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Counting and Coloring with Symmetry

A presentation of Polya's Enumeration Theorem with Applications

Amanda Noel Bjørge

Master of Science in Mathematics
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Supervisor: Lars Peter Lindqvist, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Abstract

This master's thesis explores the area of combinatorics concerned with counting mathematical objects with regards to symmetry. Two main theorems in this field are Burnside's Lemma and Pólya's Enumeration Theorem[1]. Both theorems yield a formula that count mathematical objects with regard to a group of symmetries. Burnside's Lemma utilizes the concept of orbits to count mathematical objects with regard to symmetry. As a result of the Burnside Lemma's reliance on orbits, implementation of the lemma can be computationally heavy. In comparison, Pólya's Enumeration Theorem's use of the cycle index of a group eases the computational burden. In addition, Pólya's Enumeration Theorem allows for the introduction of weights allowing the reader to tackle more complicated problems.

Building from basic definitions taken from abstract algebra a presentation of the theory leading up to Pólya's Enumeration Theorem is given, complete with proofs. Examples are given throughout to illustrate these concepts. Applications of this theory are present in the enumeration of graphs and chemical compounds.

[1] Pólya's Enumeration Theorem is also known as Redfield–Pólya's Theorem.

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Contents

1 Introduction

Pólya's Enumeration Theorem, also known as Redfield–Pólya's Theorem, is a powerful generalization of Burnside's Lemma which takes symmetry into account when counting mathematical objects. Burnside's Lemma, while powerful in its own right, can require a significant amount of computation. Pólya's Enumeration Theorem minimizes the computations needed by the use of the cycle index and explores the idea of weights which enables the reader to pursue more complex problems.

Though the theorem was first discovered and published by John Howard Redfield in 1927, it was not accordingly recognized by the mathematical community until it was proven by a Hungarian mathematician by the name of George Pólya in 1937.[2] In his paper Pólya also demonstrated several of the theorem's applications, in particular those corresponding to the enumeration of chemical compounds. With the help of basic group theory we will present and prove major propositions and lemmas needed to illustrate properties necessary to the proofs of Burnside's Lemma and Pólya's Enumeration Theorem.

Taking the basic definition of a group and its properties from abstract algebra as a foundation, we will define a permutation and describe its properties.[3] After which we will introduce the 4 main permutation groups used in correspondence with Pólya's Enumeration Theorem, the symmetric, cyclic, dihedral and alternating groups. These groups will later be used to discount symmetries when counting mathematical objects.

We will then introduce the concept of a coloring and develop a method of counting permutations with regards to symmetry aiming to answer the question: How many ways can you color the beads of a n beaded necklace black and white when you discount rotational symmetries? To do this we must first introduce 3 sets: the Invariant set, Stabilizer and Orbit. These 3 sets are crucial tools used to present and prove Burnside's Lemma and therefore answer our posed question. Burnside's Lemma and proof are then presented, resulting in the following formula that counts the number of given objects with regard to symmetry

$$\frac{1}{|G|}\sum_{\sigma\in G}|\mathrm{Inv}(\sigma)|\,. \tag{6}$$

We will introduce cycle index of a group G, Z_G, developed by Pólya to minimize the computations needed to compute the number of distinct colorings of an object with regards to symmetry. Formulas for the cycle index of the 4 main permutations groups will also be given. The concept of weights will then be introduced and applied to the Burnside's Lemma before presenting Pólya's Enumeration Theorem, resulting in the following

$$Z_G(\sum_{i=1}^{m}\omega_{c_i},\sum_{i=1}^{m}\omega_{c_i}^2,\ldots,\sum_{i=1}^{m}\omega_{c_i}^n). \tag{22}$$

Throughout the paper simple examples are used to illustrate the concepts introduced. Following the proof of Pólya's Enumeration Theorem is a series of non-trivial examples and applications of the theorem. Included are more complicated versions of the beaded necklace problem, the symmetries of a cube, as well as applications for graphs, trees and chemical compounds. Mathematica has been used to produce the computations found in the examples.

[2] Pólya's proof the theorem and its applications were published in a paper entitled *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und Chemische Verbindungen*. The paper was originally written in German and later translated into English by Ronald C. Read in 1987.[7]

2 Permutation groups

A permutation is a sequence of elements appearing in a specific order. For example, there are 6 possible permutations of the set $A = \{a, b, c\}$, given below.

$$\begin{array}{ccc} abc & acb & bac \\ bca & cab & cba \end{array}$$

Let us use group theory to redefine this concept.

2.1 Permutations and their properties

Before introducing the key terms used in Burnside's Lemma and later in Pólya's Enumeration Theorem we need the following basic definitions from group theory.

Definition 2.1 A **group** consists of a set G with a binary operator $\cdot$ defined on G such that the following properties are satisfied:
Closure If $a, b \in G$ then $a \cdot b \in G$.
Associativity If $a, b, c \in G$ then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
Identity There exists $\iota \in G$ such that $\iota \cdot a = a \cdot \iota = a$ for all $a \in G$.
Inverse For all $a \in G$ there exists $b \in G$ such that $a \cdot b = b \cdot a = \iota$.
b is called the inverse of a.

For example, the set of real numbers under addition is a group. A group G is **abelian** if in addition or multiplication to the above it also satisfies the following condition:
Commutativity For all $a, b \in G, a \cdot b = b \cdot a$.

In general, groups are not abelian as will be demonstrated by the permutation groups presented in section 2.2.2

Definition 2.2 A function $f : X \rightarrow Y$ is called **injective** if for all $x, y \in X$ $f(x) = f(y)$ implies that $x = y$.
Note that this definition is equivalent to the following statement: a function f, as above, is injective if for $x \neq y$ then $f(x) \neq f(y)$.

Definition 2.3 A function $f : X \rightarrow Y$ is called **surjective** if it maps to all values of Y.
In other words, for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. A function that is both injective and surjective is called **bijective**.

Using the group theoretical terminology given above we can define the notion of a permutation and its properties.

Definition 2.4 Given a set N of n elements a **permutation** σ of N is a bijective function $\sigma : N \rightarrow N$. The identity permutation, that maps each element in N to itself will be denoted by ι.

Example 2.1: Let $N = \{1, 2, 3, 4, 5, 6\}$ and define a permutation σ such that

$$\sigma(1) = 4 \quad \sigma(2) = 3 \quad \sigma(3) = 5 \quad \sigma(4) = 1 \quad \sigma(5) = 2 \quad \sigma(6) = 6,$$

hence

$$1 \rightarrow 4 \rightarrow 1, \qquad 2 \rightarrow 3 \rightarrow 5 \rightarrow 2, \qquad 6 \rightarrow 6.$$

We will denote such a permutation in **cyclic notation** as (14)(235)(6). Where the cycle (14) denotes that $1 \to 4 \to 1$. This permutation has 3 cycles, (14) of length 2, (235) of length 3 and (6) of length 1. Note that the cycle (235) is equivalent to (523) and (352) as all cycles send $2 \to 3 \to 5 \to 2$. Therefore (14)(235)(6) and (14)(523)(6) represent the same permutation. For the purposes of this paper, we will denote elements as integers $1, 2, \dots n$ and will write each cycle beginning with the smallest element contained in the cycle.[3]

Let us define multiplication between two permutations σ and τ by the following example. Let $\sigma = (1234)$ and $\tau = (1342)$ then

$$\sigma\tau = (1234)(1342).$$

We begin by reading the permutation right to left. The first permutation σ, sends $1 \to 2$. The second permutation, τ then sends $2 \to 1$, therefore $\sigma\tau$ sends $1 \to 1$ resulting in the cycle of length 1.

$$= (1).$$

Since $\sigma\tau$ sent $1 \to 1$ resulting in a cycle of length one, we then begin a new cycle with the smallest remaining element, 2. We see that σ sends $2 \to 3$ and τ sends $3 \to 4$, therefore $\sigma\tau$ sends $2 \to 4$, now we have a new cycle beginning with

$$= (1)(24.$$

Since this did not result in an identity we continue with element 4. The permutation σ sends $4 \to 1$ and τ sends $1 \to 3$, therefore $\sigma\tau$ sends $4 \to 3$ and we have

$$= (1)(243.$$

Lastly, σ sends $3 \to 4$ and τ sends $4 \to 2$, therefore $\sigma\tau$ sends $3 \to 2$ ending the cycle

$$= (1)(243).$$

Let G be a group of permutations of the set $N = \{1, 2, 3, 4\}$. Note that G is generally non-abelian. Define $\sigma = (123)(4)$ and $\tau = (12)(34)$. Then

$$\sigma \cdot \tau = (1)(243)$$
$$\tau \cdot \sigma = (134)(2)$$

hence $\quad \sigma \cdot \tau \neq \tau \cdot \sigma$.

Two permutations are **disjoint** if their cycles are disjoint. Two cycles are disjoint if they have no common elements. For example the two permutations given in the example above $\sigma = (123)(4)$ and $\tau = (12)(34)$ are not disjoint, hence they do not commute, as demonstrated above. However, if two permutations are disjoint they will commute. Note that every permutation can be uniquely written as the product of disjoint cycles, disregarding cycle order.[2] This is obvious from example 2.1.

Proposition 2.1 Every permutation σ can be expressed as a product of cycles of length 2 called **transpositions**.[4]

[3] Note that we will also denote elements as $a_1, a_2, \dots a_n$ and will write each cycle beginning with the element corresponding to the smallest index.

[4] Note that this product is not necessarily disjoint.[2]

Proof. By definition we know that every permutation can be expressed as the product of cycles. Hence, we need only prove that any cycle can be expressed as the product of transpositions. Consider the cycle $\sigma = (a_1 a_2 a_3 \dots a_n)$ of n elements. We claim that this cycle can be expressed as the following product of transpositions:

$$\tau = (a_1 a_2)(a_1 a_3)\dots(a_1 a_n).$$

Consider element a_i. For $1 < i < n$, a_i is invariant under the first $i-2$ transpositions of τ. The $i-1^{th}$ transposition sends $a_i \to a_1$ and the i^{th} transposition sends $a_1 \to a_{i+1}$, and since a_{i+1} appears in no other cycle $a_i \to a_{i+1}$. This leaves two cases, $i = 1$ or n. For $i = 1$, the first transposition sends $a_1 \to a_2$ and is invariant under all other transpositions as a_2 appears in no other cycle. For $i = n$, a_n is invariant under all but the last transposition where it sends $a_n \to a_1$ as desired. Hence $\sigma = \tau$. $\square$

Proposition 2.2 Every permutation σ, decomposes into either a strictly even (or odd) number of transpositions. The permutation is then called even (or odd).[2]

$$\text{even} : (12345) = (12)(13)(14)(15) \qquad \text{odd} : (1234) = (12)(13)(14)$$

Proof. Consider the formal symbol

$$\Delta_n = (2-1)(3-2)(3-1)(4-3)(4-2)(4-1)\dots(n-1),$$

where we do not perform the subtraction. It is obvious that $\Delta_n > 0$. We claim that multiplying Δ_n by any single transposition is equivalent to multiplying it by -1. For $(ab)\Delta_n$ consider the following cases:

For $c > a, b$

$$(ab)(c-a)(c-b) = (c-b)(c-a) = (c-a)(c-b).$$

For $c < a, b$

$$(ab)(a-c)(b-c) = (b-c)(a-c) = (a-c)(b-c).$$

For $a < c < b$ (similarly for $b < c < a$)

$$(ab)(c-a)(b-c) = (c-b)(a-c) = (-1)^2(c-a)(b-c) = (c-a)(b-c).$$

As shown above, these three cases result in no sign change. Lastly, for $a < b$ (similarly for $b < a$)

$$(ab)(b-a) = (a-b) = -(b-a).$$

Hence

$$(ab)\Delta_n = -\Delta_n.$$

Proposition 2.1 states that every permutation can be expressed as the product of transpositions thus $\sigma\Delta_n = \Delta_n$ or $-\Delta_n$. Therefore every permutation can be expressed as only the product of an even (or odd) number of transpositions. $\square$

Let us look at an example to clarify this proof.

Example 2.2: Consider the even permutation $\sigma = (123) = (12)(13)$ then

$$\begin{aligned}\sigma\Delta_n &= (12)(13)\Delta_n\\ &= (12)(-\Delta_n)\\ &= \Delta_n.\end{aligned}$$

Assume that σ could be expressed as the product of an odd number of transpositions, then

$$\begin{aligned}\sigma\Delta_n &= t_1t_2t_3\Delta_n\\ &= t_1t_2(-\Delta_n)\\ &= t_1\Delta_n\\ &= -\Delta_n.\end{aligned}$$

This however, contradicts our previous findings that $\sigma\Delta_n = \Delta_n$ or $-\Delta_n$. Hence σ can only be expressed as the product of an even number of transpositions.
Note that though a permutation has a unique disjoint cyclic notation, the decomposition of a permutation into transpositions is not unique. For example,

$$(123) = (12)(13) = (13)(23),$$

are both decompositions of the permutation (123) into transpositions.

2.2 Examples of permutation groups

Define S_n to be the group of all permutations of the set $N = \{1, 2, \ldots n\}$. S_n is called the **symmetric group** of n elements and $|S_n| = n!$. For example $S_3 = \{\iota, (123), (132), (12)(3), (13)(2), (1)(23)\}$. A group H is a **subgroup** of G if $H \subseteq G$ and H satisfies the above group properties with the same binary operation as G. Below we will give some examples of permutation groups, all of which are subgroups of S_n.

Cyclic group

Given a permutation $\sigma \in S_n$ and $i \in \mathbb{N}$, define σ^i to be σ composed with itself i times.[5] Then $\langle\sigma\rangle = \{\sigma^i | i \geq 0\}$ is a subgroup of S_n generated by σ and is called the cyclic group generated by σ on S_n. The **cyclic group** C_n is the subgroup of S_n generated by (123... n).

$$C_n = \langle(123\ldots n)\rangle$$

[5]Note that $\sigma^0\sigma^n = \iota$.

Note that $|C_n| = n$.

Example 2.3: $C_5 = \langle(12345)\rangle \ = \{\iota, (12345), (13524), (14253), (15432)\}$

$$\begin{aligned}
(12345)^1 &= (12345) \\
(12345)^2 &= (13524) \\
(12345)^3 &= (14253) \\
(12345)^4 &= (15432) \\
(12345)^0 = (12345)^5 &= (1)(2)(3)(4)(5) = \iota
\end{aligned}$$

The permutations in any cyclic group C_n can also be viewed as rotations. From our above example of C_5 we can compare the permutations to the rotation of the vertexes of a pentagon. Note that all rotations are considered to be clockwise. If the vertexes of the original pentagon are labeled as in figure 1, then a rotation of 72 degrees would result in the permutation (12345) as shown in (b).

(a) Original (b) (12345)

Figure 1: Example of a permutation from C_5

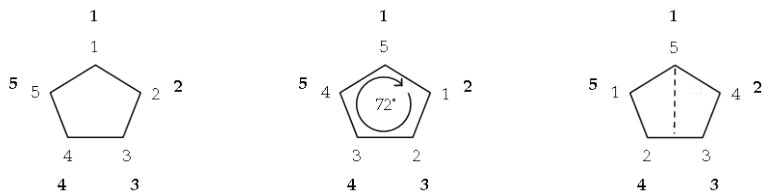
Dihedral Group

The **Dihedral Group** D_n is the group of permutations of a regular n-sided polygon including rotations and reflections. Note that $C_n \subset D_n$, and that all reflections can be found by taking each rotation $\sigma \in C_n$ and reflecting along the vertical (or horizontal) axis. Note that $|D_n| = 2n$.

Example 2.4: $D_5 = \{\iota, (12345), (13524), (14253), (15432), (13)(2)(45),$ $(15)(24)(3), (12)(35)(4), (14)(23)(5), (1)(25)(34)\}$

$$\begin{array}{rcc}
(12345)^1 = (12345) & \text{reflected } \Rightarrow & (15)(24)(3) \\
(12345)^2 = (13524) & \text{reflected } \Rightarrow & (14)(23)(5) \\
(12345)^3 = (14253) & \text{reflected } \Rightarrow & (13)(2)(45) \\
(12345)^4 = (15432) & \text{reflected } \Rightarrow & (12)(35)(4) \\
(12345)^5 = (1)(2)(3)(4)(5) = \iota & \text{reflected} \Rightarrow & (1)(25)(34)
\end{array}$$

Illustrated in figure 2.



(a) Original (b) (12345) (c) (15)(24)(3)

Figure 2: Example of permutations from D_5

Alternating Group

The **Alternating Group** A_n is a subgroup of S_n containing all even permutations, $|A_n| = \frac{n!}{2}$ for $n \geq 4$.

$$A_4 = \{\iota, (123)(4), (132)(4), (124)(3), (142)(3), (134)(2), (143)(2),\\ (1)(234), (1)(243), (12)(34), (13)(24), (14)(23)\}$$

The group A_n is abelian for $n \leq 3$ and simple for $n = 3$ and $n \geq 5$. A simple group is a non trivial group, i.e. containing more than one element, whose only normal subgroup is the trivial group.

3 Burnside's Lemma

We now need to develop a method of counting permutations with regards to symmetry. Consider a necklace consisting of n beads. How many distinct ways can you color the beads of a n bead necklace black and white when you discount rotational symmetries? For $n = 4$, there are 6 possible colorings.

Figure 3: Possible 4 bead necklaces excluding rotational symmetries given by C_4

3.1 Equivalence relations

Let N be the set of n beads we are to color, C be the set of colors, in this case black and white, and G be our permutation group. Define a coloring $k : N \to C$ to be a map that assigns each bead a color, and let K be the set of colorings. How do we determine the number of non-equivalent colorings? The Burnside's Lemma gives a formula to solve this problem. First we need to understand the concept of equivalence relations.

Definition 3.1 Let A be a set and $\sim$ a binary relation on A. We call $\sim$ an **equivalence relation** if and only if for all $a, b, c \in A$ the following conditions are satisfied:
Reflexivity $a \sim a$
Symmetry If $a \sim b$ then $b \sim a$
Transitivity If $a \sim b$ and $b \sim c$ then $a \sim c$

The **equivalence class** of a under $\sim$ is the set $[a] = \{b \in A | a \sim b\}$.

Let us define an equivalence relation $\sim$ such that for $k_1, k_2 \in K$, $k_1 \sim k_2$ if there exists $\sigma \in G$ such that $\sigma^*(k_1) = k_2$. Here σ^* denotes that the permutation is being applied to coloring of the set N.

$$\begin{aligned}\sigma^* &= \sigma \circ k(i) \\ &= k \circ \sigma^{-1}(i) \quad \forall i \in N\end{aligned}$$

Example 3.1: Let the following be a coloring of our above 4 bead necklace.

$$k(1) = white, k(2) = white, k(3) = black, k(4) = black$$

Such a coloring will be denoted $wwbb$. Consider the following permutation $\sigma = (1234)$, representing a 90 degree clockwise rotation, where the beads are labeled clockwise beginning with the top left bead representing element 1. What is $\sigma^*(k)$? For $1 \in N$ we have

$$\sigma \circ k(1) = k \circ \sigma^{-1}(1) = k(4) = black.$$

As illustrated in figure 4.

(a) k (b) $\sigma^*(k)$

Figure 4: Illustration of applying a permutation σ to a coloring k

Using our 4 bead necklace as an example and the above equivalence relation, we have 6 equivalence classes, hence the 6 possible colorings, shown in figure 5.

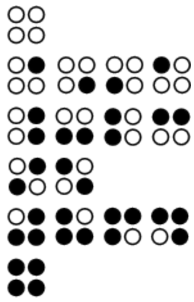


Figure 5: Equivalence classes resulting from permutation group C_4

3.2 Definitions

Before we present the Burnside's Lemma we need the following definitions. We will continue with the above example of the 4 bead necklace to illustrate the following definitions, where $N = \{1, 2, 3, 4\}, C = \{black, white\}, K$ is the group of colorings $k : N \to C$ and will use $G = \{\iota, (12), (34), (12)(34)\}$ as our permutation group.[6]

Definition 3.2 The **invariant set** of σ in K denoted $\text{Inv}(\sigma)$, is the set of elements of which the coloring k is invariant under the induced map σ^*.

$$\text{Inv}(\sigma) = \{k \in K | \sigma^*(k) = k\} \tag{1}$$

Example 3.2: Below are the four Invariant sets corresponding to the four permutations in G.

$$\begin{aligned}
\text{Inv}(\iota) &= K \\
\text{Inv}((12)(34)) &= \{bbbb, bbww, wwbb, wwww\} \\
\text{Inv}((12)) &= \{bbbb, bbwb, bbbw, bbww \\
&\quad wwbb, wwwb, wwbw, wwww\} \\
\text{Inv}((34)) &= \{bbbb, wbbb, bwbb, wwbb \\
&\quad bbww, wbww, bwww, wwww\}
\end{aligned}$$

Here the coloring $bwbb$ corresponds to coloring the first, third and fourth beads black

[6]Note that the permutation (12) is equal to the permutation (12)(3)(4), here the cycles of length 1 that send an element $i \in N$ to itself are excluded for notational purposes.

Figure 6: Illustration of the coloring *bwbb*

and the second white as seen in figure 6. From the figure it is obvious that *bwbb* is invariant under the permutation (34) since elements 3 and 4 are both colored black.

Definition 3.3 The **stabilizer** of k in G denoted St_k, is the set of permutations $\sigma \in G$ for which the coloring k is fixed.

$$St_k = \{\sigma \in G | \sigma^*(k) = k\} \tag{2}$$

Example 3.3: Below are the stabilizer sets for each coloring in K.

$$\begin{aligned}
St_{bbbb} = St_{wwww} = St_{bbww} = St_{wwbb} &= \{\iota, (12), (34), (12)(34)\} \\
St_{bbbw} = St_{bbwb} = St_{wwbw} = St_{wwwb} &= \{\iota, (12)\} \\
St_{bwbb} = St_{wbbb} = St_{bwww} = St_{wbww} &= \{\iota, (34)\} \\
St_{bwbw} = St_{wbwb} = St_{bwwb} = St_{wbbw} &= \{\iota\}
\end{aligned}$$

Here the coloring *bwbb* is left unchanged by two permutations: $\iota, (34)$.

Definition 3.4 The **orbit** of k under G denoted O_k, the set of all colorings to which k is sent by some permutation $\sigma \in G$.

$$O_k = \{\sigma(k) | \sigma \in G\} \tag{3}$$

Example 3.4: Below are the orbits corresponding to each coloring in K.

$$\begin{aligned}
O_{bbbb} &= \{bbbb\} & O_{bbbw} = O_{bbwb} &= \{bbbw, bbwb\} \\
O_{wwww} &= \{wwww\} & O_{bwbb} = O_{wbbb} &= \{bwbb, wbbb\} \\
O_{bbww} &= \{bbww\} & O_{wwwb} = O_{wwbw} &= \{wwwb, wwbw\} \\
O_{wwbb} &= \{wwbb\} & O_{wbww} = O_{bwww} &= \{wbww, bwww\}
\end{aligned}$$

$$O_{bwbw} = O_{wbwb} = O_{wbbw} = O_{bwwb} = \{bwbw, wbbw, bwwb, wbwb\}$$

Note that since G is a group, if $a \in O_b$ then $b \in O_a$. This is a result of the inverse group condition. If there exists a permutation $\sigma \in G$ such that $a \to b$ then there must exist $\sigma^{-1} = \tau \in G$ such that $b \to a$. From this we get the following proposition.

Proposition 3.1 For all $G \subseteq S_n$ then any two orbits under G, O_a and O_b are either disjoint or equal.[2]

Proof. We will proceed with a proof by contradiction. Assume that $c \in O_a$ and $c \in O_b$, hence they are not disjoint. Also assume that $x \in O_b$ but $x \notin O_a$. We will show that O_a must contain x. First, since $a \in O_b$ there exists $\sigma \in G$ such that σ

sends $a \to b$ and by the inverse condition of a group there must also exist σ^{-1} that sends $b \to a$. Similarly, there must also exist $\tau, \tau^{-1} \in G$ which send $x \to b$ and $b \to x$ respectively. Then $\sigma\tau^{-1}$ sends $a \to x$ and $\tau\sigma^{-1}$ sends $x \to a$. Hence by definition of an orbit $x \in O_a$, contradiction. Thus any two orbits are either disjoint or equal. □

Lemma 3.1 Given a set of elements $N = \{1, 2, 3 \ldots n\}$, a set of colors C, and a subgroup G of S_n, then [5]

$$|G| = |St_k| \, |O_k| \, . \tag{4}$$

Proof. First, since O_k does not contain permutations, let us define a permutation group P such that $|P| = |O_k|$ for any fixed k. Given $O_k = \{k_1, k_2, k_3 \ldots\}$ define $P = \{\rho_1, \rho_2, \rho_3 \ldots\}$ such that p_j sends $k \to k_j$ for all j. Then $|P| = |O_k|$. Note that there exists such permutations by definition of an orbit, however, the choices for each ρ_j are not necessarily unique.
Claim: Every permutation $\sigma \in G$ can be expressed uniquely as the product of a permutation from St_k and P.
Given $\sigma \in G$ it must send k to $k_j \in O_k$ for some j. By construction, ρ_j also sends $k \to k_j$. Hence,

$$\sigma\rho_j^{-1} \text{ sends } k \to k.$$

Therefore, by the definition of a stabilizer (def. 3.3), there exists $\tau \in St_k$ such that

$$\sigma\rho_j^{-1} = \tau.$$

Then

$$\sigma = \tau\rho_j.$$

Hence every permutation $\sigma \in G$ can be expressed as the product of a permutation from St_k and P. Now we must show uniqueness. Assume that

$$\tau_x\rho_i = \tau_y\rho_j. \tag{5}$$

By construction ρ_i sends $k \to k_i$ and ρ_j sends $k \to k_j$, and by definition both τ_x and τ_y send $k \to k$, then (5) implies $k_i = k_j$. Which implies that $i = j$ and hence $\rho_i = \rho_j$. Therefore $\tau_x = \tau_y$ and the product is unique. Hence we have

$$\begin{aligned} |G| &= |St_k| \, |P| \\ &= |St_k| \, |O_k| \, . \end{aligned}$$

□

Example 3.5: Let us take our 4 bead necklace from above. $N = \{1, 2, 3, 4\}$ and $G = \{\iota, (12), (34), (12)(34)\}$ with $|G| = 4$. From examples 3.3 and 3.4 we have:

$$St_{bbbb} = \{\iota, (12), (34), (12)(34)\} \qquad St_{bbbw} = \{\iota, (12)\}$$
$$O_{bbbb} = \{bbbb\} \qquad O_{bbbw} = \{bbbw, bbwb\}$$

Hence

$$\begin{gathered} |St_{bbbb}| = 4, |O_{bbbb}| = 1 \Rightarrow |St_{bbbb}| \, |O_{bbbb}| = 4 \cdot 1 = 4 = |G| \\ |St_{bbbw}| = 2, |O_{bbbw}| = 2 \Rightarrow |St_{bbbw}| \, |O_{bbwb}| = 2 \cdot 2 = 4 = |G| \, . \end{gathered}$$

3.3 Burnside's Lemma

We now have all the tools needed to present the Burnside's Lemma.

Theorem 3.1 (Burnside's Lemma) Given a group of elements N and a permutation group G that acts on N, then the number of distinct orbits, which we will call **patterns**, is given by

$$\frac{1}{|G|}\sum_{\sigma\in G}|\mathrm{Inv}(\sigma)|\,. \tag{6}$$

Before giving the proof, we want to illustrate this Theorem by an example.

Example 3.6: Continuing with our 4 bead necklace and permutation group $G = \{\iota, (12), (34), (12)(34)\}$ we have the following sets:

$$\begin{aligned}\mathrm{Inv}(\iota) &= K\\ \mathrm{Inv}((12)(34)) &= \{bbbb, bbww, wwbb, wwww\}\\ \mathrm{Inv}((12)) &= \{bbbb, bbwb, bbbw, bbww\\ &\quad wwbb, wwwb, wwbw, wwww\}\\ \mathrm{Inv}((34)) &= \{bbbb, wbbb, bwbb, wwbb\\ &\quad bbww, wbww, bwww, wwww\}\end{aligned}$$

Using Burnside's Lemma we find

$$\begin{aligned}\frac{1}{|G|}\sum_{\sigma\in G}|\mathrm{Inv}(\sigma)| &= \frac{1}{4}\left[16+4+8+8\right]\\ &= 9.\end{aligned}$$

Hence we have 9 patterns as shown above in example 3.4.

Proof. We claim that the number of distinct orbits is

$$\sum_{k\in K}\frac{1}{|O_k|}\;. \tag{$*$}$$

Define $O_{k_m} = \{k_1, k_2, \dots, k_m\}$ then $|O_{k_m}| = m$. Theorem 2.1 then gives $O_{k_1} = O_{k_2} = \cdots = O_{k_m}$. Since these orbits are equal, we want to count them as one pattern, $(*)$ then gives us

$$\frac{1}{|O_{k_1}|}+\frac{1}{|O_{k_2}|}+\cdots+\frac{1}{|O_{k_m}|} = \frac{1}{m}+\frac{1}{m}+\cdots+\frac{1}{m} = 1.$$

Hence the total sum will yield 1 for each pattern.[8] Let us define

$$\chi_k = \begin{cases}1 & \text{if } \sigma^*(k) = k\\ 0 & \text{if } \sigma^*(k) \neq k\end{cases}.$$

Then we have,

$$
\begin{aligned}
\frac{1}{|G|}\sum_{\sigma\in G}|\mathrm{Inv}(\sigma)| &= \frac{1}{|G|}\sum_{\sigma\in G}\sum_{k\in K}\chi_k \\
&= \frac{1}{|G|}\sum_{k\in K}\sum_{\sigma\in G}\chi_k \\
&= \frac{1}{|G|}\sum_{k\in k}|St_k| && \text{(by def. 3.3)} \\
&= \sum_{k\in k}\frac{1}{|St_k|\,|O_k|}\,|St_k| && \text{(by lem. 3.1)} \\
&= \sum_{k\in k}\frac{1}{|O_k|}.
\end{aligned}
$$

□

Let us apply the Burnside's Lemma to our question posed at the beginning of this section. How many n beaded necklaces are there when you discount rotational symmetries?

Example 3.7: For $n = 4$ there were 6 necklaces as shown above in figure 3. Let $N = \{1, 2, 3, 4\}$ and $G = C_4 = \{\iota, (1234), (13)(24), (1432)\}$ to count the rotational symmetries. Then Burnside's Lemma gives

$$
\begin{aligned}
&\frac{1}{|G|}\sum_{\sigma\in G}|\mathrm{Inv}(\sigma)| \\
&\quad= \frac{1}{4}\left[|\mathrm{Inv}(\iota)| + |\mathrm{Inv}((1234))| + |\mathrm{Inv}((13)(24))| + |\mathrm{Inv}((1432))|\right] \\
&\quad= \frac{1}{4}\left[16 + 2 + 4 + 2\right] \\
&\quad= 6,
\end{aligned}
$$

yielding 6 patterns, shown as equivalence classes in figure 5 above. Let us note that a pattern can be represented by a single coloring in its equivalence class. For example the coloring *bbbw* can represent the pattern given by the equivalence class $\{bbbw, bbwb, bwbb, wbbb\}$.

It is obvious that $\mathrm{Inv}(\iota) = K$ for all N and G. Also, any permutation σ consisting of i cycles $x_1, x_2 \ldots, x_i$ will be invariant under any coloring that colors all elements contained in a cycle x_j the same color. Taking the above example the permutation (1234) is only invariant under 2 colorings: *bbbb* and *wwww*. Similarly for (13)(24) if you treat the first and third bead as one bead, and the second and fourth bead as one bead, then you have $2^2 = 4$ colorings that are invariant, as shown in example 3.2.

Example 3.8: Let us take a more complicated example. Let $n = 7$ then there are $2^7 = 128$ possible colorings of a 7 bead necklace. How many patterns are possible after excluding rotational and reflectional symmetries? Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $G = D_7 = \{\iota, (1234567), (1357246), (1473625), (1526374), (1642753), (1765432),$ $(1)(27)(36)(45), (13)(2)(47)(67), (15)(24)(3)(67), (17)(26)(35)(4), (12)(37)(46)(5),$ $(14)(23)(57)(6), (16)(25)(34)(7)\}$ to count the rotational and reflectional symmetries.

Then Burnside's Lemma and the above comment gives us:

$$\begin{aligned}\frac{1}{|G|}\sum_{\sigma\in G}|\mathrm{Inv}(\sigma)| &= \frac{1}{14}\sum_{\sigma\in D_7}|\mathrm{Inv}(\sigma)|\\ &= \frac{1}{14}\left[128+6\cdot 2+7\cdot 16\right]\\ &= 18.\end{aligned}$$

As shown in figure 7.

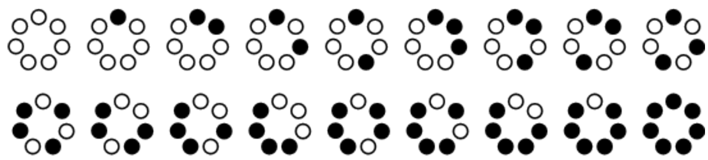


Figure 7: Possible 7 bead necklaces excluding the permutations given by D_7

4 Cycle Index

In order to compute the number of patterns using the Burnside's Lemma we must first compute the size of $\text{Inv}(\sigma)$ for all $\sigma \in G$. As $|N|$ increases so does the difficulty in computing $\text{Inv}(\sigma)$ for each $\sigma \in G$. Luckily there is a simple method to compute the size of $\text{Inv}(\sigma)$.

4.1 Cycle Index

We know that given n elements to be colored m possible colors, the total number of possible colorings is m^n.[7] As stated above, in order to ensure that a coloring k is invariant under σ, k must color all elements of each cycle in σ the same color. If we treat each cycle of σ as a single element to be colored, then we can ensure all elements in the cycle will be colored the same color. Therefore if σ consists of i disjoint cycles, then $|\text{Inv}(\sigma)| = m^i$ where m is the number of possible colors.[5] Let us illustrate this further by using permutations from the previous example 3.8 with the 7 bead necklace, 2 possible colors and $G = D_7$.

$$\begin{aligned} |\text{Inv}(\iota)| &= 2^7 = 128 \\ |\text{Inv}((1234567))| &= 2^1 = 2 \\ |\text{Inv}((1)(27)(36)(45))| &= 2^4 = 16 \end{aligned}$$

Given a permutation $\sigma \in G$ let $y_l(\sigma)$ denote the number of cycles of length l contained in σ, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where x_l represents a cycle of length l. Then the **cycle index** of G is defined by[8]

$$Z_G(\mathbf{x}) = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{l=1}^{n} x_l^{y_l(\sigma)}. \tag{7}$$

Hence the permutation $\sigma = (123)(4)(56)(78)$ would have the cycle index $x_1 x_2^2 x_3$ since σ contains 1 cycle of lengths 1 and 3, and 2 cycles of length 2. Then the cycle index from example 3.8 is

$$Z_{D_7}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \frac{1}{14}\left[x_1^7 + 7x_1^1 x_2^3 + 6x_7\right].$$

Let us apply this to the Burnside's Lemma. The Burnside's Lemma tells us that the number of ways to color n elements using m colors while discounting the symmetries given by G is $Z_G(m, m, \ldots, m)$. Using our original example of the 4 bead necklace and $G = C_4$ we have

$$Z_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{4}\left[x_1^4 + x_2^2 + 2x_4\right].$$

By substituting m for each x_j we can compute the number of distinct necklaces using m colors.

$$Z_{C_4}(m, m, m, m) = \frac{1}{4}\left[m^4 + m^2 + 2m\right]$$

[7] For each element there exists m possible colors to choose from, hence, the total number of possible colorings is $m \cdot m \cdot \ldots m = m^n$.

[8] The cycle index is denoted by the letter Z alluding to Pólya's usage of the German word Zyklenzeiger meaning cycle index.

For $m = 2$ as in our example we get 6 distinct necklaces, which corresponds to our previous example. Once the cycle index is computed, it is easy to then increase the number of possible colors, drastically reducing the amount of computations required.

$$Z_{C_4}(3,3,3,3) = 24$$
$$Z_{C_4}(4,4,4,4) = 70$$

4.2 Cycle Index for specific permutation groups

While the cycle index is a helpful tool, we still need to know the type of each permutation. Therefore we still need to write out the permutations before we are able to calculate the cycle index. This can be difficult when the group size increases. For example, $|S_4| = 4! = 24$ which is not unreasonable to write out, however, one dimension higher $|S_5| = 5! = 120$. Luckily there are formulas to compute the cycle index for certain groups.[9] Below we will calculate the formulas for the 4 major permutations groups given in section 2.2.2.

Let us begin with the symmetric group S_n. Pólya denoted the cycle index in terms of the possible partitions of the set N. A set $A = \{A_1, A_2, \ldots, A_j\}$ of non empty sets is a partition of N if the union of all A_j is equal to N and if any two sets $A_i, A_j \in A$ are disjoint. We will denote the partitions of N by the vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, where y_l denotes the number of partitions, or cycles, of length l. Then

$$|N| = \sum_{l=1}^{n} l y_l.$$

For example, let $\sigma = (123)(45)(6)$ then $|N| = 6$ and we have the following

$$\begin{aligned} |N| &= \sum_{l=1}^{n} l y_l \\ &= 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 0 \\ &= 6. \end{aligned}$$

Define $d(\mathbf{y})$ to be the number of permutations $\sigma \in S_n$ that have the cycle decomposition denoted by $\mathbf{y}$. Then

$$d(\mathbf{y}) = \frac{n!}{\prod_{l=1}^{n} l^{y_l} y_l!}. \tag{8}$$

Let us take $S_3 = \{\iota, (123), (132), (1)(23), (12)(3), (13)(2)\}$ as an example. Here are the 3 unique partitions $\mathbf{y} = (y_1, y_2, y_3)$ given by[10]

$$(3,0,0) \qquad (1,1,0) \qquad (0,0,1).$$

Then

$$\begin{aligned} d((3,0,0)) &= \frac{3!}{1^3 3!} = 1 \\ d((1,1,0)) &= \frac{3!}{(1^1 1!)(2^1 1!)} = 3 \\ d((0,0,1)) &= \frac{3!}{3^1 1!} = 2. \end{aligned}$$

[9] These formulas have been taken from [1] and [4].

[10] Note that $(1,1,0)$ corresponds to $\sigma \in S_3$ that contain 1 cycle of length 1 and 1 cycle of length 2, such as $(1)(23)$.

The **cycle index of $\mathbf{S_n}$** is given by

$$Z_{S_n} = \frac{1}{n!} \sum d(\mathbf{y}) \prod_{l=1}^{n} x_l^{y_l}, \tag{9}$$

where x_l represents a cycle of length l.

Example 4.1:

$$\begin{aligned} Z_{S_3} &= \frac{1}{3!} \sum d(\mathbf{y}) \prod_{l=1}^{3} x_l^{y_l} \\ &= \frac{1}{6}[1x_1^3 + 3x_1^1x_2^1 + 2x_3^1] \\ &= \frac{1}{6}[x_1^3 + 3x_1x_2 + 2x_3]. \end{aligned}$$

Note that the cycle index of S_n satisfies the following recurrence relation:

$$Z_{S_n} = \frac{1}{n} \sum_{i=1}^{n} x_i Z_{S_{n-i}}. \tag{10}$$

Example 4.2: Let us compute the cycle index of S_4. Using (9) we get

$$\begin{aligned} Z_{S_4} &= \frac{1}{4!} \sum d(\mathbf{y}) \prod_{l=1}^{4} x_l^{y_l} \\ &= \frac{1}{24}[x_1^4 + 6x_1^2x_3 + x_2^2 + 8x_1x_3 + 6x_4]. \end{aligned}$$

Now let us compare this using the recurrence relation (10)

$$\begin{aligned} Z_{S_4} &= \frac{1}{4} \sum_{i=1}^{4} x_i Z_{S_{n-i}} \\ &= \frac{1}{4}\left[x_1 \left(\frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3)\right) + x_2 \left(\frac{1}{2}(x_1^2 + x_2)\right) + x_3\,(x_1) + x_4(1)\right] \\ &= \frac{1}{4}\left[\frac{1}{6}\left(x_1^4 + 3x_1^2x_2 + 2x_1x_3\right) + \frac{1}{2}\left(x_1^2x_2 + x_2^2\right) + x_1x_3 + x_4\right] \\ &= \frac{1}{24}\left[x_1^4 + 3x_1^2x_2 + 2x_1x_3 + 3x_1^2x_2 + 3x_2^2 + 6x_1x_3 + 6x_4\right] \\ &= \frac{1}{24}\left[x_1^4 + 6x_1^2x_3 + x_2^2 + 8x_1x_3 + 6x_4\right]. \end{aligned}$$

Hence both formulas give the same cycle index for S_n.

The **cycle index of $\mathbf{C_n}$** is given by

$$Z_{C_n} = \frac{1}{n} \sum_{l|n} \varphi(l) x_l^{n/l}, \tag{11}$$

where φ is the Euler function given by

$$\varphi(n) = \#\{d | 1 \leq d \leq n, \gcd(d, n) = 1\}. \tag{12}$$

Example 4.3: Let us use this formula to compute the cycle index of C_4.

$$\begin{aligned}
Z_{C_4} &= \frac{1}{4}\sum_{l|4}\varphi(l)x_l^{4/l} \\
&= \frac{1}{4}\left[\varphi(1)x_1^{4/1}+\varphi(2)x_2^{4/2}+\varphi(4)x_4^{4/4}\right] \\
&= \frac{1}{4}\left[x_1^4+x_2^2+2x_4\right].
\end{aligned}$$

Which corresponds with our previous computation.

The **cycle index of $\mathbf{D_n}$** is given by

$$Z_{D_n} = \frac{1}{2}Z_{C_n} + \begin{cases} \frac{1}{2}x_1x_2^{\frac{(n-1)}{2}} & n \text{ odd} \\ \frac{1}{4}\left(x_2^{\frac{n}{2}}+x_1^2x_2^{\frac{(n-2)}{2}}\right) & n \text{ even} \end{cases}. \tag{13}$$

Example 4.4: Let us illustrate this formula by computing the cycle index of both D_4 and D_5.

$$\begin{aligned}
Z_{D_4} &= \frac{1}{2}Z_{C_4} + \frac{1}{4}\left(x_2^{\frac{n}{2}}+x_1^2x_2^{\frac{(n-2)}{2}}\right) \\
&= \frac{1}{2}\left(\frac{1}{4}(x_1^4+x_2^2+2x_4)\right) + \frac{1}{4}\left(x_2^{\frac{4}{2}}+x_1^2x_2^{\frac{(4-2)}{2}}\right) \\
&= \frac{1}{8}(x_1^4+x_2^2+2x_4) + \frac{1}{4}(x_2^2+x_1^2x_2) \\
&= \frac{1}{8}\left[x_1^4+2x_1^2x_2+3x_2^2+2x_4\right].
\end{aligned}$$

$$\begin{aligned}
Z_{D_5} &= \frac{1}{2}Z_{C_5} + \frac{1}{2}x_1x_2^{\frac{(5-1)}{2}} \\
&= \frac{1}{2}\left(\frac{1}{5}(x_1^5+4x_5)\right) + \frac{1}{2}x_1x_2^2 \\
&= \frac{1}{10}\left[x_1^5+5x_1x_2^2+4x_5\right].
\end{aligned}$$

The **cycle index of $\mathbf{A_n}$** is given by

$$Z_{A_n} = Z_{S_n} + Z_{S_n}(x_1,-x_2,x_3,-x_4,\ldots). \tag{14}$$

Example 4.5: Let us use this formula to compute the cycle index of $A_3 = \{\iota, (123), (132)\}$.

$$\begin{aligned}
Z_{A_3} &= \frac{1}{6}[x_1^3+3x_1x_2+2x_3] + \frac{1}{6}[x_1^3-3x_1x_2+2x_3] \\
&= \frac{1}{3}[x_1^3+2x_3].
\end{aligned}$$

5 Pólya's Enumeration Theorem

With the Burnside's Lemma we are able to calculate how many patterns there are given the choice of m colors. What if we want to have further restrictions? For example what if we want to know the number of possible necklaces with a specific number of beads of each color? How many 6 bead necklaces are possible with 1 white bead, 3 gray beads and 2 black beads? To solve such problems, we need to introduce weights.

5.1 Weights

Given a set of colors C we want to associate a weight ω_c for all $c \in C$. Then we will define the **weight of a coloring** $k \in K$ to be the product of the weights of the colored elements.

$$\omega(k) = \prod_{i \in N} \omega_{k(i)} \tag{15}$$

Example 5.1: Let us use our 4 bead necklace to illustrate this with $C = \{white, black\}$. Let $k = wwbb$ where

$$k(1) = white, \quad k(2) = white, \quad k(3) = black, \quad k(4) = black,$$

and let the colors have the following weights

$$\omega_{white} = W \qquad \omega_{black} = B.$$

Then the weight of the coloring k is as follows

$$\begin{aligned}
\omega(k) &= \prod_{i \in N} \omega_{k(i)} \\
&= \omega_{k(1)}\omega_{k(2)}\omega_{k(3)}\omega_{k(4)} \\
&= \omega_{white}\omega_{white}\omega_{black}\omega_{black} \\
&= W^2B^2.
\end{aligned}$$

Note that all possible colorings of N by C is

$$\left[\omega_{c_1} + \omega_{c_2} + \cdots + \omega_{c_m}\right]^n,$$

where $|N| = n$ and $|C| = m$. If $\omega_c = 1$ for all $c \in C$ then the number of possible colorings of N by C is

$$[1 + 1 + \cdots + 1]^n = m^n,$$

which corresponds to our previous thinking on the number of ways to color n objects with m colors.

Proposition 5.1 If two colorings $k_1, k_2 \in K$, are contained in the same orbit O_k then $\omega(k_1) = \omega(k_2)$.

Proof. If $k_1, k_2 \in O_k$ then there exists some permutation $\sigma \in G$ such that $k_1 = \sigma^*(k_2)$. Then

$$\begin{aligned}\omega(k_1) &= \prod_{i \in N} \omega_{k_1(i)} \\ &= \prod_{i \in N} \omega_{\sigma^*(k_2(i))} \\ &= \prod_{i \in N} \omega_{k_2(\sigma^{-1}(i))} \quad {}^{11} \\ &= \prod_{i \in N} \omega_{k_2(i)} \\ &= \omega(k_2).\end{aligned}$$

□

From this proposition, we can define the **weight of an orbit** to be the common weight of its elements.

$$\omega(O_k) = \omega(k). \tag{16}$$

Lemma 5.1 Let $C = \{c_1, c_2, \ldots, c_m\}$ be a set of m colors. Assume that a set of elements N, is the disjoint union of sets $N_1, \ldots.N_l$. Define $\tilde{K} \subseteq K$ such that for all $k \in \tilde{K}$, if $i, j \in N_l$ for some l then $k(i) = k(j)$. Then the pattern index of $\tilde{K}$ is given by

$$\begin{aligned}&[\omega_{c_1}^{|N_1|} + \omega_{c_2}^{|N_1|} + \cdots + \omega_{c_m}^{|N_1|}] \cdot [\omega_{c_1}^{|N_2|} + \omega_{c_2}^{|N_2|} + \cdots + \omega_{c_m}^{|N_2|}] \\ &\qquad \cdots [\omega_{c_1}^{|N_l|} + \omega_{c_2}^{|N_l|} + \cdots + \omega_{c_m}^{|N_l|}].\end{aligned} \tag{17}$$

Proof. By multiplying (17) out we will get a sum of colorings k of the form

$$w_{c_a}^{|N_1|} w_{c_b}^{|N_2|} \ldots w_{c_l}^{|N_l|},$$

where k colors all elements of N_1 c_a and all elements of N_2 c_b and so on. Thus (17) gives the sum of the weights of colorings k that color all elements of N_l the same color for all l. □

5.2 Pattern Index

Given a set of patterns P we will define the **pattern index** of P to be the following

$$\mathrm{Ind}(P) = \sum_{k \in P} \omega(k). \tag{18}$$

Example 5.2: Let P be the 6 patterns given by the 4 bead necklace problem, and let the following colorings represent their pattern:

$$\{wwww, wwwb, wwbb, wbwb, wbbb, bbbb\}.$$

[11] This is true since all products will run through all $i \in N$. Since the permutation will only alter the order in which it is done, it will not affect the product and thus we can remove the permutation.

Then (18) gives

$$\begin{aligned}
\mathrm{Ind}(P) &= \sum_{k\in P} \omega(k) \\
&= [\omega(wwww) + \omega(wwwb) + \omega(wwbb) + \omega(wbwb) \\
&\quad + \omega(wbbb) + \omega(bbbb)] \\
&= W^4 + W^3B + W^2B^2 + W^2B^2 + WB^3 + B^4 \\
&= W^4 + W^3B + 2W^2B^2 + WB^3 + B^4.
\end{aligned}$$

Here we can see that the coefficients denote the number of patterns with the represented color. For example, W^4 denotes that there is only one coloring with 4 beads colored white, and $2W^2B^2$ denotes that there are two colorings with 2 beads colored black and 2 beads colored white included in the set of patterns P, as seen in figure 3.[1]

5.3 Weighted Burnside's Lemma

Theorem 5.1 (Weighted Burnside's Lemma) Given a set of elements N, colors C and a subgroup $G \subseteq S_n$ of permutations, let $\bar{\omega}(\sigma)$ denote the sum of weights of all k that σ^* leaves fixed.

$$\bar{\omega}(\sigma) = \sum_{k\in \mathrm{Inv}(\sigma)} \omega(k) \tag{19}$$

Then the pattern index is given by

$$\frac{1}{|G|}\sum_{\sigma\in G} \bar{\omega}(\sigma). \tag{20}$$

Note that if all weights are equal to one we have the original Burnside's Lemma.

Proof. First, note that the right hand side of (19) sums the weight of each coloring k left fixed by each $\sigma \in G$. Hence, $w(k)$ is added to the sum $|St_k|$ times. From lemma 3.1 we have

$$|St_k| = \frac{|G|}{|O_k|}.$$

Then by substitution (20) gives

$$\begin{aligned}
\frac{1}{|G|}\sum_{\sigma\in G} \bar{\omega}(\sigma) &= \frac{1}{|G|}\sum_{k\in K} \frac{|G|}{|O_k|}\omega(k) \\
&= \left[\frac{\omega(k_1)}{|O_{k_1}|} + \frac{\omega(k_2)}{|O_{k_2}|} + \cdots\right].
\end{aligned} \tag{$*$}$$

Now we must show that $(*)$ is the sum of the weights of patterns. Given $O_{k_m} = \{k_1, k_2, \ldots, k_m\}$, $|O_{k_m}| = m$ then

$$\begin{aligned}
\frac{\omega(k_1)}{|O_{k_1}|} + \frac{\omega(k_2)}{|O_{k_2}|} + \cdots + \frac{\omega(k_m)}{|O_{k_m}|} &= \frac{\omega(O_{k_m})}{m} + \frac{\omega(O_{k_m})}{m} + \cdots + \frac{\omega(O_{k_m})}{m} \\
&= \omega(O_{k_m}).
\end{aligned}$$

Hence $(*)$ will count each distinct orbit weight once, and thus sum the weight of patterns. □

5.4 Pólya's Enumeration Theorem

We want an easier way to compute the pattern index. Currently we need to know the representative colorings of each pattern in order to calculate the pattern index. Pólya found that one could use the cycle index to compute the pattern index. Given a permutation $\sigma \in G$, if x_l appears in the cycle index then there exists a cycle of length l. If a coloring k is invariant under σ then all l elements in the cycle x_l must be colored the same color. Hence all l elements are colored $c_j \in C$ for some j. Pólya found that this can be represented by the formal sum

$$\omega_{c_1}^l + \omega_{c_2}^l + \cdots + \omega_{c_m}^l. \tag{21}$$

If we then substitute (21) for x_l for every cycle of length l in the cycle index (7) then we will compute the pattern index.[5]

Theorem 5.2 (Pólya's Enumeration Theorem) Given a set N of n elements, C of m colors and a group $G \subseteq S_n$ that acts on N. Let $Z_G(\mathbf{x})$ be the cycle index of G, then the pattern index of nonequivalent colorings of N under G using the colors of C is

$$Z_G\Big(\sum_{i=1}^{m} \omega_{c_i}, \sum_{i=1}^{m} \omega_{c_i}^2, \ldots, \sum_{i=1}^{m} \omega_{c_i}^n\Big). \tag{22}$$

Example 5.3: Let us take our 4 bead necklace with the set of colors $C = \{white, black\}$ and our permutation group $G = C_4$. Let $\omega_{white} = W$ and $\omega_{black} = B$. Then we know the cycle index is

$$Z_{C_4}(x_1, x_2, x_3, x_4) = \frac{1}{4}[x_1^4 + x_2^2 + 2x_4].$$

Pólya's Enumeration Theorem then gives the pattern index as

$$\begin{aligned}
Z_{C_4}(W + B, &W^2 + B^2, W^3 + B^3, W^4 + B^4) \\
&= \frac{1}{4}\left[(W + B)^4 + (W^2 + B^2)^2 + 2(W^4 + B^4)\right] \\
&= \frac{1}{4}[W^4 + 4W^3B + 6W^2B^2 + 4WB^3 + B^4 + W^4 + 2W^2B^2 \\
&\qquad + B^4 + 2W^4 + 2B^4] \\
&= \frac{1}{4}\left[4W^4 + 4W^3B + 8W^2B^2 + 4WB^3 + 4B^4\right] \\
&= W^4 + W^3B + 2W^2B^2 + WB^3 + B^4,
\end{aligned}$$

which corresponds to the pattern index computed in example 5.2 and show in figure 3.

Proof. Let N be a set of n elements, C be a set of m colors, and let K be the set of colorings that map $k : N \to C$. Define G to be the group of permutations σ that act on N. We have shown that the Weighted Burnside's Lemma (thm. 5.1) is equivalent to the pattern inventory as given in

$$\frac{1}{|G|} \sum_{\sigma \in G} \bar{\omega}(\sigma). \tag{20}$$

Recall that we defined $\bar{\omega}(\sigma)$ to denote the sum of weights of all k that σ^* leaves fixed (19). Given a permutation $\sigma \in G$, assume that it is composed of the following disjoint cycles $x_1, x_2, \ldots, x_p$. Then any coloring $k \in \mathrm{Inv}(\sigma)$ must color all elements in a given cycle the same color. In other words, given $i, j \in x_p$ then $k(i) = k(j)$. Thus the pattern index, or the sum of the weights of the set of colorings left invariant by σ^*, given by Lemma 5.1 is

$$\begin{aligned}[\omega_{c_1}^{|x_1|} + \omega_{c_2}^{|x_1|} + \cdots + \omega_{c_m}^{|x_1|}] \cdot [\omega_{c_1}^{|x_2|} + \omega_{c_2}^{|x_2|} + \cdots + \omega_{c_m}^{|x_2|}] \\ \cdots [\omega_{c_1}^{|x_p|} + \omega_{c_2}^{|x_p|} + \cdots + \omega_{c_m}^{|x_p|}].\end{aligned} \tag{23}$$

Hence, (23) gives $\bar{\omega}(\sigma)$ where each term of (23) is of the form

$$w_{c_1}^l + w_{c_2}^l + \cdots + w_{c_m}^l = \sum_{i=1}^{m} w_{c_i}^l, \tag{24}$$

where $l = |x_p|$. Therefore the term $\sum_{i=1}^{m} w_{c_i}^l$ occurs once for every cycle of length l σ contains. Let us denote the number of cycles of length l that σ contains by y_l.[8] Then we can write $\bar{\omega}(\sigma)$ as

$$\left[\sum_{i=1}^{m} w_{c_i}^1\right]^{y_1} \left[\sum_{i=1}^{m} w_{c_i}^2\right]^{y_2} \cdots \tag{25}$$

Substituting (25) for $\bar{\omega}(\sigma)$ in (20) we have

$$Z_G(\sum_{i=1}^{m} \omega_{c_i}, \sum_{i=1}^{m} \omega_{c_i}^2, \ldots, \sum_{i=1}^{m} \omega_{c_i}^n).$$

$\square$

6 Examples

We will now present five common example types used to illustrate Pólya's Enumeration Theorem: beaded necklaces, cubes, chemical compounds, trees and graphs.

6.1 Beaded necklace

The n bead necklace is perhaps the most used example to illustrate both Burnside's Lemma and Pólya's Enumeration Theorem. Here we will present more complex versions of the beaded necklace problem.

Example 6.1: We can now answer the question posed at the beginning of the previous section. How many 6 beaded necklaces are there with 1 white bead, 3 gray beads and 1 black bead discounting rotational symmetries? Let $G = C_6$ and the color weights be as follows:

$$\omega_{white} = W, \quad \omega_{gray} = G, \quad \omega_{black} = B,$$

then Pólya's Theorem gives

$$\begin{aligned}
&Z_{C_6}(W+G+B, W^2+G^2+B^2, \dots, W^6+G^6+B^6)\\
&\quad= \frac{1}{6}[(W+G+B)^6 + (W^2+G^2+B^2)^3 + 2(W^3+G^3+B^3)^2\\
&\qquad + 2(W^6+G^6+B^6)]\\
&\quad= \frac{1}{6}[6B^6 + 6B^5G + 18B^4G^2 + 24B^3G^3 + 18B^2G^4 + 6BG^5 + 6G^6\\
&\qquad + 6B^5W + 30B^4GW + 60B^3G^2W + 60B^2G^3W + 30BG^4W + 6G^5W\\
&\qquad + 18B^4W^2 + 60B^3GW^2 + 96B^2G^2W^2 + 60BG^3W^2 + 18G^4W^2\\
&\qquad + 24B^3W^3 + 60B^2GW^3 + 60BG^2W^3 + 24G^3W^3 + 18B^2W^4\\
&\qquad + 30BGW^4 + 18G^2W^4 + 6BW^5 + 6GW^5 + 6W^6]\\
&\quad= B^6 + B^5G + 3B^4G^2 + 4B^3G^3 + 3B^2G^4 + BG^5 + G^6 + B^5W\\
&\qquad + 5B^4GW + 10B^3G^2W + \mathbf{10B^2G^3W} + 5BG^4W + G^5W + 3B^4W^2\\
&\qquad + 10B^3GW^2 + 16B^2G^2W^2 + 10BG^3W^2 + 3G^4W^2 + 4B^3W^3\\
&\qquad + 10B^2GW^3 + 10BG^2W^3 + 4G^3W^3 + 3B^2W^4 + 5BGW^4 + 3G^2W^4\\
&\qquad + BW^5 + GW^5 + W^6.
\end{aligned}$$

Hence the number of 6 beaded necklaces with 1 white, 3 gray and 2 black beads is the coefficient of B^2G^3W, which is 10 as highlighted above.

What if we also want to discount reflectional symmetries? Let $G = D_6$ with the color weights as above, then Pólya's Theorem gives

$$
\begin{aligned}
&Z_{D_6}(W+G+B, W^2+G^2+B^2, \ldots, W^6+G^6+B^6) \\
&\quad = \frac{1}{12}[(W+G+B)^6 + 3(W+G+B)^2(W^2+G^2+B^2)^2 \\
&\qquad + 4(W^2+G^2+B^2)^3 + 2(W^3+G^3+B^3)^2 + 2(W^6+G^6+B^6)] \\
&\quad = \frac{1}{12}[12B^6 + 12B^5G + 36B^4G^2 + 36B^3G^3 + 36B^2G^4 + 12BG^5 \\
&\qquad + 12G^6 + 12B^5W + 36B^4GW + 72B^3G^2W + 72B^2G^3W + 36BG^4W \\
&\qquad + 12G^5W + 36B^4W^2 + 72B^3GW^2 + 132B^2G^2W^2 + 72BG^3W^2 \\
&\qquad + 36G^4W^2 + 36B^3W^3 + 72B^2GW^3 + 72BG^2W^3 + 36G^3W^3 \\
&\qquad + 36B^2W^4 + 36BGW^4 + 36G^2W^4 + 12BW^5 + 12GW^5 + 12W^6] \\
&\quad = B^6 + B^5G + 3B^4G^2 + 3B^3G^3 + 3B^2G^4 + BG^5 + G^6 \\
&\qquad + B^5W + 3B^4GW + 6B^3G^2W + \mathbf{6B^2G^3W} + 3BG^4W + G^5W \\
&\qquad + 3B^4W^2 + 6B^3GW^2 + 11B^2G^2W^2 + 6BG^3W^2 + 3G^4W^2 + 3B^3W^3 \\
&\qquad + 6B^2GW^3 + 6BG^2W^3 + 3G^3W^3 + 3B^2W^4 + 3BGW^4 + 3G^2W^4 \\
&\qquad + BW^5 + GW^5 + W^6.
\end{aligned}
$$

As highlighted above, the number of 6 beaded necklaces with 1 white, 3 gray and 2 black beads decreases to 6, as the number of permutations increase.

Note that by using Pólya's Theorem to compute the pattern index we have not only solved our original problem, but all 6 beaded necklace problems with 3 colors. By using the pattern index given in Z_{D_6} we are able to see that there are 3 distinct necklaces composed of 3 gray and 3 white beads. Also if we set all weights equal to 1, then we have the total number of distinct necklaces, 92.

Now let us take a more complicated example. How many ways are there to display a set of n bead necklaces? In such a situation, we have m necklaces each of which are comprised of n beads. To answer this question we need to define the composition of permutation groups.[1] Let G and H be permutation groups that act on $M = \{j_1, \ldots, j_m\}$ and $N = \{i_1, \ldots, i_n\}$ respectively. Then the composition $G[H]$ acts on the set $M \times N$. Given a permutation $\sigma \in G$ and a sequence of m permutations $\tau_1, \ldots, \tau_m \in H$ there exists a permutation $[\sigma; \tau_1, \ldots, \tau_m] \in G[H]$ such that for every pair $(i_l, j_k) \in M \times N$

$$
[\sigma; \tau_1, \ldots, \tau_m](i_l, j_k) = (\sigma(i_l), \tau_l(j_k)). \tag{26}
$$

From this it is clear that such a permutation will first permute the m necklaces by $\sigma \in G$ and afterward permute the beads in each individual necklace by $\tau_l \in H$. The cycle index of $G[H]$ is given by

$$
\begin{aligned}
&Z_{G[H]}(x_1, \ldots, x_{mn}) \\
&\qquad = Z_G(Z_H(x_1, \ldots, x_n), Z_H(x_2, \ldots, x_{2n}), \ldots, Z_H(x_m, \ldots, x_{mn})).
\end{aligned} \tag{27}
$$

Example 6.2: How many ways are there to display 2 necklaces each of which has 4 beads that can be colored white or black? Let M be the set of 2 necklaces and N be the set of 4 beads contained in each necklace. We will define C_4 to be the permutation group acting on N to discount rotation symmetries of each necklace. As we only care which 2 necklaces are being displayed and not the order in which they are displayed we will define S_2 to be the permutation group acting on M. We have

$$Z_{S_2} = \frac{1}{2}\left[x_1^2 + x_2\right] \qquad Z_{C_4} = \frac{1}{4}\left[x_1^4 + x_2^2 + 2x_4\right].$$

Now we can compute the total number of possible necklaces. First we compute the number of 4 beaded necklaces of 2 possible colors $Z_{C_4}(2,2,2,2) = 6$. Then(27) gives

$$Z_{S_2}(6,6) = \frac{1}{2}(6^2 + 6) = 21.$$

We can also weight each color, $\omega(white) = W, \omega(black) = B$ and compute the number of ways to display 2 necklaces with a specific combination of black and white beads. For example, 6 black beads and 2 white beads. Then (27) gives

$$\begin{aligned}
&Z_{S_2[C_4]}(W+B,\ldots,W^8+B^8) \\
&\quad = Z_{S_2}(Z_{C_4}(W+B,\ldots,W^4+B^4), Z_{C_4}(W^2,B^2,\ldots,W^8,B^8)) \\
&\quad = \frac{1}{2}[\left(\frac{1}{4}((W+B)^4 + (W^2+B^2)^2 + 2(W^4+B^4))\right)^2 \\
&\qquad + \frac{1}{4}((W^2+B^2)^4 + (W^4+B^4)^2 + 2(W^8+B^8))] \\
&\quad = B^8 + B^7W + \mathbf{3B^6W^2} + 3B^5W^3 + 5B^4W^4 + 3B^3W^5 + 3B^2W^6 \\
&\qquad + BW^7 + W^8.
\end{aligned}$$

Hence, of the 21 possible arrangements of 2 necklaces each with 4 beads, there are 3 with a total combination of 6 black beads and 2 white beads, as shown in figure 8.

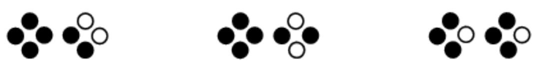
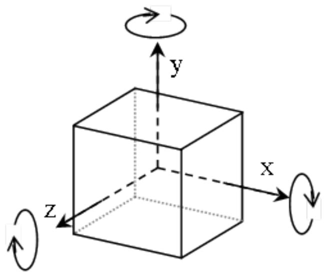


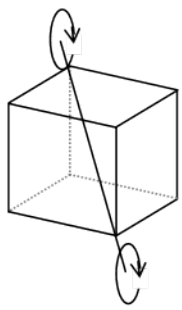
Figure 8: Possible arrangements of 2 necklaces with a total of 6 black and 2 white beads

6.2 Cube

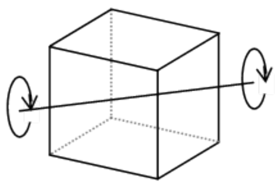
Let us now consider the rotational symmetries of the cube. There are 3 areas of the cube that we can color, the faces, vertices and line segments. There are 3 types of rotations to consider as illustrated in figure 9. There are 9 rotations about the x, y or z axes, 8 rotations by fixing opposite corners and rotating about the symmetry line created, 6 rotations by rotating about the symmetry line created by fixing opposite midpoints, and the identity.



(a) Axis rotation



(b) Corner rotation



(c) Midpoint rotation

Figure 9: Types of rotations of the cube

First consider the faces of a cube labels as in figure 10. Then the 24 cube rotations

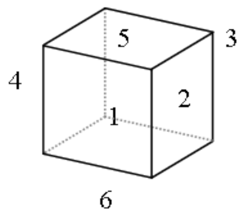


Figure 10: Faces of the cube

in terms of the faces are given as follows:

Axis rotations:

$$\begin{array}{lll} (1234)(5)(6) & (1432)(5)(6) & (13)(24)(5)(6) \\ (1536)(2)(4) & (1635)(2)(4) & (13)(2)(4)(56) \\ (1)(2546)(3) & (1)(2645)(3) & (1)(24)(3)(56) \end{array}$$

Corner rotations:

$$\begin{array}{llll} (145)(263) & (154)(236) & (152)(364) & (125)(346) \\ (146)(253) & (164)(235) & (126)(345) & (162)(354) \end{array}$$

Midpoint rotations:

$$\begin{array}{lll} (14)(23)(56) & (12)(34)(56) & (13)(25)(46) \\ (13)(26)(45) & (16)(24)(35) & (15)(24)(36) \end{array}$$

Identity:

$$\iota = (1)(2)(3)(4)(5)(6)$$

We will denote this group of rotations by G_F. The cycle index is then given by $Z_{G_F} = x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2$. Given a set of colors C, we can now use Pólya's Enumeration Theorem to compute the pattern index of the faces of the cube.

Example 6.3: Let $C = \{white, black\}$ with the usual weights associated to the colors, then Pólya's Enumeration Theorem gives

$$\begin{aligned} &Z_{G_F}(W+B,\ldots,W^6+W^6) \\ &\quad= \frac{1}{24}[(W+B)^6 + 3(W+B)^2(W^2+B^2)^2 + 6(W+B)^2(W^4+B^4) \\ &\qquad + 6(W^2+B^2)^3 + 8(W^3+B^3)^2] \\ &\quad= \frac{1}{24}[24B^6 + 24B^5W + 48B^4W^2 + 48B^3W^3 + 48B^2W^4 + 24BW^5 \\ &\qquad + 24W^6] \\ &\quad= B^6 + B^5W + 2B^4W^2 + 2B^3W^3 + 2B^2W^4 + BW^5 + W^6. \end{aligned}$$

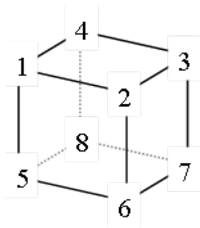


Figure 11: Vertices of the cube

Now let us consider the vertices of a cube as labeled in figure 11. Then the 24 cube rotations in terms of the vertices are given as follows:

Axis rotations:

$$\begin{array}{lll} (1234)(5678) & (1432)(5876) & (13)(24)(57)(68) \\ (1485)(2376) & (1584)(2673) & (18)(27)(36)(45) \\ (1562)(3487) & (1265)(3784) & (16)(25)(38)(47) \end{array}$$

Corner rotations:

$$\begin{array}{ll} (1)(254)(368)(7) & (1)(245)(386)(7) \\ (163)(2)(457)(8) & (136)(2)(475)(8) \\ (168)(274)(3)(5) & (186)(247)(3)(5) \\ (183)(257)(4)(6) & (138)(275)(4)(6) \end{array}$$

Midpoint rotations:

$$\begin{array}{ll} (15)(28)(37)(46) & (17)(26)(35)(48) \\ (17)(23)(46)(58) & (14)(28)(35)(67) \\ (17)(28)(34)(56) & (12)(35)(46)(78) \end{array}$$

Identity:

$$\iota = (1)(2)(3)(4)(5)(6)(7)(8)$$

We will denote this group of rotations by G_V. The cycle index is then given by $Z_{G_V} = x_1^8 + 8x_1^2x_3^2 + 9x_2^4 + 6x_4^2$. Given a set of colors C, we can now use Pólya's Enumeration Theorem to compute the pattern index of the vertices of the cube.

Example 6.4: Let $C = \{white, black\}$ with the usual weights associated to the colors, then Pólya's Enumeration Theorem gives

$$\begin{aligned}
&Z_{G_V}(W+B,\ldots,W^8+W^8)\\
&\quad=\frac{1}{24}[(W+B)^8+8(W+B)^2(W^3+B^3)^2+9(W^2+B^2)^4\\
&\qquad+6(W^4+B^4)^2]\\
&\quad=\frac{1}{24}[24B^8+24B^7W+72B^6W^2+72B^5W^3+168B^4W^4\\
&\qquad+72B^3W^5+72B^2W^6+24BW^7+24W^8]\\
&\quad=B^8+B^7W+3B^6W^2+3B^5W^3+7B^4W^4+3B^3W^5\\
&\qquad+3B^2W^6+BW^7+W^8.
\end{aligned}$$

Lastly let us consider the line segments of a cube as labeled in figure 12. Then

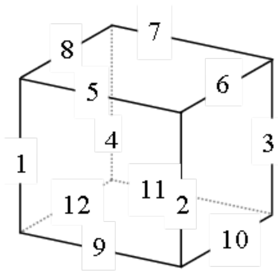


Figure 12: Line segments of the cube

the 24 cube rotations in terms of the line segments are given as follows:

Axis rotations:

(1234)(5678)(9 10 11 12)   (1432)(5876)(9 12 11 10)
(13)(24)(57)(68)(9 11)(10 12)
(184 12)(263 10)(57 11 9)   (1 12 48)(2 10 36)(59 11 7)
(14)(23)(5 11)(6 10)(79)(8 12)
(1925)(374 11)(68 12 10)   (1526)(3 11 47)(6 10 12 8)
(12)(34)(59)(6 12)(7 11)(8 10)

Corner rotations:

(185)(2 12 7)(3 10 11)(469)   (158)(27 12)(3 11 10)(496)
(17 10)(256)(398)(4 11 12)   (1 10 7)(265)(389)(4 12 11)
(1 12 9)(28 11)(367)(4 10 5)   (19 12)(2 11 8)(376)(45 10)
(16 11)(2 10 9)(3 12 5)(487)   (1 11 6)(29 10)(35 12)(478)

Midpoint rotations:

$$
\begin{gathered}
(1)(24)(3)(5\,12)(6\,11)(7\,10)(89)\\
(13)(2)(4)(5\,10)(69)(7\,12)(8\,11)\\
(1\,11)(27)(35)(49)(6)(8\,10)(12)\\
(17)(2\,11)(39)(45)(6\,12)(8)(10)\\
(1\,10)(2\,12)(38)(46)(5\,11)(7)(9)\\
(16)(28)(3\,12)(4\,10)(5)(79)(11)
\end{gathered}
$$

Identity:

$$\iota = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)$$

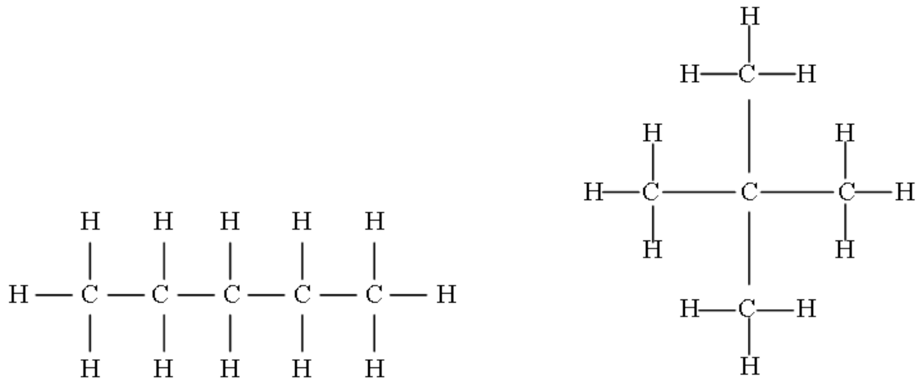
We will denote this group of rotations by G_S. The cycle index is then given by $Z_{G_S} = x_1^{12} + 6x_1^2x_2^5 + 3x_2^6 + 8x_3^4 + 6x_4^3$. Given a set of colors C, we can now use Pólya's Enumeration Theorem to compute the pattern index of the line segments of the cube.

Example 6.5: Let $C = \{white, black\}$ with the usual weights associated to the colors, then Pólya's Enumeration Theorem gives

$$
\begin{aligned}
&Z_{G_S}(W+B,\ldots,W^{12}+W^{12})\\
&\quad=\frac{1}{24}[(W+B)^{12}+6(W+B)^2(W^2+B^2)^5+3(W^2+B^2)^6\\
&\qquad+8(W^3+B^3)^4+6(W^4+B^4)^3]\\
&\quad=\frac{1}{24}[24B^{12}+24B^{11}W+120B^{10}W^2+312B^9W^3+648B^8W^4\\
&\qquad+912B^7W^5+1152B^6W^6+912B^5W^7+648B^4W^8\\
&\qquad+312B^3W^9+120B^2W^{10}+24BW^{11}+24W^{12}]\\
&\quad=B^{12}+B^{11}W+5B^{10}W^2+13B^9W^3+27B^8W^4+38B^7W^5\\
&\qquad+48B^6W^6+38B^5W^7+27B^4W^8+13B^3W^9\\
&\qquad+5B^2W^{10}+BW^{11}+W^{12}.
\end{aligned}
$$

6.3 Chemical Compounds

In chemistry, Pólya's Enumeration Theorem can be used to find isomers of a given molecule. Two molecules are said to be isomers if they are composed of the same number and types of atoms, but have different structure.[5] Let us illustrate this with C_5H_{12}. Figure 13 shows two chemical isomers that correspond to the hydrocarbon C_5H_{12}.



(a) (b)

Figure 13: Chemical isomers corresponding to the hydrocarbon C_5H_{12}

Example 6.6: Cyclobutane is a hydrocarbon constructed of 4 carbon atoms arranged cyclically with 2 hydrogen atoms attached to each carbon, as illustrated in figure 14.

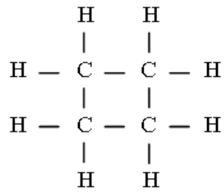


Figure 14: Cyclobutane structure

How many isomers may be obtained by replacing 2 hydrogen atoms with nitrogen? Let the 8 bonds to the carbon atoms be our elements in $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $C = \{hydrogen, nitrogen\}$ with the weights
$\omega(hydrogen) = H, \omega(nitrogen) = N$. We can graphically visualize Cyclobutane as a cube, where the 4 cyclically arranged carbon atoms are at the center of the cube and each hydrogen atom represents a vertex of the cube, therefore G_V will be used to discount reflectional and rotational symmetry. Then Pólya's Enumeration Theorem gives

$$\begin{aligned} &Z_{G_V}(H+N, \ldots, H^8+N^8) \\ &\quad = H^8 + H^7N + \mathbf{3H^6N^2} + 3H^5N^3 + 7H^4N^4 + 3H^3N^5 \\ &\qquad + 3H^2N^6 + HN^7 + N^8. \end{aligned}$$

Hence there are 3 possible isomers with 6 hydrogens and 2 nitrogens as highlighted above. Note that we have already computed this result in example 6.4, the only difference is we are now using chemical elements in place of colors.

Example 6.7: Continuing with this cyclobutane, how many isomers can be obtained by replacing 2 hydrogens with nitrogen and 3 with oxygen? Now we have three colors

$C = \{hydrogen, nitrogen, oxygen\}$ with weights $\omega(hydrogen) = H, \omega(nitrogen) = N, \omega(oxygen) = 0$. Pólya's Enumeration Theorem gives

$$\begin{aligned}
&Z_{G_V}(H+N+O,\dots,H^8+N^8+O^8)\\
&\quad= \frac{1}{24}[(H+N+O)^8 + 8(H+N+O)^2(H^3+N^3+O^3)^2+\\
&\qquad 9(H^2+N^2+O^2)^4 + 6(H^4+N^4+O^4)^2]\\
&\quad= \frac{1}{24}[24H^8 + 24H^7N + 72H^6N^2 + 72H^5N^3 + 168H^4N^4 + 72H^3N^5\\
&\qquad + 72H^2N^6 + 24HN^7 + 24N^8 + 24H^7O + 72H^6NO + 168H^5N^2O\\
&\qquad + 312H^4N^3O + 312H^3N^4O + 168H^2N^5O + 72HN^6O + 24N^7O\\
&\qquad + 72H^6O^2 + 168H^5NO^2 + 528H^4N^2O^2 + 576H^3N^3O^2\\
&\qquad + 528H^2N^4O^2 + 168HN^5O^2 + 72N^6O^2 + 72H^5O^3\\
&\qquad + 312H^4NO^3 + 576H^3N^2O^3 + 576H^2N^3O^3 + 312HN^4O^3\\
&\qquad + 72N^5O^3 + 168H^4O^4 + 312H^3NO^4 + 528H^2N^2O^4\\
&\qquad + 312HN^3O^4 + 168N^4O^4 + 72H^3O^5 + 168H^2NO^5 + 168HN^2O^5\\
&\qquad + 72N^3O^5 + 72H^2O^6 + 72HNO^6 + 72N^2O^6 + 24HO^7 + 24NO^7\\
&\qquad + 24O^8]\\
&\quad= H^8 + H^7N + 3H^6N^2 + 3H^5N^3 + 7H^4N^4 + 3H^3N^5 + 3H^2N^6\\
&\qquad + HN^7 + N^8 + H^7O + 3H^6NO + 7H^5N^2O + 13H^4N^3O\\
&\qquad + 13H^3N^4O + 7H^2N^5O + 3HN^6O + N^7O + 3H^6O^2\\
&\qquad + 7H^5NO^2 + 22H^4N^2O^2 + 24H^3N^3O^2 + 22H^2N^4O^2\\
&\qquad + 7HN^5O^2 + 3N^6O^2 + 3H^5O^3 + 13H^4NO^3 + \mathbf{24H^3N^2O^3}\\
&\qquad + 24H^2N^3O^3 + 13HN^4O^3 + 3N^5O^3 + 7H^4O^4 + 13H^3NO^4\\
&\qquad + 22H^2N^2O^4 + 13HN^3O^4 + 7N^4O^4 + 3H^3O^5 + 7H^2NO^5\\
&\qquad + 7HN^2O^5 + 3N^3O^5 + 3H^2O^6 + 3HNO^6 + 3N^2O^6\\
&\qquad + HO^7 + NO^7 + O^8.
\end{aligned}$$

Hence there are 24 possible isomers with 3 hydrogens, 2 nitrogens, and 3 oxygens as highlighted above.

We can take this example further by asking how many isomers are there with a specific number of hydrogens.[6]

Example 6.8: Let us find the number of isomers in example 6.7 with 3 hydrogens. Let us set the weights as follows: $\omega(hydrogen) = H, \omega(nitrogen) = 1, \omega(oxygen) = 1$. Pólya's Enumeration Theorem gives

$$\begin{aligned}
&Z_{G_V}(H+2,\dots,H^8+2)\\
&\quad= \frac{1}{24}[(H+2)^8 + 8(H+2)^2(H^3+2)^2 + 9(H^2+2)^4 + 6(H^4+2)^2]\\
&\quad= \frac{1}{24}[552 + 1152H + 2112H^2 + 1920H^3 + 1488H^4 + 480H^5\\
&\qquad + 216H^6 + 48H^7 + 24H^8]\\
&\quad= 23 + 48H + 88H^2 + \mathbf{80H^3} + 62H^4 + 20H^5 + 9H^6 + 2H^7 + H^8.
\end{aligned}$$

Hence there are 80 different isomers containing 3 hydrogens. From this we can also see that there are 23 isomers that contain no hydrogens at all.

Here is a similar example taken from Pólya.[7] In this example Pólya discusses 3 permutation groups that can be associated to a basic chemical compound: stereoformula, structural formula and the extended group of the stereoformula. The structural formula takes into account spatial interpretation or rotations. Structural formula is associated to the topological interpretations. The extended group of the stereoformula that accounts for rotations and reflections. We will denote these permutations groups as R, T and RR respectively.

Example 6.9: Consider the chemical compound cyclopropane (C_3H_6), illustrated in figure 15. Let us first find the cycle index to the stereoformula permutation group R.

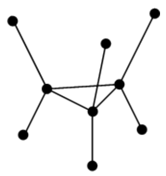


Figure 15: Cyclopropane

There are 6 permutations, 2 rotations of the triangular base of carbon atoms, 3 rotations that rotate by switching any pair of carbon atoms, and 1 identity permutation. Hence the cycle index of R is

$$Z_R = \frac{1}{6}[x_1^6 + 3x_2^3 + 2x_3^2]. \tag{28}$$

Secondly let us find the cycle index associated with the structural formula permutation group T. Above we discovered that we could permute the carbon atoms 6 ways. Once such a permutation has been chosen, the remaining hydrogen atoms of the compound can be permuted in $2^3 = 8$ ways. Therefore there are a total of $6 \cdot 8 = 48$ permutations. These permutations can be associated to those of an octahedron, and thus the cycle index of T is[12]

$$Z_T = \frac{1}{48}\left[x_1^6 + 3x_1^4x_2 + 9x_1^2x_2^2 + 7x_2^3 + 8x_3^2 + 6x_1^2x_4 + 6x_2x_4 + 8x^6\right]. \tag{29}$$

Lastly, let us find the cycle index associated with the extended permutation group of the stereoformula RR. Here we consider both the rotational and reflectional permutations and hence RR consists of 12 permutations with the following cycle index

$$Z_{RR} = \frac{1}{12}\left[x_1^6 + 4x_2^3 + 2x_3^2 + 3x_1^2x_2^2 + 2x_6\right]. \tag{30}$$

Note that the permutation groups have the following relation $R \subset RR \subset T$. Now we are able to find the number of stereoisomers (isomeric molecules with the same molecular formula but which differ in three dimensional orientation), stereoisomers taking reflections into account, and structure isomers. Hence we are able to find

[12]Note that this cycle index can be seen as the composition of S_2 and S_3. Where S_3 permutes the 3 carbon atoms corresponding to the 3 diagonals of the octahedron and S_2 permutes the 2 hydrogen atoms bonded to each carbon atom corresponding to the endpoints of an octahedral diagonal.[7]

the number of different isomeric substitutes of cyclopropane of the form $C_3X_iY_jZ_l$ where X, Y and Z are different independent radicals and $i+j+l=6$. Let $\omega(X)=X, \omega(Y)=Y, \omega(Z)=Z$.
Then the stereoisomers are give by

$$
\begin{aligned}
Z_R((X+&Y+Z),\ldots,(X^6+Y^6+Z^6))\\
&=\frac{1}{6}\left[(X+Y+Z)^6+3(X^2+Y^2+Z^2)^3+2(X^3+Y^3+Z^3)^2\right]\\
&=\frac{1}{6}[6X^6+6X^5Y+24X^4Y^2+24X^3Y^3+24X^2Y^4+6XY^5+6Y^6\\
&\quad+6X^5Z+30X^4YZ+60X^3Y^2Z+60X^2Y^3Z+30XY^4Z+6Y^5Z\\
&\quad+24X^4Z^2+60X^3YZ^2+108X^2Y^2Z^2+60XY^3Z^2+24Y^4Z^2\\
&\quad+24X^3Z^3+60X^2YZ^3+60XY^2Z^3+24Y^3Z^3+24X^2Z^4\\
&\quad+30XYZ^4+24Y^2Z^4+6XZ^5+6YZ^5+6Z^6]\\
&=X^6+X^5Y+4X^4Y^2+4X^3Y^3+4X^2Y^4+XY^5+Y^6+X^5Z\\
&\quad+5X^4YZ+10X^3Y^2Z+10X^2Y^3Z+5XY^4Z+Y^5Z+4X^4Z^2\\
&\quad+10X^3YZ^2+18X^2Y^2Z^2+10XY^3Z^2+4Y^4Z^2+4X^3Z^3\\
&\quad+10X^2YZ^3+10XY^2Z^3+4Y^3Z^3+4X^2Z^4+5XYZ^4+4Y^2Z^4\\
&\quad+XZ^5+YZ^5+Z^6.
\end{aligned}
$$

The stereoisomers taking reflections into account are given by

$$
\begin{aligned}
Z_{RR}(&(X+Y+Z),\ldots,(X^6+Y^6+Z^6))\\
&=\frac{1}{12}[(X+Y+Z)^6+4(X^2+Y^2+Z^2)^3+2(X^3+Y^3+Z^3)^2\\
&\quad+3(X+Y+Z)^2(X^2+Y^2+Z^2)^2+2(X^6+Y^6+Z^6)]\\
&=\frac{1}{12}[12X^6+12X^5Y+36X^4Y^2+36X^3Y^3+36X^2Y^4\\
&\quad+12XY^5+12Y^6+12X^5Z+36X^4YZ+72X^3Y^2Z+72X^2Y^3Z\\
&\quad+36XY^4Z+12Y^5Z+36X^4Z^2+72X^3YZ^2+132X^2Y^2Z^2\\
&\quad+72XY^3Z^2+36Y^4Z^2+36X^3Z^3+72X^2YZ^3+72XY^2Z^3\\
&\quad+36Y^3Z^3+36X^2Z^4+36XYZ^4+36Y^2Z^4+12XZ^5\\
&\quad+12YZ^5+12Z^6]\\
&=X^6+X^5Y+3X^4Y^2+3X^3Y^3+3X^2Y^4+XY^5+Y^6+X^5Z\\
&\quad+3X^4YZ+6X^3Y^2Z+6X^2Y^3Z+3XY^4Z+Y^5Z+3X^4Z^2\\
&\quad+6X^3YZ^2+11X^2Y^2Z^2+6XY^3Z^2+3Y^4Z^2+3X^3Z^3\\
&\quad+6X^2YZ^3+6XY^2Z^3+3Y^3Z^3+3X^2Z^4+3XYZ^4+3Y^2Z^4\\
&\quad+XZ^5+YZ^5+Z^6.
\end{aligned}
$$

The structure isomers are given by

$$
\begin{aligned}
&Z_T((X+Y+Z),\ldots,(X^6+Y^6+Z^6))\\
&\quad=\frac{1}{48}[(X+Y+Z)^6+3(X+Y+Z)^4(X^2+Y^2+Z^2)\\
&\qquad+9(X+Y+Z)^2(X^2+Y^2+Z^2)^2+7(X^2+Y^2+Z^2)^3\\
&\qquad+8(X^3+Y^3+Z^3)^2+6(X+Y+Z)^2(X^4+Y^4+Z^4)\\
&\qquad+6(X^2+Y^2+Z^2)(X^4+Y^4+Z^4)+8(X^6+Y^6+Z^6)]\\
&\quad=\frac{1}{48}[48X^6+48X^5Y+96X^4Y^2+96X^3Y^3+96X^2Y^4+48XY^5+48Y^6\\
&\qquad+48X^5Z+96X^4YZ+144X^3Y^2Z+144X^2Y^3Z+96XY^4Z+48Y^5Z\\
&\qquad+96X^4Z^2+144X^3YZ^2+240X^2Y^2Z^2+144XY^3Z^2+96Y^4Z^2\\
&\qquad+96X^3Z^3+144X^2YZ^3+144XY^2Z^3+96Y^3Z^3+96X^2Z^4+96XYZ^4\\
&\qquad+96Y^2Z^4+48XZ^5+48YZ^5+48Z^6]\\
&\quad=X^6+X^5Y+2X^4Y^2+2X^3Y^3+2X^2Y^4+XY^5+Y^6+X^5Z\\
&\qquad+2X^4YZ+3X^3Y^2Z+3X^2Y^3Z+2XY^4Z+Y^5Z+2X^4Z^2\\
&\qquad+3X^3YZ^2+5X^2Y^2Z^2+3XY^3Z^2+2Y^4Z^2+2X^3Z^3+3X^2YZ^3\\
&\qquad+3XY^2Z^3+2Y^3Z^3+2X^2Z^4+2XYZ^4+2Y^2Z^4+XZ^5+YZ^5+Z^6.
\end{aligned}
$$

Let us now take an example from Aigner.[1]

Example 6.10: An alcohol is an organic compound in which a hydroxyl group OH is bound to a carbon atom C. We want to determine the generating function $A(x) = \sum_{n\geq 0} a_n x^n$ where a_n is the number of alcohols with n carbon atoms. We have $a_0 = 1$, and for $n \geq 1$ let the carbon atom attached to the hydroxyl group be the root of the alcohol. Then there are three subalcohols attached to the root, which can be arbitrarily permuted. Hence the permutation group used is S_3 on $N = \{1, 2, 3\}$.

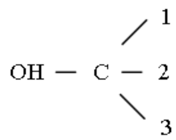


Figure 16: Alcohol structure

Let C be the set of alcohols with weight $\omega(A) = x^n$ if A contains n carbon atoms. Note that C is an infinite set, but this poses no difficulties. Any alcohol will therefore correspond to a map $k : N \to C$, and the different alcohols will correspond to the patterns under S_3. The cycle index for S_3 is

$$Z_{S_3}(x_1, x_2, x_3) = \frac{1}{6}\left[x_1^3 + 3x_1x_2 + 2x_3\right].$$

and $\sum_{A\in C} \omega(A)^k = A(x^k)$. Taking the root into account, Pólya's Enumeration Theorem gives

$$A(x) = 1 + \frac{x}{6}\left[A(x)^3 + 3A(x)A(x^2) + 2A(x^3)\right].$$

Then by comparing coefficients we have the following formula for a_n, $n \geq 1$.

$$a_n = \frac{1}{6}\left[\sum_{i+j+k=n-1} a_i a_j a_k + 3\sum_{i+2j=n-1} a_i a_j + 2a_{\frac{n-1}{3}}\right]. \tag{31}$$

Note that the values of $a_1, \dots a_{n-1}$ are needed to compute the value of a_n using this formula. The first values of which are

n	0	1	2	3	4	5	6	7	8
a_n	1	1	1	2	4	8	17	39	89 .

Hence there are 8 alcohols with 5 carbon atoms as shown in figure 17, where only the hydroxyl group (OH) and carbon atoms (C) are given.

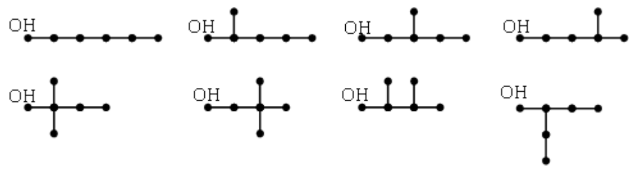


Figure 17: Eight possible alcohols for $n = 5$

6.4 Trees

In graph theory a tree is a connected graph without any cycles in which any two vertices are connected by a single path. Pólya's Enumeration Theorem can also be applied to count the number of rooted trees with p points. This method is similar to example 6.10. Now we want to determine the generating function

$$T(x) = \sum_{p=1}^{\infty} T_p x^p,$$

where T_p is the number of rooted trees with p points. The method used here is recursive. Given a root with degree n the n subtrees can be permuted S_n ways, thereafter each subtree can be permuted. First we must find the generating function which enumerates rooted trees in which the root has degree n. Thereafter we must find the generating function of the latter n secondary rooted trees corresponding to the degree of the root. This correspondence is illustrated for $n = 4$ in figure 18.

Let C be the set of trees with weight $\omega(T) = x^n$ if T contains n points. By applying Pólya's Enumeration Theorem we have $Z_{S_n}(T(x))$ as the function counting series where the coefficient of x^p corresponds to the number of rooted trees of order $p+1$ whose roots have degree n. Taking the root into account corrects the weights so that the coefficient of x^p in $xZ_{S_n}(T(x))$ is the number of trees with p points. Summing over all possible values of n gives us

$$T(x) = x\sum_{n=0}^{\infty} Z_{S_n}(T(x)).$$

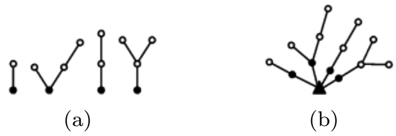


Figure 18: Four rooted trees and the corresponding tree with root of degree 4

The number of rooted trees with p points has been determined for $p \leq 39$. The following table giving the number of rooted trees with p points is taken from Harary.[4]

Number of rooted trees

p	T_p	p	T_p
1	1	14	32 973
2	1	15	87 811
3	2	16	235 381
4	4	17	634 847
5	9	18	1 721 159
6	20	19	4 688 676
7	48	20	12 826 228
8	115	21	35 221 832
9	286	22	97 055 181
10	719	23	268 282 855
11	1 842	24	743 724 984
12	4 766	25	2 067 174 645
13	12 486	26	5 759 636 510

The rooted trees of 4 or less points are shown in figure 19.

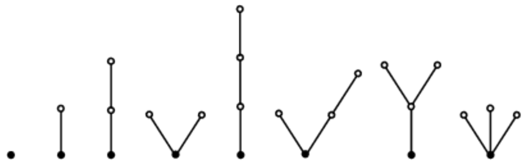


Figure 19: Rooted trees of 4 or less points

6.5 Graphs

Pólya's Enumeration Theorem can already be used to count the coloring of the vertices of any polygon as seen in the beaded necklace examples. Here we will first use Pólya's Enumeration Theorem to discover the number of graphs with p points and q lines, and after the number of bipartite graphs $K_{m,n}$ where $m \neq n$.

(p, q) Graphs

A (p, q) graph is a graph consisting of p points and q lines. We need to derive a formula for the cycle index of (p, q) graphs and then apply Pólya's Enumeration Theorem to determine the counting polynomial

$$q_p(x) = \sum_{q=0}^{\binom{p}{2}} g_{p,q} x^q, \tag{32}$$

where $g_{p,q}$ is the number of (p,q) graphs. Let $N = \{1, \ldots, p\}$ be the set of p points and S_p denote the permutation group acting on N. Then we will denote the set of all 2-subsets of N, or lines, by $N^{(2)}$. Let $C = \{0,1\}$ and let K be the set of colorings $k : N^{(2)} \to C$ where $i, j \in N$ are connected if and only if $k\{i,j\} = 1$. Then two colorings k_1, k_2 are equivalent, $k_1 \sim k_2$, if and only if there exists $\sigma \in S_p$ such that $k_1\{i,j\} = k_2\{\sigma(i), \sigma(j)\}$ for all $i, j \in N$. Let $S_p^{(2)}$ denote the pair group of S_p, which is the permutation group induced by S_p that acts on $N^{(2)}$. Then every $\sigma \in S_p$ induces a permutation σ' that acts on $N(2)$,

$$\sigma'(\{i,j\}) = \{\sigma(i), \sigma(j)\}. \tag{33}$$

Then the polynomial $g_p(x)$ is given by the following formula taken from Harary [4]

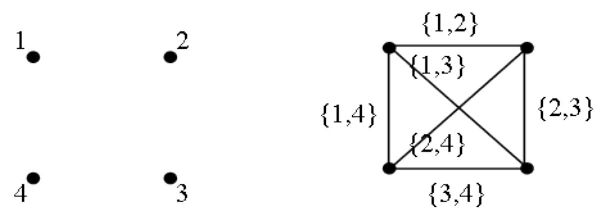
$$g_p(x) = Z_{S_p^{(2)}}(1 + x, \ldots, 1 + x^p), \tag{34}$$

where

$$Z_{S_p^{(2)}} = \frac{1}{p!} \sum_{(j)} \frac{p!}{\prod k^{j_k} j_k!} \prod_k s_{2k+1}^{k j_{2k+1}} \prod_k (x_k x_{2k}^{k-1})_{2k}^{j} x_k^{k\binom{j_k}{2}} \prod_{r<t} s_{[r,t]}^{(r,t) j_r j_t}. \tag{35}$$

Let us illustrate this with an example.

Example 6.11: Take the graph of 4 points where the permutation group is S_4 with cycle index $P_{S_4} = \frac{1}{24}[x_1^4 + 6x_1^2 x_2 + 8x_1 x_3 + 3x_2^2 + 6x_4]$. Here we have 5 types of permutations $x_1^4, x_1^2 x_2, x_1 x_3, x_2^2, x_4$ that will induce permutations in $S_4^{(2)}$. Let us choose a representative of each type of permutation from S_4 and see what type of permutation they induce in $S_4^{(2)}$.



(a) N (b) $N^{(2)}$

Figure 20: Labeling of N and $N^{(2)}$

Let the following permutations be our representatives of the above types: $\iota = (1)(2)(3)(4), \sigma_1 = (1)(23)(4), \sigma_2 = (123)(4), \sigma_3 = (12)(34), \sigma_4 = (1234)$. Then (33) gives the following.

$$\begin{array}{lll} \iota'(\{1,2\}) = \{1,2\} & \iota'(\{2,3\}) = \{2,3\} & \iota'(\{3,4\}) = \{3,4\} \\ \iota'(\{1,4\}) = \{1,4\} & \iota'(\{1,3\}) = \{1,3\} & \iota'(\{2,4\}) = \{2,4\} \end{array}$$

Then $(1)(2)(3)(4) \in S_4 \to (\{1,2\})(\{2,3\})(\{3,4\})(\{1,4\})(\{1,3\})(\{2,4\}) \in S_4^{(2)}$ and hence, $x_1^4 \to x_1^6$.

$$\begin{array}{lll} \sigma_1'(\{1,2\}) = \{1,3\} & \sigma_1'(\{2,3\}) = \{2,3\} & \sigma_1'(\{3,4\}) = \{2,4\} \\ \sigma_1'(\{1,4\}) = \{1,4\} & \sigma_1'(\{1,3\}) = \{1,2\} & \sigma_1'(\{2,4\}) = \{3,4\} \end{array}$$

Then $(1)(23)(4) \in S_4 \to (\{1,2\}\{1,3\})(\{2,3\})(\{3,4\}\{2,4\})(\{1,4\}) \in S_4^{(2)}$ and hence, $x_1^2x_2 \to x_1^2x_2^2$.

$$\begin{array}{lll} \sigma_2'(\{1,2\}) = \{2,3\} & \sigma_2'(\{2,3\}) = \{1,3\} & \sigma_2'(\{3,4\}) = \{1,4\} \\ \sigma_2'(\{1,4\}) = \{2,4\} & \sigma_2'(\{1,3\}) = \{1,2\} & \sigma_2'(\{2,4\}) = \{3,4\} \end{array}$$

Then $(123)(4) \in S_4 \to (\{1,2\}\{2,3\}\{1,3\})(\{3,4\}\{1,4\}\{2,4\}) \in S_4^{(2)}$ and hence, $x_1x_3 \to x_3^2$.

$$\begin{array}{lll} \sigma_3'(\{1,2\}) = \{1,2\} & \sigma_3'(\{2,3\}) = \{1,4\} & \sigma_3'(\{3,4\}) = \{3,4\} \\ \sigma_3'(\{1,4\}) = \{2,3\} & \sigma_3'(\{1,3\}) = \{2,4\} & \sigma_3'(\{2,4\}) = \{1,3\} \end{array}$$

Then $(12)(34) \in S_4 \to (\{1,2\})(\{2,3\}\{1,4\})(\{3,4\})(\{1,3\}\{2,4\}) \in S_4^{(2)}$ and hence, $x_2^2 \to x_1^2x_2^2$.

$$\begin{array}{lll} \sigma_4'(\{1,2\}) = \{2,3\} & \sigma_4'(\{2,3\}) = \{3,4\} & \sigma_4'(\{3,4\}) = \{1,4\} \\ \sigma_4'(\{1,4\}) = \{1,2\} & \sigma_4'(\{1,3\}) = \{2,4\} & \sigma_4'(\{2,4\}) = \{1,3\} \end{array}$$

Then $(1234) \in S_4 \to (\{1,2\}\{2,3\}\{3,4\}\{1,4\})(\{1,3\}\{2,4\}) \in S_4^{(2)}$ and hence, $x_4 \to x_4x_2$.

Therefore the cycle index of $S_4^{(2)}$ is given by

$$\begin{aligned} Z_{S_4}^{(2)} &= \frac{1}{24}\left[x_1^6 + 6x_1^2x_2^2 + 3x_1^2x_2^2 + 8x_3^2 + 6x_4x_2\right] \\ &= \frac{1}{24}\left[x_1^6 + 9x_1^2x_2^2 + 8x_3^2 + 6x_2x_4\right]. \end{aligned}$$

Now we are able to determine $g_4(x)$ by (34), illustrated by figure 21.

$$\begin{aligned} g_4(x) &= Z_{S_4}(1+x, 1+x^2, 1+x^3, 1+x^4) \\ &= \frac{1}{24}\left[(1+x)^6 + 9(1+x)^2(1+x^2)^2 + 8(1+x^3)^2 + 6(1+x^2)(1+x^4)\right] \\ &= \frac{1}{24}\left[24 + 24x + 48x^2 + 72x^3 + 48x^4 + 24x^5 + 24x^6\right] \\ &= 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6. \end{aligned}$$

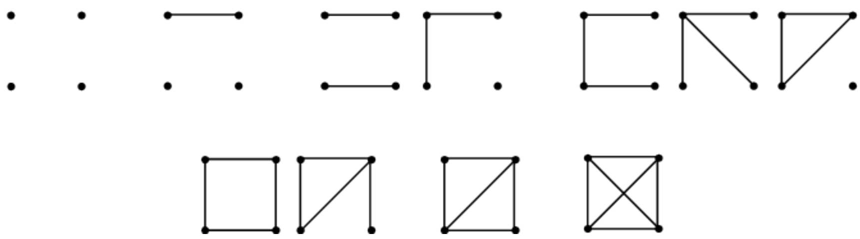


Figure 21: Possible (4,q) graphs

Bipartite Graphs

A bipartite graph $B_{m,n}$ is a graph whose vertices can be divided into 2 distinct subsets M, N such that no 2 points in one set are connected. To compute the number of bipartite graphs we need the concept of a product of permutation groups.

Given two disjoint sets N and M where $|N| = n, |M| = m$ and permutation groups G, H acting on N and M respectively, then there is a natural product $G \times H$ acting on $N \times M$. Let $\sigma \in G$ and $\tau \in H$, then we define the product to be the following

$$(\sigma, \tau)(i, j) = (\sigma(i), \tau(j)) \tag{36}$$

where $i \in N$ and $j \in M$. Then the cycle index of $G \times H$ is

$$Z_{G \times H}(x_1, \ldots x_{mn}) = \frac{1}{|G||H|} \sum_{(\sigma,\tau)} \prod_{k,l=1}^{m,n} x_{\mathrm{lcm}(k,l)}^{\gcd(k,l) y_l(\sigma) y_k(\tau)}, \tag{37}$$

taken from Aigner[1].

It is obvious that given $B_{m,n}$ where $m \neq n$, the permutation group S_m acts on the set M of m points and S_n acts on the set N of n points. Then by the above $S_m \times S_n$ acts on $B_{m,n}$. Let $C = \{0, 1\}$ and let K denote the group of colorings $k : M \times N \to C$ where $(i, j) \in M \times N$ are connected if $k(i, j) = 1$. Then the number of bipartite graphs of the form $B_{m,n}$ are given by

$$Z_{S_m \times S_n}(1 + x, \ldots, 1 + x^{mn}),$$

for $m \neq n$.

Example 6.12: Let us compute the number of bipartite graphs of the form $B_{2,3}$[13]. Then (37) gives

$$Z_{S_2 \times S_3}(x_1, \ldots, x_6) = \frac{1}{12}\left[x_1^6 + 3x_1^2 x_2^2 + 4x_2^3 + 2x_3^2 + 2x_6\right],$$

hence the number of bipartite graphs is

$$\begin{aligned} &Z_{S_2 \times S_3}(1 + x, \ldots, 1 + x^6) \\ &\quad = \frac{1}{12}[(1 + x)^6 + 3(1 + x)^2(1 + x^2)^2 + 4(1 + x^2)^3 + 2(1 + x^3)^2 \\ &\qquad + 2(1 + x^6)] \\ &\quad = \frac{1}{12}\left[12 + 12x + 36x^2 + 36x^3 + 36x^4 + 12x^5 + 12x^6\right] \\ &\quad = 1 + x + 3x^2 + 3x^3 + 3x^4 + x^5 + x^6. \end{aligned}$$

[13]This example can be found in both Aigner [1] and Harary [4].

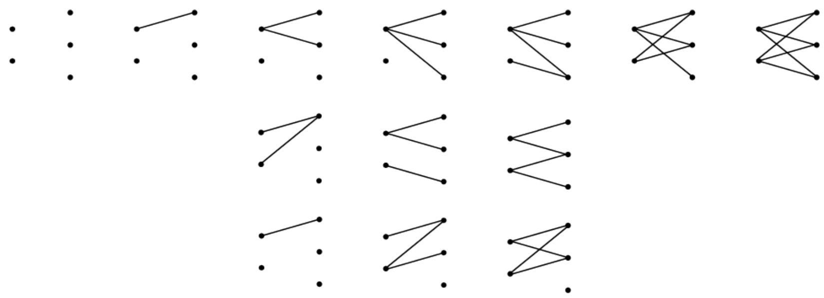


Figure 22: The bipartite graphs of the form $B_{2,3}$

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