functor $- \otimes a$ for any $a \in \mathscr{L}$ must have one. Denote its adjoint by $\mathcal{H}om(a, -)$, and define the *dual* of a to be

$$\mathrm{D}(a) := \mathcal{H}om(a, \mathbb{1})$$

The last condition we now enforce on $\{\mathscr{K}, \mathscr{L}\}$ is that all elements of $\mathscr{K}$ should be *strongly dualizable*:

Definition 3.11. An object $a \in \mathscr{L}$ is called *strongly dualizable* if there are natural isomorphisms

$$\mathrm{D}(a) \otimes b \simeq \mathcal{H}om(a, b)$$

for all $b \in \mathscr{L}$.

All these assumptions on $\{\mathscr{K}, \mathscr{L}\}$ finally enable us to invoke a result from the much cited work of Hovey, Palmieri and Strickland [18], which, reformulated to fit into our environment, reads

Theorem 3.12 (Finite localization)**.** *For any ⊗-ideal $\mathscr{J} \subset \mathscr{K}$ in the setting described above, there exists a functor $\mathrm{L}_{\mathscr{J}} : \mathscr{L} \to \mathscr{L}$ such that for any $a \in \mathscr{K}$*

i) $\mathrm{L}_{\mathscr{J}}\, a = 0 \Leftrightarrow a \in \mathscr{J}$

ii) $\mathrm{L}_{\mathscr{J}}\, a = a \otimes \mathrm{L}_{\mathscr{J}}\, \mathbb{1}$

We refer to [18, Theorem 3.3.3] for details. This provides us with a crucial step in the proof of our pre-announced isomorphism theorem:

The isomorphism theorem

Theorem 3.13 (Isomorphism of support data)**.** *Let (X, σ) be a support data on $\mathscr{K}$ such that*

i) *X is Noetherian and T_0.*

ii) *Every irreducible closed subset $Z \subset X$ can be written as the closure of a point.*

iii) *All closed subsets of* X *are on the form* $\sigma(a)$ *for an* $a \in \mathscr{K}$.

Suppose there is a $\otimes$*-triangulated category* $\mathscr{L}$ *admitting arbitrary coproducts (commuting with the tensor product), such that*

iv) $\mathscr{L} = \operatorname{loc}\langle\mathscr{K}\rangle$ *and* $\mathscr{K}$ *represents the compact objects in* $\mathscr{L}$.

v) *The objects of* $\mathscr{K}$ *are strongly dualizable in* $\mathscr{L}$.

vi) *The definition of* σ *can be extended to* $\mathscr{L}$ *such that* $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ *and* $\sigma(a) = \emptyset \Rightarrow a = 0$ *for objects in* $\mathscr{L}$.

Then $(X, \sigma) \simeq (\operatorname{Spc}\mathscr{K}, \operatorname{supp})$ *as support data.*

Proof. We have to show that the function $f(x) = \{a \in \mathscr{K} \mid x \notin \sigma(a)\}$ in Theorem 2.25 is a homeomorphism. The injectivity is the easiest part: Suppose $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then, since X is T_0, there exists a closed subset $\sigma(a)$ containing just one of them, say $x_1 \in \sigma(a)$ and $x_2 \notin \sigma(a)$. This gives $a \notin f(x_1)$ and $a \in f(x_2)$, i.e. $f(x_1) \neq f(x_2)$.

Now, to prove the surjectivity, choose a prime $\mathscr{P} \in \operatorname{Spc}\mathscr{K}$. We first show that the following holds for objects in $\mathscr{K}$:

$$\sigma(a) = \sigma(b) \text{ and } a \in \mathscr{P} \Rightarrow b \in \mathscr{P} \tag{3.1}$$

For suppose $b \notin \mathscr{P}$. Then we have from Theorem 3.12 a *finite localization*

$$\mathrm{L} := \mathrm{L}_{\mathscr{P}} : \mathscr{L} \longrightarrow \mathscr{L}$$

such that $\mathrm{L}\,a = 0$ and $\mathrm{L}\,b \neq 0$, yielding the following contradiction:

$$\emptyset = \sigma(\mathrm{L}\,a) = \sigma(a \otimes \mathrm{L}\,\mathbb{1}) = \sigma(a) \cap \sigma(\mathrm{L}\,\mathbb{1}) = \sigma(b) \cap \sigma(\mathrm{L}\,\mathbb{1}) = \sigma(b \otimes \mathrm{L}\,\mathbb{1}) = \sigma(\mathrm{L}\,b) \neq \emptyset$$

Since X is Noetherian, we have that $W := \bigcap_{c \notin \mathscr{P}} \sigma(c)$ can be rewritten as a finite intersection $\bigcap_{i=1}^{n} \sigma(c_i)$ with $c_i \notin \mathscr{P}$. Thus $W = \sigma(c_1 \otimes \cdots \otimes c_n)$. Let $\tilde{c} := c_1 \otimes \cdots \otimes c_n$.

Now, W must be irreducible. For if

$$\sigma(\tilde{c}) = \sigma(b_1) \cup \sigma(b_2) = \sigma(b_1 \oplus b_2)$$

with $\sigma(b_1)$ and $\sigma(b_2)$ proper subsets, we would from the definition of W have gotten $b_1, b_2 \in \mathscr{P}$. But this would by (3.1) have given $\tilde{c} \in \mathscr{P}$, a contradiction as $\mathscr{P}$ is prime. Thus $W = \overline{\{x\}}$ for a point $x \in X$.

We now see that $a \in \mathscr{P} \Rightarrow x \notin \sigma(a)$. This because $x \in \sigma(a)$ would yield $\sigma(\tilde{c}) = \overline{\{x\}} \subset \sigma(a)$ and $\sigma(\tilde{c}) = \sigma(\tilde{c}) \cap \sigma(a) = \sigma(\tilde{c} \otimes a)$, implying $a \notin \mathscr{P}$.

So $x \in W \subset \sigma(a)$ for $a \notin \mathscr{P}$, and $x \notin \sigma(a)$ for $a \in \mathscr{P}$. Hence

$$f(x) = \{a \in \mathscr{K} \mid x \notin \sigma(a)\} = \mathscr{P}$$

which proves the surjectivity of f.

This, in turn, gives $f(\sigma(a)) = \operatorname{supp} a$, which together with the fact that all closed subsets of X are of the form $\sigma(a)$ assures the continuity of f^{-1}. $\square$

Remark 3.14. Note that the two first requirements in Theorem 3.13 could be abbreviated to X being *spectral* and Noetherian.

Before we move on to our two examples, observe that when we have $\mathscr{K}$ as above, Theorem 3.7 will in fact enable us to classify *every* thick ⊗-ideals, as they all prove to be radical:

Proposition 3.15. *Let $\mathscr{K}$ be as in Theorem 3.13. Then all thick ⊗-ideals $\mathscr{J}$ of $\mathscr{K}$ are radical.*

Proof. If we can show that $a \otimes a \in \mathscr{J}$ implies $a \in \mathscr{J}$, we are finished, since that would yield $a^{\otimes 2^n} \in \mathscr{J} \Rightarrow a \in \mathscr{J}$ for any n, by induction. Thus suppose $a \otimes a \in \mathscr{J}$.

From the unit/counit definition of adjointness we get that $\mathbb{1} \otimes a \simeq a$ is a retract of $\mathcal{H}om(a, \mathbb{1} \otimes a) \otimes a \simeq \mathcal{H}om(a, a) \otimes a$. But since a is strongly dualizable, $\mathcal{H}om(a, a) \simeq \mathrm{D}(a) \otimes a$, implying $a \in \langle a \otimes a \rangle \subset \mathscr{J}$, as wanted. □

CHAPTER 4

EXAMPLE I: STABLE MODULE CATEGORIES

Let kG be the group algebra of a finite group G over a field k, and let $\operatorname{mod} kG$ denote the category of finitely generated left kG-modules. If we look at this category modulo the maps that factor through an injective object, the result turns out to be a $\otimes$-triangulated category. Let us first define this category more precisely:

Definition 4.1. *The stable module category* $\underline{\operatorname{mod}}\, kG$ of a finite group algebra kG is the category whose objects are inherited from $\operatorname{mod} kG$ and the morphisms are the equivalence classes of morphisms in $\operatorname{mod} kG$ modulo those factoring through an injective module.

In other words, two maps $f, g : X \to Y$ in $\operatorname{mod} kG$ are equal in $\underline{\operatorname{mod}}\, kG$ if there exists an injective module $I \in \operatorname{mod} kG$ and maps h, i such that the following diagram commutes

$$\begin{tikzcd} & I \arrow[dr, "i"] & \\ X \arrow[ur, "h"] \arrow[rr, "f-g"] & & Y \end{tikzcd}$$

Remark 4.2. Note that this actually makes the injective modules "vanish", as in $\underline{\operatorname{mod}}\, kG$ they will all be isomorphic to 0 (their identity map, which definitely factors through an injective module, will be equal to the zero map).

The stable module category $\underline{\operatorname{mod}}\, kG$ is our first example of a category satisfying the requirements of Theorem 3.13. As we will see, this will translate Theorem 3.7 into the classification of thick $\otimes$-ideals of $\underline{\operatorname{mod}}\, kG$ given by Benson, Carlson and Rickard in [7]. Before that, however, let us take a look at how $\underline{\operatorname{mod}}\, kG$ is a $\otimes$-triangulated category.

4.1 The triangulation and tensor product

In the following we will be working in $\operatorname{mod} kG$, unless explicitly stated that we are in the stable category. For an introduction to $\operatorname{mod} kG$ and group algebras in general, we refer the reader to [4, Chapter 3]. We will need the following characteristics of $\operatorname{mod} kG$:

i) The algebra kG is *self-injective*, meaning it is injective as a module over itself. This, in particular, implies that injective and projective modules coincide in $\operatorname{mod} kG$ [4, Proposition 3.1.2].

ii) Tensor products of any module by a projective/injective module in $\operatorname{mod} kG$ are still projective/injective [4, Proposition 3.1.5].

The field k is looked upon as a kG-module in the trivial way, i.e. letting $gm := m$ for any $g \in G$ and $m \in k$.

Now, to present the triangulation of $\underline{\operatorname{mod}}\, kG$ we begin with the, quite laborious, task of defining and assuring the functoriality of the translation functor. The following exposition follows and works out the details of Happel's presentation [15, Chapter I.2].

The translation functor

Since $\operatorname{mod} kG$ has enough injectives, we have, for any kG-module M, a short exact sequence

$$0 \longrightarrow M \longrightarrow I_M \longrightarrow M/I_M \longrightarrow 0$$

with I_M injective. We now define Σ on objects by $\Sigma M := M/I_M$. Later we will see that, in the stable category, ΣM is independent of what injective module I_M we choose, so this actually makes sense. To define Σ on a map $u : M \to N \in \operatorname{Hom}_{\operatorname{mod} kG}$, start with the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\iota_M"] \arrow[d, "u"] & I_M \arrow[r, "\pi_M"] & \Sigma M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r, "\iota_N"] & I_N \arrow[r, "\pi_N"] & \Sigma N \arrow[r] & 0
\end{tikzcd}$$

Since I_N is injective and ι_M is a monomorphism, there is a map $I_u : I_M \to I_N$ making the first square in the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\iota_M"] \arrow[d, "u"] & I_M \arrow[r, "\pi_M"] \arrow[d, "I_u"] & \Sigma M \arrow[r] \arrow[d, dashed, "\Sigma u"] & 0 \\
0 \arrow[r] & N \arrow[r, "\iota_N"] & I_N \arrow[r, "\pi_N"] & \Sigma N \arrow[r] & 0
\end{tikzcd}$$

Now, define Σu to be the composition $\pi_N I_u \pi_M^{-1}$. This clearly makes the diagram commute, and can easily be checked to be well-defined in $\operatorname{mod} kG$.

Call the pair $(I_u, \Sigma u)$ a *completion* of the diagram. Now, in proving the functoriality of Σ, we first suppose that we have fixed our choices $(I_M, \Sigma M)$ of short exact sequences for each kG-module M, and show that Σ can be regarded as a functor in this way. Later, we prove that any such choices yield isomorphic functors, and that it is indeed an auto-equivalence.

To see that Σ is well-defined on morphisms, we will need

Lemma 4.3. *Let $(I_u, \Sigma u)$ and $(\tilde{I}_u, \tilde{\Sigma} u)$ be two completions of*

$$\begin{tikzcd} M \arrow[r, "\iota_M"] \arrow[d, "u"] & I_M \arrow[r, "\pi_M"] & \Sigma M \\ N \arrow[r, "\iota_N"] & I_N \arrow[r, "\pi_N"] & \Sigma N \end{tikzcd}$$

Then $\Sigma u = \tilde{\Sigma} u$ in $\underline{\mathrm{mod}}\, kG$.

Proof. We show this by constructing a factorization of $\Sigma u - \tilde{\Sigma} u$ through the injective I_N. Let $\gamma := I_u - \tilde{I}_u$. By the commutativity of the first square we get

$$\gamma \iota_M = \iota_N u - \iota_N u = 0$$

Thus $\operatorname{Ker} \pi_M = \operatorname{Im} \iota_M \subset \operatorname{Ker} \gamma$, which makes $\sigma := \gamma \pi_M^{-1}$ well-defined. We then have

$$\pi_N \sigma \pi_M = \pi_N \gamma = \pi_N I_u - \pi_N \tilde{I}_u = \Sigma u \pi_M - \tilde{\Sigma} u \pi_M$$

With π_M surjective this implies $\pi_N \sigma = \Sigma u - \tilde{\Sigma} u$. □

We thus can assure

Lemma 4.4. *Σ is well-defined on morphisms in $\underline{\mathrm{mod}}\, kG$.*

Proof. We have already seen that Σu does not depend on our choice of I_u. It remains to show that it does not depend on the choice of representative in the equivalence class of u either.

So, let v be another such choice. Then $w := u - v$ must factor through an injective module I as in the following diagram

$$\begin{tikzcd} & I \arrow[dr, "f"] & \\ I_M \arrow[ur, dashed, "g"] & M \arrow[l, "\iota_M"] \arrow[u, "f'"] \arrow[r, "w"'] & N \end{tikzcd}$$

where g is constructed via the injectivity of I. Now, $(\iota_N f g, 0)$ is a completion of

$$\begin{tikzcd} M \arrow[r, "\iota_M"] \arrow[d, "w"] & I_M \arrow[r, "\pi_M"] & \Sigma M \\ N \arrow[r, "\iota_N"] & I_N \arrow[r, "\pi_N"] & \Sigma N \end{tikzcd}$$

Lemma 4.3 then gives $\Sigma w = 0$ in $\underline{\mathrm{mod}}\, kG$. Thus $\Sigma u = \Sigma v$, by the easily checked additive property of Σ. □

Hence we can settle

Corollary 4.5. *Σ is an additive functor in $\underline{\mathrm{mod}}\, kG$.*

Proof. By Lemma 4.4 we know that Σ is well-defined on morphisms. Since we have already fixed our choices $(I_M, \Sigma M)$ of short exact sequences, it is also well-defined on objects. Moreover, it is straightforward to verify that it is additive and preserves identities and compositions. $\square$

The next step is to prove that, up to isomorphism, this functor does not depend on our initial choices of short exact sequences.

Lemma 4.6. *Let $(I_M, \Sigma M)$ and $(I'_M, \Sigma' M)$ represent two choices of short exact sequences for each kG-module M. Then there is a natural isomorphism $\beta : \Sigma \to \Sigma'$ in $\underline{\mathrm{mod}}\, kG$.*

Proof. For any kG-module M define α_M, α'_M, β_M and β'_M to be maps making the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\iota_M"] \arrow[d, equal] & I_M \arrow[r, "\pi_M"] \arrow[d, "\alpha_M"] & \Sigma M \arrow[r] \arrow[d, "\beta_M"] & 0 \\
0 \arrow[r] & M \arrow[r, "\iota'_M"] \arrow[d, equal] & I'_M \arrow[r, "\pi'_M"] \arrow[d, "\alpha'_M"] & \Sigma' M \arrow[r] \arrow[d, "\beta'_M"] & 0 \\
0 \arrow[r] & M \arrow[r, "\iota_M"] & I_M \arrow[r, "\pi_M"] & \Sigma M \arrow[r] & 0
\end{tikzcd}$$

We show that $\beta := \{\beta_M\}$ is the wanted natural isomorphism. First, we see that the β_M's are isomorphisms since both $(\alpha'_M \alpha_M, \beta'_M \beta_M)$ and $(\mathrm{id}_{I_M}, \mathrm{id}_{\Sigma M})$ are completions of the diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, equal] & I_M \arrow[r] & \Sigma M \\
M \arrow[r] & I_M \arrow[r] & \Sigma M
\end{tikzcd}$$

Thus $\beta'_M \beta_M = \mathrm{id}_{\Sigma M}$ in $\underline{\mathrm{mod}}\, kG$, by Lemma 4.3. (In the same way, we have $\beta_M \beta'_M = \mathrm{id}_{\Sigma' M}$.)

And secondly, we assure that they define a natural transformation because, given a map $u : M \to N$, both $(I'_u \alpha_M, \Sigma' u \beta_M)$ and $(\alpha_N I_u, \beta_N \Sigma u)$ are completions of

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "u"] & I_M \arrow[r] & \Sigma M \\
N \arrow[r] & I'_N \arrow[r] & \Sigma' N
\end{tikzcd}$$

which implies $\Sigma' u \beta_M = \beta_N \Sigma u$ in $\underline{\mathrm{mod}}\, kG$. $\square$

Thus we can finally state the following.

Theorem 4.7. *Σ is an auto-equivalence on $\underline{\mathrm{mod}}\, kG$.*

Proof. The previous lemmas assured us that Σ is a well-defined additive functor on $\underline{\mathrm{mod}}\, kG$. To see that it is an auto-equivalence we define an inverse Σ^{-1} by the dual construction:

Define $\Sigma^{-1}M$ on a kG-module M to be the kernel of an epimorphism

$$\pi_M : P_M \to M$$

with P_M projective, giving us the short exact sequence

$$0 \longrightarrow \Sigma^{-1}M \xrightarrow{\iota_M} P_M \xrightarrow{\pi_M} M \longrightarrow 0$$

We define Σ^{-1} on morphisms in the analogue way, constructed to get commutative diagrams of short exact sequences. Dually to the proof for Σ, one can then show that Σ^{-1} is an additive functor in $\underline{\mathrm{mod}}\, kG$. Moreover, since projectives and injectives coincide in $\mathrm{mod}\, kG$, we get $\Sigma\Sigma^{-1} \simeq \Sigma^{-1}\Sigma \simeq \mathrm{id}_{\underline{\mathrm{mod}}\, kG}$. ☐

The triangulation

Now, from any map $u : M \to N$, construct triangles in $\underline{\mathrm{mod}}\, kG$ from the following diagram

$$\begin{tikzcd}
M \arrow[r, "\iota_M"] \arrow[d, "u"] & I_M \arrow[d, "c_{\iota_M}"'] \arrow[ddr, "\pi_M"] & \\
N \arrow[r, "c_u"] \arrow[drr, "0"'] & C(u) \arrow[dr, dashed, "w_u"] & \\
& & \Sigma M
\end{tikzcd}$$

where $(c_u, c_{\iota_M}, C(u))$ is the pushout of (u, ι_M) and w_u is determined via the pushout property. The resulting triangles

$$M \xrightarrow{u} N \xrightarrow{c_u} C(u) \xrightarrow{w_u} \Sigma M$$

are called *standard triangles*. We define the d.t.'s in $\underline{\mathrm{mod}}\, kG$ to be all triangles isomorphic to a standard triangle. The following is then shown for instance in Happel's book [15, Chapter I.2].

Theorem 4.8. *The stable module category $\underline{\mathrm{mod}}\, kG$ is a triangulated category with the above translation functor and triangulation.*

The triangulation turns out to have the following reassuring property, proven in [15, Chapter I.2.7]:

Theorem 4.9. *Every short exact sequence* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *in* $\operatorname{mod} kG$ *gives rise to a d.t.*

$$M' \longrightarrow M \longrightarrow M'' \xrightarrow{w} \Sigma M'$$

in $\underline{\operatorname{mod}}\, kG$ *for a suitable* w.

Moreover, all d.t.'s are isomorphic to such a d.t. □

The tensor product

For kG-modules M and N the tensor product $M \otimes_k N$ can be made into a kG-module with G acting diagonally, i.e. $g(m \otimes n) = gm \otimes gn$ for $g \in G, m \in M$ and $n \in N$. Fortunately for us, it turns out that $\underline{\operatorname{mod}}\, kG$ is a $\otimes$-triangulated category with this tensor product:

Proposition 4.10. *The stable module category* $\underline{\operatorname{mod}}\, kG$ *is a* $\otimes$*-triangulated category with tensor product given by* $\otimes := \otimes_k$.

Proof. The first three requirements in Definition 2.12 certainly hold (let $\mathbb{1} := k$). What remains is to prove that for a kG-module S the functors $- \otimes S$ and $S \otimes -$ are triangulated (where $\otimes = \otimes_k$). We will only prove the latter, as the two proofs are analogous.

Start with a standard triangle $M \xrightarrow{u} N \xrightarrow{c_u} C(u) \xrightarrow{w_u} \Sigma M$. If we can show that

$$S \otimes M \xrightarrow{1 \otimes u} S \otimes N \xrightarrow{1 \otimes c_u} S \otimes C(u) \xrightarrow{\beta_M \circ (1 \otimes w_u)} \Sigma(S \otimes M)$$

is a d.t. for a natural isomorphism $\beta_M : S \otimes \Sigma M \to \Sigma(S \otimes M)$, we are done, as every d.t. is isomorphic to a standard triangle.

Since S is k-free, we get that $0 \longrightarrow S \otimes M \xrightarrow{1 \otimes \iota_M} S \otimes I_M \xrightarrow{1 \otimes \pi_M} S \otimes \Sigma M \longrightarrow 0$ is a short exact sequence. Moreover, $S \otimes I_M$ is still injective. Thus we have from Lemma 4.6 a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S \otimes M \arrow[r] \arrow[d, equal] & S \otimes I_M \arrow[r] \arrow[d, "\alpha_M"] & S \otimes \Sigma M \arrow[r] \arrow[d, "\beta_M"] & 0 \\
0 \arrow[r] & S \otimes M \arrow[r] \arrow[d, equal] & I_{S \otimes M} \arrow[r] \arrow[d, "\alpha'_M"] & \Sigma(S \otimes M) \arrow[r] \arrow[d, "\beta'_M"] & 0 \\
0 \arrow[r] & S \otimes M \arrow[r] & S \otimes I_M \arrow[r] & S \otimes \Sigma M \arrow[r] & 0
\end{tikzcd}$$

with β_M and β'_M (natural) isomorphisms in $\underline{\operatorname{mod}}\, kG$.

Recycling notation from the previous section we define

$$f_u := \begin{pmatrix} u \\ -\iota_M \end{pmatrix} \text{ and } g_u := \begin{pmatrix} c_u & c_{\iota_M} \end{pmatrix}$$

Recall that with this notation we have $C(u) = (N \oplus I_M)/\operatorname{Im} f_u$. Look at

$$\begin{tikzcd}
S \otimes M \arrow[r, "1\otimes f_u"] \arrow[d, equal] & (S \otimes N) \oplus (S \otimes I_M) \arrow[r, "1\otimes g_u"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha_M \end{smallmatrix}\right)"] & S \otimes C(u) \arrow[r, "1\otimes w_u"] \arrow[d, dashed, "\psi_M"] & S \otimes \Sigma M \arrow[d, "\beta_M"] \\
S \otimes M \arrow[r, "f_{1\otimes u}"] \arrow[d, equal] & (S \otimes N) \oplus I_{S\otimes M} \arrow[r, "g_{1\otimes u}"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha'_M \end{smallmatrix}\right)"] & C(1 \otimes u) \arrow[r, "w_{1\otimes u}"] \arrow[d, dashed, "\psi'_M"] & \Sigma(S \otimes M) \arrow[d, "\beta'_M"] \\
S \otimes M \arrow[r, "1\otimes f_u"] & (S \otimes N) \oplus (S \otimes I_M) \arrow[r, "1\otimes g_u"] & S \otimes C(u) \arrow[r, "1\otimes w_u"] & S \otimes \Sigma M
\end{tikzcd}$$

One can check that the maps

$$\psi_M := g_{1\otimes u} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_M \end{pmatrix} (1 \otimes g_u)^{-1}$$

$$\psi'_M := (1 \otimes g_u) \begin{pmatrix} 1 & 0 \\ 0 & \alpha'_M \end{pmatrix} g_{1\otimes u}^{-1}$$

are well-defined and make the diagram commute. Now, in the stable category, the two maps turn out to be each other's inverses. For instance $\psi'_M \psi_M = \mathrm{id}_{S\otimes C(u)}$ by

$$\begin{tikzcd}
S \otimes C(u) \arrow[rr, "1-\psi'_M\psi_M"] \arrow[dr, swap, "1\otimes(1-\alpha'_M\alpha_M)c_{\iota_M}^{-1}"] & & S \otimes C(u) \\
& S \otimes I_M \arrow[ur, swap, "1\otimes c_{\iota_M}"] &
\end{tikzcd}$$

where the first map is well-defined since $(1 - \alpha'_M \alpha_M)\iota_M(m) = 0$ for any $m \in M$. Thus in $\underline{\mathrm{mod}}\, kG$ we get an isomorphism of triangles

$$\begin{tikzcd}
S \otimes M \arrow[r, "1\otimes u"] \arrow[d, equal] & S \otimes N \arrow[r, "1\otimes c_u"] \arrow[d, equal] & S \otimes C(u) \arrow[r, "\beta_M\circ(1\otimes w_u)"] \arrow[d, "\psi_M"] & \Sigma(S \otimes M) \arrow[d, equal] \\
S \otimes M \arrow[r, "1\otimes u"] & S \otimes N \arrow[r, "c_{1\otimes u}"] & C(1 \otimes u) \arrow[r, "w_{1\otimes u}"] & \Sigma(S \otimes M)
\end{tikzcd}$$

□

4.2 Classifying thick ⊗-ideals in $\underline{\mathrm{mod}}\, kG$

We are now ready to give a proof of the classification theorem of thick ⊗-ideals of $\underline{\mathrm{mod}}\, kG$ proved by Benson, Carlson and Rickard in [7] and later generalized to group schemes by Friedlander and Pevtsova in [13], using the framework of support data.

It is known that $\underline{\mathrm{mod}}\, kG$ is an essentially small category, so in particular we know that Balmer's classification (Theorem 3.7) holds here. What we thus hope for is that the support data defined in the coming section will allow us to translate 3.7 into the result of Benson, Carlson and Rickard [7].

The support data

Let us first recall some basic homological algebra (see [4, Chapter 2] for more details).

Definition 4.11. Let M and M' be two kG-modules. An *n-fold extension* of M by M' is then an exact sequence

$$0 \longrightarrow M' \longrightarrow M_{n-1} \longrightarrow M_{n-2} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

If there exist maps between two n-fold extensions making the following diagram commute

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & M_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M'_{n-1} \arrow[r] & \cdots \arrow[r] & M'_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

the two extensions are said to be *similar.*

Moreover, two extension X and Y are called *equivalent* if there exists a sequence of extensions $X = X_0, X_1, \cdots, X_{m-1}, X_m = Y$ such that X_i and X_{i+1} are similar for $0 \leq i < m$.

One can prove that this indeed defines an equivalence relation on the set of n-fold extensions of M by M', and that these equivalence classes are in one-to-one correspondence with the elements of $\mathrm{Ext}^n_{kG}(M, M')$. This leads us into defining multiplication on $\mathrm{Ext}^*_{kG}(M, M)$ by so-called *Yoneda composition*: Given $f \in \mathrm{Ext}^n_{kG}(M, M)$ and $g \in \mathrm{Ext}^m_{kG}(M, M)$ with corresponding extensions

$$0 \longrightarrow M \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow M \longrightarrow M'_{m-1} \longrightarrow \cdots \longrightarrow M'_0 \longrightarrow M \longrightarrow 0$$

we define $f \circ g \in \mathrm{Ext}^{m+n}_{kG}(M, M)$ to be the element corresponding to the extension

$$0 \longrightarrow M \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{\sigma} M'_{m-1} \longrightarrow \cdots \longrightarrow M'_0 \longrightarrow M \longrightarrow 0$$

where σ is the composition $M_0 \longrightarrow M \longrightarrow M'_{m-1}$.

It can be shown that multiplication by Yoneda composition, which clearly is well-defined, gives $\mathrm{Ext}^*_{kG}(M, M)$ the structure of a graded ring. It turns out that $\mathrm{Ext}^*_{kG}(k, k)$ will even be *graded commutative*, in the sense that, for homogeneous elements $x, y \in \operatorname{Ext}^*_{kG}(k,k)$, we have $xy = (-1)^{\deg x \deg y} yx$. If the characteristic of k is 2, we get real commutativity. If this is not the case, however, we simply drop out the odd degrees to make it so. Thus define

$$H^\bullet(G,k) := \begin{cases} \operatorname{Ext}^*_{kG}(k,k), & \text{if } \operatorname{char}(k) = 2 \\ \operatorname{Ext}^{\mathrm{ev}}_{kG}(k,k), & \text{if } \operatorname{char}(k) \neq 2 \end{cases}$$

where $\operatorname{Ext}^{\mathrm{ev}}_{kG}(k,k)$ denotes the Ext-groups of even degree. Thus $H^\bullet(G,k)$ is a commutative graded ring.

Now, to get a suitable topological space out of $H^\bullet(G,k)$, we choose its *projective prime ideal spectrum*, which is defined as follows:

Definition 4.12. The *projective prime ideal spectrum* $\operatorname{Proj} R$ of a commutative graded ring $R = \bigoplus_{i \geq 0} R_i$ is the set of its homogeneous prime ideals that does not contain the *irrelevant ideal* $\bigoplus_{i \geq 1} R_i$. $\operatorname{Proj} R$ is made into a topological space via the *Zariski topology*, i.e. defining the closed subsets to be those on the form

$$\mathrm{V}(I) = \{P \in \operatorname{Proj} R \mid I \subset P\}$$

for a homogeneous ideal I.

Recall that the *homogeneous ideals* are the ones generated by homogeneous elements. One might often want to equip $\operatorname{Proj} R$ with even more structure, turning it into a *scheme*, but here we will only need the topology.

Having our topological space defined, our next step will now be to define the "supp" subsets of this space, i.e. assigning to each kG-module M a closed subset of $\operatorname{Proj}(H^\bullet(G,k))$ satisfying the properties of Proposition 2.22. To do this we start by tensoring a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k$$

of k with M. This gives, since $- \otimes M$ is exact (M is k-free) and the $P_i \otimes M$ are still projective, a corresponding resolution for M

$$\cdots \longrightarrow P_2 \otimes M \longrightarrow P_1 \otimes M \longrightarrow P_0 \otimes M \longrightarrow M$$

In this way, each cycle $\overline{f} \in \operatorname{Ext}^n_{kG}(k,k)$ will induce a cycle $\overline{f \otimes 1} \in \operatorname{Ext}^n_{kG}(M,M)$. One can check that this assignment defines a ring homomorphism

$$\phi^*_M : H^\bullet(G,k) \longrightarrow \operatorname{Ext}^*_{kG}(M,M)$$

Now, we define

$$\mathscr{V}_{\mathrm{G}}(M) := \mathrm{V}(\operatorname{Ker} \phi^*_M)$$

which makes sense because $\operatorname{Ker} \phi^*_M$ is obviously a homogeneous ideal. This finishes the definition of our support data:

Theorem 4.13. *The pair* $(\operatorname{Proj}(H^\bullet(G,k)), \mathscr{V}_{\mathrm{G}})$ *is a support data on* $\underline{\mathrm{mod}}\, kG$.

Proof. Among the properties stated in Proposition 2.22 we here only prove the most straightforward ones, and refer the interested reader to [6] and [7] for the rest.

i) $\mathscr{V}_{\mathrm{G}}(0) = \mathrm{V}(\operatorname{Ker}\phi_0^*) = \mathrm{V}(H^\bullet(G,k)) = \emptyset$

$\mathscr{V}_{\mathrm{G}}(k) = \mathrm{V}(\operatorname{Ker}\phi_k^*) = \mathrm{V}(0) = \operatorname{Proj}(H^\bullet(G,k))$

ii) $\mathscr{V}_{\mathrm{G}}(M \oplus N) = \mathrm{V}(\operatorname{Ker}\phi^*_{M\oplus N}) = \mathrm{V}(\operatorname{Ker}\phi^*_M \cap \operatorname{Ker}\phi^*_N)$ can be seen from the following factorization of $\phi^*_{M\oplus N}$:

$$H^\bullet(G,k) \xrightarrow{\binom{\phi^*_M}{\phi^*_N}} \operatorname{Ext}^*_{kG}(M,M) \oplus \operatorname{Ext}^*_{kG}(N,N) \hookrightarrow \operatorname{Ext}^*_{kG}(M\oplus N, M\oplus N)$$

But

$$\mathrm{V}(\operatorname{Ker}\phi^*_M \cap \operatorname{Ker}\phi^*_N) = \mathrm{V}(\operatorname{Ker}\phi^*_M) \cup \mathrm{V}(\operatorname{Ker}\phi^*_N) = \mathscr{V}_{\mathrm{G}}(M) \cup \mathscr{V}_{\mathrm{G}}(N)$$

by the property of primes.

iv) Given a projective resolution $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k$ of k we look at the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \otimes \Sigma M \arrow[r] \arrow[d] & P_1 \otimes \Sigma M \arrow[r] \arrow[d] & P_0 \otimes \Sigma M \arrow[r] \arrow[d] & k \otimes \Sigma M \arrow[d] \\
\cdots \arrow[r] & \Sigma(P_2 \otimes M) \arrow[r] & \Sigma(P_1 \otimes M) \arrow[r] & \Sigma(P_0 \otimes M) \arrow[r] & \Sigma(k \otimes M)
\end{tikzcd}$$

where the downward arrows are the natural isomorphisms arising from the exactness of $\otimes$. From this we see that a map

$$f \otimes 1_{\Sigma M} : k \otimes \Sigma M \longrightarrow P_i \otimes \Sigma M$$

is a boundary if and only if $\Sigma(f \otimes 1_M) : \Sigma(k \otimes M) \to \Sigma(P_i \otimes M)$ is. But this is again equivalent to $f \otimes 1_M : k \otimes M \to P_i \otimes M$ being a boundary, as Σ is an equivalence. Thus $\operatorname{Ker}\phi^*_M = \operatorname{Ker}\phi^*_{\Sigma M}$, and $\mathscr{V}_{\mathrm{G}}(M) = \mathscr{V}_{\mathrm{G}}(\Sigma M)$.

The much more involved properties **iii)** and **v)** are stated in [7, Proposition 2.2] (note that our definition of $\mathscr{V}_{\mathrm{G}}(M)$ coincides with the one used in [6] and [7] in the case of finitely generated modules by the remarks following [6, Definition 10.2]). ☐

The classification theorem

Now, to make use of Theorem 3.13, we will need to embed $\underline{\operatorname{mod}}\, kG$ into a bigger $\otimes$-triangulated category which admits arbitrary coproducts. What then seems more appropriate than to choose $\underline{\operatorname{Mod}}\, kG$, the stable category of all, not necessarily finitely generated, kG-modules? This indeed turns out to be a good choice, as the finitely generated modules are exactly its compact objects, and we have that

$\underline{\mathrm{Mod}}\, kG = \mathrm{loc}\langle \underline{\mathrm{mod}}\, kG\rangle$, which we provide proofs for in the Appendix (Theorem A.1 and Theorem A.2). It is a ⊗-triangulated category via the same triangulation and tensor product as $\underline{\mathrm{mod}}\, kG$, and, also, the finitely generated kG-modules are strongly dualizable in $\underline{\mathrm{Mod}}\, kG$, as is shown in Theorem A.3. Thus we get:

Theorem 4.14. $(\mathrm{Proj}(H^\bullet(G,k)), \mathscr{V}_{\mathrm{G}}) \simeq (\mathrm{Spc}\, \underline{\mathrm{mod}}\, kG, \mathrm{supp})$ *as support data on* $\underline{\mathrm{mod}}\, kG$.

Proof. We here show that **i**), **ii**), **iv**) and **v**) of Theorem 3.13 are fulfilled for $(\mathrm{Proj}(H^\bullet(G,k)), \mathscr{V}_{\mathrm{G}})$. The requirements **iii**) and **vi**) are more elaborate, and we leave it for the interested reader to check it out in [5] and [7].

i) $\mathrm{Proj}(H^\bullet(G,k))$ is a Noetherian topological space because $H^\bullet(G,k)$ is a Noetherian ring (see for instance [5, Theorem 4.2.1]) and thus obeys the ascending chain condition on (homogeneous) ideals.

It is T_0 because, given $P_1, P_2 \in \mathrm{Proj}(H^\bullet(G,k))$ with $P_1 \neq P_2$, clearly either $P_1 \notin \mathrm{V}(P_2)$ or $P_2 \notin \mathrm{V}(P_1)$.

ii) Any homogeneous ideal $I \subset H^\bullet(G,k)$ with $\mathrm{V}(I) = \mathrm{V}(P) = \overline{\{P\}}$ for a prime P is clearly irreducible. Our claim is that if this is not the case, then $\mathrm{V}(I)$ must be reducible. Thus pick a prime P minimal in $\mathrm{V}(I)$, and assume that there is at least one other prime $P' \in \mathrm{V}(I)$ with $P \not\subset P'$. Let $\mathbf{S}$ denote the set of such primes. By the properties of primes and the minimality of P, we have that

$$\prod_{Q\in\mathbf{S}} Q \not\subset P, \text{ which implies } \tilde{Q} := \bigcap_{Q\in\mathbf{S}} Q \not\subset P$$

Thus we get $\mathrm{V}(I) = \mathrm{V}(P) \cup \mathrm{V}(\tilde{Q})$ with $\mathrm{V}(P)$ and $\mathrm{V}(\tilde{Q})$ proper subsets of $\mathrm{V}(I)$, confirming our claim.

iii) This follows from [5, Corollary 5.9.2] and [7, Proposition 2.2 c)].

iv)-v) See proofs in the Appendix, Theorem A.1, A.2 and A.3.

vi) Benson, Carlson and Rickard provide a generalization of $\mathscr{V}_{\mathrm{G}}$ to modules in $\underline{\mathrm{Mod}}\, kG$, which abides the two conditions (see [7, Proposition 2.2 b), f)]). □

Before we state the important corollary, we introduce the analogue to $\mathscr{K}_Y$ for the support data $(\mathrm{Proj}(H^\bullet(G,k)), \mathscr{V}_{\mathrm{G}})$:

Definition 4.15. For a subset $W \subset \mathrm{Proj}(H^\bullet(G,k)$ let $\mathscr{C}(W)$ denote the full subcategory of $\underline{\mathrm{mod}}\, kG$ with objects the kG-modules M satisfying $\mathscr{V}_{\mathrm{G}}(M) \subset W$:

$$\mathscr{C}(W) := \{M \in \underline{\mathrm{mod}}\, kG \mid \mathscr{V}_{\mathrm{G}}(M) \subset W\}$$

Corollary 4.16 (Benson-Carlson-Rickard). *There is a bijection* $F : \mathbf{S} \to \mathbf{R}$ *between the set* $\mathbf{S}$ *of specialization closed subsets of* $\mathrm{Proj}(H^\bullet(G,k))$ *and the set* $\mathbf{R}$ *of thick ⊗-ideals of* $\underline{\mathrm{mod}}\, kG$ *given by*

$$F : W \longmapsto \mathscr{C}(W)$$

with inverse

$$F^{-1} : \mathscr{C} \longmapsto \bigcup_{M \in \mathscr{C}} \mathscr{V}_{\mathrm{G}}(M)$$

Proof. Via Proposition 3.15 the above bijection is simply the bijection of Theorem 3.7 together with the order-preserving bijection between specialization closed subsets induced by the homeomorphism

$$f : \operatorname{Proj}(H^\bullet(G,k)) \to \operatorname{Spc} \underline{\mathrm{mod}}\, kG$$

of Theorem 4.14

Observe that $\mathscr{C}(W) = \mathscr{K}_{f(W)}$ and

$$f(\bigcup_{M \in \mathscr{C}} \mathscr{V}_{\mathrm{G}}(M)) = \bigcup_{M \in \mathscr{C}} \operatorname{supp} M = \operatorname{supp} \mathscr{C}$$

□

In the special case when G is a p-group with $p = \operatorname{char} k$, Benson, Carlson and Rickard [7, Corollary 3.5] point out that we in fact have classified *all* thick triangulated subcategories (not just the thick $\otimes$-ideals):

Theorem 4.17. *If P is a p-group and k is a field with characteristic p, then all thick triangulated subcategories $\mathscr{C}$ of $\underline{\mathrm{mod}}\, kP$ are $\otimes$-ideals.*

Proof. We will need to show that for $C \in \mathscr{C}$ and $M \in \underline{\mathrm{mod}}\, kP$ we have $C \otimes M \in \mathscr{C}$. It is known that k is the only simple module in $\underline{\mathrm{mod}}\, kP$ (see for instance [4, Lemma 3.14.1]). Thus given a filtration of M in $\mathrm{mod}\, kP$

$$M = M_n \supset M_{n-1} \supset \cdots \supset M_0 = 0$$

we must have $M_{i+1}/M_i \simeq k$ for $0 \leq i \leq n$. We can thus show that $C \otimes M_i \in \mathscr{C}$ for $1 \leq i \leq n$ by induction on i. First, clearly $C \otimes M_1 \simeq C \otimes k \simeq C \in \mathscr{C}$. Suppose $C \otimes M_i \in \mathscr{C}$. Then the short exact sequence

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0$$

gives rise to a d.t. $M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \xrightarrow{w} \Sigma M_i$ (see Theorem 4.9). Now, since both $C \otimes M_i$ and $C \otimes M_{i+1}/M_i \simeq C \otimes k$ are in $\mathscr{C}$, we get $C \otimes M_{i+1} \in \mathscr{C}$ by the exactness of $C \otimes -$. □

CHAPTER 5

EXAMPLE II: THE DERIVED CATEGORY OF PERFECT COMPLEXES

As another application of Theorem 3.13, we will now prove the classification first given by Hopkins [17] and later corrected by Neeman [25] of the thick subcategories of the derived category of perfect complexes over a commutative Noetherian ring.

We first provide some background on derived categories, derived functors and perfect complexes for the unacquainted reader. For a more in-depth exposition, we refer to Krause's excellent treatment in [21] or Weibel's book [33, Chapter 10].

5.1 The derived category

Let R be a ring and $\mathrm{C}(R)$ be the category of chain complexes over R. A chain map $\phi : X \to Y$ in $\mathrm{C}(R)$ is called a *quasi-isomorphism* if the induced map

$$\mathrm{H}\,\phi : \mathrm{H}(X) \to \mathrm{H}(Y)$$

is an isomorphism. The *derived category* is then the category constructed from $\mathrm{C}(R)$ by formally inverting quasi-isomorphisms. More precisely, it is defined to be a category $\mathrm{D}(R)$ equipped with a functor

$$q : \mathrm{C}(R) \longrightarrow \mathrm{D}(R)$$

such that $q\phi$ is an isomorphism whenever ϕ is a quasi-isomorphism, which, moreover, is universal with this property. It is possible to show that such a category really exists. By the universal property it is also uniquely determined (up to equivalence). One can think of $\mathrm{D}(R)$ as having as objects the chain complexes over R, any two being isomorphic if there exists a chain map between them which is an isomorphism under homology.

The category $\mathrm{D}(R)$ is called the *localization* of $\mathrm{C}(R)$ with respect to quasi-isomorphisms, and q is called the *localization functor*. The notion of derived categories is usually (and originally) generalized to any abelian category in the place of $\mathrm{Mod}\, R$.

There are three natural (triangulated) subcategories of $\mathrm{D}(R)$: The full subcategories consisting of complexes quasi-isomorphic to bounded below, bounded above and bounded (in each direction) complexes. These subcategories are denoted, respectively, by $\mathrm{D}^+(R)$, $\mathrm{D}^-(R)$ and $\mathrm{D}^b(R)$.

The triangulation

It turns out that $\mathrm{D}(R)$ is a triangulated category. Its triangulation is determined via three quick definitions:

Definition 5.1 (The translation functor)**.** The *shift* functor

$$[1] : \mathrm{D}(R) \to \mathrm{D}(R)$$

is defined to be the shifting of any complex one degree to the left and the corresponding action on morphisms, i.e $X[1]_n = X_{n-1}$ for any complex X.

Definition 5.2. From a chain map $f : X \to Y$ between two complexes define the *cone* of f to be the complex $\mathrm{M}(f)$ given by

$$\mathrm{M}(f)_n := X_{n-1} \oplus Y_n \text{ and } d_n^{\mathrm{M}(f)} := \begin{pmatrix} -d_{n-1}^X & 0 \\ f_{n-1} & d_n^Y \end{pmatrix}$$

Definition 5.3. A *standard triangle* in $\mathrm{D}(R)$ is a triangle

$$X \xrightarrow{f} Y \xrightarrow{\alpha} \mathrm{M}(f) \xrightarrow{\beta} X[1]$$

where α and β are the natural inclusion and projection.

The d.t.'s in $\mathrm{D}(R)$ are then defined to be all the triangles isomorphic to a standard triangle. For a proof that $\mathrm{D}(R)$ is triangulated with this triangulation see [33, Section 10.4]. Luckily, it turns out that the triangulation has the following convenient property:

Proposition 5.4. *All short exact sequences*

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

of complexes $X, Y, Z \in \mathrm{C}(R)$ give rise to a d.t. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *in* $\mathrm{D}(R)$*. Moreover, all d.t's are isomorphic to such a d.t.* □

From this one also quickly deduces

Proposition 5.5. *All d.t.'s* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *give rise to a long exact sequence*

$$\cdots \longrightarrow \mathrm{H}_i(X) \longrightarrow \mathrm{H}_i(Y) \longrightarrow \mathrm{H}_i(Z) \longrightarrow \mathrm{H}_{i-1}(X) \longrightarrow \cdots$$

□

Derived functors

For a commutative ring R, the bifunctors $\mathrm{Hom}_R(-,-)$ and $- \otimes_R -$ generalize neatly to bifunctors in $\mathrm{C}(R)$ via

$$(\mathrm{Hom}_R(X,Y))_n = \bigoplus_{-p+q=n} \mathrm{Hom}_R(X_p, Y_q) \text{ and } (X \otimes_R Y)_n = \bigoplus_{p+q=n} X_p \otimes_R Y_q$$

and a suitable sign rule on the differentials (see [33, Section 2.7]). However, if we wish to bring with us these further to the derived category, we soon run into difficulties, as neither functor is exact. Derived functors are designed to overcome this obstacle, i.e. the problem of extending any functor $F : \mathrm{C}(R) \to \mathrm{C}(S)$ to a functor $F^* : \mathrm{D}(R) \to \mathrm{D}(S)$.

The crucial idea here comes from the construction of ordinary derived functors, where the argument is replaced by its projective (injective) resolution. This analogy turns out to be rewarding, as one can prove the following:

Proposition 5.6. *We have that*

i) *Every* $X \in \mathrm{D}^+(R)$ *is quasi-isomorphic to a bounded below complex of projectives.*

ii) *Every* $X \in \mathrm{D}^-(R)$ *is quasi-isomorphic to a bounded above complex of injectives.*

Moreover, if our original complex consisted of finitely generated modules, then the projectives/injectives can be chosen to be finitely generated too, provided that R *is Noetherian.* □

By replacing the argument in our functor by its projective (or injective) "resolution" it turns out that we indeed get well-definedness in the derived category, as wanted. However, this only enables us to define derived functors for complexes bounded in a direction. Fortunately, Spaltenstein [30] offered a solution to this via the definition of K*-projective* and K*-injective* complexes (some places called *homotopically projective/injective* complexes):

Definition 5.7. A complex $P \in \mathrm{C}(R)$ is called K*-projective*, if we have

$$\mathrm{Hom}_{\mathrm{C}(R)}(P, X) = 0$$

for each acyclic complex X. Dually, a complex $I \in \mathrm{C}(R)$ is called K*-injective* if

$$\mathrm{Hom}_{\mathrm{C}(R)}(X, I) = 0$$

It turns out that *all* complexes are quasi-isomorphic to a K-projective (respectively injective) complex, called its *K-projective (injective) resolution*, and that Proposition 5.6 comes out as a special case of this. Furthermore, this resolution plays the same role in providing us with well-defined functors in $\mathrm{D}(R)$ as the old ones. Letting $\mathbf{p}X$ and $\mathbf{i}X$ denote the K-projective and K-injective resolutions of X respectively (these assignments turn out to be functorial) we can thus define:

Definition 5.8. Let $F : \mathrm{C}(R) \to \mathrm{C}(S)$ be a functor for rings R and S. Then its *left derived functor* $\mathbf{L}F$ is defined to be the composition

$$F \circ \mathbf{p} : \mathrm{D}(R) \longrightarrow \mathrm{D}(S)$$

Similarly, its *right derived functor* $\mathbf{R}F$ is the composition $F \circ \mathbf{i} : \mathrm{D}(R) \to \mathrm{D}(S)$.

Having defined derived functors in their generality, we now turn to the two particular functors employed in this thesis:

Definition 5.9. Let R and S be rings. For a complex of R-S-bimodules X we define

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_R(X, -) &:= \mathbf{R}(\mathrm{Hom}_R(X, -) : \mathrm{D}(R) \longrightarrow \mathrm{D}(S) \\ - \otimes^{\mathrm{L}}_R X &:= \mathbf{L}(- \otimes_R X) : \mathrm{D}(R) \longrightarrow \mathrm{D}(S) \end{aligned}$$

One can show that $\mathbf{R}\mathrm{Hom}_R(-, -)$ and $- \otimes^{\mathrm{L}}_R -$ are bifunctors

$$\mathrm{D}(R) \times \mathrm{D}(R) \to \mathrm{D}(R)$$

for commutative R (if not, replace the first $\mathrm{D}(R)$ with $\mathrm{D}(R)^{\mathrm{op}}$ in the case of $\mathbf{R}\mathrm{Hom}_R$), and that, had we chosen to define them via a K-projective resolution of their other variable instead, we had arrived at the same functors.

The following properties are inherited straight away from the module categories:

Proposition 5.10. *Let X, Y and Z be complexes of R-modules, R-S-bimodules and S-modules, respectively, where R and S are commutative rings. We then have that*

i) $\mathbf{R}\mathrm{Hom}_S(Y, -)$ *and* $- \otimes^{\mathrm{L}}_R Y$ *form an adjoint pair.*

ii) $(X \otimes^{\mathrm{L}}_R Y) \otimes^{\mathrm{L}}_S Z \simeq X \otimes^{\mathrm{L}}_R (Y \otimes^{\mathrm{L}}_S Z)$ □

Crucially for us, $\mathrm{D}(R)$ turns out to be a $\otimes$-triangulated category in the sense of Definition 2.12 with $\otimes^{\mathrm{L}}_R$ as the tensor product. This is, however, not the $\otimes$-triangulated category we are most interested in here. Our focus will rather be on the subcategory described in the following section.

Perfect complexes

For a ring R, define the *perfect* complexes over R as the complexes on the form

$$0 \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_{j+1} \longrightarrow P_j \longrightarrow 0$$

where the P's are finitely generated projective R-modules.

This leads us into defining the *derived category of perfect complexes* over R, denoted by $\mathrm{D}^{\mathrm{perf}}(R)$, as the full (triangulated) subcategory of $\mathrm{D}(R)$ having as objects the complexes isomorphic in $\mathrm{D}(R)$ to a perfect complex. This is the category where Hopkins and Neeman did their celebrated classification, and is our second example of a category satisfying the conditions of Theorem 3.13.

The perfect complexes have some nice features. First, as they are their own K-projective resolution, the functors $-\otimes^{\mathrm{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(-,X)$ (for any X) can be treated as the ordinary chain complex functors on them. Moreover, it can be shown that the perfect complexes are the compact objects of $\mathrm{D}(R)$ and that they generate $\mathrm{D}(R)$ as a triangulated category. This is proven in [20, Section 8.1.3-4]. We here also prove that they are strongly dualizable in $\mathrm{D}(R)$, making the requirements **iv**) and **v**) of Theorem 3.13 fulfilled for $\mathrm{D}^{\mathrm{perf}}(R)$, with $\mathrm{D}(R)$ as its corresponding "big" category. But first, let us verify that $\mathbf{R}\mathrm{Hom}_R$ and $\otimes^{\mathrm{L}}_R$ are still well-defined when restricted to $\mathrm{D}^{\mathrm{perf}}(R)$:

Proposition 5.11. *If $X, Y \in \mathrm{D}^{\mathrm{perf}}(R)$ for a commutative ring R, we also have*

$$\mathbf{R}\mathrm{Hom}_R(X,Y) \in \mathrm{D}^{\mathrm{perf}}(R) \;\; \textit{and}\; X \otimes^{\mathrm{L}}_R Y \in \mathrm{D}^{\mathrm{perf}}(R)$$

Proof. First, we note that if P and Q are finitely generated projective R-modules, then so are also $\mathrm{Hom}_R(P,Q)$ and $P \otimes_R Q$. As $\mathrm{Hom}_R(R,Q) \simeq Q$ and $R \otimes_R Q \simeq Q$, and we must have $P \oplus M \simeq R^n$ for some n and an R-module M, this is seen immediate, since Hom_R and $\otimes_R$ commute with finite coproducts.

We are now finished by invoking the observation above; in the case of perfect complexes, $\mathbf{R}\mathrm{Hom}_R$ and $\otimes^{\mathrm{L}}_R$ are just the ordinary Hom_R and $\otimes_R$ (the respective boundedness conditions are given by that of X and Y). $\square$

The second assertion above settles that $\mathrm{D}^{\mathrm{perf}}(R)$ is a thick $\otimes$-ideal in $\mathrm{D}(R)$, and that it indeed is a $\otimes$-triangulated category on its own. In addition to this, it can be shown to be essentially small, thus complying with the preconditions of Balmer's classification theorem.

Due to the lack of finding a direct proof in the literature of the fact that the perfect complexes are strongly dualizable in the derived category, we present one here:

Proposition 5.12. *The perfect complexes are strongly dualizable in* $\mathrm{D}(R)$.

Proof. We want to show that there is a natural isomorphism

$$\mathbf{R}\mathrm{Hom}_R(X,R) \otimes^{\mathrm{L}}_R Y \simeq \mathbf{R}\mathrm{Hom}_R(X,Y) \tag{5.1}$$

for all $X \in \mathrm{D}^{\mathrm{perf}}(R)$ and $Y \in \mathrm{D}(R)$.

Let us first verify that for R-modules P and M, with P finitely generated projective, the natural map

$$\theta : \mathrm{Hom}_R(P,R) \otimes_R M \longrightarrow \mathrm{Hom}_R(P,M)$$

given by $\theta : f \otimes_R m \mapsto mf$, is an isomorphism.

Suppose that $\{p_1, \cdots, p_n\}$ is a generating set for P, define F to be the free R-module with $\{p_1, \cdots, p_n\}$ as basis, and let $\pi : F \to P$ be the canonical projection. By the property of projectives we then have a map $\sigma : P \to F$ such that $\pi \circ \sigma = \mathrm{id}_P$. Defining $f_i : F \to R$ by $f_i(p_j) = \delta_{ij}$ for $1 \leq i \leq n$, we get maps $g_i := f_i \circ \sigma : P \to R$ such that $g_i(p_j) = \delta_{ij}$ (this is called the *dual basis property* of projective modules). Now, one can check that the map $\psi : \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, R) \otimes_R M$ defined by

$$\psi : f \longmapsto \sum_{i=1}^{n} g_i \otimes_R f(p_i)$$

is an inverse of θ.

Since $\mathrm{Hom}_R(X, R)$ is again a perfect complex, (5.1) boils down to showing

$$\mathrm{Hom}_R(X, R) \otimes_R Y \simeq \mathrm{Hom}_R(X, Y)$$

But this follows directly from the preceding discussion, as in each degree in the above complexes we get a finite coproduct of pairwise naturally isomorphic summands (whose isomorphisms commute with differentials by the naturality). □

5.2 The support data

From now on, let R always be a commutative Noetherian ring. The support data employed in the classification of Hopkins [17] and Neeman [25] is what makes their theorem so remarkably beautiful; it is simply $\mathrm{Spec}\, R$, the simplest imaginable topological space related to R, together with the following support:

Definition 5.13. Define the *support* of a complex $X \in \mathrm{D}^{\mathrm{perf}}(R)$ to be

$$\mathrm{Supp}_R(X) := \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{H}(X_\mathfrak{p}) \not\simeq 0\}$$

Note that since R is Noetherian, $\mathrm{H}(X)$ is finitely generated as an R-module. One can then check that $\mathrm{H}(X_\mathfrak{p}) \simeq \mathrm{H}(X)_\mathfrak{p} \not\simeq 0$ is equivalent to $\mathrm{ann}_R \mathrm{H}(X) \subset \mathfrak{p}$ (the first isomorphism comes from the exactness of localization).

We will now show that this indeed defines a support data on our category. To prove the tensor property, however, we will first need two short lemmas:

Lemma 5.14. *Let $X, Y \in \mathrm{D}^{\mathrm{perf}}(R)$ and $\mathfrak{p} \in \mathrm{Spec}\, R$ Then*

$$(X \otimes_R^{\mathrm{L}} Y)_\mathfrak{p} \simeq X_\mathfrak{p} \otimes_{R_\mathfrak{p}}^{\mathrm{L}} Y_\mathfrak{p}$$

Proof. Since $\otimes_R^{\mathrm{L}}$ is just the normal tensor product between complexes for objects in $\mathrm{D}^{\mathrm{perf}}(R)$ and $X_\mathfrak{p}, Y_\mathfrak{p} \in \mathrm{D}^{\mathrm{perf}}(R)$ as projectivity is a local property, the result follows from the corresponding theorem for finitely generated R-modules (see for instance [1, Proposition 3.7]). □

Define $\mathrm{i}(X) := \inf\{i \mid \mathrm{H}_i(X) \not\simeq 0\}$ for any $X \in \mathrm{D}(R)$. The following lemma is due to Foxby [12]

Lemma 5.15. *If* $X, Y \in \mathrm{D}^+(R)$, *then*

$$\mathrm{i}(X \otimes_R^{\mathrm{L}} Y) \leq \mathrm{i}(X) + \mathrm{i}(Y)$$

with equality if and only if $\mathrm{H}_{\mathrm{i}(X)}(X) \otimes_R \mathrm{H}_{\mathrm{i}(Y)}(Y) \not\simeq 0$

Proof. We may by Proposition 5.6 assume that X and Y are bounded below complexes of projectives.

The first statement is immediate. Furthermore, since projective modules are flat, it is easy to check that

$$\mathrm{H}_{\mathrm{i}(X)+\mathrm{i}(Y)}(X \otimes_R Y) \simeq \mathrm{H}_{\mathrm{i}(X)}(X) \otimes_R \mathrm{H}_{\mathrm{i}(Y)}(Y)$$

□

Theorem 5.16. *The pair* $(\operatorname{Spec} R, \operatorname{Supp}_R)$ *is a support data on* $\mathrm{D}^{\mathrm{perf}}(R)$.

Proof. We again work our way through the requirements of Proposition 2.22.

i), ii), iv) These properties are trivial, noting that

$$\begin{aligned} \mathrm{H}(R_{\mathfrak{p}}) &\simeq R_{\mathfrak{p}} \\ \mathrm{H}((X \oplus Y)_{\mathfrak{p}}) &\simeq \mathrm{H}(X_{\mathfrak{p}}) \oplus \mathrm{H}(Y_{\mathfrak{p}}) \\ \mathrm{H}(X[1]_{\mathfrak{p}}) \simeq 0 &\Leftrightarrow \mathrm{H}(X_{\mathfrak{p}}) \simeq 0 \end{aligned}$$

iii) From Lemma 5.14 and Lemma 5.15 we know that

$$\mathrm{i}((X \otimes_R^{\mathrm{L}} Y)_{\mathfrak{p}}) = \mathrm{i}(X_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}}^{\mathrm{L}} Y_{\mathfrak{p}}) \leq \mathrm{i}(X_{\mathfrak{p}}) + \mathrm{i}(Y_{\mathfrak{p}})$$

which gives us $\operatorname{Supp}_R(X \otimes_R^{\mathrm{L}} Y) \subset \operatorname{Supp}_R(X) \cap \operatorname{Supp}_R(Y)$.

On the other hand, if $\mathfrak{p} \in \operatorname{Supp}_R(X) \cap \operatorname{Supp}_R(Y)$, we know, as $R_{\mathfrak{p}}$ is local and $\mathrm{H}(X_{\mathfrak{p}})$ and $\mathrm{H}(Y_{\mathfrak{p}})$ are finitely generated as R-modules, that

$$\mathrm{H}_{\mathrm{i}(X_{\mathfrak{p}})}(X_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \mathrm{H}_{\mathrm{i}(Y_{\mathfrak{p}})}(Y_{\mathfrak{p}}) \not\simeq 0$$

(see [29, Corollary 2, p. 2]). Thus we have equality above, giving

$$\mathrm{i}((X \otimes_R^{\mathrm{L}} Y)_{\mathfrak{p}}) \neq \infty \Rightarrow \mathfrak{p} \in \operatorname{Supp}_R(X \otimes_R^{\mathrm{L}} Y)$$

v) Let $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ be a d.t. Since any d.t. is isomorphic to one arriving from a short exact sequence of complexes (see Proposition 5.4), we get, via the exactness of localization and Proposition 5.5, a corresponding long exact sequence

$$\cdots \longrightarrow \mathrm{H}_{n+1}(X_{\mathfrak{p}}) \longrightarrow \mathrm{H}_n(Y_{\mathfrak{p}}) \longrightarrow \mathrm{H}_n(Z_{\mathfrak{p}}) \longrightarrow \mathrm{H}_n(X_{\mathfrak{p}}) \longrightarrow \cdots$$

We then see that if two of $\mathrm{H}(X_{\mathfrak{p}})$, $\mathrm{H}(Y_{\mathfrak{p}})$ and $\mathrm{H}(Z_{\mathfrak{p}})$ for a prime ideal $\mathfrak{p}$ are isomorphic to zero, then so is the third. □

The small support in $\mathrm{D}(R)$

In Theorem 3.13 it is required that the support should be generalizable to the "big" category, still obeying $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ together with the condition $\sigma(a) = \emptyset \Rightarrow a = 0$. We have already suggested that in the case of $\mathrm{D}^{\mathrm{perf}}(R)$ this big category is $\mathrm{D}(R)$. In this section we prove that the *small support* introduced by Foxby in [12] is a generalization of Supp_R to $\mathrm{D}(R)$, which abides the two conditions. The small support is defined on $X \in \mathrm{D}(R)$ as

$$\mathrm{supp}_R(X) := \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{H}(X \otimes^{\mathrm{L}}_R \mathrm{k}(\mathfrak{p})) \not\simeq 0\}$$

where $\mathrm{k}(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the residue field at $\mathfrak{p}$. The following propositions are stolen from [12] and [14]:

Proposition 5.17. *If $X \in \mathrm{D}^{\mathrm{perf}}(R)$, then $\mathrm{supp}_R(X) = \mathrm{Supp}_R(X)$.*

Proof. First, note that

$$X \otimes^{\mathrm{L}}_R R_\mathfrak{p} = \mathbf{p}X \otimes_R R/\mathfrak{p} \simeq (\mathbf{p}X)_\mathfrak{p} \simeq X_\mathfrak{p}$$

in the derived category (the last isomorphism is given by the exactness of localization). Thus we have $X \otimes^{\mathrm{L}}_R \mathrm{k}(\mathfrak{p}) \simeq X_\mathfrak{p} \otimes^{\mathrm{L}}_{R_\mathfrak{p}} \mathrm{k}(\mathfrak{p})$, and Lemma 5.15 gives

$$\mathrm{i}(X \otimes^{\mathrm{L}}_R \mathrm{k}(\mathfrak{p})) = \mathrm{i}(X_\mathfrak{p} \otimes^{\mathrm{L}}_{R_\mathfrak{p}} \mathrm{k}(\mathfrak{p})) \leq \mathrm{i}(X_\mathfrak{p}) + \mathrm{i}(\mathrm{k}(\mathfrak{p})) = \mathrm{i}(X_\mathfrak{p})$$

implying $\mathrm{supp}_R(X) \subset \mathrm{Supp}_R(X)$.

Furthermore, if $\mathrm{H}(X_\mathfrak{p}) \not\simeq 0$, we have $\mathrm{H}_{\mathrm{i}(X_\mathfrak{p})}(X_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \mathrm{H}_0(\mathrm{k}(\mathfrak{p})) \not\simeq 0$, which gives equality above, and thus the other inclusion. □

Proposition 5.18. *Given $X, Y \in \mathrm{D}(R)$ we have*

$$\mathrm{supp}_R(X \otimes^{\mathrm{L}}_R Y) = \mathrm{supp}_R(X) \cap \mathrm{supp}_R(Y)$$

Proof. Bökstedt and Neeman [9, Lemma 2.17] states the (easily proved) assertion that if there is a homomorphism $\alpha : R \to k$ of a commutative ring into a field, then for any complex $X \in \mathrm{D}(R)$ we have that $X \otimes^{\mathrm{L}}_R k$ is isomorphic (in the derived category) to a direct sum of suspensions of k.

Thus, if $\mathfrak{p} \in \mathrm{supp}_R(X) \cap \mathrm{supp}_R(Y)$, we have that $Y \otimes^{\mathrm{L}}_R \mathrm{k}(\mathfrak{p})$ is isomorphic to a (non-trivial) coproduct of suspensions of $\mathrm{k}(\mathfrak{p})$. This gives us

$$\mathrm{H}((X \otimes^{\mathrm{L}}_R Y) \otimes^{\mathrm{L}}_R \mathrm{k}(\mathfrak{p})) \not\simeq 0$$

since $\otimes^{\mathrm{L}}_R$ is associative and commutes with coproducts and suspensions.

The other inclusion is given by

$$(X \otimes^{\mathrm{L}}_R Y) \otimes^{\mathrm{L}}_R \mathrm{k}(\mathfrak{p}) \simeq X \otimes^{\mathrm{L}}_R (Y \otimes^{\mathrm{L}}_R \mathrm{k}(\mathfrak{p})) \simeq Y \otimes^{\mathrm{L}}_R (X \otimes^{\mathrm{L}}_R \mathrm{k}(\mathfrak{p}))$$

□

Proposition 5.19. *If $X \in \mathrm{D}(R)$ is non-trivial, then $\mathrm{supp}_R(X) \neq \emptyset$.*

Proof. That X is non-trivial means that $\mathrm{H}(X) \not\simeq 0$. Let $\mathfrak{p}$ be maximal (with respect to inclusion) among ideals $\mathfrak{a} \subset R$ such that $\mathrm{H}(X \otimes_R^{\mathrm{L}} R/\mathfrak{a}) \not\simeq 0$. Then $\mathfrak{p}$ must be prime. For if not, there would be a (non-trivial) prime filtration

$$R/\mathfrak{p} = B_n \supset \cdots \supset B_0 = 0$$

with $B_{i+1}/B_i \simeq R/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i$ (see [22, Theorem 6.4]). But this is impossible since the following induction argument then would give $\mathrm{H}(X \otimes_R^{\mathrm{L}} B_i) \simeq 0$ for all i:

Clearly $\mathrm{H}(X \otimes_R^{\mathrm{L}} B_0) \simeq 0$, and if $\mathrm{H}(X \otimes_R^{\mathrm{L}} B_i) \simeq 0$, the short exact sequence

$$0 \longrightarrow B_i \longrightarrow B_{i+1} \longrightarrow R/\mathfrak{q}_i \longrightarrow 0$$

gives, via the exactness of $X \otimes_R^{\mathrm{L}} -$ and Proposition 5.5, rise to the long exact sequence

$$\cdots \longrightarrow \mathrm{H}_l(X \otimes_R^{\mathrm{L}} B_i) \longrightarrow \mathrm{H}_l(X \otimes_R^{\mathrm{L}} B_{i+1}) \longrightarrow \mathrm{H}_l(X \otimes_R^{\mathrm{L}} R/\mathfrak{q}_i) \longrightarrow \cdots$$

One sees easily that $\mathfrak{p} \subset \mathfrak{q}_i$, giving $\mathrm{H}(X \otimes_R^{\mathrm{L}} R/\mathfrak{q}_i) \simeq 0$ and $\mathrm{H}(X \otimes_R^{\mathrm{L}} B_{i+1}) \simeq 0$.

Having verified that $\mathfrak{p}$ is prime, we now claim that

$$\mathrm{H}(X \otimes_R^{\mathrm{L}} \mathrm{k}(\mathfrak{p})) \simeq \mathrm{H}(X \otimes_R^{\mathrm{L}} R/\mathfrak{p}) \not\simeq 0$$

making $\operatorname{supp}_R(X)$ non-trivial.

To prove this, first observe that $(X \otimes_R^{\mathrm{L}} R/\mathfrak{p})_\mathfrak{p} \simeq X_\mathfrak{p} \otimes_{R_\mathfrak{p}}^{\mathrm{L}} \mathrm{k}(\mathfrak{p}) \simeq X \otimes_R^{\mathrm{L}} \mathrm{k}(\mathfrak{p})$. Thus we settle our claim if we can show that the canonical map

$$\theta_i : \mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/\mathfrak{p}) \longrightarrow \mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/\mathfrak{p})_\mathfrak{p} \simeq \mathrm{H}_i((X \otimes_R^{\mathrm{L}} R/\mathfrak{p})_\mathfrak{p})$$

is an isomorphism for all i.

Crucial here is that, in $\mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/\mathfrak{p})$, multiplication by any element $a \in R \setminus \mathfrak{p}$ turns out to be an isomorphism: $R/\mathfrak{p} \xrightarrow{a} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + (a))$ is a short exact sequence (the first map is an injection since $R/\mathfrak{p}$ is an integral domain), and thus gives rise to a d.t.

$$R/\mathfrak{p} \xrightarrow{a} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + (a)) \longrightarrow R/\mathfrak{p}[1]$$

Since $X \otimes_R^{\mathrm{L}} -$ is exact we get, applying homology, a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/\mathfrak{p}) \xrightarrow{a^*} \mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/\mathfrak{p}) \longrightarrow \mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/(\mathfrak{p} + (a))) \longrightarrow \cdots$$

yielding the wanted isomorphism, since $\mathrm{H}(X \otimes_R^{\mathrm{L}} R/(\mathfrak{p}+(a))) \simeq 0$ by the maximality of $\mathfrak{p}$.

This, in turn, implies injectivity and surjectivity of θ_i: First, if $\theta_i(x) = x/1 = 0$ for an $x \in \mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/\mathfrak{p})$, we must have $xa = 0$ for some $a \in X \setminus \mathfrak{p}$, giving $x = 0$. Also, any $x/a \in \mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/\mathfrak{p})_\mathfrak{p}$ is equal to $\theta_i(x')$, where $x = ax'$ in $\mathrm{H}_i(X \otimes_R^{\mathrm{L}} R/\mathfrak{p})$. $\square$

5.3 Classifying thick subcategories of $\mathrm{D}^{\mathrm{perf}}(R)$

All this finally makes us ready to invoke the isomorphism theorem:

Theorem 5.20. $(\operatorname{Spec} R, \operatorname{Supp}_R) \simeq (\operatorname{Spc} \mathrm{D}^{\mathrm{perf}}(R), \operatorname{supp})$ *as support data.*

Proof. We check that the requirements of Theorem 3.13 are satisfied:

i)-ii) The corresponding proofs for $\operatorname{Proj}(H^\bullet(G,k))$ in Theorem 4.14 carry straight over to the case of $\operatorname{Spec} R$, keeping in mind that R is Noetherian.

iii) Given a closed subset $\mathrm{V}(\mathfrak{a})$ of $\operatorname{Spec} R$ for an ideal $\mathfrak{a}$, we want to find a perfect complex X such that $\operatorname{Supp}_R(X) = \mathrm{V}(\mathfrak{a})$. Since R is Noetherian, $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = \langle a_1, \cdots, a_n \rangle$. Define complexes

$$X_i : \ \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{a_i} R \longrightarrow 0 \longrightarrow \cdots$$

with the R's in degree one and zero, and let X be the (Koszul) complex

$$X := \bigotimes_{i=1}^{n} X_i$$

Since free modules are projective, this is clearly a perfect complex. By the tensor property of Supp_R we now have $\operatorname{Supp}_R(X) = \bigcap_{i=1}^n \operatorname{Supp}_R(X_i)$. Furthermore, $\operatorname{ann}_R \mathrm{H}_0(X_i) = \operatorname{ann}_R R/Ra_i = Ra_i$, and since Ra_i also annihilates $\mathrm{H}_1(X_i) = \operatorname{Ker} a_i$ we get $\operatorname{ann}_R \mathrm{H}(X_i) = Ra_i$, yielding the following equivalence:

$$\mathfrak{p} \in \operatorname{Supp}_R(X) \Leftrightarrow Ra_i \subset \mathfrak{p} \text{ for } 1 \leq i \leq n \Leftrightarrow \mathfrak{a} \subset \mathfrak{p}$$

Hence $\operatorname{Supp}_R(X) = \mathrm{V}(\mathfrak{a})$, as wanted.

iv)-v) With $\mathrm{D}(R)$ identified as the appropriate "big" category for $\mathrm{D}^{\mathrm{perf}}(R)$, this is proved in [20, Section 8.1.3-4] and Proposition 5.12.

vi) This was proved in Proposition 5.17-5.19. □

Before we move on to the classification theorem we claim that all thick triangulated subcategories of $\mathrm{D}^{\mathrm{perf}}(R)$ are $\otimes$-ideals, thus applying Theorem 3.7 to $\mathrm{D}^{\mathrm{perf}}(R)$ gives us a classification of *all* thick triangulated subcategories. This claim is settled in the Appendix (Theorem A.8). By the same arguments as in Corollary 4.16 we thus get

Corollary 5.21 (Hopkins-Neeman)**.** *There is a bijection $F : \mathbf{S} \to \mathbf{R}$ between the set $\mathbf{S}$ of specialization closed subsets of* $\operatorname{Spec} R$ *and the set $\mathbf{R}$ of thick triangulated subcategories of* $\mathrm{D}^{\mathrm{perf}}(R)$ *given by*

$$F : W \longmapsto \{X \in \mathrm{D}^{\mathrm{perf}}(R) \mid \operatorname{Supp}_R(X) \subset W\}$$

with inverse

$$F^{-1} : \mathscr{C} \longmapsto \bigcup_{X \in \mathscr{C}} \operatorname{Supp}_R(X)$$

Remark 5.22. This result has been generalized by Thomason in [31] to hold when R is replaced by a more general entity, a Noetherian *scheme*.

CLOSING REMARKS

As already mentioned in the introduction of Section 3.2, we needed to introduce quite a few preconditions to be able to prove the existence of an isomorphism between $\{\operatorname{Spc} \mathscr{K}, \operatorname{supp}\}$ and another given support data. It is perfectly possible, and also highly probable, that a simpler set of requirements would still be enough for the isomorphism to hold, and hence still enable a translation of Balmer's classification. To dig further into the question of what we really need to prove this isomorphism would be an interesting study. In particular, it would be very exciting if one could give a proof of the surjectivity without needing to involve the result of Hovey, Palmieri and Strickland [18], and thus getting rid of the "big" category and "compactly generated" constraints.

Also, as another direction for further research, we strongly suspect that Theorem 3.13 can be used in giving proofs of classification theorems in other categories than our two examples, especially in those already having a classification that is comparable to those of our examples. Perhaps, if one succeeds with getting rid of some of the conditions, it could even provide totally new classifications in tensor triangulated categories not yet having a classification theorem.

APPENDIX A

In this Appendix we give proofs for $\underline{\mathrm{Mod}}\, kG$ being the appropriate "big" category to $\underline{\mathrm{mod}}\, kG$ in the sense of Theorem 3.13, and that all thick triangulated subcategories of $\mathrm{D}^{\mathrm{perf}}(R)$ are $\otimes$-ideals.

A.1 $\underline{\mathrm{Mod}}\, kG$ is compactly generated by $\underline{\mathrm{mod}}\, kG$

Let kG be a group algebra where G is a finite group and k is a field. We here prove that $\underline{\mathrm{mod}}\, kG$, the stable category of finitely generated kG-modules, represents the compact objects in $\underline{\mathrm{Mod}}\, kG$, the stable category of all, not necessarily finitely generated, kG-modules. We also show that $\underline{\mathrm{Mod}}\, kG = \mathrm{loc}\langle \underline{\mathrm{mod}}\, kG \rangle$, and that the objects in $\underline{\mathrm{mod}}\, kG$ are strongly dualizable in $\underline{\mathrm{Mod}}\, kG$. The proofs of Theorem A.1 and Theorem A.2 are taken from [24].

Theorem A.1. *The compact objects in $\underline{\mathrm{Mod}}\, kG$ are exactly those which are isomorphic to a finitely generated kG-module.*

Proof. First note that since kG is obviously Artinian as a ring, we have projective covers and know that M/MJ is a direct sum of simple modules for any kG-module M, where J is the Jacobson radical of kG (which also is nilpotent).

If M is finitely generated, any map $f : M \to \bigoplus_{i \in I} N_i$ factors through the finite coproduct $\bigoplus_{i \in I'} N_i$ where $I' \subset I$ is chosen such that the N_i's with $i \in I'$ are those in which $f(e_j)$ has a non-zero component for a generator e_j. Thus it is compact.

To prove the converse, suppose that M is compact. Since $M/MJ \simeq \bigoplus_{i \in I} S_i$ for a set of simple modules S_i, the projection $p : M \twoheadrightarrow M/MJ$ must factor through

$$T_1 := \bigoplus_{i \in I'} S_i$$

for some finite subset $I' \subset I$. Letting $T_2 := \bigoplus_{i \in I \setminus I'} S_i$ we must thus have an f making the following diagram commute in the stable category

$$\begin{tikzcd}
M \arrow[rr, two heads, "\left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right)"] \arrow[dr, "f"'] & & T_1 \oplus T_2 \\
& T_1 \arrow[ur, hook] &
\end{tikzcd}$$

where p_1, p_2 are the projections of p onto the respective summands. This means that

$$\left(\begin{smallmatrix} p_1 - f \\ p_2 \end{smallmatrix}\right) : \ M \twoheadrightarrow T_1 \oplus T_2$$

factors through a projective module. Let $\pi_i : P_i \to T_i$ for $i \in \{1, 2\}$ be projective covers. We can then, via the property of projectives, replace this module by $P_1 \oplus P_2$, yielding a commutative diagram

$$\begin{tikzcd}
M \arrow[rr, two heads, "\left(\begin{smallmatrix} p_1 - f \\ p_2 \end{smallmatrix}\right)"] \arrow[dr, "\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)"'] & & T_1 \oplus T_2 \\
& P_1 \oplus P_2 \arrow[ur, two heads, "\left(\begin{smallmatrix} \pi_i & 0 \\ 0 & \pi_2 \end{smallmatrix}\right)"'] &
\end{tikzcd}$$

Now, since p_2 is surjective, y must be so too (since π_2 is an essential epimorphism). Furthermore it must split, as P_2 is projective. Thus $M \simeq M' \oplus P_2$ for some M', which gives

$$T_1 \oplus T_2 \simeq M/MJ \simeq M'/M'J \oplus P_2/P_2J$$

Since T_2 is a coproduct of simple modules we must have $P_2/P_2J \simeq T_2$. Thus we get $M'/M'J \simeq T_1$, which is finitely generated by definition.

But this implies that M' too must be finitely generated: First, it gives us that $M'J^i/M'J^{i+1}$ must be finitely generated for $i \geq 0$ (as J is). Then, since J is nilpotent, an induction argument yields that, indeed, $M'J^i$ must be finitely generated for all i.

We are now finished by observing that $M \simeq M'$ in the stable category. $\square$

Theorem A.2. *Let $\mathcal{S}$ be a complete set of non-isomorphic simple kG-modules. Then*

$$\underline{\mathrm{Mod}}\, kG = \mathrm{loc}\langle \mathcal{S} \rangle$$

Proof. Since kG is Artinian as a ring, the Jacobson radical J is nilpotent. Let r be the least positive integer such that $J^r = 0$, and choose an $M \in \underline{\mathrm{Mod}}\, kG$. We prove by induction that $MJ^i \in \mathrm{loc}\langle \mathcal{S} \rangle$ for $0 \leq i \leq r$.

Clearly $MJ^r = 0 \in \mathrm{loc}\langle \mathcal{S} \rangle$. Now, suppose $MJ^i \in \mathrm{loc}\langle \mathcal{S} \rangle$, and look at the d.t.

$$MJ^i \longrightarrow MJ^{i-1} \longrightarrow MJ^{i-1}/MJ^i \longrightarrow \Sigma MJ^i$$

which exists by Theorem 4.9. Since $MJ^{i-1}/MJ^i = (MJ^{i-1})/(MJ^{i-1})J$ is a coproduct of simple modules, we must thus have $MJ^{i-1} \in \mathrm{loc}\langle \mathcal{S} \rangle$, completing the proof. $\square$

Theorem A.3. *The objects of* $\underline{\mathrm{mod}}\, kG$ *are strongly dualizable in* $\underline{\mathrm{Mod}}\, kG$*.*

Proof. The functor $\mathrm{Hom}_k(M, -)$ is a right adjoint of $- \otimes M$ in $\underline{\mathrm{Mod}}\, kG$ (where $\mathrm{Hom}_k(M, N)$ is made into a kG-module via $(g\phi)(m) := g(\phi(g^{-1}m))$ for $g \in G$, $m \in M$ and $\phi \in \mathrm{Hom}_k(M, N)$). Thus we have to show that there is a natural isomorphism

$$\mathrm{Hom}_k(M, k) \otimes N \simeq \mathrm{Hom}_k(M, N)$$

for $M \in \underline{\mathrm{mod}}\, kG$ and $N \in \underline{\mathrm{Mod}}\, kG$. Indeed, defining the ($k$-bilinear) natural map $g : \mathrm{Hom}_k(M, k) \times N \to \mathrm{Hom}_k(M, N)$ by

$$g : (\psi, n) \longmapsto n\psi$$

the induced (kG-linear) map $\overline{g} : \mathrm{Hom}_k(M, k) \otimes N \to \mathrm{Hom}_k(M, N)$ is such an isomorphism.

To see this, let $(e_1, \cdots, e_n)$ be a basis for M as a k-vector space and $(f_1, \cdots, f_n)$ the corresponding dual basis (given by $f_i(e_j) = \delta_{ij}$). Then the map

$$h : \mathrm{Hom}_k(M, N) \longrightarrow \mathrm{Hom}_k(M, k) \otimes N$$

defined by

$$h : \phi \longmapsto \sum_{i=1}^{n} f_i \otimes \phi(e_i)$$

is an inverse. $\square$

A.2 Thick subcategories in $\mathrm{D}^{\mathrm{perf}}(R)$ are $\otimes$-ideals

As announced in Section 5.3, we here prove that all thick triangulated subcategories of $\mathrm{D}^{\mathrm{perf}}(R)$ are $\otimes$-ideals, where R is a commutative Noetherian ring as usual. We will need to introduce some machinery to succeed, as the proof will involve DG-algebras and homotopy colimits. The idea of the proof is taken from [19, Theorem 2].

DG-algebras

Most of the results we presented in Section 5.1 can be transferred to what can be thought of as a generalization of the setting of complexes over a ring, namely the case of *DG-modules* over a *DG-algebra*, which we here give a brief introduction to (see [20, Section 8.2] for more).

Definition A.4. For a commutative ring R, a *differential graded* R-algebra (DG-algebra) is a graded R-algebra

$$A = \bigoplus_{p \in \mathbb{Z}} A_p$$

endowed with an R-linear differential

$$\cdots \xrightarrow{d} A_{i+1} \xrightarrow{d} A_i \xrightarrow{d} A_{i-1} \xrightarrow{d} \cdots$$

such that we have $d^2 = 0$, and, for $a \in A_p$ and $b \in A_q$, the Leibniz rule

$$d(ab) = (da)b + (-1)^p a(db)$$

A trivial example is any ordinary R-algebra B (for instance R itself), defining $A_0 := B$ and $A_i = 0$ everywhere else. The R-algebra $\mathrm{Hom}_R(X, X)$ for a complex $X \in \mathrm{C}(R)$ is another standard example, with multiplication composition of chain maps. In the same way, $\mathbf{R}\mathrm{Hom}_R(X, X)$ can be thought of as a DG-algebra.

Now, fixing a DG-algebra A, we define

Definition A.5. A *differential graded module* (DG-module) over A is a graded right A-module

$$M = \bigoplus_{p \in \mathbb{Z}} M_p$$

with an R-linear differential

$$\cdots \xrightarrow{d} M_{i+1} \xrightarrow{d} M_i \xrightarrow{d} M_{i-1} \xrightarrow{d} \cdots$$

such that we have $d^2 = 0$, and, for $m \in M_p$ and $a \in A_q$, the Leibniz rule

$$d(ma) = (dm)a + (-1)^p m(da)$$

A *morphism* $f : M \to N$ between DG A-modules is a homogeneous (A-linear) map of degree 0 which commutes with the differentials.

In the trivial case of $A = R$ we get that the DG A-modules are just the complexes over R, settling the claim that DG-modules over DG-algebras are a generalization of this setting. Other examples include $\mathrm{Hom}_R(X, Y)$ for any complex $Y \in \mathrm{C}(R)$, which is viewed as a DG-module over $\mathrm{Hom}_R(X, X)$ under composition. Similarly, $\mathbf{R}\mathrm{Hom}_R(X, Y)$ is a DG $\mathbf{R}\mathrm{Hom}_R(X, X)$-module.

It turns out that, for a DG-algebra A, we can define $\mathrm{D}(A)$, the derived category of DG A-modules, in the same way as in Section 5.1, by inverting quasi-isomorphisms (morphisms turning into isomorphisms when applying homology with respect to the module differential). This category is triangulated as well, via the very same structure as the derived category over a ring. Moreover, all the results of Section 5.1 carry over, mutatis mutandis, to the case of DG-algebras. In particular we have, for two DG R-algebras A and B and a DG A-B-bimodule X (defined in the obvious way), an adjoint pair of functors

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_B(X, -) &: \mathrm{D}(B) \to \mathrm{D}(A) \\ - \otimes^{\mathrm{L}}_A X &: \mathrm{D}(A) \to \mathrm{D}(B) \end{aligned}$$

These functors are constructed via *K-projective DG A-modules*, to which all DG-modules are isomorphic. We also get the special case of this in the bounded case, as in Proposition 5.6: Any DG A-module $\bigoplus_{p \in \mathbb{Z}} M_p$ with $\mathrm{H}(M_p) = 0$ when p is small enough is isomorphic in $\mathrm{D}(A)$ to a DG A-module $\bigoplus_{p \in \mathbb{Z}} Q_p$ with each Q_p projective (as A-module) and $Q_p = 0$ for small p.

Homotopy colimits

Let $\mathscr{K}$ be a triangulated category and let

$$X^0 \xrightarrow{\phi_1} X^1 \xrightarrow{\phi_2} X^2 \xrightarrow{\phi_3} \cdots$$

be a direct system of objects in $\mathscr{K}$. The homotopy colimit of this system, denoted $\underrightarrow{\mathrm{hocolim}}\, X^n$, is defined to be the *cone* of the map

$$1 - \text{shift} := \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -\phi_1 & 1 & 0 & \cdots \\ 0 & -\phi_2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : \bigoplus_{n\geq 0} X^n \longrightarrow \bigoplus_{n\geq 0} X^n$$

In other words, it is the object constructed via TR1 such that

$$\bigoplus_{n\geq 0} X^n \xrightarrow{1-\text{shift}} \bigoplus_{n\geq 0} X^n \xrightarrow{\quad\chi\quad} \underrightarrow{\mathrm{hocolim}}\, X^n \xrightarrow{\quad\psi\quad} (\bigoplus_{n\geq 0} X^n)[1]$$

is a d.t. for some morphisms χ and ψ. This object is uniquely determined up to (non-canonical) isomorphism. In the case of $\mathrm{D}(R)$ the homotopy colimit coincides, up to isomorphism, with the direct limit of the same system (hence the name).

The homotopy colimit plays an important role in the proof of the *Brown representability theorem*, but here, we will need it in proof of the $\otimes$-closedness of the thick subcategories in $\mathrm{D}^{\mathrm{perf}}(R)$ via the following proposition:

Proposition A.6. *All bounded below complexes of finitely generated modules in* $\mathrm{D}(R)$*, where R is a commutative Noetherian ring, are isomorphic to a homotopy colimit of a direct system of objects in* $\mathrm{thick}_{\mathrm{D}(R)}\langle R\rangle$*, where* $\mathrm{thick}_{\mathrm{D}(R)}\langle R\rangle$ *denotes the smallest thick triangulated subcategory of* $\mathrm{D}(R)$ *containing R.*

Moreover, the corresponding assertion also holds for finitely generated DG-modules over a DG-algebra which is Noetherian as an R-algebra.

Proof. Let $X \in \mathrm{D}(R)$ be a bounded below complex of finitely generated R-modules. For convenience, suppose $X_n = 0$ for $n < 0$. Then we know from Proposition 5.6 that X is isomorphic in $\mathrm{D}(R)$ to a bounded below complex

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

of finitely generated projective R-modules. Define complexes

$$X^i :\ 0 \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$$

for $i \geq 0$. The X^i's form a direct system $X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$ via inclusion. We claim that $X \simeq \underrightarrow{\mathrm{hocolim}}\, X^i$, and that $X^i \in \mathrm{thick}_{\mathrm{D}(R)}\langle R\rangle$, for each i.

The former is immediate, as in $\mathrm{D}(R)$ we have $\underrightarrow{\mathrm{hocolim}}\, X^i \simeq \underrightarrow{\lim}\, X^i$. We prove the second assertion inductively. First, since we have $P_0 \oplus M = R^n$ for some M

and n, we get $P_0 \in \mathrm{thick}_{\mathrm{D}(R)}\langle R\rangle$. Now, suppose $X^i \in \mathrm{thick}_{\mathrm{D}(R)}\langle R\rangle$, and look at the following morphism f between objects in $\mathrm{thick}_{\mathrm{D}(R)}\langle R\rangle$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{i+1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_i \arrow[r] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which commutes as the P_i's form a complex. A quick computation then shows that $\mathrm{M}(f) = X^{i+1}$, giving $X^{i+1} \in \mathrm{thick}_{\mathrm{D}(R)}\langle R\rangle$.

The above proof translates directly to the case of DG-algebras, noting that a finitely generated DG-module must be bounded in each direction. □

We will also need the following lemma:

Lemma A.7. *Suppose that* $\underrightarrow{\mathrm{hocolim}}\, X^n$ *for a direct system*

$$X^0 \xrightarrow{\phi_1} X^1 \xrightarrow{\phi_2} X^2 \xrightarrow{\phi_3} \cdots$$

of objects in a triangulated category is compact. Then in the d.t.

$$\bigoplus_{n\geq 0} X^n \xrightarrow{1-\mathrm{shift}} \bigoplus_{n\geq 0} X^n \xrightarrow{\quad\chi\quad} \underrightarrow{\mathrm{hocolim}}\, X^n \xrightarrow{\quad\psi\quad} (\bigoplus_{n\geq 0} X^n)[1]$$

ψ must be the zero morphism.

Furthermore, $\underrightarrow{\mathrm{hocolim}}\, X^n$ *must be a retract of* $\bigoplus_{n\in I} X^n$ *for some finite subset* $I \subset \mathbb{N}_0$.

Proof. Define $H := \underrightarrow{\mathrm{hocolim}}\, X^n$. Since H is compact and suspension is an additive functor, ψ factors as

$$H \xrightarrow{\overline{\psi}} \bigoplus_{n\in J} X^n[1] \longrightarrow (\bigoplus_{n\geq 0} X^n)[1]$$

for a finite subset $J \subset \mathbb{N}_0$. Define $\psi_i : H \xrightarrow{p_i\circ\overline{\psi}} X^i[1]$, where the p_i's are the respective projections (projections exist since finite products and coproducts coincide). Then the map

$$H \xrightarrow{\left(\begin{smallmatrix}\psi_i \\ \psi_{i+1}\end{smallmatrix}\right)} X^{i+1}[1] \oplus X^i[1] \xrightarrow{(1\ -\phi_{i+1})[1]} X^{i+1}[1]$$

is zero for $i \geq 0$ (since the composition $(1-\mathrm{shift})[1] \circ \psi$ is). This gives, as ψ_0 clearly must be the zero morphism, that $\psi_i = 0$ for all i, by induction. Thus ψ is zero.

Now, we have the following morphism of d.t.'s

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H \arrow[r, equal] \arrow[d, dashed, "g"] & H \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
\bigoplus\limits_{n\geq 0} X^n \arrow[r, "1-\text{shift}"] & \bigoplus\limits_{n\geq 0} X^n \arrow[r] & H \arrow[r, "\psi"] & (\bigoplus\limits_{n\geq 0} X^n)[1]
\end{tikzcd}$$

where g is constructed via TR2, proving the second assertion (as H is compact). $\square$

The proof

Now, after having introduced the right tools, the proof of our theorem is actually quite appealing:

Theorem A.8. *All thick triangulated subcategories of* $\mathrm{D}^{\mathrm{perf}}(R)$ *are* $\otimes$*-ideals.*

Proof. Fixing a perfect complex $X \in \mathrm{D}^{\mathrm{perf}}(R)$, we are finished if we can show $Y \otimes^{\mathrm{L}}_R X \in \mathrm{thick}_{\mathrm{D}(R)}\langle X \rangle$ for any $Y \in \mathrm{D}^{\mathrm{perf}}(R)$.

Start by defining the DG-algebra $\varepsilon := \mathbf{R}\mathrm{Hom}_R(X,X)$. Since X is naturally a DG ε-module, we have that the functors

$$\mathbf{R}\mathrm{Hom}_R(X,-) : \mathrm{D}(R) \longrightarrow \mathrm{D}(\varepsilon)$$
$$- \otimes^{\mathrm{L}}_{\varepsilon} X : \mathrm{D}(\varepsilon) \longrightarrow \mathrm{D}(R)$$

form an adjoint pair.

The unit/counit definition of adjoint functors gives us that

$$Y \otimes^{\mathrm{L}}_R X = Y \otimes^{\mathrm{L}}_R (\varepsilon \otimes^{\mathrm{L}}_{\varepsilon} X) = (Y \otimes^{\mathrm{L}}_R \varepsilon) \otimes^{\mathrm{L}}_{\varepsilon} X$$

is a retract of

$$\mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathrm{L}}_R X) \otimes^{\mathrm{L}}_{\varepsilon} X$$

Now, since ε is Noetherian as an R-algebra and $\mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathrm{L}}_R X)$ is finitely generated over ε (it is even finitely generated over R), we get from Proposition A.6 that

$$\mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathrm{L}}_R X) \simeq \underrightarrow{\mathrm{hocolim}}\, X^n$$

in $\mathrm{D}(\varepsilon)$, for a direct system

$$X^0 \xrightarrow{\phi_1} X^1 \xrightarrow{\phi_2} X^2 \xrightarrow{\phi_3} \cdots$$

of DG ε-modules with $X^n \in \mathrm{thick}_{\mathrm{D}(\varepsilon)}\langle \varepsilon \rangle$

Furthermore, since $- \otimes^{\mathrm{L}}_{\varepsilon} X$ is a triangle functor and commutes with coproducts, we get

$$\mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathrm{L}}_R X) \otimes^{\mathrm{L}}_{\varepsilon} X \simeq \underrightarrow{\mathrm{hocolim}}(X^n \otimes^{\mathrm{L}}_{\varepsilon} X)$$

Noting that $\mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathrm{L}}_R X) \otimes^{\mathrm{L}}_\varepsilon X$ is a perfect complex and hence compact, we get from Lemma A.7 that $\mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathrm{L}}_R X) \otimes^{\mathrm{L}}_\varepsilon X$, and thus $Y \otimes^{\mathrm{L}}_R X$, must be a direct summand of $\bigoplus_{n \in J}(X^n \otimes^{\mathrm{L}}_\varepsilon X)$ for some finite subset $J \subset \mathbb{N}_0$.

It is known that all objects in $\mathrm{thick}_{\mathrm{D}(\varepsilon)}\langle \varepsilon \rangle$ can be constructed via a finite number of coproducts, retracts, suspensions and formations of d.t.'s (see [8, Section 3.2]). Since $- \otimes^{\mathrm{L}}_\varepsilon X$ commutes with all these actions, we must thus have $X^n \otimes^{\mathrm{L}}_\varepsilon X \in \mathrm{thick}_{\mathrm{D}(R)}\langle \varepsilon \otimes^{\mathrm{L}}_\varepsilon X \rangle = \mathrm{thick}_{\mathrm{D}(R)}\langle X \rangle$ for each n, finally giving $Y \otimes^{\mathrm{L}}_R X \in \mathrm{thick}_{\mathrm{D}(R)}\langle X \rangle$, as wanted. $\square$

BIBLIOGRAPHY

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[2] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[3] P. Balmer. Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebr. Geom. Topol.*, 10(3):1521–1563, 2010.

[4] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. Basic representation theory of finite groups and associative algebras.

[5] D. J. Benson. *Representations and cohomology. II*, volume 31 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. Cohomology of groups and modules.

[6] D. J. Benson, J. F. Carlson, and J. Rickard. Complexity and varieties for infinitely generated modules. II. *Math. Proc. Cambridge Philos. Soc.*, 120(4):597–615, 1996.

[7] D. J. Benson, J. F. Carlson, and J. Rickard. Thick subcategories of the stable module category. *Fund. Math.*, 153(1):59–80, 1997.

[8] P. A. Bergh, S. B. Iyengar, H. Krause, and S. Oppermann. Dimensions of triangulated categories via Koszul objects. *Math. Z.*, 265(4):849–864, 2010.

[9] M. Bökstedt and A. Neeman. Homotopy limits in triangulated categories. *Compositio Math.*, 86(2):209–234, 1993.

[10] E. S. Devinatz, M. J. Hopkins, and J. H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.

[11] A. Dold and D. Puppe. Homologie nicht-additiver Funktoren. Anwendungen. *Ann. Inst. Fourier Grenoble*, 11:201–312, 1961.

[12] H.-B. Foxby. Bounded complexes of flat modules. *J. Pure Appl. Algebra*, 15(2):149–172, 1979.

[13] E. M. Friedlander and J. Pevtsova. Π-supports for modules for finite group schemes. *Duke Math. J.*, 139(2):317–368, 2007.

[14] G. Garkusha. R-supports in tensor triangulated categories. *J. Algebra Appl.*, 9(6):1001–1014, 2010.

[15] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Math. Soc. Lecture Note Ser.* Cambridge University Press, Cambridge, 1988.

[16] T. Holm and P. Jørgensen. Triangulated categories: definitions, properties, and examples. In *Triangulated categories*, volume 375 of *London Math. Soc. Lecture Note Ser.*, pages 1–51. Cambridge Univ. Press, Cambridge, 2010.

[17] M. J. Hopkins. Global methods in homotopy theory. In *Homotopy theory (Durham, 1985)*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 73–96. Cambridge Univ. Press, Cambridge, 1987.

[18] M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.

[19] S. B. Iyengar. Thick subcategories of perfect complexes over a commutative ring. In *Thick Subcategories - Classifications and Applications*, volume 3, pages 461–509. Oberwolfach Reports, 2006.

[20] B. Keller. On the construction of triangle equivalences. In *Derived equivalences for group rings*, volume 1685 of *Lecture Notes in Math.*, pages 155–176. Springer, Berlin, 1998.

[21] H. Krause. Derived categories, resolutions, and Brown representability. In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 101–139. Amer. Math. Soc., Providence, RI, 2007.

[22] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[23] J. P. May. The additivity of traces in triangulated categories. *Adv. Math.*, 163(1):34–73, 2001.

[24] J. Miyachi. Compact objects in stable module categories. *Arch. Math. (Basel)*, 89(1):47–51, 2007.

[25] A. Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by M. Bökstedt.

[26] A. Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *J. Amer. Math. Soc.*, 9(1):205–236, 1996.

[27] A. Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[28] A. Neeman. The homotopy category of flat modules, and Grothendieck duality. *Invent. Math.*, 174(2):255–308, 2008.

[29] J.-P. Serre. *Local algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000. Translated from the French by C. Chin and revised by the author.

[30] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[31] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math.*, 105(1):1–27, 1997.

[32] J.-L. Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by L. Illusie, Edited and with a note by G. Maltsiniotis.

[33] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.