Reliability of Technical Systems Estimated by Enhanced Monte Carlo Simulation

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ABSTRACT

Computation of the reliability of large technical systems is usually a very difficult problem for realistic systems, and it is generally not possible to calculate the exact reliability. There are many techniques for approximate calculations, but they are often complicated and difficult to implement. In this paper the development of a new method based on Monte Carlo simulation for efficient calculation of system reliability is described. Standard Monte Carlo simulation forms a simple and robust alternative for calculating system reliability. If one can generate large samples, the law of large numbers ensures that the estimated reliability will be accurate as well. This may, however, be a very time consuming operation. The new method introduces a parametrized system that corresponds to the given system for a specific parameter value. By using regularity of the system reliability as a function of the introduced parameter, the system reliability for our original system can be predicted accurately from relatively small samples.

INTRODUCTION

Standard Monte Carlo simulation often forms a simple and robust alternative for estimating the reliability of mechanical systems. One of the problems with the standard method is, however, its slow convergence. The standard Monte Carlo method normally needs large samples to get accurate results for highly reliable systems, and this is a time and memory consuming operation. (Huseby et al. 2004) used conditional Monte Carlo methods to provide estimates of system reliability. In this paper a Monte Carlo simulation method is introduced that allows the investigation of system reliability via a parametrized cascade of systems. This allows the use of reduced sample size for reliability estimation by exploiting the regularity of the parametrized simulation results as a function of the parameter. To estimate the reliability of the original system, an extrapolation technique based on a least squares error optimization between the simulation results and parametric curves that represent the reliability of the parametrized system. The result is an efficient way to determine system reliability, both for dependent and independent systems.

SYSTEM RELIABILITY

Reliability Block Diagram

It is noted that the standard ISO 8402 defines reliability as

- The ability of an item to perform a required function, under given environmental and operational conditions and for a stated period of time.

In this paper the notation used in (Rausand and Hoyland 2004) is followed, and the term "item" denotes any component, subsystem, or system that can be considered as an entity. A function may be a single function or a combination of functions that is necessary to provide a specified service. By using a reliability block diagram, deterministic models of structural relationships may be established. When the components are in series, all of the components need to function for the system to be functioning. When all the components are in parallel, however, it is sufficient that one component functions for the system to be functioning. A way to combine components in series and parallel is to establish k-out-of-s *systems* (Birolini 2004; Rausand and Hoyland 2004). For these systems, k out of the s components in the system need to function for the system to be functioning. In Figure 1, a structure with 9 components is given. This structure has two k-out-of-s sub-systems, both with $k = 2$ and $s = 3$. These are combined in series with three other components.

Structure Function

Given a system consisting of s components where each component has two distinguishable states, one functioning and one failed state. The state of component i, $i = 1, 2, ..., s$ is defined by

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is in a failed state} \end{cases}$$

The state of the system can be described by the function

$$\phi(\mathbf{x}) = \phi(x_1, x_2, ..., x_s),$$

where $\mathbf{x} = (x_1, x_2, ..., x_s)$ is called the *state vector* and

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is in a failed state} \end{cases}$$

$\phi(\mathbf{x})$ is called the *structure function* of the system.

Since it cannot be predicted with certainty whether or not a component will be in a failed state after t time units, random variables are introduced for the components of the state vector by $X_1(t), X_2(t), \ldots, X_s(t)$. The corresponding random state vector will be denoted by

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_s(t)), \tag{1}$$

and the corresponding structure function is $\phi(\mathbf{X}(t))$. With this state vector, the following probabilities are defined:

$$p_i(t) = \Pr(X_i(t) = 1) \quad \text{for} \ \ i = 1, 2, \ldots, s \ ; \tag{2}$$

$$p_S(t) = \Pr(\phi(\mathbf{X}(t)) = 1), \tag{3}$$

where $p_i(t)$ is the probability that component i will be functioning at time t and $p_S(t)$ is the probability that the system will be functioning at time t.

Cascading Failures

Cascading failures are multiple failures initiated by a failure of one component, referred to as a "domino effect" by (Rausand and Hoyland 2004). These failures may occur when components share a common load, and failure of one component increases the load on the remaining components. When the cascading failures are implemented, the probability of failure for the different components are dependent on the time, t. The stochastic variable that determines the state of component i is represented by

$$X_i(t, \mathbf{x}_{-i}) : \begin{cases} \Pr(X_i(t, \mathbf{x}_{-i}) = 1) \\ = p_i(t, \mathbf{x}_{-i}) = 1 - 10^{-z_i(t, \mathbf{x}_{-i})} \\ \Pr(X_i(t, \mathbf{x}_{-i}) = 0) \\ = 1 - p_i(t, \mathbf{x}_{-i}) = 10^{-z_i(t, \mathbf{x}_{-i})}. \end{cases} \tag{4}$$

The vector $\mathbf{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s)$ represents the state vector without the $i'th$ entry.

The system reliability is given as $p_S(t) = \mathrm{E}(\phi(\mathbf{X}(t))$, and the probability of failure for the system is defined as $p_F(t) = 1 - p_S(t)$.

Two ways of constructing a realistic time dependent probability of failure $p_i(t, \mathbf{x}_{-i})$ will be implemented. By modelling cascading failures, previous behaviour will affect the probability to fail forward in time. To construct such systems in a good way, a repair interval or a condition that forces the repair of the components back to their initial state is needed. Otherwise, the system would end up failing every time when it is run $n \to \infty$ times. So the scenario in this paper is systems for which $p_F(t)$ would be the long run proportion of time when the system is in a failed state.

The different systems with cascading failures comply with the following:

- If one component fails, it is removed from the system until the system fails or all components are repaired

- If one component fails, the probability of other components to fail increases

The two steps in the procedure are combined for the different components in a way that represent realistic systems.

Markov Chains

Some of the dependent systems may be represented by Markov chains. Let the stochastic process $Y_n, \quad n = 0, 1, 2, \ldots$ represent the different states the system is in at different times, $t = n$. If $Y_n = i$, then the system is in state i. For the Markov chain to be valid, there must be a fixed probability P_{ij} that the system will go from state i to state j in the next time step. This is expressed in (Ross 2010) as

$$\begin{aligned} &Pr(Y_{n+1} = j | Y_n = i, Y_{n-1} = i_{n-1}, \ldots, Y_0 = i_0) \\ &= Pr(Y_{n+1} = j | Y_n = i) = P_{ij} \end{aligned} \tag{5}$$

for all states $i_0, i_1, \ldots, i_{n-1}, i, j$ and $n \geq 0$.

The transition probabilities in a Markov chain is conveniently represented in matrix form. The matrix of one step transition probabilities for a Markov chain with S states is given in Equation (6)

$$\mathbf{P} = \begin{bmatrix} P_{SS} & P_{S(S-1)} & \ldots & P_{S0} \\ P_{(S-1)S} & P_{(S-1)(S-1)} & \ldots & P_{(S-1)0} \\ \vdots & \vdots & \vdots & \vdots \\ P_{0S} & P_{0(S-1)} & \ldots & P_{00} \end{bmatrix} \tag{6}$$

Figure 2 may serve to illustrate the flow of transiitons, with associated transition probabilities, that can occur between the S states of the Markov chain.

The matrix in Equation (6) can be used to calculate the limiting probabilities of the Markov chain (Ross 2010). Let $P_{ij}^{(n)}$ denote the n-step transition probabilities. Then the following theorem applies.

Theorem [Limiting Probabilities] For an irreducible ergodic Markov chain $\lim_{n\to\infty} P_{ij}^{(n)}$ exists and is independent of i. Furthermore, letting

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}, \quad j \geq 0$$

then π_j is the unique nonnegative solution of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j \geq 0,$$
$$\sum_{j=0}^{\infty} \pi_j = 1$$

If we have systems of components defined as a Markov chain, the limiting probabilities can be used to find the long run probability of failure for the system. This is done by adding π_j for the j states where the system is not functioning.

To find the limiting probabilities of the states, the conditions in the Theorem and Equation (5) needs to be satisfied. This means that the Markov chain needs to be aperiodic, all states needs to communicate with each other with fixed transition probabilities, and if starting in state i, the expected time until the process returns to state i should be finite. If the necessary conditions are satisfied, the long run probability of failure in the system would be $p_F = \pi_0$.

ENHANCED MONTE CARLO

Sample Estimates

By applying the Monte Carlo method on the system reliability p_S from Equation (3), an estimator of p_S for N trials is obtained,

$$\hat{p}_{SN} = \frac{1}{N}\sum_{j=1}^{N} \phi(\mathbf{x}_j), \tag{7}$$

where $\hat{p}_{SN}$ is the estimator of p_S obtained with N trials. $\mathbf{x}_j$ are independent replicas of the state vector defined in Equation (1), and ϕ is the structure function of the system. By the Law of Large

Numbers, the estimator $\hat{p}_{SN}$ is unbiased. The variance of the estimator is estimated by

$$\hat{\sigma}_N^2 = \frac{1}{N-1}\Big[\frac{1}{N}\sum_{j=1}^{N}\phi(\mathbf{x}_j)^2 - \hat{p}_{SN}^2\Big], \tag{8}$$

which can be simplified to

$$\hat{\sigma}_N^2 = \frac{\hat{p}_{SN}(1-\hat{p}_{SN})}{N-1}. \tag{9}$$

Approximate confidence intervals of the estimator can be defined by applying the Central Limit Theorem (Weiss 2006), which yields

$$CI = [\hat{p}_{SN} - z_\alpha \hat{\sigma}_N, \hat{p}_{SN} + z_\alpha \hat{\sigma}_N], \tag{10}$$

where z_α is found from the tables in (Weiss 2006). $\alpha = 2.5\%$ provides a 95% confidence interval

$$CI_{95} = [\hat{p}_{SN} - 1.96\hat{\sigma}_N, \hat{p}_{SN} + 1.96\hat{\sigma}_N] \tag{11}$$

With $\hat{\sigma}_N$ from Equation (9), it is seen that the convergence rate of the estimator is $O(1/\sqrt{N})$.

Parametrization

Since Monte Carlo simulation has a slow convergence rate, a parametrization of the stochastic variables defined in Equation (4) will be introduced. The idea behind the parametrization is to investigate the system for different failure probabilities. We want to increase the failure probabilities for each component in order to take advantage of the robustness of the standard Monte Carlo method. When the failure rate increases, we need fewer simulations to get a descent result from Monte Carlo simulations. The goal is that it should be possible to fit a curve to the simulation results obtained for increased failure rates, and by extrapolation draw conclusions about the original system reliability.

The parametrization of the stochastic variable, $X_{i,\lambda}(t, \mathbf{x}_{-i})$, for cascading failures becomes

$$X_{i,\lambda}(t, \mathbf{x}_{-i}) : \begin{cases} \Pr(X_{i,\lambda}(t, \mathbf{x}_{-i}) = 1) \\ = p_{i,\lambda}(t, \mathbf{x}_{-i}) = 1 - 10^{-\lambda z_i(t,\mathbf{x}_{-i})} \\ \Pr(X_{i,\lambda}(t, \mathbf{x}_{-i}) = 0) \\ = 1 - p_{i,\lambda}(t, \mathbf{x}_{-i}) = 10^{-\lambda z_i(t,\mathbf{x}_{-i})}. \end{cases} \tag{12}$$

where $0 < \lambda \leq 1$.

By inserting $\lambda = 1$ in Equation (12), it follows that $X_{i,\lambda=1}(t, \mathbf{x}_{-i}) = X_i(t, \mathbf{x}_{-i})$, which is the same stochastic variable as was defined in equation (4) for the initial system. When λ goes to zero the following limit is obtained,

$$X_{i,\lambda\to 0}(t, \mathbf{x}_{-i}) : \begin{cases} \Pr(X_{i,\lambda\to 0}(t, \mathbf{x}_{-i}) = 1) \\ = 1 - 10^{-0\cdot z_i(t,\mathbf{x}_{-i})} = 0 \\ \Pr(X_{i,\lambda\to 0}(t, \mathbf{x}_{-i}) = 0) \\ = 10^{-0\cdot z_i(t,\ x_{-i})} \qquad = 1 \end{cases} \tag{13}$$

The results from simulations of a parametrized system is shown in Figure 3. The system is a dependent system with cascading failures of a 2-out-of-3 system as defined in the section on Example Systems below, cf. Figure 4. It is the first example system discussed in the next section. Since the range of the estimated probability of failure, $\hat{p}_{FN}(\lambda)$, is from 0.1 to 10^{-5}, a logarithmic y-axis is used to present the results. The original system is obtained for $\lambda = 1$, and the behavior of the $\log(\hat{p}_{FN}(\lambda))$ is remarkably close to linear, which, of course, would be the expected behavior for a single component. The estimates of $\hat{p}_{FN}(\lambda)$ were calculated for a sample of size $N = 10^8$ for each λ. By decreasing the sample size to $N = 10^5$, the number of failures when $\lambda \to 1$ will basically be 0, but good estimates will be obtained for $\hat{p}_{FN}(\lambda)$ for the smaller values of λ. These good estimates will be used to predict how the system will behave for the values of λ with typically no observed failures.

When results are obtained for a given system for the different values of λ in the parametrization, a curve will be fitted to these results in order to obtain the probability of failure for the non-parametrized system. To do this curve fitting, $m = 10$ simulations of size n are performed for each value of λ, so the total sample size is $N = mn$. This is carried out for a suitably chosen range of λ-values, $\lambda_1, \ldots, \lambda_l$. The mean of the 10 estimated failure probabilities over the range of λ-values constitute the data that enter the curve fitting by using minimization of least squares. The following family of functions will be used to represent the fitted curve:

$$\tilde{p}_F(\lambda) = 10^{-a(b+\lambda)^c + d}, \tag{14}$$

where $\tilde{p}_F(\lambda)$ denotes the fitted probability of failure, and a, b, c and d are parameters in $\mathbb{R}$. The least squares optimization of parameter fitting is achieved as follows:

$$\min_{a,b,c,d} \sum_{i=1}^{l} w(\lambda_i)\big(- a(b + \lambda_i)^c + d - \log_{10}(\hat{p}_{FN}(\lambda_i))\big)^2, \tag{15}$$

where $w(\lambda_i)$ is a weight factor that reflects the level of uncertainty of the estimate $\hat{p}_{FN}(\lambda_i)$. The minimization procedure chosen for the problems discussed here is based on the trust region method (Forst and Hoffmann 2010).

One way to represent the weights is by the inverse logarithmic difference of the endpoints of a specified confidence interval of $p_F(\lambda)$ for the different λs. By constructing a 95 % confidence interval, the following approximate representation is obtained.

$$\mathrm{CI}_{\pm}(\lambda) = \hat{p}_{FN}(\lambda)(1 \pm 1.96\, CV(\lambda)), \tag{16}$$

where the coefficient of variation of our Bernoulli trials may be written as

$$CV(\lambda) = \sqrt{\frac{1 - \hat{p}_{FN}(\lambda)}{(N-1)\hat{p}_{FN}(\lambda)}}, \tag{17}$$

Then the weights can be defined as

$$w(\lambda) = \frac{1}{\left(\log_{10}(\mathrm{CI}_+(\lambda)) - \log_{10}(\mathrm{CI}_-(\lambda))\right)^2}, \tag{18}$$

This choice of weight factors is convenient, but somewhat arbitrary. However, it has proven to be a suitable choice for the class of problems in this paper. In (Naess et al. 2013) it is shown that the least squares optimization can be expressed as a weighted linear regression. Then the best choice of weight factor will be the inverse of the empirical variance for each value of λ (Montgomery et al. 2001). Notice that the effect of introducing the weight factors is the following: The higher the accuracy of the estimated failure probability $\hat{p}_{FN}(\lambda)$, the more emphasis is put on this point in the optimization. The practical consequence and importance of this can be seen in Figures 5, 6 and 8. If equal weight had been given to all points in these plots, the fitted curves would clearly miss the target value.

EXAMPLE SYSTEMS

Cascading Failures of 2-out-of-3 Systems

Consider the 2-out-of-3 system in Figure 4. Let the components be defined by the stochastic variable in Equation (4). The system can represent a case where the components each share a common load. When one of the components fail, the other components need to take a larger share of the load.

The system is functioning when 2 components are functioning. When the first components in the system fail, the probability to fail for the two other components increase with 50%. The component that failed remains failed until it gets repaired. In the implemented system, the components only get repaired when the system has failed. That is, when 2 or 3 of the components are not functioning.

The one step transition probability matrix **P** introduced in Equation (6) is established, and the long run probability of failure can be calculated. It is obtained that

$$p_F \approx \pi_0 = \frac{q}{q + \frac{2}{3}}, \tag{19}$$

where q denotes the common one step failure probability for all components. With $q = 1-p = 10^{-7}$, it is found at $p_F \approx 1.50 \cdot 10^{-7}$. The results obtained by the proposed enhanced Monte Carlo simulation technique with a total sample size of $N = 10^5$ is shown in Figure 5. The relative error $(\tilde{p}_F(1) - \text{target value})/\text{target value}$ is 0.012.

Cascading Failures of two 2-out-of-3 Systems and Three Independent Components in Series.

This system is of the same form as Figure 1, where the 2-out-of-3 subsystems are identical to the 2-out-of-3 system defined in Figure 4. The other three components in the system act independently. This system is also possible to monitor by Markov chains, to get an analytical solution for the probability to fail, p_F. Let p_4, p_5 and p_9 be the reliability for the three independent components in series, 4,5 and 9. The long run probability of system failure, p_F, for this system can be expressed by

$$p_F = 1 - (1 - \pi_0)_1(1 - \pi_0)_2(p_4)(p_5)(p_9), \tag{20}$$

where $(1 - \pi_0)_1$ is the reliability of the first 2-out-of-3 subsystem and $(1 - \pi_0)_2$ the reliability of the second. With $q = 10^{-7}$, it is found that $p_F \approx 3.30 \cdot 10^{-7}$. The results obtained by the proposed enhanced Monte Carlo simulation technique with a total sample size of $N = 10^5$ is shown in Figure 6. The relative error is -0.052.

Cascading Failures with Repair Interval Combined in Series

The reliability block diagram for this system is shown in Figure 7. The single components, 3 and 6 are independent, but the other four components are implemented with dependencies. When one of the dependent components fail, it is taken out of the system until it is repaired. The dependent components 1 and 2 are repaired simultaneously when both fail, and when at least one of the two components have been functioning for $n = 1/q$ runs, where q denotes the common one step failure probability for these two components. The dependent components 4 and 5 are only repaired when both of them have failed. For the numerical calculations, the one step failure probability $q = 10^{-7}$ for all dependent components, while $q = 10^{-6}$ for the independent components. No analytical solution is available for this example, so a massive sample of size $N = 10^{11}$ was used to establish the long run failure probability of the system. It was found that $p_F \approx 2.085 \cdot 10^{-6}$. The results obtained by the proposed enhanced Monte Carlo simulation technique with a total sample size of $N = 10^5$ is shown in Figure 8. The relative error here is -0.048.

CONCLUSIONS

The preliminary results presented in this paper indicate that it is possible to estimate the probability of failure efficiently and accurately by using Monte Carlo simulations combined with the proposed parametrized systems. The sample size can then be reduced substantially, e.g. from 10^8 with standard Monte Carlo simulation to 10^5 with the proposed method, and still achieve results with the same precision. The parametrization would seem to work well for a wide range of model types beyond the simple models presented here. In fact, the authors believe that the complexity and size of the system has only a minor influence on the efficiency and accuracy of the proposed method.

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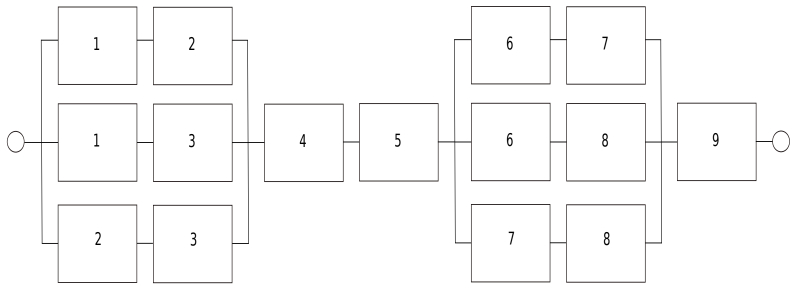

Fig. 1. Structure with 9 components combined in parallel and series.

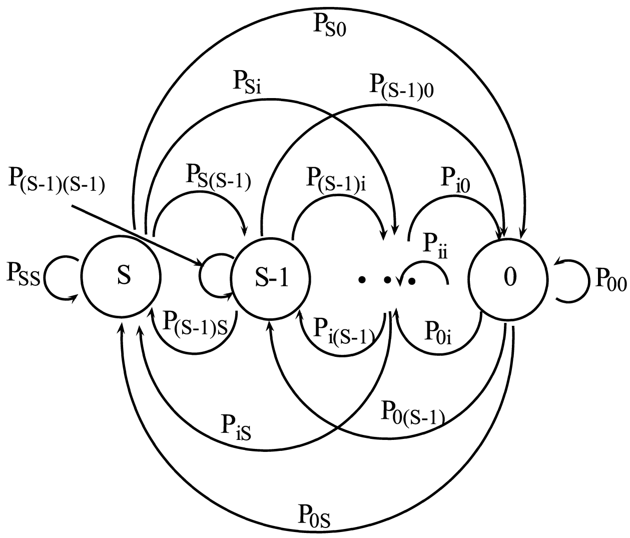

Fig. 2. Markov chain of a system with S states. P_{ij} denotes the fixed probability defined in Equation (5)

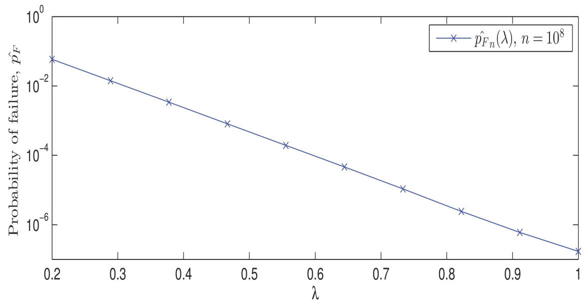

Fig. 3. Simulated probability failure, $\hat{p}_{FN}(\lambda)$, as a function of λ. Simulations are done with $N = 10^8$ for the model with cascading failures of a 2-out-of-3 system. The common one step failure probability is $q = 10^{-5}$ for each component. The original system is obtained for $\lambda = 1$.

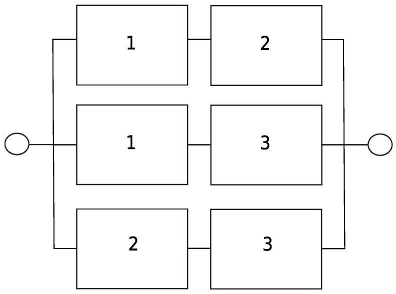

Fig. 4. k-out-of-s system with $k = 2$ and $s = 3$.

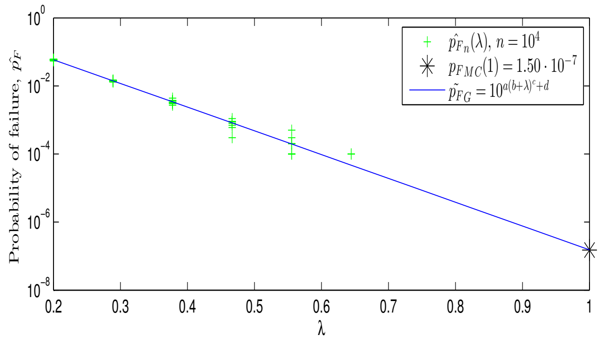

Fig. 5. Cascading failures of 2-out-of-3 system. Logarithmic plot of the fit of the simulated probability failure, $\tilde{p}_F(\lambda)$. Original model is obtained for $\lambda = 1$, and the target value is marked by an asterisk.

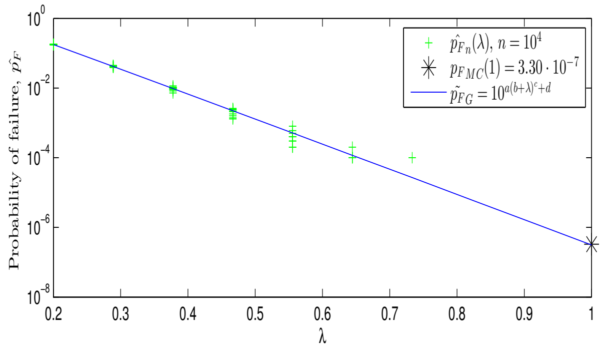

Fig. 6. Cascading failures of 2-out-of-3 system and three independent components in series. Logarithmic plot of the fit of the simulated probability failure, $\tilde{p}_F(\lambda)$. Original model is obtained for $\lambda = 1$, and the target value is marked by an asterisk.

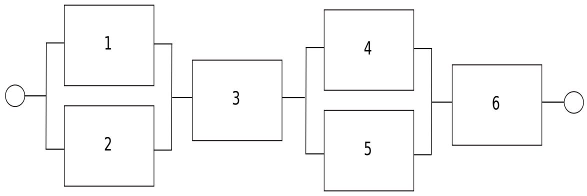

Fig. 7. Structure with 6 components combined in parallel and series.

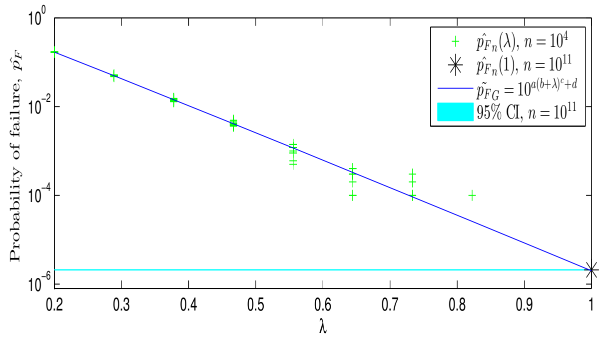

Fig. 8. Cascading failures with repair interval combined in series. Logarithmic plot of the fit of the simulated probability failure, $\tilde{p}_F(\lambda)$. Original model is obtained for $\lambda = 1$, and the target value is marked by an asterisk.