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Evaluating Different Simulation-Based Estimates for Value and Risk in Interest Rate Portfolios

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Problem Description

This master thesis will focus on estimation of value and risk in interest rate portfolios. The estimates will be found using statistics and simulation by considering different modell assumptions and data from different periods.

Assignment given: 15. March 2010
Supervisor: Jacob Laading, MATH

Preface

This thesis was carried out at the Department of Mathematical Sciences and Technology at the Norwegian University of Science and Technology (NTNU) during the period March 2010 to August 2010.

I would like to thank my supervisor Jacob Laading for providing guidance and giving constructive feedback. I would also like to thank Anders Schmelck and Yngve Borgan for their contributions.

Trondheim, August 2010
Kaja Kierulf

Abstract

This thesis evaluates risk measures for interest rate portfolios. First a model for interest rates is established: the LIBOR market model. The model is applied to Norwegian and international interest rate data and used to calculate the value of the portfolio by using Monte Carlo simulation. Estimation of volatility and correlation is discussed as well as the two risk measures value at risk and expected tail loss. The data used is analysed before the results of the backtesting evaluating the two risk measures are presented.

Contents

Chapter 1

Introduction

Finance is a field becoming increasingly important. This was clearly illustrated during the recent financial crisis spreading throughout the world. The fundamental role that finance plays in our society makes it particularly important to attain deeper insight in this field. Mathematics is an excellent tool that helps quantify and interpret financial data. One of many uses of mathematical finance is exploring the vast amount of data through estimation and modelling. However, it should be noted that treating the data as if it is physics is debatable. In the end, the data is created by humans and their choices and they are all but rational. One should therefore keep this in mind and all information attained should be used accordingly.

The models are implemented in C++ which is an efficient computer language when performing large calculations. The rest of this chapter will be used to give a brief discussion on some of the key concepts in the financial world.

1.1 Assets

An asset is a resource with economic value and the term can be used for any financial product whose value is quoted or can in principle be measured. This thesis will mainly discuss the two asset classes fixed-income (bonds) and cash equivalents (money market instruments) which combined are modelled as interest rates.

This thesis does not discuss speculation of asset prices because it is assumed that future asset prices are not known and cannot be predicted. According to the efficient market hypothesis which is stated in [8], asset prices must move randomly:

- The past history is fully reflected in the present time, which does not hold any further information.
- Markets respond immediately to any new information about an asset.

Thus the modelling of asset prices is really about modelling the arrival of new information which affects the price. This does not mean that the historic values of the asset price contains no information. On the contrary, the time series based on the historic data can be used to estimate the volatility and the mean as well as the most likely distribution for the future asset price.

1.2 Derivatives

A derivative is a security whose price is dependent upon or derived from one or more underlying assets. Hence the value of the derivative will change as the value of the underlying assets change. A derivative can be used as an underlying asset for another derivative and the derivative itself is a contract between two or more parties. The derivative market is vast; the notional principal amount exceeds trillions of US$ and is growing fast. The most common types of derivatives are: forward contracts, futures contracts, options and swaps. A forward contract is an agreement between two parties where one agrees to buy an asset for a given price at a given date in the future. A futures contract is much like a forward contract, but a futures contract is standardised and traded on an exchange. An option gives the holder the right to buy or sell an asset for a given price at a given date in the future. Notice that an option gives the holder a right, but not the obligation to buy or sell the asset. A swap is a contract between two parties agreeing to exchange or swap future cashflows. How to calculate the sizes of the cashflows are decided upon the agreement of the contract.

1.3 Hedging

Hedging is the reduction of risk by taking advantage of the correlation between derivatives and the underlying assets. The main use of derivatives are in fact for the use of hedging and not for the use of speculation. The reduction of risk can lead to an improved risk/return ratio and is widely used. A perfect hedge reduces the risk to zero, but note that this will greatly reduce the expected return as well.

Hedging is used in many other areas than finance. For example a farmer growing potatoes knows that the price of potatoes fluctuates throughout the year. He will therefore like to sell some of the harvest before the fall to ensure some economic stability. This is possible by signing a forward contract which specifies the price he will get for the potatoes he sells when he harvests.

1.4 Arbitrage

Arbitrage is one of the fundamental concepts in finance. The concept states that there are never any opportunities to make an instantaneous risk-free profit. In practice this means that such an opportunity never exists for a significant period of time. Assume that there exists a risk-free investment with a guaranteed return. An approximation to such an investment is a government bond or a deposit in a sound bank. The greatest risk-free return anyone can make is the return gained in any of the two examples mentioned. An investment in a financial instrument will possibly have a greater return, but it is not guaranteed: greater return comes with greater risk. If a risk-free investment offered greater return than the risk-free return, no investor would want to put their money in the bank or buy a government bond. On the contrary, investors would want to borrow money to invest in the risk-free investment having greater return and by doing so exploit the arbitrage opportunity. This would cause the risk-free interest rate to increase and the arbitrage opportunity would disappear. As noted earlier, in practice an arbitrage opportunity will not exist in a significant period of time. Suppose an arbitrage opportunity exists, then arbitragers and special computers programmed to find such mispricings act quickly and the arbitrage opportunity will vanish.

1.5 Risk Neutral Pricing

When pricing an asset the concepts of hedging and arbitrage should be used. The assumption of no arbitrage opportunities and the use of hedging leads to the fact that there is no return above the risk-free return. If an asset was to be valued in the real world, the expectation would be found and then adjusted for risk. Instead the probabilities of future payoff can be changed in such a way that they incorporate the effect of risk before the expectation is found. A so called risk-neutral world where the investors do not care about risk, is created to find the future payoff incorporating the effect of risk. The following qualities characterizes the risk-neutral world:

- Investors do not care about risk. They do not expect any extra return for taking unnecessary risk.
- Investors do not need statistics for estimating probabilities of events happening.
- Investors believe everything is priced using simple expectations.

This is in strong contrast to what was discussed in the section concerning hedging where risk was a highly unwanted quality of a financial instrument.

Chapter 2

Interest Rates and Interest Rate Derivatives

A bond is an agreement in which an investor loans money to a company or a government. The variation of bonds is vast where factors as duration of the bond as well as the size of the repayment affects the characteristics of the bond. The main works of reference for this chapter are [8] and [12].

2.1 Bond and Bond Pricing

A more formal definition of a bond is: A bond is a contract paid for up-front that yields a known amount on specified dates in the future. The simplest form of a bond is a zero-coupon bond. This is a contract paying a fixed amount of money called the principal, at a given date in the future called the maturity date T. A coupon-bearing bond pays smaller quantities called coupons, up to and including the maturity date in addition to the principal. The coupons are usually pre-specified fractions of the principal.

The value of a zero-coupon bond $V(t)$, is a known function of time if the interest rate $r(t)$ is a known function of time. In a time-step dt the value of the bond changes by

$$\frac{dV}{dt}dt. \tag{2.1}$$

The change in value must depend on the interest rate and by using the arbitrage principle the value is equal to

$$\frac{dV}{dt} = r(t)V. \tag{2.2}$$

The solution of this ordinary differential equation is

$$V(t;T) = Pe^{-\int_t^T r(\tau)d\tau}, \tag{2.3}$$

where the value of the bond at time T is P. Let there be zero-coupon bonds quoted for all possible maturity dates T. If $V(t;T)$ is differentiable with respect to T, solving (2.3) and differentiating gives

$$r(T) = \frac{-1}{V(t;T)}\frac{\partial V}{\partial T}. \tag{2.4}$$

This equation gives the value of the interest rate at future dates if the market of zero-coupon bonds reflects a deterministic interest rate. Another interesting observation is the value of $\frac{\partial V}{\partial T}$, which is negative since the interest rate is positive. Thus a bond's current value decreases the longer it has to live.

2.2 The Yield Curve

The rate of return on an investment is called the yield and for a zero-coupon bond it is defined by

$$Y(t;T) = -\frac{log(V(t;T)/V(T;T))}{T-t}, \tag{2.5}$$

where V is the value of the zero-coupon bond. This definition has two important advantages compared to (2.4): The bond prices V do not have to be differentiable and continuous distribution of bonds with all maturities is not required. The two measures are identical when the interest rates are constant. Plotting the values of Y against time to maturity $(T-t)$, gives the yield curve. The dependence of the yield curve on the time to maturity is called "the term structure of interest rates".

Due to non-deterministic interest rates, the shape of the yield curve varies. There are three distinct shapes often seen in the market: The increasing yield curve is most common. This shape is characterized by higher values for interest rates with longer time to maturity than for those with short time to maturity. Under normal market conditions the return should be higher the longer the money is tied up which is consistent with the increasing yield curve. The decreasing and humped yield curves are typical when the short rate is currently high but expected to fall. Examples of increasing and decreasing yield curves can be seen in figure (2.1).

2.3 Interest Rate Models

Modelling the interest rate can be done by introducing a random variable and letting the interest rate follow a random walk. The simplest interest rate models have only one source of randomness and are therefore called one-factor models. The interest rate modelled is the spot rate which is the rate received by the shortest possible deposit. Over a small period of time dt it is best modelled by both a deterministic and a random part, which is common for several financial assets. The interest rate r is given by the equation

$$dr = w(r,t)dX + u(r,t)dt, \tag{2.6}$$

where dX is the random element modelled by a Brownian motion and different functions for $w(r,t)$ and $u(r,t)$ will give the interest rate different behaviours. A Brownian motion has the following properties:

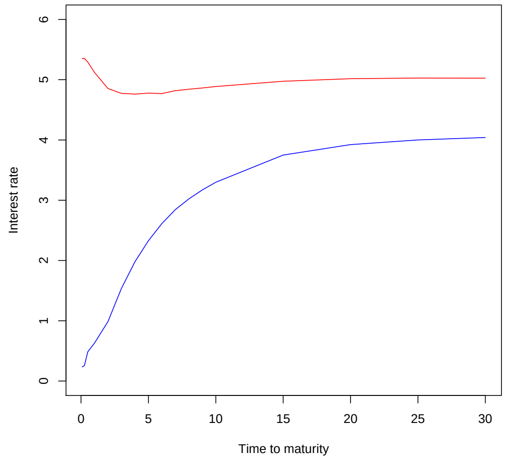


Figure 2.1: Yield curves for the US interest rate displaying a decreasing and an increasing yield curve. The red yield curve (4th Dec 2006) is decreasing while the blue (29th Nov 2009) is increasing. Normally the yield curve is increasing, but under certain market conditions it may be decreasing.

- $dX(0) = 0$
- the mapping $t \mapsto W(t)$ is, with probability 1, a continuous function on [0,T]
- the increments $W(t_1) - W(t_0), W(t_2) - W(t_1), ..., W(t_k) - W(t_{k-1})$ are independent of any k and any $0 \leq t_0 < t_1 < t_2 ... < t_k \leq T$
- $W(t) - W(s) \sim N(0, t-s)$ for any $0 \leq s < t \leq T$

as can be seen in [5]. The random element dX can therefore be written $dX = \sqrt{dt}Z$ where Z is a standard normal variable. The use of a standard normal variable as the random element is debatable and the validity of this assumption will be tested later.

One of the most sought after qualities for an interest rate model is the mean reverting behaviour. The value of many financial assets e.g. stocks, have no upper limit and can in theory tend to infinity when time tends to infinity. This is in contrast to interest rates where extreme values rarely are seen. The mean reverting property ensures that the interest rate tends towards the mean. It is also important to avoid negative interest rates. Even though negative interest rates have occurred in some parts of the world it is not common and negative interest rates should be avoided.

When pricing an equity option the underlying asset is used to hedge the derivative to find a fair price. When pricing a bond, there is no underlying asset with which to hedge. This makes pricing a bond more difficult than pricing an equity option and the only alternative is to hedge a bond with a another bond maturing at a different date.

This is used when the bond pricing equation is derived as can be seen in appendix (A.1). The bond pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\omega^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda\omega)\frac{\partial V}{\partial r} - rV = 0, \tag{2.7}$$

where V is the value of the zero-coupon bond, T is the maturity date, ω is the volatility, u is the drift and λ is the market price of risk. Several interest rate models are chosen so that the solution of (2.7) can be found analytically. Parameter estimation can be done both by using historic values and by using values given by the market.

If a one-factor model is used for describing the whole yield curve, the curve will be given from a specified interest rate at a specified time. This yield curve cannot capture the diversity seen in a yield curve given by the market, thus the multi-factor models are introduced. A multi-factor interest rate model comprises two or more sources of randomness. Commonly the sources of randomness are different interest rates, for example a short-term and a long-term interest rate, but some models use other measures such as the yield curve slope or the volatility of the spot rate. When pricing a derivative dependent on the difference of yields of different maturities, a one-factor model cannot be used, but when pricing a derivative only dependent on the level of the yield curve, a one-factor model may be sufficient. Examples of multi-factor models are the Heath, Jarrow and Morton (HJM) framework and the LIBOR market model (LMM). The LMM will be used for simulation in this thesis. It will be discussed in greater detail in the next section.

When the Heath, Jarrow and Morton (HJM) approach was introduced it drastically changed the pricing of fixed income products. Unlike most interest rate models at that time the framework describes the evolution of the whole forward rate curve and was a great improvement compared to the one-factor models which mostly models the spot rate. Another advantage is that yield-curve fitting occurs naturally because of the properties of a multi-factor model. A drawback is that the general model is not a Marcov process and an infinite number of variables are needed to write the model as a partial differential equation.

As many other multi-factor interest rate models, the HJM describes the evolution of forward rates. As can be seen in [5], a forward rate is an interest rate set today for both borrowing and lending some time in the future. If $F(t, T_1, T_2)$ denotes the forward rate, it will be fixed at time t for the interval $[T_1, T_2]$ where $t < T_1 < T_2$. An investor borrowing at this rate will enter into the agreement at time t, borrow the money at time T_1, repay the loan at time T_2 and pay interest at rate $F(t, T_1, T_2)$.

2.4 Simple Compounded Forward Rates and the LIBOR Market Model

The reference for this section is [5].

Simple Compounded Forward Rates

The London Inter-Bank Offered Rates (LIBOR) are among the most important benchmark interest rates. The rates are quoted daily for different maturities and currencies and are based on simple interest. Even though the LIBOR rates are not completely risk-free, they will be treated as if they are in this thesis. The forward LIBOR rate L(0,T) is set at time 0 for the interval $[T, T+\delta]$. It is given by the bond prices B to be

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)}. \tag{2.8}$$

A finite set of tenor dates are introduced. They specify the maturities by which the interest rates are modelled

$$0 = T_0 < T_1 < ... < T_M < T_{M+1}. \tag{2.9}$$

The length of the intervals between the tenor dates are denoted

$$\delta_i = T_{i+1} - T_i, i = 0, 1, ..., M. \tag{2.10}$$

These are often equal to a fixed interval, e.g. half a year. Let $B_n(t)$ denote the price of a bond at time t maturing at time T_n $(0 \leq t \leq T_n)$, instead of the notation used earlier $B(t,T)$. Similarly let $L_n(t)$ denote the forward rate at time t for the interval $[T_n, T_{n+1}]$. $L_n(t)$ is then given by

$$L_n(t) = \frac{B_n(t) - B_{n+1}(t)}{\delta_n B_{n+1}(t)}, 0 \leq t \leq T_n, n = 0, 1, ..., M. \tag{2.11}$$

The inverted relationship gives the value of the bond $B_n(T_i)$ in terms of the forward rate L_n

$$B_n(T_i) = \prod_{j=i}^{n-1} \frac{1}{1 + \delta_j L_j(T_i)}, n = i+1, ..., M+1. \tag{2.12}$$

When (2.12) is used to price bonds they can only be determined at the maturity dates because the discount factors are valid only for the specified time intervals. Further work can be done to determine bond prices between the tenor dates.

The LIBOR Market Model under the Forward Measure

The LIBOR Market Model (LMM) describes the evolution of the arbitrage-free forward rates. More precisely it describes simple compounded interest rates which, as seen previously in this section, are easily observed in the market. The model is therefore called a "market model". The LMM can be formulated both under spot measure and under forward measure. The model is described by a system of stochastic differential equations (SDEs) of the form

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \sigma_n(t)\top dW(t), 0 \leq t \leq T_n, n = 1, ..., M, \tag{2.13}$$

where W is a d-dimensional standard Brownian motion, μ_n is the drift and σ_n is the volatility. The LMM can be formulated both under spot measure and under forward measure. The forward measure for maturity T_{M+1} uses the bond B_{M+1} as numeraire asset. The deflated bond prices are defined to be the ratios

$$D_n(t) = \frac{B_n(t)}{B_{M+1}(t)} = \prod_{j=n+1}^{M} (1 + \delta_j L_j(t)). \tag{2.14}$$

The evolution of the forward LIBOR rates can be found by requiring that D_n from (2.14) are martingales and by the use of induction, see [5]. It is given by

$$\frac{dL_n(t)}{L_n(t)} = -\sum_{j=n+1}^{M} \frac{\delta_j L_j(t)\sigma_n(t)^\top \sigma_j(t)}{1 + \delta_j L_j(t)} dt + \sigma_n(t)^\top dW^{M+1}(t), 0 \leq t \leq T_n, \tag{2.15}$$

where W^{M+1} is a standard d-dimensional Browninan motion.

2.5 Interest Rate Derivatives

There is a vast number of different interest rate products. An interest rate derivative derives its value from the interest rate or from another interest rate derivative. A bond is one of the simplest and most common interest rate derivatives. Three other common interest rate derivatives are studied in this section: swaps, caps and floors. It is important to emphasize that when valuing interest rate derivatives, only the risk-free interest rate should be used.

Swaps

An interest rate swap is a contract between two parties agreeing to exchange or swap future cashflows represented by the interest on a notional principal. The principal is notional in the sense that it is never paid by either party, it is only used to determine the magnitudes of the payments. One party pays the other a fixed interest rate multiplied by the principal while the other pays a floating interest rate multiplied by the principal. Thus, the swap has the following payoff seen from the payer of the fixed cashflow

$$S = r - r_s \tag{2.16}$$

multiplied by the principal. r is the floating interest rate and r_s is the fixed interest rate. When the contract is entered into it is usual for the deal to have no value to either party. This is done by choosing the fixed interest rate in such a way that the net present value of the two sides equal one another and no money changes hands on the day the of the agreement.

Caps and Floors

The owner of a cap contract pays several cashflows determined by the floating interest rate and the notional principal at specified dates. However, the owner is guaranteed that the floating interest rate will not exceed a specified value, called the cap. Each of the individual payments is called a caplet, thus a cap is the sum of several caplets. The payoff for a caplet is

$$C = (r - r_c)^+, \tag{2.17}$$

multiplied by the principal where r is the floating interest rate and r_c is the cap. Thus a caplet is a call option on the floating interest rate r as can be seen in [7].

A floor is similar to a cap except that the interest rate is bounded below by the floor. Each of the individual cashflows is called a floorlet and the payoff for a floorlet is

$$F = (r_f - r)^+, \tag{2.18}$$

multiplied by the principal where r is the floating interest rate and r_f is the floor. A floorlet is similar to a put option on the floating interest rate r as can be seen in [7].

The cap-floor parity expresses the relationship between a cap, a floor and a swap. Let a portfolio Π consist of a long caplet and a short floorlet where $r_c = r_f$. The value of this portfolio is

$$\Pi = (r - r_c)^+ - (r_c - r)^+ = r - r_c. \tag{2.19}$$

The last term is recognised as one of the cashflows of a swap. Thus there is a no-arbitrage relationship between a cap, a floor and a swap:

$$\text{swap} = \text{cap} - \text{floor}. \tag{2.20}$$

Options

An option gives the holder the right to buy or sell an asset for a given price at a given date in the future. Notice than an option gives the holder the right but not the obligation to buy or sell the asset. Examples of interest rate options are bond options, swaptions, captions and floortions. A bond option is valued as an equity option except that the underlying asset is a bond. For more information of how to value an equity option, see [7]. Swaptions, captions and floortions are valued as swaps, caps and floors except that the holder only exercises the option if it has positive value.

Chapter 3

Estimation of Volatility and Correlation

Volatility measures the dispersion of the value of a given asset. In this thesis volatility is defined as the standard deviation of the returns of an asset and this chapter discusses the estimation of volatility for prediction purposes. The main work of reference for this chapter is [1].

3.1 Implied vs Historical Information

Estimation of volatility and correlation can be done using both implied and historical information. Implied information means taking advantage of the relationship between derivative prices and the volatility, as well as other variables that are used in analytical formulas describing this relationship. Originally these formulas were used to estimate the value of the derivative, but they might be used "the other way around" and estimate the volatility or correlation given the derivative price. When using historic information to estimate the future volatility, the historic prices of the underlying asset are used to produce the estimate. This is more traditional and originates from the classical statistics.

Using implied volatility or correlation leads to several problems. One of the fundamental problems is that the analytical formulas are not exact. This implies that using different models will lead to different estimates of the volatility or correlation. Many assumptions are made to calculate an analytical solution of the derivative price. An implied estimate should not be used if one of the assumptions in the formula used contradicts an assumption of the model itself. I.e. an investor might want to avoid the normality assumption in his model, but most analytical relationships are based on this assumption. Using an implied estimate based on a model assuming normality would then not be advisable. Another problem is that most models assume constant volatility or correlation during the lifespan of the derivative, thus the estimation will have a fixed forecast horizon. I.e. an investor might want to estimate the volatility for the next day. If he uses a derivative with a lifespan of one year to estimate the implied volatility, this will most likely be a poor estimate. Yet another problem concerning implied estimates is that it would require observable derivative prices on all instruments contained in a portfolio. Generally the derivative prices are not liquid enough to produce consistent estimates. At last it should be noted that the implied volatility is subject to what is expected by the market. In comparison historical values used to calculate the historical volatility contains both data that is expected by the market as well as unexpected. Thus one can argue that historical information gives a richer and more realistic view of the volatility and the correlation.

As concluded in [1], some research point towards implied estimation performing better than historical estimation while other point towards historical estimation outperforming implied estimation. An alternative of choosing one of the two estimation methods would be to combine the two, but this is beyond the scope of this thesis. For the purposes of this analysis historical estimates of the volatility and correlation will be used.

3.2 Simple Moving Average Model

The simple moving average (SMA) estimate is based on the traditional method of defining variances and covariances as can be seen in [6]. Let $r_{i,m}$ be the i'th measurement of the m'th component of the sample. The variance of a component is estimated by

$$\hat{s}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(r_i - \overline{r})^2, \tag{3.1}$$

where N is the total number of samples and $\overline{r}$ is the estimated mean of the component. Similarly, the covariance between the m'th and n'th component is estimated by

$$\hat{s}_{m,n} = \frac{1}{N-1}\sum_{i=1}^{N}(r_{i,m} - \overline{r}_m)(r_{i,n} - \overline{r}_n). \tag{3.2}$$

By letting r be the return of any asset, this method can be used to estimate the variance and covariance of a multivariate time series. The return $r_{i,m}$ will be the return of the i'th period of time for the m'th component.

Correlation is another measure of the influence two components have on each other. It is defined by the standard deviation and the covariance to be

$$\hat{\rho}_{m,n} = \frac{\hat{s}_{m,n}}{\hat{s}_m \hat{s}_n}. \tag{3.3}$$

The coefficient measures the linear association between the two random variables r_m and r_n.

3.3 Exponentially Weighted Moving Average Model

The US forward rate for the period of 1-2 years can be seen in figure (3.1). When calculating the volatility and correlation of the interest rate, the log return of the time

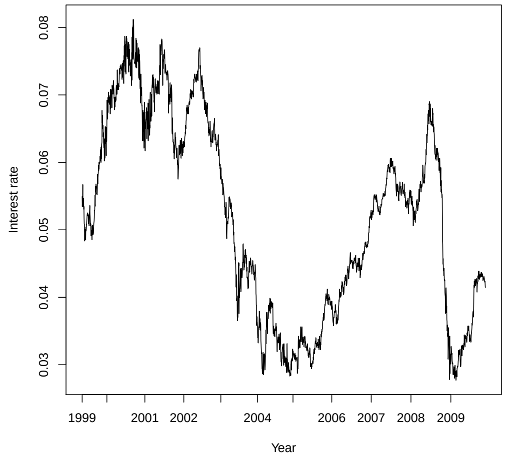


Figure 3.1: The US forward rate for the period of 1-2 years.

series is used. Figure (3.2) shows the log return of the time series in figure (3.1) and it is clear that the volatility of the log return is not constant during the period of time chosen.

As the variance and covariance is not constant over time, another model is suggested to handle the time-dependence better. Using the exponentially weighted moving average (EWMA) model to estimate the matrix allows for time-dependence and lets recent data have greater impact on the estimate. The estimate of the variance using this model is

$$\hat{\sigma}^2 = (1-\lambda)\sum_{i=1}^{N}\lambda^{i-1}(r_i - \overline{r})^2. \tag{3.4}$$

Notice that compared to the SMA, the EWMA depends on the parameter λ $(0 < \lambda < 1)$ which is called the decay factor. Both the relative weights of the returns and the effective amount of data used in estimating the volatility is dependent on the size of this factor. The decay factor is chosen to be 0.94 when estimating daily volatility and 0.97 when estimating monthly volatility. The reason for this seemingly arbitrary choice can be found in [1].

Assuming that the sample mean $\overline{r}$ is zero, a recursive form of (3.4) can be obtained

$$\hat{\sigma}^2_{i+1|i} = \lambda\hat{\sigma}^2_{i|i-1} + (1-\lambda)r_i^2. \tag{3.5}$$

The subscript "$i+1|i$" can be interpreted as the forecast at time $i+1$ given all information up to and including time i. The subscript "$i|i-1$" can be interpreted similarly. The ability to obtain the estimate recursively is also an advantage when it comes to computing time.

The EWMA estimate of the covariance for the m'th and n'th component is found

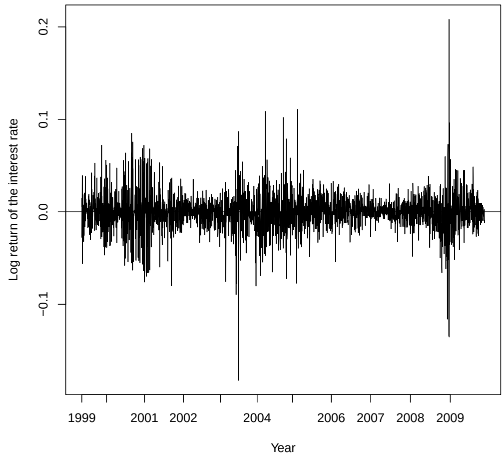


Figure 3.2: The log return of the US forward rate for the period of 1-2 years. The log return of the interest rate will be used for calculating the variance and the covariance. Notice that the volatility of the sample is not constant.

similarly

$$\hat{\sigma}_{m,n} = (1-\lambda)\sum_{i=1}^{N} \lambda^{i-1}(r_{i,m} - \overline{r}_m)(r_{i,n} - \overline{r}_n). \tag{3.6}$$

By making the assumption of zero mean, a recursive formula can be obtained for the covariance as well

$$\hat{\sigma}_{i+1|i,m,n} = \lambda\hat{\sigma}_{i|i-1,m,n} + (1-\lambda)r_{i,m}r_{i,n}. \tag{3.7}$$

The correlation is defined by the relationship between the covariance and standard deviations and is therefore

$$\hat{\rho}_{i+1|i,m,n} = \frac{\hat{\sigma}_{i+1|i,m,n}}{\hat{\sigma}_{i+1|i,m}\hat{\sigma}_{i+1|i,n}}. \tag{3.8}$$

Estimates of the volatility is made by using both SMA and EWMA to be able to compare the two methods, the plot can be seen in figure (3.3). It is clear that a shock effects the SMA and EWMA estimate differently. The EWMA estimate reacts faster to the shock and peaks higher than the SMA estimate. It should also be mentioned that a shock affects the SMA estimate over a longer period of time than it affects the EWMA estimate. The SMA estimate remains quite large as long as the data from the shock is in the sample while the EWMA estimate decreases earlier and more gradually. The sample size is of great importance for the SMA estimate because the peaks last as many days as the sample size is large. This is easily seen in figure (3.3) where the estimate using 3 months of data has thinner peaks than the estimate using 6 months of data.

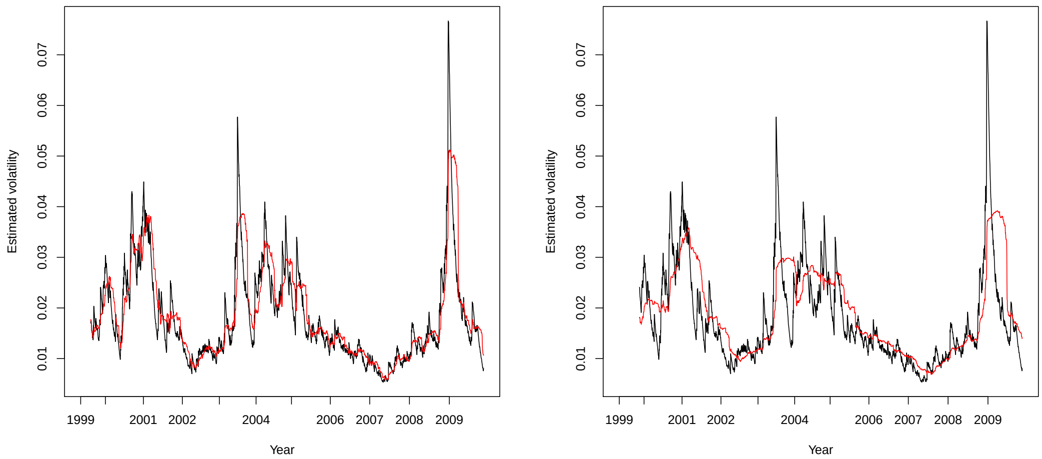


Figure 3.3: The SMA and the EWMA estimate of the volatility is calculated using 3 and 6 months of data in the sample. The red line is the SMA estimate while black line is the EWMA estimate. The value of the decay factor is set to be 0.94. Notice that the EWMA estimate peaks higher and adjusts faster whenever a shock occurs. In comparison the SMA estimate has a lower value over a longer period of time after a shock. This effect is strengthened when the size of the data in the sample increases and the value of the SMA estimates are lower but lasts longer when using 6 months of data in the sample.

3.4 Multiple Days

The variances and covariances estimated so far are defined over the period of time from i to $i+1$, where each step represents one business day. Often an estimate over several days is wanted and this can be estimated by

$$\hat{\sigma}_M^2 = M\hat{\sigma}^2 \tag{3.9}$$

and

$$\hat{\sigma}_{m,n,M} = M\hat{\sigma}_{n,m} \tag{3.10}$$

where M is the number of days for which the estimate is chosen to be valid, see [1]. The approximation is valid for both the equally and the exponentially weighted estimation. Remember that the value of the decay factor changes when the estimate is valid for a longer period of time. In particular, as was commented in the last section, the decay factor is set to 0.94 when the estimate is valid for a day and 0.97 when the estimate is valid for a month. The standard deviation over a period of time N is estimated by $\hat{\sigma}_N = \sqrt{N}\hat{\sigma}$ and this is often called the "square root of time" relationship. It should be noted that the correlation does not change because the multiples will cancel each other out.

A closer look at the "square root of time rule" reveals that the variance and covariance are assumed to be constant over time. As previously shown, this is a poor assumption and is a serious flaw of the model. In addition there are three scenarios in particular where the the model performs poorly:

- When the time series is mean reverting.
- When boundaries limit the movement of the time series.
- When the estimate optimized for a particular time horizon is used for another horizon.

As both the first two scenarios listed are characteristics of the interest rate it might be tempting to reject this model, but finding a good replacement is not easy.

Both the SMA and the EWMA model have several flaws and efforts have been made to make better models. Some models do seem to perform better, but they are far more complicated and will not be discussed in this thesis. The SMA and EWMA models provide adequate estimates and will be used for the purpose of this thesis.

Chapter 4

Measurement of Risk

Investing in assets exposes the investor to risk. Several models are made to describe the risk, and in this chapter two of the most popular models will be described. The reference for this chapter is [3].

4.1 Risk Measurement

According to [3], financial risk is defined as the prospect of financial loss - or gain - due to unforeseen changes in underlying risk factors. The formal work of risk measurement started in the middle of the 20th century, but has developed rapidly since then. The rapid development is due to several factors, one being the more volatile environment in which the firms operate. As more volatile environments exposed the firms to greater financial risk the firms responded by improving their risk management. The factors contributing to the volatile environment are volatility in the stock and commodity markets and the volatile exchange and interest rates. Another factor contributing to the fast development of risk measurement is the enormous growth in trading activity. The activity in the stock exchange markets have increased tenfold. At the same time vast amounts of new instruments, among them derivatives, have been created and the trading volumes in these new instruments has grown rapidly as well. A third factor contributing to the development is the improvements in information technology. Because of increases in computing power and computing speed as well as reduction in computing costs, all calculations relating to risk measurements are now done using information technology. Financial risk can be separated into several forms of risk, among them market risk, credit risk and operational risk. This chapter will discuss the measurement of market risk.

Even though risk measurement has improved the management of risk greatly, there are some concerns that should be noted. All risk measures are based on models, and all models are based on assumptions. If these assumptions are incorrect or fail to capture important behaviour observed in the real world, the model will perform poorly. The risk of a model performing poorly in this way is called model risk. Another risk issue is the implementation risk which arises when a system is implemented. The same system can be implemented in several ways and will perform differently depending on how it is done. Other flaws could be discussed, but the bottom line is that risk measurements improve the monitoring of risk for the firms.

4.2 Value at Risk

The concept of value at risk (VaR) was created when the need for a better risk measurement grew at the end of the 20th century. Previously the risk measures used were crude and measures used in one area could not necessarily be compared with measures used in another area. Several models were developed in this period and most of them were based on VaR, which became the new standard of risk measurement.

As can be seen in [12], a common definition of VaR is:

> Value at Risk is an estimate, with a given degree of confidence, of how much one can lose from one's portfolio over a given time horizon.

When the degree of confidence increases, the value of VaR increases as well. It should carefully be noted that VaR often increases at an increasing rate which means that the possible losses can be large. When the time horizen increases, VaR often increases with the square root of the holding period. However, this is only a rule of thumb, and the VaR might increase in a different way or even fall as the time horizon increases. It should now be clear that the combination of increasing degree of confidence and increasing time horizon might produce large VaR's.

The confidence level is typically 95%, 97.5%, 99% etc. However, it should be noted that for backtesting purposes a relatively low confidence level is an advantage to get a reasonable proportion of excess-loss observations. The usual holding periods are one day or one month, but any arbitrary holding period can be chosen. The time horizon is amongst other things dependent upon the liquidity of the markets in which the assets are bought and sold. Other things being equal, the ideal holding period in any given market is the time it takes to ensure orderly liquidation of positions in that market. However, a short holding period is favoured by other factors: The portfolio is assumed not to change over the holding period and clearly this holds better for a short than a long holding period. It is also preferred to have a short holding period when backtesting or validating models because of the large amounts of data that is needed. To get a better understanding of how the VaR depends on the confidence level and the holding period, it is suggested that whenever applicable the point values of the variables should be replaced by intervals of the variables. The result will be a VaR-surface which will describe the risk more thoroughly than a point estimate. It should also be noted that VaR is calculated assuming normal market conditions. Thus extreme market conditions such as crashes are not considered and should be examined separately.

One of the advantages of the VaR is that the measure is consistent across different positions and risk factors. Institutions can improve their management of the overall risk by using VaR if their previous measure could not compare risk across the different positions. Another important characteristic is that VaR takes the correlations between different risk factors into account: If two risks offset each other, the value of the overall

VaR will be fairly low. If two risks do not offset each other, the value of the overall VaR will be greater.

A drawback of the VaR estimates is that they are rather imprecise. Different VaR models can give large differences in the estimates. Even theoretically similar models can produce different estimates because of different implementation. If an investor uses an inaccurate VaR measurement and believes it to be correct, the risk and the loss can be larger than he expected it to be. It is worse to believe one's inaccurate estimate of VaR is accurate than to not have an estimate of the VaR at all.

One of the other disadvantages of the VaR measure is that it gives no idea of the behaviour of the tail. If a tail event occurs the loss is expected to be greater than the VaR, but there is no information on how large the possible loss is. The lack of this information is a large drawback of the VaR. It is in fact possible to construct different portfolios with the same VaR, but where the loss in a possible tail event is much larger for one of the portfolios. Even though the VaR is equal for the two portfolios the risk is not and this can easily mislead investors. This drawback can even be exploited by traders to construct portfolios with greater risk than first presumed. If applicable, this problem can be avoided by the use of several confidence levels or by the use of VaR-surfaces as earlier described.

Another drawback of the VaR measure is that it is in general, not sub-additive. Sub-additivity means that adding up individual risks does not increase the overall risk. Let ρ be a risk measure and let A and B be positions. Sub-additivity can then be expressed

$$\rho(A + B) \leq \rho(A) + \rho(B). \tag{4.1}$$

Sub-additivity is important because it gives an overestimate of combined risk which often is convenient. If the returns are normally, or more generally elliptically, distributed, the VaR is sub-additive. However, if we cannot assume an elliptical distribution, the measure is not sub-additive. Sub-additivity will be discussed in greater depth in the next section.

4.3 Coherent Measure of Risk

It is clear that VaR has several weaknesses and the discussion of what a good risk measure really is started in the late 1990's. Philippe Artzner et al. proposed a theory of the properties of a good risk measure: the theory of coherent risk measures. Let X and Y represent any two portfolios and let ρ be a measure of risk over a chosen horizon. As can be seen in [4], the properties of a coherent measure is:

1. $Monotonicity : Y \geq X \Rightarrow \rho(Y) \leq \rho(X)$.
2. $Subadditivity : \rho(X + Y) \leq \rho(X) + \rho(Y)$.
3. $Positive\ homogeneity : \rho(hX) = h\rho(X)$ for $h > 0$.
4. $Translational\ invariance : \rho(X + n) = \rho(X) - n$ for some certain amount n.

Monotonicity means that if a portfolio Y is always greater than X, it should have lower risk as well. Positive homogeneity implies that the risk of a position is proportional to its size, e.g. double the investment and the risk is doubled as well. Translational invariance means that adding risk free capital to the portfolio reduces the risk. The reduction of risk is at the same rate as the addition of the risk-free capital.

Subadditivity implies that the risk of a portfolio made up of subportfolios will be no greater, and in some cases less than, the sum of the risks of the subportfolios. It is an important property because non-subadditivity has some awkward characteristics. Non-subadditivity might suggest that diversification increases the risk, which means that a risk manager following this "rule" might end up betting all his money on one horse which would be anything but a safe bet. Another characteristic of a non-subadditivity measure of risk is that it might create extra risk when adding two subportfolios. This risk did not exist before the merging of the subportfolios and one might wonder where this risk should come from if it existed. Some consequences of non-subadditivity:

- Adding risk together would not give an overestimate of the combined risk. On the contrary, adding the risk would give an underestimate which would be useless. This means that a risk manager can not use the sum of risks reported to him as a conservative measure of risk. It follows that decisions made on a decentralised level is more risky than presumed and the consequence is that decisions should be centralised.

- Traders using non-subadditive risk measures can break up their accounts at an exchange to reduce the risk which will reduce the margin requirements. The exchange will itself be exposed to possible loss because the separate accounts would no longer cover the combined risk.

- Financial institutions are required to have a certain amount of capital to ensure that that they do not increase the risk of default in the market. If regulators deciding the size of the capital requirement use non-subadditive risk measures, a financial institution might be tempted to break itself up to reduce the capital requirements. The sum of the capital requirements for the smaller units will be less than the capital requirement for the institutions as a whole and the institution will then make more money.

4.4 Expected Tail Loss

Expected tail loss (ETL) is a coherent measure and is also called expected shortfall (ES). It is the expected loss if the loss exceeds VaR. Let the loss be denoted L, ETL is then defined by

$$ETL = E[L|L > VaR]. \tag{4.2}$$

While the VaR estimates the maximum loss if a tail event does not occur, the ETL estimates what is expected to loose if a tail event does occur. ETL is a consistent measure of risk across different positions and also takes correlations into account. ETL increases when the level of confidence increases. Often the ETL increases at an increasing rate which means that possible tail losses can be large. ETL is also dependent on the holding period, usually the ETL increases when the holding period increases. The discussion of the choice of parameters for ETL is similar as the discussion of the choice for parameters for VaR. As the same arguments may be applied for ETL the arguments are not repeated but can be found in the section on VaR. It should also be emphasised that the ETL-surface, as the VaR-surface, gives more insight into the risk and provides more information than a point estimate and therefore should be used whenever appropriate. Since the ETL is a coherent measure, but still entails many of the good properties of VaR, ETL is considered to be a better risk measure than VaR. The two most important reasons are:

- The ETL estimates what to expect if a tail event occurs, e.g. how bad the situation might turn out to be. VaR on the contrary, gives no more information than to expect a loss greater than the VaR itself.
- The ETL is coherent and satisfies the subadditivity condition, while the VaR does not. The consequence is that the VaR measure has some awkward characteristics which is a major drawback of the risk measure.

Chapter 5

Implementation

This chapter describes implementation of methods and models used in this thesis.

5.1 The LIBOR Market Model

The reference for this section is [5]. When simulating multi dimensional interest rates, both time and maturity arguments must be discrete. When using the LIBOR market model (LMM) the maturity is already discrete due to the use of a finite set of maturities and it is only necessary to discretize the time argument. The Euler scheme is applied to $\log L_n$, and the LIBOR rates can be simulated using

$$\tilde{L}_n(t_{i+1}) = \tilde{L}_n(t_i)\exp\left(\left[\mu_n(\tilde{L}(t_i), t_i) - \frac{1}{2}\sigma_n(t_i)^2\right][t_{i+1} - t_i] + \sqrt{t_{i+1} - t_i}\sigma_n(t_i)Z_n(t_{i+1})\right) \tag{5.1}$$

where

$$\mu_n(\tilde{L}(t_i), t_i) = -\sum_{j=n+1}^{M} \frac{\delta_j \tilde{L}_j(t_i)\sigma_n(t_i)\sigma_j(t_i)}{1 + \delta_j \tilde{L}_j(t_i)}. \tag{5.2}$$

The tildes are used to emphasize that these variables are discretized. The $Z_n(t_i)$'s are normal correlated variables, but the vectors $\boldsymbol{Z}(t_i)$'s are independent of each other. This relationship will be discussed in more detail later. Notice that $\mu_M \equiv 0$. If σ_M is deterministic and constant between the t_i's, then (5.1) simulates the forward LIBOR rates without discretization error. Another way to look at (5.1) is that it approximates L_n by a geometric Brownian motion over the time-interval $[t_i, t_{i+1}]$ where drift and volatility parameters are fixed at time t_i. So far no restrictions have been imposed on the volatility parameter but a deterministic σ_n will cause the L_n to be close to lognormal. It is also worth mentioning that $\hat{L}_n$ always stays non-negative.

Generation of correlated random variables can be done using Cholesky factorisation. Let the matrix $\boldsymbol{M}$ be a triangular matrix satisfying

$$\boldsymbol{M}\boldsymbol{M}^\top = \boldsymbol{\rho} \tag{5.3}$$

where $\boldsymbol{\rho}$ is the correlation matrix and $\boldsymbol{M}^\top$ means the transpose of the matrix $\boldsymbol{M}$. The matrix $\boldsymbol{M}$ can be found using Cholesky decomposition as can be seen in [11]. This decomposition is not unique and the Cholesky factorization is just one of several ways to decide $\boldsymbol{M}$.

With $\boldsymbol{M}$ decided, the correlated d-dimensional vector $\boldsymbol{Z}$ can be found by letting $\boldsymbol{\epsilon}$ be a random d-dimensional vector of independent standard normal variables. $\boldsymbol{Z}$ is then given by

$$\boldsymbol{Z} = \boldsymbol{M}\boldsymbol{\epsilon}. \tag{5.4}$$

The algorithm for simulating forward rates by the LMM can be seen in algorithm (1).

Algorithm 1 LMM - Simulation of forward rates for one time-step

---

Input:
forward rates, $\hat{L}_n(t_i), n = 1, ..., M$
volatilities, $\hat{\sigma}_n(t_i), n = 1, ..., M$
time between tenors, $\delta_n, n = 1, ..., M$
Do:
for $n = 1$ to M **do**
  for $j = n + 1$ to M **do**
    $\hat{\mu}_n(\hat{L}(t_i), t_i) = \hat{\mu}_n(\hat{L}(t_i), t_i) - \frac{\delta_j \hat{L}_j(t_i)\hat{\sigma}_n(t_i)\hat{\sigma}_j(t_i)}{1+\delta_j \hat{L}_j(t_i)}$
  end for
  $\hat{\mu}_M(\hat{L}(t_i), t_i) = 0$
  randomly calculate $Z(t_{i+1})$
  $\hat{L}_n(t_{i+1}) = \hat{L}_n(t_{i+1})\cdot$
    $\exp\left(\left[\hat{\mu}_n(\hat{L}_n(t_i), t_i) - \frac{1}{2}\hat{\sigma}(t_i)^2\right][t_{i+1} - t_i] + \sqrt{t_{i+1} - t_i}\hat{\sigma}_n(t_i)Z(t_{i+1})\right)$
end for
Return:
$\hat{L}_n(t_{i+1}), n = 1, ..., M$

5.2 Estimation of Volatility and Correlation

When estimating the variances and covariances used in the LMM, the log return and not the return itself is used as basis for the calculations. This is due to the fact that the variances and covariances are used to describe the Brownian motion and not the interest rate itself. The log return is only an approximation of the behaviour of the Brownian motion and is not theoretically correct. It is still a common approximation often used for this purpose. As the LMM does not specify interest rates between the tenor dates, the variances and covariances are assumed to be constant between these dates.

The log return for the forward rates is calculated by $\ln \frac{L_n(t^{i+1})}{L_n(t_i)}$. Notice that using N observations of the interest rate to calculate the returns will give $N - 1$ returns. Two estimates will be calculated: the SMA estimate and the EWMA estimate.

The SMA estimate of the variance and covariance is

$$\hat{s}_{m,n} = \frac{1}{N-2}\sum_{i=1}^{N-1}(\ln\left(\frac{L_m(t_{i+1})}{L_m(t_i)}\right) - \hat{\mu}_m)(\ln\left(\frac{L_n(t_{i+1})}{L_n(t_i)}\right) - \hat{\mu}_n), \tag{5.5}$$

where

$$\hat{\mu}_n = \frac{1}{N-1}\sum_{i=1}^{N-1}\ln\left(\frac{L_n(t_{i+1})}{L_n(t_i)}\right) = \frac{1}{N-1}\ln\left(\frac{L_n(t_N)}{L_n(t_1)}\right), \tag{5.6}$$

N is the observed number of the interest rates L_n in the sample and $\hat{s}_{m,m}$ is the variance. The covariance matrix can easily be calculated using matrix operations

$$\hat{\boldsymbol{S}} = \frac{1}{N-2}\boldsymbol{X}^\top\left(\boldsymbol{I} - \frac{1}{N-1}\boldsymbol{1}\boldsymbol{1}^\top\right)\boldsymbol{X}. \tag{5.7}$$

$\hat{\boldsymbol{S}}$ is the covariance matrix, $\boldsymbol{X}$ is the log return matrix, $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{1}$ is a vector with only ones and N is the number of interest rates in the sample. The work of reference for the SMA estimate is [6].

When calculating the EWMA estimate, the mean log return is assumed to be zero. The reference for the EWMA estimate is [1] and the estimate is calculated by

$$\hat{\sigma}_{m,n} = (1-\lambda)\sum_{i=1}^{N-1}\left(\ln\left(\frac{L_m(t_{i+1})}{L_m(t_i)}\right)\right)\left(\ln\left(\frac{L_n(t_{i+1})}{L_n(t_i)}\right)\right). \tag{5.8}$$

The covariance matrix can be calculated by using matrix operations

$$\hat{\boldsymbol{\Sigma}} = (1-\lambda)EP(\boldsymbol{\Lambda}^\top\boldsymbol{X}^\top)\boldsymbol{X}, \tag{5.9}$$

where $\hat{\boldsymbol{\Sigma}}$ is the covariance matrix, $EP(\mathbf{AB})$ is the elementwise product of the matrix $\mathbf{A}$ and $\mathbf{B}$ and $\boldsymbol{\Lambda}$ is the matrix

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda^0 & \lambda^0 & \lambda^0 & \dots & \lambda^0 \\ \lambda^1 & \lambda^1 & \lambda^1 & \dots & \lambda^1 \\ \lambda^2 & \lambda^2 & \lambda^2 & \dots & \lambda^2 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ \lambda^{N-1} & \lambda^{N-1} & \lambda^{N-1} & \dots & \lambda^{N-1} \end{bmatrix}. \tag{5.10}$$

By using the recursive formula, the variance and covariances can easily be obtained

$$\hat{\sigma}_{i+1|i,m,n} = \lambda\hat{\sigma}_{i|i-1,m,n} + (1-\lambda)\left(\ln\left(\frac{L_m(t_{i+1})}{L_m(t_i)}\right)\right)\left(\ln\left(\frac{L_n(t_{i+1})}{L_n(t_i)}\right)\right). \tag{5.11}$$

The correlation is defined by the relationship between the covariance and the standard deviation and is the same for the two estimates

$$\rho_{m,n} = \frac{\sigma_{m,n}}{\sigma_m\sigma_n}. \tag{5.12}$$

5.3 Interpolation

The interest rates used are quoted for 6 months, 1 year, 2 years, 3 years, etc. Since cashflows every half a year is sometimes preferred, interpolation is used to give an estimate of these cashflows. The oscillating nature of high-degree polynomials make piecewise-polynomial approximation attractive. As the name suggests, this approach divide the interval into subintervals and constructs different approximating polynomials on each subinterval. The reference for this section is [2].

The most common piecewise-polynomial approximation is called cubic spline interpolation. Let $[x_0, x_n]$ be the entire interval where interpolations is needed. The cubic spline method fits cubic polynomials between each successive pair of nodes: One cubic polynomial on $[x_0, x_1]$ agrees with the function at x_0 and x_1, the next cubic polynomial on $[x_1, x_2]$ agrees with the function at x_1 and x_2 etc. A general cubic polynomial has four arbitrary constants: the constant term, the coefficient of x, the coefficient of x^2 and the coefficient of x^3. Fitting the polynomial to the endpoints of the interval only requires two constants, so the remaining two can be used to ensure that the interpolant has continuous first and second derivatives on the entire interval $[x_0, x_n]$.

Let a function f be defined on the interval $[a, b]$ and a set of nodes $a = x_0 < x_1 < ... < x_n = b$. A cubic spline interpolant S for f is a function that satisfies the following conditions:

- $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, ..., n-1$
- $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, ..., n-1$
- $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$
- $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$
- $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, ..., n-2$
- Boundary conditions: $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$.

Cubic splines may be defined with other boundary conditions, but the clamped conditions are preferred. Compared to other boundary conditions they lead to a more accurate approximation because they include information that other boundary conditions may not include: the values of the derivative at the endpoints. The algorithm for constructing the interpolant can be seen in algorithm (2). Note that since both the node at half a year and the node at one year is given, interpolation is not needed for the interest rate during the first year. However, it is necessary for the interest rates the other years. Thus the interval used for interpolation only contains equally spaced intervals of i.e. one year, simplifying the calculations. The distance between the nodes is called h.

Since clamped cubic spline is chosen, values of the derivative at the endpoints must be approximated. Let x_0 be the node at the end of the interval containing the shortest interest rates. Let x_{-1} be the node to the interest rate shorter than x_0, i.e $x_{-1} = 6$

Algorithm 2 Interpolation - Clamped Cubic Spline

Input:
distance between the nodes, h
values of function f at the nodes, $a_0 = f(x_0), a_1 = f(x_0 + h), ..., a_n = f(x_0 + nh)$
derivative at the endpoints, $FPO = f'(x_0)$ and $FPN = f'(x_0 + nh)$
Do:
$\alpha_0 = 3(a_1 - a_0)/h - 3FPO$
$\alpha_n = 3FPN - 3(a_n - a_{n-1})/h$
for i = 1, 2, ..., n-1 **do**
  $\alpha_i = \frac{3}{h}(a_{i+1} - a_i) - \frac{3}{h}(a_i - a_{i-1})$
end for
$l_0 = 2h$
$\mu_0 = 0.5$
$z_0 = \alpha_0/l_0$
for i = 1,2, ...,n-1 **do**
  $l_i = 4h - h\mu_{i-1}$
  $\mu_i = h/l_i$
  $z_i = (\alpha_i - hz_{i-1})/l_i$
end for
$l_n = h(2 - \mu_{n-1})$
$z_n = (\alpha_n - hz_{n-1})/l_n$
$c_n = z_n$
for j=n-1, n-2, ..., 0 **do**
  $c_j = z_j - \mu_j c_{j+1}$
  $b_j = (a_{j+1} - a_j)/h - h(c_{j+1} + 2c_j)/3$
  $d_j = (c_{j+1} - c_j)/(3h_j)$
end for
Return:
a_j for $j = 0, 1, ..., n - 1$
b_j for $j = 0, 1, ..., n - 1$
c_j for $j = 0, 1, ..., n - 1$
d_j for $j = 0, 1, ..., n - 1$
(Note: $S(x) = S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$ for $x_j \leq x \leq x_{j+1}$.)

months and x_1 be the first interest rate longer than x_0, i.e. $x_1 = 2$ years. The value of the interest rates at these nodes together with Lagrange's polynomial can be used to approximate the derivative at x_1. Lagrange's polynomial is needed because of the unequal space between the three nodes, otherwise a simpler method could have been used. Let P' be the approximated value of the derivative. Proof of the following result can be found in [10]:

$$P'(x_0) = \frac{-h_0}{h_{-1}(h_{-1}+h_0)}f(x_{-1}) + \frac{h_0 - h_{-1}}{h_{-1}h_0}f(x_0) + \frac{h_{-1}}{h_0(h_{-1}+h_0)}f(x_1). \tag{5.13}$$

h_{-1} is the length of the interval between x_{-1} and x_0 and h_0 is the length of the interval between x_0 and x_1.

The approximation of the derivative at the other endpoint cannot be found in the same way. As opposed to the endpoint previously discussed, the length of the subintervals at the end of the long-term interest rates are equal which is important for the method chosen for approximating the derivative. As can be seen in [2], the approximation of the derivative P' can be found by

$$P'(x_n) = \frac{1}{12h}[-25f(x_n)+48f(x_n-h)-36f(x_n-2h)+16f(x_n-3h)-3f(x_n-4h)]. \tag{5.14}$$

5.4 Pricing Derivatives

The references for this section are both [5] and [12]. The value of an interest rate derivative is the expected present value of its payoff under risk neutral conditions. The value can be found using Monte Carlo simulation, for more information on Monte Carlo simulation see [9]. Let $E[presentValuePayoff]$ be the expected present value of the payoff. The value can be found by generating numerous realisations of the interest rate by using the LMM, calculate the present value of the derivative and use the strong law of large numbers:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{presentValuePayoff_i}{n} = E[presentValuePayoff]. \tag{5.15}$$

A drawback of the Monte Carlo simulation is the poor convergence, an illustrating example will later be provided.

The principal used for calculation of the cashflows is assumed to be 1. Let $g(\hat{L}(T_n))$ be the payoff of the derivative at time T_n. Under the forward measure the deflated payoff is

$$g(\hat{L}(T_n)) \cdot B_{M+1}(0)\prod_{j=n}^{M}(1+\delta_j\hat{L}_j(T_j)), \tag{5.16}$$

where $B_{M+1}(0)$ is the value of a bond at time $t=0$ maturing at T_{M+1}. The algorithm for pricing a derivative can be seen in algorithm (3). It should carefully be noted that when valuing derivatives generating cashflows every 6 months, the bond B_{M+1} is the bond ending 6 months after the last cashflow of the derivative. If this bond is not quoted, its value must be estimated by the use of interpolation. Let r_{fix} be the coupon of the bond or the fixed interest rate used in the swap, cap or floor. The bond is assumed to be paid back at the end of the period specified. The deflated payoff-functions for the the derivatives are:

$$\sum_{n=1}^{M}\left[\delta_{n-1}r_{fix}\cdot B_{M+1}(0)\prod_{j=n}^{M}(1+\delta_j\hat{L}_j(T_j))\right]+1\cdot B_{M+1}(0)(1+\delta_M\hat{L}_M(T_M)) \tag{5.17}$$

for a bond,

$$\sum_{n=1}^{M}\left[\delta_{n-1}(L_M(T_{n-1})-r_{fix})\cdot B_{M+1}(0)\prod_{j=n}^{M}(1+\delta_j\hat{L}_j(T_j))\right], \tag{5.18}$$

for a swap,

$$\sum_{n=1}^{M}\left[\delta_{n-1}(L_M(T_{n-1})-r_{fix})^{+}\cdot B_{M+1}(0)\prod_{j=n}^{M}(1+\delta_j\hat{L}_j(T_j))\right], \tag{5.19}$$

for a cap and

$$\sum_{n=1}^{M}\left[\delta_{n-1}(r_{fix}-L_M(T_{n-1}))^{+}\cdot B_{M+1}(0)\prod_{j=n}^{M}(1+\delta_j\hat{L}_j(T_j))\right], \tag{5.20}$$

for a floor. Thousands of simulations are performed to estimate the derivative values. If a portfolio of several derivatives is to be valued, all the derivatives may be valued at the same time using the the same estimated interest rates to shorten the time of computation. Let a swap, cap and floor have the same maturity, paying cashflows at the same dates and having the same "fixed" interest rate r_{fix}. If the values of all the three derivatives are to be estimated, the value of either of them may be found by exploiting the cap-floor parity to reduce the time of computation.

To give an idea of the convergence of the Monte Carslo simulation a plot describing the absolute relative error when valuing a caplet is made, see figure (5.1). The value of reference of the caplet is found by the use of Blacks equation, see [5]. The caplet valued matures in 5 years, have a cap rate of 5.0% and is otherwise dependent on the American interest rates quoted on 17 Nov 2009. Note that due to discretization error, the simulated price will not in general converge exactly to the price given by Blacks equation even though the number of simulations increase towards infinity. After 10 000 simulations, the discrepancy between the two values is 0.59%.

5.5 Multinational LIBOR Market Model

So far the discussion has mainly concerned the modelling of interest rates for only one currency. However, with a few adjustments, the LMM described can be expanded to

Algorithm 3 Pricing derivatives using the LMM

Input:
variables needed for the calculation of the payoff
forward rates, $\hat{L}_n(t_i), n = 1, ..., M$
volatilities, $\hat{\sigma}_n(t_i), n = 1, ..., M$
time between tenors, $\delta_n, n = 1, ..., M$
Do:
for $k = 1$ to $numberOfSimulations$ **do**
  calculate the forward rates according to algorithm (1)
  calculate the $deflatedPayoff$
  $sumDerivative = sumDerivaive + deflatedPayoff$
end for
$valueDerivative = sumDerivative/numberOfSimulations$
Return:
$valueDerivative$

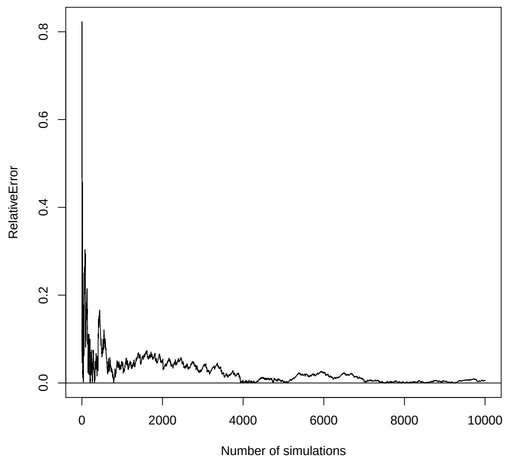


Figure 5.1: The absolute value of the relative error of a cap maturing in 5 years, having a cap rate of 5.0% and otherwise be dependent on the American interest rates quoted on 17 Nov 2009. The relative error decreases as the number of simulations increases.

be applicable for several currencies at the same time. The adjustments will result in a multinational model which will be more realistic as the interest rates for a currency is dependent on interest rates for other currencies.

Some remarks should be noted about the expansion of the LMM:

- The drift μ in (5.1) is only dependent on other interest rates of the same currency in the multinational as well as the regular LMM. I.e. the drift of a European interest rate is only dependent on other European interest rates. It follows that the calculation of the drift does not change when the model is adjusted to accept several currencies.
- The covariance matrix of the multinational model includes all the interest rates from all the currencies. This is the biggest difference between the multinational and the regular LMM and it is what makes the model multinational as all the normal correlated random variables $Z_n(t_i)$'s from all the currencies are dependent on each other.

It should also be noted that the currencies are interpolated individually. This means that when interpolating, the only variables determining the value of an intermediate interest rate are other interest rates of the same currency. At last it should be noted that derivatives in this thesis will only be dependent on interest rates based on one currency even though the contrary is possible.

5.6 Value at Risk and Expected Tail Loss

The references for this section are [3] and [4]. Calculating the Value at Risk (VaR) and the expected tail loss (ETL) is done by simulation as the distribution of the profit/loss (P/L) only can be found in this way. The P/L over the time period s for the asset A is

$$P/L_s = A_{t+s} + a_t - A_t, \tag{5.21}$$

where A_{t+s} is the value of the asset at time $t + s$ and a_{t+s} is any interim payments. The definition can be used for a portfolio as well letting A be several assets and not only one. Incorporating the time value of money is necessary to make the definition theoretically correct. It is done by valuing the portfolios either at time t or at time $t + s$. As will be clear later, this might be difficult for some time periods. Letting P be the value of a portfolio, the distribution of P/L can be found by this stepwise procedure.

1. Find the value of P at time t, P_t, as described in section (5.4), but do not use the values for finding the mean. Instead the values should be kept for further calculations.
2. Find the value of $PSim_{t+s}$: Simulate all the interest rates $L_n(t_i)$ from time t to time $t + s$ by using the real and not the risk neutral drift μ. This is done by replacing the risk neutral drift $\mu_n(\tilde{L}(t_i), t_i)$ in (5.2) with the real drift

$$\mu_n(t + s) = \lambda\mu_n(t) + (1 - \lambda)(\tilde{L}(t_i) - \tilde{L}(t_{i-s})), \tag{5.22}$$

where λ $(0 < \lambda < 1)$ is the decay factor. The value of $PSim_{t+s}$ is found as it was described in section (5.4), but by using the simulation of the real interest rate as basis. One path of the simulated interest rates $L_n(t_{i+s})$, should be used for basis for one path of the simulated value of $PSim_{t+s}$. The values of the portfolio should not be used for finding the mean, but rather kept for the calculations of the P/L's.

3. Calculate the P/L for each pair of P_t and $PSim_{t+s}$. The values are called $P/LSim_{t+s}$.

The accuracy of the P/L distribution increases as the number of paths and number of P/L pairs increases, but so does the computing time. A compromise must therefore be made between the accuracy of the calculations and the computing time.

Some drawbacks of the stepwise procedure described should be mentioned. The portfolio is valued at two different times, t and $t+s$ and the time value of money is not incorporated. As mentioned earlier, this is not correct, but the time period s is often quite short: a day or a month. As bonds maturing later than a year are only quoted every whole year, it is difficult to correct this flaw. It should also be mentioned that a portfolio generating the next cashflow in 6 months should after the period s generate the next cahsflow in 6-s months. This will not be taken into account and the portfolio will be valued at time $t+s$ as if the next cashflow occurs in 6 months. Another drawback is the calculation of the bond B_{M+1}. At time t, this is obtained by the interest rates given, but at time $t+s$ they have to be obtained by the estimated forward rates using the real drift. As all these interest rates are forward rates, they cannot calculate the bond $B_{M+1}(t+s)$ and therefore a rather rough estimate is used: When forward rates are calculated every 6 months, the forward rate for the period 6-12 months is used as an estimate for the interest rate for the period 0-6 months. When forward rates are only calculated every year, the forward rate for the period 1-2 years is used as an estimate for the interest rate for the period 0-1 year. At last it should be mentioned that the covariance matrix used at time $t+s$ for valuing the portfolio is the same as used at time t. This is a simplification making the calculation easier.

Finding the VaR and ETL is relatively easy when the distribution of P/L is calculated. Assuming n simulations and letting the confident level be c, the risk measures can be found:

1. Remove the $n\frac{c}{100}$ lowest values from the sample.

2. The VaR is the lowest value left in the sample.

3. The ETL is the average of the sample removed.

5.7 Backtesting the Value at Risk and the Expected Tail Loss

The principle of backtesting is to test whether the model used provides satisfying results. It is carried out using historic values and tests whether the model performs well or not.

The main focus of the backtesting in this thesis is the two risk risk measures VaR and ETL.

The bactesting is performed in the following way:

1. Estimate the VaR and ETL at time t over a time horizon s as described in section (5.6).

2. Estimate the real value of the portfolio P at time $t + s$, P_{t+s}, using real interest rates as it was described in section (5.4). Do not use the values of the portfolio to estimate the mean, but keep the values for calculation of the P/L.

3. The P/L for each pair of P_t and P_{t+s} is found and called P/L_{t+s}.

4. The values of P/L_{t+s} are compared to the estimated VaR and ETL.

5. Move a time s forward in the data and repeat the procedure.

It is important to remember to update the covariance matrix used in the calculations each time one returns to step one in the procedure. It s clear that the variance and covariance is not stable and therefore this update is important. It is also important to remember that if the risk measures with multiple days s are tested, each period of length s cannot overlap another period of length s. This is because each period is supposed to be independent and overlapping would contradict this assumption. The result is that fewer periods are tested when the horizon is multiple days than when the horizon is only one day. I.e. if the data used for testing a method consist of 252 days, a one day horizon would have 252 periods for testing while a horizon of one month would have 12 periods available for testing assuming 21 days in one month. In this thesis it will be assumed that there are 252 business days in one year and 21 in one month.

Chapter 6

Preliminary Data Analysis

The data used for backtesting the risk measures is analysed in this chapter.

6.1 Volatility and Correlation

The volatility and correlation is assumed to be constant over time. A part of the preliminary data analysis is to test this assumption, and as was seen in chapter (3) this assumption does not hold. When it comes to the future volatilities and correlations, they are not known. The best way to estimate them is therefore by calculating the present volatilities and correlations and use them as an estimate, even though they are known not to be constant over time. For more details on volatility and correlation, see chapter (3).

6.2 Independence

The reference for this section is [1]. The random variables in equation (5.1) are assumed to be dependent on other random variables occurring at the same time, but independent of all random variables occurring at another time. Independence of random variables occurring at different times will be discussed in this section. A random variable is autocorrelated if its returns are statistically dependent over time. It is therefore possible to test whether a random variable is statistically independent by testing if it is autocorrelated with other random variables. Autocorrelation is defined by

$$\varrho_k = \frac{\sigma_{t,t-k}}{\sigma_t \sigma_{t-k}}, \tag{6.1}$$

where ϱ_k is the autocorrelation of order k, $\sigma_{t,t-k}$ is the covariance between the t'th and $t-k$'th time steps and σ_t and σ_{t-k} is the standard deviation at time t and $t-k$ respectively. Note that since the autocorrelation operates only on one time series, the subscript refers to the time index and not a specific time series. The autocorrelation of order k is also called the lag-k autocorrelation. As the data sample tested for independence is large compared to the lags tested, the standard deviations are assumed to be equal which results in

$$\varrho_k = \frac{\sigma_{t,t-k}}{\sigma_t \sigma_{t-k}} = \frac{\sigma_{t,t-k}}{\sigma_t^2}. \tag{6.2}$$

The sample autocorrelation is given by

$$\hat{\varrho}_k = \frac{\frac{1}{T-(k-1)} \sum_{t=k+1}^{T} (r_t - \overline{r})(r_{t-k} - \overline{r})}{\frac{1}{T-1} \sum_{t=1}^{T} (r_t - \overline{r})^2}, \tag{6.3}$$

where r_t is the return at time t, k is the lag and $\overline{r}$ is the sample mean. When using the autocorrelation to test the independence of the residuals in the LMM, r is replaced by Z. Z is calculated by

$$Z_n(t_{i+1}) = \left[\ln \left(\frac{\tilde{L}_n(t_{i+1})}{\tilde{L}_n(t_i)} \right) - \left(\mu_n(\tilde{L}(t_i), t_i) - \frac{1}{2} \sigma_n(t_i)^2 \right) (t_{i+1} - t_i) \right] / \sqrt{t_{i+1} - t_i} \sigma_n(t_i), \tag{6.4}$$

where

$$\mu_n(\tilde{L}(t_i), t_i) = - \sum_{j=n+1}^{M} \frac{\delta_j \tilde{L}_j(t_i) \sigma_n(t_i) \sigma_j(t_i)}{1 + \delta_j \tilde{L}_j(t_i)}. \tag{6.5}$$

A time series with randomly generated steps will not be autocorrelated and the values of $\hat{\varrho}$ will not differ significantly from zero. The autocorrelation of the US forward rate for the period $0.5 - 1$ year can be seen in figure (6.1). The first lag is noticeable larger in absolute value than the other lags and most likely there is a dependence between r_t and r_{t-1}.

Another measure illustrating the dependency of the random variables in the time series is the autocorrelation of squared returns which is also called the autocorrelation of the variances of returns. Commonly the autocorrelation of squared returns imply dependency even though the autocorrelation of returns do not, and it is therefore common to analyse both measures. The sample autocorrelation of the squared return is

$$\hat{\varrho}Sq_k = \frac{\frac{1}{T-(k-1)} \sum_{t=k+1}^{T} (r_t^2 - \overline{r}^2)(r_{t-k}^2 - \overline{r}^2)}{\frac{1}{T-1} \sum_{t=1}^{T} (r_t^2 - \overline{r}^2)^2}. \tag{6.6}$$

The autocorrelation of squared returns of the US forward rate for the period $0.5 - 1$ year can be found in figure (6.2). The dependence of the first lag is more distinct than was seen in figure (6.1). There is no clear evidence of dependence except from the first lag.

Even though there is some evidence of dependency, independence will be assumed. Dependency between lags complicate the calculations and is beyond the scope of this thesis.

6.3 Normality

The residuals are calculated according to (6.4) and assessed for multi normality as suggested in [6]. The normality behaviour will be checked in one and two dimensions because of the lack of a "good" overall test of joint normality in more than two dimensions.

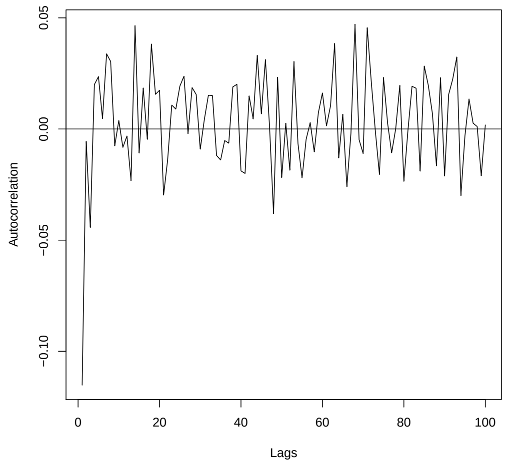


Figure 6.1: The autocorrelation of the US forward rate for the period $0.5 - 1$ year using the EWMA model to estimate the volatility and correlation. 3 months of data is used to estimate the covariance matrix. Significant values of the autocorrelation imply dependency between the lags. The first lag is noticeably larger in absolute value than the other lags and is the only lag that may be dependent.

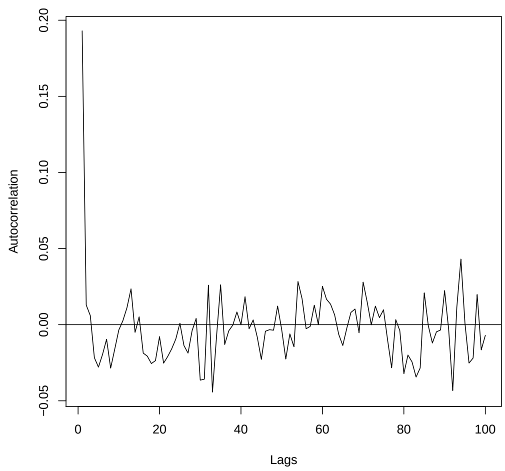


Figure 6.2: The autocorrelation of squared returns for the US forward rate for the period $0.5 - 1$ year using the EWMA model to estimate the volatility and correlation. 3 months of data is used to estimate the covariance matrix. Significant values of the autocorrelation of squared returns imply dependency between the lags. The first lag is noticeably larger than the other lags, no other lag is likely to be dependent.

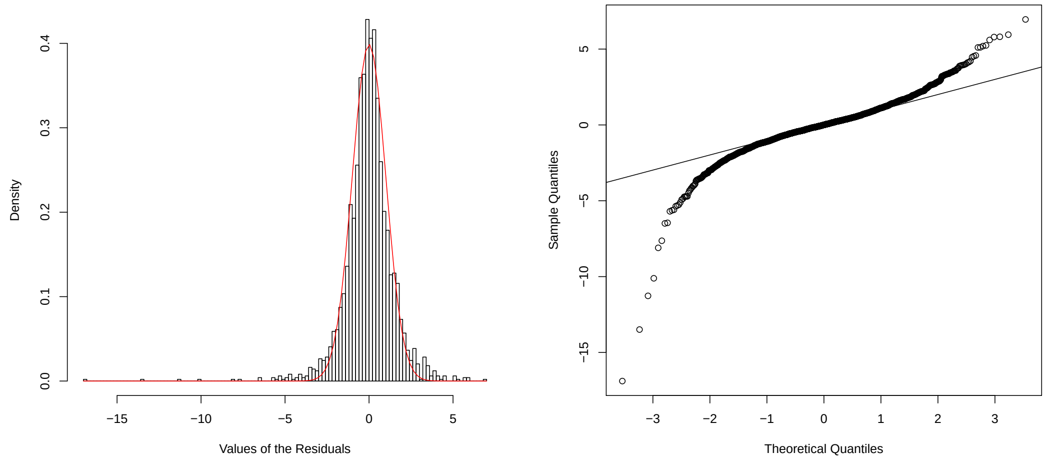


Figure 6.3: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $0,5-1$ year. The correlation and the volatility is estimated by the use of 3 months of data and the SMA model. Notice especially that the tails are thicker than the tails of the normal distribution.

This means that there is always a possibility that nonnormal behaviour only revealed in higher dimensions will never be detected. Fortunately many types of nonnormality are often reflected in the marginal distribution, and for most practical work one- and two-dimensional investigations are sufficient.

Histograms and QQ plots have been constructed for all the 18 interest rates, using both the SMA model and the EWMA model for estimating the variances and covariances. All the plots can be seen in appendix (B). Normally the distribution of financial data has fatter tails than the normal distribution. In addition the distribution is higher and narrower that the normal distribution. These characteristics are also seen in the residuals of the LMM as can be seen in the appendix. The distributions seem to have some differences dependent on which estimate is used as can be seen in the figures (6.3) and (6.4). The "SMA-distribution" has more outliers than the "EWMA-distribution", however the "EWMA-distribution" is more centralized around zero.

Three scatter plots are constructed to investigate the bivariate behaviour, each plot contains the interest rate of one currency. The correct use of scatter plots would be to include all the interest rate in one plot, but as this would be a 18×18 matrix three smaller plots are preferred. Scatter plots from normal distributions have elliptical shapes. The plot containing the American interest rates can be seen in figure (6.5) while all the plots can be seen in appendix (B). All the plots are close to having an elliptical shape, except from some outliers.

The assessment of the normal behaviour has made it is clear that the normal distribution does not describe the distribution of the residuals perfectly, but it may be sufficient

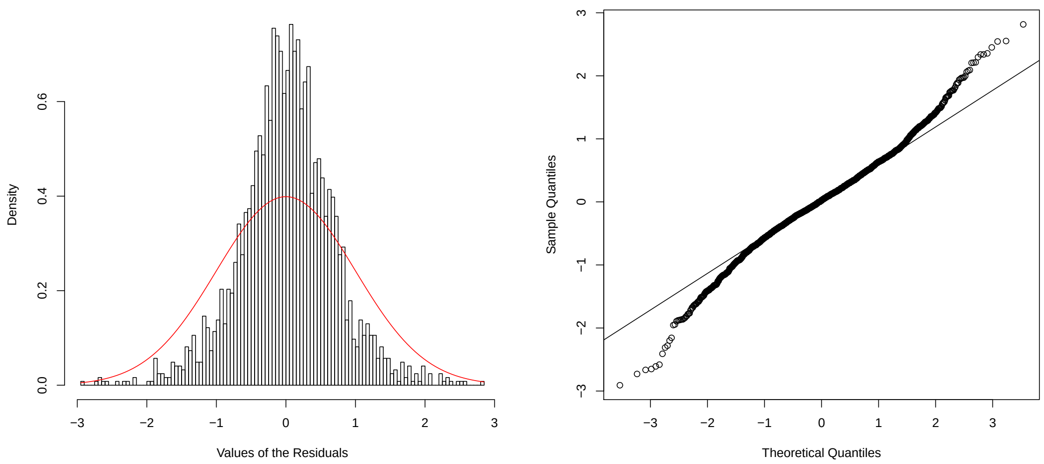


Figure 6.4: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $0.5 - 1$ year. The correlation and the volatility is estimated by the use of 3 months of data and the EWMA model. Notice in particular that the residuals are more centralized around the zero than the normal distribution.

for the purposes at hand. Thus the distribution is chosen because of its relative good fit and because it is easy to implement.

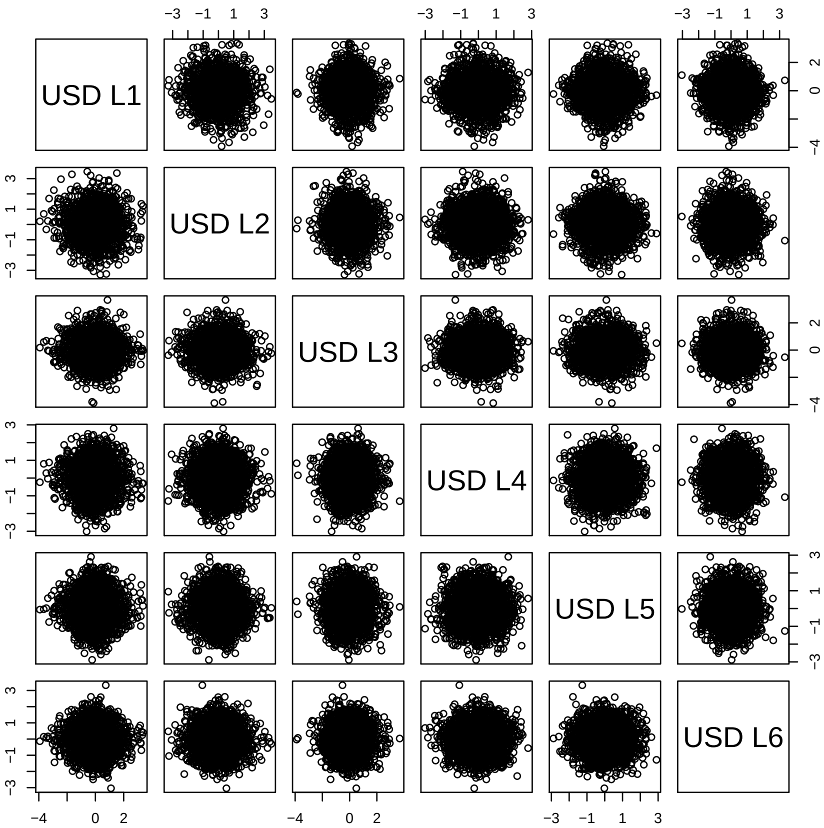


Figure 6.5: The scatter plot of the residuals of the American interest rates. The covariance matrix is estimated by using 3 months of data and the EWMA model. The ij'th scatter plot contains the residuals of L_i plotted against L_j. L_1 is the forward rate for the period 6-12 months, L_2 is the forward rate for the period 1-2 years, L_3 is the forward rate for the period 2-3 years etc. Scatter plots of normal distributed variables have elliptical shapes which these plots seem to have.

Chapter 7

Results

The results of the backtesting are presented in this chapter.

7.1 Description of Data

The data used in this thesis consists of interest rates quoted from 4 Jan 1999 to 17 Nov 2009 for the three currencies USD, EURO and NOK.[1] The interest rates are valid from the day they are quoted to the maturity date and are therefore not forward rates. Forward rates are obtained by first calculating the values of zero-coupon bonds and then use (2.11) to calculate the forward rates. The forward rates for the period 5-6 years for the three currencies can be seen in figure (7.1).

7.2 Portfolio

The portfolio used for testing is intended to resemble a multinational bank's portfolio of mortgages. All the three currencies will be used, but the main focus will be on American mortgages. The portfolio will consist of two types of derivatives: The first derivative is a coupon-bearing bond. This derivative imitates fixed-rate mortgages even though the principal is repaid on the maturity date and not partly throughout the term of the loan. The other derivative is a cap but with a minor alteration: Instead of a notional principal, the principal is repaid at the maturity date. Another way of describing this derivative is as a mix between a cap and a zero-coupon bond. The cap ensures payments both before and on the maturity date while the bond ensures the repayment of the loan at the maturity date. This derivative imitates adjustable-rate mortgages with a cap even though as for the first derivative, the principal is repaid on the maturity date and not partly throughout the term of the loan. Two last remarks should be noted about both the derivatives: The principal is always set to 1 unit and the maturity of the derivatives will be 5 years but no longer as the maximum time to maturity of the interest rate in the data used is 5 years. The terms of loans are usually longer than 5 years, so this is

[1]The data is provided by my supervisor Jacob Laading.

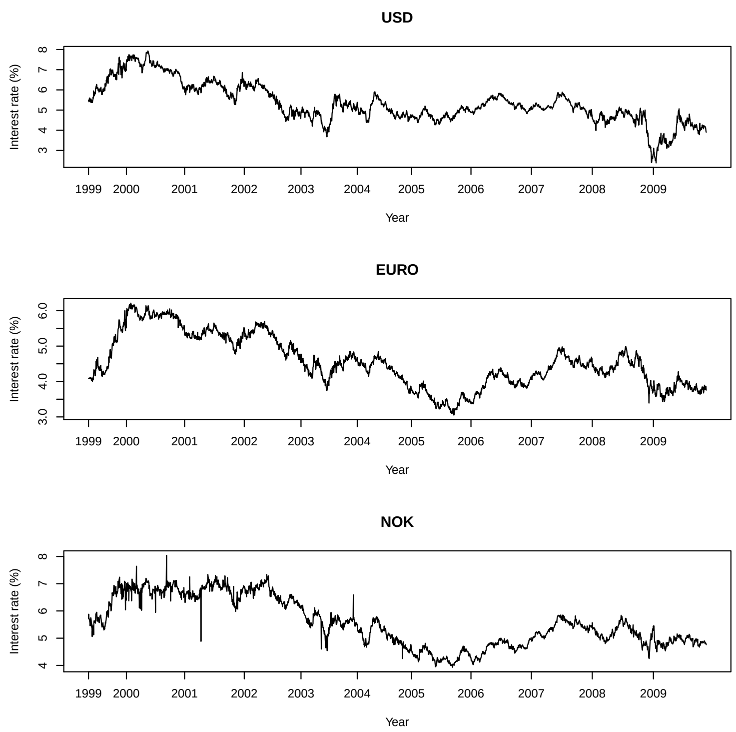


Figure 7.1: Forward rates for the period 5-6 years for the three currencies USD, EURO and NOK.

Number	Currency	Derivative	Fixed interest rate (%)
1	USD	Coupon Bond	$r_t + 0.25$
2	USD	Coupon Bond	$r_t + 0.50$
3	USD	Coupon Bond	$r_t + 0.75$
4	USD	Coupon Bond	$r_t + 1.00$
5	USD	Cap and Zero-coupon Bond	$r_t + 1.50$
6	USD	Cap and Zero-coupon Bond	$r_t + 2.00$
7	EURO	Coupon Bond	$r_t + 0.50$
8	EURO	Coupon Bond	$r_t + 1.00$
9	EURO	Cap and Zero-coupon Bond	$r_t + 2.00$
10	NOK	Cap and Zero-coupon Bond	$r_t + 2.00$

Table 7.1: An overview of the derivatives in the portfolio used for testing the VaR and ETL. The derivatives resemble a multinational bank's portfolio of mortgages and derivatives using the American, European and Norwegian interest rates are included. The fixed interest rates used in the derivatives are dependent on the interest rates of each day of the backtesting.

a drawback of the derivatives chosen. The exact portfolio chosen can be found in table (7.1). The fixed interest rates used to decide the size of the cashflows for the derivatives are dependent on the interest rate of the day of the backtesting. I.e. if the interest rate for USD at time t for the forward rate between 5 and 6 years is 4.00%, the coupon of the first bond in table (7.1) is 4.25%.

The value of the portfolio is calculated with both derivatives generating cashflows every whole year and every 6 months. The values are first calculated for the portfolio with annual cashflows then for the portfolio with semi-annual cashflows. As interpolation is necessary when cashflows are generated semi-annually the estimates are more rough and less precise. An example of the values of the derivatives in the portfolio can be found in table (7.2). Notice that the values of the derivatives with annual cashflows are generally worth more than the derivatives with semi-annual cashflows. As the bank receives the payments earlier when the cashflows are generated semi-annually, these derivatives should have a greater value than the derivatives generating annual cashflows. This odd behaviour is an example that illustrates the need for including the portfolio with annual cashflows in this thesis. The calculations are simpler and the measures are not influenced by the effects of the interpolation.

7.3 The Backtesting

The backtesting will test the performance of the VaR and the ETL using all the data provided. Two time horizons are tested, one day and one month, and three confidence levels, 95%, 97.5% and 99%. 3 months of data is used to estimate the covariance matrix

Number	Annual Cashflows	Semi-Annual Cashflows
1	1.09	1.08
2	1.11	1.09
3	1.12	1.11
4	1.13	1.12
5	1.07	1.06
6	1.07	1.06
7	1.09	1.07
8	1.11	1.09
9	1.06	1.04
10	1.04	1.02
Total	10.89	10.74

Table 7.2: The table shows the values of the derivatives described in table (7.1) the last day of the dataset. Data from the last 3 months is used for the calculation of the covariance matrix based on the EWMA estimate and 10 000 simulations are used to calculate the values of the portfolio. Notice the odd values of the derivatives: The values of the derivatives generating annual cashflows are larger than the derivatives generating semi-annual cashflows.

either based on the SMA model or the EWMA model and 10 000 simulations are carried out each day.

The VaR measure is tested by verifying that the amount of data within the VaR is correct. The confidence level determines how many percentages of the real profits and losses that are expected to be within the measure. I.e. if the confidence level is set to 95%, 95% of the profits and losses are expected to be within the VaR and only 5% of the losses are expected to be greater. The values in the last 5% will cause the investor a greater loss than the VaR. The VaR can be examined by plotting the expected percentage and the actual percentage of values within the VaR. The mean of how many percentages of the real values that are within the VaR also gives information on the overall performance of the measure.

The ETL is assessed by calculating the mean of the real losses greater than the VaR, the tail losses, each day and compare them to the ETL of each day. The relative error between the ETL and the mean of the tail losses each day are calculated in the following way

$$\text{Relative error at time } t = \frac{\text{Mean}(TL_t) - ETL_t}{ETL_t}, \tag{7.1}$$

where the TL_t's are the tail losses. The errors are displayed in a histogram to give an overview of the distribution of the relative errors. Some extreme values are excluded from the plots to better display the rest of the data. Another interesting measure is the variance of the tail losses each day. The ETL does not describe this measure, never the less it is interesting to investigate it to get an idea of the actual possible losses. The variances are plotted in a histogram to give an idea of the different variances produced each day. Some extreme values are excluded from the plots to better display the rest of the data.

7.4 Value at Risk and Expected Tail Loss with a Daily Horizon

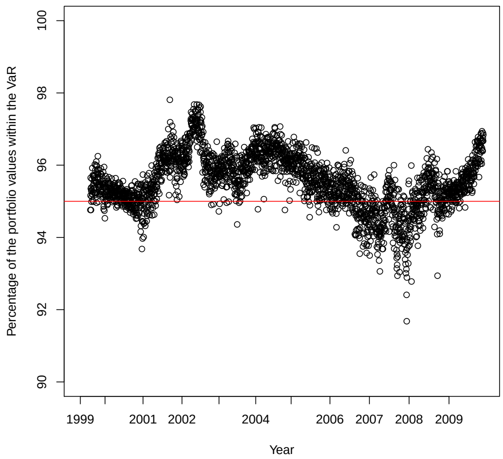
VaR

The accuracy of the VaR of the portfolio generating annual cashflows is tested and displayed in figure (7.2). Only the two confidence levels 95% and 99% are displayed in the figure, but all the plots can be seen in appendix (C). The accuracy of the VaR varies and there is a clear difference between the performance of the measure depending on which estimate is used for the covariance matrix. The model using the EWMA estimate performs better than the model using the SMA estimate. The model is both more accurate and more stable than the SMA model. Another interesting observation is that the VaR performs better when the confidence level increases. This could be caused by the poor fit of the normal distribution to the actual distribution. It is often observed that the fit is particularly poor where the normal density function bends and the tail begins. This might be why the 95% confidence level performs poorly. The combination of the EWMA model and the 99% confidence level is a particularly good match and the VaR performs better for this combination than any other tested.

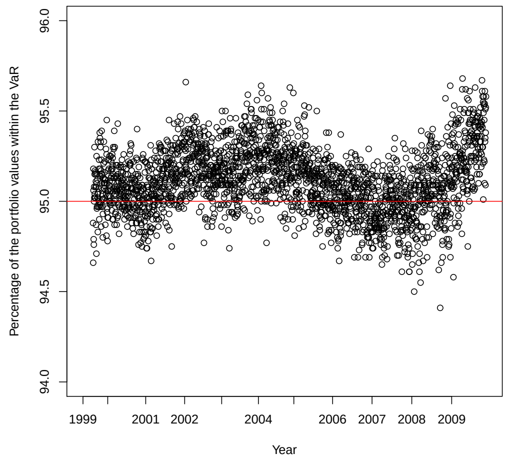
An interesting observation is made by comparing the interest rates in figure (7.1) with the results in figure (7.2): The accuracy of the VaR seems to be affected by rapid changes in the interest rate. When the interest rates change rapidly in the negative direction, the percentage of the portfolio's profits and losses within the VaR declines. When the interest rates change rapidly in the positive direction, the percentage of the portfolio's profits and losses within the VaR increases. This behaviour is especially apparent for the SMA model, but some of the same tendency is observed for the EWMA model at the 95% confidence level.

The mean percentage of the profits and losses within the VaR each day is calculated and displayed in table (7.3). Some of the same tendencies described above are also seen in this table: The EMWA model performs better than the SMA model even though the mean percentages of the model using the SMA estimate are higher. The means should be as close to the given confidence levels as possible, and it is clear that the model using the EWMA estimate performs best.

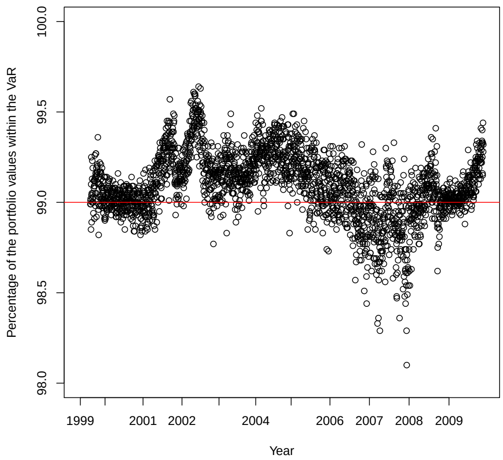
The VaR of the portfolio with semi-annual cashflows is also tested and the results can be seen in figure (7.3) for the two confidence levels 95% and 99%. All the plots can be seen in appendix (C). Both the EWMA model and the SMA model performs poorly, but the SMA model performs worst and it is extremely unstable. The EWMA model is both more accurate and more stable than the SMA model. The same but less apparent tendency is observed for the portfolio with annual cashflows. It is also clear that the



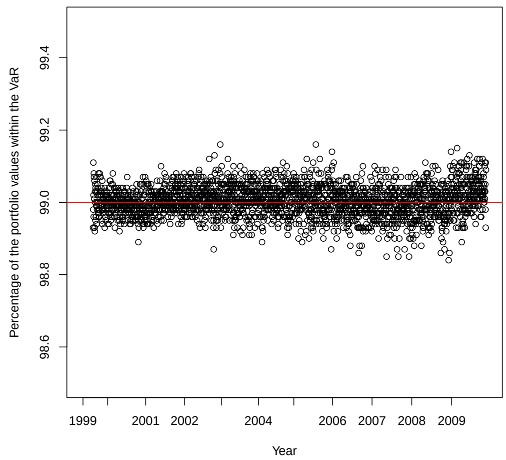
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level



(c) SMA estimate, 99% confidence level



(d) EWMA estimate, 99% confidence level

Figure 7.2: Percentages of the portfolio's profits and losses within the VaR using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies. Both the model using SMA estimate and the model using the EWMA estimate perform reasonably well, but the model using the EWMA estimate generally performs better.

Confidence level (%)	**Mean Percentage Within the VaR, SMA estimates (%)**	**Mean Percentage Within the VaR, EWMA estimates (%)**
95.0	95.5	95.1
97.5	97.8	97.5
99.0	99.1	99.0

Table 7.3: The table displays the means of the percentages of the portfolio's profits and losses within the VaR each day as well as the given confidence level. The portfolio tested is generating annual cashflows and has a time horizon of one day. The model using the EWMA estimate is preferred as these results are more accurate than the models using the SMA estimate.

Confidence level (%)	**Mean Percentage Within the VaR, SMA estimates (%)**	**Mean Percentage Within the VaR, EWMA estimates (%)**
95.0	38.9	88.8
97.5	52.6	95.9
99.0	69.0	98.7

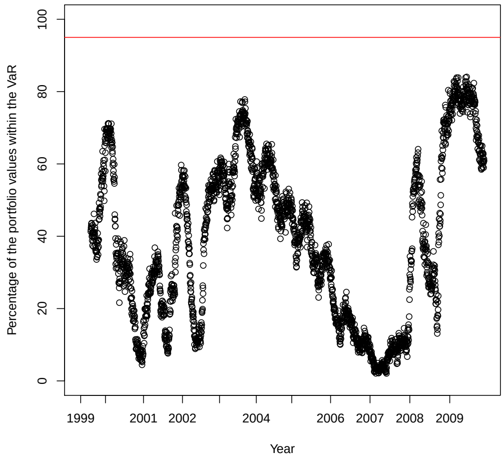
Table 7.4: The table displays the means of the percentages of the portfolio's profits and losses within the VaR each day as well as the given confidence level. The portfolio tested generates semi-annual cashflows and the time horizon is one day. Notice that the model using the EWMA estimate performs significantly better than the model using the SMA estimate and that both the models perform better when the confidence level increases.

VaR performs better when the confidence level increases. This tendency is also more apparent for the semi-annual portfolio than the annual portfolio. In general both the EWMA model and the SMA model performs worse for the semi-annual portfolio than the annual portfolio.

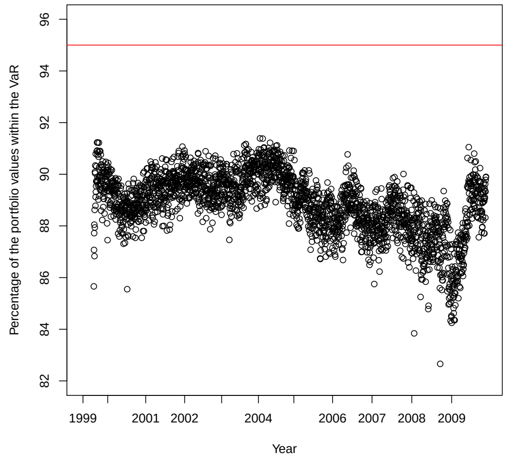
The mean percentage of profits and losses within the limit of the VaR each day for the semi-annual portfolio is calculated and displayed in table (7.4). Some of the same tendencies described above are also seen in this table: The method using the EWMA estimates outperforms the method using the SMA estimates and the accuracy of the VaR becomes better as the confidence level increases. Again it is made clear that both models perform worse for the semi-annual portfolio than the annual portfolio.

ETL

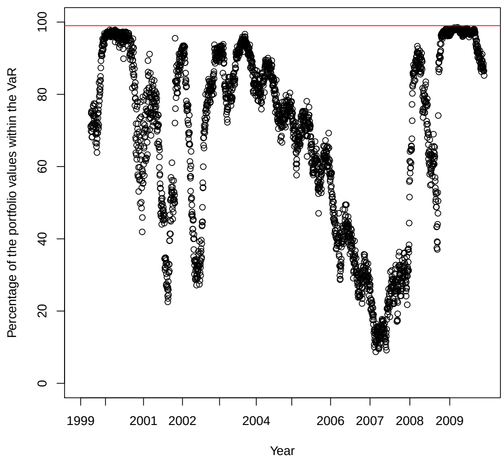
The ETL is assessed by examining the mean of the tail losses, the losses larger than VaR, each day and compare it to the ETL. The results for the portfolio with annual cashflows are seen in figure (7.4), for the 95% confidence level. All the plots can be seen



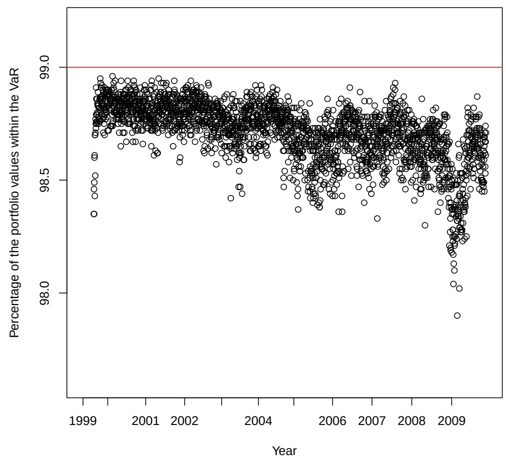
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level

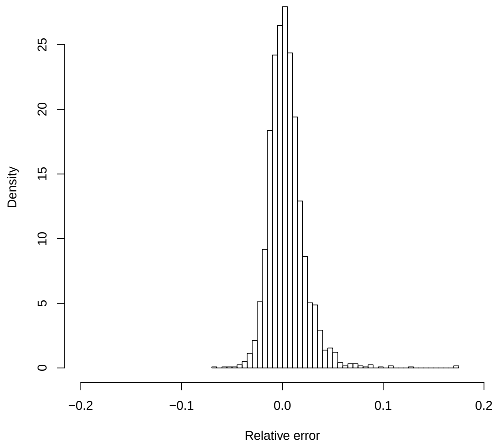


(c) SMA estimate, 99% confidence level

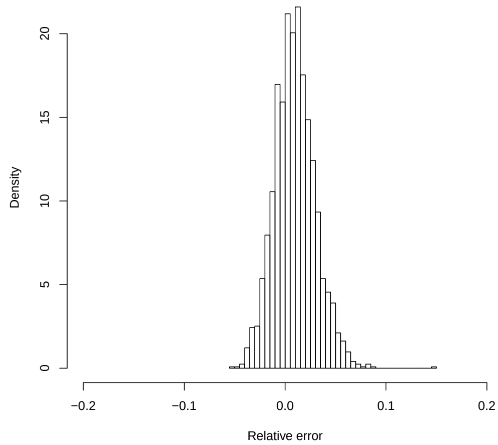


(d) EWMA estimate, 99% confidence level

Figure 7.3: Percentages of the portfolio's profits and losses within the VaR using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates semi-annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies. The estimated VaR using the SMA model is quite poor while the VaR using the EWMA estimate generally performs better.



(a) SMA estimate, 95% confidence level



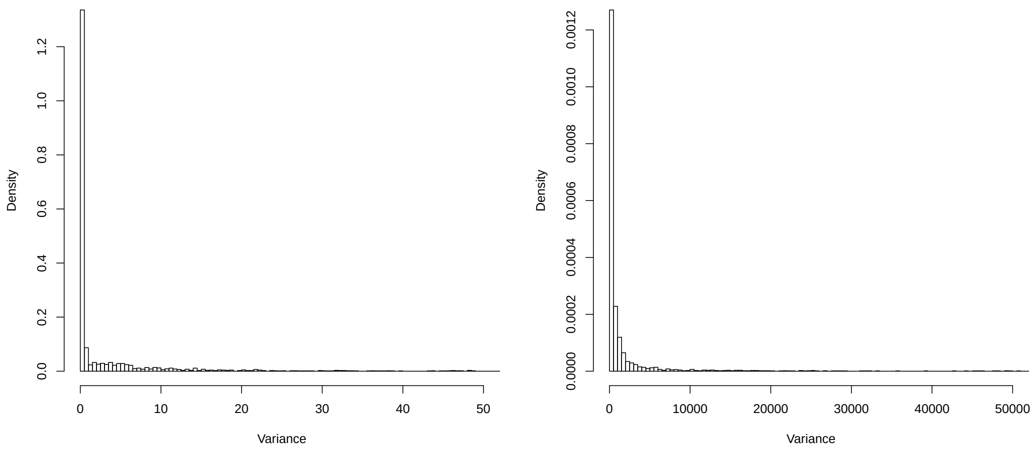
(b) EWMA estimate, 95% confidence level

Figure 7.4: Histograms of the relative error between the ETL and the mean of the tail losses each day. The portfolio tested generates annual cashflows and both the SMA model and the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to better display the rest of the data. Both the models seem to perform quite well as the mean of the relative error seems to be about zero and the variance is relatively low.

in appendix (C). Both the SMA model and the EWMA model perform quite well. The relative error seems to be centralised around zero for both models, and the variance is not too large. The performance of the ETL is connected to the performance of the VaR. If the VaR does not perform well, either more or less than expected tail losses will occur and this might affect the performance of the ETL. The relatively good performance of the ETL is therefore partly due to the relatively good performance of the VaR.

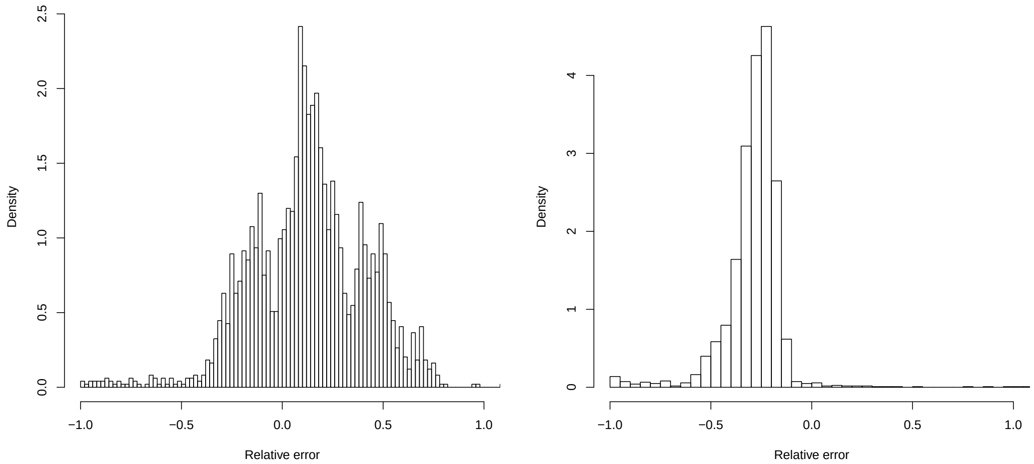
The variance of the tail losses each day for the annual portfolio are displayed in figure (7.5) for the 95% confidence interval. All the plots can be seen in appendix (C). The variance of the EWMA model is large compared to the SMA model which implies that the EWMA model produces more extreme values than the SMA model. It is not possible to say if this is good or bad as the objective is to imitate the real world. Further studies are needed to draw a conclusion, but this is beyond the scope of this thesis. The same behaviour is found for the confidence levels 97.5% and 99%. The plots emphasise that extreme variances occur which means that extreme losses occur even though they are rare. One should therefore keep in mind that even though both VaR and ETL perform well, larger losses are possible.

The mean of the tail losses are calculated for each day and compared to the ETL for the portfolio with semi-annual cashflows. The result of the test with 95% confidence



(a) SMA estimate, 95% confidence level (b) EWMA estimate, 95% confidence level

Figure 7.5: Histograms of the variances of the tail losses each day for the portfolio with annual cashflows. Some extreme values are left out of the histograms to better display the remaining data. Notice that the variance of the EWMA model is large compared to the SMA model. It should also be commented that even though most of the variances are relatively low, a few are extremely large.



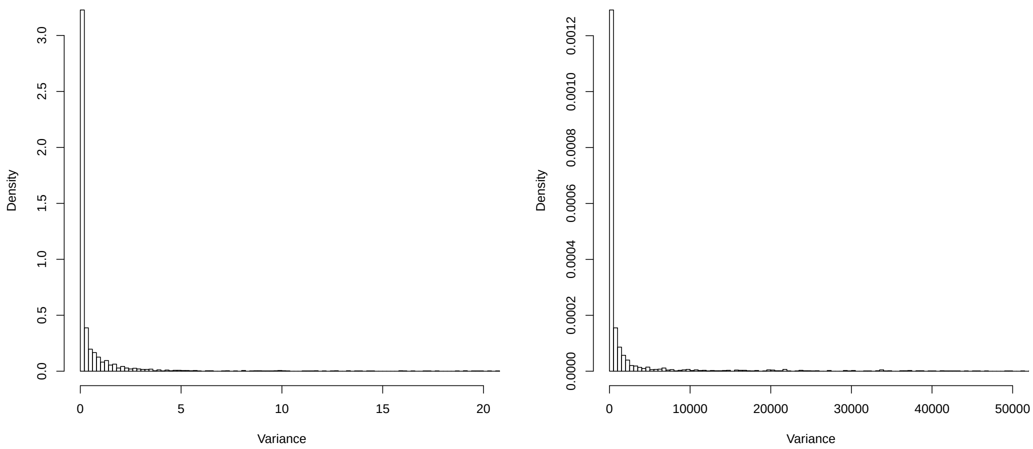
(a) SMA estimate, 95% confidence level

(b) EWMA estimate, 95% confidence level

Figure 7.6: Histograms of the relative error between the ETL and the mean of the tail losses each day. The portfolio tested generates semi-annual cashflows and both the SMA model and the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to better display the rest of the data. Notice that the variance of the relative error is greater for the SMA model. Notice also the difference of the mean of the relative errors for the two models.

level can be seen in figure (7.6), while all the plots can be found in appendix (C). There is a great difference between the results depending on which estimate is used for the covariance matrix. The mean of the relative error for the EWMA model is negative and relative few values at all are greater than zero. The mean of the relative error for the SMA model is positive and the variance is clearly larger. It is important to remember that the VaR estimate for the semi-annual portfolio performs poorly and that the performance of the ETL is affected. It is likely that at least some of the poor performance of the ETL is due to the poor performance of the VaR. The similar plot with a 99% confidence interval performs better in the sense that the mean of the relative error is closer to zero and that the variance is smaller. It is clear that the ETL for the annual portfolio performs better than the ETL for the semi-annual portfolio.

The variances of the tail losses each day are also examined for the semi-annual portfolio. The plots generated from the 95% confidence level can be seen in (7.7) while all the plots can be seen in appendix (C). The model using the EWMA estimate has in general larger variance of its tail losses than the model using the SMA estimate as was seen for the annual portfolio. Some variances are extremely large and one should therefore keep in mind that large losses may occur.



(a) SMA estimate, 95% confidence level (b) EWMA estimate, 95% confidence level

Figure 7.7: Histograms of the variances of the tail losses each day for the portfolio with semi-annual cashflows. Some extreme values are left out of the histograms to better display the remaining data. Notice that the variance of the EWMA model is large compared to the SMA model. It should also be commented that even though most of the variances are relatively low, a few are extremely large.

Confidence level (%)	Mean Percentage Within the VaR, SMA estimates (%)	Mean Percentage Within the VaR, EWMA estimates (%)
95.0	95.6	95.3
97.5	97.8	97.6
99.0	99.1	99.0

Table 7.5: The table displays the means of the percentages of the portfolio's profits and losses within the VaR each day as well as the given confidence level. The portfolio tested is generating annual cashflows and has a time horizon of one month. The model using the EWMA estimate is preferred as these results are more accurate than the models using the SMA estimate.

7.5 Value at Risk and Expected Tail Loss with a Monthly Horizon

All plots of the results where the horizon is one month are left out of this chapter because they contribute little new information. The same tendencies are discovered when the monthly horizon is tested as when the daily horizon is tested. All the plots are displayed in appendix (C).

VaR

The VaR for the portfolio with annual cashflows generally performs quite well. The EWMA model performs better than the SMA model in the sense that it is more accurate and more stable. There are in general too many profits and losses within the VaR for the SMA model and some of the same behaviour is observed for the EWMA model. The effect decreases as the confidence level rises. The same tendencies are seen in similar tests with a daily horizon. The VaR seems to perform worse when the horizon is one month instead of one day. This is to be expected: The further into the future one wishes to make a prediction, the more inaccurate the estimate becomes due to the fact that some of the key assumptions become less and less true as the horizon increases. The means of the percentages within the VaR are calculated and displayed in table (7.5). The means show that too many profits and losses are within the VaR for the SMA model.

The VaR of the semi-annual portfolio is tested and the performance of the estimate varies greatly depending on whether the SMA model or the EWMA model is used to estimate the covariance matrix. The SMA model performs terribly as both the accuracy and the stability is poor. The VaR for both the EWMA model and the SMA model improves when the confidence level increases. The means of the percentages within the VaR are calculated and displayed in table (7.6). Comparing this table to the similar table for the daily horizon, table (7.4), makes it clear that the VaR performs better with a daily horizon than a monthly horizon.

Confidence level (%)	Mean Percentage Within the VaR, SMA estimates (%)	Mean Percentage Within the VaR, EWMA estimates (%)
95.0	32.0	88.9
97.5	45.9	95.8
99.0	62.7	98.7

Table 7.6: The table displays the means of the percentages of the portfolio's profits and losses within the VaR each day as well as the given confidence level. The portfolio tested generates semi-annual cashflows and the time horizon is one month. Notice that the model using the EWMA estimate performs significantly better than the model using the SMA estimate and that both the models perform better when the confidence level increases. It is clear that the VaR performs far better for the annual portfolio than the semi-annual portfolio.

ETL

The mean of the tail losses are compared to the ETL for each day and it is clear that the ETL for the portfolio with annual cashflows performs well even though the horizon is a month. The relative error is centralised around zero and the variance is not too large. The variance of the tail losses with a monthly horizon behaves as the variances of the tail losses with a daily horizon: The variances of the EWMA model are large compared to the variances of the SMA model. However, it should be noted that both models occasionally produce large variances and that large losses therefore are possible.

A similar comparison is performed for the portfolio with semi-annual cashflows. The mean of the relative error is negative for the EWMA model while it is positive for the SMA model. The variance of the SMA model is larger than the variance of the EWMA model. It is important to remember that the VaR of both models perform poorly and that this will affect the performance of the ETL. Both models perform better at the 99% confidence level in the sense that the mean of the relative error is closer to zero. The variances of the tail losses behave exactly as the other variances, and it should therefore be clear that large losses may occur.

Chapter 8

Conclusion

This thesis has discussed the calculation of the two risk measures value at risk (VaR) and expected tail loss (ETL). The calculations are based on simulations of interest rates using the LIBOR market model, and two different models are used to estimate the covariance matrix: the SMA model and the EWMA model.

The performance of the VaR varies greatly depending on which estimate is used for the covariance matrix. In general the EWMA model produces better estimates of the VaR than the SMA model. The VaR based on the EWMA model is more accurate and more stable. It is clear that the interpolation has a negative effect on the performance of the VaR because the estimates perform significantly worse when interpolation is used. The VaR's for the portfolio with the semi-annual cashflows were expected to perform worse because of the uncertainty that arises from the calculations, but the difference is surprisingly large. The performance of the ETL varies in the same manner as the VaR. The annual portfolio performs quite well for both the SMA model and the EWMA model. The mean of the relative error is approximately zero and the variance is relatively low. The performance of the semi-annual portfolio is worse, but some of the bad performance may be due to the relatively poor performance of the VaR for this portfolio.

In statistics the SMA model is traditionally used to estimate the covariance matrix, but this is not the best estimate when it comes to the time series studied in this thesis. Since the variance of the time series investigated in this thesis is not constant with respect to time, the EWMA model gives a better estimate. The EWMA estimate adjusts quicker to changes in the variance, and quick changes are common when it comes to interest rates. The effect of the estimation model chosen is in particular clear when the VaR is tested.

Throughout this thesis many assumptions have been made and many of them do not hold as well as hoped. It is therefore no surprise that neither the VaR nor the ETL is a perfect risk measure. However, under the right circumstances both perform well and are sufficient risk measures. It is often debated which risk measure is the best, but one should not need to choose only one as they both contribute different information: The VaR estimates the maximum loss at a given confidence level while the ETL estimates the likely loss if the loss exceeds the VaR. The conclusion is therefore that both measures should be calculated whenever possible, but the values should not be treated as absolutes but more as a frame of reference.

Further Work

Two estimates for the covariance matrix have been calculated in this thesis. There are however several other estimates and combinations of them that have not been tested. Exploring the different models used for the estimation as well as test the effect it has on the VaR and ETL would be interesting. It is also clear that the performance of the risk measures is dependent upon whether a portfolio with annual or semi-annual cashflows is tested. In general the VaR and the ETL performs better for the annual portfolio than the semi-annual. It would therefore be interesting to explore different interpolation methods and see if the performance improves. It would also be interesting to explore the interest rate model, the LMM, and its variations and how they affect the two risk measures.

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Appendix A

Proof

A.1 The bond pricing equation

This derivation can be found in [8]. A portfolio consisting of two bonds maturing at different dates is constructed. The bond maturing at time T_1 has the value V_1 and the bond maturing at time T_2 has the value V_2. The portfolio is constructed in such a way that its value is

$$\Pi = V_1 - \Delta V_2. \tag{A.1}$$

The value of the portfolio over time-step dt changes by

$$\begin{aligned} d\Pi = &\frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial r}dr + \frac{1}{2}\omega^2\frac{\partial^2 V_1}{\partial r^2}dt \\ &- \Delta\left(\frac{\partial V_2}{\partial t}dt + \frac{\partial V_2}{\partial r}dr + \frac{1}{2}\omega^2\frac{\partial^2 V_2}{\partial r^2}dt\right). \end{aligned} \tag{A.2}$$

This result is found by applying Itô's lemma to functions of r and t. The value of Δ is chosen to be

$$\Delta = \frac{\partial V_1}{\partial r}/\frac{\partial V_2}{\partial r}, \tag{A.3}$$

to eliminate the randomness. Substituting (A.3) into (A.2) gives

$$d\Pi = \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\omega^2\frac{\partial^2 V_1}{\partial r^2} - \frac{\partial V_1/\partial r}{\partial V_2/\partial r}\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\omega^2\frac{\partial^2 V_2}{\partial r^2}\right)\right)dt. \tag{A.4}$$

The arbitrage principle states that the value of the portfolio must equal the return given by the risk-free interest rate

$$\begin{aligned} d\Pi &= r\Pi dt \\ &= r\left(V_1 - \frac{\partial V_1}{\partial r}/\frac{\partial V_2}{\partial r}V_2\right)dt. \end{aligned} \tag{A.5}$$

Gathering all V_1 terms on the left-hand side and all V_2 terms on the right-hand side gives

$$\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\omega^2\frac{\partial^2 V_1}{\partial r^2} - rV_1\right)/\frac{\partial V_1}{\partial r} = \left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\omega^2\frac{\partial^2 V_2}{\partial r^2} - rV_2\right)/\frac{\partial V_2}{\partial r}. \tag{A.6}$$

Since the left-hand side is only a function of T_1 and the right-hand side is only a function of T_2, both sides have to be independent of the maturity date. Thus the function can be written

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\omega^2\frac{\partial^2 V}{\partial r^2} - rV\right)/\frac{\partial V}{\partial r} = a(r,t). \tag{A.7}$$

Later development will show that it is convenient to write the function $a(r,t)$ as

$$a(r,t) = \omega(r,t)\lambda(r,t) - u(r,t), \tag{A.8}$$

where $\omega(r,t)$ is not identically zero. Inserting (A.8) into (A.7) gives the zero-coupon bond pricing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\omega^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda\omega)\frac{\partial V}{\partial r} - rV = 0. \tag{A.9}$$

Appendix B

Normal Probability Plots

This chapter contains histograms and QQ plots of the forward rates described in section (6.3).

B.1 Histograms and QQ Plots using the SMA Estimate

This section contains histograms and QQ plots where the volatility and correlation is based on the SMA estimate.

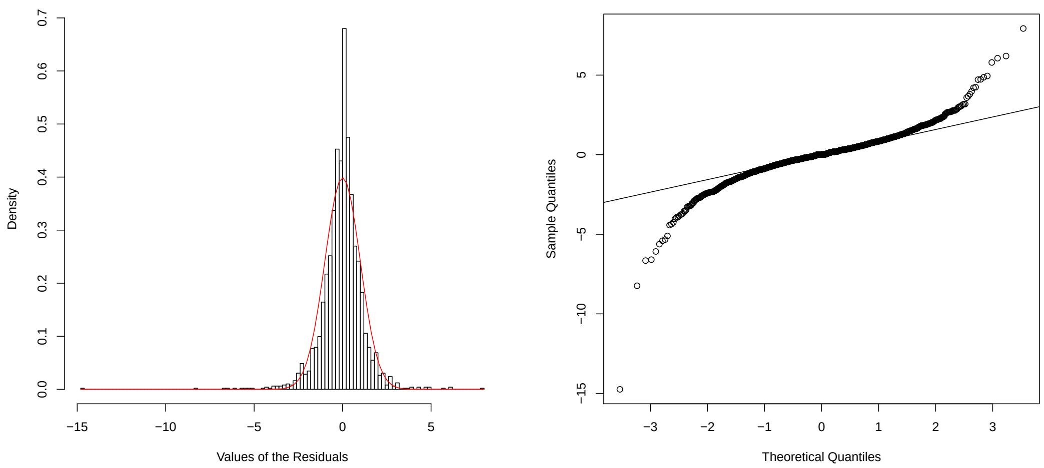


Figure B.1: Histogram and QQ plot of the residuals of the American forward rate for the period $0.5 - 1$ year. The correlation and the volatility is estimated by the use of the SMA model.

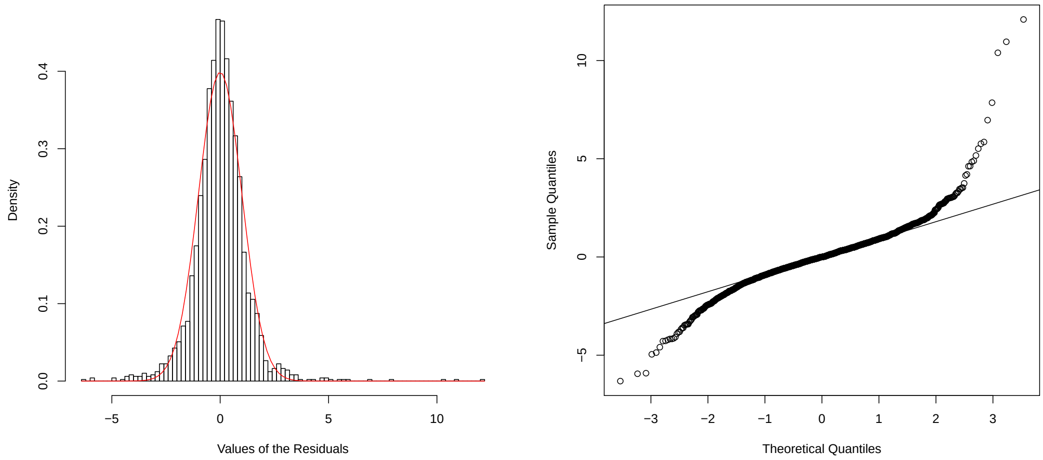


Figure B.2: Histogram and QQ plot of the residuals of the American forward rate for the period $1 - 2$ years. The correlation and the volatility is estimated by the use of the SMA model.

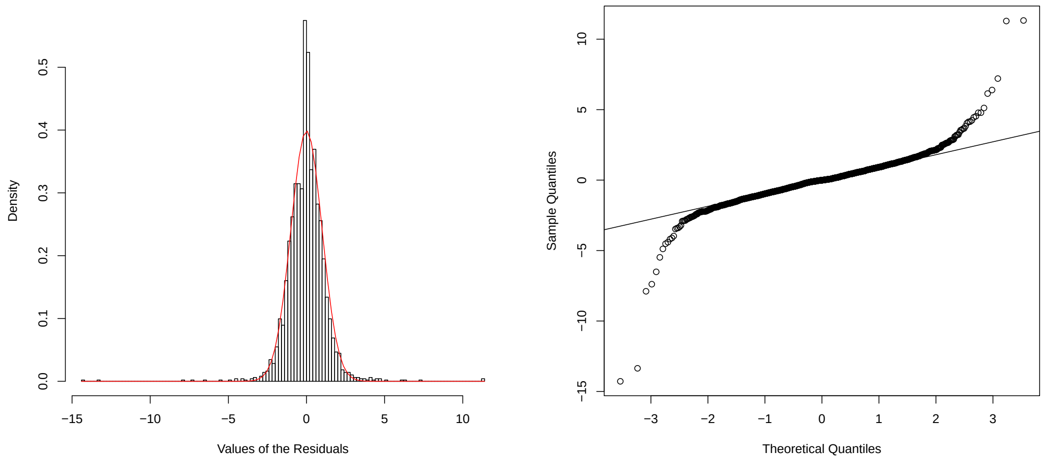


Figure B.3: Histogram and QQ plot of the residuals of the American forward rate for the period $2 - 3$ years. The correlation and the volatility is estimated by the use of the SMA model.

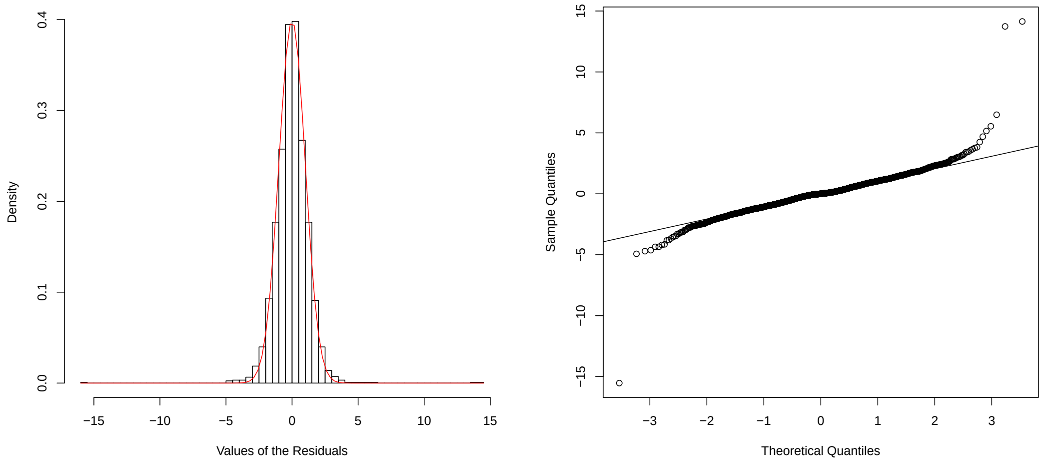


Figure B.4: Histogram and QQ plot of the residuals of the American forward rate for the period $3-4$ years. The correlation and the volatility is estimated by the use of the SMA model.

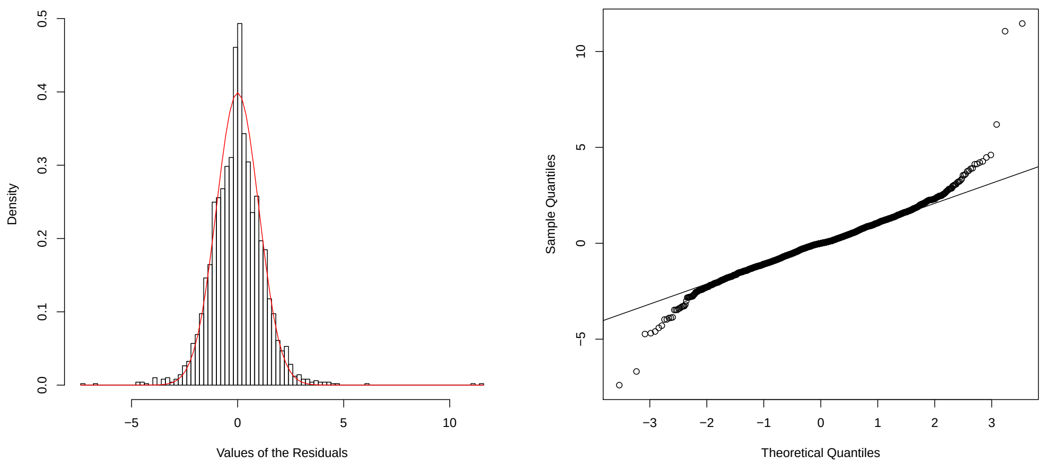


Figure B.5: Histogram and QQ plot of the residuals of the American forward rate for the period $4-5$ years. The correlation and the volatility is estimated by the use of the SMA model.

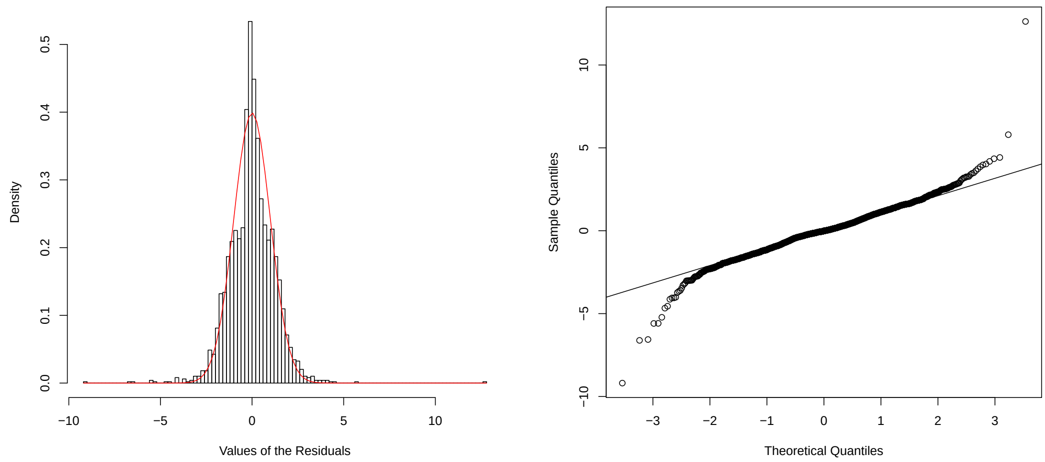


Figure B.6: Histogram and QQ plot of the residuals of the American forward rate for the period $5 - 6$ years. The correlation and the volatility is estimated by the use of the SMA model.

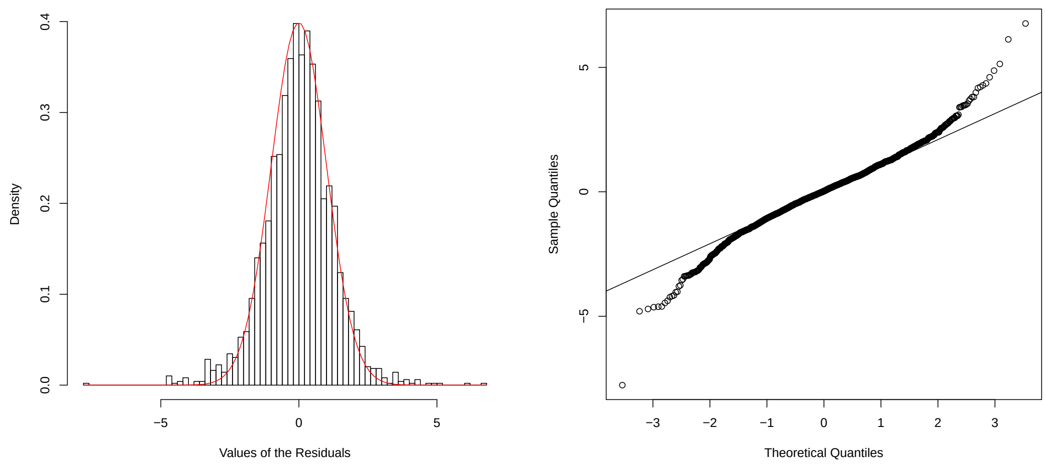


Figure B.7: Histogram and QQ plot of the residuals of the European forward rate for the period $0.5 - 1$ year. The correlation and the volatility is estimated by the use of the SMA model.

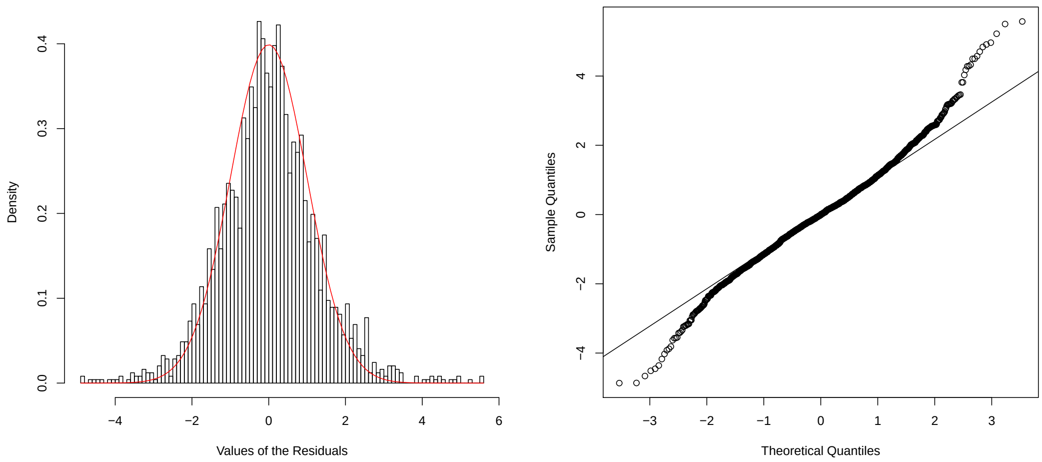


Figure B.8: Histogram and QQ plot of the residuals of the European forward rate for the period $1 - 2$ years. The correlation and the volatility is estimated by the use of the SMA model.

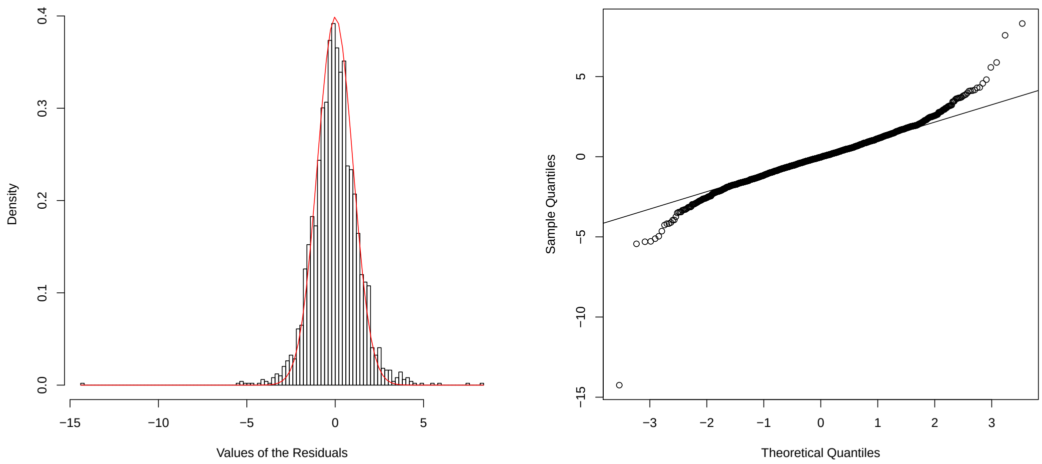


Figure B.9: Histogram and QQ plot of the residuals of the European forward rate for the period $2 - 3$ years. The correlation and the volatility is estimated by the use of the SMA model.

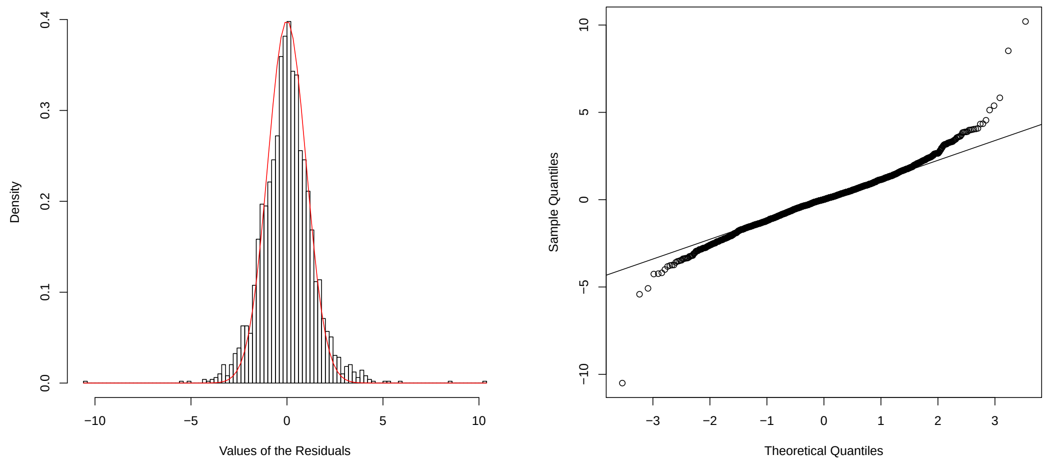


Figure B.10: Histogram and QQ plot of the residuals of the European forward rate for the period $3-4$ years. The correlation and the volatility is estimated by the use of the SMA model.

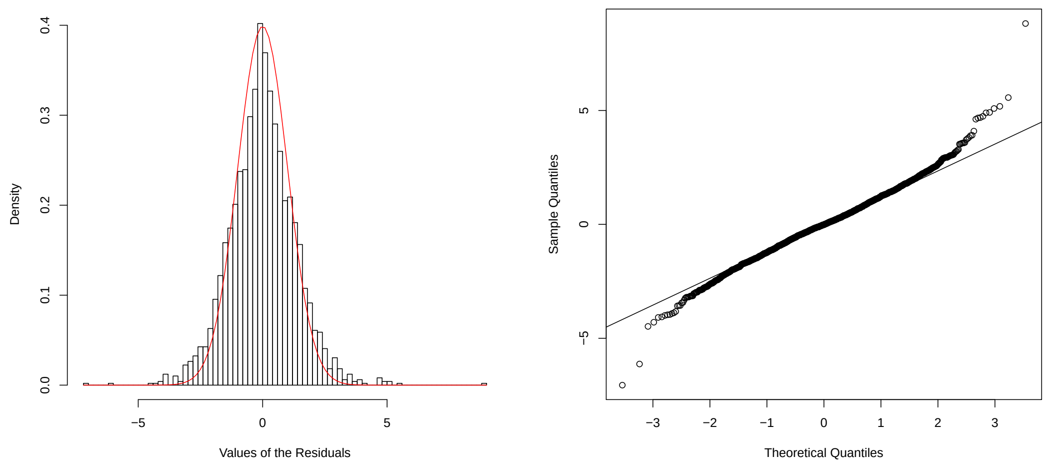


Figure B.11: Histogram and QQ plot of the residuals of the European forward rate for the period $4-5$ years. The correlation and the volatility is estimated by the use of the SMA model.

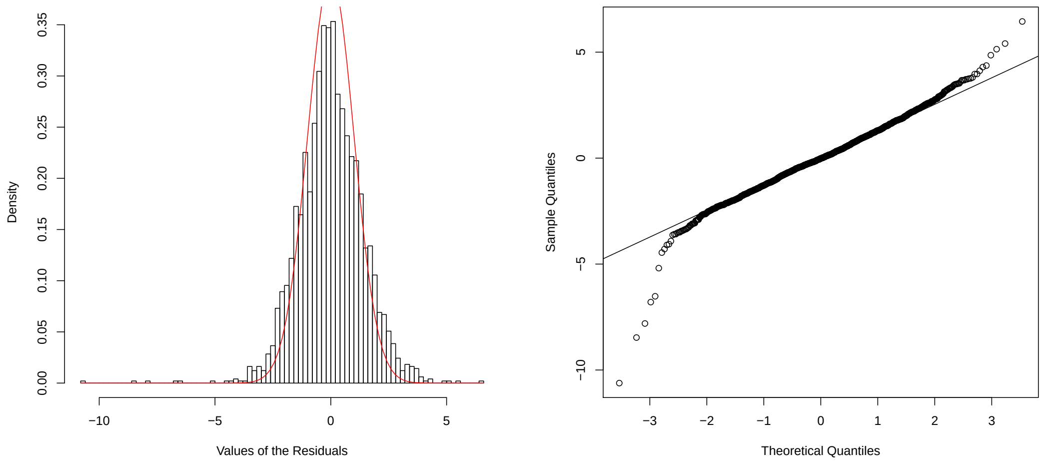


Figure B.12: Histogram and QQ plot of the residuals of the European forward rate for the period $5 - 6$ years. The correlation and the volatility is estimated by the use of the SMA model.

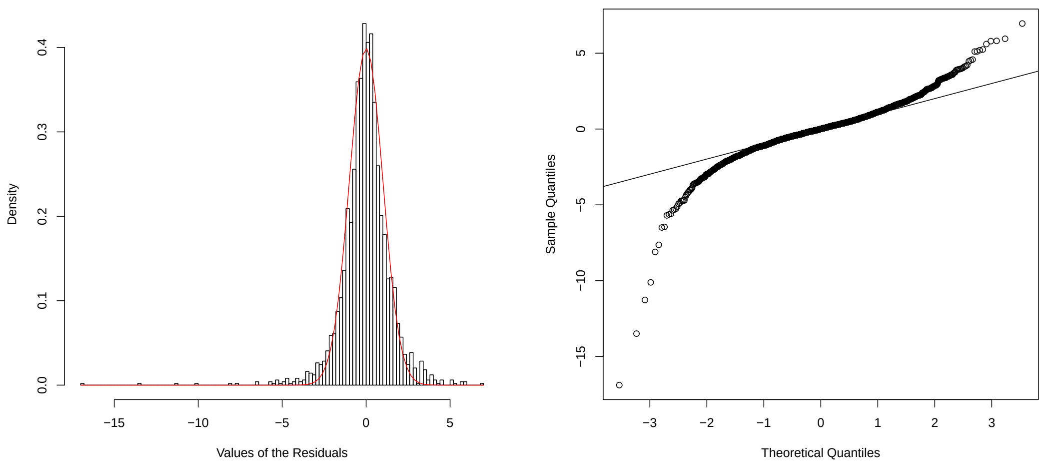


Figure B.13: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $0.5 - 1$ year. The correlation and the volatility is estimated by the use of the SMA model.

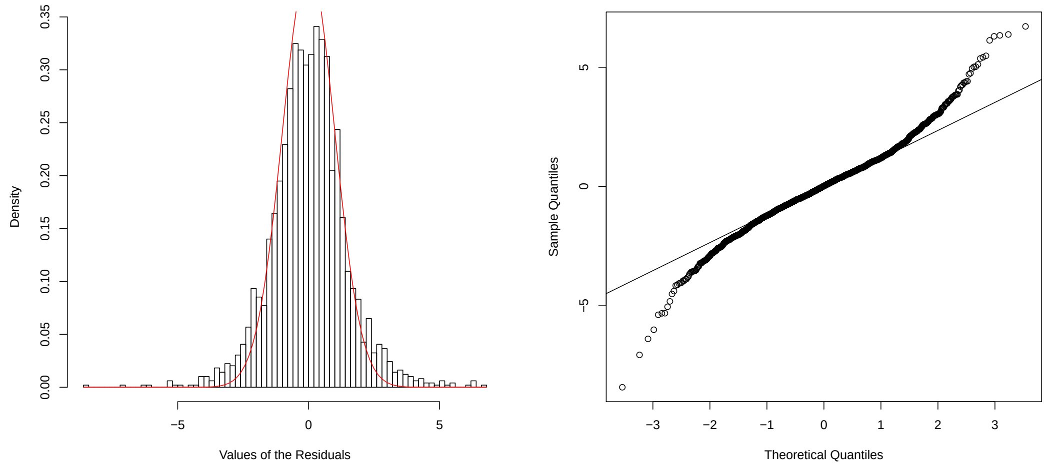


Figure B.14: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $1 - 2$ years. The correlation and the volatility is estimated by the use of the SMA model.

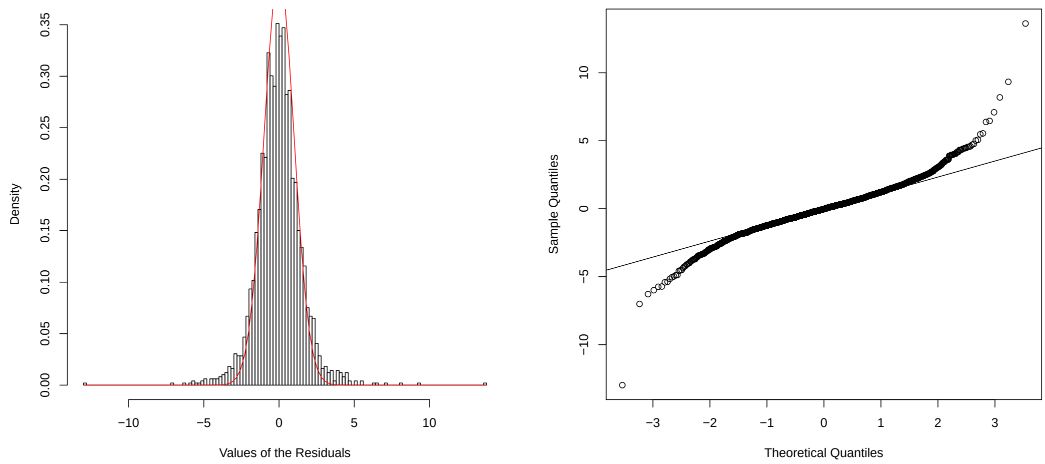


Figure B.15: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $2 - 3$ years. The correlation and the volatility is estimated by the use of the SMA model.

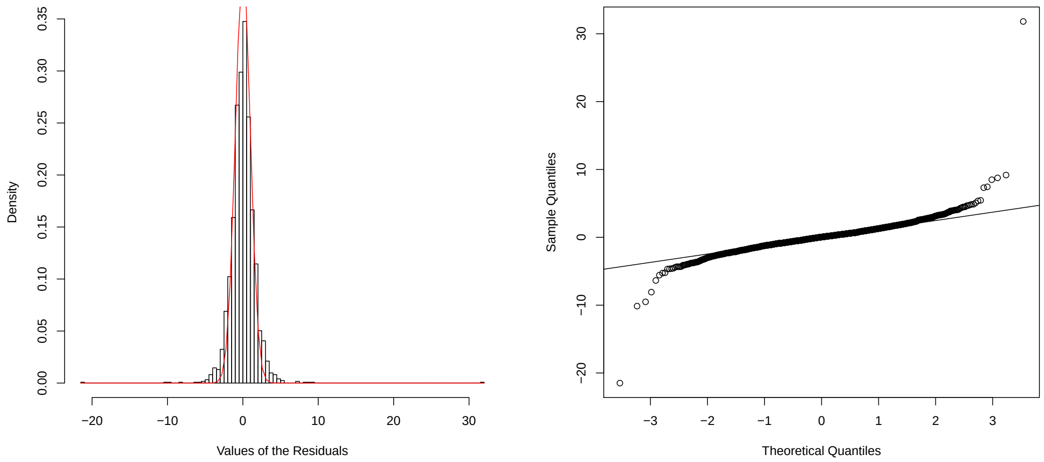


Figure B.16: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $3-4$ years. The correlation and the volatility is estimated by the use of the SMA model.

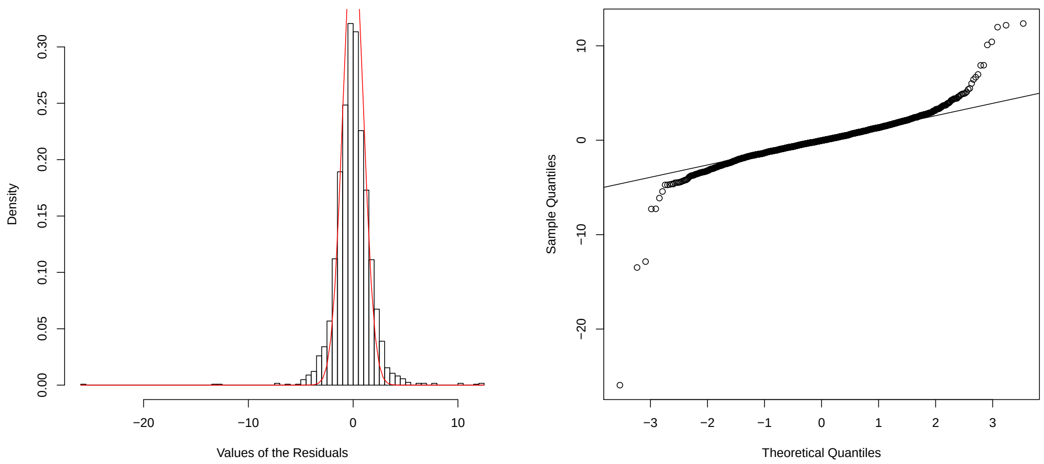


Figure B.17: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $4-5$ years. The correlation and the volatility is estimated by the use of the SMA model.

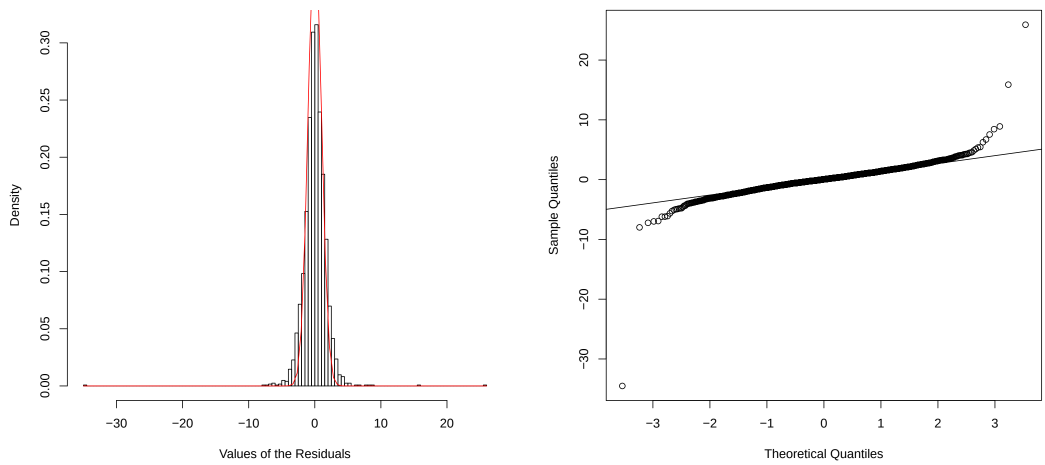


Figure B.18: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $5 - 6$ years. The correlation and the volatility is estimated by the use of the SMA model.

B.2 Histograms and QQ Plots using the EWMA Estimates

This section contains histograms and QQ plots where the volatility and correlation is based on the EWMA estimate.

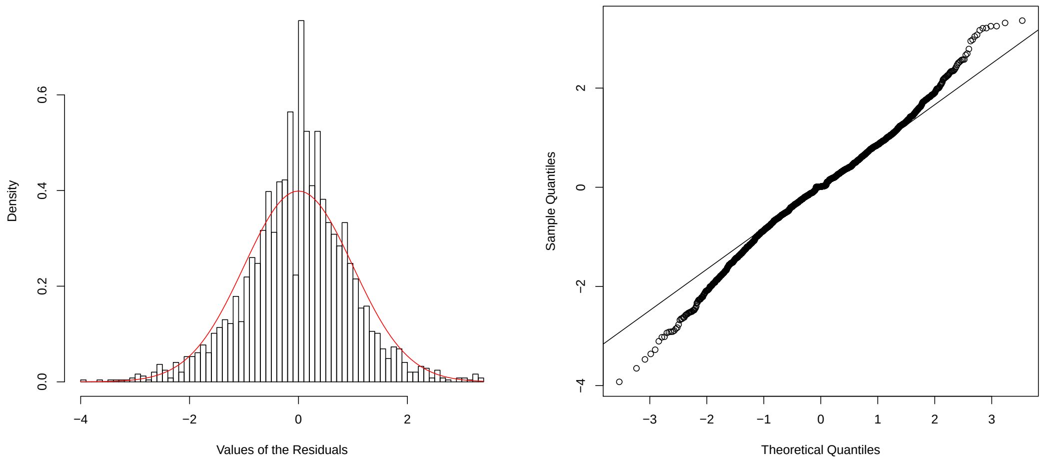


Figure B.19: Histogram and QQ plot of the residuals of the American forward rate for the period $0.5 - 1$ years. The correlation and the volatility is estimated by the use of the EWMA model.

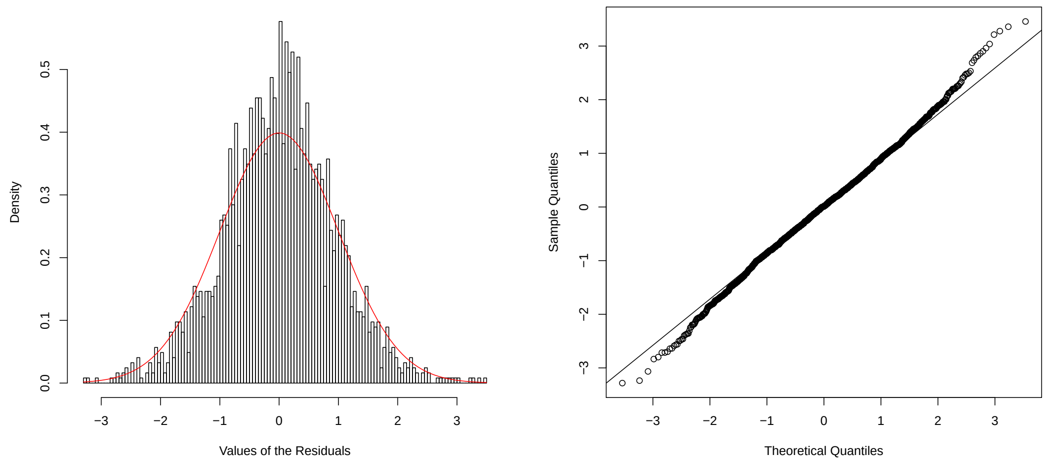


Figure B.20: Histogram and QQ plot of the residuals of the American forward rate for the period $1-2$ years. The correlation and the volatility is estimated by the use of the EWMA model.

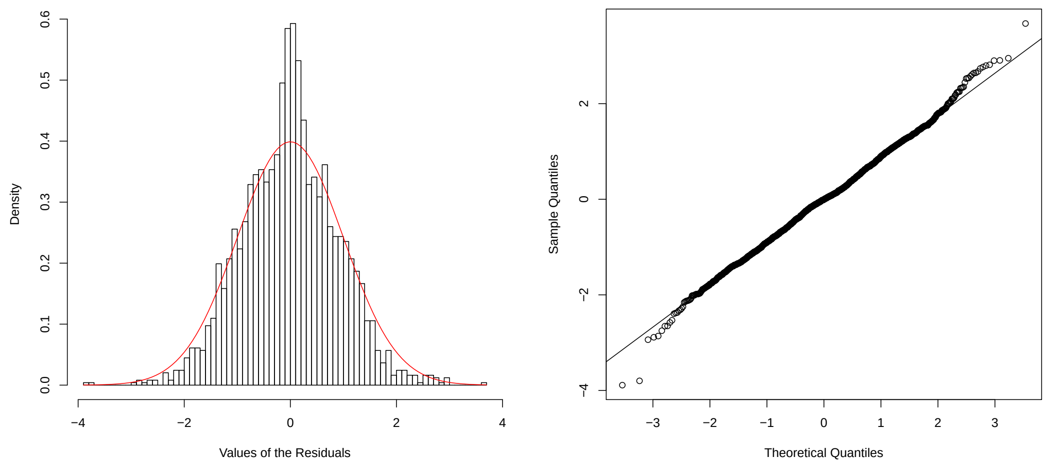


Figure B.21: Histogram and QQ plot of the residuals of the American forward rate for the period $2-3$ years. The correlation and the volatility is estimated by the use of the EWMA model.

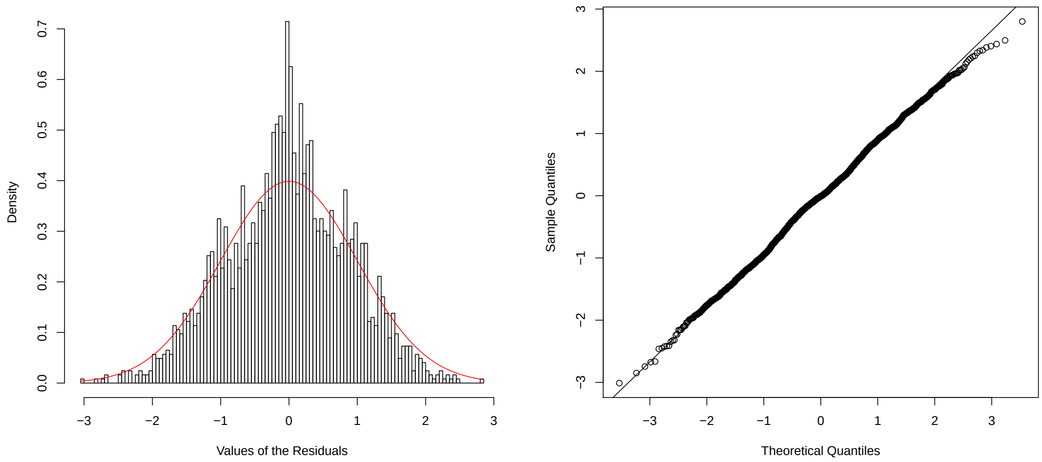


Figure B.22: Histogram and QQ plot of the residuals of the American forward rate for the period $3-4$ years. The correlation and the volatility is estimated by the use of the EWMA model.

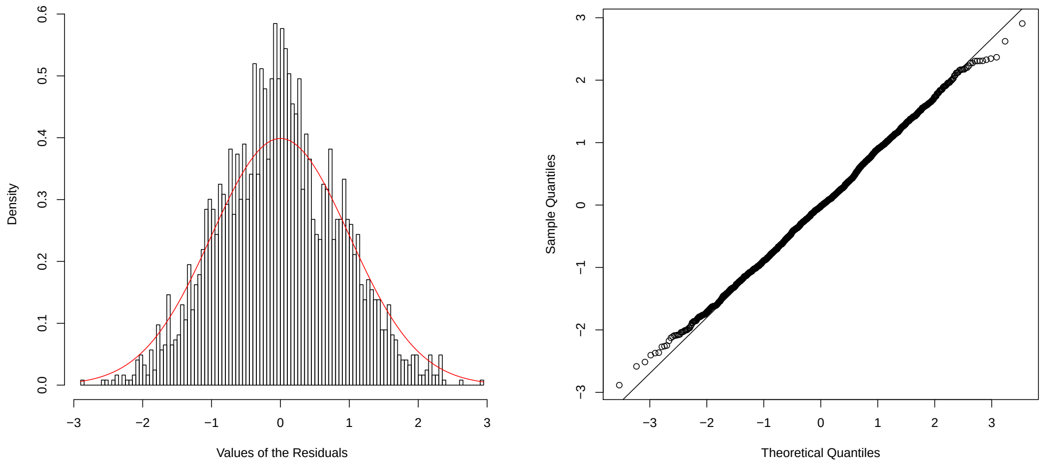


Figure B.23: Histogram and QQ plot of the residuals of the American forward rate for the period $4-5$ years. The correlation and the volatility is estimated by the use of the EWMA model.

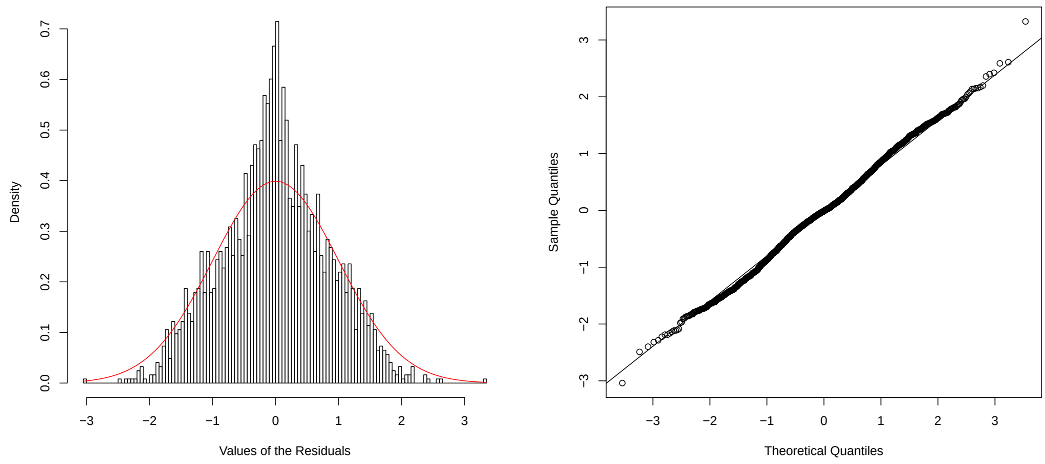


Figure B.24: Histogram and QQ plot of the residuals of the American forward rate for the period $5-6$ years. The correlation and the volatility is estimated by the use of the EWMA model.

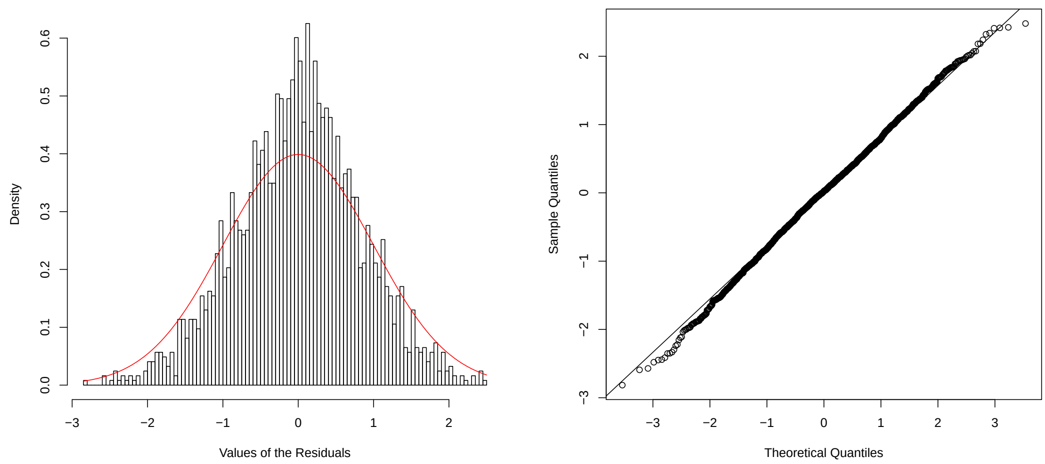


Figure B.25: Histogram and QQ plot of the residuals of the European forward rate for the period $0.5-1$ year. The correlation and the volatility is estimated by the use of the EWMA model.

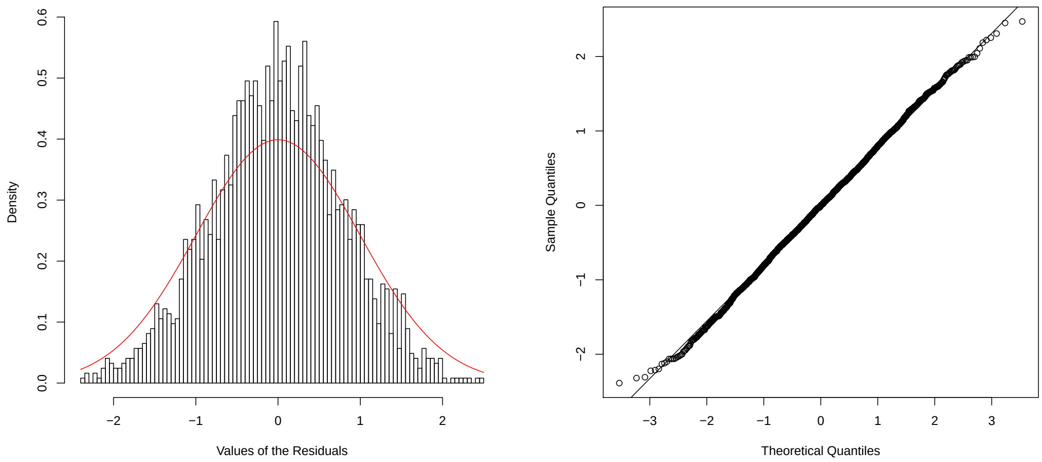


Figure B.26: Histogram and QQ plot of the residuals of the European forward rate for the period $1-2$ years. The correlation and the volatility is estimated by the use of the EWMA model.

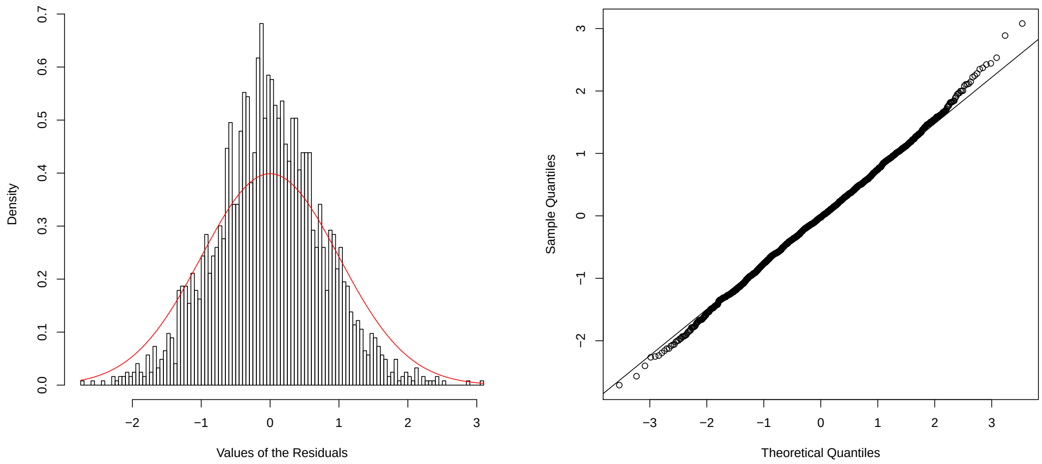


Figure B.27: Histogram and QQ plot of the residuals of the European forward rate for the period $2-3$ years. The correlation and the volatility is estimated by the use of the EWMA model.

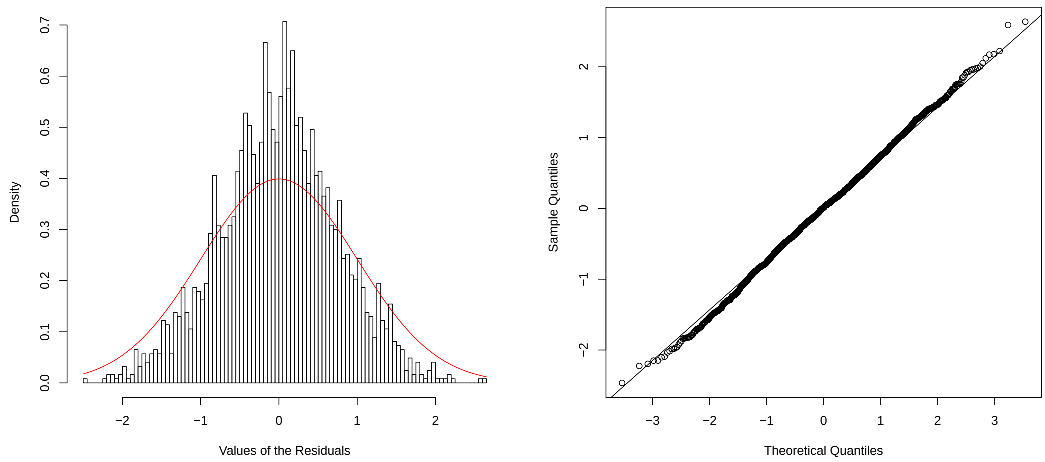


Figure B.28: Histogram and QQ plot of the residuals of the European forward rate for the period $3-4$ years. The correlation and the volatility is estimated by the use of the EWMA model.

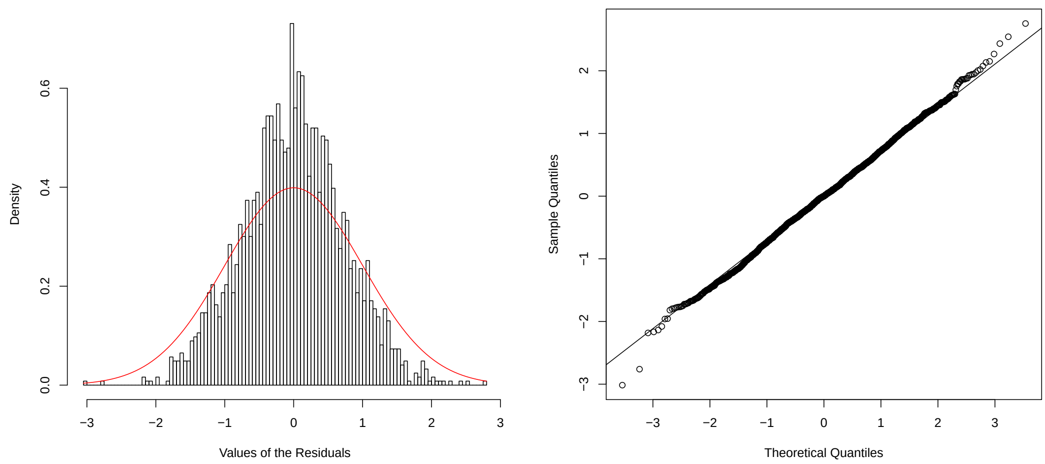


Figure B.29: Histogram and QQ plot of the residuals of the European forward rate for the period $4-5$ years. The correlation and the volatility is estimated by the use of the EWMA model.

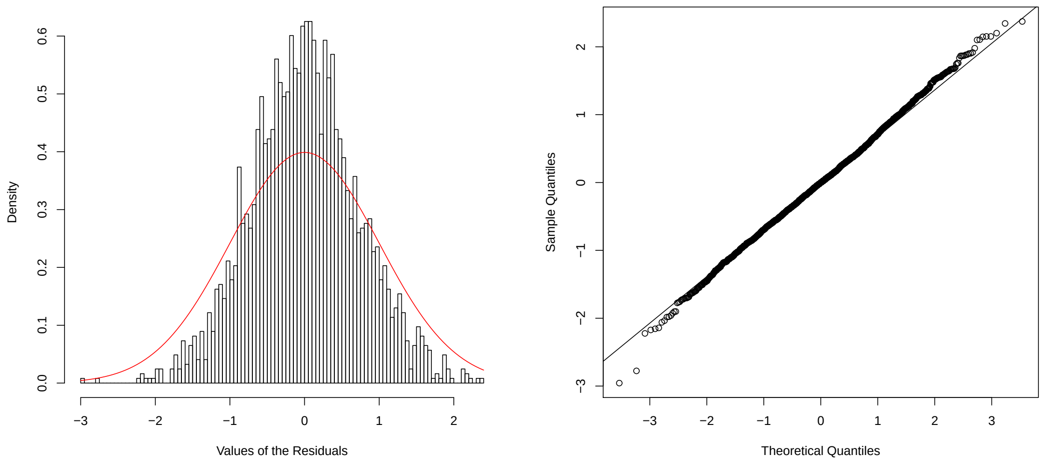


Figure B.30: Histogram and QQ plot of the residuals of the European forward rate for the period $5 - 6$ years. The correlation and the volatility is estimated by the use of the EWMA model.

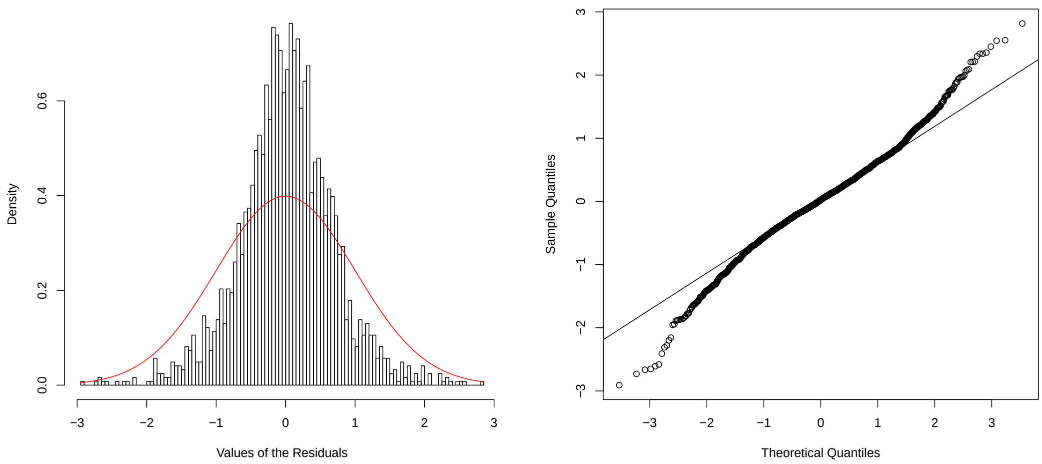


Figure B.31: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $0.5 - 1$ year. The correlation and the volatility is estimated by the use of the EWMA model.

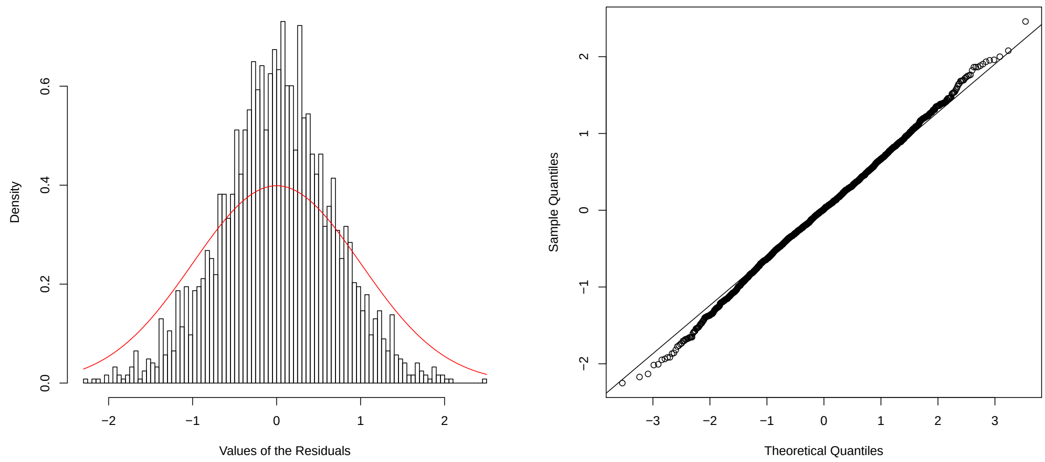


Figure B.32: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $1-2$ years. The correlation and the volatility is estimated by the use of the EWMA model.

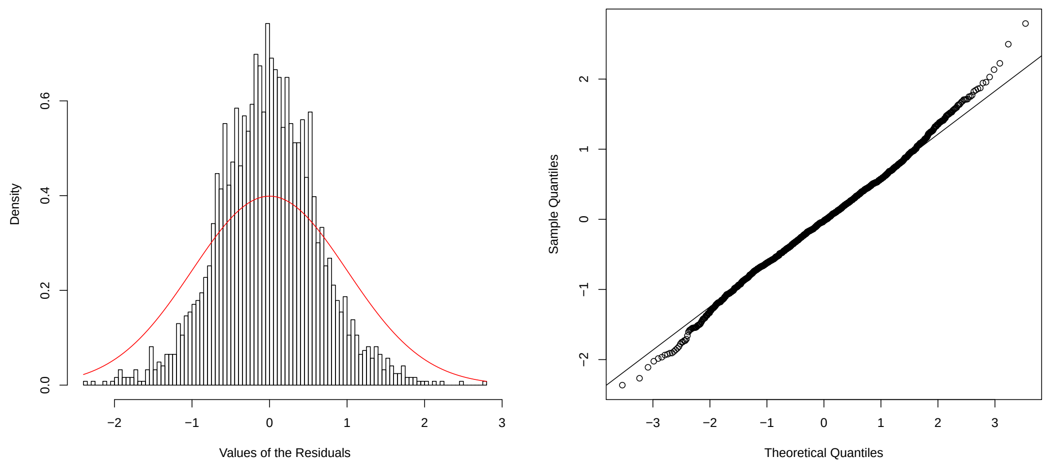


Figure B.33: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $2-3$ years. The correlation and the volatility is estimated by the use of the EWMA model.

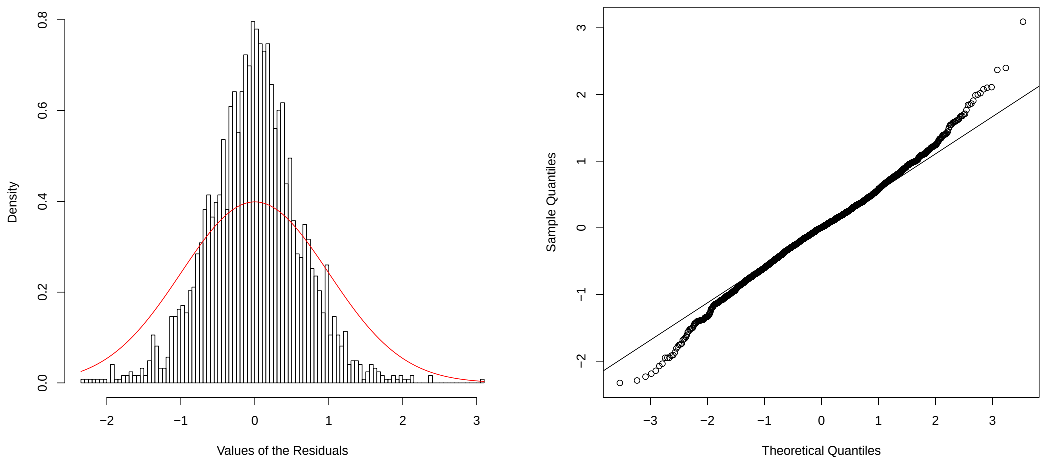


Figure B.34: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $3-4$ years. The correlation and the volatility is estimated by the use of the EWMA model.

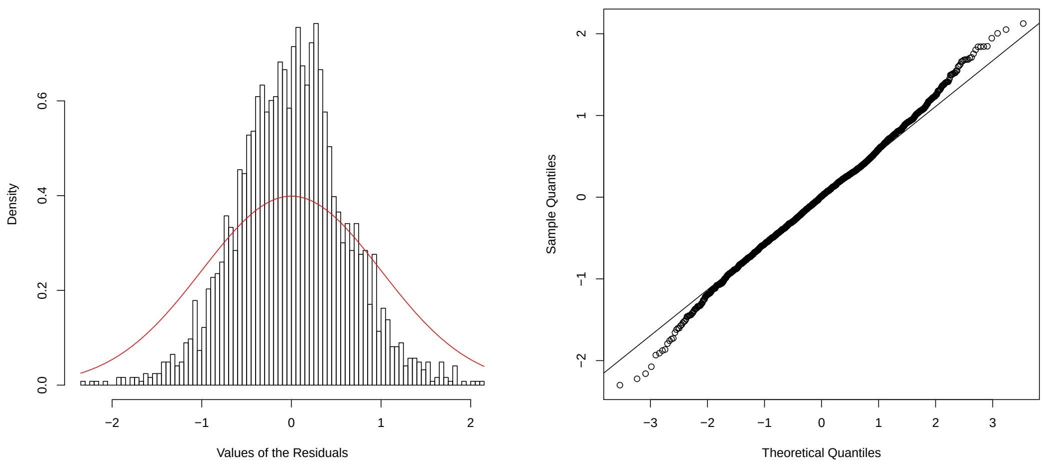


Figure B.35: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $4-5$ years. The correlation and the volatility is estimated by the use of the EWMA model.

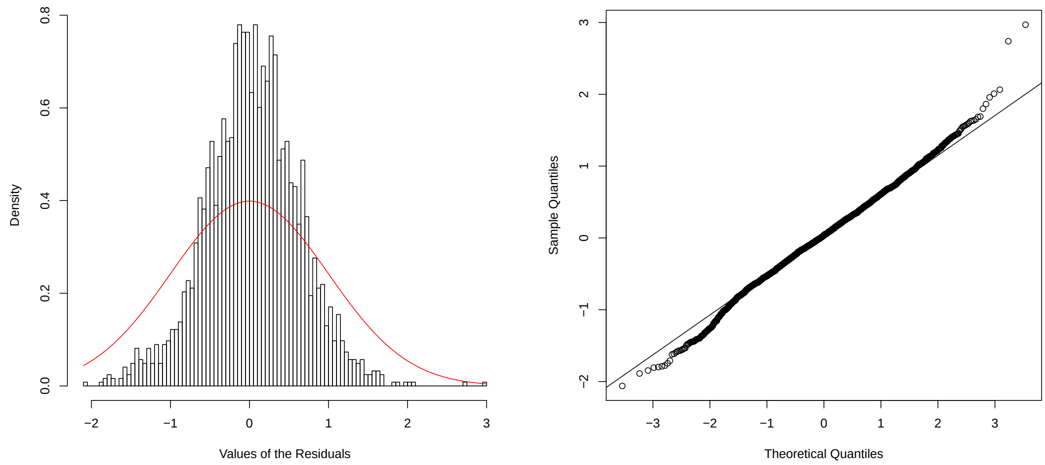


Figure B.36: Histogram and QQ plot of the residuals of the Norwegian forward rate for the period $5-6$ years. The correlation and the volatility is estimated by the use of the EWMA model.

B.3 Scatter Plots

This section contains scatter plots for the three currencies USD, EURO and NOK.

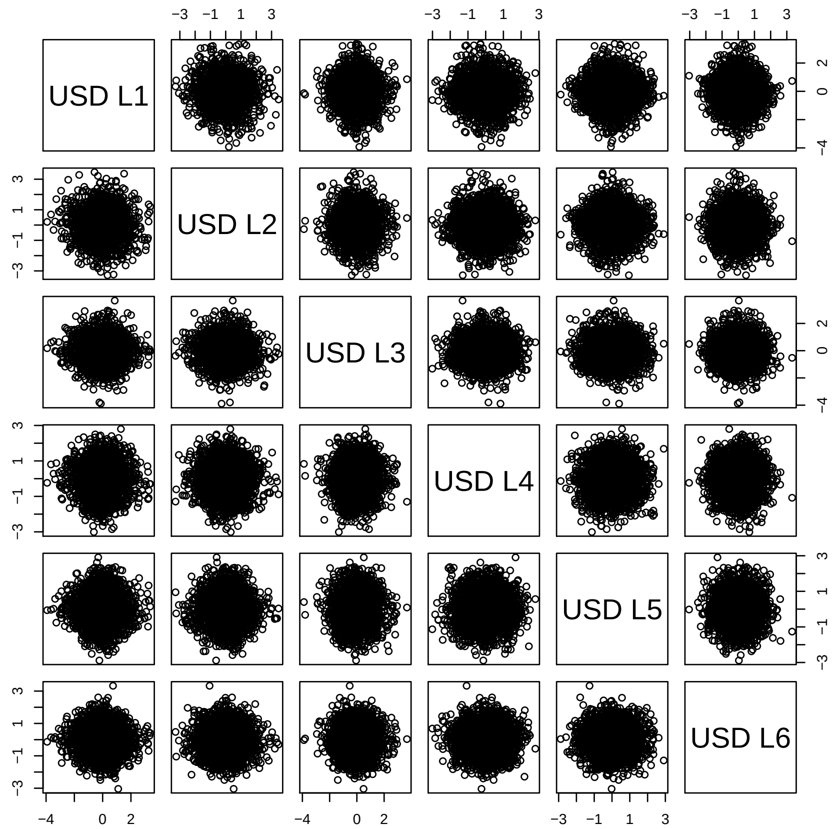


Figure B.37: The scatter plot of the residuals of the American interest rates. The ij'th scatter plot contains the residuals of L_i plotted against L_j. L_1 is the forward rate for the period 6-12 months, L_2 is the forward rate for the period 1-2 years, L_3 is the forward rate for the period 2-3 years etc.

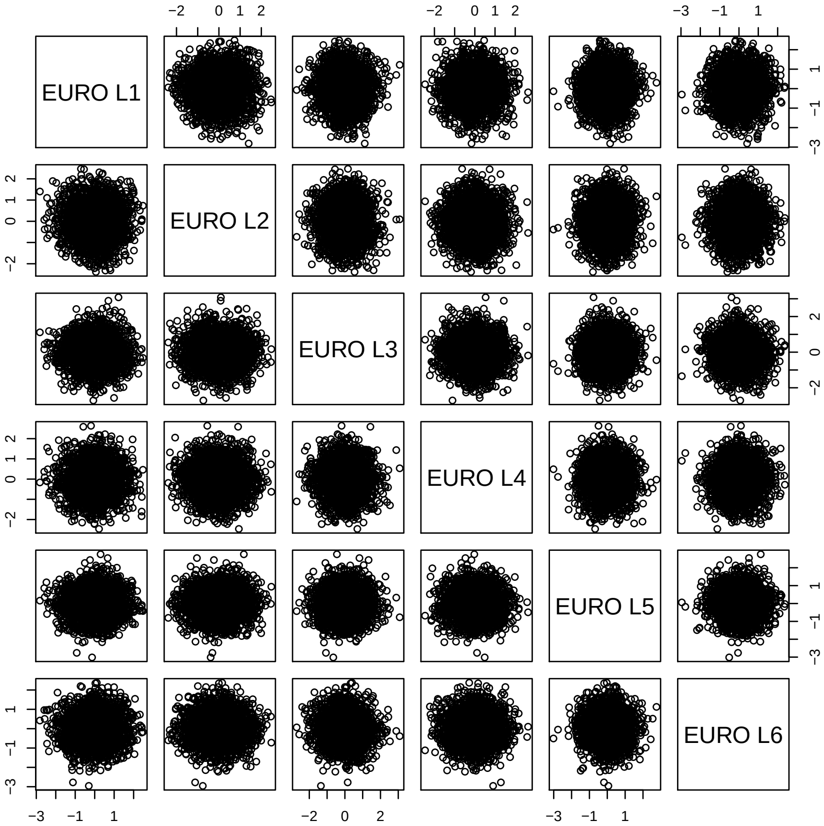


Figure B.38: The scatter plot of the residuals of the European interest rates. The ij'th scatter plot contains the residuals of L_i plotted against L_j. L_1 is the forward rate for the period 6-12 months, L_2 is the forward rate for the period 1-2 years, L_3 is the forward rate for the period 2-3 years etc.

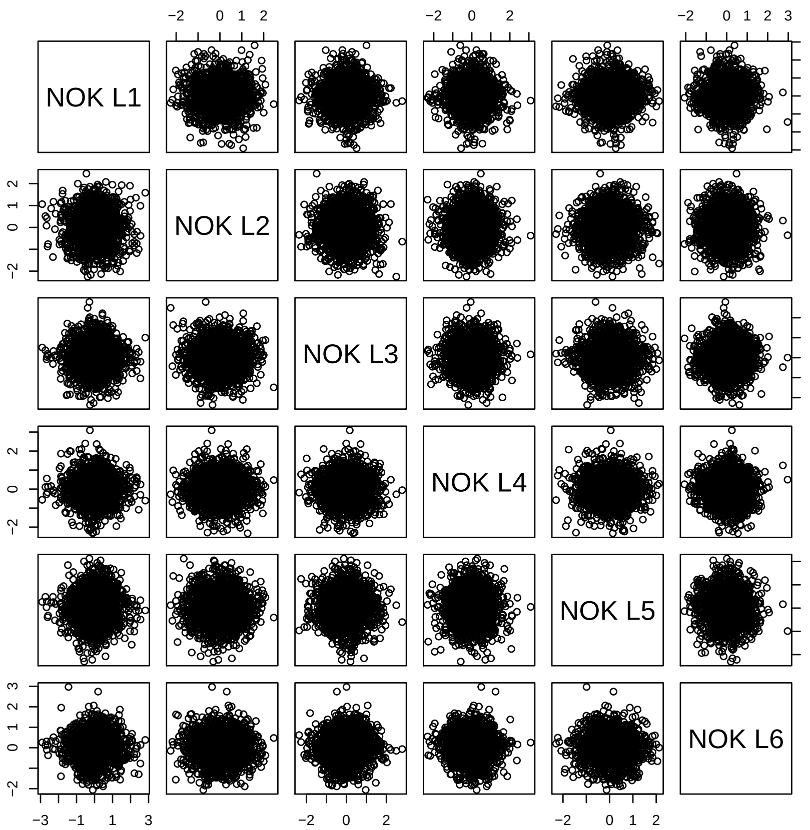


Figure B.39: The scatter plot of the residuals of the Norwegian interest rates. The ij'th scatter plot contains the residuals of L_i plotted against L_j. L_1 is the forward rate for the period 6-12 months, L_2 is the forward rate for the period 1-2 years, L_3 is the forward rate for the period 2-3 years etc.

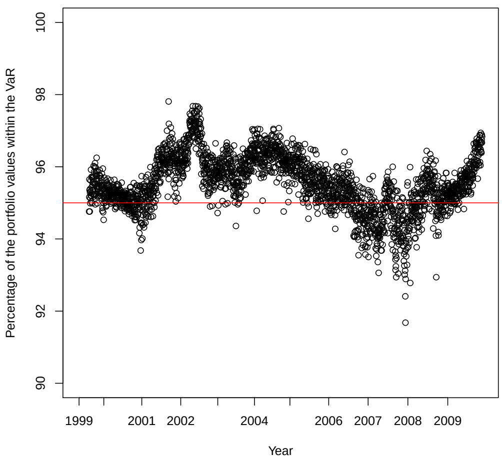
Appendix C

Plots describing the Results

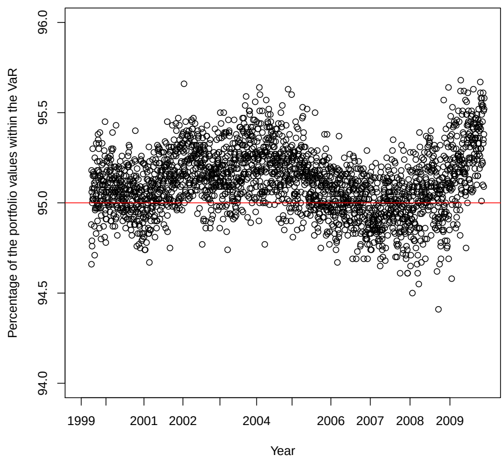
This chapter contains plots describing the results found in this thesis.

C.1 Value at Risk with a Daily Horizon

This section contains plots describing the performance of the VaR with a daily horizon.

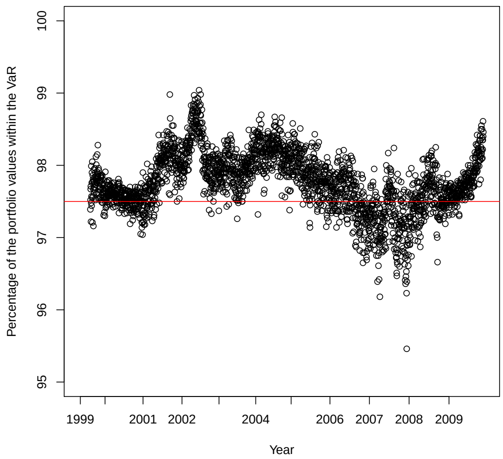


(a) SMA estimate, 95% confidence level

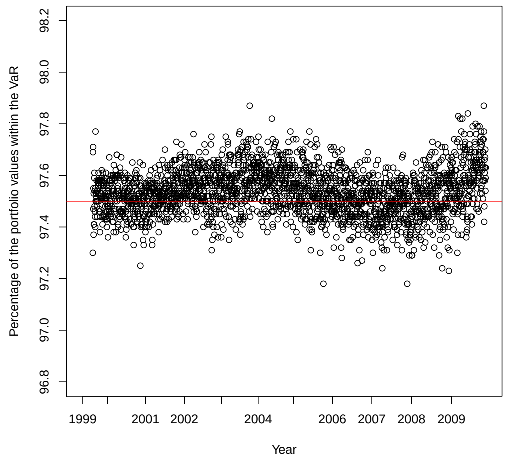


(b) EWMA estimate, 95% confidence level

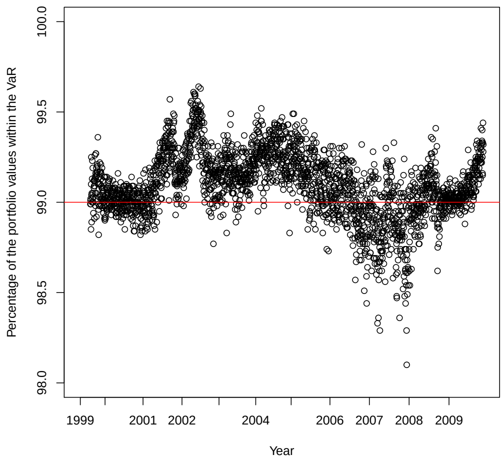
Figure C.1: Percentages of the portfolio's profits and losses within the VaR with a daily horizon using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies.



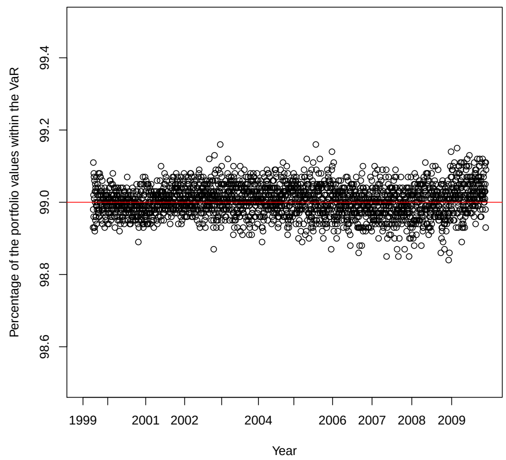
(c) SMA estimate, 97.5% confidence level



(d) EWMA estimate, 97.5% confidence level

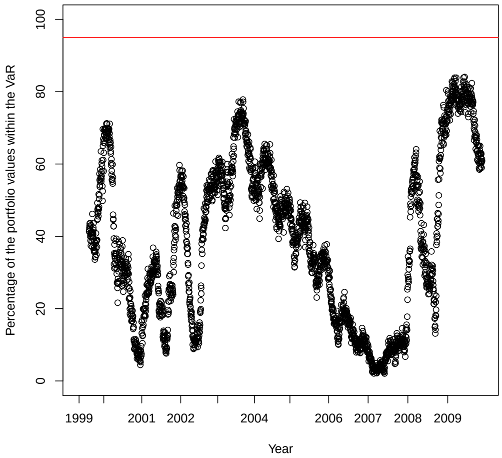


(e) SMA estimate, 99% confidence level

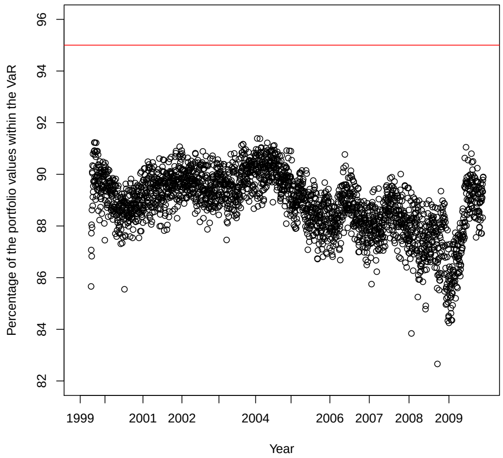


(f) EWMA estimate, 99% confidence level

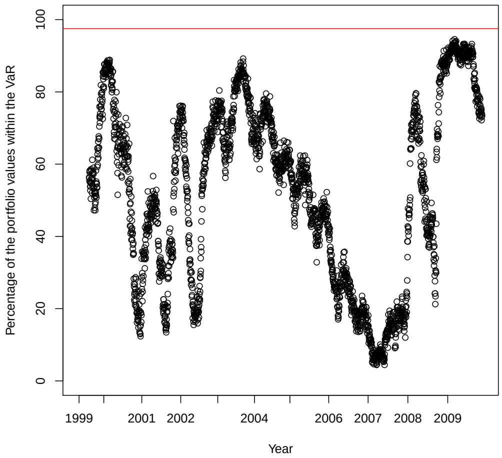
Figure C.1: Percentages of the portfolio's profits and losses within the VaR with a daily horizon using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies.



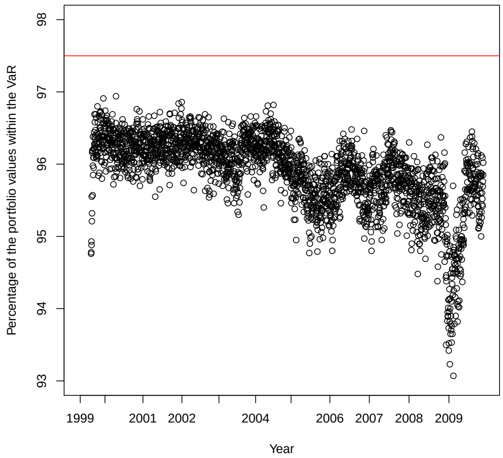
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level

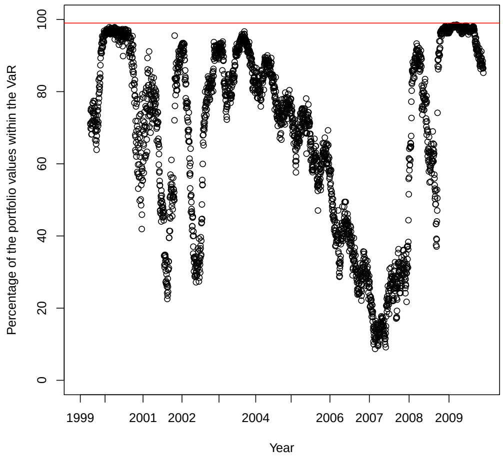


(c) SMA estimate, 97.5% confidence level

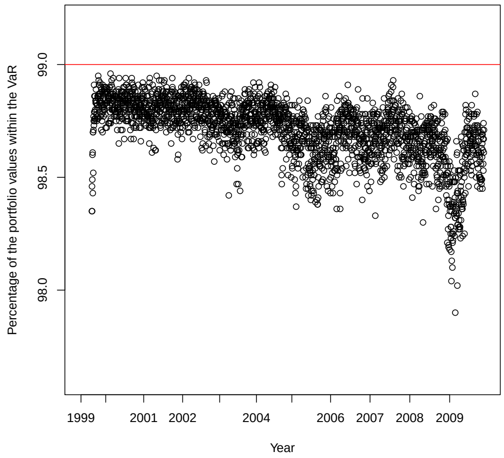


(d) EWMA estimate, 97.5% confidence level

Figure C.2: Percentages of the portfolio's profits and losses within the VaR with a daily horizon using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates semi-annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies.



(e) SMA estimate, 99% confidence level

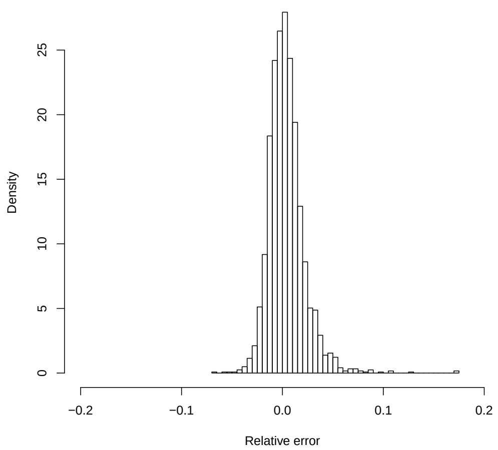


(f) EWMA estimate, 99% confidence level

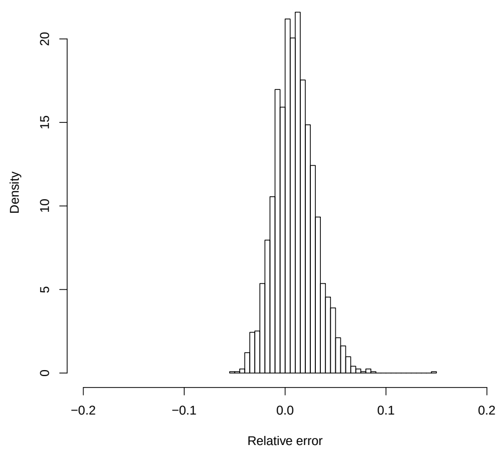
Figure C.2: Percentages of the portfolio's profits and losses within the VaR with a daily horizon using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates semi-annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies.

C.2 Expected Tail Loss with a Daily Horizon

This section contains plots describing the performance of the ETL with a daily horizon.

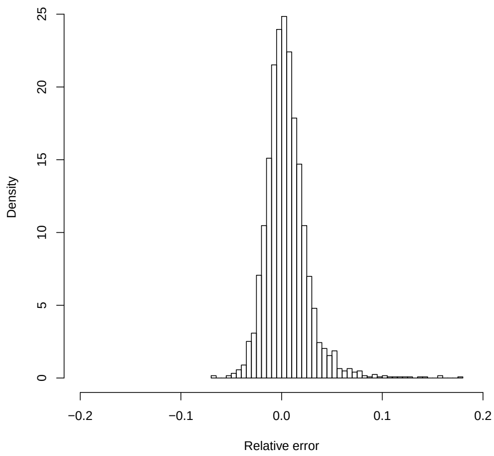


(a) SMA estimate, 95% confidence level

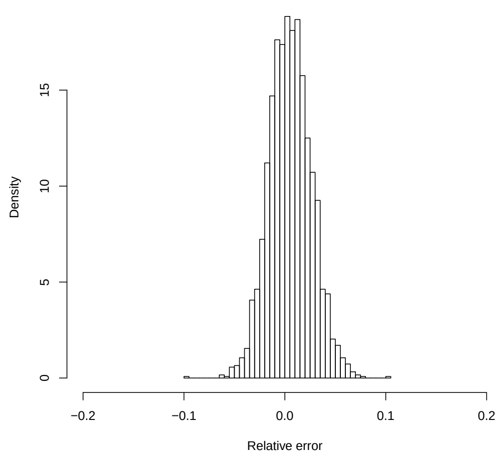


(b) EWMA estimate, 95% confidence level

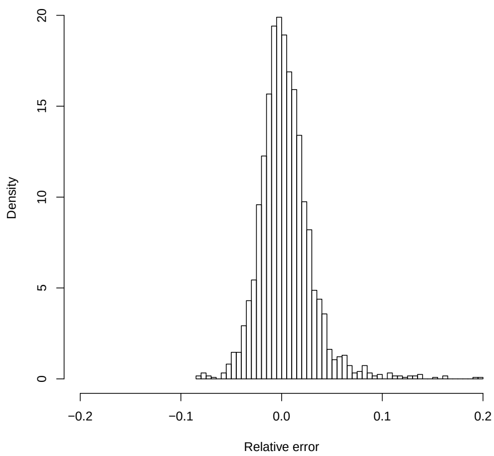
Figure C.3: Histograms of the relative error between the ETL with a daily horizon and the mean of the tail losses each day. The portfolio tested generates annual cashflows and both the SMA model and the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to to better display the remaining data.



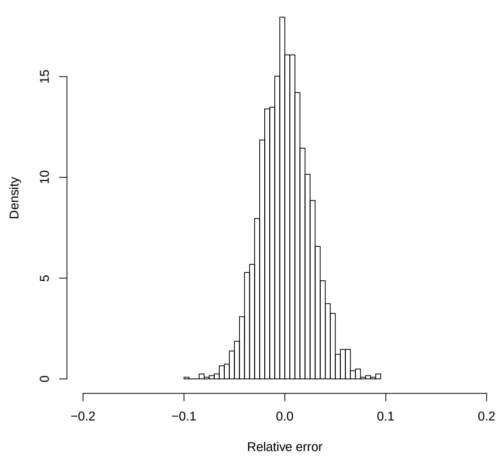
(c) SMA estimate, 97.5% confidence level



(d) EWMA estimate, 97.5% confidence level

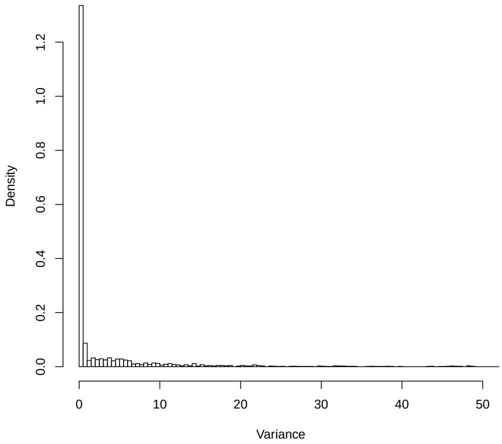


(e) SMA estimate, 99% confidence level

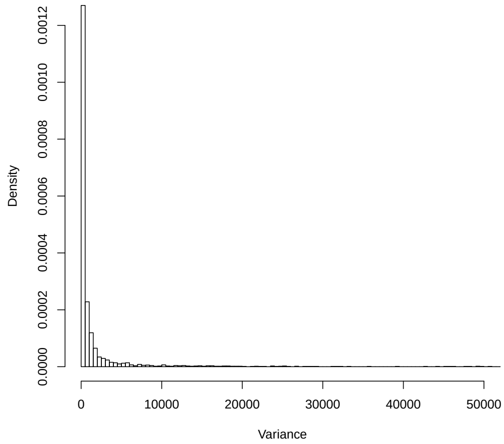


(f) EWMA estimate, 99% confidence level

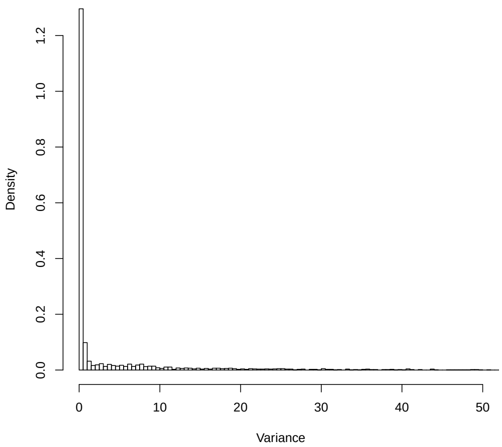
Figure C.3: Histograms of the relative error between the ETL with a daily horizon and the mean of the tail losses each day. The portfolio tested generates annual cashflows and both the SMA model and the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to to better display the remaining data.



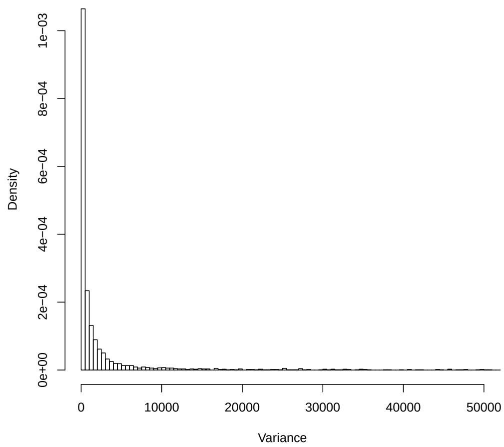
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level

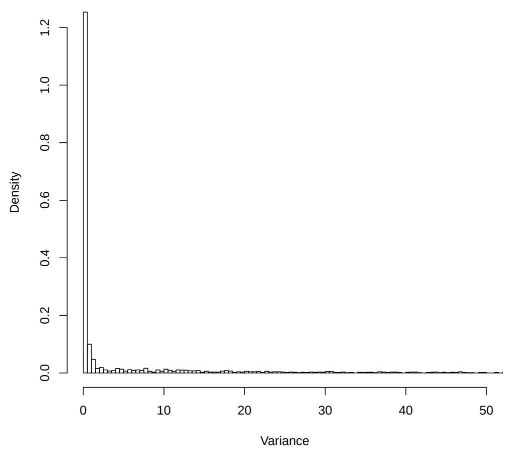


(c) SMA estimate, 97.5% confidence level

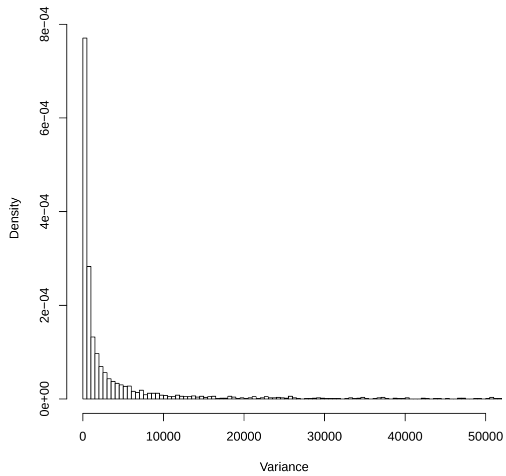


(d) EWMA estimate, 97.5% confidence level

Figure C.4: Histograms of the variances of the tail losses each day for the portfolio with annual cashflows and a daily horizon. Some extreme values are left out of the histograms to better display the remaining data.

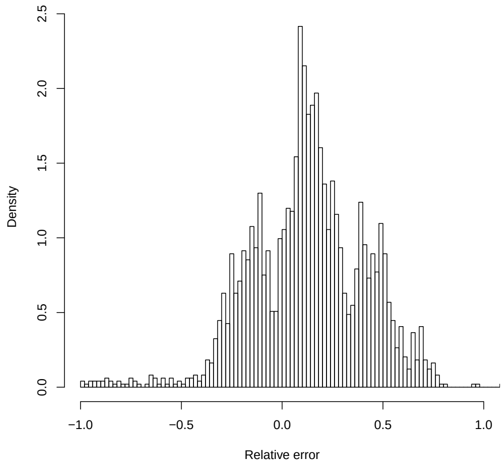


(e) SMA estimate, 99% confidence level

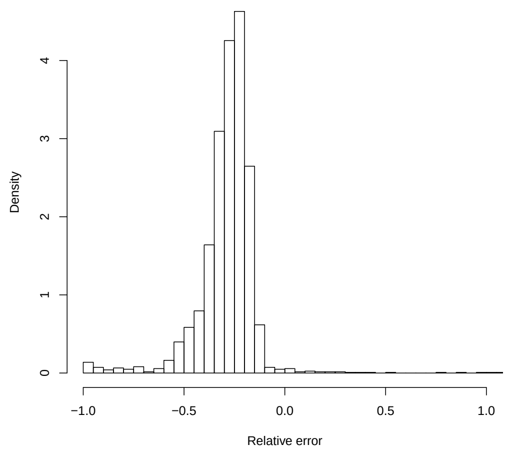


(f) EWMA estimate, 99% confidence level

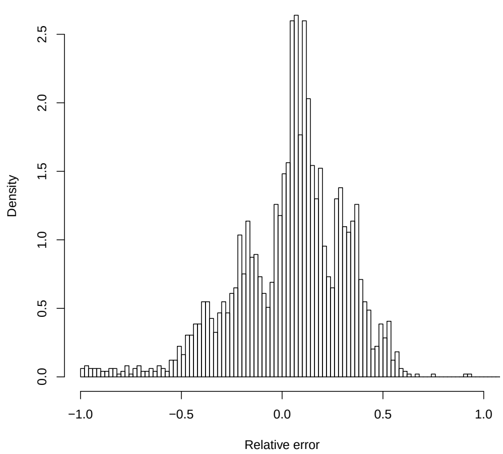
Figure C.4: Histograms of the variances of the tail losses each day for the portfolio with annual cashflows and a daily horizon. Some extreme values are left out of the histograms to better display the remaining data.



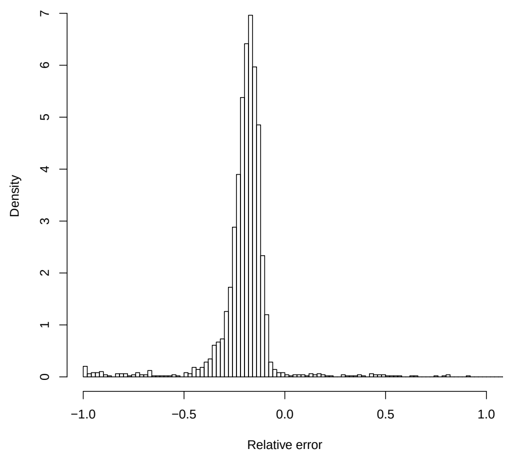
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level

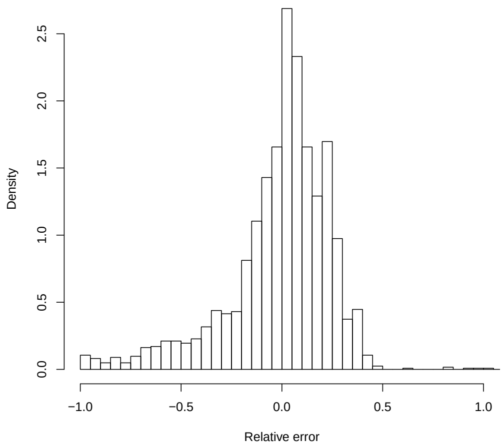


(c) SMA estimate, 97.5% confidence level

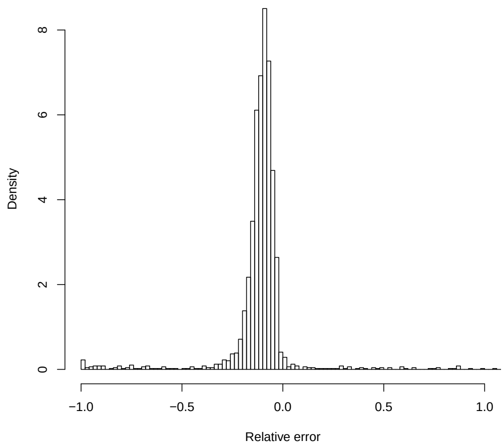


(d) EWMA estimate, 97.5% confidence level

Figure C.5: Histograms of the relative error between the ETL with a daily horizon and the mean of the tail losses each day. The portfolio tested generates semi-annual cashflows and both the SMA model and the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to to better display the remaining data.

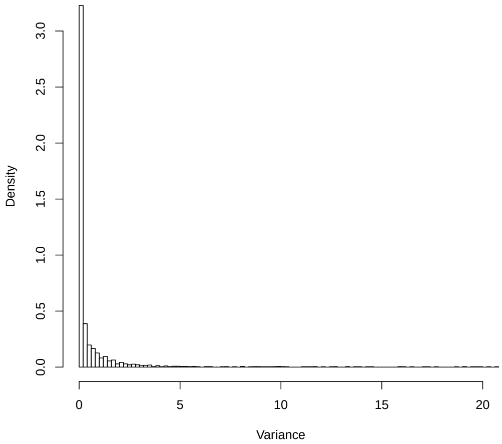


(e) SMA estimate, 99% confidence level

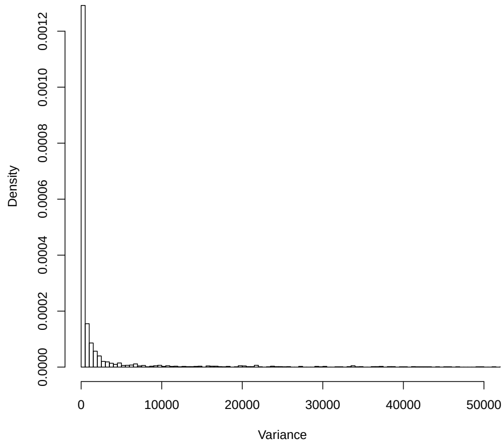


(f) EWMA estimate, 99% confidence level

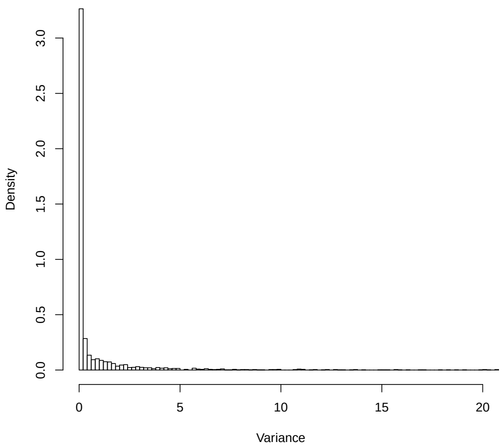
Figure C.5: Histograms of the relative error between the ETL with a daily horizon and the mean of the tail losses each day. The portfolio tested generates semi-annual cashflows and both the SMA model and the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to to better display the remaining data.



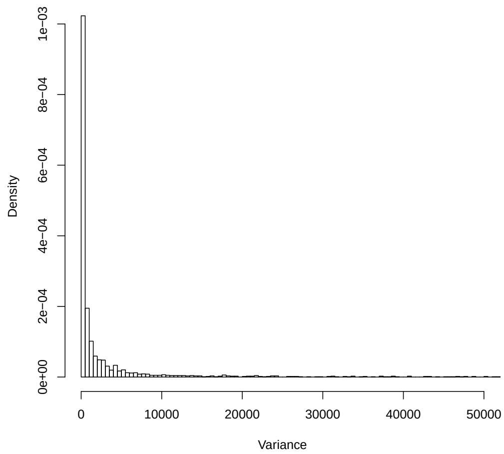
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level

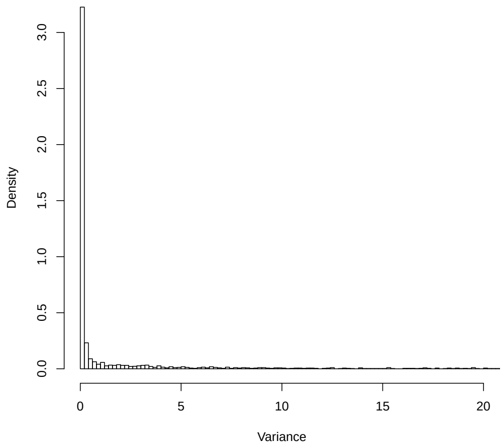


(c) SMA estimate, 97.5% confidence level

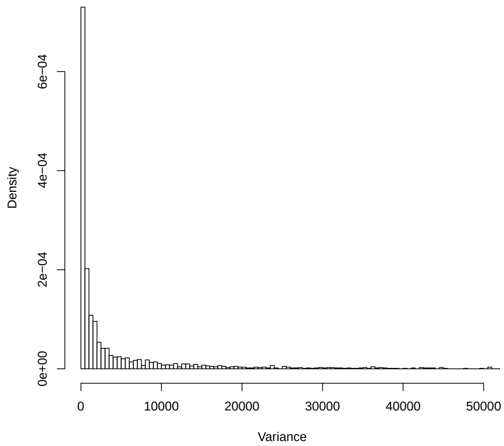


(d) EWMA estimate, 97.5% confidence level

Figure C.6: Histograms of the variances of the tail losses each day for the portfolio with semi-annual cashflows and a daily horizon. Some extreme values are left out of the histograms to better display the remaining data.



(e) SMA estimate, 99% confidence level

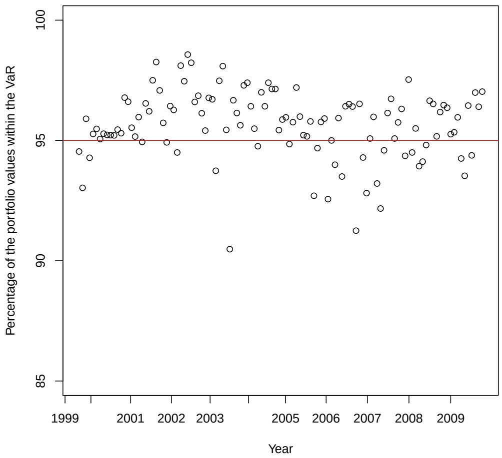


(f) EWMA estimate, 99% confidence level

Figure C.6: Histograms of the variances of the tail losses each day for the portfolio with semi-annual cashflows and a daily horizon. Some extreme values are left out of the histograms to better display the remaining data.

C.3 Value at Risk with a Monthly Horizon

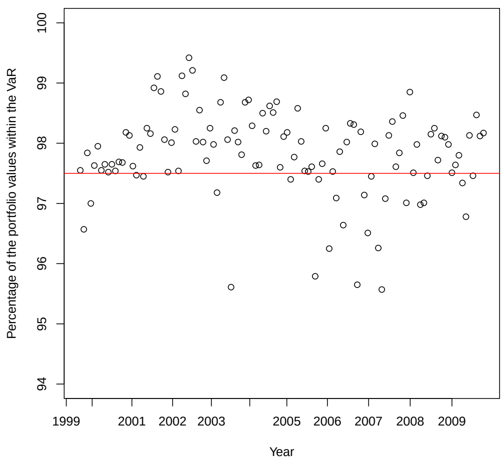
This section contains plots describing the performance of the VaR with a monthly horizon.



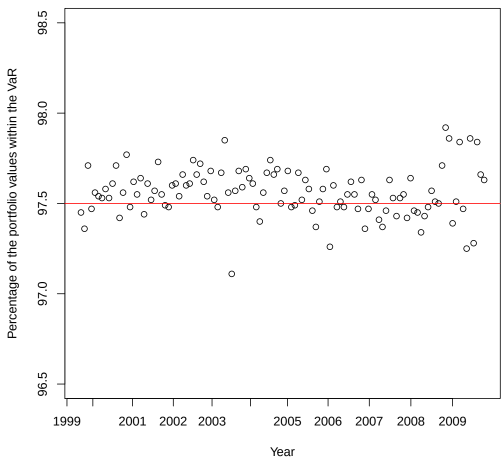
(a) SMA estimate, 95% confidence level

(b) EWMA estimate, 95% confidence level

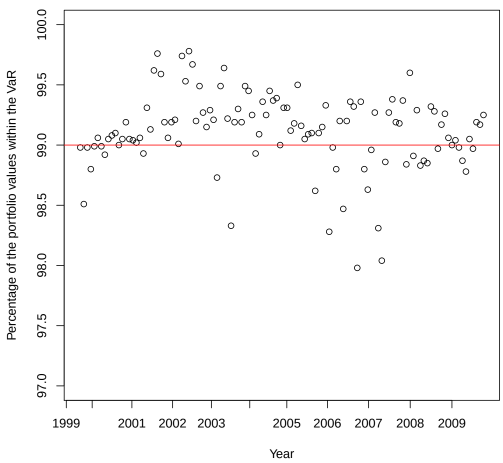
Figure C.7: Percentages of the portfolio's profits and losses within the VaR with a monthly horizon using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies.



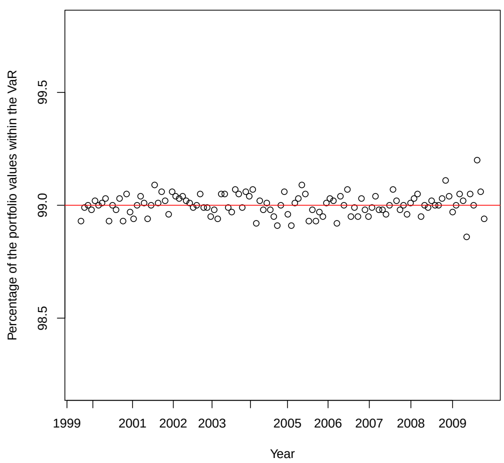
(c) SMA estimate, 97.5% confidence level



(d) EWMA estimate, 97.5% confidence level

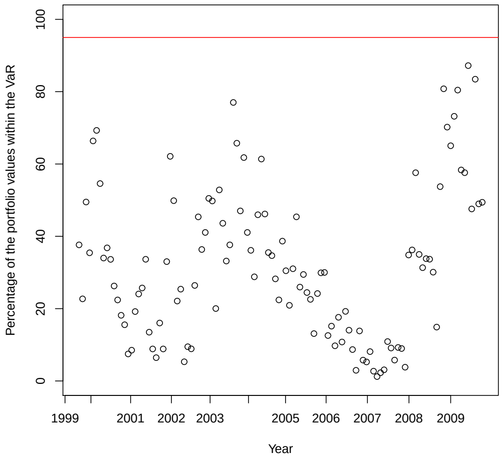


(e) SMA estimate, 99% confidence level

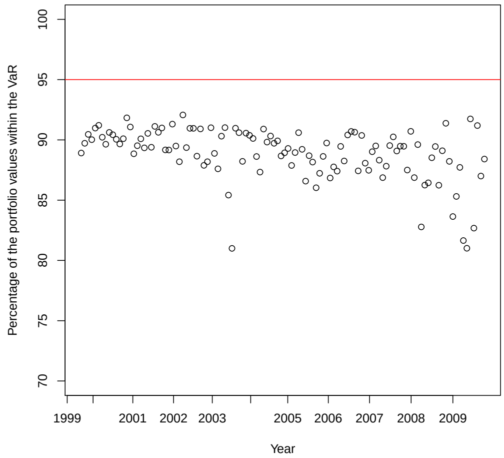


(f) EWMA estimate, 99% confidence level

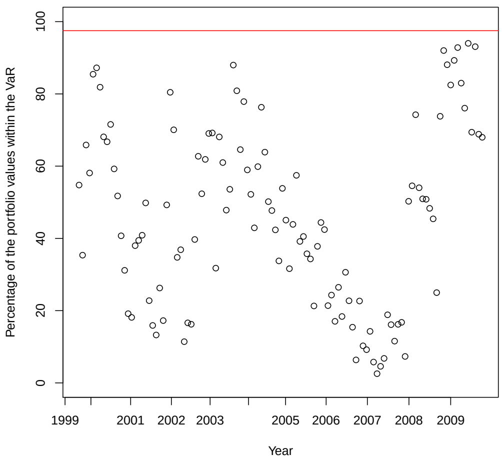
Figure C.7: Percentages of the portfolio's profits and losses within the VaR with a monthly horizon using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies.



(a) SMA estimate, 95% confidence level



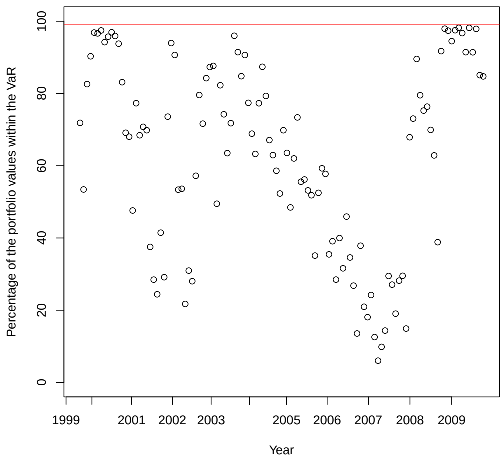
(b) EWMA estimate, 95% confidence level



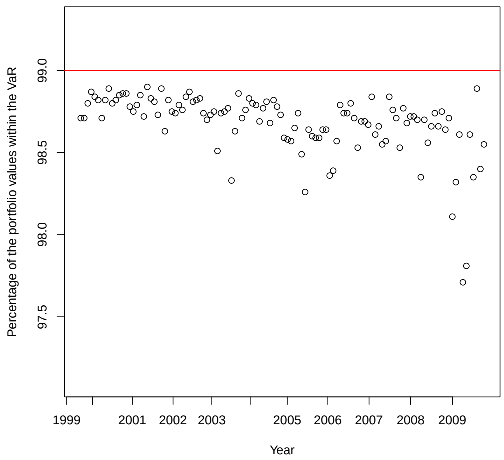
(c) SMA estimate, 97.5% confidence level

(d) EWMA estimate, 97.5% confidence level

Figure C.8: Percentages of the portfolio's profits and losses within the VaR with a monthly horizon using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates semi-annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies.



(e) SMA estimate, 99% confidence level

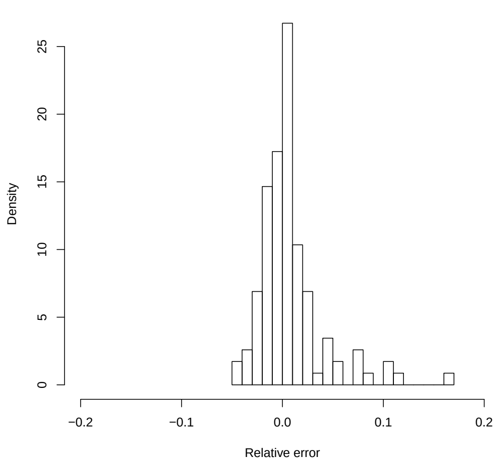


(f) EWMA estimate, 99% confidence level

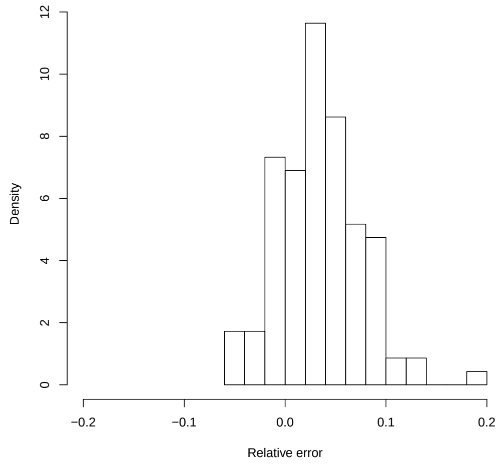
Figure C.8: Percentages of the portfolio's profits and losses within the VaR with a monthly horizon using either the SMA model or the EWMA model when estimating the covariance matrix. The portfolio tested generates semi-annual cashflows and the percentage for each day is displayed as a black circle while the given confidence level is displayed as a red line. Note that the range of the values on the y-axis varies.

C.4 Expected Tail Loss with a Monthly Horizon

This section contains plots describing the performance of the ETL with a monthly horizon.

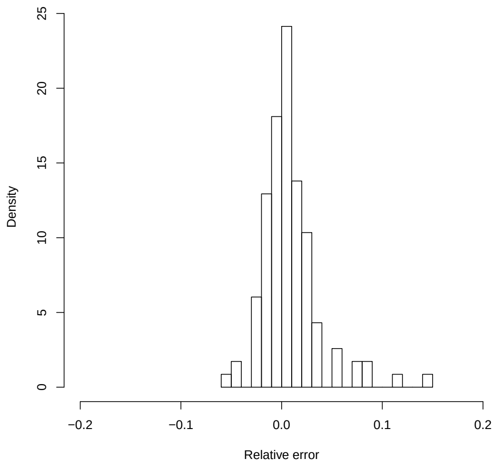


(a) SMA estimate, 95% confidence level

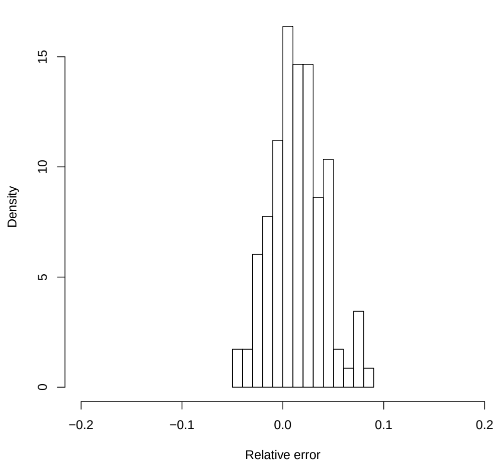


(b) EWMA estimate, 95% confidence level

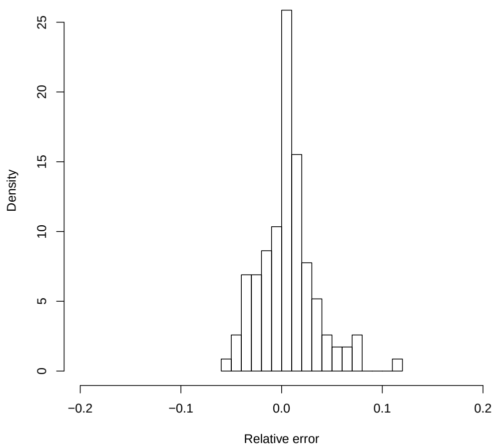
Figure C.9: Histograms of the relative error between the ETL with a monthly horizon and the mean of the tail losses each day. The portfolio tested generates annual cashflows and either the SMA model or the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to better display the remaining data.



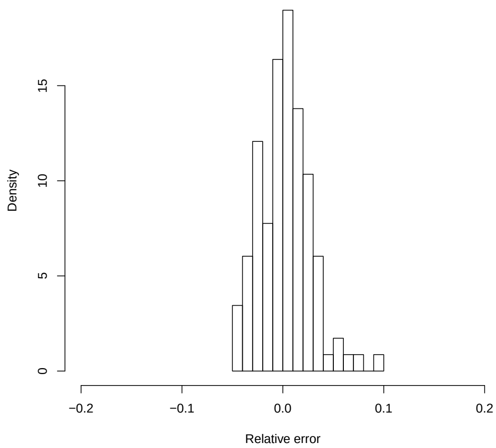
(c) SMA estimate, 97.5% confidence level



(d) EWMA estimate, 97.5% confidence level

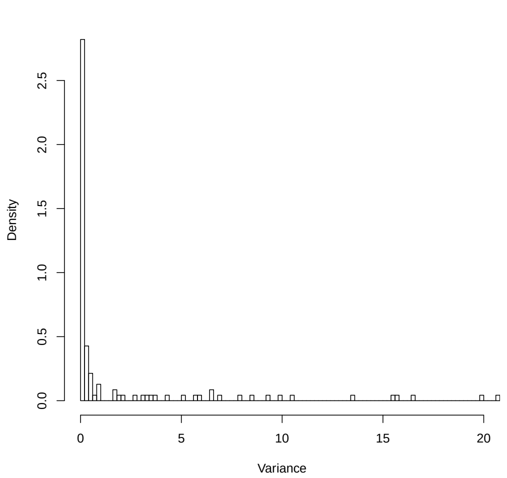


(e) SMA estimate, 99% confidence level

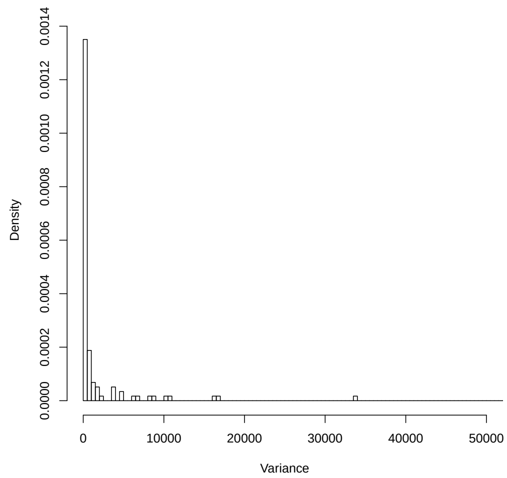


(f) EWMA estimate, 99% confidence level

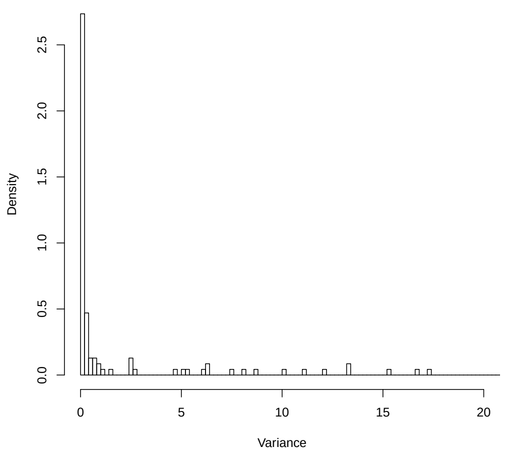
Figure C.9: Histograms of the relative error between the ETL with a monthly horizon and the mean of the tail losses each day. The portfolio tested generates annual cashflows and either the SMA model or the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to better display the remaining data.



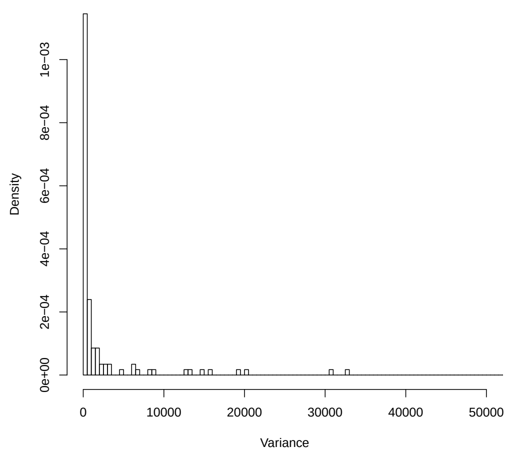
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level

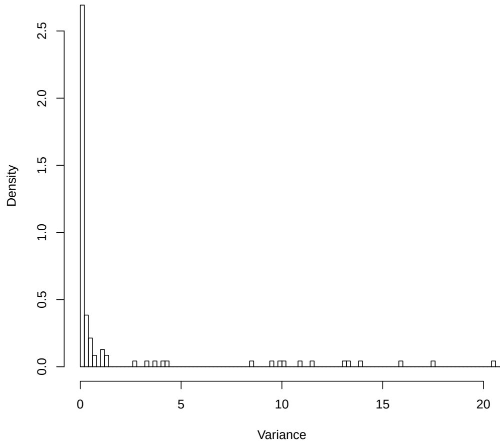


(c) SMA estimate, 97.5% confidence level

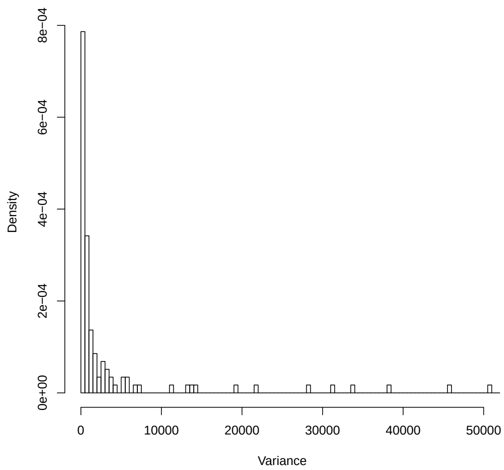


(d) EWMA estimate, 97.5% confidence level

Figure C.10: Histograms of the variances of the tail losses each day for the portfolio with annual cashflows and a monthly horizon. Some extreme values are left out of the histograms to better display the remaining data.

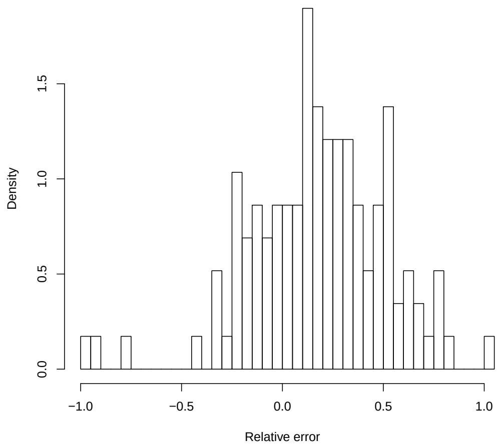


(e) SMA estimate, 99% confidence level

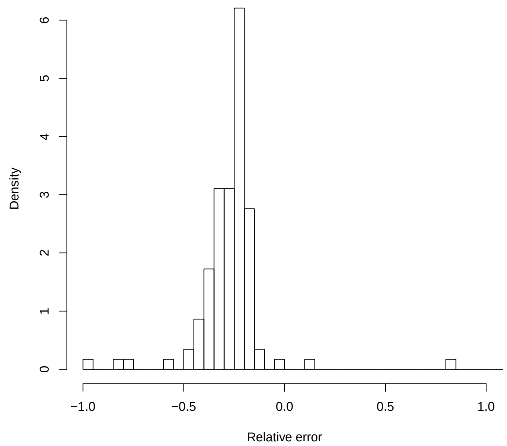


(f) EWMA estimate, 99% confidence level

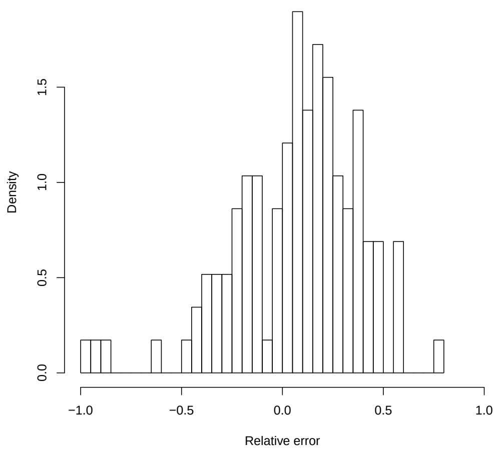
Figure C.10: Histograms of the variances of the tail losses each day for the portfolio with annual cashflows and a monthly horizon. Some extreme values are left out of the histograms to better display the remaining data.



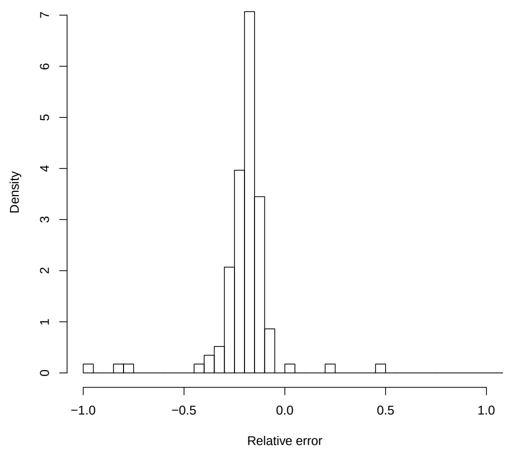
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level

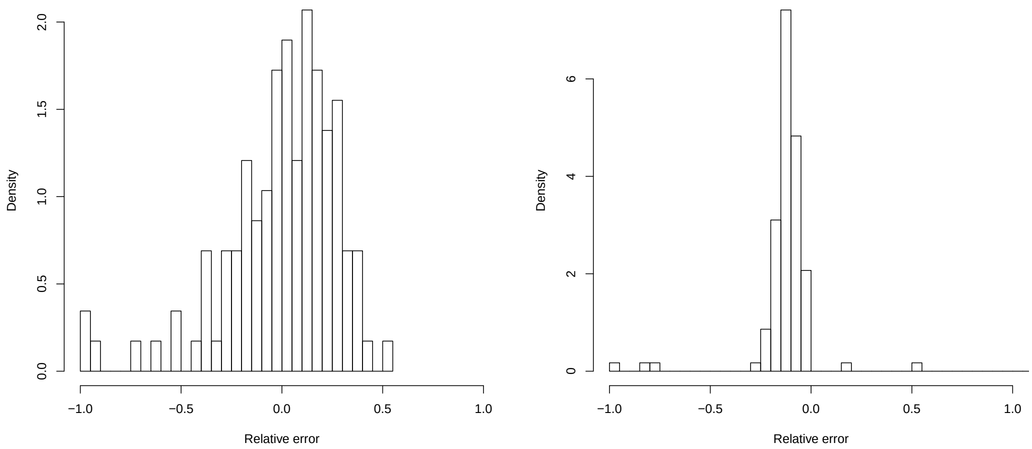


(c) SMA estimate, 97.5% confidence level



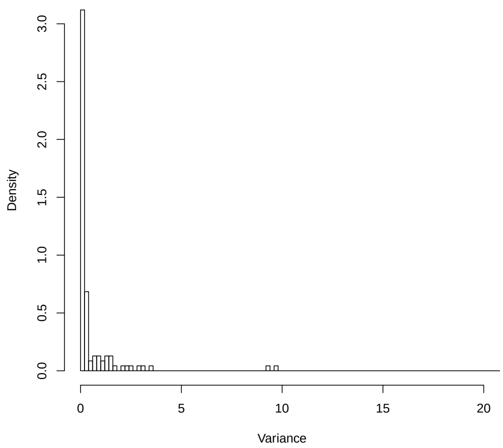
(d) EWMA estimate, 97.5% confidence level

Figure C.11: Histograms of the relative error between the ETL with a monthly horizon and the mean of the tail losses each day. The portfolio tested generates semi-annual cashflows and either the SMA model or the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to better display the remaining data.

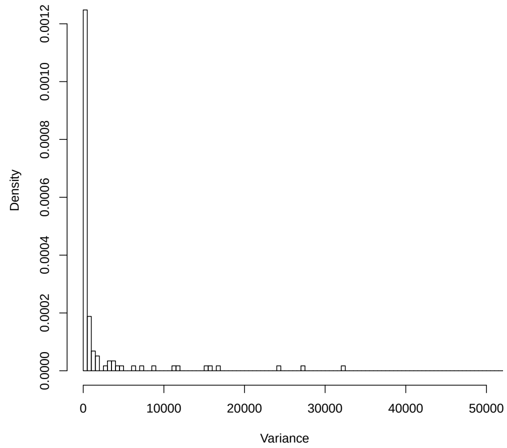


(e) SMA estimate, 99% confidence level (f) EWMA estimate, 99% confidence level

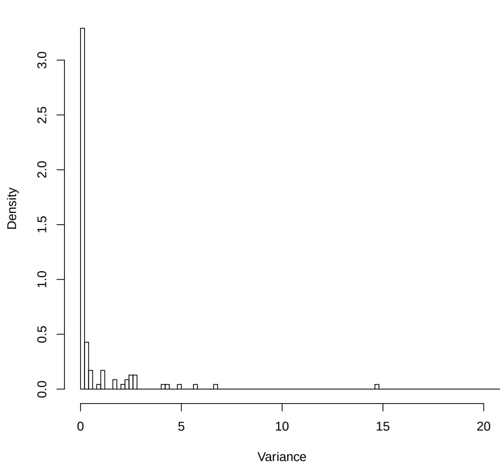
Figure C.11: Histograms of the relative error between the ETL with a monthly horizon and the mean of the tail losses each day. The portfolio tested generates semi-annual cashflows and either the SMA model or the EWMA model is used when the covariance matrix is estimated. Some extreme values are left out to better display the remaining data.



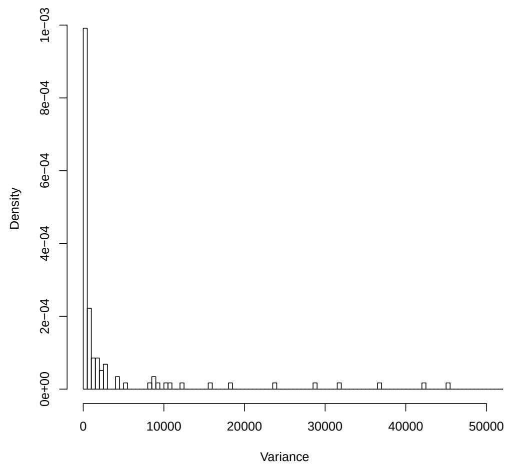
(a) SMA estimate, 95% confidence level



(b) EWMA estimate, 95% confidence level

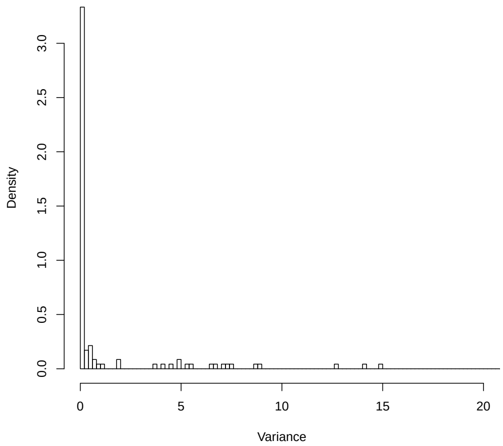


(c) SMA estimate, 97.5% confidence level

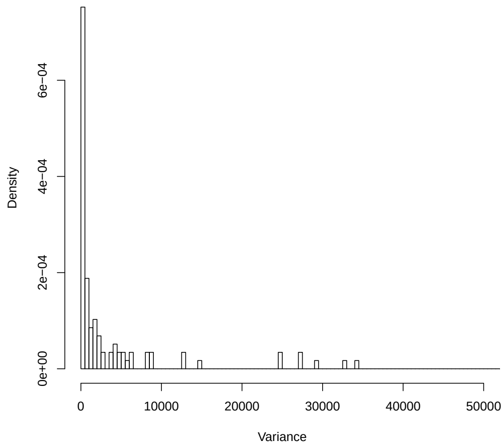


(d) EWMA estimate, 97.5% confidence level

Figure C.12: Histograms of the variances of the tail losses each day for the portfolio with semi-annual cashflows and a monthly horizon. Some extreme values are left out of the histograms to better display the remaining data.



(e) SMA estimate, 99% confidence level



(f) EWMA estimate, 99% confidence level

Figure C.12: Histograms of the variances of the tail losses each day for the portfolio with semi-annual cashflows and a monthly horizon. Some extreme values are left out of the histograms to better display the remaining data.