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MOMENT-BASED CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION IN THE CLASS OF DISTRIBUTIONS WITH MONOTONE HAZARD RATE

N.G. Ushakov[1] and V.G. Ushakov[2]

Some new characterizations of the exponential distribution in the class of distributions with monotone hazard rate are obtained. The characterizations are formulated in terms of expectations of order statistics.

1. Introduction

Let X be a nonnegative absolutely continuous random variable with cumulative distribution function $F(x)$ and probability density function $f(x)$. The function

$$h(x) = \frac{f(x)}{1 - F(x)},$$

defined for those x-s where $F(x) < 1$, is called the hazard rate function. In reliability theory there are several functions which are of special interest in describing the evolution of the risks to which a given unit is subjected over time. The hazard rate function, also known as the failure rate, is perhaps most popular due to its intuitive interpretation as the amount of risk associated with an item at time x. For a small increment Δx, the hazard rate in x, multiplied by Δx, is approximately equal to the conditional probability that a failure occurs in the interval $(x, x + \Delta x]$ given that no failure has occurred before x:

$$h(x)\Delta x \approx \mathsf{P}(x < X \leqslant x + \Delta x | X > x).$$

Another reason for the popularity of the hazard rate is that it is a special case of the intensity function for a non-homogeneous Poisson process. The hazard rate function goes by several aliases: in actuarial science it is also known as the force of mortality or the force of decrement; in point process and extreme value theory it is known as the rate or intensity function; in vital statistics it is known as the age-specific death rate; in economics its reciprocal is known as Mill's ratio.

In many applications it is reasonable to suppose that the hazard rate is monotone in x. The increasing hazard rate function is probably the most likely situation. In this case, items are more likely to fail as time passes. In other words, items wear out or degrade with time. This is almost certainly the case with mechanical items that undergo wear or fatigue. The decreasing hazard rate function is less common. In this case, the item is less likely to fail as time passes. Items with this type of hazard function improve with time. Some metals, for example, work harden through use and thus have increased strength as time passes. Such widely used distributions as the gamma distribution, Weibull distribution, and the generalized exponential distribution are distributions with monotone hazard rate. The increase or decrease of the hazard rate of these three distributions is determined by the value of the shape parameter. Distributions with monotone hazard rate have received considerable attention in the literature; in particular, various inequalities for distributions with monotone hazard rate have been obtained by a number of authors. One can mention, for example, [1–3].

[1] Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway, e-mail: ushakov@math.ntnu.no

[2] Moscow State University, Moscow, Russia, e-mail: vgushakov@mail.ru

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Let $X_1, \ldots, X_n$ be a random sample consisting of independent and identically distributed random variables. Denote the corresponding order statistics by $X_{(1)}, \ldots, X_{(n)}$. Among the results related to order statistics, there are various inequalities. Let n and k be natural numbers, $k \leqslant n$. Denote

$$\gamma_{k,n} = \left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{k}\right)^{-1}.$$

For a fixed n the numbers $\gamma_{k,n}$ increase in k, $\gamma_{1,n} < 1$, $\gamma_{n,n} = n$. So, there exists a minimal k for which $\gamma_{k,n} > 1$.

The results of [4] imply the following inequalities:

$$\gamma_{1,n}\mathsf{E}X_{(n)} \leqslant \gamma_{2,n}\mathsf{E}X_{(n-1)} \leqslant \ldots \leqslant \gamma_{n,n}\mathsf{E}X_{(1)}$$

for distributions with increasing hazard rate average, and

$$\gamma_{1,n}\mathsf{E}X_{(n)} \geqslant \gamma_{2,n}\mathsf{E}X_{(n-1)} \geqslant \ldots \geqslant \gamma_{n,n}\mathsf{E}X_{(1)}$$

for distributions with decreasing hazard rate average. These classes include the classes of distributions with increasing and decreasing hazard rate respectively. We show that for distributions with monotone hazard rate the inequalities are strict, if the exponential distribution is excluded. Similar strict inequalities are obtained for the expectations of order statistics and observations. Thus these inequalities can be used for obtaining characterizations of the exponential distribution in these classes.

It is known (see, for example, [5,6]) that in the class of distributions with monotone hazard rate, the equality of moments $\mathsf{E}X_1 = n\mathsf{E}X_{(1)}$ or $\mathsf{E}X_1 = (n-r+1)\left(\mathsf{E}X_{(r)} - \mathsf{E}X_{(r-1)}\right)$ implies the exponentiality of the distribution of X_1. At first sight, these characterizations seem to be moment-based. In fact, the coincidence of these moments is equivalent to coincidence of distributions. It was proved in [7] that the equality $\mathsf{E}X_1 = \gamma_{1,n}\mathsf{E}X_{(n)}$ implies the exponentiality of the distribution of X_1. This characterization is purely moment-based because distributions of random variables X_1 and $\gamma_{1,n}\mathsf{E}X_{(n)}$ as well as other moments do not coincide. In the present paper this result is essentially generalized.

The usual terminology, when dealing with monotone hazard rate, is to use the term "increasing" for "nondecreasing" and "decreasing" for "nonincreasing." We will also use this convention but with one exception: we do not include exponential distribution (the hazard rate is constant) into the classes with increasing and decreasing hazard rate. Thus, "increasing" means "nondecreasing" and "nonconstant"; "decreasing" means "nonincreasing" and "nonconstant." The reason is that all the inequalities obtained in the paper become strict. Denote the union of the set of distributions with increasing hazard rate and the set of distributions with decreasing hazard rate by $\mathcal{M}$. The set of all exponential distributions will be denoted by $\mathcal{E}$. We call the union $\mathcal{M} \cup \mathcal{E}$ the class of distributions with monotone hazard rate.

In what follows, without loss of generality, we will assume that $F(x) < 1$ for all x.

The paper is organized as follows. In Section 2 we present the inequalities for the expectation of order statistics for samples from distributions with monotone hazard rate. In Section 3 the inequalities are applied to the problem of characterization of the exponential distribution.

2. Inequalities

First, we obtain some auxiliary results.

Lemma 1. *Let $G(u)$ be a continuous function defined on the interval $[0,1]$ and satisfying the following conditions*:

$$G(0) < 0, \quad G(1) \geqslant 0, \quad \int_0^1 G(u)du = 0,$$

the solution of $G(u) = 0$ on $(0,1)$ is unique.

If $F(x)$ is a distribution function with increasing hazard rate, then

$$\int_0^\infty (1-F(x))G(F(x))dx < 0.$$

If $F(x)$ is a distribution function with decreasing hazard rate, then

$$\int_0^\infty (1-F(x))G(F(x))dx > 0.$$

Proof. It follows from the conditions of the lemma that there exists a unique point $u_0 \in (0,1)$ such that $G(u_0) = 0$. We have

$$\int_0^\infty (1-F(x))G(F(x))dx = \int_0^\infty \frac{1-F(x)}{f(x)} G(F(x))f(x)dx.$$

Let x_0 be such a value that $F(x_0) = u_0$. Then

$$\int_0^\infty \frac{1-F(x)}{f(x)} G(F(x))f(x)dx =$$

$$= \int_{x_0}^\infty \frac{1-F(x)}{f(x)} G(F(x))f(x)dx - \int_0^{x_0} \frac{1-F(x)}{f(x)} (-G(F(x)))f(x)dx.$$

If $F(x)$ has a decreasing hazard rate, then the function $(1-F(x))/f(x)$ increases and, hence,

$$\int_{x_0}^\infty \frac{1-F(x)}{f(x)} G(F(x))f(x)dx - \int_0^{x_0} \frac{1-F(x)}{f(x)} (-G(F(x)))f(x)dx >$$

$$> \int_{x_0}^\infty \frac{1-F(x_0)}{f(x_0)} G(F(x))f(x)dx - \int_0^{x_0} \frac{1-F(x_0)}{f(x_0)} (-G(F(x)))f(x)dx =$$

$$= \frac{1-F(x_0)}{f(x_0)} \int_0^\infty G(F(x))f(x)dx = \frac{1-F(x_0)}{f(x_0)} \int_0^1 G(u)du = 0.$$

If $F(x)$ has an increasing hazard rate, the opposite inequalities hold.

Lemma 2. *Let $F(x)$ and $f(x)$ be the cumulative distribution function and the density of a positive, absolutely continuous random variable X. If $g(x)$ is a differentiable function such that $g(0) = 0$ and*

$$\lim_{x\to\infty} (1-F(x))g(x) = 0, \tag{1}$$

then

$$\mathsf{E}g(X) = \int_0^\infty (1-F(x))g'(x)dx. \tag{2}$$

Proof. Integrating by parts the right-hand side of (2) and taking (1) into account, we obtain

$$\int_0^\infty (1-F(x))g'(x)dx = \int_0^\infty (1-F(x))dg(x) =$$

$$= \lim_{x\to\infty}(1-F(x))g(x) - \int_0^\infty g(x)d(1-F(x)) =$$

$$= \int_0^\infty g(x)f(x)dx = \mathsf{E}g(X).$$

We will also use the following elementary equalities:

$$\int_0^\infty (1-n(1-F(x))^{n-1})f(x)dx = 0 \tag{3}$$

and

$$\int_0^\infty \left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+F^2(x)+\ldots+F^{n-1}(x))\right)f(x)dx = 0, \tag{4}$$

which hold for any $n=1,2,\ldots$. Both (3) and (4) are obtained by direct integration after the change of variable $u=F(x)$.

Lemma 3. *If $F(x)$ has an increasing hazard rate, then*

$$\gamma_{k+1,n}\mathsf{E}X_{(n-k)} > \gamma_{k,n}\mathsf{E}X_{(n-k+1)}$$

for all $k=1,\ldots,n-1$.

If $F(x)$ has a decreasing hazard rate, then

$$\gamma_{k+1,n}\mathsf{E}X_{(n-k)} < \gamma_{k,n}\mathsf{E}X_{(n-k+1)}$$

for all $k=1,\ldots,n-1$.

In other words,

$$\gamma_{1,n}\mathsf{E}X_{(n)} < \gamma_{2,n}\mathsf{E}X_{(n-1)} < \ldots < \gamma_{n,n}\mathsf{E}X_{(1)}$$

for the distributions with increasing hazard rate and

$$\gamma_{1,n}\mathsf{E}X_{(n)} > \gamma_{2,n}\mathsf{E}X_{(n-1)} > \ldots > \gamma_{n,n}\mathsf{E}X_{(1)}$$

for the distributions with decreasing hazard rate.

Proof. Since the distribution function of $X_{(k)}$ is

$$\mathsf{P}(X_{(k)} \leqslant t) = \sum_{j=k}^{n} \mathrm{C}_n^j F^j(t)(1-F(t))^{n-j} = 1-\sum_{j=0}^{k-1} \mathrm{C}_n^j F^j(t)(1-F(t))^{n-j},$$

we have

$$\mathsf{E}X_{(n-k)} = \sum_{j=0}^{n-k-1} \mathrm{C}_n^j \int_0^\infty F^j(t)(1-F(t))^{n-j}dt,$$

$$\mathsf{E}X_{(n-k+1)} = \sum_{j=0}^{n-k} \mathrm{C}_n^j \int_0^\infty F^j(t)(1-F(t))^{n-j} dt.$$

Consider the difference

$$\gamma_{k+1,n}^{-1}\mathsf{E}X_{(n-k+1)} - \gamma_{k,n}^{-1}\mathsf{E}X_{(n-k)} =$$

$$= \int_0^\infty (1-F(t))\Big(\gamma_{k+1,n}^{-1}\sum_{j=0}^{n-k}\mathrm{C}_n^j F^j(t)(1-F(t))^{n-j-1} -$$

$$-\gamma_{k,n}^{-1}\sum_{j=0}^{n-k-1}\mathrm{C}_n^j F^j(t)(1-F(t))^{n-j-1}\Big)dt.$$

Denote

$$\psi_k(u) = \gamma_{k+1,n}^{-1}\sum_{j=0}^{n-k}\mathrm{C}_n^j u^j(1-u)^{n-j-1} - \gamma_{k,n}^{-1}\sum_{j=0}^{n-k-1}\mathrm{C}_n^j u^j(1-u)^{n-j-1},\ u \in [0,1].$$

Since

$$\int_0^1 u^j(1-u)^{n-j-1}du = \frac{1}{(n-j)\mathrm{C}_n^j},$$

we have

$$\int_0^1 \psi_k(u)du = 0.$$

Further,

$$\psi_k(0) = -\frac{1}{k} < 0,\ \psi_k(1) = 0,\ k = 2,\dots,n-1,\ \psi_1(1) = \gamma_{2,n}^{-1},\ n > 0.$$

Prove now that in the interval $(0,1)$, the function $\psi_k(u)$ has exactly one zero. Rewrite it in the form

$$(1-u)^{k-1}\left(\gamma_{k+1,n}^{-1}\mathrm{C}_n^{n-k}u^{n-k} - \frac{1}{k}\sum_{j=0}^{n-k-1}\mathrm{C}_n^j u^j(1-u)^{n-k-j}\right).$$

The multiplier $(1-u)^{k-1}$ has no influence on the number of zeros in the interval $(0,1)$; therefore it suffices to find the number of zeros of the function

$$\gamma_{k+1,n}^{-1}\mathrm{C}_n^{n-k}u^{n-k} - \frac{1}{k}\sum_{j=0}^{n-k-1}\mathrm{C}_n^j u^j(1-u)^{n-k-j},$$

which can be written as

$$\gamma_{k+1,n}^{-1}\mathrm{C}_n^{n-k}u^{n-k} + \frac{1}{k}\mathrm{C}_{n-1}^{n-k-1}u^{n-k} - \frac{1}{k}\sum_{l=0}^{n-k-1}\mathrm{C}_{k+l-1}^l u^l =$$

$$= \gamma_{k+1,n}^{-1}\mathrm{C}_n^{n-k}u^{n-k} + \frac{1}{k}\left(\sum_{l=0}^{n-k-1}\mathrm{C}_{k+l-1}^l\right)u^{n-k} - \frac{1}{k}\sum_{l=0}^{n-k-1}\mathrm{C}_{k+l-1}^l u^l.$$

It follows from this representation that the function itself and its $n-k-1$ first derivatives take negative values at zero and positive values at one. The derivative of order $n-k-1$ is a linear function with a positive coefficient at u. Thus, this derivative has one zero and is negative on the left and positive on the right. This implies that all other derivatives (of smaller order) and the function itself have the same properties: unique zero, negativity on the left and positivity on the right. Now the statement of the theorem follows from Lemma 1.

Since for the exponential distribution the equalities

$$\gamma_{1,n}\mathsf{E}X_{(n)} = \gamma_{2,n}\mathsf{E}X_{(n-1)} = \ldots = \gamma_{n,n}\mathsf{E}X_{(1)}$$

hold, Lemma 3 is sharp.

Due to applications, the relationship between the expectations of order statistics, especially of the extreme order statistics, and the expectation of the population is very important. It follows from the above theorem that

$$\gamma_{1,n}\mathsf{E}X_{(n)} < \mathsf{E}X_1 < \gamma_{n,n}\mathsf{E}X_{(1)} \tag{5}$$

for the distributions with increasing hazard rate and

$$\gamma_{1,n}\mathsf{E}X_{(n)} > \mathsf{E}X_1 > \gamma_{n,n}\mathsf{E}X_{(1)} \tag{6}$$

for the distributions with decreasing hazard rate. It turns out that much stronger results can be obtained. Relations (5) and (6) can be sharpened in two directions, as is shown in the following three lemmas.

Lemma 4. *If* $\gamma_{k,n} > 1$ *and* $F(x)$ *has an increasing hazard rate, then*

$$\gamma_{1,n}\mathsf{E}X_{(n)} < \mathsf{E}X_1 < \gamma_{k,n}\mathsf{E}X_{(n-k+1)}.$$

If $\gamma_{k,n} > 1$ *and* $F(x)$ *has a decreasing hazard rate, then*

$$\gamma_{1,n}\mathsf{E}X_{(n)} > \mathsf{E}X_1 > \gamma_{k,n}\mathsf{E}X_{(n-k+1)}.$$

Proof. The inequalities $\gamma_{1,n}\mathsf{E}X_{(n)} < \mathrm{E}X_1$ for the distributions with an increasing hazard rate and $\gamma_{1,n}\mathsf{E}X_{(n)} > \mathsf{E}X_1$ for the distributions with a decreasing hazard rate were obtained in [7]. To prove other inequalities, consider the difference

$$\gamma_{k,n}^{-1}\mathsf{E}X_1 - \mathsf{E}X_{(n-k+1)} =$$

$$= \int_0^\infty (1-F(t)) \left(\gamma_{k,n}^{-1} - \sum_{j=0}^{n-k} \mathrm{C}_n^j F^j(t)(1-F(t))^{n-j-1} \right) dt.$$

The function

$$\chi_k(u) = \gamma_{k,n}^{-1} - \sum_{j=0}^{n-k} \mathrm{C}_n^j u^j (1-u)^{n-j-1}, \ u \in [0,1],$$

has the following properties:

$$\int_0^1 \chi_k(u)du = 0, \ \chi_k(0) = \gamma_{k,n}^{-1} - 1 < 0, \ \chi_k(1) = \gamma_{k,n}^{-1} > 0.$$

The number of zeros of the function $\chi_k(u)$ in the interval $(0,1)$ coincides with the number of zeros of the function

$$\frac{\gamma_{k,n}^{-1}}{(1-u)^{k-1}} - \sum_{j=0}^{n-k} \mathrm{C}_n^j u^j (1-u)^{n-k-j} = \frac{\gamma_{k,n}^{-1}}{(1-u)^{k-1}} - \sum_{l=0}^{n-k} \mathrm{C}_{k+l-1}^l u^l.$$

All derivatives of this function of order from 1 to $n-k$ are negative at $u=0$ and converge to $+\infty$ as $u \to 1$. The derivative of order $n-k$ increases on the interval $(0,1)$ and, hence, has a unique zero. The rest of the proof repeats the corresponding part of the proof of Lemma 3.

The lemma is sharp in the following sense: the order of the order statistic on the right-hand side cannot be made greater. Without the condition $\gamma_{k,n} > 1$, the relationship between $\mathsf{E}X_1$ and $\gamma_{k,n}\mathsf{E}X_{(n-k+1)}$ can be arbitrary: the first variable can be less than, greater than, or equal to the second one. Let, for example, $F(x)$ be the gamma distribution with the shape parameter 2 (increasing hazard rate) and the scale parameter 1. If $n=4$, then $\gamma_{2,4} < 1$, and $\gamma_{2,4}\mathsf{E}X_{(3)} > 2 = \mathrm{E}X_1$. If $n=6$, then $\gamma_{2,6} < 1$, and $\gamma_{2,6}\mathrm{E}X_{(5)} < 2 = \mathsf{E}X_1$.

Lemma 5. *Let $g(x)$ be a convex differentiable function such that*

$$\lim_{x\to\infty}(1-F(x))g(x)=0, \quad g(0)=0.$$

If $F(x)$ has a decreasing hazard rate, then

$$\gamma_{1,n}\mathsf{E}g(X_{(n)}) > \mathsf{E}g(X_1) > \gamma_{n,n}\mathsf{E}g(X_{(1)}).$$

Proof. Without loss of generality, one can suppose that $g(x)$ is twice differentiable. Then the derivative $g'(x)$ is a nondecreasing function. Consider the difference $\mathrm{E}g(X_1) - \gamma_{n,n}\mathsf{E}g(X_{(1)})$. Since the distribution function of the random variable $X_{(1)}$ is $1-(1-F(x))^n$, and due to Lemma 2, the following equality holds:

$$\mathsf{E}g(X_1) - \gamma_{n,n}\mathsf{E}g(X_{(1)}) = \int_0^\infty g'(x)(1-F(x))dx - n\int_0^\infty g'(x)(1-F(x))^n dx =$$

$$= \int_0^\infty g'(x)\frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx. \tag{7}$$

Since, due to (3),

$$\int_0^\infty g'(x)\frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx =$$

$$= \int_0^\infty \left(g'(x)\frac{1-F(x)}{f(x)} + A\right)(1-n(1-F(x))^{n-1})f(x)dx$$

for any A, without loss of generality we can suppose that

$$g'(x)\frac{1-F(x)}{f(x)} > 0.$$

Consider the function $\psi(u) = 1-n(1-u)^{n-1}$. It increases in the interval $[0,1]$ and $\psi(0) = 1-n < 0$, $\psi(1) = 1 > 0$. Hence, there exists the unique $u_0 \in (0,1)$ such that $\psi(u_0) = 0$, $\psi(u) < 0$, $u \in [0, u_0)$, $\psi(u) > 0$, $u \in (u_0, 1]$. Let x_0 be such that $F(x_0) = u_0$. The integral in (7) is represented as the difference of two integrals of nonnegative functions:

$$\int_0^\infty g'(x)\frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx =$$

$$= \int_{x_0}^{\infty} g'(x)\frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx -$$

$$- \int_{0}^{x_0} g'(x)\frac{1-F(x)}{f(x)}(n(1-F(x))^{n-1}-1)f(x)dx.$$

Since the function

$$g'(x)\frac{1-F(x)}{f(x)}$$

increases, we have

$$\int_{x_0}^{\infty} g'(x)\frac{1-F(x)}{f(x)}(1-n(1-F(x))^{n-1})f(x)dx >$$

$$> g'(x_0)\frac{1-F(x_0)}{f(x_0)}\int_{x_0}^{\infty}(1-n(1-F(x))^{n-1})f(x)dx,$$

and

$$\int_{0}^{x_0} g'(x)\frac{1-F(x)}{f(x)}(n(1-F(x))^{n-1}-1)f(x)dx <$$

$$< g'(x_0)\frac{1-F(x_0)}{f(x_0)}\int_{0}^{x_0}(n(1-F(x))^{n-1}-1)f(x)dx.$$

This implies

$$\mathsf{E}g(X_1) - \gamma_{n,n}\mathsf{E}g(X_{(1)}) >$$

$$> g'(x_0)\frac{1-F(x_0)}{f(x_0)}\Big(\int_{x_0}^{\infty}(1-n(1-F(x))^{n-1})f(x)dx -$$

$$- \int_{0}^{x_0}(n(1-F(x))^{n-1}-1)f(x)dx\Big) =$$

$$= g'(x_0)\frac{1-F(x_0)}{f(x_0)}\int_{0}^{\infty}(1-n(1-F(x))^{n-1})f(x)dx = 0.$$

Now consider the difference

$$\frac{1}{\gamma_{1,n}}\mathsf{E}g(X_1) - \mathsf{E}g(X_{(n)}).$$

Since the distribution function of $X_{(n)}$ is $F^n(x)$, using Lemma 2, we obtain

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\mathsf{E}g(X_1) - \mathsf{E}g(X_{(n)}) =$$

$$= \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\int_{0}^{\infty} g'(x)(1-F(x))dx - \int_{0}^{\infty} g'(x)(1-F^n(x))dx =$$

$$= \int_0^\infty g'(x)\frac{1-F(x)}{f(x)}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+F^2(x)+\ldots+F^{n-1}(x))\right)f(x)dx.$$

As in the first part of the proof, since due to (9),

$$\int_0^\infty g'(x)\frac{1-F(x)}{f(x)}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+F^2(x)+\ldots+F^{n-1}(x))\right)f(x)dx =$$

$$= \int_0^\infty \left(g'(x)\frac{1-F(x)}{f(x)}+A\right)\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+\ldots+F^{n-1}(x))\right)f(x)dx$$

for any A, without loss of generality we can suppose that

$$g'(x)\frac{1-F(x)}{f(x)} > 0.$$

The function

$$\chi(u) = \left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+u+u^2+\ldots+u^{n-1})\right)$$

satisfies the conditions $\chi(0) = \frac{1}{2}+\ldots+\frac{1}{n} > 0$, $\chi(1) = 1+\frac{1}{2}+\ldots+\frac{1}{n}-n < 0$, $\chi(u)$ decreases in the interval $[0,1]$. Let u_0 be the unique zero of the function $\chi(u)$ in the interval $(0,1)$, and $F(x_0) = u_0$. We have

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\mathsf{E}g(X_1) - \mathsf{E}g(X_{(n)}) =$$

$$= \int_0^{x_0} g'(x)\frac{1-F(x)}{f(x)}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+\ldots+F^{n-1}(x))\right)f(x)dx -$$

$$- \int_{x_0}^\infty g'(x)\frac{1-F(x)}{f(x)}\left((1+F(x)+\ldots+F^{n-1}(x))-1-\frac{1}{2}-\ldots-\frac{1}{n}\right)f(x)dx,$$

and therefore

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\mathsf{E}g(X_1) - \mathsf{E}g(X_{(n)}) <$$

$$< g'(x_0)\frac{1-F(x_0)}{f(x_0)}\int_0^{x_0}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+\ldots+F^{n-1}(x))\right)f(x)dx -$$

$$-g'(x_0)\frac{1-F(x_0)}{f(x_0)}\int_{x_0}^\infty\left((1+F(x)+\ldots+F^{n-1}(x))-1-\frac{1}{2}-\ldots-\frac{1}{n}\right)f(x)dx =$$

$$= g'(x_0)\frac{1-F(x_0)}{f(x_0)}\int_0^\infty\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-(1+F(x)+\ldots+F^{n-1}(x))\right)f(x)dx = 0.$$

Lemma 6. *Let $g(x)$ be a concave differentiable function such that*

$$\lim_{x\to\infty}(1-F(x))g(x) = 0, \quad g(0) = 0.$$

If $F(x)$ has an increasing hazard rate, then

$$\gamma_{1,n}\mathsf{E}g(X_{(n)}) < \mathsf{E}g(X_1) < \gamma_{n,n}\mathsf{E}g(X_{(1)}).$$

The proof of Lemma 6 is completely analogous to the proof of Lemma 5.

3. Characterizations of the exponential distribution in the class of distributions with monotone hazard rate

The above inequalities can be applied to the problem of characterization of the exponential distribution in the class of distributions with monotone hazard rate.

Theorem 1. *Let $F(x)$ belong to the class $\mathcal{M} \cup \mathcal{E}$. $F(x)$ is an exponential distribution function if and only if*

$$\gamma_{i+1,n} \mathsf{E} X_{(n-i)} = \gamma_{j+1,n} \mathsf{E} X_{(n-j)} \tag{8}$$

for at least one pair of i and j, $i \neq j$.

Thus, if (8) holds for one pair $i \neq j$, then it holds for all pairs.

Theorem 2. *Let $F(x)$ belong to the class calM $\cup \mathcal{E}$. $F(x)$ is an exponential distribution function if and only if*

$$\mathsf{E} X_1 = \gamma_{1,n} \mathsf{E} X_{(n)}.$$

Theorem 3. *Let $F(x)$ belong to the class $\mathcal{M} \cup \mathcal{E}$. $F(x)$ is an exponential distribution function if and only if the equality*

$$\mathsf{E} X_1 = \gamma_{k,n} \mathsf{E} X_{(n-k+1)} \tag{9}$$

holds for at least one k, such that

$$\gamma_{k,n} > 1. \tag{10}$$

Note that as in Theorem 1, if (9) holds for one k satisfying (10), then (9) holds for all k-s, satisfying (10).

Theorem 4. *Let $F(x)$ belong to the class $\mathcal{M} \cup \mathcal{E}$. Let $G(u)$ be a continuous function defined on the interval $[0,1]$ and satisfying the following conditions*:

$$G(0) < 0, \quad G(1) \geqslant 0, \quad \int_0^1 G(u)du = 0,$$

solution of $G(u) = 0$ on $(0,1)$ is unique. Then $F(x)$ is an exponential distribution function if and only if

$$\int_0^\infty (1 - F(x))G(F(x))dx = 0.$$

A typical way to obtain a characterization of the exponential distribution in the class of distributions with monotone hazard rate on the basis of inequalities is as follows. If for some Expression 1 and Expression 2, the first expression is greater than the second one if $F(x)$ has increasing hazard rate and less if it has decreasing hazard rate, then Expression 1 is equal to Expression 2 if and only if $F(x)$ is an exponential distribution function. Thus Theorem 1 follows directly from Lemma 3, Theorems 2 and 3 from Lemma 4, and Theorem 4 from Lemma 5.

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REFERENCES

1. R.E. Barlow and A.W.Marshall, "Bounds for distributions with monotone hazard rate," *Ann. Math. Stat.*, **35**, 1234–1274 (1964).
2. R. Shimizu, "Inequalities for a distribution with monotone hazard rate," *Ann. Inst. Stat. Math.*, **38**, Part A, 195–204 (1986).

3. B.O. Oluyede, "On some length biased inequalities for reliability measures," *J. Inequal. Appl.*, **5**, 447–466 (2000).

4. R.E. Barlow and F. Proschan, "Inequalities for linear combinations of order statistics from restricted families," *Ann. Math. Stat.*, **37**, 1574–1592 (1966).

5. T.A. Azlarov and N.A. Volodin, *Characterization Problems Associated with the Exponential Distribution*, Springer-Verlag, New York (1986).

6. M. Ahsanullah, "On a conjecture of Kakosyan, Klebanov, and Melamed," *Stat. Papers*, **29**, 151–157 (1988).

7. N.G. Ushakov and V.G. Ushakov, "A note on characterizations of the exponential distribution," *J. Math. Sci.*, **214**, 132–138 (2016).