Switching from Oil to Gas Production in a Depleting Field*

Kristian Støre[a,*], Stein-Erik Fleten[b], Verena Hagspiel[b], Cláudia Nunes[c]

[a] *Nord University Business School, 8049 Bodø, Norway*
[b] *Norwegian University of Science and Technology (NTNU), Department of Industrial Economics and Technology Management, 7491 Trondheim, Norway*
[c] *CEMAT, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

Abstract

We derive an optimal decision rule with regards to making an irreversible switch from oil to gas production. The approach can be used by petroleum field operators to maximize the value creation from a petroleum field with diminishing oil production and remaining gas reserves. Assuming that both the oil and gas prices follow a geometric Brownian motion we derive an analytical solution for the exercise threshold. We also propose an explicit solution for the option value that is new to the literature. Numerical examples are used to demonstrate the threshold and option value for a generic petroleum field. Both the threshold and option value solutions are relevant for application to other real options cases with similar features (e.g. other types of switching options or a perpetual spread option).

Keywords: *OR in energy, Switching option, Petroleum, Investment under uncertainty*

1. Introduction

At the Prudhoe Bay field in Alaska, one of the largest oil fields in North America, operators have increased the recovery factor substantially due to gas injection, together with other techniques (see e.g. Ning et al. (2016) or Szabo & Meyers (1993)). The associated gas being produced together with the oil is re-injected into the reservoir. As oil production from the field falls, a gas pipeline

*The authors gratefully acknowledge comments from three anonymous referees and would also like to thank Øystein Dahl Hem, Alexander Svendsen, and Vidar Gunnerud for discussions on reservoir dynamics and petroleum field cases.
*Corresponding author: kristian.store@nord.no

to export the gas is being discussed; necessary infrastructure for large-scale gas export is not currently present. In the North Sea, on the Norwegian Continental Shelf (NCS), substantial investments have been made in the Statfjord Latelife project on the Statfjord field. The investments, including a new pipeline which connects the gas exports of the field to the UK market, have changed the primary function of the production facilities from predominantly oil production to gas production. On the Oseberg field, also located on the NCS, natural gas has been imported from the nearby Troll field and used for injection to enhance oil production. The field has been in a phase of declining oil production for many years, often refered to as the "tail production phase". Discussions are ongoing as to what the optimal course for future action should be and producing a significant portion the injected gas (a small portion is already being produced and sold every year) is one of the considered alternatives.

Injection of natural gas is one of a number of techniques employed by operators of petroleum fields to increase the recovery rate of oil. The gas used for injection may be associated gas produced with the oil, gas transported to the field from other sources, or a combination of the two. From a business point of view this makes sense as long as the value of continuing oil production under the gas injection scheme is higher than the alternative value of stopping the gas injection and investing in producing and exporting the gas that has been injected (the term "export" here means the transportation of the gas to a market). As the oil field matures, and the amount of oil in the reservoir as well as the oil production rate decline, it may become optimal to export the gas rather than continuing the injection scheme. This could involve substantial investments in both the production facilities and in export solutions for the gas, as well as having a strong adverse effect on the oil production. Therefore, determining the optimal timing to start gas production and export is relevant for a number of stakeholders in a petroleum field. For the operators and owners of petroleum fields such decision models can contribute to maximizing the value of the asset both for themselves and the society in which they operate. Also, policymakers can make use of such models to avoid value-erosive regulations or approval decisions. Furthermore, the option valuation approach could serve as a tool for the petroleum field owner(s) seeking to fund or sell an interest in the switching venture (which may require substantial investments), and also assist engineers and suppliers of conversion equipment seeking early project development[1].

The type of optionality considered here falls naturally into a category of real options often refered to as switching options. There are many examples of real options applications with switching features in the literature, with the work of Brennan & Schwartz (1985) being one of the earliest. Using a copper mine as an example they value the combined options to temporarily shut down, re-opening after a temporary shutdown, and abandoning entirely. In a more recent example, Tsekrekos & Yannacopoulos (2016) derive a closed form approximate solution to a class of optimal switching problems where the underlying prices follow stochastic mean-reverting volatility models. Studying a switching case similar to the one described herein, Hahn & Dyer (2008) propose a binomial lattice approach for modeling an oil-to-gas switching option when the underlying uncertainty factors follow correlated one-factor mean reverting processes. Using the Prudhoe Bay field previously mentioned as their case (including a research and development program with uncertain outcome) they apply their proposed approach to value the asset. The focus in their study is to approximate the asset value, rather than on a tractable decision rule for making a switch. Adkins & Paxson (2011a) propose an analytical approach to an optimal asset replacement case when operating costs and revenues are stochastic (which is similar to a switching option) and arrive at what they term a "quasi-analytical solution" to the decision rule problem. This approach has been applied by the same authors to a range of real options cases with multiple sources of uncertainty and with switching-like features (see e.g. Adkins & Paxson (2011b) and Adkins & Paxson (2017)). They study cases where there is a single opportunity to make a switch (or replacement) and cases where there is a perpetual string of sequential switching opportunities. By assuming that asset prices follow geometric Brownian motions and that a smooth pasting condition[2] holds, their approach results in an equation set that the authors solve numerically. Gahungu & Smeers (2011)

[1] We thank an anonymous referee for bringing up this point.

[2] This principle is sometimes called *high contact* or *smooth fit*. See Brekke & Øksendal (1991) for an introduction to the concept as well as a proof of sufficient and necessary conditions for the smooth pasting condition to produce the optimal solution to the stopping problem.

study in a more general manner the same type of problem as the "single opportunity" switching case; they find the optimal time to exercise an option which gives the right to exchange a basket of assets for another, assuming the asset prices follow correlated geometric brownian motions. They show that an equation set such as the ones Adkins & Paxson (2011a,b) solve numerically can be determined in closed form. Specific examples of real options applications where such closed form solutions are presented can be found in Heydari et al. (2012) and Rohlfs & Madlener (2011), who both derive decision rules related to investmens in emission-reduction technologies. Where Gahungu & Smeers (2011) use simulation tecniques to determine option value inside of the continuation region, Adkins & Paxson (2011a,b) and Heydari et al. (2012) make simplifying assumptions about the solution in order to approximate option value (Rohlfs & Madlener (2011) and Adkins & Paxson (2017) do not calculate option values inside of the continuation region). However, neither of them provide explicit solutions for how to determine the option value *inside* of the continuation region (in all cases the option value exactly at the exercise threshold is expressed explicitly).

We model the switching option as a perpetual American style option and the decision to switch is considered irreversible. Although the negative effects on oil production from starting gas production depend on the characteristics of the oil field, we assume that the remaining oil is lost if the decision to switch is made[3]. This is a conservative assumption which will emphasize the trade-off effect between the two resources in the model. On the basis of a parameter set that describes a representative large size oil field (initial reserves of 100–500 mill. barrels of oil) in the North Sea, we derive the region of oil and gas prices for which it is optimal to undergo a switch. We contribute to the existing literature by determining and applying an analytical solution to the decision to change from oil to gas production in the tail production phase of a petroleum field. Moreover, we propose an explicit solution which is new to the literature

[3]The effect of gas injection on the oil production rate is dependent on the reservoir properties of each field, and placement of injecting and producing wells. Assuming that oil production drops to zero when the gas is produced might be a fair approximation if the oil layer in the reservoir is thin, where many wells can move below the oil-water contact if this shifts slightly upwards. In fields where gas is mostly used for moving the oil towards the wells this might be a poor approximation and more complex reservoir models may be necessary.

for determining the option value (inside of the continuation region).

The remainder of the paper is organized as follows. Section 2 formulates and develops the model setup for the switching option and presents the solutions for the exercise boundary and option value. The section also contains comparative statics for price process parameters. In section 3, numerical examples for a base case as well as a sensitivity analysis is presented. Lastly, section 4 concludes the paper.

2. Switching option

To achieve a tractable model for the switching option some simplying assumptions are made about the petroleum field and the nature of the switching option. Firstly, the switching is assumed to happen instantaneously with all of the switching costs incurring immediately and it is not possible to reverse the switch once it has been made. Secondly, the operational costs are assumed to be known and fixed. Thirdly, the "potential" initial gas production rate, after a switch is made, is assumed fixed (i.e. unaffected by injection and oil being produced) and the gas used for injection is assumed to be costless (i.e. we exclude any potential cost from importing gas to use for injection). This assumption makes the example case more relevant for fields where the injection gas is only re-injected associated gas (rather than imported from an external source) and the potential gas production is unaffected by decreasing oil reserves. Lastly, the rate at which oil is being produced is assumed to be deterministically declining, and the same is assumed for the gas once a switch has been made.

Although the production profile for an oil field depends on the field's physical characteristics and the chosen depletion strategy, there are in general three phases of production; build-up, plateau and decline (see e.g. Wallace et al. (1987) for a discussion of aggregate production profiles and examples). As can be seen in Figure 1, both the Oseberg and Prudhoe Bay fields are examples of fields whose production profiles[4] exhibit the typical characteristics of these three phases. When we consider the option to switch to gas production we

[4]Sources for the production numbers are the Norwegian Petroleum Directorate for the Oseberg field and the State of Alaska, Department of Revenue for Prudhoe Bay. The Prudhoe data is for the fiscal year July-June and is converted from daily average in thousand barrels by assuming it is averaged across 365 days per year.

assume that this is only relevant in the decline phase. Although it is possible to consider stopping oil production during the build-up or plateau phase, it is highly unlikely to be considered as a viable alternative. The model we propose therefore needs to include a decline in the oil production rate in order to capture the characteristics of a representative field. We assume in the following that the production rate is exponentially declining, very much in line with the shape of the production curves in Figure 1. An exponentially declining production rate is a standard simplifying assumption used in literature addressing decision making related to petroleum extraction (see e.g. Paddock et al. (1988) for an early example). For each commodity $I \in \{1, 2\}$ (with oil given as $I = 1$ and gas as $I = 2$), we assume that when production is ongoing the production rate $R_{I,t}$ is exponentially declining over time, i.e. $R_{I,t} = R_{I,0}e^{-\theta_I t}$. Here $R_{I,0}$ and θ_I are constants and the former is the initial production rate while the latter is the exponential decline factor of the production. Furthermore, we assume that the production costs, E_I, are independent of the production rate, i.e. that the total costs of operation are fixed. Thus, the cash flow from production, when producing commodity I, is given by $(X_{I,t}R_{I,t} - E_I)dt$. Note that for simplicity the effects of taxes and royalties are ignored.

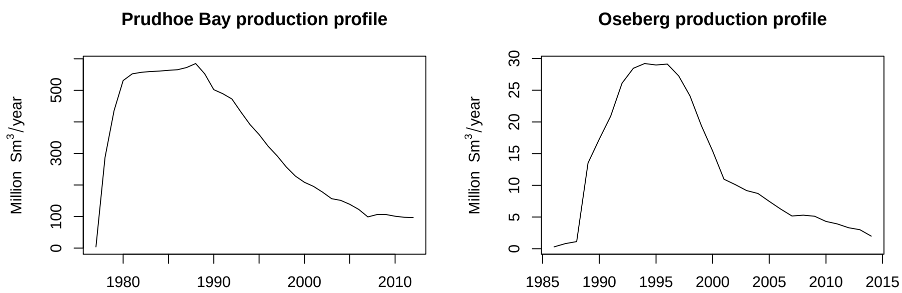

Figure 1: Historical oil production profiles for the Prudhoe and Oseberg fields

Under the assumptions described above, let $F(\tau, x_1, x_2)$ denote the value of a petroleum field - with current oil price x_1 and gas price x_2 - if it is decided to switch from oil to gas production at time $t = \tau$:

$$\begin{aligned} F(\tau, x_1, x_2) =& \mathbb{E}\bigg[\int_0^\tau (X_{1,t}R_{1,t} - E_1)\, e^{-rt}dt \\ &+ \int_\tau^\infty (X_{2,t}R_{2,t} - E_2)\, e^{-rt}dt - e^{-r\tau}S\bigg] \\ =& \mathbb{E}\bigg[\int_0^\tau X_{1,t}e^{-\theta_1 t}R_{1,0}e^{-rt}dt + \int_\tau^\infty X_{2,t}e^{-\theta_2 t}R_{2,0}e^{-rt}dt \\ &- e^{-r\tau}\left(S + \frac{E_2 - E_1}{r}\right)\bigg] - \frac{E_1}{r}. \end{aligned} \tag{1}$$

Here $X_{I,t}$[5] is the spot price of oil ($I = 1$) and gas ($I = 2$) at time t, S denotes the switching cost of converting from oil to gas production, and r is the risk free rate (this assumes that the dynamics of $X_{1,t}$ and $X_{2,t}$ are described under the risk neutral measure). Note also that as long as oil is being extracted from the field, the oil production rate declines exponentially (at rate θ_1); however, when the switching occurs and gas production starts, the production rate for gas starts declining exponentially at rate θ_2. This means that the "potential" gas production rate is constant as long as no gas is being produced. The optimal value of the field is now given by

$$V(x_1, x_2) = sup_\tau F(\tau, x_1, x_2). \tag{2}$$

To find a solution for the optimal exercise threshold and option value given by (2) it's necessary to formulate the price dynamics for the spot price of oil and gas under the risk neutral measure. For simplicity, we assume that both of these prices follow a geometric Brownian motion (GBM). Although this may be a simplifying assumption, it was noted by Pindyck (2001), and confirmed by Postali & Picchetti (2006), that the half-life of oil price shocks is sufficiently long to justify using GBM. If we deviate from the GBM assumption, for example by incorporating mean reversion, we suspect that the change will be small compared to the GBM case. Schwartz (1997, 1998) offer insight into why this may be; the GBM captures the persistence of commodity price shocks, whereas mean reversion, capturing transitory shocks, has little bearing on decisions and valuation due to the averaging effects of long lifetime and required time for construction

[5] In the rest of the paper we denote random variables by an uppercase letter, while their realizations will be denoted by a lowercase letter.

once an investment decision is made (although we assume for simplicity that a switch happens instantaneously in our model)[6].

With X_I denoting the spot price for oil ($I = 1$) and gas ($I = 2$), respectively, their dynamics under the risk neutral measure are described by the following stochastic differential equation:

$$dX_{I,t} = \alpha_I X_{I,t} dt + \sigma_I X_{I,t} dZ_{I,t}. \tag{3}$$

Here α_I is the risk adjusted drift (we assume $r > \alpha_2$ and $r > \alpha_1 - \theta_1$ to ensure that the option has a well-defined exercise threshold), σ_I is the volatility, and $dZ_{I,t}$ is the increment of a standard Brownian process. We allow the prices of oil and gas to be dependent, introducing the correlation parameter ρ, where $Cov[dZ_1, dZ_2] = \rho\sigma_1\sigma_2 dt$ represents the covariance between the two Brownian motions (Z_1 and Z_2), and with $|\rho| \leq 1$.

Standard techniques found in the literature (see, e.g., Øksendal (2013)) show that $V(x_1, x_2)$ in (2) must be a solution to the following Hamilton-Jacobi-Bellman equation:

$$\max\Bigg(- rV(x_1, x_2) + LV(x_1, x_2) + x_1 R_{1,0} - E_1, \tag{4}$$
$$\frac{x_2 R_{2,0}}{r + \theta_2 - \alpha_2} - \frac{E_2}{r} - S - V(x_1, x_2)\Bigg) = 0,$$

where L represents the infinitesimal generator of the process (X_1, X_2), that, according to our assumptions, is given by:

$$LV(x_1, x_2) = \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 V(x_1, x_2)}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 x_2^2 \frac{\partial^2 V(x_1, x_2)}{\partial x_2^2} \tag{5}$$
$$+ \rho\sigma_1\sigma_2 x_1 x_2 \frac{\partial^2 V(x_1, x_2)}{\partial x_1 \partial x_2} + (\alpha_1 - \theta_1) x_1 \frac{\partial V(x_1, x_2)}{\partial x_1} + \alpha_2 x_2 \frac{\partial V(x_1, x_2)}{\partial x_2}.$$

The above equation (4) should be interpreted as follows: in the continuation region, that we denote by $\mathcal{D}$, $V(x_1, x_2) > \frac{x_2 R_{2,0}}{r+\theta_2-\alpha_2} - \frac{E_2}{r} - S$, and it is a solution

[6]Schwartz uses different stochastic processes, including a two-factor model that incorporates both geometric commodity price movements and a mean reverting factor, and finds that for so-called long-term assets, the mean reversion aspect is of little importance. One can as well approximate the problem with a GBM (Schwartz (1998) allows for a time-varying volatility in the approximation). By long-term assets he means investments with a long lifetime, with some time to build, and that has operational characteristics that give rise to cash flow patterns that are unaffected by volatility and price reversions. These are in practice the kinds of assets we are discussing in this paper. The intuition behind these results is that the long-term (geometric) part of the process is carrying the persistence in prices, essentially shouldering the value of waiting for more information. The mean reversion effect is dissipated over time (lifetime of the asset, and time from decision to invest to the cash flow starts).

to the partial differential equation

$$-rV(x_1,x_2) + LV(x_1,x_2) + x_1R_{1,0} - E_1 = 0. \tag{6}$$

Thus in the continuation region the value of the field (with continued oil extraction) is larger than the value that results from switching and therefore the operators should postpone the switching decision. We remark again that the production rate for gas only declines when one starts extracting gas, and therefore in the continuation region (i.e., before the switching) there is no declining behavior for the gas. Consequently, in the continuation region, the drift for the product of the gas price and the gas production rate, $X_{2,t}R_{1,0}$, is α_2 (with no decline in the potential/initial gas production rate), whereas for the product of the oil price and the oil production rate, $X_{1,t}R_{1,t}$, it is $\alpha_1 - \theta_1$.

2.1. Exercise boundary

For the remainder of the paper, let x_1^* and x_2^* denote the threshold switching values for the processes X_1 and X_2 respectively. At the exercise boundary the value from continuation must be equal to the value of switching, as the solution of (2) must be a continuous function in all its domain (Øksendal (2013)). Therefore we propose the following as the solution for (6), with $x_1 = x_1^*$ and $x_2 = x_2^*$ at the threshold boundary:

$$v(x_1,x_2) = A(x_1,x_2)x_1^{\beta(x_1,x_2)}x_2^{\eta(x_1,x_2)} + \frac{x_1R_{1,0}}{r+\theta_1-\alpha_1} - \frac{E_1}{r}, \tag{7}$$

where A, β and η are parameters that still need to be derived and that may depend on x_1 and x_2. For convenience the notation A, β and η will sometimes be used instead of $A(x_1,x_2), \beta(x_1,x_2)$ and $\eta(x_1,x_2)$, respectively.

Note that in (7) the term $Ax_1^{\beta}x_2^{\eta}$ corresponds to the switching option value and it is the homogeneous solution of (6), whereas the second and third terms, $\frac{x_1R_{1,0}}{r+\theta_1-\alpha_1} - \frac{E_1}{r}$, represent the present value of perpetual oil production and it is the particular solution of (6). Moreover, based on economical arguments it is necessary that the option value goes towards zero when the oil price goes towards infinity ($\lim_{x_1\to\infty} v(x_1,x_2) = 0$) and the value should go towards infinity if the gas price goes towards infinity ($\lim_{x_2\to\infty} v(x_1,x_2) = \infty$). Consequently, it means that $\beta < 0$ and $\eta > 0$ (trivially, it must also hold true that $A > 0$).

In order to determine the parameters A, β and η we must derive the necessary conditions that ensure that the option value, $k(x_1,x_2) \equiv Ax_1^{\beta}x_2^{\eta}$, is indeed a solution of the homogeneous part of (6). As part of this, it is necessary to compute the partial derivatives stated in (5) and verify that (6) always holds. As we let A, β and η depend on x_1 and x_2, then the partial derivatives of $k(x_1, x_2)$ with respect to x_1 and x_2 should also include the derivatives of A, β and η with respect to x_1 and x_2. However, as we will argue in the next section, all the derivatives with respect to these parameters cancel out. Therefore we end up with simple expressions for the partial derivatives which are identical to a case where the parameters are constants (e.g. $\frac{\partial k(x_1,x_2)}{\partial x_1} = A\beta x_1^{\beta-1} x_2^{\eta}$). Using this result, trivial calculations lead to β and η being the roots of the following equation:

$$\frac{1}{2}\sigma_1^2\beta(\beta-1) + \frac{1}{2}\sigma_2^2\eta(\eta-1) + \rho\sigma_1\sigma_2\beta\eta + (\alpha_1-\theta_1)\beta + \alpha_2\eta - r = 0 \tag{8}$$

and A is a parameter that still needs to be determined.

In order to determine A and the switching thresholds x_1^* and x_2^* we assume that the standard value-matching and smooth-pasting conditions must hold, resulting in:

$$A{x_1^*}^{\beta}{x_2^*}^{\eta} + \frac{x_1^* R_{1,0}}{r+\theta_1-\alpha_1} - \frac{E_1}{r} = \frac{x_2^* R_{2,0}}{r+\theta_2-\alpha_2} - \frac{E_2}{r} - S, \tag{9}$$

and

$$A\beta{x_1^*}^{\beta-1}{x_2^*}^{\eta} + \frac{R_{1,0}}{r+\theta_1-\alpha_1} = 0, \tag{10}$$

$$A\eta{x_1^*}^{\beta}{x_2^*}^{\eta-1} = \frac{R_{2,0}}{r+\theta_2-\alpha_2}. \tag{11}$$

This implies that

$$-\frac{R_{1,0}x_1^*}{\beta(r+\theta_1-\alpha_1)} = \frac{R_{2,0}x_2^*}{\eta(r+\theta_2-\alpha_2)}, \tag{12}$$

and therefore

$$x_1^* = -\frac{\beta(r+\theta_1-\alpha_1)}{\eta(r+\theta_2-\alpha_2)}\frac{R_{2,0}}{R_{1,0}}x_2^*, \tag{13}$$

$$A = -\frac{R_{1,0}}{\beta(r+\theta_1-\alpha_1){x_1^*}^{\beta-1}{x_2^*}^{\eta}}. \tag{14}$$

Substituting equations (13) and (14) into the value-matching relationship (9) we derive the following useful relation:

$$x_1^*\frac{R_{1,0}}{r+\theta_1-\alpha_1}\left(\frac{\eta+\beta-1}{\beta}\right) + S - \frac{E_1-E_2}{r} = 0. \tag{15}$$

Combining this with condition (8) and (13) leads to the following equation set which must be solved to find the switching threshold:

$$\frac{1}{2}\sigma_1^2\beta(\beta-1)+\frac{1}{2}\sigma_2^2\eta(\eta-1)+\rho\sigma_1\sigma_2\beta\eta+(\alpha_1-\theta_1)\beta+\alpha_2\eta-r=0, \quad (16)$$

$$x_1^*=-\frac{\beta(r+\theta_1-\alpha_1)}{\eta(r+\theta_2-\alpha_2)}\frac{R_{2,0}}{R_{1,0}}x_2^*, \quad (17)$$

$$x_1^*\frac{R_{1,0}}{r+\theta_1-\alpha_1}\left(\frac{\eta+\beta-1}{\beta}\right)+S-\frac{E_1-E_2}{r}=0. \quad (18)$$

This equation set is very similar to those stated in Adkins & Paxson (2011a, eq. 2.4, 3.3 and 3.5) and Adkins & Paxson (2011b, eq. 4, 15 and 20). It was shown for a more general case (of switching baskets consisting of sums of geometric Brownian motion prices) by Gahungu & Smeers (2011), and particular two- and three-dimensional real options cases by Heydari et al. (2012) and Rohlfs & Madlener (2011), that the set should have an analytical solution. Note that there are four unknowns (x_1^*, x_2^*, β, and η) and three equations in this equation set. Although this seemingly makes the solution indetermined, that is not the case. The solution we are looking for is not a particular point, but rather pairs of critical oil and gas prices. When determining whether it is optimal to switch it only makes sense to consider the two prices jointly and therefore we can first assume a critical oil/gas price and find the corresponding critical gas/oil price. Analytical solutions for A, η and β could be expressed in terms of either x_1^* or x_2^*. However, one can select the alternative which ensures that the solution can be interpreted unambiguously for all prices. To determine whether x_1^* or x_2^* should be used to achieve this, consider the following expression:

$$C(x_1^*)\equiv 1+\left[\frac{r+\theta_1-\alpha_1}{x_1^*R_{1,0}}\right]\left[S-\frac{(E_1-E_2)}{r}\right]. \quad (19)$$

When $\left[S-\frac{(E_1-E_2)}{r}\right]>0$ it means that $C(x_1^*)>1$ and also that the switching threshold intercepts the gas price axis. The reason for this is that $S-\frac{(E_1-E_2)}{r}$ represents the total fixed cost component associated with making a switch; when this is positive there must be some interval of low gas prices for which it is never optimal to make a switch regardless of how low the oil price becomes. Therefore the threshold is in such a case defined for all positive threshold oil prices, but not for all gas prices. We assume this condition ($C(x_1^*)>1$) is satisfied in the following, but will also show a solution for the alternative case. Under this assumption it can then be shown that the solution to $\beta(x_1^*)$ from the above equation set must be

$$\beta(x_1^*) = \frac{f(x_1^*)}{2g(x_1^*)} - \sqrt{\left(\frac{f(x_1^*)}{2g(x_1^*)}\right)^2 + 2\frac{(r-\alpha_2)}{g(x_1^*)}}, \tag{20}$$

where $f(x_1^*) \equiv \sigma_1^2 - 2(\alpha_1 - \theta_1) - 2\rho\sigma_1\sigma_2 + C(x_1^*)(2\alpha_2 + \sigma_2^2)$ and $g(x_1^*) \equiv \sigma_1^2 + \sigma_2^2 C(x_1^*)^2 - 2\rho\sigma_1\sigma_2 C(x_1^*)$. Assuming that $r > \alpha_2$ (otherwise it is never optimal to exercise the option) it must always be true that

$$\frac{f(x_1^*)}{2g(x_1^*)} < \sqrt{\left(\frac{f(x_1^*)}{2g(x_1^*)}\right)^2 + 2\frac{(r-\alpha_2)}{g(x_1^*)}}, \tag{21}$$

when $g(x_1^*) > 0$. Recognizing that $g(x_1^*)$ is equivalent to a weighted variance expression, $\mathrm{Var}\,(x_2 C(x_1^*) - x_1 |\, x_1^*)$, and that variances for non-constant variables are stricly positive (i.e. $g(x_1^*) > 0$), then it must also be true that $\beta(x_1^*) < 0$ for all values of x_1^*. Rearranging (18) shows that

$$\eta(x_1^*) = 1 - \beta(x_1^*)C(x_1^*) \tag{22}$$

and that the parameter η no longer needs to explicitly be part of the analytical solution. It follows that $\eta(x_1^*) > 1$ (since $\beta(x_1^*) < 0$ and $C(x_1^*) > 1$) and that $\eta(x_1^*) + \beta(x_1^*) > 1$:

$$\eta(x_1^*) + \beta(x_1^*) = 1 - \beta(x_1^*)(C(x_1^*) - 1) > 1 \tag{23}$$

This result is the same as for the particular switching option cases studied by Adkins & Paxson (2011a,b). The analytical solutions for $x_2^*(x_1^*)$ and $A(x_1^*)$, expressed as functions of x_1^*, are found by substituting η with $1 - \beta(x_1^*)C(x_1^*)$ in (14) and (17) and rearranging the latter expression:

$$x_2^*(x_1^*) = -\frac{(1 - \beta(x_1^*)C(x_1^*))(r + \theta_2 - \alpha_2)R_{1,0}}{\beta(x_1^*)(r + \theta_1 - \alpha_1)R_{2,0}} x_1^* \tag{24}$$

$$A(x_1^*) = -\frac{R_{1,0}}{\beta(x_1^*)(r + \theta_1 - \alpha_1){x_1^*}^{\beta(x_1^*)-1} x_2^*(x_1^*)^{1-\beta(x_1^*)C(x_1^*)}}. \tag{25}$$

Note that if $\left[S - \frac{(E_1 - E_2)}{r}\right] < 0$ (i.e. $C(x_1^*) < 1$ and the threshold has an intercept on the oil price axis) the solution is defined for all threshold gas prices, but not all threshold oil prices. If this is the case, and to make sure that the solution is defined for all prices, similar expressions can be found for a given x_2^* (see Appendix A for this version of the solution). In the special case that

$\left[S - \frac{(E_1 - E_2)}{r}\right] = 0$ the problem collapses to a version of the solution derived by McDonald & Siegel (1986), where all the parameters are constant. When this term is zero the problem can be simplified by reducing it to a one-dimensional case. Nunes & Pimentel (2017) further extends this result by deriving an analytical solution to the optimal stopping problem when jumps are added to the price processes. A special version of the suggested solution is found when the present values of producing oil or gas (rather than the prices themselves) are assumed to follow geometric Brownian motions, and the production decline rates are set to zero (this ensure that the dynamics of the present value of gas is the same in the stopping and continuation region). In this scenario, the payout from the option is equal to the difference between the value of two assets following a geometric Brownian motion, minus a fixed switching cost. In the finance literature this is often refered to as a spread option. Using the same approach as outlined above for the switching option, an analytical solution can be expressed for the exercise threshold of a perpetual spread option (see Appendix B for this version of the solution).

2.2. Value of the switching option

In this section we derive the option value function for the continuation region, using the results derived in the previous section.

Gahungu & Smeers (2011) use Monte Carlo simulation techniques to find the option value (termed by them as the "performance" of their exercise rule) for specific starting points inside the continuation region, using a set of examples of options where a basket of GBMs is exchanged for another. Attempting to use the exercise threshold more directly, Adkins & Paxson (2011b) assume that the parameters of the solution (A, β and η according to our notation) are constant along one of the asset prices. However, no reasoning is given for why the parameters should be constant across one of the asset prices rather than the other and this should therefore be viewed as an approximation. In the solution derived by McDonald & Siegel (1986), with no cost of exercising the option, the parameters are constants and the unique set of parameters can be used directly to determine the option value anywhere inside of the continuation region. It is clear that the parameters A, β and η must change inside the continuation region in our model setup (and along the switching threshold) and in the following we argue how to determine these parameters for a given oil and gas price, (x_1, x_2).

Before presenting the main result of this section we state three useful lemmas:

Lemma 1 *For a given set $\{A, \beta, \eta\}$ the function*

$$k(x_1, x_2) = A x_1^{\beta} x_2^{\eta}$$

is excessive with respect to (X_1, X_2).

Proof. Using definition 2 of Alvarez (2003), in order to prove that f is excessive, we need to prove that

$$E[e^{-rs} k(X_{1,s}, X_{2,s}) | X_{1,0} = x_1, X_{2,0} = x_2] \leq k(x_1, x_2), \quad \forall x_1, x_2, s$$

as the other conditions hold trivially (namely, k is a nonnegative and measurable function, such that $\lim_{t\to 0} E[e^{-rt} k(X_1(t), X_2(t)) | X_1(0) = x_1, X_2(0) = x_2] = k(x_1, x_2)$, which follows from the fact that the GBM has continuous sample paths and the function k is also continuous). Using the fact that X_1 and X_2 are (correlated) GBMs, it follows that:

$$\begin{aligned}
E[e^{-rs} k(X_{1,s}, & X_{2,s}) | X_{1,0} = x_1, X_{2,0} = x_2] = \\
& e^{-rs} A x_1^{\beta} x_2^{\eta} e^{(\alpha_1 - \theta_1 - 0.5\sigma_1^2)\beta s} e^{(\alpha_2 - 0.5\sigma_2^2)\eta s} \\
& \times E\left[e^{\beta \sigma_1 W_s^{X_1} + \eta \sigma_2 W_s^{X_2}} \right] \\
= & A x_1^{\beta} x_2^{\eta} e^{\left(-r + (\alpha_1 - \theta_1 - 0.5\sigma_1^2)\beta + (\alpha_2 - 0.5\sigma_2^2)\eta + 0.5\left(\beta^2\sigma_1^2 + \eta^2\sigma_2^2 + 2\rho\beta\eta\sigma_1\sigma_2\right)\right)s} \\
= & A x_1^{\beta} x_2^{\eta} = k(x_1, x_2)
\end{aligned}$$

using the definition of β and η (the roots of (8)). $\square$

Based on lemma 1 and Theorem 10.1.6 of Øksendal (2013) the following must hold:

Lemma 2 *For a given set $\{A, \beta, \eta\}$, the function $k(x_1, x_2) = A x_1^{\beta} x_2^{\eta}$ is super-harmonic with respect to (X_1, X_2).*

Returning to the problem of derivation of the value function in the continuation region; for $(x_1, x_2) \in \Re_+^2$, and for any $\hat{x} > 0$, we define the following function:

$$k_{\hat{x}}(x_1, x_2) = A(\hat{x}) x_1^{\beta(\hat{x})} x_2^{\eta(\hat{x})}$$

where β, η and A are computed using (20), (22) and (25), respectively, and with $x_1^* = \hat{x}$. Then in view of lemma 2, for each $\hat{x}$, the function $k_{\hat{x}}(x_1, x_2)$ is super-harmonic. Moreover, using lemma 10.1.3. c) of Øksendal (2013), the following important result holds:

Lemma 3 *The function* $k_{\mathcal{D}}(x_1, x_2) = \inf_{\hat{x}} \{k_{\hat{x}}(x_1, x_2)\}$ *is super-harmonic with respect to* (X_1, X_2).

Defining $k_{\mathcal{D}}(x_1, x_2) = \inf_{\hat{x}} \{k_{\hat{x}}(x_1, x_2)\}$ as the option value in the continuation region and combining it with the intrinsic value of oil production we get the following:

Theorem 1 *The value function V - the solution of the optimization problem (2) - is given by:*

$$V(x_1, x_2) = \begin{cases} k_{\mathcal{D}}(x_1, x_2) + \frac{x_1 R_{1,0}}{r+\theta_1-\alpha_1} - \frac{E_1}{r} & x_2 \leq x_2^*(x_1) \\ \frac{x_2 R_{2,0}}{r+\theta_2-\alpha_2} - \frac{E_2}{r} - S & x_2 > x_2^*(x_1) \end{cases}, \tag{26}$$

where $k_{\mathcal{D}}(x_1, x_2) = \inf_{\hat{x}} \left\{ A(\hat{x}) x_1^{\beta(\hat{x})} x_2^{\eta(\hat{x})} \right\}$.

Proof. First, we need to prove that in the continuation region $\mathcal{D}$, the value function that we propose is a solution of the partial differential equation (6). As A, β and η depend on the state variables x_1 and x_2, and in order to check that the differential equation (6) holds with the proposed solution, we would need to compute derivatives of A, β and η with respect to x_1 and x_2. However, the following argument can be used to prove that these derivatives must be zero: when one is computing $\inf_{\hat{x}} \{k_{\hat{x}}(x_1, x_2)\}$, we may see $A(\hat{x}), \beta(\hat{x})$ and $\eta(\hat{x})$ as the *choice parameters*, whereas x_1 and x_2 are the *state parameters* (using the terminology of Milgrom & Segal (2002), who describe versions of the "envelope theorem" for an arbitrary choice set). Since the function $A(\hat{x}) x_1^{\beta(\hat{x})} x_2^{\eta(\hat{x})}$ is continous for each (x_1, x_2) we can use a result from a standard version of the envelope theorem (see e.g. Benveniste et al. (1979)) which imply that the total derivative of the value function with respect to any choice variable must be equal to zero, and that you can treat the choice parameters as though they are constant (and therefore with derivatives equal to zero). In particular this means that the proposed solution can be verified to be correct at the exercise threshold and the value function for the pair $(x_1^*, x_2^*(x_1^*))$ is given by $A(x_1^*) {x_1^*}^{\beta(x_1^*)} x_2^*(x_1^*)^{\eta(x_1^*)} + \frac{x_1^* R_{1,0}}{r+\theta_1-\alpha_1} - \frac{E_1}{r}$. Now we have to prove that also in the strictly continuation region (i.e. not including the threshold), the value function is given by $k_{\mathcal{D}}(x_1, x_2) + \frac{x_1 R_{1,0}}{r+\theta_1-\alpha_1} - \frac{E_1}{r}$. It follows from Theorem 10.1.9 from Øksendal (2013) that the value function for the option in the continuation region is given by the least superharmonic majorant of the payoff function. As it is given from lemma 3 that $k_{\mathcal{D}}(x_1, x_2)$ is super-harmonic (and trivially the infimum is the *least* of the potential solutions) it remains to show that it is a majorant of the payoff to the option. We make the following definition

$$h(x_1, x_2) = k_{\mathcal{D}}(x_1, x_2) - \left[\left(\frac{x_2 R_{2,0}}{r + \theta_2 - \alpha_2} - \frac{E_2}{r} \right) - \left(\frac{x_1 R_{1,0}}{r + \theta_1 - \alpha_1} - \frac{E_1}{r} \right) - S \right] \tag{27}$$

for any given x_1 and for $x_2 < x_2^*(x_1)$. Since $h(x_1^*, x_2^*) = 0$ (value-matching) and $\frac{dk_{\mathcal{D}}(x_1^*, x_2^*)}{dx_2} = -\frac{R_{2,0}}{r+\theta_2-\alpha_2}$ (smooth-pasting) at the threshold it is sufficient to show that $k_{\mathcal{D}}(x_1, x_2)$ is convex in x_2, for any given x_1 and for $x_2 < x_2^*(x_1)$. Clearly, $k(x_1, x_2) = Ax_1^{\beta} x_2^{\eta}$ is convex in x_2 for a fixed set $\{A, \beta, \eta\}$ (since $\eta > 1$) and if it is also convex in $(x_2, \{A(\hat{x}), \beta(\hat{x}), \eta(\hat{x})\})$, and C is a convex set, then the function $k_{\mathcal{D}}(x_1, x_2) = \inf_{\hat{x} \in C} \{k_{\hat{x}}(x_1, x_2)\}$ is convex (see e.g Boyd & Vandenberghe (2004), section 3.2.5, for a proof). Under this assumption it must follow that the proposed solution for the option value is a majorant of the payoff to the option since $h(x_1, x_2) > 0$ for any given x_1 and for $x_2 < x_2^*(x_1)$. Consequently, the value function for the continuation region is given by Theorem (26). For the stopping region, the value function follows trivially from the definition of the problem. □

Based on Theorem 1, and under the assumption of convexity in $\hat{x}$, the option value inside of the continuation region can be calculated by finding $\min_{\hat{x}} \left\{ A(\hat{x}) x_1^{\beta(\hat{x})} x_2^{\eta(\hat{x})} \right\}$. The minimum can be determined by substituting $A(\hat{x})$, $\beta(\hat{x})$ and $\eta(\hat{x})$ with expressions (25), (20) and (22) (with $x_1^* = \hat{x}$), respectively, and finding $\frac{d\left(A(\hat{x}) x_1^{\beta(\hat{x})} x_2^{\eta(\hat{x})}\right)}{d\hat{x}} = 0$. Using standard calculus and simplifying gives the following expression:

$$0 = \left(\ln \left(\frac{(\beta(\hat{x})C(\hat{x}) - 1)(r + \theta_2 - \alpha_2)R_{1,0}}{\beta(\hat{x})(r + \theta_1 - \alpha_1)R_{2,0}} \right) - \ln(x_2) \right)$$

$$\left(C(\hat{x})\hat{x}\frac{\frac{\partial\beta(\hat{x})}{\partial\hat{x}}}{\beta(\hat{x})} + 1 - C(\hat{x}) \right) \tag{28}$$

$$+ \ln(\hat{x}) \left((1 - C(\hat{x})) \left(1 - \hat{x}\frac{\frac{\partial\beta(\hat{x})}{\partial\hat{x}}}{\beta(\hat{x})} \right) \right) + \ln(x_1) \left(\hat{x}\frac{\frac{\partial\beta(\hat{x})}{\partial\hat{x}}}{\beta(\hat{x})} \right),$$

where

$$\frac{\partial\beta(\hat{x})}{\partial\hat{x}} = \frac{f'(\hat{x})g(\hat{x}) - f(\hat{x})g'(\hat{x})}{2g(\hat{x})^2} - \frac{1}{2}\left[\left(\frac{f(\hat{x})}{2g(\hat{x})} \right)^2 + 2\frac{(r - \alpha_2)}{g(\hat{x})} \right]^{-\frac{1}{2}} \times$$

$$\frac{1}{g(\hat{x})^2} \left(\frac{1}{2} \left(\frac{f(\hat{x})}{g(\hat{x})} \right) [f'(\hat{x})g(\hat{x}) - f(\hat{x})g'(\hat{x})] - 2(r - \alpha_2)g'(\hat{x}) \right) \tag{29}$$

and as before $f(\hat{x}) \equiv \sigma_1^2 - 2(\alpha_1 - \theta_1) - 2\rho\sigma_1\sigma_2 + C(\hat{x})(2\alpha_2 + \sigma_2^2)$ and $g(\hat{x}) \equiv \sigma_1^2 + \sigma_2^2 C(\hat{x})^2 - 2\rho\sigma_1\sigma_2 C(\hat{x})$.

The option value for any point (x_1, x_2) can now be found using the following procedure: solve the one-dimensional non-linear equation in (28) for $\hat{x}$ using any standard numerical algorithm. Use the solution for $\hat{x}$ in (20), (22), and (25) (with $x_1^* = \hat{x}$) to determine the parameter set $\{A,\ \beta,\ \eta\}$ for the point (x_1, x_2). Finally, find the option value by applying the determined parameter set in $k_{\hat{x}}(x_1, x_2) = A(\hat{x})x_1^{\beta(\hat{x})}x_2^{\eta(\hat{x})}$. A similar procedure to determine option value for the related perpetual spread option is outlined in the appendix.

2.3. Comparative statics for price process parameters

Comparative statics for the exercise threshold with regards to the price process parameters are presented in this section. Only the conclusions are included here while the proofs are relegated to Appendix C.

Proposition 1 *Assuming that the covariance $\sigma_{1,2}$ ($\sigma_{1,2} = \rho\sigma_1\sigma_2$) is held fixed, or the correlation ρ is fixed and $\rho \leq 0$, the threshold gas price x_2^* (for a given x_1^*) is increasing in the volatility σ_1 and σ_2 of the oil and gas price respectively. If the correlation ρ is assumed fixed and $\rho > 0$, the effect on x_2^* (for a given x_1^*) is non-monotonic. Below some critical values for σ_1 and σ_2, an increase in either of the volatilities will decrease x_2^* (for a given x_1^*). However, above these critical values for σ_1 and σ_2, an increase in either of the two volatilities will monotonically result in an increase in x_2^* (for a given x_1^*).*

Increasing either σ_1 or σ_2 generally increases the volatility of the payout to the option and therefore increases option value. In turn this makes the continuation region larger (i.e. increases x_2^*). However, for this effect to be entirely monotonic one has to assume that the covariance ($\sigma_{1,2} = \rho\sigma_1\sigma_2$) is fixed or that the correlation is fixed and $\rho < 0$. If this is not the case the volatility of the payout to the option can actually decrease when σ_1 or σ_2 increase (this is only true for very low values of σ_1 and σ_2). This non-monotonic behavior was also noted by Adkins & Paxson (2011a,b). It was shown by Adkins & Paxson (2011b), for an option with similar characteristics to the switching option considered here - albeit with somewhat different notation, that the "turning point" for σ_2 (here denoted $\hat{\sigma}_2$) can be determined by $\hat{\sigma}_2 = \rho\sigma_1\frac{\beta}{(1-\eta)}$. We add to this result by substituting β and η with our analytical expressions (20) and (22), and the expression then simplifies to $\hat{\sigma}_2 = \frac{\rho\sigma_1}{C(x_1^*)}$ (with $C(x_1^*)$ as defined in (19)). Increases in σ_2 when $\sigma_2 > \hat{\sigma}_2$ will lead to an increase in x_2^* (for a given x_1^*), but the opposite is true when $\sigma_2 < \hat{\sigma}_2$. Similar derivations can determine the critical value for σ_1 and this is shown in Appendix C.

Increasing ρ unambiguously decreases the volatility of the payout to the option and therefore the following result must hold true:

Proposition 2 *The threshold gas price x_2^* (for a given x_1^*) is decreasing in the correlation ρ in the change in oil and gas prices.*

Intepreting the effects of changing the drift rates for either oil or gas must be done with caution. This is due to the fact that it does not only change the dynamics of the prices, it effectively also changes the net present values of perpetual oil/gas production as well. This gives a non-monotonic behavior when changing the drift rate for the gas price, but not for the gas production decline rate (since this does not change the dynamics in the continuation region):

Proposition 3 *The threshold gas price x_2^* (for a given x_1^*) is increasing in the gas production decline rate θ_2.*

Increasing the gas price drift rate α_2 montonically increases x_2^* (for a given x_1^*) when α_2 is higher than a certain level, and monotonically decreases when α_2 is below this level. Where this change in behavior occurs can be determined exactly and is included in Appendix C.3. A non-monotonic behavior is not observed for the oil price drift rate or oil production decline rate due to the assumption that $r > \alpha_1 - \theta_1$ (see Appendix C.2 for details):

Proposition 4 *The threshold gas price x_2^* (for a given x_1^*) is increasing in the drift rate for the oil prices α_1 and decreasing with the oil production decline rate θ_1.*

3. Numerical Examples

The numerical examples are constructed around a base case for the switching option. Parameter values for the base case are chosen to reflect a "representative" case for a large size (initial reserves of 100–500 mill. barrels of oil) oilfield in the North Sea. This means that oil and gas prices from this region are used to estimate price process parameters. The example case is considered to be an offshore field in the decline phase. Therefore, the decline rate of production should be realistic for a representative offshore field in the North Sea. The International Energy Agency (IEA (2008)) estimates the average decline rate post-plateau to be 15.5% for OECD Europe (only North Sea fields included). Based on this study we assume a 15.5% decline rate for both oil and gas in the base case.

3.1. Price process parameters

The data used for estimating the price process parameters are daily observations of futures prices from the Intercontinental Exchange (ICE) for the time period August 12th 2010 to June 16th 2015. For the oil prices the Brent crude futures are used and for the gas prices we use UK Natural gas futures. As a proxy for the spot price for oil and gas the front month contract price is used. The gas prices, which are quoted in GBP, are converted to USD using USD/GBP forward rates quoted by Thomson Reuters. When annualizing the volatility estimates, 251 tradings days per year is assumed. Moreover, since the estimates for volatility are conducted using log returns on the data, we adjust for rollover effects. Table 1 summarizes the estimation results.

Table 1: Estimated price process parameters

	Estimated values(S.E.)
α_1	0.004(0.0013)
σ_1	0.338(0.0056)
α_2	0.005(0.0007)
σ_2	0.267(0.0054)
ρ	0.184(0.0278)

To estimate the risk adjusted drifts, a pair of futures were chosen for each commodity such that the difference in time to maturity between the two contracts is constant. We use the 12th position future relative to the observation day (approximately one year to maturity) and the 36th position (approximately 3 years to maturity) with a constant 2 year timespan between them in terms of time to maturity. Using no-arbitrage arguments, it is assumed that futures prices are equal to the risk adjusted expected spot prices. Since we assume geometric Brownian motion, the following must therefore hold true: $\alpha_i = \frac{\ln\left(\frac{F_{s,T}}{F_{s,t}}\right)}{T-t}$. Here α_i is the risk adjusted drift for commodity i, T and t are times of maturity with $T > t$, so that $F_{s,T}$ is a contract with a longer time to maturity than $F_{s,t}$, and finally $s < t$ is the time of observation. Using this relationship to calculate observed α_i for both oil and gas the risk adjusted drifts α_1 and α_2 are estimated as the mean of each observed set respectively.

3.2. Switching Option

For the numerical results a set of parameters, summarized in Table 2 (and with price process parameters as stated in Table 1), are assumed as a base case for the switching option. The current production rate for oil is measured in million standard cubic meters (Sm^3). However, to calculate the revenue stream while producing oil a conversion[7] is made to million barrels (bbl). A similar conversion is made for the gas, where the production rate is listed in billion standard cubic meters and converted to 100 mill. therms. Using prices of USD/bbl for the oil and 0.01 USD/therm for the oil and gas respectively, this means that the product of the production rates and the prices are in mill. USD.

The current switching threshold and the option value for the base case are depicted for a range of combinations of oil and gas prices in Figure 2. The thresholds should be interpreted such that for a given oil price, it is optimal to switch to gas production if the market price of gas is above the corresponding critical gas price. Alternatively, for a given gas price it is optimal to switch from oil to gas production if the price of oil drops below the critical price. Numerical

[7]Conversion factor for oil from mill. Sm^3 to mill. barrels is 6.29, and for gas from bill. Sm^3 to 100 mill. therms the conversion factor is 3.79121 (this assumes the following standard conversion rates for oil and gas: Sm^3 crude oil = 6.29 barrels. 1 Sm^3 natural gas = 40 MJ. 1 MJ = 947.80 Btu (British Thermal Unit). 1 Therm = 100 000 Btu.)

Table 2: Base case parameters switching option

	Values	Units	Description
$R_{1,0}$	2.0	mill. Sm^3	Yearly oil production
$R_{2,0}$	15	bill. Sm^3	Yearly gas production
θ_1	0.155		Oil production decline rate
θ_2	0.155		Gas production decline rate
E_1	500	mill. USD	Yearly oil production costs
E_2	500	mill. USD	Yearly gas production costs
r	0.03		Risk free rate
S	1000	mill. USD	Cost of switching

values for critical prices for a range of points on the threshold, as well as the associated parameters, are reported in Table 3. For a specific point inside of the continuation region, $x_1 = x_2 = 100$, the $\hat{x}$ found by solving (28) is 47.44. The option value at this point is 25428 mill. USD (with $\beta = -0.0984$, $\eta = 1.1283$, and $A = 221.61$). Note that the seemingly very high option values should be interpreted with caution for the following reason: there is no option to abandon the oil directly and therefore the option to switch also includes the option value from avoiding production of oil in perpetuity where this could potentially have a large negative present value.

As the production rate of oil declines deterministically the threshold also has to change. The thresholds one and five years ahead are shown in Figure 3. The changing threshold across time due to the deterministic decrease in production is similar to the effect of changing the initial production $R_{1,0}$. Changing the oil production (either the initial production or as an effect of the deterministic decline rate) produces a monotonic change in the entire threshold, decreasing the size of the continuation region as production decreases.

Table 3: Numerical values for a range of points on the exercise threshold

x_1^*	x_2^*	β	η	A
1.0	12.4	-0.0245	1.3775	88.80
10.0	32.6	-0.0809	1.1972	159.93
30.0	79.0	-0.0953	1.1411	206.74
50.0	125.6	-0.0987	1.1271	223.10
70.0	172.2	-0.1002	1.1208	231.64
90.0	218.8	-0.1011	1.1172	236.94
110.0	265.5	-0.1016	1.1149	240.57
130.0	312.1	-0.1020	1.1133	243.23

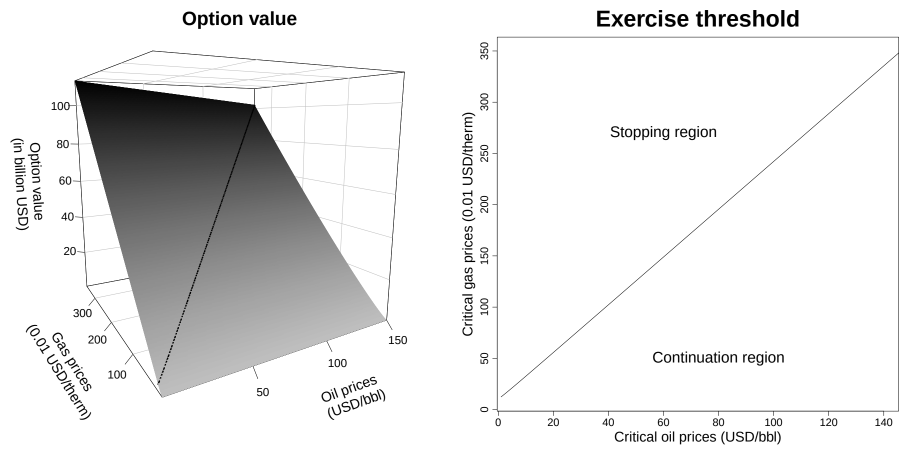

Figure 2: Exercise threshold and option value for the base case

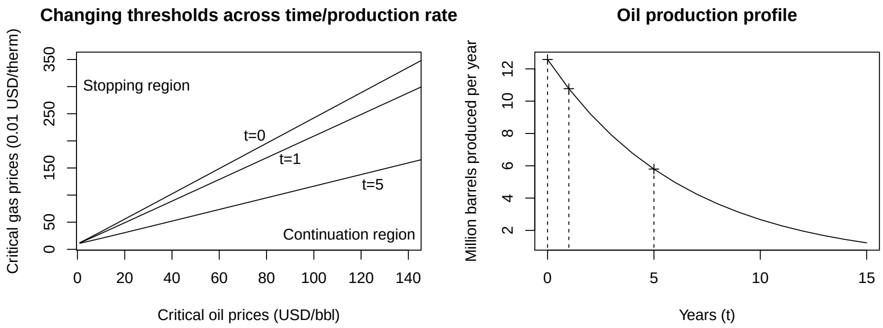

Figure 3: Switching threshold across time/production rate for the base case

In the following, the effects of changing key parameters of the model is demonstrated through a sensitivity analysis. Unless otherwise noted, only one parameter at the time is allowed to change and the other parameter values are assumed equal to those set in the base case. Consider now changes in the drift rates α_1 and α_2 of the oil and gas prices. For the range of values illustrated in Figure 4 the continuation region always increase when either of the drift rates increase. However, while this monotonic behavior is always true for α_1 this was shown not to be the case for α_2. As the drift rate of gas decreases the continuation region also always decreases given $\alpha_2 > 0$, but for some negative value of α_2 the behavior is reversed. This occurs due to two competing effects when α_2

is decreasing; the present value of gas production decreases (switch "later"), and expected future gas price decreases (switch "earlier"; standard option pricing result). The absolute changes in the threshold values are also much more sensitive to changes in α_2 than in α_1 for these parameter ranges.

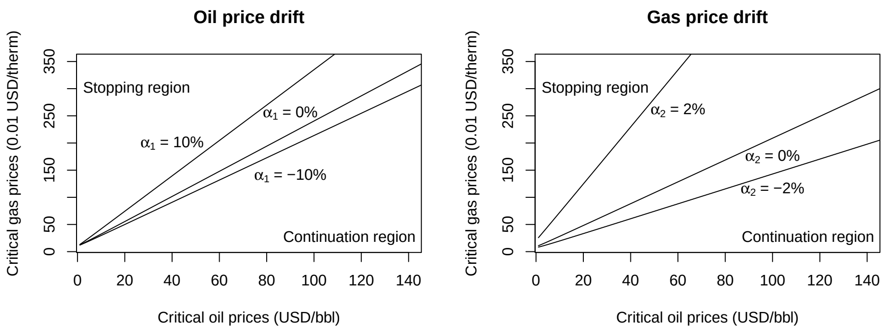

Figure 4: Effects on the switching threshold from changing the drift rate parameters

Increasing the volatilities of either the gas or oil price generally increases the volatility of the payout from the switching option. This is always the case when the correlation $\rho \leq 0$. However, when $\rho > 0$, increasing one of the volatilities can have a negative effect for very low volatility values. This effect is due to the fact that the payout of the switching option is a function of the difference between two stochastic elements (the present values of gas and oil) and the variance expression for such a payout has a negative term for the covariance/correlation. However, if the covariance (rather than the correlation coefficient) is assumed to be fixed the effect is a montonic increase when either of the volatilities increase. In general, when the volatility of the payout of the option increases the value of the option increases and consequently the continuation region for the option should increase. These effects are in line with the observations made by Adkins & Paxson (2011b) and McDonald & Siegel (1986). The effects on the switching threshold of changing the volatility levels are illustrated in Figure 5, both for the base case and for $\rho = 1$ (to demonstrate the non-monotonic behavior).

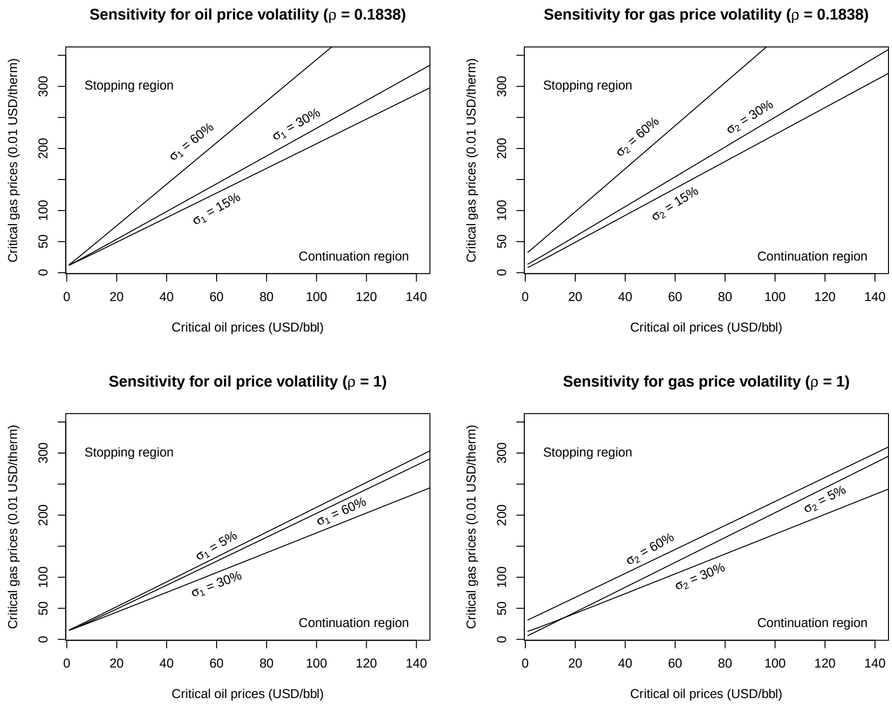

Figure 5: Effects on the switching threshold from changing the volatility parameters

The effects of changing some of the other key parameters to the model; θ_2, r, S, and ρ, are summarized and depicted in Figure 6. Increasing either the switching cost S or the gas production decline rate θ_2 both increase the size of the continuation region. The intuition is straightforward; both of these effects decrease the value received when switching, making a switch to gas less valuable in general. Increasing the correlation ρ or the risk free discount rate r decreases the size of the continuation region. The effect from correlation can be interpreted as a volatility effect; increasing the correlation decreases the volatility of the payout of the option and therefore the continuation region shrinks. Although the effect of increasing r is also a monotonically shrinking continuation region, the interpretation is not straightforward. This effect changes both the present value of gas production and oil production, as well as the discount rate for the option payout.

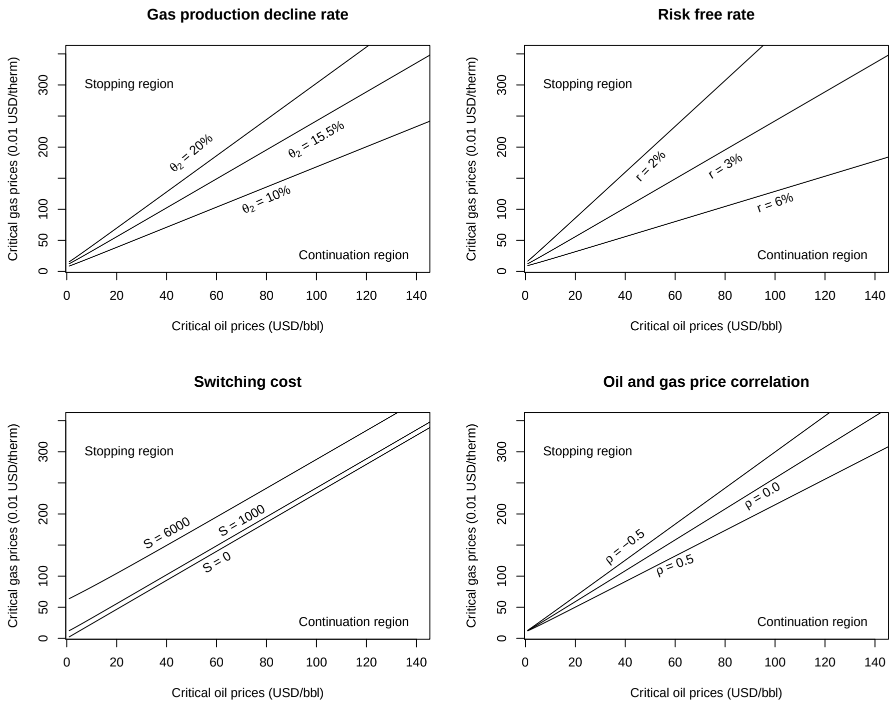

Figure 6: Effects on the switching threshold from changing key parameters

4. Conclusion

We propose a model to determine the optimal exercise boundary for making a switch from oil to gas production. Assuming that the oil and gas prices follow geometric Brownian motions with correlated increments, we derive an analytical solution for the switching strategy. This approach can be used to maximize the value-creation from aging oil fields with remaining gas reserves. Moreover, we propose an explicit solution for determining the option value which is new to the literature. The solutions for both the threshold and the option value may be applicable to other options applications with similar features to the switching option considered here. The proposed solution for the option value may also contribute to shed light on how one can determine the exercise thresholds and option value for more complicated compound switching options.

Although the negative effects on oil production from starting gas production depend on the characteristics of each oil field, we have assumed that the remaining oil is lost if the decision to switch is made. This is a conservative assumption which emphasizes the trade-off effect between the two resources in the model. Relaxing this assumption to model a more complicated relationship between oil and gas production would expand the applicability of our model to a broader range of oil fields. This seems like a valuable extension of our model and an interesting idea to pursue in further research. As long as adding such effects does not lead to a time-dependent optimal switching strategy, it may be possible to find solutions using the same approach as herein.

References

Adkins, R., & Paxson, D. (2011a). Reciprocal energy-switching options. *Journal of Energy Markets*, *4*, 91–120.

Adkins, R., & Paxson, D. (2011b). Renewing assets with uncertain revenues and operating costs. *Journal of Financial and Quantitative Analysis*, *46*, 785–813.

Adkins, R., & Paxson, D. (2017). Replacement decisions with multiple stochastic values and depreciation. *European Journal of Operational Research*, *257*, 174–184.

Alvarez, L. H. (2003). On the properties of r-excessive mappings for a class of diffusions. *Annals of Applied Probability*, *13*, 1517–1533.

Benveniste, L., Scheinkman, J. et al. (1979). On the differentiability of the value function in dynamic models of economics. *Econometrica*, *47*, 727–32.

Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

Brekke, K. A., & Øksendal, B. (1991). The high contact principle as a sufficiency condition for optimal stopping. In D. Lund, & B. Øksendal (Eds.), *Stochastic Models and Option Values: Applications to Resources, Environment, and Investment Problems* (pp. 187–208). North-Holland.

Brennan, M., & Schwartz, E. (1985). Evaluating natural resource investments. *Journal of Business*, *58*, 135–157.

Gahungu, J., & Smeers, Y. (2011). Optimal time to invest when the price processes are geometric brownian motions. A tentative based on smooth fit. Université catholique de Louvain, Center for Operations Research and Econometrics (CORE), Discussion Paper (2011/34).

Hahn, W. J., & Dyer, J. S. (2008). Discrete time modeling of mean-reverting stochastic processes for real option valuation. *European Journal of Operational Research*, *184*, 534–548.

Heydari, S., Ovenden, N., & Siddiqui, A. (2012). Real options analysis of investment in carbon capture and sequestration technology. *Computational Management Science*, *9*, 109–138.

IEA (2008). *World Energy Outlook*. International Energy Agency.

McDonald, R., & Siegel, D. (1986). The value of waiting to invest. *The Quarterly Journal of Economics*, *101*, 707–727.

Milgrom, P., & Segal, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, *70*, 583–601.

Ning, S., Jhaveri, B., Fueg, E., Stechauner, G., Jemison, J., & Hoang, T. (2016). Optimizing the utilization of miscible injectant at the greater Prudhoe Bay fields. In *SPE Western Regional Meeting*. Society of Petroleum Engineers.

Nunes, C., & Pimentel, R. (2017). Analytical solution for an investment problem under uncertainties with shocks. *European Journal of Operational Research*, *259*, 1054–1063.

Øksendal, B. (2013). *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media.

Paddock, J. L., Siegel, D. R., & Smith, J. L. (1988). Option valuation of claims on real assets: The case of offshore petroleum leases. *The Quarterly Journal of Economics*, *103*, 479–508.

Pindyck, R. S. (2001). The dynamics of commodity spot and futures markets: a primer. *The Energy Journal*, *22*, 1–29.

Postali, F. A., & Picchetti, P. (2006). Geometric brownian motion and structural breaks in oil prices: a quantitative analysis. *Energy Economics*, *28*, 506–522.

Rohlfs, W., & Madlener, R. (2011). Valuation of ccs-ready coal-fired power plants: a multi-dimensional real options approach. *Energy Systems*, *2*, 243–261.

Schwartz, E. (1998). Valuing long-term commodity assets. *Journal of Energy Finance & Development*, *3*, 85–99.

Schwartz, E. S. (1997). The stochastic behavior of commodity prices: Implications for valuation and hedging. *The journal of finance*, *52*, 923–973.

Szabo, J., & Meyers, K. (1993). Prudhoe Bay: development history and future potential. In *SPE Western Regional Meeting*. Society of Petroleum Engineers.

Tsekrekos, A. E., & Yannacopoulos, A. N. (2016). Optimal switching decisions under stochastic volatility with fast mean reversion. *European Journal of Operational Research*, *251*, 148–157.

Wallace, S. W., Helgesen, C., & Nystad, A. N. (1987). Generating production profiles for an oil field. *Mathematical Modelling*, *8*, 681–686.

Appendix A. Analytical solution expressed in gas prices

Defining

$$C_2(x_2^*) \equiv 1 + \left[\frac{r + \theta_2 - \alpha_2}{x_2^* R_{2,0}}\right]\left[\frac{(E_1 - E_2)}{r} - S\right] \tag{A.1}$$

then $\beta(x_2^*)$ is given analytically by

$$\beta(x_2^*) = \frac{m(x_2^*)}{2n(x_2^*)} - \sqrt{\left(\frac{m(x_2^*)}{2n(x_2^*)}\right)^2 + \frac{(2(C_2(x_2^*)^2 r - \alpha_2 C_2(x_2^*)) + C_2(x_2^*)\sigma_2^2 - \sigma_2^2)}{n(x_2^*)}}, \tag{A.2}$$

where $m(x_2^*) \equiv C_2(x_2^*)^2\sigma_1^2 - 2(\alpha_1 - \theta_1) - 2C_2(x_2^*)\rho\sigma_1\sigma_2 + 2\alpha_2 + \sigma_2^2$ and $n(x_2^*) \equiv C_2(x_2^*)^2\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$. Assuming that $C_2(x_2^*) > 1$ (i.e. $C(x_1^*) < 1$) and $r > \alpha_2$ then it follows that $\beta(x_2^*) < 0$ as long as $n(x_2^*) > 0$. Recognizing that $n(x_2^*) \geq \mathrm{Var}\,(x_2 - x_1 |\, x_2^*)$, and that variances for non-constant variables are stricly positive (i.e. $n(x_2^*) > 0$), then it must also be true that $\beta(x_2^*) < 0$ for all values of x_2^*. Using the same assumptions it can be shown that $\eta(x_2^*) = \frac{1-\beta(x_2^*)}{C_2(x_2^*)}$. The analytical solutions for $x_1^*(x_2^*)$ and $A(x_2^*)$, expressed as functions of x_2^*, are:

$$x_1^*(x_2^*) = -\frac{\beta(x_2^*)(r + \theta_1 - \alpha_1)R_{2,0}}{((1 - \beta(x_2^*))/C(x_2^*))(r + \theta_2 - \alpha_2)R_{1,0}} x_2^*, \tag{A.3}$$

$$A(x_2^*) = -\frac{R_{1,0}}{\beta(x_2^*)(r + \theta_1 - \alpha_1) x_1^*(x_2^*)^{\beta(x_2^*)-1} x_2^{*(1-\beta(x_2^*))/C_2(x_2^*)}}. \tag{A.4}$$

Appendix B. Analytical solution for the spread option

Assume that two asset prices X_1 and X_2 follow correlated geometric Brownian motions under the risk neutral measure:

$$dX_{I,t} = \alpha_I X_{I,t} dt + \sigma_I X_{I,t} dZ_{I,t}, \tag{B.1}$$

using similar notation as in section 2. Consider an option with no maturity date that would give a payout of $(X_2 - X_1 - S)$ if exercised, where S is a constant strike price. Assume further that the value of such an option can be expressed in the functional form

$$v_s(x_1, x_2) = B x_1^{\gamma} x_2^{\nu}, \tag{B.2}$$

where the subscript s is used to signify that this is the value of a spread option. Following the same type of argument as in section 2, the exercise threshold for the option can be determined by the characteristic root equation

$$\frac{1}{2}\sigma_1^2\gamma(\gamma-1) + \frac{1}{2}\sigma_2^2\nu(\nu-1) + \rho\sigma_1\sigma_2\gamma\nu + \alpha_1\gamma + \alpha_2\nu - r = 0. \tag{B.3}$$

and the value-matching and smooth-pasting conditions:

$$B {x_1^*}^{\gamma} {x_2^*}^{\nu} = x_2^* - x_1^* - S \tag{B.4}$$

$$B\gamma {x_1^*}^{\gamma-1} {x_2^*}^{\nu} = -1 \tag{B.5}$$

$$B\nu {x_1^*}^{\gamma} {x_2^*}^{\nu-1} = 1, \tag{B.6}$$

where x_1^* and x_2^* denote the threshold switching values for the processes X_1 and X_2 respectively. Solving this equation set, the analytical solution to γ, given a value for x_1^*, is:

$$\gamma(x_1^*) = \frac{p(x_1^*)}{2q(x_1^*)} - \sqrt{\left(\frac{p(x_1^*)}{2q(x_1^*)}\right)^2 + 2\frac{(r-\alpha_2)}{q(x_1^*)}}, \tag{B.7}$$

where $p(x_1^*) \equiv \sigma_1^2 - 2\alpha_1 - 2\rho\sigma_1\sigma_2 + (1 + S/x_1^*)(2\alpha_2 + \sigma_2^2)$ and $q(x_1^*) \equiv \sigma_1^2 + \sigma_2^2(1+S/x_1^*)^2 - 2\rho\sigma_1\sigma_2(1+S/x_1^*)$. Given the assumption that $r > \alpha_2$ (otherwise the option is never exercised) it must always be true that $\gamma(x_1^*) < 0$ for all values of x_1^*. It can also be shown that $\nu(x_1^*) = 1 - \gamma(x_1^*)(1 + S/x_1^*) > 1$ and $\gamma(x_1^*) + \nu(x_1^*) > 1$. The analytical solutions for $x_2^*(x_1^*)$ and $B(x_1^*)$, expressed as functions of x_1^*, are:

$$x_2^*(x_1^*) = -\frac{1-\gamma(x_1^*)(1+S/x_1^*)}{\gamma(x_1^*)}x_1^* \tag{B.8}$$

$$B(x_1^*) = -\frac{1}{\gamma(x_1^*){x_1^*}^{\gamma(x_1^*)-1}x_2^*(x_1^*)^{1-\gamma(x_1^*)(1+S/x_1^*)}}. \tag{B.9}$$

The value of the spread option - assuming it is inside of the continuation region - is given by:

$$v_s(x_1,x_2) = B(\hat{x})x_1^{\gamma(\hat{x})}x_2^{1-\gamma(\hat{x})\left(1+\frac{S}{\hat{x}}\right)}, \tag{B.10}$$

where

$$\gamma(\hat{x}) = \frac{p(\hat{x})}{2q(\hat{x})} - \sqrt{\left(\frac{p(\hat{x})}{2q(\hat{x})}\right)^2 + 2\frac{(r-\alpha_2)}{q(\hat{x})}}, \tag{B.11}$$

$$B(\hat{x}) = -\gamma(\hat{x})^{-1}\hat{x}^{\gamma(\hat{x})\frac{S}{\hat{x}}}\left[\frac{\gamma(\hat{x})\left(1+\frac{S}{\hat{x}}\right)-1}{\gamma(\hat{x})}\right]^{\gamma(\hat{x})\left(1+\frac{S}{\hat{x}}\right)-1}, \tag{B.12}$$

and $p(\hat{x}) \equiv \sigma_1^2 - 2\alpha_1 - 2\rho\sigma_1\sigma_2 + (1+S/\hat{x})(2\alpha_2+\sigma_2^2)$ and $q(x_1^*) \equiv \sigma_1^2 + \sigma_2^2(1+S/\hat{x})^2 - 2\rho\sigma_1\sigma_2(1+S/\hat{x})$. The parameter $\hat{x}$ is determined implicitly by

$$\begin{aligned} 0 =& \Bigg[\hat{x}\ln(x_1) - \ln(x_2)(\hat{x}+S) + S\ln\left(1+\frac{S}{\hat{x}}-\frac{1}{\gamma(\hat{x})}\right) \\ &+ \hat{x}\ln\left(1+\frac{S}{\hat{x}}-\frac{1}{\gamma(\hat{x})}\right) + S\ln(\hat{x})\Bigg] \\ &+ \frac{S\gamma(\hat{x})}{\hat{x}\frac{\partial\gamma(\hat{x})}{\partial\hat{x}}}\left[\ln(x_2) - \ln\left(1+\frac{S}{\hat{x}}-\frac{1}{\gamma(\hat{x})}\right) - \ln(\hat{x})\right], \end{aligned} \tag{B.13}$$

or, alternatively;

$$\begin{aligned} 0 =& \left(\ln\left(1+\frac{S}{\hat{x}}-\frac{1}{\gamma(\hat{x})}\right) - \ln(x_2)\right)\left(S+\hat{x}-\frac{S}{\hat{x}}\frac{\gamma(\hat{x})}{\frac{\partial\gamma(\hat{x})}{\partial\hat{x}}}\right) \\ &+ \ln(\hat{x})\left(S-\frac{S}{\hat{x}}\frac{\gamma(\hat{x})}{\frac{\partial\gamma(\hat{x})}{\partial\hat{x}}}\right) + \ln(x_1)(\hat{x}). \end{aligned} \tag{B.14}$$

To find the option value equation B.13 (or B.14) is solved numerically and the result is plugged back into equation B.12, B.11 and B.10.

Appendix C. Comparative statics

Appendix C.1. Comparative statics for σ_1, σ_2 and $\rho_{1,2}$.

Consider the analytical expression for x_2^*

$$x_2^*(x_1^*) = -\frac{(1-\beta(x_1^*)C(x_1^*))(r+\theta_2-\alpha_2)R_{1,0}}{\beta(x_1^*)(r+\theta_1-\alpha_1)R_{2,0}}x_1^*, \tag{C.1}$$

which can be rewritten as

$$x_2^*(x_1^*) = \left(\frac{-1}{\beta(x_1^*)} + C(x_1^*)\right)\frac{(r+\theta_2-\alpha_2)R_{1,0}}{(r+\theta_1-\alpha_1)R_{2,0}}x_1^*. \tag{C.2}$$

Using standard calculus it is straightforward to show that

$$\frac{\partial x_2^*}{\partial \sigma_1}_{|x_1^*} = \left(\frac{1}{(\beta(x_1^*))^2}\frac{\partial \beta(x_1^*)}{\partial \sigma_1}\right)\frac{(r+\theta_2-\alpha_2)R_{1,0}}{(r+\theta_1-\alpha_1)R_{2,0}}x_1^*. \tag{C.3}$$

Consequently, the sign of $\frac{\partial x_2^*}{\partial \sigma_1}_{|x_1^*}$ is the same as the sign of $\frac{\partial \beta(x_1^*)}{\partial \sigma_1}$. One can use the rule of total derivatives in the following manner to determine the sign:

$$\frac{d\beta}{d\sigma_1} = -\frac{\frac{\partial Q}{\partial \sigma_1}}{\frac{\partial Q(\beta)}{\partial \beta}}, \tag{C.4}$$

where Q is the characteristic root equation as defined in equation (8). Replacing η with $(1-\beta C(x_1^*))$ in Q and defining $f(x_1^*) \equiv (\sigma_1^2 - 2(\alpha_1-\theta_1) - 2\rho\sigma_1\sigma_2 + C(x_1^*)(2\alpha_2+\sigma_2^2))$ and $g(x_1^*) \equiv (\sigma_1^2 + \sigma_2^2 C(x_1^*)^2 - 2\rho\sigma_1\sigma_2 C(x_1^*))$ we get

$$\frac{\partial Q(\beta)}{\partial \beta} = \beta g(x_1^*) + \frac{f(x_1^*)}{2}. \tag{C.5}$$

With β defined as follows

$$\beta(x_1^*) = \frac{f(x_1^*) - \sqrt{f(x_1^*)^2 + 8g(x_1^*)(r-\alpha_2)}}{2g(x_1^*)} \tag{C.6}$$

we can substitute this into the expression for $\frac{\partial Q(\beta)}{\partial \beta}$ and we get

$$\frac{\partial Q(\beta)}{\partial \beta} = -\sqrt{f(x_1^*)^2 + 8g(x_1^*)(r-\alpha_2)} \tag{C.7}$$

which must be negative. Consequently, the sign of $\frac{d\beta}{d\sigma_1}$ must be the same as for $\frac{\partial Q}{\partial \sigma_1}$. The sign of this derivative depends on the assumptions regarding the covariance or the correlation when the volatility is allowed to change. If we keep the covariance (which is given by $\rho\sigma_1\sigma_2$) fixed, then $\frac{\partial Q}{\partial \sigma_1} = \sigma_1\beta(\beta-1) + \frac{\partial}{\partial \sigma_1}(\rho\sigma_1\sigma_2\beta\eta)$ is positive (since $\beta < 0$ and the second term is 0). Therefore, assuming a fixed covariance, the sign of $\frac{d\beta}{d\sigma_1}$ and $\frac{\partial x_2^*}{\partial \sigma_1}_{|x_1^*}$ must be positive, and thus the effect on the gas price x_2^* (for a fixed oil price) of increasing the variance of the oil price is monotonic. The effect of σ_1 on x_2^* in the case where the correlation is assumed fixed is somewhat different. If correlation is assumed fixed and $\rho \leq 0$ then the results are the same as when the covariance is fixed. However, if correlation is assumed fixed and $\rho > 0$ the effect is non-monotonic and this behavior was also noted by Adkins & Paxson (2011b). Simple calculations show that the threshold for the gas price x_2^*, for a fixed oil price x_1^*, decreases with σ_1 for values below a certain critical value, hereby denoted by $\hat{\sigma}_1$, and that it increases above this value, where the critical value is given by

$$\hat{\sigma}_1 = \rho\sigma_2 \frac{1-\beta(x_1^*)}{\eta(x_1^*)} = \frac{\rho\sigma_2}{C_2(x_2^*)} \tag{C.8}$$

and with $C_2(x_2^*)$ defined as in Appendix A[8]. Following a similar argument, and still assuming that the correlation is fixed, one can show that the results are qualitatively the same for σ_2, where now the critical value is given by

$$\hat{\sigma}_2 = \rho\sigma_1 \frac{\beta(x_1^*)}{1-\eta(x_1^*)} = \frac{\rho\sigma_1}{C_1(x_1^*)} \tag{C.9}$$

with $C_1(x_1^*)$ defined in (19).

Assuming that the covariance is fixed rather than the correlation, a similar argument as was made for σ_1 can be made for both σ_2 and $\rho_{1,2}$. Given that $\frac{\partial Q(\beta)}{\partial \beta}$ is negative then the sign of $\frac{\partial x_2^*}{\partial \sigma_2}_{|x_1^*}$ is the same as for $\frac{\partial Q}{\partial \sigma_2}$. Since $\frac{\partial Q}{\partial \sigma_2} = \eta(\eta-1) > 0$ (assuming again that the covariance is held constant) then the sign of $\frac{\partial x_2^*}{\partial \sigma_2}_{|x_1^*}$ is positive. Similarly, since $\frac{\partial Q}{\partial \rho_{1,2}} = \sigma_1\sigma_2\beta(1-\beta C(x_1^*)) > 0$, the sign of $\frac{\partial x_2^*}{\partial \rho_{1,2}}_{|x_1^*}$ is negative. The signs of $\frac{\partial x_2^*}{\partial \sigma_1}_{|x_1^*}$, $\frac{\partial x_2^*}{\partial \sigma_2}_{|x_1^*}$ and $\frac{\partial x_2^*}{\partial \rho_{1,2}}_{|x_1^*}$ for a fixed covariance coincide with the well known result from the options literature that the continuation region becomes larger (because the option value increases) when the volatility of the pay-out to the option increases.

Appendix C.2. Comparative statics for α_1 and θ_1

Consider the analytical expression for x_2^*

$$x_2^*(x_1^*) = \left(\frac{-1}{\beta(x_1^*)} + C(x_1^*)\right) \frac{(r+\theta_2-\alpha_2)R_{1,0}}{(r+\theta_1-\alpha_1)R_{2,0}} x_1^*. \tag{C.10}$$

[8]For ease of notation, we omit the dependency of both β and η on σ_1, but some straightforward manipulations leads to the fact that the quotient $\frac{1-\beta(x_1^*)}{\eta(x_1^*)}$ can be written as a function of x_2^* which only indirectly depends on σ_1, as we present in the last term of equation (C.8).

Using standard calculus it is straightforward to show that

$$\frac{\partial x_2^*}{\partial \alpha_1}_{|x_1^*} = \frac{(r+\theta_2-\alpha_2)R_{1,0}}{(r+\theta_1-\alpha_1)R_{2,0}} x_1^* \Bigg[\frac{\partial C(x_1^*)}{\partial \alpha_1} + \frac{C(x_1^*)}{(r+\theta_1-\alpha_1)} + \frac{\frac{-\partial \beta(x_1^*)}{\partial \alpha_1}}{(\beta(x_1^*))^2} + \frac{-\frac{1}{(\beta(x_1^*))}}{(r+\theta_1-\alpha_1)}\Bigg]. \tag{C.11}$$

By assumption $(r+\theta_2-\alpha_2)>0$ and $(r+\theta_1-\alpha_1)>0$ so the sign of $\frac{\partial x_2^*}{\partial \alpha_1}_{|x_1^*}$ is the same as for the entire bracket term. Using similar arguments as in the previous section one can show that

$$\frac{\partial \beta(x_1^*)}{\partial \alpha_1} = -\frac{\frac{\partial Q}{\partial \alpha_1}}{\frac{\partial Q(\beta)}{\partial \beta}} < 0 \tag{C.12}$$

since $\frac{\partial Q(\beta)}{\partial \beta} < 0$ and $\frac{\partial Q}{\partial \alpha_1} = \beta < 0$. Therefore the two last terms must be positive. It remains to determine the sign of

$$\frac{\partial C(x_1^*)}{\partial \alpha_1} + \frac{C(x_1^*)}{(r+\theta_1-\alpha_1)}$$

which can be reformulated as

$$-\frac{1}{x_1^* R_{1,0}}\left[S - \frac{E_1-E_2}{r}\right] + \frac{1+\frac{(r+\theta_1-\alpha_1)}{x_1^* R_{1,0}}\left[S-\frac{E_1-E_2}{r}\right]}{r+\theta_1-\alpha_1} = \frac{1}{r+\theta_1-\alpha_1}.$$

This has to be positive (since $(r+\theta_1-\alpha_1)>0$) and consequently $\frac{\partial x_2^*}{\partial \alpha_1}_{|x_1^*}$ must be positive. The parameters α_1 and θ_1 always appear together in expressions as $(\theta_1-\alpha_1)$ and therefore $\frac{\partial x_2^*}{\partial \theta_1}_{|x_1^*}$ must be negative.

Appendix C.3. Comparative statics for α_2 and θ_2

Consider the analytical expression for x_2^*

$$x_2^*(x_1^*) = \left(\frac{-1}{\beta(x_1^*)} + C(x_1^*)\right)\frac{(r+\theta_2-\alpha_2)R_{1,0}}{(r+\theta_1-\alpha_1)R_{2,0}} x_1^*. \tag{C.13}$$

Using standard calculus it is straightforward to show that

$$\frac{\partial x_2^*}{\partial \alpha_2}_{|x_1^*} = \frac{R_{1,0}}{(r+\theta_1-\alpha_1)R_{2,0}} x_1^* \left[\frac{\frac{\partial \beta(x_1^*)}{\partial \alpha_2}}{(\beta(x_1^*))^2}(r+\theta_2-\alpha_2) + \frac{1}{(\beta(x_1^*))} - C(x_1^*)\right]. \tag{C.14}$$

By assumption $(r+\theta_2-\alpha_2)>0$ and $(r+\theta_1-\alpha_1)>0$ so the sign of $\frac{\partial x_2^*}{\partial \alpha_2}_{|x_1^*}$ is the same as for the entire bracket term. It can be shown that $\frac{\partial \beta(x_1^*)}{\partial \alpha_2} > 0$ and the sign of the bracket term is therefore ambiguous. If

$$\frac{\partial \beta(x_1^*)}{\partial \alpha_2} > \frac{-\beta(x_1^*)\eta(x_1^*)}{(r+\theta_2-\alpha_2)} \tag{C.15}$$

it holds that x_2^* is increasing in α_2 (for a given x_1^*), otherwise it decreases. The turning point can be determined by finding when this is an equality, which is equivalent to

$$\beta(x_1^*) = -\frac{(r+\theta_2-\alpha_2)}{\sqrt{\left(\frac{f(x_1^*)}{2}\right)^2 + 2(r-\alpha_2)g(x_1^*)}} \tag{C.16}$$

After some cumbersome algebra it can be shown that the turning point is

$$\begin{aligned}\alpha_2 = & \frac{1}{2g(x_1^*) - 4\theta_2 C(x_1^*)^2}\Bigg[4\theta_2 h(x_1^*)C(x_1^*) + 4g(x_1^*)(r-\theta_2) \\ & \pm \sqrt{\begin{aligned}&(4\theta_2 h(x_1^*)C(x_1^*) + 4g(x_1^*)(r-\theta_2))^2 \\ &-4\left(2g(x_1^*) - 4\theta_2 C(x_1^*)^2\right)\left(2g(x_1^*)(r-\theta_2)^2 - \theta_2 h(x_1^*)^2\right)\end{aligned}}\Bigg]\end{aligned} \tag{C.17}$$

with $h(x_1^*) \equiv \sigma_1^2 - 2(\alpha_1 - \theta_1) - 2\rho\sigma_1\sigma_2 + C(x_1^*)\sigma_2^2$, and with $g(x_1^*)$ and $f(x_1^*)$ defined as before.

Deriving $\frac{\partial x_2^*}{\partial \theta_2}_{|x_1^*}$ in a similar way we get

$$\frac{\partial x_2^*}{\partial \alpha_2}_{|x_1^*} = \frac{R_{1,0}}{(r+\theta_1-\alpha_1)R_{2,0}} x_1^* \left[\frac{\frac{\partial \beta(x_1^*)}{\partial \theta_2}}{(\beta(x_1^*))^2}(r+\theta_2-\alpha_2) + \frac{-1}{(\beta(x_1^*))} + C(x_1^*)\right]. \tag{C.18}$$

Noting that $\frac{\partial \beta(x_1^*)}{\partial \theta_2} = 0$ it is clear that the sign of $\frac{\partial x_2^*}{\partial \theta_2}_{|x_1^*}$ must be positive.