Rotating shaft's non-linear response statistics under biaxial random excitation, by path integration

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Abstract

The response of rotating shaft to random excitations is of practical concern for various rotor type engine design applications, with high level of potential external forces of stochastic nature.

Authors have studied extreme value statistics of random vibrations of a Jeffcott type rotor, modeled as multidimensional dynamic system with non-linear restoring forces, under biaxial white noise excitation. The latter type of dynamic system is of wide use in stability studies of rotating machinery – from automotive to rocket turbo engine design. In particular, the design of liquid-propellant turbo pump rocket engines may be a potential application area of the studied system, due to the biaxial nature of the mechanical excitation, caused by surrounding liquid turbulent pressure field.

In this paper, the extreme statistics of the rotor's non-linear oscillations has been studied by applying an enhanced implementation of the numerical path integration method, benchmarked by a known analytical solution. The obtained response probability distributions can serve as input for a wide range of system reliability issues, for example in extreme response study of turbo-pumps for liquid-propellant rocket engines. Predicting extreme transverse random vibrations of shafts in rotating machinery is of importance for applications with high environmental dynamic loads on Jeffcott type rotor supports.

The major advantage of using path integration technique rather than direct Monte Carlo simulation is ability to estimate the probability distribution tail with high accuracy. The latter is of critical importance for extreme value statistics and first passage probability calculations.

The main contribution of this paper is a reliable and independent confirmation of the path integration technique as a tool for assessing the dynamics of the kind of stochastic mechanical models considered in this paper. By the latter authors mean application of a unique and nontrivial analytical solution, that yields exact dynamic system response distribution, therefore it provides absolutely reliable reference to be compared to. The potential for providing a good qualitative understanding of the behavior of such systems is therefore available.

Keywords: Jeffcott rotor; Path integration; Monte Carlo; FFT; non-linear restoring force; Fokker–Planck-

Kolmogorov equation.

1 Introduction

Jeffcott type rotating machinery can be found in a wide range of industrial areas: from automotive to airspace and marine engines, see e.g. [3], [6]-[9]. The primary feature of Jeffcott type rotating machinery design is to deliver stability under random and even extreme type of external excitations. The nature of stochastic exciting forces, acting from outside and affecting rotor stability, is also quite different between say vehicle, airplane turbo fan, turbo-pump rocket engine or submarine engine. From all above mentioned application areas, this paper applies mostly to turbo-pump rocket engines, since biaxial random excitations have been studied.

The response of a rotating shaft to a biaxial random excitation is of practical concern for the design of liquid-propellant turbo-pump rocket engines. Biaxial random excitation of the rocket engine shaft due to turbulence is an important practical concern, especially for the purpose of improved stability; see [3], [6], [15]. The assumption of zero correlation between two transverse components of biaxial excitation is always satisfied in the important case of excitations generated by a turbulent pressure field, which is delta-correlated in circumferential direction [6].

The nature of hydrodynamic turbulent pressure field is quite complex, for recent studies on VIV (vortex induced vibrations) see e.g. [23], [24], where for the first time, the method developed in [23] provided the possibility to get accurate distributions of hydrodynamics coefficients along a flexible shaft under turbulent pressure field conditions, and became indispensable reference for that field.

Design of a turbo-pump for a liquid-propellant rocket engine may be quoted as an example whereby operation of the shaft close to its instability threshold is of concern because of increased sensitivity of the whole system of shaft-machine-vehicle to such loads [15]. On the other hand, small random vibration components may sometimes be observed in turbulence affected rotating machinery (turbines, fans, etc.), [6], [15]. Figure 1 from [22] presents the spectral density of the vibration signal from the bearing of a large fan with a dominant peak at the rotational frequency 9.92 Hz and neighboring peak at the shaft's resonance at 11.80 Hz.

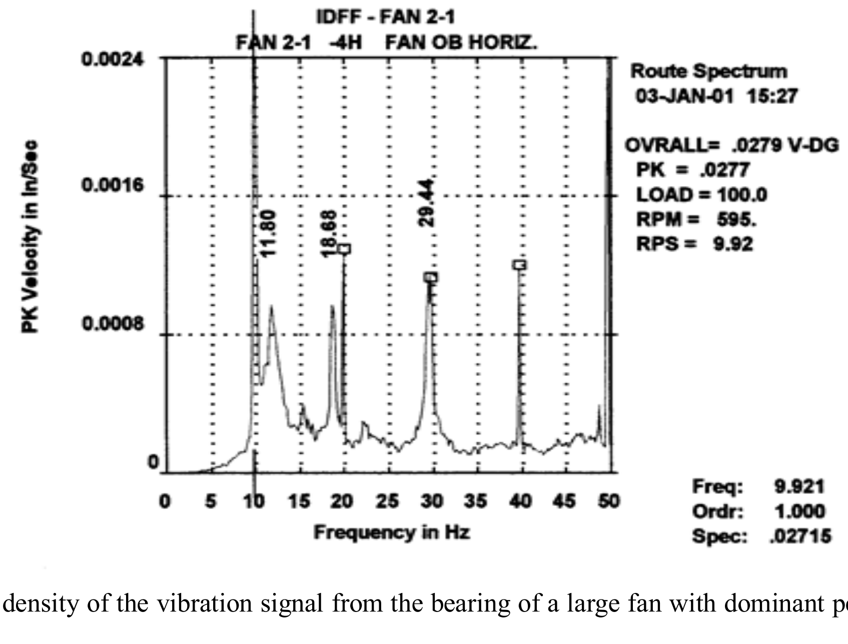

Figure 1 Spectral density of the vibration signal from the bearing of a large fan with dominant peak at the rotational

frequency 9.92 Hz and neighboring peak at the shaft's resonance at 11.80 Hz.

Since the fan rotational response itself is of narrow band stable nature, the resonant multi-peaked excitation is due to a wide band white noise-like ambient turbulence. As long as any modal generalized force is derived from the turbulence field by proper spatial integration, the Gaussian model seems reasonable. The latter point is in view of the basic theorems of asymptotic Gaussianity for linear systems' response with a high excitation/system bandwidth ratio.

Because Jeffcott type systems present significant engineering interest, this paper aims at contributing to the study of Jeffcott-type mechanical systems, namely studying statistics of the system response, specifically the transverse/radial shaft displacement. Radial motions are of importance for system stability and design issues. The restoring force in the shaft of the Jeffcott rotor is modelled as elastic non-linear, which corresponds to a realistic engineering case. Both internal (rotating) and external damping have been included in the studied rotor mechanical model.

Statistics of the rotor's non-linear oscillations is studied by applying the path integration (PI) method, which relies on the Markov property of the coupled dynamical system. By Markov property one means that the probabilistic behaviour of the future response increment depends only on the present response value and discards its past behaviour, see e.g. [1], [4]. The Jeffcott rotor response statistics was obtained by solving the Fokker–Planck-Kolmogorov equation [4] for the corresponding 4D dynamic system. In order to study probability distribution of the system response, an efficient modification of the PI algorithm is introduced, based on the fast Fourier transform (FFT), to calculate the response statistics of a dynamic system with additive noise. The latter allows a significant reduction of the computational time, compared to the classical PI. The excitation force is modelled as Gaussian white noise; however, any noise process with stationary, independent increments with known distribution, can be implemented with analogous PI technique. It should be noted, however, that the path integration (PI) methodology presented in this paper, equally applies to a uniaxial excitation force. The latter makes the PI technique relevant for a wide range of Jeffcott type rotating shaft applications: not only in fluid machinery, but also in transport, [5], [12], [21].

The numerical path integration technique (see e.g. [1], [4], [21]) is well known as an efficient approximation for solving the Fokker–Planck-Kolmogorov equation, and giving accurate estimates of stationary response PDF (probability density functions) for dynamic systems. The base of PI method is the Markov property of dynamic system (the latter is expressed as an equivalent system of first order differential equations). Then the evolution of the response PDF is estimated via a step-by-step solution scheme, based on reasonably short time steps. In more detail, the response PDF at a current time instant can be estimated, if the response PDF at an earlier discrete time instant, and the conditional PDF (which in this paper is of Gaussian form) are already known.

The PI solution suggested in this paper has been accelerated by using a pre-estimated joint probability density function (PDF) as an initial input. Pre-estimation of the stationary PDF has been done by Monte Carlo (MC) simulation.

The main objective of studying statistics of the non-linear case by applying a 4D path integration (PI) method, was to obtain a sufficiently high resolution of the stationary response PDF tail (specifically at quite low probability levels). A major advantage of PI method compared to e.g. direct Monte Carlo Simulation (MCS) has been demonstrated in details in this paper. The latter suggests PI methodology as a practical engineering/design tool for performing reliability analysis of Jeffcott type rotating shafts (and other related mechanical systems).

Finally, in order to validate the accuracy of the proposed numerical PI scheme, a comparison has been done with an available exact analytical solution of the joint response probability density function (PDF), see [12].

2 Dynamic analysis

Numerous studies have been done to study non-linear effects in rotor dynamics [8], [9], [16]-[20]. This paper considers a Jeffcott rotor with non-linear restoring force. Then the linearized axisymmetric combined shaft/support lateral stiffness is denoted K; the shaft angular velocity is denoted ν.
The shaft carries a disk of mass m located at its mid-span and has both external ("non-rotating") damping and internal ("rotating") damping. The shaft external and internal damping coefficients in this paper are denoted as c_n and c_r respectively.
The shaft's Z-axis is modelled to be horizontal and external exciting forces are applied along both transverse directions X and Y. Then, by neglecting gravity forces for sufficiently high rotation speeds and adding the transverse forces, the following equations of the disk lateral motion [5], [12] can be obtained

$$\ddot{X} + 2\kappa\dot{X} + f_X(X,Y) + 2\beta\nu Y = \gamma\dot{W}_1(t);\ \ddot{Y} + 2\kappa\dot{Y} + f_Y(X,Y) - 2\beta\nu X = \gamma\dot{W}_2(t) \tag{1}$$

with $\kappa = \alpha + \beta$, $\alpha = c_n/2m$, $\beta = c_r/2m$. $\dot{W}_k(t)$ being stationary, independent zero-mean Gaussian white noises, being formal derivatives of Wiener processes $W_k(t)$ with normally distributed increments with $\mathrm{E}[dW_k(t)] = 0$ and $\mathrm{E}[dW_k(t)dW_k(s)] = dt$ for $t = s$, and $= 0$ for $t \neq s$; $k = 1,2$. Noise intensities are assumed to be the same for both transverse directions, and are denoted by γ. Note that the dynamic system (1) is an axisymmetric (biaxial) Jeffcott system, see [12], where the analytical solution to the Fokker–Planck-Kolmogorov equation has been obtained.

The non-linear elastic restoring force components along X and Y axes are denoted $f_X(X,Y)$ and $f_Y(X,Y)$, respectively. The non-linear version of the restoring forces studied in this paper, is assumed to be given as in [9]:

$$f_X(X,Y) = \Omega^2 X \cdot [1 + f(r)];\ f_Y(X,Y) = \Omega^2 Y \cdot [1 + f(r)];\ f(r) = \left[1 - \frac{1}{\sqrt{1+R^2}}\right]\tilde{\varepsilon} \tag{2}$$

with $\Omega^2 = K/m$, and $r(X,Y) = \sqrt{X^2+Y^2}$, $\varepsilon = T/EA \ll 1$, $\tilde{\varepsilon} = \varepsilon^{-1} - 1 > 0$, with T being the reference (constant) tension in the shaft, corresponding to its horizontal position, i.e. $T = EA\varepsilon$, thus ε is actually the shaft inverse stiffness parameter. It is worth noting that in this paper the shaft is assumed to be perfectly balanced. As mentioned above, X and Y are being non-dimensionalized displacements, representing actual radial shaft transverse displacements u and v by scaling, $X = u/L$, $Y = v/L$ with L being the half distance between two shaft supports [9].
Since the main focus of this paper is extreme response statistics, linearization of the restoring force was not appropriate. Therefore, an efficient PI technique has been applied, enabling high accuracy estimation of the PDF tails, without simplifying system nonlinearities.

The dynamic system represented by Eq. (1) can be equivalently re-written as a four dimensional (4D) first order differential system by introducing the 4D state space variable $\boldsymbol{x} = (x_1, y_1, x_2, y_2)$, where $x_1 = X$, $y_1 = Y$, $x_2 = \dot{X}$, $y_2 = \dot{Y}$,

$$
\begin{aligned}
dx_1 &= x_2 dt \\
dy_1 &= y_2 dt \\
dx_2 &= (-2\kappa x_2 - f_Y(x_1, y_1) + 2\beta v y_1)dt + \gamma dW_1 \\
dy_2 &= (-2\kappa y_2 - f_X(x_1, y_1) - 2\beta v x_1)dt + \gamma dW_2
\end{aligned}
\tag{3}
$$

3 Path integration method

The dynamic system given by equation (3) belongs to a class of Markov diffusion processes [10]. Thus, system (3) can be studied using Itô stochastic differential equation (SDE)

$$d\boldsymbol{x} = \boldsymbol{\mu}(\boldsymbol{x}, t)dt + \boldsymbol{\sigma}(t)d\boldsymbol{W}(t) \tag{4}$$

where $\boldsymbol{x}(t) = (x_1, .., x_n)^T$ is an n-dimensional vector, as is $\boldsymbol{\mu}(\boldsymbol{x}, t)$; $\boldsymbol{\sigma}(t)$ is an $n \times n$ matrix, and $\boldsymbol{W}(t)$ being an n-dimensional standard Wiener vector process, with dimension $n = 4$. Then the state space vector $\boldsymbol{x}(t)$ being a Markov process and its transition probability density (TPD) $p(\boldsymbol{x}, t|\boldsymbol{x}', t')$ will satisfy the following well known Fokker–Planck-Kolmogorov (FPK) equation [16], [17].

$$
\begin{aligned}
\frac{\partial}{\partial t} p(\boldsymbol{x}, t|\boldsymbol{x}', t') = &-\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\mu_k(\boldsymbol{x}, t) p(\boldsymbol{x}, t|\boldsymbol{x}', t')\right) \\
&+\frac{1}{2}\sum_{k.l=1}^{n} \frac{\partial}{\partial x_k}\left(\sigma_{kl}(\boldsymbol{x}, t)\sum_{m=1}^{n}\left[\frac{\partial}{\partial x_m}\sigma_{ml}(\boldsymbol{x}, t) p(\boldsymbol{x}, t|\boldsymbol{x}', t')\right]\right)
\end{aligned}
\tag{5}
$$

PI method estimates the probabilistic evolution of $\boldsymbol{x}(t)$ by means of taking advantage of the system Markov property. The numerical PI is an approximate approach and the PDF of the process $\boldsymbol{x}(t)$ can be estimated by referring to the following equation

$$p(\boldsymbol{x}, t) = \int_{\mathbb{R}^n} p(\boldsymbol{x}, t|\boldsymbol{x}', t') p(\boldsymbol{x}', t') d\boldsymbol{x}' \tag{6}$$

with $d\boldsymbol{x}' = \prod_{k=1}^{n} dx_k$. The PDF $p(\boldsymbol{x}, t)$ at time t can be calculated from Eq. (7) based on knowledge of the TPD $p(\boldsymbol{x}, t|\boldsymbol{x}', t')$ and the values of the previous PDF $p(\boldsymbol{x}', t')$ at time t'. For a numerical solution of the stochastic differential equation (4), a time-discretized approximation has been employed. In [4] the fourth-order Runge–Kutta–Maruyama (RKM) discretization approximation has been presented

$$\boldsymbol{x}(t) = \boldsymbol{x}(t') + \boldsymbol{h}(\boldsymbol{x}(t'), \Delta t') + \boldsymbol{\sigma}(t')\Delta\boldsymbol{W}(t') \tag{7}$$

where the vector $\boldsymbol{h}(\boldsymbol{x}(t'), \Delta t')$ being an explicit fourth-order Runge–Kutta (RK4) approximation. Since $\boldsymbol{W}(t)$ is a Wiener type process, the noise increment $\Delta\boldsymbol{W}(t') = \boldsymbol{W}(t) - \boldsymbol{W}(t')$ is a Gaussian variable for every time instant t' and t. The time sequence $\boldsymbol{x}(k\Delta t'), k = 0, \dots, \infty$, is a Markov chain approximating the time-continuous Markov process solution of the stochastic differential equation (4), while the time increment $\Delta t' = t - t'$ being small enough, and the spatial mesh for the $\boldsymbol{x}$-variable being fine enough.

Note that, as mentioned in [11], the interpolation error at each time step will be about the same order of magnitude and will be independent of the time step magnitude, thus the final solution may give a larger final error for shorter time steps, because more iterations are required to reach stationary response state. In practical applications, for higher dimensional systems, a very fine mesh cannot be numerically afforded, thus the time step can not be chosen arbitrarily small.
Since the expression for the TPD is known (two independent one-dimensional Gaussian distributions in this paper), the time evolution of the PDF of $\boldsymbol{x}(t)$ can be estimated by the following iterative equation

$$p(\boldsymbol{x},t)=\int\limits_{\mathbb{R}^n}\dots\int\limits_{\mathbb{R}^n}\prod_{k=1}^{N}p\left(\boldsymbol{x}^{(k)},t_k|\boldsymbol{x}^{(k-1)},t_{k-1}\right)\,p\left(\boldsymbol{x}^{(0)},t_0\right)d\boldsymbol{x}^{(0)}\dots d\boldsymbol{x}^{(\boldsymbol{N}-1)} \tag{8}$$

with $p\left(\boldsymbol{x}^{(0)},t_0\right)$ being an initial PDF, and $\boldsymbol{x}=\boldsymbol{x}^{(N)};\ \ t=t_N=t_0+N\Delta t$. In this paper authors has proposed to estimate initial PDF from prior Monte Carlo (MC) simulation, in order to accelerate solution convergence, specifically to reduce a number of necessary time steps N before the converged solution can be obtained.

4 Efficient numerical implementation

The iterative solution technique of the PI approach is presented by Eq. (8). For numerical implementation of the PI method, a proper computational domain and the corresponding computational grid (carrier domain) have to be estimated at first. For more detailed description of the numerical iterative algorithm see [19], [20]. Larger higher dimensional grids naturally lead to a larger size of variables and to a longer computational time. In [18] it is suggested to use GPU for a computational speed up, as compared to CPU based implementation. The problem with the latter suggestion is that the GPU memory capacity (VRAM) is much less than the CPU memory capacity (RAM).

In this paper, the initial PDF $p\left(\boldsymbol{x}^{(0)},t_0\right)$ and its carrier domain have been estimated from the prior extensive MC simulation of the dynamical system of type (3) over a sufficiently long time period T. The latter choice of the pre-estimated initial PDF by MCS is proven to enable faster convergence of the iterative algorithm (8) within a reasonably shorter number of time steps N. Note, that the initially pre-estimated joint PDF by MCS is not accurate in the PDF tail area (namely for low probability levels); thus it is a task of the subsequent PI calculation to develop a converged solution with accurate PDF tails.

Numerically, the joint PDF at any time t is being approximated using interpolated parabolic B-spline surface. Then, the RK4 approximation scheme is applied to estimate the deterministic trajectories backward for each $\boldsymbol{x}$-grid point. Thus, each $\boldsymbol{x}$-grid point at each time t is mapped backwards with the corresponding starting point at the previous time t_0 and the PDF values at the backward-mapped points, i.e., the starting points, can be given by using the B-spline surface. Finally, the PDF at time t can be calculated by substituting the TPD $p(\boldsymbol{x},t|\boldsymbol{x}',t')$ (normally distributed in the case of this paper) into Eq. (6). Furthermore, the capability of the PI method in producing accurate and reliable solutions for stochastic dynamic systems was demonstrated by numerous studies, see e.g. [1], [4].

The mean up-crossing rate is an important parameter for estimation of the large and extreme response statistics as well as for evaluation of the associated reliability of dynamic structures, subjected to random external forces [2]. The radial displacement r of the disk when approaching the casing circular wall of the turbo fan can be regarded as a high response level. Alternatively, one can study von Mises stresses in the shaft as a response of interest. For the latter purpose, the same methodology as described here will apply. The estimation of the mean up-crossing rate of the radial response R of the level ξ is usually based on the Rice equation

$$\nu^{+}(\xi,t)=\int_{0}^{\infty}\dot{r}\,p_{R\dot{R}}(\xi,\dot{r})d\dot{r} \tag{9}$$

The 2D joint PDF $p_{R\dot{R}}$ can be extracted from the 4D PI joint PDF $p(\boldsymbol{x},t)$ in (8). In case when the assumption of statistically independent up-crossing is valid for a certain level in this region, the corresponding crossing events are Poisson distributed. Under the latter assumption, the reliability evaluation is usually phrased in terms of the probability that the disk vertical displacement X exceeds the specific threshold at least once during a time interval of length T. Thus, the exceedance probability for duration of exposure time T can be approximated by the following widely used approach

$$P_{\xi}(T)=1-\exp\left(-\int_{0}^{T}\nu^{+}(\xi,t)dt\right)\approx 1-\exp(-\nu^{+}(\xi)T)\approx\nu^{+}(\xi)T\ll 1 \tag{10}$$

where $\nu^{+}(\xi)$ represents the mean up-crossing rate of the level ξ (radial transverse displacement r in this paper) at a suitable reference point in time, which can be estimated directly by the 4D PI method and the Rice Eq. (9).

5 Numerical results

In this section the non-linear system (1), (2) statistical results are compared with corresponding analytical solution that is available. Also, the stability margin is well known for the linear case. The linearized (namely linearized restoring force) system (1), (2) possesses the following dynamic stability condition $\nu<\nu_{*}$, provided in [3], [7]

$$\nu<\nu_{*}=\frac{\kappa\Omega}{\beta}=\Omega\left(1+\frac{c_n}{c_r}\right) \tag{11}$$

For the case $\Omega=1,\ \alpha=\beta=0.02,\ \gamma=0.02$, Eq. (11) gives a linearized stability margin $\nu^{*}/\Omega=2$. In this paper a computational mesh for the 4D variable $\boldsymbol{x}$ in the equivalent system (3) has been chosen with following grid resolution 128×128×128×128. Regarding the small stiffness parameter $\varepsilon=T/EA$ in Eq. (2), for the non-linear restoring force it is chosen to be equal 0.1.

In brief the effect of varying the force stiffness parameter $\varepsilon=T/EA$ on the transverse radial displacement mean up-crossing rate for the non-linear dynamic system (1), (2). Keeping the shaft tension T constant, while increasing the shaft stiffness EA, will decrease the shaft stiffness parameter ε, and yield a decrease in the transverse radial amplitude. The latter phenomenon is clearly seen from Figure 2.

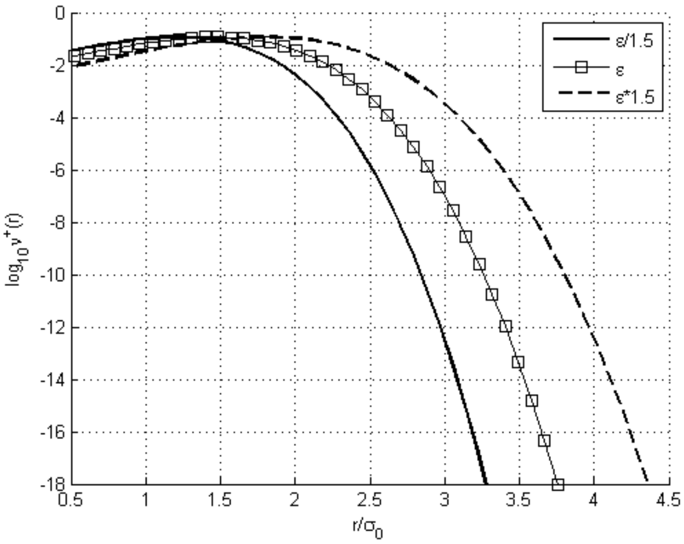

Figure 2 Effect of the shaft stiffness parameter ε on the radial displacement up-crossing rate, for the non-linear dynamic system (1), (2).

In the case of purely additive white noise, independently acting in two different dimensions, the PI algorithm can be implemented by performing the PI integration by means of the Fourier transformation, see e.g. [16]. The FFT (fast Fourier transform) numerical implementation is based on the path integral (8), Section 3, being represented as two pure consequent convolutions (for the case of additive white noise entering two different system dimensions). This requires the full Jacobian method that is used in this paper. Since the Fourier transform of the convolution is a simple product, and computing a product is numerically much more efficient than a convolution integral, it is CPU time-saving to use the FFT algorithm for the additive noise PI. Since the FFT method is not exact, its numerical accuracy can be further enhanced by choosing a sufficiently high grid resolution and mesh span.

5.1 Analytical solution

In this subsection the following parameters are chosen for the system (1), (2): $\Omega = 1, \alpha = \beta = 0.02, \nu = 2.2$, and the linear dynamic stability condition (11) $\nu < \nu_*$ does not hold. In [12] an analytical solution $p(x_1, y_1, x_2, y_2)$ was established that satisfies the Fokker-Planck-Kolmogorov equation

$$\begin{aligned} x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\{\Omega^2[1+f(r)]x_1 p + 2\kappa x_2 p + 2\beta\nu y_1 p\} - y_2 \frac{\partial p}{\partial y_1} \\ + \frac{\partial}{\partial y_2}\{\Omega^2[1+f(r)]y_1 p + 2\kappa y_2 p - 2\beta\nu x_1 p\} + \frac{\gamma^2}{2}\left(\frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial y_2^2}\right) = 0 \end{aligned} \tag{12}$$

with $r = \sqrt{x_1^2 + y_1^2}$. Direct substitution shows that the partial differential equation (12) has the following exact analytical solution

$$p(x_1, y_1, x_2, y_2) = C \cdot \exp\left[-\frac{4\kappa}{\gamma^2} H + \frac{4\beta\nu}{\gamma^2}(x_1 y_2 - x_2 y_1)\right],\ H = \frac{\Omega^2}{2}\left(r^2 + F(r)\right) + \frac{1}{2}\left(x_2^2 + y_2^2\right), \text{ where } \frac{dF}{dz} = f \text{ ,and } z = r^2 \tag{13}$$

with C being a normalization constant which is obtained by requiring that the integral of the expression (13) over $\mathbb{R}^4$ is equal to 1.0; function f is defined in (2). The solution of the form (13) is well known for the case of the symmetric apparent nonlinear stiffness matrix in the original equation of motion (1), that is for $\nu = 0$, see [12], [13], [14].

5.2 Comparison between path integration results and the analytical solution

In this subsection the following parameters are chosen in system (1)-(2): $\Omega = 1, \alpha = \beta = 0.02, \nu = 2.2$, and the linear dynamic stability condition (11), $\nu < \nu_*$, is not satisfied. Since the biaxial excitation force, studied in this paper, has radial symmetry (as opposed to the uniaxial excitation force), it is expedient to consider the transverse radial displacement $r = \sqrt{X^2 + Y^2}$. In particular, the mean up-crossing rate will be studied. For this purpose, it is convenient to change coordinates from (x_1, y_1, x_2, y_2) to $(r, \dot{r}, \theta, \dot{\theta})$ with (r, θ) being polar coordinates for $(x_1, y_1) \equiv (X, Y)$. Then, a coordinate transformation leads to the following PDF relation $p(r, \dot{r}, \theta, \dot{\theta}) = p(x_1, y_1, x_2, y_2) \cdot r^2$. This allows the calculation of the mean up-crossing rate $\nu^+(r)$ of the radial displacement by invoking Rice's formula (9).

Figure 3 presents the mean up-crossing rate of the transverse radial response for the non-linear dynamic system (1), (2) obtained by the three methods available here. It is seen from Figure 3 that PI offers highly accurate estimates of the radial response at low probability levels, compared to extensive MC simulation. Note that CPU time spent in order to obtain both MC and PI simulation results, presented in Figure 3, is about the same order, while PI gives upto 8 orders of magnitude deeper crossing rate tail resolution. The latter effectively means that PI provides much more accurate probability tail distribution, than MC, provided same CPU time spent. Thus the PI may offer a significant advantage, with respect to the straightforward MC, since in many engineering design applications multiple numerical simulations, corresponding to different dynamic system numerical parameter setups, often have to be run.
Regarding error analysis for numerical methods such as MC and PI, since Figure 3 presents good match with analytical (exact) probability tail, it is clear that numerics has been performed accurately. Both PI and MC could have been compared in terms of the relative error for the predicted response level at a given low probability of interest, since an exact analytical solution is available. However since MC simulation provides much more "shallow" probability distribution tail, for the same CPU time spent as for PI, comparison between only PI and exact solution was done at the probability level 10^{-15}; the relative error of PI method for the latter response level was less than 5%.
The reference value σ_0 is the corresponding standard deviation of r-response for the case of linear restoring force. Reaching probability levels of order 10^{-15} is simply unaffordable with direct MC simulation, given the requirement of a small time-step and limited CPU time. PI however manages the latter task within a relatively small number of time steps (due to initial PDF pre-calculation by MC) and at low computational PI cost (due to the FFT based implementation). Therefore, PI becomes an efficient tool for a study of the reliability or extreme value statistics of the Jeffcott systems of type (1).

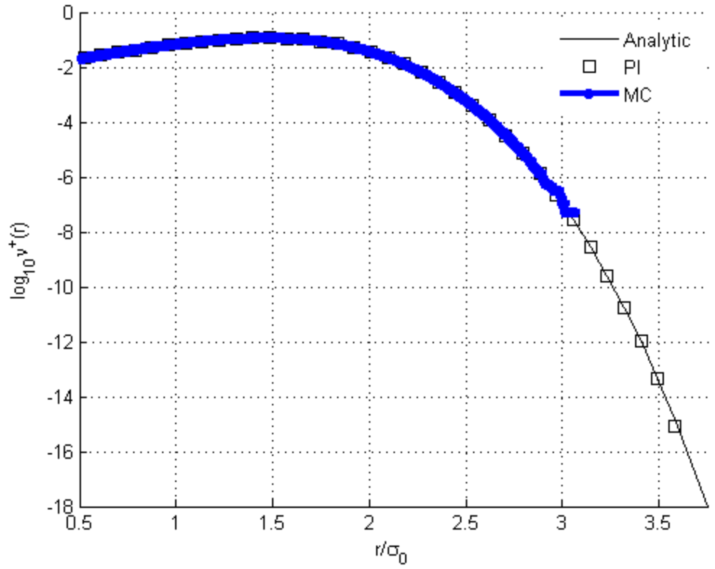

Figure 3 Mean up-crossing rates for non-linear system, transverse radial displacement: MC (*) versus PI (□) versus analytical solution (—).

Figure 4 presents the stationary joint 2D PDF(X, Y) of the non-linear dynamic system (1), (2), extracted from the 4D PDF solution $p(\boldsymbol{x}, t)$ of the PI equation (8), where $\boldsymbol{x} = (x_1, y_1, x_2, y_2)$, see Section 5.1. Figure 5 presents the contour lines of the joint PDF(X, Y), corresponding to Figure 4. It is seen from Figure 5 that the PDF contour lines are axisymmetric, as expected due to axial symmetry of the dynamic system (1), (2) itself.

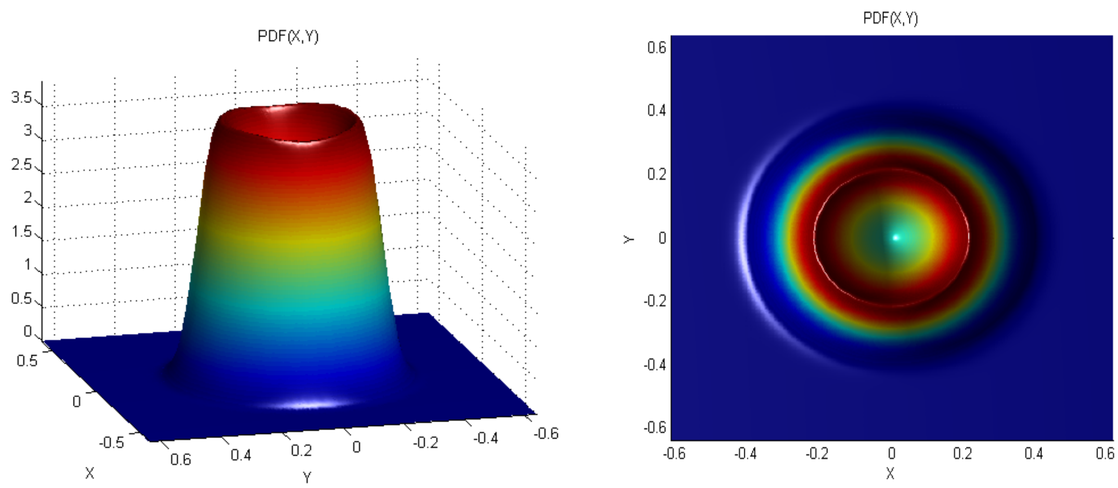

Figure 4 Stationary joint PDF(X, Y) for non-linear dynamic system (1), (2); top view on the right.

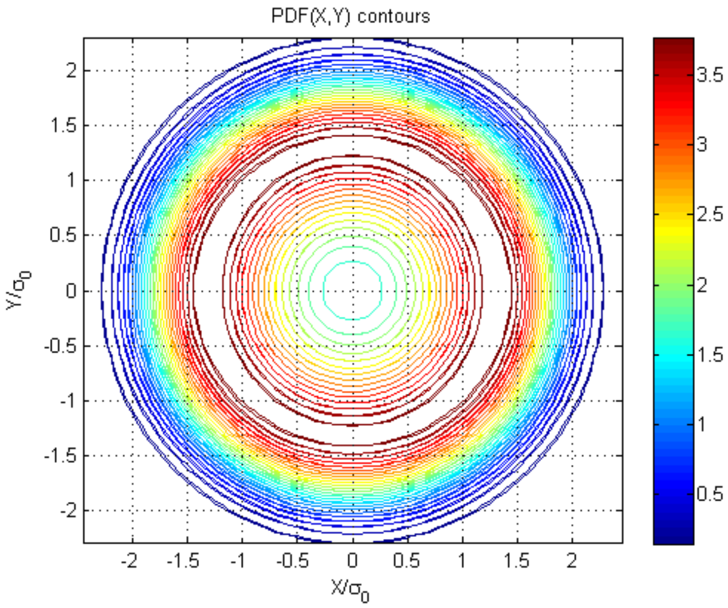

Figure 5 Contour lines for the stationary joint PDF(X, Y) presented in Figure 4 for the non-linear dynamic system (1), (2).

Note that for the parameter choice made in this subsection, the linearized system is unstable. As one can see from the presented figures, the non-linear system is still stable due to restoring force non-linearity. This is an important practical observation, since in reality restoring forces are indeed non-linear. The existence of a stationary response PDF implies stability of the self-excited response. The instability threshold may be estimated easily by the switch from unimodal to bimodal PDF.

6 Conclusions

In this work, the Jeffcott type rotor with non-linear restoring force subject to biaxial white noise excitation has been studied. Statistics of the rotor shaft transverse response was studied. The theory of Markov processes was used in modelling the rotor dynamics in a state space formulation, and the 4D PI approach based on the Markov property of the dynamic system was found to be in a good agreement with available analytical solution, proving high accuracy of the suggested numerical technique. Comparison with analytical results demonstrated efficiency, robustness and reliability of the 4D PI method.

In this paper, the PI approach has been reinforced by using MC and FFT, which significantly reduces both computational effort and the number of iterations needed for convergence to a stationary solution. In overall, the proposed technique is proven to be highly accurate and more efficient than direct MC estimation of extreme response with low probability levels.
The accuracy of PI is demonstrated by estimating the mean up-crossing rate at low probability levels. For ergodic random processes the mean up-crossing rate is known to be a key factor for studying reliability and extreme value statistics.

Naturally, often question about advantage of PI over direct MC arises. With regard to extreme value statistics, first passage probability, defining design values, etc. for rotating machinery – it is clear that direct MC is computationally

expensive when very low probabilities are targeted, moreover when numerous system parameters are to be optimized at the mechanical system design stage.
Another point of skepticism about PI is that it is seldom used for systems with number of degrees of freedom higher than four (4D). Here it is worth noting that recent advances in GPUs (Graphic Processing Unit) do enable 6D PI solutions, and obviously this is not a final limit for PI.

As shown in this paper, the non-linear system is still stable for the parameter set where the linearized system is unstable (here authors refer to the dynamic instability condition from [7]). In this paper the restoring force non-linearity (and therefore dynamic system non-linearity as a whole) was responsible for the system dynamic stability. This observation is of practical importance, since in practice restoring forces are non-linear. Also, the structure of the response joint PDF is observed to be strongly non-Gaussian, when exceeding the stability margin. Finally, convergence of the PI solution to the stationary PDF implies stability of the (hardening) nonlinear system's response in case of the instability of the linearized system.

Conflict of interests statement

The authors declare that they have no conflict of interest.

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