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Topological Dynamics and Algebra in the Spectrum of L infinity of a locally compact Group

With Application to Crossed Products

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Problem Description

To study an approach to the classical Kadison-Singer problem using dynamical systems and algebra in the Stone-Cech compactification of a discrete group, as well as trying to develop the theory in a crossed product setting following recent work by Vern Paulsen.

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Purpose of the work: To study an approach to the classical Kadison-Singer problem using dynamical systems and algebra in the Stone-Čech compactification of a discrete group, as well as trying to develop the theory in a crossed product setting following recent work by Vern Paulsen.

This diploma thesis is to be carried out at the Department of Mathematical Sciences under guidance of Professor Magnus B. Landstad.

Trondheim, January 19, 2009.

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Preface

The topic of this master thesis was mainly motivated by the still open Kadison-Singer problem, which I wrote about in my specialization project this last autumn. The Kadison-Singer problem is a 50 year old problem in operator theory that has lately been found to be equivalent to open problems in a wider variety of mathematical subfields such as in Hilbert and Banach space geometry, as well as signal coding. Two of these formulations are the Feichtinger and Bourgain-Tzafriri conjectures. I was fascinated by this problem that seems to hold the key to an understanding of some of the deeper aspects of these theories and also what may in a loose sense be called the discrepancy inherit in certain types of bases or sampling procedures.

In my specialization project I covered some of these equivalences, to a great extent following the paper [5] by Casazza, Fickus, Tremain and Weber. In this text, I will look at some new approaches that may shed some light on the theory, mainly one instigated by Vern Paulsen in [23]. In order to do this, I will try to develop the theory in a crossed product setting, and look at some aspects of it that may hold interest of their own. This article will be much more specialized than my previous one, but hopefully it may be of interest to some.

I would like to thank my advisor Professor Magnus B. Landstad for some helpful discussions that ultimately led to an improvement of the article, and Professor Christian Skau who helped me with a few difficult questions.

Magnus Dahler Norling,
Trondheim, June 12, 2009

Contents

1 Introduction

As was noted in the preface, this article was motivated by the Kadison-Singer problem. In short, the problem asks if any character, or pure state, on a discrete maximal abelian subalgebra of $B(\ell^2(\mathbb{Z}))$ has a unique extension to a pure state on all of $B(\ell^2(\mathbb{Z}))$. Such a maximal abelian subalgebra is isomorphic to $\ell^\infty(\mathbb{Z})$ in a natural way. To study the problem, we want to develop a broader variety of tools. Most of the theorems in this text are due to other authors, but sometimes we put them into a context. For completeness we have included many background results even when they are well known within the field.

In the first section, about amenable locally compact groups, we try to establish what kind of sets different invariant means on the same group will disagree on. We look at invariant means as invariant measures on the spectrum $\mathfrak{M}_G$ of L^∞ of the group, which is the same as the Stone-Čech compactification βG when the group is discrete. We also look more specifically at the dynamics and algebra of βG when acted upon by the group itself. These topics will prove important later on because βG is homeomorphic to the pure state space of $\ell^\infty(G)$.

In the section about crossed products, we look at both C^* and von Neumann crossed products by discrete groups. In particular, we study the reduced crossed product of $\ell^\infty(G)$ by G. The von Neumann crossed product of these two objects is because of the Stone-von Neumann Theorem known to be isomorphic to $B(\ell^2(G))$, and thus $\ell^\infty(G)$ sits inside it as a maximal abelian subalgebra in a natural way. We then look at the right group action of G on the crossed product and extend it to a right semigroup action of βG. We proceed to see what this action does to the C^* crossed product.

Finally, we come to the Kadison-Singer problem itself, and following Paulsen's paper [23], we relate it to problems concerning extensions of several completely positive maps, in particular the right action of βG discussed in the previous section. We also relate the problem to some concerning invariant means on the dual group $\hat{G}$, following earlier work by Halpern, Kaftal and Weiss [11]. Using these approaches, we try to narrow down the search for a potential counterexample or solution to the problem.

This text relies on some extensive background theory, and to ease the flow of the text, we have tried to put a good deal of it into the appendices. However, if the reader feels uncomfortable with some of the background topics, she or he should probably make a quick read through of some of the appendices before trying for the main text. Even with the background theory provided here, the text will hard be to read without any previous knowledge in functional analysis and operator theory, but it shoud be accessible to anybody with some experience in these topics. We recommend using the index at the back of the article to look up unfamiliar words.

2 Dynamics in the spectrum of $L^\infty(G)$

2.1 Preliminaries

Let G be a locally compact group. In this section, we will look at dynamical systems in the spectrum of $L^\infty(G) = L^\infty(G,\mu)$. Here $\mu = \mu_G$ denotes the left Haar measure on G. If G is abelian, we will denote its dual group by $\hat{G}$. We assume that the Haar measures are always normalized to make the Plancherel transform an isometry. See appendix B for a short introduction to this theory. Although it may be a little confusing, we will let 1 denote the identity element in G unless G is a specific group such as $\mathbb{Z}$. $\mathcal{B}(G)$ is the σ-algebra of Borel sets in G, and $\mathcal{B}_0(G)$ is the Boolean algebra of Borel sets modulo the null sets.

By Theorem E.2 we know that the pure state space $PS(L^\infty(G))$ is homeomorphic to $\mathfrak{M}_G = \mathfrak{M}(\mathcal{B}(\mu_G))$, the Stonean space of ultrafilters on $\mathcal{B}_0(G)$. See also appendix C for reference. It should be mentioned that if G is discrete, $\mathfrak{M}_G$ is just the Stone-Čech compactification βG of G. We will use this to see how the left action of G on itself affects $L^\infty(G)$, which we by the Gelfand transform and above mentioned homeomorphism know is isometrically isomorphic to $C(\mathfrak{M}_G)$. Each $\omega \in \mathfrak{M}_G$ will be associated to the pure state s_ω given by evaluation at ω.

2.2 Invariant means and amenability

Let G be a locally compact group with left Haar measure μ. For any function f on G and any $g \in G$, denote by $\lambda_g f$ the function $\lambda_g f(h) = f(g^{-1}h)$. Note that $\lambda_g \chi_E = \chi_{gE}$, where χ_E is the characteristic function of a Borel subset $E \subset G$. By the invariance of the Haar measure, the operation λ_g preserves μ-nullsets, and is therefore well-defined for functions in $L^\infty(G)$ as well.

A (left) *invariant mean* on $L^\infty(G)$ is a state $m : L^\infty(G) \to \mathbb{C}$ satisfying

$$m(\lambda_g f) = m(f) \tag{2.1}$$

for all $g \in G$ and $f \in L^\infty(G)$. This can also be written $m \circ \lambda_g = m$. We say that G is *amenable* if $L^\infty(G)$ has an invariant mean.

Note that the definition of amenability is not entirely consistent between different books. In some earlier texts, it was customary to say that G was amenable if and only if $\ell^\infty(G)$ had an invariant mean, regardless of G being discrete or not [27]. We will say that G is *discretely amenable* if $\ell^\infty(G)$ posesses an invariant mean. The definition we use coincides with the one in [10], where it is shown that discrete amenability implies amenability (Theorem A.13). The opposite is not the case, as there are compact groups, such as the Lie group of three-dimensional isometries $O(3)$, that are not discretely amenable [10]. But all compact groups are amenable by integration with the Haar measure.

Example 2.1 Let G be discrete and countable. We say that a *Følner sequence*[1] in G is a sequence of finite sets $\{F_j\}_{j=0}^{\infty}$ with $F_j \subset G$ such that for each $g \in G$ there is an $n \in \mathbb{N}$ with $g \in F_j$ for all $j \geq n$. Also, for each $g \in G$ we want that

$$\lim_{j \to \infty} \frac{|gF_j \Delta F_j|}{|F_j|} = 0.$$

Since the sequence of functionals given by

$$m_j(f) = \frac{1}{|F_j|} \sum_{g \in F_j} f(g)$$

is bounded, we can by the Banach-Alaoglu Theorem pick a subsequence that converges in the weak*-topology to a functional m. Then m is an invariant mean on $\ell^\infty(G)$, because given $f \in \ell^\infty(G)$, $g \in G$ and $\varepsilon > 0$, we can pick $n \in \mathbb{N}$ such that

$$\frac{|gF_j \Delta F_j|}{|F_j|} < \frac{\varepsilon}{2\|f\|}.$$

for all $j \geq n$. Then

$$\frac{1}{|F_j|} \left| \sum_{h \in F_j} f(g^{-1}h) - f(h) \right| \leq 2\|f\| \frac{|gF_j \Delta F_j|}{|F_j|} < \varepsilon$$

for all $j \geq n$, so we get $|m(\lambda_g f - f)| = 0$.

The typical example of a Følner sequence in $\mathbb{Z}$ is of course $F_k = [-k, k]$. The corresponding means then coincide with the Cesàro mean whenever it exists.

2.3 Borel sets with special structure

A few special families of subsets of the locally compact group G will be of interest to us when we study the dynamical properties of points in $\mathfrak{M}_G$.

Let $E \subset G$ be a Borel subset. We say that E is *thick* if for every finite sequence $g_0 \dots g_r$ in G,

$$\mu\left(\bigcap_{j=0}^{r} g_r E\right) > 0.$$

[1] If G is uncountable, we can use a Følner net instead

Example 2.2 If G is compact, then every thick set is dense, and in any locally compact group, every dense open set is thick. The last remark follows from the fact that the intersection of an open set with a dense set is nonempty, and that the intersection of two open sets is open and does therefore have nonzero measure. Moreover, the intersection of two dense open sets must be dense since the intersection clearly is dense relatively to either of them, and therefore also to all of G.

Example 2.3 For $G = \mathbb{Z}$, a typical thick subset can be given as

$$E = \bigcup_{j=0}^{\infty} [2^{2^j-1}, 2^{2^j}]. \tag{2.2}$$

The same formula also gives a thick subset of $\mathbb{R}$.

Example 2.4 Constructing a nontrivial thick subset of a compact group is a little more subtle. We will create a thick subset of $\mathbb{T}$ that is the complement of a Cantor set with nontrivial measure. Identify $\mathbb{T}$ with the interval $[0,1]$, and let $\{a_j\}_{j\in\mathbb{N}}$ be a positive real sequence with

$$0 < \sum_{j\in\mathbb{N}} 2^j a_j < 1 \tag{2.3}$$

Construct the "middle third" Cantor set by removing an open interval of measure a_1 from the middle of $[0,1]$, and proceed to remove two open intervals of length a_2 from the middle of the remaining intervals and so on. Our thick set E will be the union of the removed intervals. Then the measure of E equals the number in (2.3), and to see that E is thick, it is enough to see that it is dense. But this is obvious, because any point in $[0,1]$ is arbitrarily close to a point of the form $\sum_{j=1}^{N} b_j 2^{-j}, b_j \in \{0,1\}$.

The set in (2.2) is actually an example of a thick set whose complement is thick. As we shall see, such sets can also be found in compact groups. The next two theorems are from [27]. We say that a locally compact group is *compactly generated* if there is a compact subset $E \subset G$ with $G = \cup_{n=1}^{\infty}(F \cup F^{-1})^n$.

Theorem 2.5 (Rudin) If G is a locally compact group which is not discrete and which is compactly generated, and $\varepsilon > 0$, then G contains a dense open set with $\mu(E) < \varepsilon$

Proof. Since G is not discrete, the identity of G has neighbourhoods U_n with compact closure and $\mu(U_n) < 1/n$. It follows from Theorem 8.7 in [13] that G has a compact normal subgroup $N \subset \cap_{n=1}^{\infty} U_n$ such that $\mu(N) = 0$ and such that G/N is separable. The union of a countable number of cosets of

N is therefore dense in G, and by the regularity of μ, these can be covered by open sets whose union has measure less than ε. $\square$

Theorem 2.6 (Rudin) Let G be an infinite compact group. Then G has a thick subset with thick complement.

Proof. Metrize $\mathcal{B}_0(G)$ with the metric $d(E,F) = \mu(E\Delta F)$. It is straightforward to check that this makes $\mathcal{B}_0$ into a complete metric space. For each $n \in \mathbb{N}$, let $Q_n \subset \mathcal{B}_0$ be the collection of sets E having the property that for some sequence $g_1 \dots g_n$ of length n,

$$\mu\left(\bigcap_{j=1}^{n} g_j E\right) = 0. \tag{2.4}$$

Let Q'_n be the collection of sets whose complement belongs to Q_n. The map sending a set to its complement is an isometry of $\mathcal{B}_0$ in the given metric, so Q_n and Q'_n are homeomorphic. We shall show that Q_n, and thus Q'_n, is closed and has empty interior. By Baire's Category Theorem, $\mathcal{B}_0$ can therefore not be the union of the Q_ns and the Q'_ns, but must contain something more, i.e. a set satisfying the claim of this theorem.

Let $E \in Q_n$. By the regularity of μ and the previous theorem, there are dense open sets $V_k \supset E$ such that $d(E, V_k) < 1/k$ for all $k \in \mathbb{N}$. Each V_k is outside Q_n, thus Q_n has empty interior.

Let $F \in \mathcal{B}_0$ be in the closure of Q_n, and fix $\varepsilon > 0$. Let $E \in Q_n$ with $d(E,F) < \varepsilon/n$, and let $g_1 \dots g_n$ be such that (2.4) holds. Now,

$$\begin{aligned} \mu\left(\bigcap_{j=1}^{n} g_j F\right) &\leq d\left(\bigcap_{j=1}^{n} g_j F, \bigcap_{j=1}^{n} g_j E\right) \\ &\leq \sum_{j=1}^{n} d(g_j F, g_j E) = n d(F,E) < \varepsilon. \end{aligned}$$

The left hand side is a continuous function of $g_1 \dots g_n$ on the space G^n, and since G^n is compact, the function attains its infimum, that is 0. So $F_n \in Q_n$, and Q_n is closed. $\square$

We will need the next lemma later.

Lemma 2.7 Let G be a compact group, and let $E \subset G$ be thick. Then

$$\inf\{\int_G f \,\mathrm{d}\mu \mid f \in C(G), f \geq \chi_E \text{ a.e.}\} = 1$$

Proof. Clearly, the expression is less than or equal to 1. Let $f \geq \chi_E$ be continuous. Then $\{f < 1\} \subset E^c$ is open, so a finite number of translates of it cover G since G is compact. But then E^c is syndetic, so E is not thick. It follows that the infimum is 1. $\square$

If E is a Borel subset of G whose complement is not thick, we call E *syndetic*. So E is syndetic if and only if there exists a finite sequence $g_0 \dots g_r$ in G satisfying

$$\mu\left(G \setminus \bigcup_{j=0}^{r} g_r E\right) = 0.$$

I.e. a finite number of translates of E cover almost all of G.

A *piecewise syndetic* set is one that is the intersection of a syndetic set and a thick set. We see that by definition, piecewise syndetic sets are necessarily nonnull, because the complement of a syndetic set is not thick and can therefore not contain a thick set.

2.4 Dynamics in $\mathfrak{M}_G$

We will study a natural action of G on $\mathfrak{M}_G$ and in particular describe the invariant subsets and orbits of special points under this action.

For $g \in G$, define $\kappa_g \in \mathrm{Homeo}(\mathfrak{M}_G)$ by

$$\kappa_g(\omega) = \{gE \mid E \in \omega\},$$

where the right hand side defines a new ultrafilter on $\mathcal{B}_0(G)$. It is straightforward to see that for a fixed g, κ_g is continuous in the Stone topology. Alternatively, the action can be defined on $PS(L^\infty(G))$ by the map $\phi \mapsto \phi \circ \lambda_g$. For simplicity, we will use the notation

$$\begin{aligned} g \cdot \omega &= \kappa_g(\omega) \\ G \cdot \omega &= \mathcal{O}_\kappa(\omega). \end{aligned}$$

This action has been studied to a great extent when G is discrete, in particular when $G = \mathbb{Z}$, and has several applications in combinatorial number theory [3][14]. Note that the map $g \mapsto g \cdot \omega$ is usually far from continuous when G is nondiscrete, and thus this system does not qualify to be a dynamical system if using the definition often found in the literature. We, however use a looser definition only requireing the map $\omega \mapsto g \cdot \omega$ to be continuous.

An action of G on a space X is said to be *free* if every $x \in X$ has a trivial stabilizer. That is if for each $x \in X$, the set

$$\mathrm{St}_\kappa(x) = \{g \in G \mid \kappa_g(x) = x\}$$

consists solely of the identity.

Lemma 2.8 Let G be a locally compact abelian group that has the property that for every $g \in G$, there is a syndetic set $E \subset G$ such that $gE \cap E^{\complement}$ has zero measure. Then the action of G on $\mathfrak{M}_G$ is free.

Proof. Let $\omega \in \mathfrak{M}_G$ and $g \in G$, and let $E \subset G$ be as in the statement of the lemma. Then since some finite translates of E cover almost all of G, there is a $h \in G$ with $hE \in \omega$. But since $ghE \cap (Eh)^{\complement}$ is a null set, we can't have $ghE \in \omega$, thus $g \cdot \omega \neq \omega$. □

Example 2.9 It is easy to devise such sets in some of the most common groups. For instance if $G = \mathbb{Z}$, given $k \in \mathbb{Z}$, the syndetic set $(k+1)\mathbb{Z}$ will do. For $t \in \mathbb{R}$, let

$$E = \{x \in \mathbb{R} \mid 0 \leq x \pmod{2t} < t\}.$$

In any compact group, we can simply find a neighborhood E of the identity that is small enough.

We have not been able to find out if the action is free in general, but it is when G is discrete. Veech's Theorem [31] says that every locally compact group G acts freely on its left uniformly continuous compactification. This can be viewed as the spectrum of the C^*-algebra of left uniformly continuous functions on G, which of course coincides with βG when G is discrete.

Recall from appendix D that a minimal element of a dynamical system is one that topologically generates a minimal subsystem. That is, the closure of its orbit contains no smaller closed subsystems.

Lemma 2.10 Let (G, X) be a discrete dynamical system where X is compact Hausdorff. If U is an open neighbourhood of a minimal $x \in X$, then the set

$$E = \{g \in G \mid g \cdot x \in U\}$$

is syndetic.

Proof. Note first that for each $h \in G$, $hE = \{g \in G \mid g \cdot x \in h \cdot U\}$. Since $\overline{\mathcal{O}_\kappa(x)} = \overline{G \cdot x}$ is a minimal subsystem, the orbit of U covers $\overline{G \cdot \omega}$, and since this closure is compact we can pick out a finite subcover $\{h_j \cdot U\}_{j=0}^{r}$. Then

$$\bigcup_{j=0}^{r} h_j E = \{g \in G \mid g \cdot x \in \bigcup_{j=0}^{r} h_j \cdot U\} = G,$$

so E is syndetic. □

Lemma 2.11 Let G be a locally compact group, and let $\omega \in \mathfrak{M}_G$ be minimal. Then every $E \in \omega$ is piecewise syndetic.

Proof. We have that $\widetilde{E} \subset \mathfrak{M}_G$ is open, and that $\omega \in \widetilde{E}$, so by minimality of ω there are $g_1 \ldots g_r \in G$ such that

$$\overline{G \cdot \omega} \subset \bigcup_{j=1}^{r} g_j \widetilde{E} = \widetilde{\bigcup_{j=1}^{r} E}.$$

Let F denote the union of these particular translates of E. Then $g \cdot \omega \in \widetilde{F}$ for all $g \in G$, so $gF \in \omega$ for all g and thus the collection $\{gF\}_{g \in G}$ has the finite intersection property, i.e. F is thick. So a finite union of translates of E is thick, and E is therefore piecewise syndetic. □

See Greenleaf's Theorem 3.3.5 for a much more general statement of the next lemma [10], which in some sense generalizes the Markov-Kakutani Theorem.

Lemma 2.12 Let G be a discrete group. Then G is amenable if and only if for every dynamical system (X, G, κ) there is a G-invariant state on $C(X)$.

Proof. If G has this property, let $X = \mathfrak{M}_G$ with κ as above. Since $C(\mathfrak{M}_G) \simeq \ell^\infty(G)$, a G-invariant state on $C(\mathfrak{M}_G)$ carries over to a G-invariant state, i.e an invariant mean on $\ell^\infty(G)$.

Now, if G is amenable, pick $x \in X$ and define $\pi_x : C(X) \to \ell^\infty(G)$ by $\pi_x(f)(g) = f \circ \kappa_g(x)$. If m is an invariant mean on $\ell^\infty(G)$, we see now that $m \circ \pi_x$ is a G-invariant state on $C(X)$. □

Lemma 2.13 Every abelian locally compact group is (discretely) amenable.

Proof. The proof is the same for discrete and non-discrete groups. Let $\alpha_g : C(\mathfrak{M}_G) \to C(\mathfrak{M}_G)$ be the translation map, and let $\alpha_g^* : S(C(\mathfrak{M}_G)) \to S(C(\mathfrak{M}_G))$ be given by $\alpha_g^*(\phi) = \phi \circ \alpha_g$. Then $\{\alpha_g^*\}_{g \in G}$ is a commuting family of linear self-maps of the compact convex set $S(C(\mathfrak{M}_G))$, so by the Markov-Kakutani Theorem D.4 they have a common fixed point, i.e. a G-invariant state. □

2.5 Algebra in βG

The results in this section are taken from the book "Algebra in the Stone-Čech compactification" by Hindman and Strauss [14].

Let G be a discrete group or semigroup. It turns out that some of the algebraic operations in G can be extended to βG. For $\omega, \rho \in \beta G$, define

$$\omega \cdot \rho = \lim_{g \to \omega} g \cdot \rho.$$

By Lemma C.4, this is well-defined since $g \mapsto g \cdot \rho$ maps G into βG, which is a compact Hausdorff space.

Lemma 2.14 With the operation $(\omega, \rho) \mapsto \omega \cdot \rho$, βG is a right topological semigroup. Here right topological means that for every $\omega \in \beta G$, the map $\rho \mapsto \rho \cdot \omega$ is continuous.

Proof. Right continuity follows directly from the definition, so we only have to prove associativity. We have for $\omega, \rho, \varrho \in \beta G$,

$$\begin{aligned}
\omega \cdot (\rho \cdot \varrho) &= \lim_{g \to \omega} g \cdot (\lim_{h \to \rho} h \cdot \varrho) \\
&= \lim_{g \to \omega} \lim_{h \to \rho} (gh) \cdot \varrho \\
&= \lim_{g \to \omega} (g \cdot \rho) \cdot \varrho \\
&= (\omega \cdot \rho) \cdot \varrho.
\end{aligned}$$

$\square$

However, even when G is commutative, βG is generally far from being so. In fact, the centre of $\beta \mathbb{Z}$ is $\mathbb{Z}$. Write $G^* = \beta G \setminus G$. Since $\mathbb{N}^*$ and $(-\mathbb{N})^*$ can easily be verified to be left ideals of $\beta \mathbb{Z}$, we have that given $\omega \in \mathbb{N}^*$ and $\rho \in (-\mathbb{N})^*$, $\rho + \omega \in \mathbb{N}^*$, and $\omega + \rho \in (-\mathbb{N})^*$, so $\omega + \rho \neq \rho + \omega$.

Lemma 2.15 (The Ellis-Nakamura Lemma) Let $X \subset \beta G$ be a closed invariant subsystem. Then there is an $\omega \in X$ such that $\omega \cdot \omega = \omega$.

Proof. Let $Y \subset X$ be a compact subsemigroup that is minimal with respect to set inclusion, and let $\omega \in Y$. Such minimal subsemigroups can be found using Zorn's Lemma. Then $Y \cdot \omega$, is a compact subsemigroup of Y, and so must equal Y. In addition, the set $\{\rho \in Y \mid \rho \cdot \omega = \omega\}$ is nonempty, and it is also a closed subsemigroup, thus it must equal Y. Then $\omega \cdot \omega = \omega$ since $\omega \in Y$ by choice. $\square$

We call such elements *idempotents*. If $\omega \in \beta G$ is minimal and idempotent, we call it (surprisingly enough) a *minimal idempotent*. The principal ultrafilter of the identity of G is always idempotent, but for instance Lemma 2.10 guarantees us that unless G is finite, any minimal idempotent is nontrivial. Their existence follows from the existence of minimal subsystems in any dynamical system (see Lemma D.1).

By Lemma C.4, we also see that the closure of the orbit of an element $\omega \in \beta G$ can be written as $\beta G \cdot \omega$. This becomes a closed left ideal in the semigroup βG, and it is straightforward to see that the closed minimal left ideals in βG are exactly the minimal systems $\beta G \cdot \omega$, where ω is minimal.

2.6 Nonuniqueness of invariant means

The next Theorem is basically just a modification of Theorem 3.4 in [27], but Rudin does not mention the (albeit trivial) necessity of the condition, only its sufficiency.

Theorem 2.16 Let G be a locally compact (not necessarily discrete) group, that is discretely amenable, and let $E \subset G$ be a Borel subset. For there to be a left invariant mean m on $L^\infty(G)$ such that $m(\chi_E) = 1$, it is necessary and sufficient for E to be thick.

Proof. First, if E is not thick, then its complement is syndetic, so there is a finite sequence $g_0 \dots g_r$ in G such that $\{g_j E^c\}_{j=0}^r$ covers almost all of G. Any invariant mean m is finitely additive, and it follows that $m(\chi_{E^c}) \geq 1/r$. Then $m(\chi_E) = 1 - m(\chi_{E^c}) < 1$. This proves necessity.

Now, let $E \subset G$ be a thick subset. Then the set

$$Y = \bigcap_{g \in G} \widetilde{gE}$$

is an invariant closed subsystem, so we can pick a minimal $\omega \in Y$, and define $\pi_\omega : L^\infty(G) \to \ell^\infty(G)$ by $\pi_\omega(f)(g) = \hat{f}(g \cdot \omega)$. If m is any invariant mean on $\ell^\infty(G)$, then $m \circ \pi_\omega$ is an invariant mean on $L^\infty(G)$, and $\pi_\omega(\chi_E) = 1$. $\square$

Corollary 2.17 Let G be a locally compact discretely amenable group, and let $E \subset G$ be Borel. For E to have the property that $m(\chi_E) > 0$ for every invariant mean on $L^\infty(G)$, it is necessary and sufficient for E to be syndetic.

The next theorem comprises a few results from [11] and [23].

Theorem 2.18 Let G be a compact discretely amenable group, and let $f \in L^\infty(G)$ be a real-valued function. The following two properties of f are equivalent.

(i) $m(f) = \int_G f \, \mathrm{d}\mu$ for every invariant mean m on $L^\infty(G)$.

(ii) For every $\varepsilon > 0$, there is an $r \in \mathbb{N}$ and $g_1 \dots g_r \in G$ with

$$\left\| \frac{1}{r} \sum_{j=1}^{r} \lambda_{g_j} f - \int_G f \, \mathrm{d}\mu \right\|_\infty < \varepsilon.$$

Moreover, the next property implies (i)-(ii).

(iii) For every $\varepsilon > 0$, there are $f_1, f_2 \in C(G)$ with $f_1 \leq f \leq f_2$ almost everywhere and $\int_G (f_2 - f_1)\,\mathrm{d}\mu < \varepsilon$.

If $G = \mathbb{T} = [0,1)(\bmod 1)$, then f satisfies (iii) if and only if it is almost everywhere equal to a Riemann-integrable function.

Proof. (ii)⇒(i) is trivial, and (iii)⇒(i) follows from the fact that every invariant mean must agree with the Haar integral on $C(G)$ since every state on $C(G)$ is a measure on G, and the Haar measure is the unique invariant probability measure on G.

(i)⇒(ii): Assume without loss of generality that $\int_G f\,\mathrm{d}\mu = 0$, and suppose (ii) fails. Then we can pick a nonzero $x \in \mathbb{R}$ such that

$$\sup_{\substack{r\in\mathbb{N}\\ \{g_j\}_{j=1}^r \subset G}} \operatorname*{ess\,inf}_{g\in G} \frac{1}{r}\sum_{j=1}^r f(g_j^{-1}g) \leq x \leq \inf_{\substack{r\in\mathbb{N}\\ \{g_j\}_{j=1}^r \subset G}} \operatorname*{ess\,sup}_{g\in G} \frac{1}{r}\sum_{j=1}^r f(g_j^{-1}g),$$

where the suprema and infima are taken over all finite subsequences of G. Denote by μ the functional on $C(G)$ given by Haar integration, and create an extension m' of μ to the space spanned by all translates of f given by $m'(f'+t\lambda_g f) = \int_G f'\,\mathrm{d}\mu + tx$. By the last equation, this extension is positive, and by construction it is G-invariant. As we will come back to in Krein's Extension Theorem (Lemma 4.1), the space X of positive extensions of m' to all of $L^\infty(G)$ is nonempty, and one can check that it is a closed G-invariant subset of the dual of $L^\infty(G)$. Then since G is discretely amenable, we can as in Lemma 2.12 pick an invariant mean k on $\ell^\infty(G)$ and a point $m_0 \in X$, and construct the functional m on $L^\infty(G)$ given by $m(f') = k(g \mapsto m_0(\lambda_g f'))$. Then m is an invariant mean, and $m(f) = m_0(f) = m'(f) = x \neq 0$, which contradicts (i).

We leave the last comment about Riemann-integrability as an exercise to the reader. The proof can be found in [11] or [23]. Note that being almost everywhere equal to a Riemann-integrable function is not the same as being Riemann-integrable. The characteristic function of the rationals is a good example of this. □

2.7 Notes

Invariant means were first introduced by von Neumann in [32]. Greenleaf's book probably has the largest collection of results in this theory [10]. Williams also gives a good overview about their importance for operator theory in [34] (appendix A).

The nonuniqueness of invariant means on discrete infinite groups were first established in [9] and expanded among other places in [10] (Lemma A.1.2). Rudin first proved that any infinite locally compact discretely amenable group has nonunique twosided invariant means on $L^\infty(G)$ [27]. He used thick sets, which he called "permanently positive" sets, to prove this. We have chosen to label them "thick" because this corresponds better with the terminology used in combinatorics and the theory of dynamical systems. Other terminology has also been used about what we call syndetic and piecewise syndetic sets. For instance in [30], syndetic sets are called relatively accumulating.

The reader may have noticed that we only manage to prove some of the central results for groups that are discretely amenable. We have not succeeded in finding out if they can be stated for amenable groups in general, and indeed Rudin seemed to have the same problem in his article [27]. The problem boils down to whether or not the sets

$$\{g \in G \mid gE \in \omega\}$$

are measurable, where $E \subset G$ is Borel, and $\omega \in \mathfrak{M}_G$ is an ultrafilter. Someone more well versed in descriptive set theory may be better able to tackle this question than we are.

3 Crossed products

3.1 The reduced crossed product

Let G be a locally compact group, and let A be a C^*-algebra. An action of G on A is a homomorphism $\alpha : G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of continuous automorphisms of A. A *C^*-dynamical system* is a triple (A, G, α) of such objects. It is customary to write $\alpha(g) = \alpha_g$ for the automorphism corresponding to each $g \in G$.

Example 3.1 One important family of examples of C^*-dynamical systems can be constructed using a classical dynamical system (X, G, κ). Let $A = C(X)$, and let G act on A by $\alpha_g(f) = f \circ \kappa_g^{-1}$. One can check that for each $g \in G$, α_g is continuous if and only if κ_g is continuous.

We will not introduce the entire crossed product construction here, but restrict ourselves to crossed products with discrete amenable groups. In fact, what we construct will be the reduced crossed product, but this coincides with the full crossed product when the group is amenable ([34] Theorem 7.13). The construction presented here is pretty standard, but we have closely followed the one given in [25].

A *covariant representation* of a C^*-dynamical system (A, G, α) is a pair (π, u) where $\pi : A \to B(H)$ is a representation of A on a Hilbert space H and where $g \mapsto u_g$ is a unitary representation of G on H such that for every $g \in G$ and $a \in A$,

$$\pi(\alpha_g(a)) = u_g \pi(a) u_g^*.$$

It is common to demand that the map $g \mapsto u_g$ is continuous into the weak operator topology, but when G is discrete, this condition is of course vacuous.

We will construct the *regular covariant representation* of (A, G, α) as follows. Let $\pi_0 : A \to B(H_0)$ be an injective representation, and let $\ell_0(G, H_0)$ be the set of finitely supported functions from G to H_0. Let $H = \ell^2(G, H_0)$ be the completion of $\ell_0(G, H_0)$ in the inner product

$$\langle \xi, \eta \rangle = \sum_{g \in G} \langle \xi(g), \eta(g) \rangle.$$

We can then define a covariant representation (π, u, H) of (A, G, α) with respect to (π_0, H_0) to be

$$\begin{aligned}(\pi(a)\xi)(g) &= \pi_0(\alpha_{g^{-1}}(a))\xi(g)\\ (u_g\xi)(h) &= \xi(g^{-1}h).\end{aligned}$$

Example 3.2 When $A = \ell^\infty(G)$ and $\alpha = \lambda$, we can make this construction even simpler. Let $H = \ell^2(G)$, and let $\ell^\infty(G)$ and G be represented by

$$(\pi(f)\xi)(g) = f(g)\xi(g)$$
$$(u_g\xi)(h) = \lambda_g\xi(h) = \xi(g^{-1}h).$$

Then

$$(u_g\pi(f)u_g^*\xi)(h) = f(g^{-1}h)\xi(h) = (\pi(\alpha_g(f))\xi)(h)$$

as desired. As an orthonormal basis for $\ell^2(G)$, we will sometimes use $\{e_g\}_{g\in G}$, where $e_g(h) = \delta_{g,h}$.

To define the reduced crossed product, we introduce the algebra $\ell_0(G, A, \alpha)$ to be the algebra of functions from G to A with finite support, and with product rule

$$(ab)(g) = \sum_{h\in G} a(h)\alpha_h(b(h^{-1}g))$$

defined for a, b in $\ell_0(G, A, \alpha)$. Associativity of this product is a straightforward calculation. We also introduce a $*$-operation on $\ell_0(G, A, \alpha)$ as follows.

$$a^*(g) = \alpha_g((a(g^{-1}))^*).$$

Another calculation shows that $(ab)^* = b^*a^*$.

If (π, u) is the regular covariant representation of (A, G, α) on H, define a representation $\sigma : \ell_0(G, A, \alpha) \to B(H)$ by

$$\sigma(a)\xi = \sum_{g\in G} \pi(a(g))u_g\xi.$$

Clearly, σ is linear, but we also want to check that it is a $*$-homomorphism. Let $a, b \in \ell_0(A, G, \alpha)$. Then

$$\begin{aligned}
\sigma(a)\sigma(b) &= \sum_{g,h\in G} \pi(a(h))u_h\pi(b(g))u_g \\
&= \sum_{g,h\in G} \pi(a(h))u_h\pi(b(g))u_h^*u_hu_g \\
&= \sum_{g,h\in G} \pi(a(h)\alpha_h(b(g)))u_{hg} \\
&= \sum_{g,h\in G} \pi(a(h)\alpha_h(b(h^{-1}g)))u_g \\
&= \sigma(ab),
\end{aligned}$$

and

$$\begin{aligned}
\sigma(a^*) &= \sum_{g\in G} \pi(\alpha_g(a(g^{-1})^*))u_g \\
&= \sum_{g\in G} u_g\pi(a(g^{-1}))^* u_g^* u_g \\
&= \sum_{g\in G} u_g^*\pi(a(g))^* \\
&= \sigma(a)^*.
\end{aligned}$$

We let $C_r^*(A, G, \alpha)$ be the completion of $\ell_0(G, A, \alpha)$ under the norm induced by the the regular covariant representation. We will let $W_r^*(A, G, \alpha)$ be the weak closure (or double commutant) of $\ell_0(G, A, \alpha)$ in the same representation. Note that there is a natural embedding of A in $C_r^*(A, G, \alpha) \subset W^*(A, G, \alpha)$ by sending $a \in A$ to the function that takes the value a at the identity and 0 for all other $g \in G$.

Lemma 3.3 The imbedding of A into $C_r^*(A, G, \alpha)$ is isometric.

Proof. It is sufficient to prove that the imbedding into $\ell_0(A, G, \alpha)$ is an isometry. Let $\sigma : \ell_0(A, G, \alpha) \to B(\ell^2(G, H_0))$ be a regular representation. Let $a \in A$, and let f_a be the function taking the value a at 1 and 0 elsewhere. Then

$$\begin{aligned}
\|\sigma(f_a)\|^2 &= \sup_{\substack{\xi\in\ell^2(G,H_0)\\ \|\xi\|=1}} \sum_{g\in G} \|\pi_0(f_a(g))\xi(g)\|^2 \\
&= \sup_{\substack{\xi\in H_0\\ \|\xi\|=1}} \|\pi_0(a)\xi\|^2 \\
&= \|a\|^2.
\end{aligned}$$

□

Subsequently, we will identify A with its imbedding in the reduced crossed product.

Again, when the C^* dynamical system is $(\ell^\infty(G), G, \alpha)$, we can make the crossed product construction much simpler. By using the covariant representation from Example 3.2, we can define the $*$-algebra $\ell_0(G, \ell^\infty(G)) \subset B(\ell^2(G))$ to be the set of all sums

$$\sum_{g\in F} a_g u_g, \quad F \subset G \text{ finite}$$

with $a_g \in \ell^\infty(G)$ for each $g \in F$. Here we associate $\ell^\infty(G)$ with its representation on $\ell^2(G)$. We then define the reduced C^* crossed product $C_r^*(\ell^\infty(G), G)$ to be the completion of $\ell_0(G, \ell^\infty(G))$ in the uniform topology[2]. Similarly, we define the reduced von Neumann crossed product $W_r^*(\ell^\infty(G), G)$ to be the weak closure of $\ell_0(G, \ell^\infty(G))$.

Two other important objects are the reduced C^* and W^* group algebras written $C_r^*(G)$ and $W_r^*(G)$. We define these to be the crossed products $C_r^*(\mathbb{C}1, G)$ and $W_r^*(\mathbb{C}1, G)$ respectively, with $\mathbb{C}1 \subset \ell^\infty(G)$ being the subalgebra of constant functions. That is, they are generated by the set of finite sums

$$\sum_{g \in F} a_g u_g,$$

where each a_g is a constant.

If G is a discrete group, let $c_0(G) \subset \ell^\infty(G)$ be the bounded functions that vanish at infinity. These are trivially just the compact $\ell^\infty(G)$ operators. Let $K(\ell^2(G))$ be all the compact operators on $\ell^2(G)$. The next theorem can also be stated for non-discrete groups, but that falls outside our discussion. It originally arose to describe the uniqueness of the canonical commutation relations between the position and momentum operators in quantum mechanics, but has later been generalized and reformulated in the language of crossed products. We will mostly need the corollary that says that $\ell_0(G, \ell^\infty(G))$ is strongly dense in $B(\ell^2(G))$.

Theorem 3.4 (Stone-von Neumann) Let G be a discrete group. Then $C^*(c_0(G), G) = K(\ell^2(G))$.

Proof. See Theorem 4.24 in [34]. □

Corollary 3.5 Let G be a discrete group. Then $W_r^*(\ell^\infty(G), G) = B(\ell^2(G))$.

Theorem 3.6 Let G be a discrete abelian group with dual $\hat{G}$, and let $\mathscr{F} : \ell^2(G) \to L^2(\hat{G})$ be the Plancherel transform. Identify $C(\hat{G})$ and $L^\infty(\hat{G})$ with their representations as multiplication algebras on $L^2(\hat{G})$. Then

(i) $\mathscr{F} C_r^*(G) \mathscr{F}^* = C(\hat{G})$

(ii) $\mathscr{F} W_r^*(G) \mathscr{F}^* = L^\infty(\hat{G})$.

Proof. The functions $\hat{e}_g \in C(\hat{G})$ given by $\hat{e}_g(\gamma) = \gamma(g^{-1})$ separate the points

[2] We will sometimes be lazy and omit the subscripted r when G is amenable, but we will always talk about the particular construction defined here.

in $\hat{G}$, and form an orthonormal basis for $L^2(\hat{G})$. Now,

$$\begin{aligned}
\mathscr{F} u_g \mathscr{F}^* \hat{e}_h &= \mathscr{F} u_g \left(k \mapsto \int_{\hat{G}} \hat{e}_h(\gamma)\gamma(k^{-1})\,\mathrm{d}\hat{\mu}(\gamma) \right) \\
&= \mathscr{F} \left(k \mapsto \int_{\hat{G}} \gamma(h^{-1})\gamma(g^{-1}k^{-1})\,\mathrm{d}\hat{\mu}(\gamma) \right) \\
&= \zeta \mapsto \sum_{k\in G} \zeta(k^{-1}) \int_{\hat{G}} \gamma(h^{-1}g^{-1}k^{-1})\,\mathrm{d}\hat{\mu}(\gamma) \\
&= \zeta \mapsto \sum_{k\in G} \zeta(h^{-1}g^{-1}k^{-1}) \int_{\hat{G}} \gamma(k^{-1})\,\mathrm{d}\hat{\mu}(\gamma) \\
&= \zeta \mapsto \zeta(h^{-1}g^{-1}) \\
&= \hat{e}_g \hat{e}_h.
\end{aligned}$$

So $\mathscr{F} u_g \mathscr{F}^* = \hat{e}_g$ as a multiplication operator. Since these span a uniformly dense subalgebra of $C(\hat{G})$ by the Stone-Weierstrass Theorem we have proved point (i). Point (ii) then follows from Lemma E.3. $\square$

Lemma 3.7 Let G be discrete and abelian, and let $a \in W_r^*(\ell^\infty(G), G)$. Then $a \in W_r^*(G)$ if and only if $au_g = u_g a$ for all $g \in G$.

Proof. This is the same as saying that $W_r^*(G)$ is a maximal abelian von Neumann algebra. We follow the proof of Theorem 3.5.2 in [18].

The if part is clear, so suppose a commutes with every u_g. For $f \in L^\infty(G)$, let $M_f \in W_r^*(G)$ be the corresponding multiplication operator. Let $f \in L^\infty(G)$, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of trigonometric polynomials that converge to f in the L^2 norm. Then a commutes with every M_{f_n}, and

$$\|af_n - af\| \to 0.$$

Choosing a subsequence, we can assume f_n converges to f pointwise almost everywhere, and that af_n converges to af almost everywhere. Let $g = ae_1$. Then $af_n = aM_{f_n}e_1 = M_{f_n}ae_1 = f_n g$ a.e, so $af = fg = gf$ a.e. Let

$$E_n = \{x \in G \mid |g(x)| > \|a\| + 1/n\}.$$

Then E_n is measurable, and

$$\begin{aligned}
\|a\|^2 \|\chi_{E_n}\|_2^2 &\geq \|a\chi_{E_n}\|_2^2 \\
&= \|g\chi_{E_n}\|_2^2 \\
&\geq (\|a\| + 1/n)^2 \|\chi_{E_n}\|_2^2
\end{aligned}$$

so $\mu(E_n) = 0$, and g is in $L^\infty(G)$. Since $L^\infty(G)$ is L^2-dense in $L^2(G)$, we get that $af = gf$ for all $f \in L^2(G)$. $\square$

3.2 The dual action

When G is locally compact abelian, there is a natural action of $\hat{G}$ on any crossed product by G. We give a brief introduction to it here, but only in the case when G is discrete. A more comprehensive account can for instance be found in [19] or [24].

Let (A, G, α) be a C^* or W^* dynamical system, and let $\gamma \mapsto \theta_\gamma$ be the homomorphism of $\hat{G}$ into $\mathrm{Aut}(\ell_0(G, A, \alpha))$ given by

$$(\theta_\gamma(a))(g) = \gamma(g)a(g).$$

For a fixed a, this is easily seen to be continuous from $\hat{G}$ into the uniform topology, and hence extends to an action of $\hat{G}$ on $C_r^*(A, G, \alpha)$. It is important to note that it does not necessarily extend to a uniformly continuous action on $W_r^*(A, G, \alpha)$, but it can still be extended to an action that is continuous into the weak topology. We will look at this action in our favourite special case.

Let $A = \ell^\infty(G)$, and let $\gamma \mapsto v_\gamma$ be the unitary representation of $\hat{G}$ on $\ell^2(G)$ given by

$$v_\gamma \xi(g) = \gamma(g)\xi(g).$$

We see that v_γ is an element of $\ell^\infty(G)$ for each $\gamma \in \hat{G}$. We can create an action $\theta : \hat{G} \to \mathrm{Aut}(\ell_0(G, \ell^\infty(G)))$ given by

$$\theta_\gamma(a) = v_\gamma a v_\gamma^*.$$

Now,

$$\begin{aligned}
v_\gamma \left(\sum_{g \in F} a_g u_g \right) v_\gamma^* \xi &= v_\gamma \left(\sum_{g \in F} a_g u_g \right) (h \mapsto \gamma(h^{-1})\xi(h)) \\
&= v_\gamma \sum_{g \in F} (h \mapsto a_g(h)\gamma(gh^{-1})\xi(g^{-1}h)) \\
&= \sum_{g \in F} (h \mapsto \gamma(g) a_g(h) \xi(g^{-1}h)) \\
&= \sum_{g \in F} \gamma(g) a_g u_g \xi.
\end{aligned}$$

Which appropriately shows that this θ is a special case of the θ we defined above.

Interestingly, we can also show that θ acts by translation on $L^\infty(\hat{G})$. By Theorem 3.6, we can identify the weakly dense subset of trigonometric polynomials in $L^\infty(\hat{G})$ by elements of $W^*(G)$ of the form

$$f = \sum_{g \in F} a_g u_g,$$

where $F \subset G$ is finite and $a_g \in \mathbb{C}$ for each $g \in F$. Remember that u_g is mapped to the $C(\hat{G})$ function given by $\gamma \mapsto \gamma(g^{-1})$. Clearly, θ is a self-map of $W_r^*(G)$, and

$$\begin{aligned}\left(\mathscr{F} u_\gamma \mathscr{F}^* \sum_{g\in F} a_g \hat{e}_g \mathscr{F} u_\gamma \mathscr{F}^*\right)(\zeta) &= \sum_{g\in F} \gamma(g) a_g \zeta(g^{-1}) \\ &= \sum_{g\in F} a_g \hat{e}_g(\gamma^{-1}\zeta).\end{aligned}$$

It is now easy to see why θ is not uniformly continuous in general. If we let a be the $W^*(G)$ operator corresponding to a noncontinous function in $L^\infty(\hat{G})$, we know that translation of this function is not continuous in the essential supremum topology on $L^\infty(\hat{G})$, and hence θ fails to affect a in a uniformly continuous manner. However, it is an easy exercise to show that translation is weakly continuous on $L^\infty(\hat{G})$.

3.3 Conditional expectations

Conditional expectations are an important part of the theory of discrete crossed products. In some sense they are the natural operator-theoretical concept of retracts. Let A be a unital C^*-algebra and let $B \subset A$ be a C^*-subalgebra. See appendix A for a definition of a completely positive map.

A *conditional expectation* of A onto B is a completely positive linear projection $\Psi : A \to B$ that fixes B pointwise. Using Lemma A.3 and Theorem A.5, we see that such maps have the following additional properties.

(i) $\Psi(b_1 a b_2) = b_1 \Psi(a) b_2$ for all $b_1, b_2 \in B$ and $a \in A$.

(ii) $\|\Psi\| = 1$.

Point (i) says that Ψ is what we call a *B-bimodule map*.

There is a natural conditional expectation of $W_r^*(A, G, \alpha)$ or $C_r^*(A, G, \alpha)$ onto A called the *canonical conditional expectation*. On $\ell_0(G, A, \alpha)$, this can be defined by $\Phi(a) = a(1)$. It is best defined on the crossed product by

$$\langle \Phi(a)\xi, \eta\rangle = \langle a\xi', \eta'\rangle,$$

where $\xi' \in \ell^2(G, H_0)$ is given by $\xi'(1) = \xi$ and $\xi'(g) = 0$ for all other g. η' is defined in the same way.

Theorem 3.8 Let (A, G, α) be a W^*-dynamical system where G is discrete. The canonical conditional expectation $\Phi : W_r^*(A, G, \alpha) \to A$ is a faithful. Moreover, Φ is normal and unique when $A = \ell^\infty(G)$.

Proof. First, Φ is clearly positive and completely positive, which is easy to see by looking at the definition. This shows that it is indeed a conditional expectation[3].

To see that it is faithful, suppose $\Phi(a) = 0$ and $a \geq 0$. Then, for all $\xi, \eta \in H_0$,

$$0 = \langle \Phi(a)\xi, \eta \rangle = \langle a\xi', \eta' \rangle.$$

So, a equals zero restricted to the subspace spanned by the $\ell^2(G, H_0)$ vectors with support on the identity. We note that it is important for a to be positive for this argument to work.

Also, if ξ', η' is supported on $g \neq 1$, we have

$$\langle a\xi', \eta' \rangle = \langle \Phi(u_g a u_g^*) u_g \xi'(g), u_g \eta'(g) \rangle = 0$$

since α_g is a faithful action. So a is therefore also zero restricted to vectors with support on any one other group element, and since these span $\ell^2(G, H_0)$, b must be 0.

That Φ is ultraweakly continuous on $W_r^*(\ell^\infty(G), G)$ is obvious. To show uniqueness, let χ_g be the projection onto the span of e_g, and let $\Psi : W_r^*(\ell^\infty(G), G) \to \ell^\infty(G)$ be a conditional expectation. By Theorem A.5, $\Psi(\chi_g a \chi_g) = \chi_g \Psi(a) \chi_g$ for all $a \in W_r^*(\ell^\infty(G), G)$ and all $g \in G$, so

$$\langle \Psi(a) e_g, e_g \rangle = \langle \chi_g \Psi(a) \chi_g e_g, e_g \rangle = \langle \Psi(\chi_g a \chi_g) e_g, e_g \rangle = \langle \chi_g a \chi_g e_g, e_g \rangle = \langle a e_g, e_g \rangle.$$

So $\Psi = \Phi$. □

For $a \in W_r^*(G)$, when G is abelian, we see that $\Phi(au_g^*)$ gives us the g'th Fourier coefficient of $\mathscr{F} a \mathscr{F}^*$, and it is well known that any $f \in L^\infty(\hat{G})$ is completely determined by its Fourier coefficients. There is a similar useful result for general elements of the reduced crossed product.

Corollary 3.9 Let (A, G, α) be a discrete W^*-dynamical system, and let $a, b \in W_r^*(A, G, \alpha)$. Then $a = b$ if and only if $\Phi(au_g^*) = \Phi(bu_g^*)$ for all $g \in G$.

That is, every element in the von Neumann-crossed product is uniquely determined by its Fourier-like series expansion. However, it is important to note that the expression

$$\sum_{g \in G} \Phi(au_g^*) u_g$$

does not necessarily converge even in the weak topology. It does converge in a strictly weaker topology called the Bures topology. See [17] for more about this.

[3] Some authors also demand faithfulness in the definition of an expectation.

3.4 Extending the right group action

Let G be a discrete group, and define the right action ρ of G on $\ell^\infty(G)$ and $\ell^2(G)$ by

$$\rho_g(f)(h) = f(hg).$$

This can also be extended to an action on $W_r^*(\ell^\infty(G), G)$ by

$$\langle \rho_g(a)\xi, \eta \rangle = \langle a\rho_{g^{-1}}(\xi), \rho_{g^{-1}}(\eta) \rangle,$$

just as we did with the left action.

We wish to look at a way of extending ρ to an action of βG on $W_r^*(\ell^\infty(G), G)$. Let $a \in W_r^*(\ell^\infty(G), G)$, and look at the map

$$\begin{aligned} \rho(\cdot)(a) : G &\rightarrow W_r^*(\ell^\infty(G), G) \\ \rho(g)(a) &= \rho_g(a). \end{aligned}$$

$\rho(\cdot)(a)$ has a bounded image, so because the unit ball of $W_r^*(\ell^\infty(G), G)$ is compact in the weak topology, we can use Lemma C.4 to construct an extension of ρ to a continous map

$$\hat{\rho}(\cdot)(a) : \beta G \rightarrow W_r^*(\ell^\infty(G), G).$$

Write $\hat{\rho}(\omega)(a) = \hat{\rho}_\omega(a) = \lim_{g \to \omega} \rho_g(a)$. As we shall se later, $\hat{\rho}_\omega$ is exactly the map ψ_s that Paulsen discusses in [23], but his way of constructing it is a bit different. See also [22].

Lemma 3.10 Let G be discrete. The map

$$\hat{\rho}_\omega : W_r^*(\ell^\infty(G), G) \rightarrow W_r^*(\ell^\infty(G), G)$$

is unital and completely positive. If G is abelian, it is a $W_r^*(G)$-bimodule map that fixes $W_r^*(G)$ pointwise.

Proof. Linearity follows from the weak continuity of the linear operations. If G is abelian, $\hat{\rho}_\omega$ fixes $W_r^*(G)$ pointwise since δ_g does for every $g \in \beta G$ and since G is dense in βG and $W_r^*(G)$ is weakly closed.

To show complete positivity, let $n \in \mathbb{N}$ be arbitrary, and let $\xi_1 \dots \xi_n, \eta_1 \dots \eta_n \in H$. We need to show that

$$\sum_{j,k=1}^{n} \langle \hat{\rho}_\omega(a_{j,k})\xi_j, \eta_k \rangle \geq 0$$

whenever $\{a_{j,k}\} \in W_r^*(\ell^\infty(G), G)$ are such that

$$\sum_{j,k=1}^{n} \langle a_{j,k}\xi'_j, \eta'_k \rangle \geq 0$$

for all $\xi'_1 \dots \xi'_n, \eta'_1 \dots \eta'_n \in H$. But we have

$$\begin{aligned}\sum_{j,k=1}^{n} \langle \hat{\rho}_\omega(a_{j,k})\xi_j, \eta_k\rangle &= \lim_{g\to\omega} \sum_{j,k=1}^{n} \langle \alpha_g(a_{j,k})\xi_j, \eta_k\rangle \\ &= \lim_{g\to\omega} \sum_{j,k=1}^{n} \langle a_{j,k}u_g^*\xi_j, u_g^*\eta_k\rangle \geq 0.\end{aligned}$$

That $\hat{\rho}_\omega$ is a $W_r^*(G)$-bimodule map now follows from Theorem A.5. $\square$

The next Lemma shows that $\hat{\rho}$ is in fact a semigroup action of βG on $W^*(\ell^\infty(G), G)$.

Lemma 3.11 Let $\omega, \varrho \in \beta G$. Then $\hat{\rho}_\omega \circ \hat{\rho}_\varrho = \hat{\rho}_{\omega\cdot\varrho}$.

Proof. Let $a \in W^*(\ell^\infty(G), G)$. Since $\rho(\cdot)(a)$ is weakly continuous, we have

$$\begin{aligned}\hat{\rho}_\omega(\hat{\rho}_\varrho(a)) &= \lim_{g\to\omega} \hat{\rho}_{g\cdot\varrho}(a) \\ &= \hat{\rho}_{\omega\cdot\varrho}(a).\end{aligned}$$

$\square$

It follows that if $\omega \in \beta G$ is idempotent, $\hat{\rho}_\omega$ is in fact a conditional expectation onto its image.

The map has another special interpretation. Let $\omega \in \beta G$, and let s_ω be the pure state on $\ell^\infty(G)$ given by $s_\omega(f) = \hat{f}(\omega)$. Now,

$$\hat{\rho}_\omega(f)(1) = \lim_{g\to\omega} \langle \rho_g(f)e_1, e_1\rangle = \lim_{g\to\omega} f(g) = \hat{f}(\omega).$$

We see that $a \mapsto \langle \hat{\rho}_\omega(a)e_1, e_1\rangle$ provides a state extension of s_ω to $W^*(\ell^\infty(G), G, \alpha)$. Moreover,

$$\hat{\rho}_\omega(f)(h) = \lim_{g\to\omega} f(hg) = \hat{f}(h\cdot\omega).$$

In fact, we see that $\hat{\rho}_\omega$ restricts to the map π_ω that we discussed in the previous chapter. We can also describe what $\hat{\rho}_\omega$ does to any $a \in W^*(\ell^\infty(G), G)$. Let a be given by

$$a \sim \sum_{g\in G} a_g u_g \quad a_g \in \ell^\infty(G)$$

where this kind of notation is justified by Corollary 3.9. Since Φ is normal and commutes with ρ, we get that

$$\Phi \circ \hat{\rho}_\omega = \hat{\rho}_\omega \circ \Phi.$$

Moreover, $\hat{\rho}_\omega(a_g u_g) = \pi_\omega(a_g)u_g$ for every $g \in G$ since $\rho_h(a_g u_g) = \rho_h(a_g)u_g$ as G is abelian. Then

$$\hat{\rho}_\omega(a) \sim \sum_{g\in G} \pi_\omega(a_g)u_g.$$

3.5 The C^* crossed product of $\ell^\infty(G)$

In this section, we wish to study the structure of the C^* crossed product $C_r^*(\ell^\infty(G), G) \simeq C_r^*(C(\beta G), G)$ when G is a discrete amenable group, and in particular look at some of its ideal structure. The techniques we are going to use are fairly well known, and are studied in a more general setting, for instance in [8] and [34].

By Lemma D.5, we know that every closed G-invariant ideal in $\ell^\infty(G)$ is of the form

$$I_X = \{f \in \ell^\infty(G) \mid \hat{f}|_X = 0\},$$

where $X \subset \beta G$ is a closed G-invariant subset. These ideals are important because they generate closed ideals in $C_r^*(\ell^\infty(G), G)$.

Lemma 3.12 If $I_X \subset \ell^\infty(G)$ is a closed invariant ideal, then $C_r^*(I_X, G) \subset C_r^*(\ell^\infty(G), G)$ is a closed ideal.

Proof. First, let $b \in \ell_0(G, I_X)$ and $a \in \ell_0(G, \ell^\infty(G))$ be finite sums. Then

$$\begin{aligned} ba &= \sum_{g\in F_k}\sum_{h\in F_a} b_g u_g a_h u_h \\ &= \sum_{g\in F_k}\sum_{h\in F_a} b_g u_g a_h u_g^* u_{gh} \\ &= \sum_{g\in F_k}\sum_{h\in F_a} b_g \alpha_g(a_h) u_{gh} \end{aligned}$$

which is in $C_r^*(I_X, G)$ since $\alpha_g(a_h) \in \ell^\infty(G)$ for each h and since $b_g \in I_X$ which is an ideal. Similarly,

$$\begin{aligned} ab &= \sum_{h\in F_a}\sum_{g\in F_k} a_h u_h b_g u_g \\ &= \sum_{h\in F_a}\sum_{g\in F_k} a_h \alpha_h(b_g) u_{hg} \end{aligned}$$

which is also in $C^*(I_X, G)$ since I_X is invariant. Since the operation of taking the product of two elements is a jointly continuous operation in the C^* norm, we get that $C_r^*(I_X, G)$ is a two-sided closed ideal. □

Let $\hat{\varphi}_\omega$ be the restriction of $\hat{\rho}_\omega$ to $C_r^*(\ell^\infty(G), G)$, and let π_ω be its restriction to $\ell^\infty(G)$.

Lemma 3.13 Let $\omega \in \beta G$ where G is discrete and abelian. Then the kernel of π_ω is $I_{\beta G\cdot\omega}$. The kernel of $\hat{\varphi}_\omega$ is $C_r^*(I_{\beta G\cdot\omega}, G)$.

Proof. First off, for $f \in \ell^\infty(G)$, we see that $\pi_\omega(f) = 0$ if and only if $\hat{f}(g \cdot \omega) = 0$ for every $g \in G$ if and only if $\hat{f}|_{\beta G \cdot \omega} = 0$ if and only if $\hat{f} \in I_{\beta G \cdot \omega}$.

It now follows that $C_r^*(I_{\beta G \cdot \omega}, G)$ is contained in the kernel of $\hat{\varphi}_\omega$. To see equality, let $\hat{\varphi}_\omega(a) = 0$. Then $\hat{\varphi}_\omega(au_g) = 0$ for all $g \in G$, and since $\hat{\varphi}_\omega$ commutes with Φ, we have $\hat{\varphi}_\omega(\Phi(au_g)) = 0$ for all g, so each $\Phi(au_g)$ is in the kernel of $\hat{\varphi}_\omega$, and the claim is proved. □

Let $A_\omega = \hat{\varphi}_\omega(C_r^*(\ell^\infty(G), G))$, and let Φ denote the restriction to $C_r^*(\ell^\infty(G), G)$ of the unique conditional expectation of $W_r^*(\ell^\infty(G), G)$ onto $\ell^\infty(G)$.

Lemma 3.14 Let $\omega \in \beta G$ be idempotent. Then $A_\omega \simeq C_r^*(C(\beta G \cdot \omega), G)$.

Proof. First, we would like to observe that $\pi_\omega(\ell^\infty(G)) \simeq C(\beta G \cdot \omega)$. This follows from the fact that we can identify $f \in C(\beta G \cdot \omega)$ with the function $g \mapsto f(g \cdot \omega)$, and that this identification is injective since $G \cdot \omega$ is dense in $\beta G \cdot \omega$. By extending f continuously to all of βG using the Tietze Extension Theorem, we also get that it is onto.

It remains to see that $C_r^*(\hat{\varphi}_\omega(\ell^\infty(G)), G) = \hat{\varphi}_\omega(C_r^*(\ell^\infty(G), G))$. Inclusion of the second into the first is obvious. But if we let $a \in C_r^*(\hat{\varphi}_\omega(\ell^\infty(G)), G)$, we get that $\hat{\varphi}_\omega(a) = a$ since $\hat{\varphi}_\omega$ is idempotent. This is because $\hat{\varphi}_\omega$ works independently on each of the "Fourier coefficients" a as noted at the end of the last subsection. Thus $a \in A_\omega$. □

Theorem 3.15 Let $\omega \in \beta G$ be minimal, where G is discrete and abelian. Then A_ω is a simple C^*-subalgebra of $C_r^*(\ell^\infty(G), G)$.

Proof. Let $I \subset A_\omega$ be a closed ideal. Let $a \in I$, and let b be such that $\hat{\varphi}_\omega(a) = b$. As we shall see later in Lemma 4.5, there is for every $\varepsilon > 0$ an $E \in \omega$ such that

$$\|\chi_E(b - \Phi(b))\chi_E\| < \varepsilon.$$

We have $\pi_\omega(\chi_E^2) = \pi_\omega(\chi_E)^2 = \chi_F$, for some $F \subset G$, so

$$\|\hat{\varphi}_\omega(\chi_E(b - \Phi(b))\chi_E)\| = \|\chi_F(\hat{\varphi}_\omega(b) - \Phi(\hat{\varphi}_\omega(b))\chi_F)\| < \varepsilon,$$

by Theorem A.5. Since I is closed, $\Phi(a) \in I$, and so is $\Phi(au_g)$ for all $g \in G$. So if a is nonzero and positive, there is also a nonzero positive element of $\ell^\infty(G)$, say f, in I. Let $U = \{f > 0\} \subset \beta G$. Then since $f = \pi_\omega(f_0)$ for some $f_0 \in \ell^\infty(G)$, a finite number of translates of U cover βG as per Lemma 2.10. Say $g_1 U \dots g_s U$. Then

$$\sum_{j=1}^{s} u_{g_j} f u_{g_j}^* = \sum_{j=1}^{s} \alpha_{g_j}(f) \in I$$

is strictly positive and invertible, and thus $I = A_\omega$. □

4 Extensions of pure states from $L^\infty(G)$.

4.1 Preliminaries

The investigation of the uniqueness of pure state extensions from a maximal abelian subalgebra (*masa*) A of $B(H)$ to all of $B(H)$ was first instigated by Kadison and Singer in [15], where they proved that unless A was discrete, the pure states of A do not have unique extensions to $B(H)$ in general. The question of whether or not the pure states of a discrete masa have unique extensions has so far remained open, and the problem is commonly known as the Kadison-Singer problem, or the Kadison-Singer conjecture. In the next section, we will investigate this problem a little further. It is standard knowledge that any discrete masa in $B(H)$, where H is separable, is isometrically isomorphic to $\ell^\infty(\mathbb{Z})$, so it is no loss of generality to study the problem for $\ell^\infty(G) \subset W^*(\ell^\infty(G), G)$, where G is a discrete countable group.

4.2 Extensions of completely positive maps

Recall that an *operator system* is an involutive (i.e. self-adjoint) linear subspace Q of a unital C^*-algebra A with $1_A \in Q$. The first version of the next lemma made its appearance in [16].

Lemma 4.1 (Krein's Extension Theorem) Let A be a unital C^*-algebra, $Q \subset A$ an operator system, and let $\phi : Q \to \mathbb{C}$ be a linear functional with $\phi(Q \cap A_+) \subset \mathbb{R}_+$. Given $a \in A_{sa}$ and $x \in \mathbb{R}$ such that

$$\sup\{\phi(q) \mid q \in Q_{sa}, q \leq a\} \leq x \leq \inf\{\phi(q) \mid q \in Q_{sa}, q \geq a\}, \qquad (4.1)$$

there is a positive linear functional $\psi : A \to \mathbb{C}$ that extends ϕ, and satisfies $\psi(a) = x$. Note that such an x always exists.

Proof. Since a positive linear functional is completely determined by its values on the self-adjoint elements, we only have to look at extensions of $\phi|_{Q_{sa}}$ on the real vector subspaces between Q_{sa} and A_{sa}. When we have obtained an extension to A_{sa}, we can extend it to A by using the standard decomposition of any $a \in A$ into two self-adjoint elements. Suppose that we have extended $\phi|_{Q_{sa}}$ to a positive linear functional ϕ' on the a closed subspace Q' containing $Q|_{sa}$, and suppose $b \notin Q'$. Replace Q with Q' and ϕ with ϕ' in (4.1). Clearly some $x \in \mathbb{R}$ satisfying (4.1) exists, and we can define $\phi'' : Q' + \mathbb{R}b$ by

$$\phi''(q + tb) = \phi'(q) + tx.$$

We need to check that this extension is positive for the three cases $t < 0$, $t = 0$ and $t > 0$. We'll only do the first one, as the two others are similar.

Let $t < 0$, and let $q \in Q'$ be such that $q + tb \geq 0$. Then $b \leq t^{-1}q$, so by the way we picked x, we get

$$\begin{array}{rcl} x & \leq & t^{-1}\phi'(q) \\ tx + \phi'(q) & \geq & 0 \\ \phi''(q + tx) & \geq & 0. \end{array}$$

The rest is just the Hahn-Banach Theorem. We use Zorn's lemma to obtain an extension ψ to all of A_{sa}. To get $\psi(a) = x$ for a particular a and x, we only have to remember to extend to $Q + \mathbb{R}a$ before anywhere else. □

Before we prove the next Theorem, we would like to introduce the *bounded weak topology* on the space $B(A, B(H))$, where A is a C^*-algebra, and H is a Hilbert space. Define linear functionals υ on $B(A, B(H))$ by

$$\upsilon_{a,\xi,\eta}(\psi) = \langle \psi(a)\xi, \eta \rangle$$

with $a \in A$ and $\xi, \eta \in H$. Let Z be the closure of the linear span of all these functionals in $B(A, B(H))^*$. We define the bounded weak topology on $B(A, B(H))$ to be the weak*-topology on $B(A, B(H))$ induced by the canonical map into Z^*. Then $B(A, B(H))$ is closed in this topology, because if the net $\{\psi_k\}$ converges to ψ' in Z^*, we can define $\psi : A \to B(H)$ by

$$\langle \psi(a)\xi, \eta \rangle = \psi'(\upsilon_{a,\xi,\eta}).$$

This map has the desired properties. By the Banach-Alaoglu Theorem, the unit ball of $B(A, B(H))$ is compact, and we see that a net ψ_k converges to $\psi \in B(A, B(H))$ in the bounded weak topology if and only if $\langle \psi_k(a)\xi, \eta \rangle$ converges to $\langle \psi(a)\xi, \eta \rangle$ for all $a \in A$ and $\xi, \eta \in H$.

The next Theorem first appeared in [2]. We have taken our proof from [21].

Theorem 4.2 (Arveson's Extension Theorem) Let A be a C^*-algebra H a Hilbert space, and let $Q \subset A$ be an operator system. Then every completely positive map

$$\phi : Q \to B(H)$$

has a completely positive extension with domain A and range in $B(H)$.

Proof. We will extend the map to finite-dimensional subspaces of $B(H)$ and proceed by induction. Let $K \subset H$ be a finite-dimensional subspace, and let $\phi_K : Q \to B(K)$ be the compression $a \mapsto p_K\phi(a)|_K$ with p_K being the orthogonal projection of H onto K. Choose an orthonormal basis $e_1 \ldots e_n$ for K, and let $e_{i.k}$ be the corresponding matrix units for $B(K)$. Let $x \in K^n$ be $\oplus_{k=1}^n e_k$, and let s_ϕ be the linear functional on $M_n(Q)$ given by

$$s_\phi((q_{i,j})) = \langle 1_n \otimes \phi((q_{i,j}))x, x \rangle = \sum_{i,j=1}^n \langle \phi(q_{i,j})e_i, e_j \rangle.$$

We see that s_ϕ is positive, and we can extend it to a positive linear functional s on $M_n(A)$ by the previous Lemma. Now, define $\psi_K : A \to B(K)$ by

$$\langle \psi_K(a)e_i, e_j\rangle = s(a \otimes e_{i,j}).$$

A quick computation shows that ψ_K is an extension of ϕ_K, so it remains to see that it is completely positive. Let $a = (a_{i,j}) \in M_m(A)_+$, and let $b = (b_{k,l}) \in M_m(M_{n^2}(A)) \simeq M_{mn^2}(A)$ be the matrix given by the blocks $b_{k,l} = a_{k,l} \otimes \mathbf{e}$ with $\mathbf{e} \in M_n(M_n(\mathbb{C}))$ being the matrix with block entries $\mathbf{e}_{i,j} = e_{i,j}$. Then b is positive, as it is unitarily conjugate to the direct sum of a with itself n^2 times.

Now,

$$\begin{aligned}
\psi_K \otimes 1_m((a_{i,j})) &= \begin{bmatrix} \psi_K(a_{1,1}) & \cdots & \psi_K(a_{1,m}) \\ \vdots & \ddots & \vdots \\ \psi_K(a_{m,1}) & \cdots & \psi_K(a_{m,m}) \end{bmatrix} \\
&= \begin{bmatrix} s \otimes 1_n(a_{1,1} \otimes \mathbf{e}) & \cdots & s \otimes 1_n(a_{1,m} \otimes \mathbf{e}) \\ \vdots & \ddots & \vdots \\ s \otimes 1_n(a_{m,1} \otimes \mathbf{e}) & \cdots & s \otimes 1_n(a_{m,m} \otimes \mathbf{e}) \end{bmatrix} \\
&= s \otimes 1_n \otimes 1_m(b),
\end{aligned}$$

which is positive since s is a positive linear functional and is therefore completely positive by Lemma A.2.

Put the inclusion order on all the finite subspaces K of H. Assume that we have defined the ψ_K's in a compatible way so that ψ_{K_2} extends ψ_{K_1} whenever $K_1 \subset K_2$, and let the maps $\psi'_K : A \to B(H)$ be defined by extending each $\psi_K(a)$ to be 0 on $K^\perp$. By Lemma A.3, we have that $\|\psi'_K\| \leq \|\phi\|$. For simplicity, assume $\|\phi\| = 1$. Then $\{\psi'_K\}$ is a net in the unit ball of $B(A, B(H))$. By compactness, we can choose a subnet of $\{\psi'_K\}$ that converges to a map $\psi \in B(A, B(H))$ in the bounded weak topology. It remains to see that ψ is a completely positive extension of ϕ. Let $q \in Q$, $\xi, \eta \in H$, and let $L = \operatorname{span}\{\xi, \eta, \psi(q)\xi, \phi(q)\xi\}$. Then

$$\begin{aligned}
\langle \psi(q)\xi, \eta\rangle &= \lim_{K \to H} \langle \psi_K(q)\xi, \eta\rangle \\
&= \langle \psi_L(q)\xi, \eta\rangle \\
&= \langle \phi_L(q)\xi, \eta\rangle \\
&= \langle \phi(q)\xi, \eta\rangle.
\end{aligned}$$

So $\psi(q) = \phi(q)$ since ξ and η were arbitrary. Finally, let $(a_{i,j}) \in M_n(A)_+$, and let $\xi = (\xi_1 \ldots \xi_n) \in H^n$. Let $L \subset H$ be the subspace

$$L = \operatorname{span}\{\xi_j, \psi(a_{i,j})\xi_j\}_{i,j=1}^n$$

Then

$$\sum_{i,j=1}^{n} \langle \psi(a_{i,j})\xi_j, \xi_i \rangle = \sum_{i,j=1}^{n} \langle \psi_L(a_{i,j})\xi_j, \xi_i \rangle \geq 0$$

since ψ_L was completely positive. So ψ is completely positive. $\square$

Theorem 4.3 Let A be a unital C^*-algebra, $Q \subset A$ an operator system, and let $\phi : Q \to B(H)$ be completely positive. The extension to a completely positive map $\psi : A \to B(H)$ guaranteed by the last Theorem is unique only if for every $\xi \in H$ and every $a \in A_{sa}$, we have

$$\sup\{\langle \phi(q)\xi, \xi \rangle \mid q \in Q_{sa}, q \leq a\} = \inf\{\langle \phi(q)\xi, \xi \rangle \mid q \in Q_{sa}, q \geq a\} \quad (4.2)$$

Proof. We return to the central step in Arveson's Theorem, where we extend the compression $\phi_K : Q \to B(K)$ to a completely positive map $\psi_K : A \to B(K)$. Let $a \in A_{sa}$, and $\xi \in K$. Choose an orthonormal basis $\{e_i\}_{i=1}^n$ with $e_1 = \xi$. As before, we define $s_\phi : M_n(Q) \to B(K)$ by

$$s_\phi((q_{i,j})) = \sum_{i,j=1}^{n} \langle \phi_K(q_{i,j})e_i, e_j \rangle$$

and extend it to a positive linear functional $s : M_n(A) \to B(H)$. Now $a \otimes e_{1,1}$ is self-adjoint, and we see that $(q_{i,j}) \leq a \otimes e_{1,1}$ if and only if $(q_{i,j}) = q_0 \otimes e_{1,1}$ with $q_0 \leq a$. By Lemma 4.1, we can then choose $s(a \otimes e_{1,1})$ to be any value in the range (4.2). Hence we can define ψ_K such that $\langle \psi_K(a)\xi, \xi \rangle$ takes any value in the same range. The same can be done for any finite-dimensional subspace of H containing ξ, and hence we can force the final extension ψ to be such that $\langle \psi(a)\xi, \xi \rangle$ takes any given value in this range. $\square$

Extensions of pure states

Halpern, Kaftal and Weiss made some extensive research in the eighties of a property that is called the relative Dixmier property of the imbedding of a von Neumann algebra A into its crossed product with a discrete group [12]. This property is closely related to the Kadison-Singer problem.

We say that A has the *relative Dixmier property* in a superalgebra B if there is a conditional expectation Φ of B onto A and for every $b \in B$ and $\varepsilon > 0$, there is an $r \in \mathbb{N}$ and $u_1 \ldots u_r \in A_u$, such that

$$\left\| \frac{1}{r} \sum_{j=1}^{r} u_j b u_j^* - \Phi(b) \right\| < \varepsilon.$$

From [15] and [1], we get the following theorem, which is also related to several so-called paving results. Anderson proves versions of the theorem for more general settings, but that is outside our scope and requires some more work.

Theorem 4.4 Let B be a unital C^*-algebra, and let $L^\infty(\mu) \subset B$ where $(X, \mathcal{B}, \mu)$ is a locally compact Hausdorff measure space. Suppose that there is a conditional expectation $\Phi : B \to L^\infty(\mu)$. The following are equivalent.

(i) $L^\infty(\mu)$ has the relative Dixmier property in B

(ii) Every pure state of $L^\infty(\mu)$ has a unique extension to a pure state on B.

(iii) For every $b \in B$ and $\varepsilon > 0$, there is a finite partition $\{E_j\}_{j=1}^r$ of X into Borel subsets such that

$$\|\chi_{E_j}(b - \Phi(b))\chi_{E_j}\| < \varepsilon, \quad j = 1 \ldots r.$$

Proof. (i)⇒(ii): Let $\varepsilon > 0$. Suppose $L^\infty(\mu) \subset B$ has the relative Dixmier property, and let $u_1 \ldots u_r$ be as in the definition. If ϕ is a pure state on $L^\infty(\mu)$ and ψ is any extension to B, then

$$1 = \psi(u_j^* u_j) = \psi(u_j)^* \psi(u_j).$$

So by Theorem A.5, $\psi(u_j b u_j^*) = \psi(b)$ for all $b \in B$. So

$$|\psi(b) - \phi(\Phi(b))| = \left| \frac{1}{r} \sum_{j=1}^{r} \psi(u_j b u_j^* - \Phi(b)) \right| < \varepsilon,$$

So $\psi = \phi \circ \Phi$.

(ii)⇒(iii): Next, suppose every pure state on $L^\infty(\mu)$ has a unique extension, and let $b \in B$ be self-adjoint. By Krein's Extension Theorem, we get that given $\varepsilon > 0$ and any $\phi \in PS(L^\infty(\mu))$, there are $f = f_{\phi,\varepsilon} \in L^\infty(\mu)$ with $-f \leq b - \Phi(b) \leq f$ and $\phi(f) < \varepsilon/2$. Then since ϕ can be associated to an ultrafilter on $\mathcal{B}_0(\mu)$, there is a Borel set $E = E_{\phi,\varepsilon} \subset X$ with $\phi(\chi_E) = 1$ and $|f(x) - \phi(f)| < \varepsilon/2$ for almost every $x \in E$. By hypothesis,

$$\{\tilde{E}_{\phi,\varepsilon}\}_{\phi \in PS(L^\infty(\mu))}$$

is a clopen covering of $PS(L^\infty(\mu))$. Reduce it to a finite covering $\{\tilde{E}_1 \ldots \tilde{E}_r\}$, and then to a finite clopen partition $\{Y_1 \ldots Y_r\}$ of $PS(L^\infty(\mu))$ by appropriately removing each intersection from one of two overlapping sets. Then for $1 \leq j \leq r$,

$$\begin{aligned} \|\chi_{Y_j}(b - \Phi(b))\chi_{Y_j}\| &\leq \|\chi_{Y_j} a \chi_{Y_j}\| \\ &\leq \|\chi_{\tilde{E}_j}(a_j - \phi_j(a_j))\chi_{\tilde{E}_j}\| + \phi_j(a_j) < \varepsilon. \end{aligned}$$

The result easily generalizes to the non-self-adjoint case.

(iii)⇒(i): The unimodular functions

$$u_j = \sum_{k=1}^{r} e^{2\pi i(j-k)/r} \chi_{Y_k}, \quad j = 1 \dots r$$

in $C(PS(L^\infty(\mu)))$ can be associated to unitaries in $L^\infty(\mu)$, and we have

$$\begin{aligned}
\left\| \frac{1}{r} \sum_{j=1}^{r} u_j b u_j^* - \Phi(b) \right\| &= \left\| \frac{1}{r} \sum_{j=1}^{r} u_j (b - \Phi(b)) u_j^* \right\| \\
&= \left\| \sum_{j=1}^{r} \chi_{Y_j} (b - \Phi(b)) \chi_{Y_j} \right\| \\
&= \max_{j=1\dots r} \|\chi_{Y_j} (b - \Phi(b)) \chi_{Y_j}\| < \varepsilon.
\end{aligned}$$

□

An operator with the third property is sometimes called *pavable*. The next theorem may for instance be found in [12].

Lemma 4.5 Let G be an abelian discrete group. Then $\ell^\infty(G)$ has the relative Dixmier property in $C_r^*(\ell^\infty(G), G)$.

Proof. Pick $\varepsilon > 0$. Let $a \in C_r^*(\ell^\infty(G), G)$, and let $b \in \ell_0(G, \ell^\infty(G))$ be ε close to a in norm. The map $\hat{G} \to C_r^*(\ell^\infty(G), G)$ given by $\gamma \mapsto v_\gamma a v_\gamma^*$ is then norm-continuous. Given $g \in G$, we have

$$v_\gamma b_g u_g v_\gamma^* = b_g v_\gamma u_g v_\gamma^*.$$

Since u_g can be identified with the $C(\hat{G})$ function given by evaluation at γ, we know by Theorem 2.18 that there is an $r \in \mathbb{N}$ and $\gamma_1 \dots \gamma_r \in \hat{G}$ such that

$$\left\| \frac{1}{r} \sum_{j=1}^{r} v_{\gamma_j} u_g v_{\gamma_j}^* - \Phi(u_g) \right\| < \varepsilon,$$

such that

$$\left\| \frac{1}{r} \sum_{j=1}^{r} v_{\gamma_j} b_g u_g v_{\gamma_j}^* - \Phi(b_g u_g) \right\| < \varepsilon \|b_g\|.$$

Since b is a finite sum of bounded elements of this type, it is straightforward to find a convex sum that works for all of b. This then also works for a by adjusting ε appropriately. □

It would of course also be tempting to try and use the unitaries from the dual action to do the approximations necessary for any element in $W^*(\ell^\infty(G), G)$, but by Theorem 2.18, this does not work. The counterexamples are $W^*(G)$-operators for which invariant means on $L^\infty(\hat{G})$ disagree.

As far as we know, the following facts were first realized by Halpern, Kaftal and Weiss in [11] (in a slightly less general setting).

Theorem 4.6 Let G be a countable abelian discrete group, and let ϕ be a state on $W^*(\ell^\infty(G), G)$.

(i) If ϕ restricts to a pure state on $\ell^\infty(G)$, then $\phi \circ \mathrm{Ad}_{\mathscr{F}^*}$ restricts to an invariant mean on $L^\infty(\hat{G})$.

(ii) If ϕ restricts to a pure state on $W^*(G)$, then ϕ restricts to an invariant mean on $\ell^\infty(G)$.

Proof. First, suppose ϕ restricts to a pure state on $\ell^\infty(G)$. Let $\gamma \mapsto v_\gamma$ be the dual representation of $\hat{G}$ we studied in section 3.2. Since each v_γ is an element of $\ell^\infty(G)$, and since ϕ is a homomorphism when restricted to $\ell^\infty(G)$, we get that $\phi(v_\gamma)^*\phi(v_\gamma) = \phi(v_\gamma^* v_\gamma) = 1$. So by Theorem A.5, $\phi(v_\gamma a v_\gamma^*) = \phi(a)$ for every $a \in W^*(\ell^\infty(G), G)$. But conjugation by v_γ (through $\mathscr{F}$) is exactly the action of translation on $L^\infty(\hat{G})$.

The proof of point (ii) is the same. Every u_g is a $W^*(G)$ unitary, and hence any state that is pure on $W^*(G)$ is invariant to conjugation by it. But this is exactly the action of translation on $\ell^\infty(G)$. □

Kadison and Singer proved that the pure states on a continuous maximal abelian subalgebra of $B(H)$, such as $L^\infty([0,1]) \simeq L^\infty(\mathbb{T}) \simeq W^*(\mathbb{Z})$ do not have unique extensions to $B(H)$. The way they proved it was to show that there are many different conditional expectations of $B(H)$ onto $L^\infty([0,1])$. If ϕ is a pure state on $L^\infty([0,1])$, and $\Psi : B(H) \to L^\infty([0,1])$ is a conditional expectation, then it is easy to see that $\phi \circ \Psi$ is an extension of ϕ to $B(H)$. So different expectations lead to different state extensions. Motivated by the previous theorem, we give a new description of all the conditional expectations of this type.

Theorem 4.7 Let G be a countable abelian discrete group. Then for every invariant mean m on G there is a conditional expectation $\Psi_m : W^*(\ell^\infty(G), G) \to W^*(G)$ such that

$$\Phi \circ \Psi_m(a) = m \circ \Phi(a)1 \tag{4.3}$$

for all $a \in W^*(\ell^\infty(G), G)$. Moreover, for every conditional expectation Ψ of $W^*(\ell^\infty(G), G)$ onto $W^*(G)$, there is an invariant mean m on $\ell^\infty(G)$ such that Ψ satisfies (4.3).

Before we go on to the proof, we would like to say a little about integration of Banach-space valued functions. Usually, if $f : X \to B$ is a continuous compactly supported function on a measure space $(X, \mathcal{B}, \mu)$, and B is a Banach space, one defines the integral

$$\int_X f \,\mathrm{d}\mu$$

to be the element $b \in B$ such that

$$\int_X \phi(f(x)) \,\mathrm{d}\mu(x) = \phi(b)$$

for every $\phi \in X^*$. There are several ways to show that this element exists, depending on different properties of the measure space, but this is often quite complicated. Fortunately, what we are going to need is a little simpler. Let X be compact Hausdorff with a measure μ, and let $f : X \to B(A, B(H))$ be continuous into the bounded weak topology. We define

$$\int_X f \,\mathrm{d}\mu$$

by

$$\left\langle \left[\int_X f \,\mathrm{d}\mu\right](a)\xi, \eta \right\rangle = \int_X \langle f(x)(a)\xi, \eta\rangle \,\mathrm{d}\mu(x)$$

for every $a \in A$ and $\xi, \eta \in H$. Since we already know that $B(A, B(H))$ is closed in Z^*, this actually converges to a well-defined element of $B(A, B(H))$ in the bounded weak topology.

Proof. (Of Theorem 4.7). If m is an invariant mean on G, it can be associated to a G-invariant state on $C(\beta G)$, and by Riesz' Representation Theorem there is a unique G-invariant probability measure ν_m on βG such that for every $f \in \ell^\infty(G)$,

$$m(f) = \int_{\beta G} \hat{f}(\omega) \,\mathrm{d}\nu_m(\omega).$$

Define $\Psi_m : W^*(\ell^\infty(G), G) \to W^*(\ell^\infty(G), G)$ by

$$\Psi_m = \int_{\beta G} \hat{\rho}_\omega \,\mathrm{d}\nu_m(\omega)$$

where the integration limit is taken in the bounded weak topology. I.e.

$$\langle \Psi_m(a)\xi, \eta\rangle = \int_{\beta G} \langle \hat{\rho}_\omega(a)\xi, \eta\rangle \,\mathrm{d}\nu_m(\omega)$$

for every $\xi, \eta \in \ell^2(G)$. To see that Ψ_m is completely positive, let $(a_{i,j}) \in M_n(W^*(\ell^\infty(G), G))_+$, and let $\xi_1 \ldots \xi_n \in \ell^2(G)$. Then

$$\sum_{i,j=1}^n \int_{\beta G} \langle \hat{\rho}_\omega(a_{i,j})\xi_i, \xi_j\rangle \,\mathrm{d}\nu_m(\omega) \geq 0$$

since $\hat{\rho}_\omega$ was completely positive by Lemma 3.10. Pointwise fixation of $W^*(G)$ follows from the same lemma.

To see that its image is entirely contained in $W^*(G)$, let $g \in G$ be arbitrary. Then

$$\begin{aligned} u_g \Psi_m(a) u_g^* &= u_g \int_{\beta G} \hat{\rho}_\omega(a) \, \mathrm{d}\nu_m(\omega) \, u_g^* \\ &= \int_{\beta G} u_g \hat{\rho}_\omega(a) u_g^* \, \mathrm{d}\nu_m(\omega) \\ &= \int_{\beta G} \hat{\rho}_{g\cdot\omega}(a) \, \mathrm{d}\nu_m(\omega) \\ &= \Psi_m(a). \end{aligned}$$

since ν_m is invariant. So $\Psi_m(a) \in W^*(G)$ by Lemma 3.7.

Moreover, for every $f \in \ell^\infty(G)$ and $g \in G$,

$$\begin{aligned} \langle \Psi_m(f) e_g, e_g \rangle &= \int_{\beta G} \langle \hat{\rho}_\omega(f) e_g, e_g \rangle \, \mathrm{d}\nu_m(\omega) \\ &= \int_{\beta G} \langle \hat{\rho}_{g\cdot\omega}(f) e_1, e_1 \rangle \, \mathrm{d}\nu_m(\omega) \\ &= \int_{\beta G} \hat{f}(g \cdot \omega) \, \mathrm{d}\nu_m(\omega) \\ &= m(f). \end{aligned}$$

So $\Psi_m(f) = m(f)1$ as desired. To see that every conditional expectation $\Psi : W^*(\ell^\infty(G), G) \to W^*(G)$ satisfies (4.3), let $g \in G$. Then $\Psi(f) = \alpha_g(\Psi(f))$. So $f \mapsto \langle \Psi(f) e_1, e_1 \rangle$ defines an invariant mean on $\ell^\infty(G)$. $\square$

We now see that if $E \subset \mathbb{Z}$ is a thick subset with thick complement, such as the one in equation (2.2), then by Theorem 2.16 there are invariant means m_1, m_2 on $\mathbb{Z}$ such that $m_1(\chi_E) = 1$ and $m_2(\chi_E) = 0$. So by the last theorem, every pure state on $W^*(\mathbb{Z})$ has extensions that evaluate χ_E to both 0 and 1. In fact, every pure state of $W^*(\mathbb{Z})$ has at least as many extensions as there are invariant means on $\mathbb{Z}$.[4] It is interesting to note that χ_E is exactly the operator that Kadison and Singer used to show the nonuniqueness of conditional expectations, however their proof did not employ invariant means or any group structure at all [15].

This argument can not be directly adopted to show that the pure states of $\ell^\infty(G)$ have nonunique extensions, because we know that the conditional

[4] According to [14], Theorem 6.44, there are at least 2^c distinct minimal left ideals in βG when G is infinite. Here c is the cardinality of the continuum. This implies that there are also at least 2^c different invariant means on $\ell^\infty(G)$. This method of determining the number of invariant means was employed by Chou already in 1969 [7].

expectation onto $\ell^\infty(G)$ is unique. However, there is another class of completely positive maps that are interesting to look at. Remember that for $\omega \in \beta G$, the pure state s_ω given by $f \mapsto \hat{f}(\omega)$ had an extension given by $a \mapsto \langle \hat{\rho}_\omega(a)e_1, e_1\rangle = s_\omega \circ \Phi(a)$. We call this the *regular extension* of s_ω. The next theorem is from [23].

Theorem 4.8 (Paulsen) Let G be a discrete group, and let $\omega \in \beta G$. If s_ω is the pure state on $\ell^\infty(G)$ given by evaluation at ω, then s_ω has a unique extension to $W^*(\ell^\infty(G), G)$ if and only if $\hat{\rho}_\omega$ is the only completely positive extension of $\hat{\varphi}_\omega$ to $W^*(\ell^\infty(G), G)$.

Proof. Suppose s_ω does not have unique extensions, and let s be an extension that is nonregular. Let $\pi : W^*(\ell^\infty(G), G) \to B(H)$ be the GNS representation of s with cyclic vector ξ_1 such that $s(a) = \langle \pi(a)\xi_1, \xi_1\rangle$. By Lemma 4.5, s restricts to the regular extension on $C^*(\ell^\infty(G), G)$, and hence $s(u_g^* u_h) = \delta_{g,h}$. So the vectors $\{\pi(u_g)\xi_1\}_{g\in G} = \{\xi_g\}_{g\in G}$ form an orthonormal set in H. Let $w : \ell^2(G) \to H$ be the partial isometry given by $we_g = \xi_g$. Then $\hat{\rho}^s_\omega \in B(W^*(\ell^\infty(G), G))$ given by

$$\hat{\rho}^s_\omega(a) = w^*\pi(a)w$$

is completely positive by Theorem A.1, and for any $a \in C^*(\ell^\infty(G), G)$ and $g, h \in G$, we have

$$\begin{aligned}
\langle \hat{\rho}^s_\omega(a)e_g, e_h\rangle &= \langle \pi(a)\xi_g, \xi_h\rangle \\
&= s_\omega \circ \Phi(u_h^* a u_g) \\
&= \langle \hat{\varphi}_\omega(u_h^* a u_g)e_1, e_1\rangle \\
&= \langle u_h^* \hat{\varphi}_\omega(a) u_g e_1, e_1\rangle \\
&= \langle \hat{\varphi}_\omega(a)e_g, e_h\rangle.
\end{aligned}$$

So $\hat{\rho}^s_\omega$ extends $\hat{\varphi}_\omega$.

Next, suppose that $\hat{\rho}'_\omega \neq \hat{\rho}_\omega$ is any completely positive extension of $\hat{\varphi}_\omega$. Then as noted earlier,

$$s = (a \mapsto \langle \hat{\rho}'_\omega(a)e_1, e_1\rangle)$$

is a state extension of s_ω. If we use s to construct a positive extension of $\hat{\varphi}_\omega$ as above, we see that we get $\hat{\rho}^s_\omega = \hat{\rho}'_\omega$ back, so the correspondence between state extensions of s_ω and completely positive extensions of $\hat{\varphi}_\omega$ is (affinely) bijective. □

Let G be a countable and discrete abelian group. Then $\hat{G}$ is compact, and by Theorem 2.6 we can find a set $E \subset \hat{G}$ that is thick and has thick complement. By Lemma 2.7, we have

$$\begin{aligned}
\inf\{\langle fe_1, e_1\rangle \mid f \in C(\hat{G}), f \geq \chi_E \text{ a.e}\} &= 1 \\
\sup\{\langle fe_1, e_1\rangle \mid f \in C(\hat{G}), f \leq \chi_E \text{ a.e}\} &= 0.
\end{aligned}$$

It follows by Theorem 4.3 that the inclusion map of $C(\hat{G}) \simeq C_r^*(G)$ into $B(\ell^2(G))$ has completely positive extensions to maps of $L^\infty(\hat{G})$ into $B(\ell^2(G))$ that do not fix χ_E. This is basically Proposition 24 of [4]. This makes the $W_r^*(G)$-operator $a = \mathscr{F}^*\chi_E\mathscr{F}$ a likely counterexample to the Kadison-Singer conjecture[5]. Another indication is of course that E is exactly the kind of set that invariant means tend to differ on.

Since $\langle\hat{\rho}_\omega(a)e_1, e_1\rangle = s_\omega(a)$, we know by Lemma 4.1 that all completely positive extensions of $\hat{\varphi}_\omega$ agree on a if and only if for every $\varepsilon > 0$, there are $q_1, q_2 \in C^*(\ell^\infty(G), G)$ with $q_1 \leq a \leq q_2$ and

$$\langle\hat{\varphi}_\omega(q_2 - q_1)e_1, e_1\rangle < \varepsilon.$$

That is, there are $q_1', q_2' \in A_\omega = \hat{\varphi}_\omega(C_r^*(\ell^\infty(G), G))$ with $q_1' \leq a \leq q_2'$ and

$$\langle(q_2' - q_1')e_1, e_1\rangle < \varepsilon.$$

The last observation follows from the fact that $\hat{\rho}_\omega(a) = a$. Since $\hat{\varphi}_\omega$ fixes $C_r^*(G)$ pointwise, we know that the elements of $C_r^*(G)$ are useless for this approximation, so one would like to know essentially how much larger A_ω is than $C_r^*(G)$. That was one of the main points put forth in Paulsen's paper [23].

Suppose that $\omega \in \beta G$ is minimal. The set

$$U = \{\rho \in \beta G \mid \langle\hat{\varphi}_\rho(q_2 - q_1)e_1, e_1\rangle < \frac{1}{2}\}$$

is open, and hence a finite number of translates of it cover $\beta G \cdot \omega$. Let m be an invariant mean with support on $\beta G \cdot \omega$. Then $\nu_m(U) > 0$. We have $\Psi_m(q_1), \Psi_m(q_2) \in C_r^*(G)$, $\Psi_m(a) = a$, and

$$\begin{aligned} 1 &\leq \langle(\Psi_m(q_2) - \Psi_m(q_1))e_1, e_1\rangle \\ &= \int_{\beta G} \langle\hat{\varphi}_\rho(q_2 - q_1)e_1, e_1\rangle \, \mathrm{d}\nu_m(\rho) \\ &\leq \frac{1}{2}\nu_m(U) + \int_{\beta G \setminus U} \langle\hat{\varphi}_\rho(q_2 - q_1)e_1, e_1\rangle \, \mathrm{d}\nu_m(\rho). \end{aligned}$$

So if one can prove that the hypothesized existence of q_1 and q_2 implies that they can be chosen such that $\langle\hat{\varphi}_\omega(q_2 - q_1)e_g, e_g\rangle$ is nowhere much larger than 1, this will create a contradiction.

We will look at a few more ways to study the pavability of $W_r^*(G)$ operators. The next theorem is also from [23].

[5]We would like to point out that it is not really necessary to look at a set with thick complement in this argument. We always have $\langle fe_1, e_1\rangle \leq \mu(E)$ for $f \leq \chi_E$, so the difference between the infimum and supremum will be at least $1 - \mu(E)$ even if only the set itself is thick.

Theorem 4.9 (Paulsen) Let G be a discrete abelian group, and let $a \in W_r^*(G)$. Then the following are equivalent.

(i) a is pavable.

(ii) There is a minimal idempotent $\omega \in \beta G$ such that any state extension of s_ω agrees with the regular extension of s_ω on a.

(iii) For every $\varepsilon > 0$ there is a syndetic set $E \subset G$ with

$$\|\chi_E(a - \Phi(a))\chi_E\| < \varepsilon.$$

Proof. (i)⇒(ii) follows from Theorem 4.4.

(ii)⇒(iii): Let $\varepsilon > 0$, and let $\omega \in \beta G$ be a minimal idempotent. Then $\hat{\rho}_\omega$ is a conditional expectation of $W^*(\ell^\infty(G), G)$ onto its image. If every state extension of s_ω maps a to $s_\omega(\Phi(a))$, we know as in the proof of Theorem 4.4 that there is a set $J \in \omega$ such that $\|\chi_J(a - \Phi(a))\chi_J\| < \varepsilon$. Let

$$E = \{g \in G \mid g \cdot \omega \in J\}.$$

Then $\hat{\rho}_\omega(\chi_J) = \chi_E$, and since ω is idempotent, we also have $\hat{\rho}_\omega(\chi_E) = \chi_E$, so

$$\hat{\rho}_\omega(\chi_J(a - \Phi(a))\chi_J) = \hat{\rho}_\omega(\chi_J)\hat{\rho}_\omega(a - \Phi(a))\hat{\rho}_\omega(\chi_J) = \chi_E(a - \Phi(a))\chi_E.$$

The result now follows from the fact that $\hat{\rho}_\omega$ is norm-reducing.

(iii)⇒(i): Let $\varepsilon > 0$, and let $E \subset G$ be a syndetic set satisfying the conditions of (iii). If ϕ is any pure state on $\ell^\infty(G)$, we have that there is some $g \in G$ such that $\phi(\chi_{gE}) = 1$. Let ψ be an extension of ϕ. Then

$$\begin{aligned}
\psi(a - \Phi(a)) &= \psi(\chi_{gE}(a - \Phi(a))\chi_{gE}) \\
&= \psi(\alpha_g(\chi_E)(a - \Phi(a))\alpha_g(\chi_E)) \\
&= \psi(u_g\chi_E\alpha_{g^{-1}}(a - \Phi(a))\chi_E u_g^*) \\
&= \psi(u_g\chi_E(a - \Phi(a))\chi_E u_g^*) \\
&\leq \|\chi_E(a - \Phi(a))\chi_E\| < \varepsilon.
\end{aligned}$$

So every extension of ϕ takes the same value on a. □

In $\mathbb{Z}$, an important family of syndetic sets are the the infinite arithmetic progressions of the form $n\mathbb{Z} + k$ with $n, k \in \mathbb{N}$. We will now present a theorem from [11] classifying the $W_r^*(\mathbb{Z})$ operators that can be paved by such sets.

Theorem 4.10 Let $a \in W_r^*(\mathbb{Z})$, and let $f \in L^\infty(\mathbb{T})$ be the corresponding function by which a acts by multiplication. The following are equivalent.

(i) For every $\varepsilon > 0$, there is a $n \in \mathbb{N}$ such that

$$\|\chi_{n\mathbb{Z}}(a - \Phi(a))\chi_{n\mathbb{Z}}\| < \varepsilon.$$

(ii) For every $\varepsilon > 0$ there is a $n \in \mathbb{N}$ such that

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} \lambda_{e^{\frac{-2\pi i j}{n}}}(f) - \int_{\mathbb{T}} f \,\mathrm{d}\mu \right\|_{\infty} < \varepsilon. \tag{4.4}$$

Proof. We clearly have that

$$\begin{aligned} \|\chi_{n\mathbb{Z}}(a - \Phi(a))\chi_{n\mathbb{Z}}\| &= \|\chi_{(n\mathbb{Z}+k)}(a - \Phi(a))\chi_{(n\mathbb{Z}+k)}\| \\ &= \left\| \sum_{j=0}^{n-1} \chi_{(n\mathbb{Z}+j)} a \chi_{(n\mathbb{Z}+j)} - \Phi(a) \right\| \end{aligned}$$

for any $k \in \mathbb{Z}$. It remains to see that this also equals the left hand quantity in equation (4.4). Write $z = e^{-2\pi i/n}$ We have

$$v_{z^j} = \sum_{k=0}^{n-1} \chi_{(n\mathbb{Z}+k)} z^{kj},$$

and

$$\sum_{j=0}^{n-1} z^{kj} = \begin{cases} n & \text{if } k = 0 \\ 0 & \text{otherwise,} \end{cases}$$

so

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-1} v_{z^{kj}} a v_{z^{kj}}^{*} &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=0}^{n-1} \sum_{j=0}^{n-1} z^{(k-m)j} \chi_{(n\mathbb{Z}+k)} a \chi_{(n\mathbb{Z}+m)} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=0}^{n-1} n\delta_{kj} \chi_{(n\mathbb{Z}+k)} a \chi_{(n\mathbb{Z}+m)} \\ &= \sum_{k=0}^{n-1} \chi_{(n\mathbb{Z}+k)} a \chi_{(n\mathbb{Z}+k)}. \end{aligned}$$

$\square$

Operators satisfying Theorem 4.10 are sometimes said to be *uniformly pavable*, and as noted earlier, there are many operators that do not have property (4.4).

5 Discussion

When we first set out to write this article we had many ideas about which directions we could investigate, but the short time limit of half a year and the large amount of background material needed to be established has allowed us to properly look at only a few of these. We are however still satisfied with the result.

The only really new technique we have come up with is the construction of conditional expectations onto $W_r^*(G)$ using invariant means on $\ell^\infty(G)$. We have also given a description of the perhaps most likely counterexamples to the Kadison-Singer conjecture, that is $W_r^*(G)$-projections corresponding to thick sets in G. Something similar has been done before, but the specific characteristics needed have maybe not been this articulated. In [11], dense open sets are used, while in [4], a more specific construction is used. Of course, concrete examples are useful since they allow one to do quantitative calculations.

There are a few things we would have liked to study further given more time. One of these is the invariant measures on βG, that is the invariant means on $\ell^\infty(G)$. For instance, we wanted to see if there was a unique invariant mean on $\pi_\omega(\ell^\infty(G))$ when ω was minimal. We have some indications that it may be so, but the problem turned out to rely heavily on difficult combinatorics. If it turned out to be so, we might perhaps have described the invariant means on $\ell^\infty(G)$ as convex combinations of such minimal means. We would also have got that $A_\omega = \hat{\varphi}_\omega(C_r^*(\ell^\infty(G), G))$ had a unique and faithful trace.

A C^*-algebras and completely positive maps

Recall that a C^* algebra is a Banach $*$-algebra satisfying the C^*-identity

$$\|a^*a\| = \|a\|^2, \quad a \in A.$$

Let $\sigma_A(a)$ denote the spectum of an element $a \in A$. We will use the following notation for the respective subsets of self-adjoint, positive, unitary elements and orthogonal projections in A

$$\begin{aligned} A_{sa} &= \{a \in A \mid a = a^*\} \\ A_+ &= \{a \in A \mid \sigma_A(a) \subset \mathbb{R}_+\} \\ A_u &= \{a \in A \mid a^*a = aa^* = 1\} \\ A_p &= \{a \in A \mid a = a^* = a^2\}. \end{aligned}$$

Let A be a C^*-algebra. A *positive linear functional* on A is a linear functional $\phi : A \to \mathbb{C}$ such that $\phi(a) \geq 0$ whenever $a \in A_+$. It is standard knowledge that all such functionals are continuous, and if A is unital, $\|\phi\| = \phi(1_A)$. We say that ϕ is a *state* if ϕ is positive and $\|\phi\| = 1$.

Let $A \neq 0$, and denote by $S(A)$ set of states on A. This is a compact convex subset of the dual space of A, and by the Krein-Milman theorem it has many extreme points. If f is such an extreme point it is called a *pure state* of A. Denote the set of pure states by $PS(A)$.

If A is abelian, then it is well known that the pure states of A are exactly the nonzero multiplicative $*$-algebra homomorphisms $A \to \mathbb{C}$. These are also called *characters*, and $PS(A)$ is called the character space or maximal ideal space or *spectrum* of A. We will adopt the last name for this space.

Let A and B be a C^*-algebras. We say that a map $\phi : A \to B(H)$ is *completely positive* if for every $n \in \mathbb{N}$, the map

$$\begin{aligned} M_n(A) &\to M_n(B) \\ (a_{i,j})_{i,j=1}^n &\mapsto (\phi(a_{i,j}))_{i,j=1}^n \end{aligned}$$

is positive. For people familiar with tensor products, it makes sense to denote this map by $\phi \otimes 1_n$, where 1_n is the identity automorphism of $M_n(\mathbb{C})$. As we shall see, completely positive maps are in some sense the natural generalization of states. The next Theorem generalizes the GNS construction [28].

Theorem A.1 (Stinespring's Factorization Theorem) Let A be a unital C^*-algebra, and let H be a Hilbert space. A map

$$\phi : A \to B(H)$$

is completely positive if and only if there exists a Hilbert space K, a bounded linear operator $v : H \to K$ and a $*$-homomorphism $\pi : A \to B(K)$ such that

$$\phi(a) = v\pi(a)v^*$$

for all $a \in A$.

Lemma A.2 (Stinespring) Every positive linear functional $\phi : A \to \mathbb{C}$ on a C^*-algebra A is completely positive.

Proof. Let $(a_{i,j}) \in M_n(A)_+$, and write $(a_{i,j}) = (b_{i,j})^*(b_{i,j})$ with $(b_{i,j}) \in M_n(A)$. For any $(x_1 \dots x_n) \in \mathbb{C}^n$, we have

$$\sum_{i,j=1}^{n} \phi(a_{i,j})x_j\overline{x_i} = \phi\left(\sum_{i,j=1}^{n} a_{i,j}x_j\overline{x_i}\right).$$

But,

$$\begin{aligned}\sum_{i,j=1}^{n} a_{i,j}x_j\overline{x_i} &= \sum_{i,j=1}^{n} x_j\overline{x_i}\sum_{k=1}^{n} b_{k,i}^* b_{k,j} \\ &= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} x_i b_{k,i}\right)^*\left(\sum_{j=1}^{n} x_j b_{k,j}\right) \geq 0.\end{aligned}$$

□

A map between unital C^*-algebras is a *unital map* if it maps the unit of the first algebra to the unit of the second. We lift the proofs of the next three results from Paulsen's book [21].

Lemma A.3 Let A and B be unital C^*-algebras, and let $\phi : A \to B$ be completely positive. Then $\|\phi\| = \|\phi(1)\|$.

Proof. Clearly $\|\phi\| \leq \|\phi(1)\|$. Let $a \in A$ with $\|a\| \leq 1$. Suppose A is represented on the Hilbert space H_1 and B on H_2, and let $\xi, \eta \in H$ be arbitrary. Then

$$\begin{aligned}\left\langle \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}, \begin{bmatrix} \xi \\ \eta \end{bmatrix}\right\rangle &= \langle \xi, \xi\rangle + \langle a\eta, \xi\rangle + \langle \eta, a\xi\rangle + \langle \eta, \eta\rangle \\ &\geq \|\xi\|^2 + 2\|\xi\|\|\eta\| + \|\eta\|^2 \geq 0.\end{aligned}$$

Then for all $\xi, \eta \in H_2$,

$$\langle \phi(1)\xi, \xi\rangle + \langle \phi(a)\eta, \xi\rangle + \langle \eta, \phi(a)\xi\rangle + \langle \phi(1)\eta, \eta\rangle \geq 0.$$

But if $\|\phi(a)\| > \|\phi(1)\|$, we can pick unit vectors ξ and η such that $\mathfrak{Re}\langle \phi(a)\eta, \xi\rangle < -\|\phi(1)\|$ and the expression becomes negative. □

Lemma A.4 (The Cauchy-Schwarz inequality) Let A and B be C^*-algebras, and let $\phi : A \to B$ be completely positive. Then $\phi(a)^*\phi(a) \leq \phi(a^*a)$ for all $a \in A$.

Proof. We see that

$$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ a^* & a^*a \end{bmatrix} \geq 0.$$

So

$$\begin{bmatrix} 1 & \phi(a) \\ \phi(a)^* & \phi(a^*a) \end{bmatrix} \geq 0.$$

Represent B on H, and let $\xi \in H$. Then

$$\begin{aligned} & \left\langle \begin{bmatrix} 1 & \phi(a) \\ \phi(a)^* & \phi(a^*a) \end{bmatrix} \begin{bmatrix} \phi(a)\xi \\ -\xi \end{bmatrix}, \begin{bmatrix} \phi(a)\xi \\ -\xi \end{bmatrix} \right\rangle \\ = \quad & \langle \phi(a)\xi - \phi(a)\xi, \phi(a)\xi \rangle + \langle \phi(a)^*\phi(a)\xi - \phi(a^*a)\xi, -\xi \rangle \geq 0, \end{aligned}$$

so $\langle \phi(a^*a)\xi, \xi \rangle \geq \langle \phi(a)^*\phi(a)\xi, \xi \rangle$. □

The next Theorem first appeared in [6]. See also [20].

Theorem A.5 (Choi's Theory for Multiplicative Domains) Let A and B be unital C^*-algebras, and let $\phi : A \to B$ be completely positive and unital. Then the sets

$$\begin{aligned} \{a \in A \mid \phi(a)^*\phi(a) = \phi(a^*a)\} &= \{a \in A \mid \phi(ba) = \phi(b)\phi(a) \text{ for all } b \in A\} \\ \{a \in A \mid \phi(a)\phi(a)^* = \phi(aa^*)\} &= \{a \in A \mid \phi(ab) = \phi(a)\phi(b) \text{ for all } b \in A\} \end{aligned}$$

are C^*-subalgebras of A, and ϕ is a homomorphism when restricted to them.

Proof. We now prove the first equality, but skip the second, as the proof is just the same. By putting $b = a^*$ and using the fact that ϕ is $*$-linear we get that the first set is included in the second.

Apply the Cauchy-Schwarz inequality to the map $\phi \otimes 1_2 : M_2(A) \to M_2(B)$ and the matrix

$$\begin{bmatrix} a & b^* \\ 0 & 0 \end{bmatrix}$$

We get

$$\begin{bmatrix} \phi(a) & \phi(b^*) \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} \phi(a) & \phi(b^*) \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} \phi(a^*a) & \phi(a^*b) \\ \phi(ba) & \phi(bb^*) \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} \phi(a^*a) - \phi(a)^*\phi(a) & \phi(a^*b^*) - \phi(a)^*\phi(b^*) \\ \phi(ba) - \phi(b)\phi(a) & \phi(bb^*) - \phi(b)\phi(b^*) \end{bmatrix} \geq 0.$$

Since the upper left entry is zero and the lower left entry is the adjoint of the upper right entry, a computation shows that it is necessary for both of them to be zero in order for the matrix to be positive.

The rest of the claims are now imminent. □

B Locally compact groups

In this section, we will provide a short introduction to the theory of locally compact groups. More material on abstract harmonic analysis and the theory of locally compact groups can be found in [13] and [26], and it is from these two sources we pull the following results.

A *topological group* is a group G together with a topology on G such that the operations of group multiplication and inversion

$$\begin{aligned} G \times G \to G, \qquad (g,h) &\mapsto gh \\ G \to G, \qquad g &\mapsto g^{-1} \end{aligned}$$

are continuous. A topological space is *locally compact* if every point has a compact neighbourhood. In a topological group, it is clearly sufficient to check that the identity has a compact neighborhood.

Example B.1 Any discrete group is clearly locally compact. The most basic example is perhaps $\mathbb{Z}$. The other example we will use as a standard tool of analysis is $\mathbb{T}$. Other examples include matrix groups such as the unitary groups $U(n) \subset M_n(\mathbb{C})$ and others.

One of the most useful results about locally compact groups is the existence of an invariant measure μ. This is a regular σ-finite measure defined on the Borel σ-algebra of G, and it satisfies

$$\mu(gE) = \mu(E)$$

for all $g \in G$ and $E \subset G$. μ is usually called the (left) *Haar measure* on G, and it is the unique translation invariant Borel measure on G up to the scaling with a constant. When G is compact, we will assume that μ is scaled such that $\mu(G) = 1$. If G is discrete, we set $\mu_G(\{g\}) = 1$ for every $g \in G$ when G. These two scales do of course contradict each other if G is finite and nontrivial, but that will not cause problems. If other cases are needed, the scaling will be specified separately.

One central topic in abstract harmonic analysis is the Pontryagin duality and Fourier analysis of locally compact abelian (*lca*) groups.

Let G be a lca group. We define its *dual group* $\hat{G}$ to be the set of all continous homomorphisms $G \to \mathbb{T}$ with pointwise multiplication as the group operation. $\hat{G}$ can be identified with a subspace of the dual of $L^1(G)$ where $\gamma \in \hat{G}$ defines the functional

$$f \mapsto \int_G f(g)\gamma(g^{-1})\,\mathrm{d}\mu(g).$$

We give $\hat{G}$ the topology induced from the weak*-topology on $L^1(G)^*$. We will now state a few results without proof.

Theorem B.2 Let G be a locally compact abelian group.

(i) $\hat{G}$ is the character space of the convolution algebra $L^1(G)$.

(ii) If G is discrete, $\hat{G}$ is compact and if G is compact, $\hat{G}$ is discrete.

(iii) (The Pontryagin Duality Theorem). The dual group of $\hat{G}$ is G.

(iv) The dual of $\mathbb{Z}$ is $\mathbb{T}$. The dual of $\mathbb{R}$ is $\mathbb{R}$.

(v) Let $f \in L^1(G)$. Then the Gelfand transform of f is a continous function $\hat{f} : \hat{G} \to \mathbb{C}$ given by

$$\hat{f}(\gamma) = \int_G f(g)\gamma(g^{-1})\,\mathrm{d}\mu(g).$$

We call this the Fourier transform of f, and it coincides with the usual Fourier transform when $G = \mathbb{T}$ or $G = \mathbb{R}$.

(vi) (The Plancherel Invertibility Theorem). The Fourier transform extends to a unitary operator $\mathscr{F} : L^2(G) \to L^2(\hat{G})$ that we call the Plancherel transform. (Note that the Haar measures on the groups must be properly scaled for $\mathscr{F}$ to be an isometry).

C Boolean algebra

The book [14] is a good reference for this section, as it constructs the Stone-Čech compactification of a discrete group just the way we need it. We will however prove the results for general Boolean algebras instead of proving them just for the power set Boolean algebra of some set S. This will allow us to describe the spectrum of $L^\infty(G)$ when G is a nondiscrete locally compact group as well.

A *Boolean algebra* is a set $\mathcal{B}$ together with binary operations $\vee$ and $\wedge$, and a unary operation $^\perp$ such that for all $p, q, r \in \mathcal{B}$.

$$\begin{aligned} p \vee q &= q \vee p & p \wedge q &= q \wedge p \\ p \vee (q \vee r) &= (p \vee q) \vee r & p \wedge (q \wedge r) &= (p \wedge q) \wedge r \\ p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r) & p \wedge (q \vee r) &= (p \wedge q) \vee (p \wedge r) \\ p \vee (p \wedge q) &= p & p \wedge (p \vee q) &= p. \end{aligned}$$

In addition, there are unique distinct elements $0, 1 \in \mathcal{B}$ such that

$$p \vee p^\perp = 1 \qquad p \wedge p^\perp = 0$$

for all $p \in \mathcal{B}$.

Example C.1 The simplest Boolean algebra is $\{0, 1\}$ with all operations uniquely determined by the relation $0^\perp = 1$. If S is a set, then the operations of union, intersection and complementation makes the power set of S a Boolean algebra. Another important example is the Boolean algebra of clopen subsets of a topological space X.

A *Boolean algebra homomorphism* is a map $\omega : \mathcal{B} \to \mathcal{C}$ between Boolean algebras $\mathcal{B}, \mathcal{C}$ such that for all $p, q \in \mathcal{B}$,

$$\begin{aligned} \omega(p \vee q) &= \omega(p) \vee \omega(q) \\ \omega(p \wedge q) &= \omega(p) \wedge \omega(q) \\ \omega(0) = 0 \quad & \omega(1) = 1. \end{aligned}$$

A bijective homomorphism is called an isomorphism.

If $\mathcal{B}$ is any Boolean algebra and $\omega : \mathcal{B} \to \{0, 1\}$ is a homomorphism, we will sometimes identify ω with a subset of $\mathcal{B}$ and write $p \in \omega$ whenever $\omega(p) = 1$. Sets defined in this way are also called *ultrafilters*. We see that ultrafilters are characterized by the following properties

(i) $0 \notin \omega$.

(ii) $p \in \omega$ implies $p \vee q \in \omega$ for every $q \in \mathcal{B}$.

(iii) $p \wedge q \in \omega$ whenever p and q are both in ω.

(iv) If $p \vee q = 1$, then $p \in \omega$ or $q \in \omega$.

Let $\mathfrak{M}(\mathcal{B})$ be the set of all homomorphisms $\omega : \mathcal{B} \to \{0, 1\}$. We say that a subset $\mathcal{C} \subset B$ has the *finite intersection property* if for any finite sequence $p_0, \ldots, p_r$ in $\mathcal{C}$, we have

$$\bigwedge_{j=0}^{r} p_j \neq 0$$

It is clear that any homomorphism $\omega \in \mathfrak{M}(\mathcal{B})$ has the finite intersection property when viewed as a subset of $\mathcal{B}$. The converse is sometimes called the *Boolean Prime Ideal Theorem*.

Lemma C.2 Let $\mathcal{B}$ be a Boolean algebra, and let $\mathcal{C} \subset B$ have the finite intersection property. Then there is an $\omega \in \mathfrak{M}(\mathcal{B})$ with $\mathcal{C} \subset \omega$.

Proof. Let $\mathfrak{F}$ be the collection of all $\mathcal{D}$ with the finite intersection property such that $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$. Inclusion defines a partial ordering on $\mathfrak{F}$. If $\mathfrak{K}$ is a chain in $\mathfrak{F}$, then $\cup\mathfrak{K} \in \mathfrak{F}$, because if $p_0, \ldots, p_r \in \cup\mathfrak{K}$ is a finite sequence, there is a $\mathcal{D} \in \mathfrak{K}$ such that $p_0, \ldots, p_r \in \mathcal{D}$, and hence

$$\bigwedge_{j=0}^{r} p_j \neq 0$$

since $\mathcal{D}$ has the finite intersection property. So $\mathfrak{K}$ has an upper bound $\cup\mathfrak{K}$, and by Zorn's Lemma $\mathfrak{F}$ has at least one maximal element ω.

Identify ω with a function $\mathcal{B} \to \{0, 1\}$. It remains to see that this is a homomorphism. Clearly, $\omega(0) = 0$ and $\omega(1) = 1$, because if $\mathcal{D} \in \mathfrak{F}$ has the finite intersection property, then so does $\mathcal{D} \cup \{1\}$, while $0 \wedge p = 0$ for all p. We prove that $\omega(p)^{\perp} = \omega(p^{\perp})$. If $\omega(p) = 1$, then $\omega(p^{\perp}) = 0$ since $p \wedge p^{\perp} = 0$, and hence a set with the finite intersection property can not contain both p and $p^{\perp}$. Assume that $\omega(p) = 0$. There is an $r \in \mathcal{C}$ with $p \wedge r = 0$. Then $p^{\perp} \wedge r \wedge q = r \wedge q \neq 0$ and hence $p^{\perp} \wedge q \neq 0$ for all $q \in \mathcal{C}$, so $\omega(p^{\perp}) = 1$ by the maximality of ω.

Now, by maximality, $\omega(p \vee q) = 1$ whenever $\omega(p) = 1$ or $\omega(q) = 1$. If both are zero, then as before, there must be a r with $\omega(r) = 1$ and $(q \vee p) \wedge r = 0$, so $\omega(q \vee p) = 0$. All the other relations follow. $\square$

For $p \in \mathcal{B}$, let $\tilde{p} \subset \mathfrak{M}(\mathcal{B})$ be the set

$$\tilde{p} = \{\omega \in \mathfrak{M}(\mathcal{B}) \mid p \in \omega\}.$$

It follows directly from the definition that for $p, q \in \mathcal{B}$,

$$\begin{aligned} \widetilde{p \vee q} &= \tilde{p} \cup \tilde{q} \\ \widetilde{p \wedge q} &= \tilde{p} \cap \tilde{q} \\ \widetilde{p^{\perp}} &= \tilde{p}^{\complement} \\ \emptyset &= \tilde{0} \\ \mathfrak{M}(\mathcal{B}) &= \tilde{1}. \end{aligned}$$

The collection $\tau = \{\tilde{p} \mid p \in \mathcal{B}\}$ is the basis for a topology on $\mathfrak{M}(\mathcal{B})$ that we call the *Stone topology*. The next theorem is in fact equivalent to the

Boolean Prime Ideal Theorem, but we will only prove the implication in one direction [29]. Our statement of the theorem is weaker than Stone's result, but it is still stronger than what we will need for our applications.

Theorem C.3 (Stone's Representation Theorem) Let $\mathcal{B}$ be a Boolean algebra. Then $\mathfrak{M}(\mathcal{B})$ with the Stone topology is a totally disconnected compact Hausdorff space whose Boolean algebra of clopen subsets is isomorphic to $\mathcal{B}$.

Proof. To show that the space is totally disconnected, let $A \subset \mathfrak{M}(\mathcal{B})$ be a subset containing more than one point. Then we can pick $\omega \in A$ and $p \in \omega$ such that $\tilde{p}$ does not contain all of A. Then $\tilde{p} \cap A$ and $\tilde{p}^{\complement} \cap A$ are disjoint and open in the subspace topology relative to A, so A is disconnected.

Let $\omega, \rho \in \mathfrak{M}(\mathcal{B}), \omega \neq \rho$. Then there is a $p \in \mathcal{B}$ with $\omega(p) = 1, \rho(p) = 0$. So $\tilde{p}$ and $\tilde{p}^{\complement}$ are disconnected neighborhoods of ω and ρ respectively, and hence the topology is Hausdorff.

To prove compactness it is sufficient to show that every covering of basis sets has a finite subcovering. Let $\mathcal{C} \subset \mathcal{B}$ be such that

$$\bigcup_{p \in \mathcal{C}} \tilde{p} = \mathcal{B}.$$

We need to prove that there is a finite $\mathcal{D} \subset \mathcal{C}$ with the same property. Assume not. Then $\{p \mid p^{\perp} \in \mathcal{C}\}$ has the finite intersection property, so there is a $\omega \in \mathfrak{M}(\mathcal{B})$ with $\omega(p) = 0$ for every $p \in \mathcal{C}$. This negates the assumption that $\{\tilde{p} \mid p \in \mathcal{C}\}$ is a cover of $\mathfrak{M}(\mathcal{B})$.

Finally, the isomorphism between $\mathcal{B}$ and the algebra of clopen subsets is just the map $p \mapsto \tilde{p}$. It is clearly an injective homomorphism, so it remains to see that it is onto. Suppose that $A \subset \mathfrak{M}(\mathcal{B})$ is any clopen subset. Since A is open, it is the union of basis sets. Since it is closed, it is compact, and it is therefore the union of a finite number of basis sets, i.e.

$$A = \bigcup_{j=0}^{r} \tilde{p_j} = \widetilde{\bigvee_{j=0}^{r} p_j}.$$

So the homomorphism is onto. $\square$

When S is a discrete set and $\mathcal{B}$ is its power set, $\mathfrak{M}(\mathcal{B})$ is known to be the Stone-Čech compactification βS of S. We can then imbed S in βS by sending each $x \in S$ to the ultrafilter

$$\omega_x = \{E \subset S \mid x \in E\}.$$

Ultrafilters constructed this way are sometimes called *principal ultrafilters*, while the rest are called *free ultrafilters*. Moreover, if we associate $E \subset S$ to a subset of βS this way, then $\overline{E} = \tilde{E}$, where the closure is taken in the topology on βS. We see that S is dense in βS, because given any basic open set $\overline{E} \subset \beta S$, the principal ultrafilter corresponding to any element of E is contained in $\overline{E}$. The compactness of βS shows that the nonprincipal ultrafilters exist.

Lemma C.4 Let S be a discrete set, and X be locally compact Hausdorff. Then every function $f \in \ell^\infty(S, X)$ has a unique extension to a continuous function $\hat{f} \in C(\beta S, X)$. Here $\ell^\infty(S, X)$ denotes the set of functions $f : S \to X$ whose image $f(S)$ is contained in a compact subset of X. The image of $\hat{f}$ is the closure of the image of f.

Proof. Given $\omega \in \beta S$, pick any x in the intersection

$$\bigcap_{E \in \omega} \overline{f(E)}.$$

Being the intersection of a family of compact sets with the finite intersection property, this is nonempty. Let $\hat{f}(\omega) = x$. We will show that $\hat{f}$ is continuous, such that the chosen x is necessarily unique by the density of S in βS. If $Y \subset X$ is closed, we have

$$\begin{aligned} \hat{f}^{-1}(Y) &= \{\omega \in \beta S \mid \exists Z \subset Y, f^{-1}(Z) \in \omega\} \\ &= \{\omega \in \beta S \mid f^{-1}(Y) \in \omega\} \\ &= \overline{f^{-1}(Y)}, \end{aligned}$$

which is of course closed. The fact that $\hat{f}(\beta S) = \overline{f(S)}$ follows from $\hat{f}$ being the continuous image of a compact set in which S is dense. □

It will sometimes be convenient to write

$$\lim_{x \to \omega} f(x) = \lim_{x \to \omega} \hat{f}(x) = \hat{f}(\omega).$$

D Classical dynamical systems

Most of the results in this section are taken from [33].

Let X be a compact Hausdorff space, and let $\mathrm{Homeo}(X)$ be the group of homeomorphisms $X \to X$. A *classical dynamical system* is a triple (X, G, κ), where X is a compact Hausdorff space, G is a locally compact group, and $\kappa : G \to \mathrm{Homeo}(X)$ is a homomorphism. It is customary to denote $\kappa(g)$ by κ_g for $g \in G$.

The *orbit* of $x \in X$ is the set $\mathcal{O}_\kappa(x) = \{\kappa_g(x) \mid g \in G\}$. We also write

$$\mathcal{O}_\kappa(U) = \bigcup_{x\in U} \mathcal{O}_\kappa(x)$$

for the orbit of a subset $U \subset X$ We say that a closed subset $Y \subset X$ is an *invariant subsystem* if Y equals its own orbit. In that case, $(Y, G, \kappa|_Y)$ is a dynamical system in its own right. A *minimal subsystem* is one that does not have any nontrivial invariant subsystems.

Lemma D.1 Every dynamical system (X, G, κ) has a minimal subsystem.

Proof. Let $\mathcal{E}$ be the collection of all closed invariant nonempty subsets of X. Any chain in $\mathcal{E}$ has a minimal element Y, which is the intersection of all elements in the chain. Y is nonempty since it is the intersection of non-disjoint compact sets. By Zorn's lemma, X does then have a minimal subsystem. $\square$

We also see that if Y is a minimal subsystem, then for every $x \in Y$, $\mathcal{O}_\kappa(x)$ is dense in Y. If not, then $\overline{\mathcal{O}_\kappa(x)}$ is a smaller invariant subsystem. We call such elements *minimal*.

Lemma D.2 Let (X, G, κ) be a classical dynamical system. Then $x \in X$ is minimal if and only if for every neighbourhood U of x, $\overline{\mathcal{O}_\kappa(x)} \subset \mathcal{O}_\kappa(U)$.

Proof. Let x be minimal. Then

$$\overline{\mathcal{O}_\kappa(x)} \setminus \mathcal{O}_\kappa(U)$$

is a closed invariant proper subset of $\overline{\mathcal{O}_\kappa(x)}$, and hence it is empty. Suppose conversely that for every $y \in \overline{\mathcal{O}_\kappa(x)}$, and every neighbourhood U of x, $y \in \kappa_g(U)$ for some g. Then $\kappa_{g^{-1}}(y) \in U$. So $\mathcal{O}_\kappa(y)$ is dense in $\overline{\mathcal{O}_\kappa(x)}$. $\square$

A weaker property than this is that of being non-wandering. We say that $x \in X$ is *wandering* if for some open neighborhood U of x, $\kappa_g(U) \cap \kappa_h(U) = \emptyset$ for every $g \neq h \in G$. The non-wandering set $\Omega(X, G, \kappa) \subset X$ is the set of all points that are not wandering.

Lemma D.3 Let (X, G, κ) be a dynamical system. Every minimal $x \in X$ is non-wandering.

Proof. Let U be an open neighbourhood of x. Since x is minimal, $\{\kappa_g(U)\}_{g\in G}$ is an open cover of $\overline{\mathcal{O}_\kappa(x)}$. By compactness, the cover can be reduced to a finite subcover, but this implies that not all the elements of the cover are disjoint. $\square$

Theorem D.4 (The Markov-Kakutani Fixed Point Theorem) Let X be a locally compact topological vector space, and let $C \subset X$ be a compact convex subset. If $\mathcal{T} \subset B(X)$ is a family of commuting operators that map C into itself, then there is an $x \in C$ such that $Tx = x$ for all $T \in \mathcal{T}$.

Proof. See for instance [8] Theorem VII.2.1. $\square$

Lemma D.5 Let (X, G, α) be a dynamical system. Then every closed invariant ideal in $C(X)$ is of the form $I_Y = \{f \in C(X) \mid f|_Y = 0\}$, where $Y \subset X$ is a closed invariant subset.

E Abelian von Neumann algebras and their spectra

A well known fact about abelian von Neumann algebras is the following Theorem.

Theorem E.1 Let A be an abelian von Neumann algebra on a separable Hilbert space H. Then there exists a second countable compact Hausdorff space X and a positive Borel measure μ on X such that A is $*$-isomorphic to $L^\infty(X, \mu)$.

Proof. Se for instance [18] (Theorem 4.4.4). $\square$

We will give a description of the spectrum of abelian von Neumann algebras. It is clear that any σ-algebra is a Boolean algebra under the normal operations. If $(X, \mathcal{B}, \mu)$ is a measure space, define an equivalence relation on $\mathcal{B}$ by $E \sim F$ if $\mu(E \Delta F) = 0$. It is easy to check that the quotient of $\mathcal{B}$ under this equivalence relation is a well-defined Boolean algebra by defining the algebra operations on equivalence class representations. We call this quotient $\mathcal{B}_0 = \mathcal{B}_0(\mu)$, or the algebra of measurable sets in X modulo the null sets.

Theorem E.2 Let $(X, \mathcal{B}, \mu)$ be a measure space. Then the map

$$\begin{aligned} \Theta : PS(L^\infty(\mu)) &\rightarrow \mathfrak{M}(\mathcal{B}_0) \\ \Theta(\phi)(E) &= \phi(\chi_E) \end{aligned}$$

is a homeomorphism between the weak*-topology on the spectrum of $L^\infty(\mu)$ and the Stone topology on $\mathfrak{M}(\mathcal{B}_0)$.

Proof. First we check that Θ maps into $\mathfrak{M}(\mathcal{B}_0)$, i.e. that for any $\phi \in PS(L^\infty(\mu))$, $\Theta(\phi)$ is a Boolean algebra homomorphism from $\mathcal{B}_0$ to $\{0, 1\}$. That the range of $\Theta(\phi)$ is $\{0, 1\}$ follows from the fact that ϕ is multiplicative and χ_E is idempotent, i.e.

$$\phi(\chi_E) = \phi(\chi_E^2) = \phi(\chi_E)^2.$$

Moreover, $\phi(\chi_X) = \phi(1) = 1$ and $\phi(\chi_\emptyset) = \phi(0) = 0$. Since $\chi_{E\cap F} = \chi_E\chi_F$ and $\chi_{E\cup F} \leq \chi_E + \chi_F$, we also have

$$\begin{aligned} \Theta(\phi)(E\cap F) &= \phi(\chi_E)\phi(\chi_F) = \Theta(\phi)(E) \wedge \Theta(\phi)(F) \\ \Theta(\phi)(E\cup F) &= \phi(\chi_{E\cup F}) = \Theta(\phi)(E) \vee \Theta(\phi)(F). \end{aligned}$$

So $\Theta(\phi)$ is a homomorphism.

Since the characteristic functions of $\mathcal{B}_0$ elements are dense in $L^\infty(\mu)$, they are a separating set for $PS(L^\infty(\mu))$, and hence Θ is injective.

To show that Θ is onto, we pick an arbitrary $\omega \in \mathfrak{M}(\mathcal{B}_0)$, and construct a corresponding character ϕ. Let $K = \omega^{-1}(0) \subset \mathcal{B}_0$, and let $J \subset L^\infty(\mu)$ be the closed subalgebra spanned by the characteristic functions of elements in K. Then J is an ideal in $L^\infty(\mu)$. Let ϕ be a character that annihilates J. Then $\Theta(\phi) \subset \omega$. But since $\Theta(\phi)$ is a homomorphism, we get that it must equal ω.

Now let $U = \{\psi \mid |\psi - \phi|(\chi_{E_j}) < 1, j = 0\ldots r\}$ be a typical neighborhood in the weak* topology on $PS(L^\infty(\mu))$, We see that

$$\Theta(U) = \bigcap_{j=0}^{r} \tilde{E}$$

which is open. A similar argument goes to show that Θ is continuous. □

Remark. If X is discrete, then $L^\infty(\mu) = \ell^\infty(X,\mathbb{C})$, so if $\hat{f} \in C(\beta S,\mathbb{C})$ is the unique extension described in Lemma C.4, $\hat{f}\circ\Theta$ is the Gelfand transform of f.

It will sometimes be useful to represent $L^\infty(\mu)$ as a multiplication algebra on $L^2(\mu)$, and it is interesting to know what topologies are induced on it from the standard topologies on $B(L^2(\mu))$.

Lemma E.3 Let $f \in L^\infty(\mu)$, and let $\pi : L^\infty(\mu) \to B(L^2(\mu)$ be given by $(\pi(f)\xi)(x) = f(x)\xi(x)$. Then $\|\pi(f)\| = \|f\|_\infty$, and the weak topology induced by π is equivalent to the weak*-topology on L^∞ viewed as the dual of $L^1(\mu)$. This is the topology induced by the seminorms

$$f \mapsto \sum_{j=0}^{r} \int_X |f(x)\xi_j(x)|\,\mathrm{d}\mu(x)$$

with each $\xi_j \in L^1(\mu)$. $C(X)$ is dense in $L^\infty(\mu)$ in this topology.

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