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Product of Hyperfunctions with Disjoint Support

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Problem Description

In the article "Product of Hyperfunctions on the Circle" in Israel Journal of Mathematics, 2000, J. Esterle and R. Gay discuss the definition of the product of hyperfunctions on the circle; in particular they show that for two hyperfunctions with disjoint support the product is defined and equals zero. Their approach uses the structure of the sheaf of hyperfunctions. They mention in the introduction that the problem comes from the article by Y. Domar of 1997 and the later comments to this article. J. Esterle and R. Gay also claim that their first approach to the problem was based on the theory of entire functions.

The first aim of this project is to study the original article by Y. Domar, fill in the details of his construction of an invariant subspace of the weighted l^p-space and see how this result is connected to the product of hyperfunctions. Then we want to see how the technique based on the theory of entire functions can be adjusted to show that the product of two hyperfunctions with support in two disjoint arcs is zero in the sense of Esterle and Gay.

Further aims of this project are to give a new proof of the result by Esterlie and Gay about hyperfunctions on the circle with disjoint support and consider the corresponding problem for hyperfunctions on the real line.

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Abstract

We prove that if two hyperfunctions T_1 and T_2 on the unit circle have disjoint support, then

$$\lim_{r \to 1^-} \sum_{n \in \mathbb{Z}} r^{|n|} c_n(T_1) c_{m-n}(T_2) = 0 \qquad (m \in \mathbb{Z})$$

where c_k are the Fourier coefficients of the hyperfunctions. We prove this by using the Fourier-Borel transform and the G-transform of analytic functionals. The proof is inspired by an article by Yngve Domar. In the end of his article he proves the existence of a translation-invariant subspace of a certain weighted l^p-space. This proof has similarities to our proof, so we compare them. We also look at other topics related to Domar's article, for example the existence of entire functions of order ≤ 1 under certain restrictions on the axes. We will see how the Beurling-Malliavin theorem gives some answers to this question. Finally, we prove that if T and S are hyperfunctions on $\mathbb{R}$ with compact and disjoint support, then

$$\lim_{a \to 0} \int_{-\infty}^{\infty} FT(z) FS(w-z) e^{-a|x|} \, \mathrm{d}x = 0.$$

Preface

This paper has been written in the spring of 2008 as the final part of the Master of Technology degree within the Industrial Mathematics Program at the Norwegian University of Science and Technology.

I want to thank my fellow student Maria Skartsæterhagen for our discussions about hyperfunctions and for being a very good friend. I also want to express my gratitude to my supervisor Eugenia Malinnikova for her excellent guidance. She always has time to answer my questions, and her ideas and good advice have been of great value to the development of this paper.

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Kjersti Solberg Eikrem

Contents

Chapter 1

Introduction

In my project in the fall of 2007 I gave an introduction to hyperfunctions and analytic functionals on the unit circle, and showed that there exists a bijection between these sets. I also showed that the product of an analytic function h and a hyperfunction f is well defined and that we have the following formula for the Fourier coefficients of the product:

$$\widehat{hf}(m) = \sum_n \hat{h}(n)\hat{f}(m-n).$$

In an article by Esterle and Gay [5] it is shown that if two hyperfunctions T_1 and T_2 have disjoint support, then

$$\lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} r^{|n|} c_n(T_1) c_{m-n}(T_2) = 0 \qquad (m \in \mathbb{Z}) \tag{1.1}$$

where c_k are the Fourier coefficients of the hyperfunctions. They proved this by first showing that it is possible to take the product of two hyperfunctions with disjoint support and then they drew conclusions about the Fourier coefficients. One of the aims of this project is to prove this formula directly using the Fourier-Borel transform and the G-transform of analytic functionals. That was what Esterle and Gay did first, but they did not write about it in the article. Their first proof was inspired by a paper by Yngve Domar [4] and a preprint by Atzmon. We will go through the details of the proof by Domar that inspired Esterle and Gay.

Before we do that, we will look at some other topics related to Domar's article. The article proves the existence of entire functions of order ≤ 1 under certain restrictions on the axes. We will see how the Beurling-Malliavin theorem given in [3] gives some answers to this question. Domar proves two theorems, and one of them is used to show that there exists a translation-invariant subspace of a weighted l^p-space. It is this part that has similarities to the proof of formula (1.1). When we have proved (1.1), we will compare Domar's proof to our proof of formula (1.1).

Finally, we will give an introduction to hyperfunctions on $\mathbb{R}$ and see if it is possible to find a formula similar to (1.1) here.

Chapter 2

Hyperfunctions and analytic functionals

2.1 Hyperfunctions on $\mathbb{T}$

We will let $\mathscr{H}(W)$ be the space of holomorphic functions on an open subset W of $\mathbb{C}$. If V_1 and V_2 are open subsets of $\mathbb{C}$ and $V_1 \subseteq V_2$, we let $R_{V_1,V_2} : \mathscr{H}(V_2) \to \mathscr{H}(V_1)$ be the restriction map. Let L be a nonempty open subset of $\mathbb{T}$, then we denote by $\mathcal{U}_L$ the set of all open subsets W of $\mathbb{C}$ such that $W \cap \mathbb{T} = L$. If $W \in \mathcal{U}_L$ we define the quotient space

$$\mathcal{B}_W(L) := \mathscr{H}(W \setminus L)/R_{W\setminus L,W}(\mathscr{H}(W)).$$

It can be shown that this definition is independent of W, so we may define $\mathcal{B}(L) := \mathcal{B}_W(L)$ [5, page 273].

Definition 2.1.1. *The space of hyperfunctions on L is the complex vector space defined as the quotient*

$$\mathcal{B}(L) := \mathscr{H}(W \setminus L)/R_{W\setminus L,W}(\mathscr{H}(W)).$$

A hyperfunction on L is an element of $\mathcal{B}(L)$.

This means that the elements of $\mathcal{B}(L)$ are represented by pairs of holomorphic functions (f^+, f^-) in $W^+ = W \cap \mathbb{D}$ and $W^- = W \cap (\mathbb{C} \setminus \bar{D})$, respectively. Two pairs of functions (f^+, f^-) and (g^+, g^-) are equivalent if there exists a $U \in \mathscr{H}(W)$ such that $f^+ - g^+ = U|W^+$ and $f^- - g^- = U|W^-$.

Let $\mathscr{H}_0(\mathbb{C} \setminus \bar{\mathbb{D}}) = \{g \in \mathscr{H}(\mathbb{C} \setminus \bar{\mathbb{D}}) | \lim_{|z|\to\infty} g(z) = 0\}$. We will need the following fact a few times: If $T \in \mathcal{B}(\mathbb{T})$ there exists a unique $f^+ \in \mathscr{H}(\mathbb{D})$ and a unique $f^- \in \mathscr{H}_0(\mathbb{C} \setminus \bar{\mathbb{D}})$ such that T is represented by (f^+, f^-) [5, page 273].

Definition 2.1.2. *Let $L_1 \subset L_2$ be two nonempty subsets of $\mathbb{T}$. Let $T \in \mathcal{B}(L_2)$ be represented by $f \in \mathscr{H}(V_2 \setminus L_2)$, where $V_2 \in \mathcal{U}_{L_2}$. The restriction of T to L_1, denoted $T|L_1$, is the hyperfunction in $\mathcal{B}(L_1)$ which is associated to $f|((V_1 \cap V_2) \setminus L_1)$, for any open $V_1 \in \mathcal{U}_{L_1}$.*

Definition 2.1.3. *Let $T \in \mathcal{B}(\mathbb{T})$. The support of T on $\mathbb{T}$, $\operatorname{supp} T$, is the complement on $\mathbb{T}$ of the largest open set $U \subseteq \mathbb{T}$ such that $T|U = 0$.*

This means that supp T is the complement on $\mathbb{T}$ of the largest open set $U \subseteq \mathbb{T}$ such that f^+ and f^- extend each other analytically across U.

2.2 Analytic functionals

We now define the restriction map $R_{V_1,V_2} : \mathscr{H}(V_2) \to C(V_1)$ where $C(V_1)$ is the set of continuous functions on V_1, and we no longer require V_1 to be open. The space of holomorhic functions on a compact set K is then

$$\mathscr{H}(K) = \bigcup_{K \subseteq \Omega} R_{K,\Omega}(\mathscr{H}(\Omega))$$

where the union is taken over all open sets containing K.

Definition 2.2.1. *Let $K \subset \mathbb{C}$ be compact. We define $\mathscr{H}'(K)$, the space of analytical functionals carried by K, to be the space of linear functionals Γ on $\mathscr{H}(K)$ such that for all open ω containing K there exists C_ω for which*

$$|\langle \Gamma, \varphi \rangle| \leq C_\omega \sup_\omega |\varphi|$$

holds for each $\varphi \in \mathscr{H}(K)$ analytic in ω.

2.3 Fourier coefficients

The Fourier coefficients of a function $f \in L^1(\mathbb{T})$ are given by

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, \mathrm{d}\theta.$$

We may also define the Fourier coefficients of an analytic functional.

Definition 2.3.1. *Let $T \in \mathscr{H}'(\mathbb{T})$. The nth Fourier coefficient of T is given by*

$$c_n(T) = \left\langle T, \zeta^{-n-1} \right\rangle$$

for all $n \in \mathbb{Z}$.

There is a definition in [5] which we will also use.

Definition 2.3.2. *The Fourier coefficients $\hat{f}(n)$ of a hyperfunction $T \in \mathcal{B}(\mathbb{T})$ represented by (f^+, f^-) where $f^+ \in \mathscr{H}(\mathbb{D})$ and $f^- \in \mathscr{H}_0(\mathbb{C} \setminus \bar{\mathbb{D}})$, are defined by the formulae*

$$f^+(z) = \sum_{n \geq 0} \hat{f}(n) z^n \qquad |z| < 1$$

and

$$f^-(z) = -\sum_{n < 0} \hat{f}(n) z^n \qquad |z| > 1.$$

2.4 Bijection between $\mathcal{B}(\mathbb{T})$ and $\mathscr{H}'(\mathbb{T})$

I showed in my project that there exists a bijection between $\mathcal{B}(\mathbb{T})$ and $\mathscr{H}'(\mathbb{T})$. Here we will use a slightly different bijection which has the advantage of preserving the Fourier coefficients.

We let

$$H : \mathscr{H}'(\mathbb{T}) \to \mathcal{B}(\mathbb{T})$$

be given by

$$\Gamma \mapsto [\tilde{\Gamma}_1, -\tilde{\Gamma}_2]$$

where $[\tilde{\Gamma}_1, -\tilde{\Gamma}_2]$ is the equivalence class of $(\tilde{\Gamma}_1, -\tilde{\Gamma}_2)$ and

$$\tilde{\Gamma}_1(z) = \langle \Gamma, \frac{1}{\zeta - z} \rangle \qquad |z| < 1$$

$$\tilde{\Gamma}_2(z) = \langle \Gamma, \frac{1}{z - \zeta} \rangle \qquad |z| > 1.$$

It follows from Proposition 2.4.1, which is part of Proposition 1.6.10 in [2], that $\tilde{\Gamma}_1 \in \mathscr{H}(\mathbb{D})$ and $\tilde{\Gamma}_2 \in \mathscr{H}_0(\mathbb{C} \setminus \bar{\mathbb{D}})$.

Proposition 2.4.1. *Let $\Gamma \in \mathscr{H}'(\mathbb{T})$. Then $\tilde{\Gamma}_1 \in \mathscr{H}(\mathbb{D})$ and $\tilde{\Gamma}_2 \in \mathscr{H}_0(\mathbb{C} \setminus \bar{\mathbb{D}})$ and*

$$\begin{aligned} \tilde{\Gamma}_1(z) &= \left\langle \Gamma_\zeta, \frac{1}{\zeta - z} \right\rangle = \sum_{n \geq 0} c_n(\Gamma) z^n \quad (|z| < 1), \\ \tilde{\Gamma}_2(z) &= \left\langle \Gamma_\zeta, \frac{1}{z - \zeta} \right\rangle = \sum_{n > 0} c_{-n}(\Gamma) z^{-n} \quad (|z| < 1). \end{aligned}$$

By comparing this proposition with definition 2.3.2 we see that H preserves the Fourier coefficients.

We also define the inverse of H. Let $T = [f^+, f^-] \in \mathcal{B}(\mathbb{T})$. Then there exists a unique $g^+ \in \mathscr{H}(\mathbb{D})$ and a unique $g^- \in \mathscr{H}_0(\mathbb{C} \setminus \bar{\mathbb{D}})$ such that T is represented by (g^+, g^-) [5, page 273]. We then define

$$F : \mathcal{B}(\mathbb{T}) \to \mathscr{H}'(\mathbb{T})$$

by

$$[f^+, f^-] \mapsto \Gamma_T$$

where

$$\langle \Gamma_T, h \rangle = -\frac{1}{2\pi i} \int_{\gamma_1} g^-(\zeta) h(\zeta) \, \mathrm{d}\zeta + \frac{1}{2\pi i} \int_{\gamma_2} g^+(\zeta) h(\zeta) \, \mathrm{d}\zeta$$

and γ_1 and γ_2 are cycles outside and inside the unit circle, respectively, and $h \in \mathscr{H}(\mathbb{T})$. The cycles are contained in the domain where h is analytic. Γ_T is in $\mathscr{H}'(\mathbb{T})$ since

$$\begin{aligned} |\langle \Gamma_T, h \rangle| &\leq \frac{1}{2\pi} \int_{\gamma_1} |g^-(\zeta)| |\mathrm{d}\zeta| \sup_{z \in \gamma_1} |h(\zeta)| + \frac{1}{2\pi} \int_{\gamma_2} |g^+(\zeta)| |\mathrm{d}\zeta| \sup_{z \in \gamma_2} |h(\zeta)| \\ &\leq C_\omega \sup_{z \in \omega} |h(\zeta)| \end{aligned}$$

where ω is the annulus between γ_1 and γ_2.

Chapter 3

Domar's article

We need some definitions before we can formulate the problem in [4], which is Domar's article.

3.1 Entire functions

We will now define the order and the type of an entire function. This is taken from [8].

Definition 3.1.1. *The entire function f is of order ρ if*

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho \qquad (0 \leq \rho \leq \infty)$$

where $M(r) = \max_{|z|=r} |f(z)|$.

Definition 3.1.2. *The entire function f of positive order ρ is of type τ if*

$$\limsup_{r\to\infty} \frac{\log M(r)}{r^\rho} = \tau \qquad (0 \leq \rho \leq \infty).$$

An entire function is said to be of exponential type if its order is ≤ 1 and its type is finite.

3.1.1 Example

The function e^{az^2} is of order 2 and type $|a|$.

3.2 The problem

Domar's article gives some answers to the following question:

Question 3.2.1. *Is there a not identically vanishing entire function f, of order ≤ 1, such that*

$$\log|f(x)| \leq k(x), \qquad x \in \mathbb{R},$$
$$\log|f(iy)| \leq a|y|, \qquad y \in \mathbb{R},$$

where a is a positive number and k is a continuous real-valued function on $\mathbb{R}$.

We will study this problem, both for its own interest, and for its links to hyperfunctions. One of Domar's theorems leads to results about translation-invariant subspaces of weighted l^p-spaces, so we will study this topic too. It is through this topic we will find similarities to the product of hyperfunctions.

First we will take a look at a famous theorem by Beurling and Malliavin, and see how this theorem gives us some answers to question 3.2.1. Then we will state Domar's generalisation. We will define what we mean by weighted l^p-spaces and see how Domar's theorem leads to results about translation-invariant subspaces of these spaces.

3.3 The Beurling-Malliavin theorem

We will now see how the original theorem by Beurling and Malliavin in [3] leads to the one in Domar's article. We will need some new definitions.

Let $\mathfrak{M}$ be the set of measures with compact support on the real line. For $a > 0$ we denote by $\mathfrak{M}_a$ the set of measures with support contained in $[-a, a]$. We do not include the identically vanishing measure in these sets. The Fourier transform of a measure with support in K is

$$\hat{\mu}(z) = \int_K e^{-iz\zeta}\, \mathrm{d}\mu(\zeta).$$

$\hat{\mathfrak{M}}$ and $\hat{\mathfrak{M}}_a$ are the sets of Fourier transforms of the measures belonging to $\mathfrak{M}$ and $\mathfrak{M}_a$, respectively. For $\mu \in \mathfrak{M}_a$ we have

$$|\hat{\mu}(z)| = \left|\int_{-a}^{a} e^{-iz\zeta}\, \mathrm{d}\mu(\zeta)\right| \leq \left|\int_{-a}^{a} \mathrm{d}\mu(\zeta)\right| \sup_{\zeta\in[-a,a]} \left|e^{-iz\zeta}\right| \leq Ce^{a|y|}$$

so we see that $\hat{\mu}$ is an entire function of order 1 and type $\leq a$.

Let $w(x) \geq 1$ be a measurable function on $\mathbb{R}$. We let f be a measurable function and define the norm

$$||f||_p = \left(\int_{-\infty}^{\infty} |f(x)|^p w(x)^p \,\mathrm{d}x \right)^{1/p}$$

for $1 \leq p < \infty$ and

$$||f||_\infty = \operatorname*{ess\,sup}_{-\infty<x<\infty} |f(x)|w(x) = \inf\{a \in \mathbb{R} : |f(x)|w(x) \leq a \,\text{for a.e.}\, x\}$$

for $p = \infty$. We let L^p_w be the space of measurable functions that are finite in this norm.

Let W_p be the set of all weight functions $w(x) \geq 1$ that fulfil the following requirements:

- The translation operators $f(x) \to f(x+t)$ are bounded in L^p_w.
- For each $a > 0$, L^p_w contains elements of $\hat{\mathfrak{M}}_a$.

Then we have the following result:

Theorem 3.3.1. Beurling-Malliavin *The sets W_p, where $1 \leq p \leq \infty$, are independent of p and consists of all weight functions $w(x) \geq 1$ satisfying*

$$\operatorname*{ess\,sup}_{-\infty<x<\infty} |\log w(x+t) - \log w(x)| < \infty$$

and

$$\int_{-\infty}^{\infty} \frac{\log w(x)}{1+x^2} \,\mathrm{d}x < \infty.$$

The next theorem is stated in Domar's article.

Theorem 3.3.2. *The answer to question 3.2.1 is yes for every a if k, outside some compact interval, is absolutely continuous with bounded derivative, and*

$$\int_{-\infty}^{\infty} \frac{\min(0, k(x))}{1+x^2} \,\mathrm{d}x > -\infty.$$

We want to show that this theorem follows from Theorem 3.3.1. We define the weight

$$w(x) = \begin{cases} e^{-k(x)} & k(x) \leq 0 \\ 1 & k(x) > 0 \end{cases}$$

where k is defined like in theorem 3.3.2. If we assume that

$$\int_{-\infty}^{\infty} \frac{\min(0, k(x))}{1+x^2} \,\mathrm{d}x > -\infty,$$

we also have

$$\int_{-\infty}^{\infty} \frac{\max(0, -k(x))}{1+x^2} \, \mathrm{d}x < \infty,$$

and since $\log w(x) = \max(0, -k(x))$, we have

$$\int_{-\infty}^{\infty} \frac{\log w(x)}{1+x^2} \, \mathrm{d}x < \infty.$$

Now we need to show that the first assumption in Theorem 3.3.1 is fulfilled. Let K be the interval from the assumption of Theorem 3.3.2 and fix $t > 0$. Let $I = K + [-t, t]$. Then

$$\operatorname*{ess\,sup}_{x \in I} |\log w(x+t) - \log w(x)| < \infty$$

since k is assumed to be continuous in question 3.2.1. Since k is continuous and has bounded derivative outside K, we have

$$|k(x+t) - k(x)| \leq Ct$$

for $x \in \mathbb{R} \setminus K$ where $C = \sup_{\mathbb{R} \setminus K} |k'(x)|$. Then we have

$$\operatorname*{ess\,sup}_{-\infty < x < \infty} |\log w(x+t) - \log w(x)| < \infty.$$

Then, by Theorem 3.3.1, there is an entire function f in $\hat{\mathfrak{M}}_a \cap L^\infty_w$ such that

$$\operatorname*{ess\,sup}_{-\infty < x < \infty} |f(x)| w(x) \leq C.$$

Since f and k are continuous we have $|f(x)|w(x) = |f(x)|e^{-k(x)} \leq C$ for all $x \in \mathbb{R}$. Then $|f(x)| \leq Ce^{k(x)}$. We also know that $|f(iy)| \leq \tilde{C}e^{a|y|}$ since $f \in \hat{\mathfrak{M}}_a$. If we divide by $\max\{C, \tilde{C}\}$ we get a new function that fulfils $|f(iy)| \leq e^{a|y|}$ and $|f(x)| \leq e^{k(x)}$, so the requirements of question 3.2.1 are fulfilled.

3.4 Domar's theorem

Domar proves two theorems in his article. We will need the following one here. This is Theorem 2' in his article.

Theorem 3.4.1. *Let k be odd on $\mathbb{R}$, absolutely continuous in some interval $[b, \infty)$, and with its derivative equivalent to a function of bounded variation. Then the answer to question 3.2.1 is yes, for every a.*

This theorem shows that for an odd weight the conditions of Theorem 3.3.2 can be weakened. We will not give the proof of this result, but we will see how it is used for a construction of a translation-invariant subspace of a weighted l^p-space. Before we do that we will state some theorems that we will need later.

3.5 Necessary theorems

The first theorem is given as an exercise on page 130 in [11], and is a version of the Phragmén-Lindelöf theorem.

Theorem 3.5.1. Phragmén-Lindelöf theorem
Let S be a sector whose vertex is the origin, and forming an angle of π/β. Let F be a holomorphic function in S that is continuous in the closure of S, so that $|F(z)| \leq 1$ on the boundary of S and

$$|F(z)| \leq Ce^{c|z|^{\alpha}}$$

for all $z \in S$ and some $c, C > 0$ and $0 < \alpha < \beta$. Then $|F(z)| \leq 1$ for all $z \in S$.

The sector in this theorem can be rotated and the result will remain the same [11]. If we let $\alpha = 1$ and $\beta = 2$ we get the version we will need.

We use the following Fourier transform in the next theorems:

Definition 3.5.2. *If f is in $L^1(\mathbb{R})$, its Fourier transform is*

$$\hat{f}(w) = \int_{\mathbb{R}} e^{-2i\pi wx} f(x)\,\mathrm{d}x.$$

If f is in $L^2(\mathbb{R})$, we define its Fourier transform to be

$$\hat{f}(w) = \lim_{n\to\infty} \int_{-n}^{n} e^{-2i\pi wx} f(x)\,\mathrm{d}x.$$

The Paley-Wiener theorem is taken from page 122 in [11], and the Poisson summation formula is from page 345 in [7].

Theorem 3.5.3. Paley-Wiener theorem
Suppose f is continuous and of moderate decrease on $\mathbb{R}$, that is

$$|f(x)| \leq \frac{C}{1+x^2}$$

for some $C > 0$. Then f has an extension to the complex plane that is entire with

$$|f(z)| \leq Ae^{2\pi M|z|}$$

for some $A > 0$ if and only if $\hat{f}$ is supported in the interval $[-M, M]$.

Theorem 3.5.4. Poisson summation formula
Let f be a distribution with compact support. Then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where we may interpret the left-hand sum as a distribution where the specific values of the function do not appear:

$$\sum_{n=-\infty}^{\infty} \tau_n f$$

Here τ_n is the translation operator.

3.6 Weighted l^p-spaces

We will now define $l^p(w,\mathbb{Z})$, which is the weighted l^p-spaces.

Definition 3.6.1. *For $1 \le p < \infty$, $l^p(w,\mathbb{Z})$ is the Banach space of complex sequences $c = \{c_n\}_{n\in\mathbb{Z}}$ with*

$$||c||^p = \sum_{n=-\infty}^{\infty} |c_n|^p w_n^p < \infty.$$

The completeness of this space follows from the completeness of $\mathbb{C}$. The translation operator τ is defined by $\{c_n\} \mapsto \{c_{n-1}\}$. We have

$$\begin{aligned}
||\tau|| = \sup_{||\{c_n\}||=1} \left(\sum_{n=-\infty}^{\infty} |c_{n-1}|^p w_n^p \right)^{\frac{1}{p}} &= \sup_{||\{c_n\}||=1} \left(\sum_{n=-\infty}^{\infty} |c_{n-1}|^p w_n^p \frac{w_{n-1}^p}{w_{n-1}^p} \right)^{\frac{1}{p}} \\
= \sup_{||\{c_n\}||=1} \left(\sum_{n=-\infty}^{\infty} |c_{n-1}|^p w_{n-1}^p \frac{w_n^p}{w_{n-1}^p} \right)^{\frac{1}{p}} \\
\le \sup_{||\{c_n\}||=1} \underbrace{\left(\sum_{n=-\infty}^{\infty} |c_{n-1}|^p w_{n-1}^p \right)^{\frac{1}{p}}}_{1} \left(\sup_n \frac{w_n^p}{w_{n-1}^p} \right)^{\frac{1}{p}} \\
= \sup_n \frac{w_n}{w_{n-1}},
\end{aligned}$$

so it is bounded if $\sup_n \frac{w_n}{w_{n-1}}$ is bounded. We also see from this that τ is a well-defined operator from $l^p(w,\mathbb{Z})$ to itself if $\sup_n \frac{w_n}{w_{n-1}}$ is bounded, since $||\tau(c)|| \le ||\tau|| ||c|| < \infty$. Similarly, its inverse defined by $\{c_n\} \mapsto \{c_{n+1}\}$ is well-defined and bounded if $\sup_n \frac{w_n}{w_{n+1}}$ is bounded.

Definition 3.6.2. *Let $T : V \to V$ be a linear mapping from some vector space V to itself. A subspace W of V is an invariant subspace if $T(W)$ is contained in W.*

Description of invariant subspaces is a famous topic in analysis. In particular it is interesting if the given translation operator in the Banach space has a nontrivial closed subspace. In the end of his article Domar uses Theorem 3.4.1 to show that there exists a nontrivial translation-invariant subspace of $l^p(w, \mathbb{Z})$, where w satisfies some conditions. We will write the details of this last part and see how this relates to the product of hyperfunctions.

3.7 Translation-invariant subspaces of $l^p(w, \mathbb{Z})$

We will now look at the details of the last page of Domar's article. What we will show is the following:

Fact 3.7.1. *Let $\{k(n)\}_{n\in\mathbb{Z}}$ be a real odd sequence such that $\{k(n+1) - k(n)\}_{n\in\mathbb{Z}}$ is bounded and*

$$\sum_{n=-\infty}^{\infty} |k(n+1) - 2k(n) + k(n-1)| < \infty.$$

We consider the Banach space $l^p(w, \mathbb{Z})$ with $w_n = e^{k(n)}$ and the translation operator $\tau : l^p(w, \mathbb{Z}) \to l^p(w, \mathbb{Z})$, which is well-defined and bounded since $\sup_n \frac{w_n}{w_{n-1}} = \sup_n e^{k(n)-k(n-1)} < +\infty$. Then there exists a nontrivial closed translation-invariant subspace of $l^p(w, \mathbb{Z})$.

By proving this we come across another fact which is of more interest to proving formula 1.1:

Fact 3.7.2. *If f is an entire function such that*

$$|f(x)| \leq \frac{e^{k(x)}}{1+x^2}, \qquad x \in \mathbb{R},$$

$$|f(iy)| \leq e^{a|y|}, \qquad y \in \mathbb{R},$$

with $a < \frac{1}{2}\pi$, then

$$\sum_{n\in\mathbb{Z}} f(n)(-1)^n f(m-n) = 0.$$

3.7.1 Proof of fact 3.7.1 and fact 3.7.2

We extend k to an odd continuous function on $\mathbb{R}$ by letting it be linear in each interval in $\mathbb{R} \setminus \mathbb{Z}$. Then k satisfies the conditions of Theorem 3.4.1. It is absolutely continuous since $\{k(n+1) - k(n)\}_{n\in\mathbb{Z}}$ is bounded. Its derivative is piecewice constant and has bounded variation since the sum of the jumps is the sum of $|k(n+1) - 2k(n) + k(n-1)|$, which is finite. Let $a \in (0, \frac{1}{2}\pi)$. Then by Theorem 3.4.1 there exists an entire function of order ≤ 1, such that $\log|f(x)| \leq k(x)$ for $x \in \mathbb{R}$ and $\log|f(iy)| \leq a|y|$ for $y \in \mathbb{R}$.

It follows from the proofs of the theorems in Domar's article that after dividing away some zeros if necessary, we may assume

$$|f(x)| \leq \frac{e^{k(x)}}{1+x^2} \tag{3.1}$$

for $x \in \mathbb{R}$.

For $m \in \mathbb{Z}$ we define

$$g_m(z) = f(z)f(m-z).$$

g_m is an entire function and $|g_m(x)| \leq (1+x^2)^{-1}(1+(m-x)^2)^{-1}e^{k(x)+k(m-x)}$ for $x \in \mathbb{R}$. We know that $f(iy) \leq e^{a|y|}$. It can be shown with a method similar to what we use below for g_m, that $f(m-iy) \leq K_m e^{a|y|}$ for some $K_m > 0$. Then we have $g_m(iy) \leq K_m e^{2a|y|}$ for $y \in \mathbb{R}$. We now want to show that $|g_m(z)| \leq C_m e^{2a|y|}$ for all $z \in \mathbb{C}$. We have $k(x) + k(m-x) = k(x) + k(-x) + k(1-x) - k(-x) + k(2-x) - k(1-x) + ... + k(m-x) - k(m-1-x) \leq B_m$ for some constant B_m since $k(x) = -k(-x)$ and $\{k(n+1) - k(n)\}_{n \in \mathbb{C}}$ is bounded. We consider the function $h(z) = g_m(z)e^{2aiz}$ in the first quadrant. h is analytic and bounded on the real ray by $A_m = e^{B_m}$ since $|h(x)| = |g_m(x)e^{2aix}| = |g_m(x)| \leq e^{B_m} = A_m$. It is also bounded on the imaginary ray by K_m since $|h(iy)| = |g_m(iy)||e^{-2ay}| \leq K_m e^{2a|y|}|e^{-2ay}| = K_m$. h is of order 1 and bounded type since both g_m and e^{2aiz} are of order 1 and bounded type. Then by the Phragmén-Lindelöf theorem h is bounded by $C_m = \max\{K_m, A_m\}$ in the first quadrant. Then $|g_m(z)| \leq C_m|e^{-2aiz}| = C_m e^{2ay} = C_m e^{2a|y|}$.

We may use the same h to show that $|g_m(z)| \leq C_m e^{2a|y|}$ in the second quadrant. In the third and fourth quadrants we use $h(z) = g_m(z)e^{-2aiz}$. Then we get $|h(iy)| = |g_m(iy)||e^{2ay}| \leq K_m e^{2a|y|}|e^{2ay}| = K_m$ on the imaginary ray since y is negative. Then $|g_m(z)| \leq C_m|e^{2aiz}| = C_m e^{-2ay} = C_m e^{2a|y|}$ in the lower half-plane as well, and then $|g_m(z)| \leq C_m e^{2a|y|}$ for all $z \in \mathbb{C}$. This means that g_m is of exponential type $\leq 2a$. We also have $|g_m(x)| \leq C_m(1+x^2)^{-1}(1+(m-x)^2)^{-1}$ by (3.1). Then by the Paley-Wiener theorem the Fourier transform of g_m has support in $[-\frac{a}{\pi}, \frac{a}{\pi}] \subset (-\frac{1}{2}, \frac{1}{2})$ since $a \in (0, \frac{1}{2}\pi)$.

We will apply the Poisson summation formula to the function

$$x \mapsto g_m(x)e^{\pi ix}.$$

The Fourier transform of $g_m(x)e^{\pi ix}$ is

$$\int_{\mathbb{R}} e^{-2i\pi wx} g_m(x)e^{\pi ix}\,\mathrm{d}x = \int_{\mathbb{R}} e^{-2i\pi x(w-\frac{1}{2})} g_m(x)\,\mathrm{d}x = \hat{g}_m(w - \frac{1}{2}).$$

Then by the Poisson summation formula we get

$$\sum_{n=-\infty}^{\infty} \hat{g}_m(n-\frac{1}{2}) = \sum_{n=-\infty}^{\infty} g_m(n)e^{\pi i n},$$

and since the support of $\hat{g}_m$ is in $(-\frac{1}{2}, \frac{1}{2})$, this sum must be equal to 0. Since $g_m(n)e^{\pi in} = f(n)(-1)^n f(m-n)$, we get

$$\sum_{n\in\mathbb{Z}} f(n)(-1)^n f(m-n) = 0.$$

For every p, $\{f(m-n)\}_{n\in\mathbb{Z}}$ is in $l^p(w,\mathbb{Z})$, since by (3.1) we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |f(m-n)|^p w_n^p &= \sum_{n\in\mathbb{Z}} |f(m-n)|^p e^{k(n)p} \\ &\le \sum_{n\in\mathbb{Z}} \frac{1}{(1+(m-n)^2)^p} e^{k(m-n)p} e^{k(n)p} \\ &\le \sum_{n\in\mathbb{Z}} \frac{1}{(1+(m-n)^2)^p} e^{B_m p} < \infty.\end{aligned}$$

We also have that $\{(-1)^n f(n)\}_{n\in\mathbb{Z}}$ belongs to the dual space of $l^p(w,\mathbb{Z})$: Since $|(-1)^n f(n)| \le \frac{1}{1+n^2} e^{k(n)}$ we have by Hölder's inequality [6, page 182]

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}} c_n(-1)^n f(n)\right| &\le \sum_{n\in\mathbb{Z}} |c_n||(-1)^n f(n)| \\ &\le \left(\sum_{n\in\mathbb{Z}} |c_n|^p w^p\right)^{\frac{1}{p}} \left(\sum_{n\in\mathbb{Z}} \frac{|(-1)^n f(n)|^q}{w^q}\right)^{\frac{1}{q}} \\ &\le ||c||_{p,w} \left(\sum_{n\in\mathbb{Z}} \left(\frac{e^{k(n)}}{(1+n^2)e^{k(n)}}\right)^q\right)^{\frac{1}{q}} \\ &= ||c||_{p,w} \left(\sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^q}\right)^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then the norm of $\{(-1)^n f(n)\}_{n\in\mathbb{Z}}$ as a functional is

$$\sup_{c\in l^p(w,\mathbb{Z})} \frac{\left|\sum_{n\in\mathbb{Z}} c_n(-1)^n f(n)\right|}{||c||_{p,w}} \le \left(\sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^q}\right)^{\frac{1}{q}} < \infty$$

so $\{(-1)^n f(n)\}_{n\in\mathbb{Z}}$ belongs to the dual space.

Let A be the closed linear span of the translates of $\{f(-n)\}$. We claim that this is a nontrivial closed translation-invariant subspace of $l^p(w, \mathbb{Z})$. Since we may choose f such that it does not vanish identically at the integers, the subspace is not equal to $\{0\}$. Since $\sum_{n \in \mathbb{Z}} f(n)(-1)^n f(m-n) = 0$ for all $m \in \mathbb{Z}$, A is contained in the kernel of the functional $\{(-1)^n f(n)\}_{n \in \mathbb{Z}}$. This is a nonzero functional, so A is properly contained in $l^p(w, \mathbb{Z})$. The space is translation-invariant by the way it is defined. Then the closed linear span of the translates of $\{f(-n)\}$ is a nontrivial closed translation-invariant subspace. □

Chapter 4

Product of hyperfunctions

I showed in my project that it is possible to define the product of an analytic function and a hyperfunction and that the product of two hyperfunctions is not well defined in general. We also saw that the Fourier coefficients of the product of a hyperfunction and a function $h \in \mathscr{H}(\mathbb{T})$ are given by

$$\widehat{hf}(m) = \sum_n \hat{h}(n)\hat{f}(m-n).$$

We want to show that a similar formula is valid for two hyperfunctions with disjoint support. We want to prove the following theorem:

Theorem 4.0.3. *Let $T_1, T_2 \in \mathcal{B}(\mathbb{T})$. If $\operatorname{supp} T_1 \cap \operatorname{supp} T_2 = \emptyset$, then*

$$\lim_{r \to 1^-} \sum_{n \in \mathbb{Z}} r^{|n|} c_n(T_1) c_{m-n}(T_2) = 0 \qquad (m \in \mathbb{Z}).$$

If, further, $\lim_{n\to\infty} c_n(T_1)c_{m-n}(T_2) + c_{-n}(T_1)c_{m+n}(T_2) = 0$, then

$$\lim_{p \to \infty} \sum_{|n| \le p} c_n(T_1) c_{m-n}(T_2) = 0.$$

This is proved in [5], but here we will use another method. We first need to define the Fourier-Borel transform and the G-transform. The next two sections are taken from chapter 1.3 and 4.1 in [2].

4.1 The Fourier-Borel transform

Definition 4.1.1. *Let Ω be a subset of $\mathbb{C}$. The Fourier-Borel transform of an analytic functional $T \in \mathscr{H}'(\Omega)$ is the function*

$$\mathfrak{F}(T)(z) = \langle T_\zeta, e^{\zeta z} \rangle.$$

We will state two theorems that we will need later.

Proposition 4.1.2. *Let Ω be a convex set and let $S \in \mathscr{H}'(\Omega)$ have as convex support the set K, then for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that*

$$|\mathfrak{F}(S)(z)| \leq C_\epsilon e^{H_K(z)+\epsilon|z|} \qquad (z \in \mathbb{C}),$$

where $H_K(z) = \sup_{\zeta \in K} \operatorname{Re}(z\zeta)$. In particular, $\mathfrak{F}(S)$ is an entire function of exponential type.

Theorem 4.1.3. *Let K be a compact convex subset of a convex open set Ω. Let f be an entire function of exponential type such that for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ so that f satisfies everywhere the estimate*

$$|f(z)| \leq C_\epsilon e^{H_K(z)+\epsilon|z|}.$$

Then there is a unique analytic functional $S \in \mathscr{H}'(\Omega)$ such that $\mathfrak{F}(S) = f$.

4.2 The G-transform

Let K be a convex compact subset of the strip $\Omega := \{z \in \mathbb{C} : |Imz| < \pi\}$. Ω can also be moved up or down, as long as its width is less than 2π. We define $e^{-K} := \{e^{-z} : z \in K\}$ and $\Omega(K) = \mathbb{C} \setminus e^{-K}$. Then let $\mathscr{H}_0(\Omega(K))$ denote the space of those holomorphic functions in $\Omega(K)$ which vanish at ∞. If T is an analytic functional carried by such a compact set K we define its G-transform to be

$$G(T)(z) := \langle T_\zeta, \frac{1}{1 - ze^\zeta} \rangle.$$

The following proposition expresses a relation between the Fourier-Borel transform and the G-transform.

Proposition 4.2.1. *If T is an analytic functional carried by the compact convex set $K \subset \Omega$, then $G(T)$ belongs to $\mathscr{H}_0(\Omega(K))$. Moreover, its Taylor series development about $z = 0$ is given by*

$$G(T)(z) = \sum_{n \geq 0} \mathfrak{F}(T)(n) z^n,$$

and its Laurent development about $z = \infty$ is

$$G(T)(z) = -\sum_{n>0} \frac{\mathfrak{F}(T)(-n)}{z^n}.$$

We will also need the following fact.

Proposition 4.2.2. *Let K be a compact convex subset of Ω, then the map $G : \mathscr{H}'(K) \to \mathscr{H}_0(\Omega(K))$, given by $T \mapsto G(T)$, is bijective.*

4.3 A necessary theorem

The following theorem is stated as an example following the Mittag-Leffler theorem on page 225 in [1].

Theorem 4.3.1. *Let* Ω_1 *and* Ω_2 *be two open subsets of* $\mathbb{C}$*, and let* $f \in \mathscr{H}(\Omega_1 \cap \Omega_2)$*. Then there are* $f_1 \in \mathscr{H}(\Omega_1)$ *and* $f_2 \in \mathscr{H}(\Omega_2)$ *such that*

$$f = (f_1|(\Omega_1 \cap \Omega_2)) - (f_2|(\Omega_1 \cap \Omega_2)).$$

4.4 Proof of Theorem 4.0.3, first statement

We will first prove a lemma that we will need in the proof of Theorem 4.0.3.

4.4.1 A lemma

Lemma 4.4.1. *Let* T *be an analytic functional on* $\mathbb{T}$ *with support contained in an arc* $U = \{e^{i\theta} : \theta \in [\alpha_1, \alpha_2]\}$ *which is not all of* $\mathbb{T}$*. Then there exists an analytic functional* S *on* $K = [-i\alpha_2, -i\alpha_1]$ *such that* $\mathfrak{F}(S)(n) = c_n(T)$ *for* $n \in \mathbb{Z}$ *and for every* $\epsilon > 0$ *there is a constant* $C_\epsilon > 0$ *such that*

$$\mathfrak{F}(S)(z) \leq C_\epsilon e^{\alpha_2 y + \epsilon |z|} \qquad \operatorname{Im} z \geq 0,$$

$$\mathfrak{F}(S)(z) \leq C_\epsilon e^{\alpha_1 y + \epsilon |z|} \qquad \operatorname{Im} z < 0.$$

Conversely, if F *is an entire function such that for every* $\epsilon > 0$ *there is a constant* $C_\epsilon > 0$ *such that*

$$F(z) \leq C_\epsilon e^{\alpha_2 y + \epsilon |z|} \qquad \operatorname{Im} z \geq 0,$$

$$F(z) \leq C_\epsilon e^{\alpha_1 y + \epsilon |z|} \qquad \operatorname{Im} z < 0,$$

where $\alpha_2 - \alpha_1 < 2\pi$*, then there exists an analytic functional* T *on* $\mathbb{T}$ *with support contained in* $[e^{i\alpha_1}, e^{i\alpha_2}]$ *such that* $c_n(T) = F(n)$*.*

Proof. Let T be an analytic functional on $\mathbb{T}$ with support contained in an arc $U = \{e^{i\theta} : \theta \in [\alpha_1, \alpha_2]\}$. The corresponding hyperfunction under the bijection H is $[\tilde{T}_1, -\tilde{T}_2]$, and since $\operatorname{supp} T \subset U$ we know that $\tilde{T}_1$ and $-\tilde{T}_2$ can be extended across $\mathbb{T} \setminus U$ to a function $\tilde{T}$ in $\mathscr{H}_0(\mathbb{C} \setminus U)$. If we let $\Omega(K) = \mathbb{C} \setminus U$, then $U = e^{-K}$ and $K = [-i\alpha_2, -i\alpha_1]$. From Proposition 4.2.2 we know that the map $G : \mathscr{H}'(K) \to \mathscr{H}_0(\Omega(K))$ given by $S \mapsto G(S)$ is bijective. Since $\tilde{T}$ is in $\mathscr{H}_0(\mathbb{C} \setminus U)$ there must exist an analytic functional S on $K = [-i\alpha_2, -i\alpha_1]$ such that $G(S) = \tilde{T}$.

We then take the Fourier-Borel transform of S. From Proposition 4.2.1 we see that $G(S)(z) = \sum_{n\geq 0} \mathfrak{F}(S)(n) z^n$ for $|z| < 1$ and $G(S)(z) = -\sum_{n>0} \frac{\mathfrak{F}(S)(-n)}{z^n}$ for $|z| > 1$. We also have from Proposition 2.4.1 that $\tilde{T}_1(z) = \sum_{n\geq 0} c_n(T) z^n$ and $\tilde{T}_2(z) = \sum_{n>0} c_{-n}(T) z^{-n}$. Since $G(S) = \tilde{T}_1$ for $|z| < 1$ and $G(S) = -\tilde{T}_2$ for $|z| > 1$, we see by comparing the expressions above that $\mathfrak{F}(S)(n) = c_n(T)$ for $n \in \mathbb{Z}$.

Now we will use Proposition 4.1.2. Since $K = [-i\alpha_2, -i\alpha_1]$ and $\mathrm{Re}((x + iy)i\alpha) = -y\alpha$ we get

$$H_K(z) = \begin{cases} \alpha_2 y & y = \operatorname{Im} z \geq 0 \\ \alpha_1 y & y = \operatorname{Im} z < 0 \end{cases}$$

Then

$$\mathfrak{F}(S)(z) \leq C_\epsilon e^{\alpha_2 y + \epsilon|z|} \qquad \operatorname{Im} z \geq 0$$

and

$$\mathfrak{F}(S)(z) \leq C_\epsilon e^{\alpha_1 y + \epsilon|z|} \qquad \operatorname{Im} z < 0.$$

Conversely, let F be an entire function such that for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$F(z) \leq C_\epsilon e^{\alpha_2 y + \epsilon|z|} \qquad \operatorname{Im} z \geq 0,$$

$$F(z) \leq C_\epsilon e^{\alpha_1 y + \epsilon|z|} \qquad \operatorname{Im} z < 0.$$

By Proposition 4.1.3 there is a unique analytic functional S such that $\mathfrak{F}(S) = F$, and the support of S is contained in $[-i\alpha_2, -i\alpha_1]$. By using the G-transform on S we get a function in $\mathscr{H}(\mathbb{C} \setminus [e^{i\alpha_1}, e^{i\alpha_2}])$. This function can be considered as a hyperfunction or analytic functional T on $\mathbb{T}$. Then $\operatorname{supp} T \subset [e^{i\alpha_1}, e^{i\alpha_2}]$. We have $\tilde{T}_1 = \sum_{n\geq 0} c_n(T) z^n$ and $\tilde{T}_2 = \sum_{n<0} c_n(T) z^n$. We also have $G(S)(z) = \sum_{n\geq 0} \mathfrak{F}(S)(n) z^n$ for $|z| < 1$ and $G(S)(z) = -\sum_{n>0} \frac{\mathfrak{F}(S)(-n)}{z^n}$ for $|z| > 1$. Then $c_n(T) = F(n)$. □

4.4.2 Support contained in two disjoint arcs

We consider first the case where two analytic functionals T_1 and T_2 have support contained in two disjoint arcs $\{e^{i\theta} : \theta \in [\alpha_1, \alpha_2]\}$ and $\{e^{i\theta} : \theta \in [\beta_1, \beta_2]\}$ on the unit circle. The angles are chosen so that $\alpha_1 > \beta_2$ and $\alpha_2 < 2\pi + \beta_1$, see figure 4.1.

We define a new functional $\check{T}_2$ by $\check{T}_2(e^{i\theta}) = T_2(e^{i(\pi+\theta)})$ or equivalently $\check{T}_2(\zeta) = T_2(-\zeta)$. Then

$$c_n(\check{T}_2) = \langle \check{T}_2, \zeta^{-n-1} \rangle = \langle T_2, (-\zeta)^{-n-1} \rangle = (-1)^{-n-1} c_n(T_2).$$

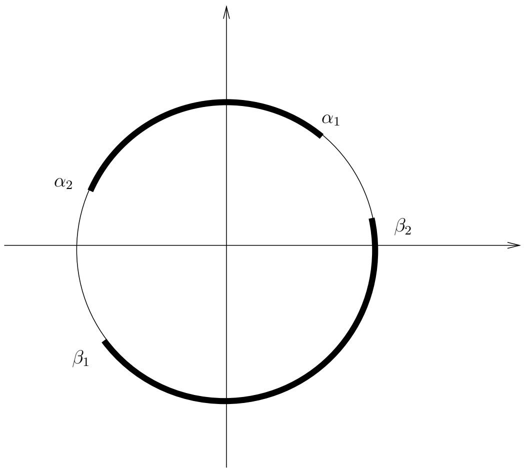

Figure 4.1: Hyperfunctions with support in disjoint arcs

Using Lemma 4.4.1 we find the analytic functionals S_1 and S_2 corresponding to T_1 and $\check{T}_2$, respectively. Let $F_1 = \mathfrak{F}(S_1)$ and $F_2 = \mathfrak{F}(S_2)$. By the same lemma we have

$$F_1(z) \leq C_\epsilon e^{\alpha_2 y + \epsilon|z|} \qquad \operatorname{Im} z \geq 0$$

and

$$F_1(z) \leq C_\epsilon e^{\alpha_1 y + \epsilon|z|} \qquad \operatorname{Im} z < 0.$$

Similarly, for F_2 we get

$$F_2(z) \leq C_\epsilon e^{(\pi+\beta_2) y + \epsilon|z|} \qquad \operatorname{Im} z \geq 0$$

and

$$F_2(z) \leq C_\epsilon e^{(\pi+\beta_1) y + \epsilon|z|} \qquad \operatorname{Im} z < 0.$$

Then we define $G_m(z) = F_1(z)F_2(m-z)$ where $m \in \mathbb{Z}$. G_m is an entire function and $G_m(n) = c_n(T_1)c_{m-n}(\check{T}_2) = c_n(T_1)c_{m-n}(T_2)(-1)^{-m+n-1}$. By combining the above expressions we get

$$|G_m(z)| \leq B_\epsilon e^{\alpha_2 y - (\pi+\beta_1)y + \epsilon|z|} \qquad \operatorname{Im} z \geq 0$$

and

$$|G_m(z)| \leq B_\epsilon e^{\alpha_1 y - (\pi+\beta_2)y + \epsilon|z|} \qquad \operatorname{Im} z < 0.$$

We want to make

$$|G_m(z)| \leq B_\epsilon e^{A|y| + \epsilon|z|}$$

with $A < \pi$, then we need $\alpha_2 - \pi + \beta_1 < \pi$ and $\alpha_1 - \pi + \beta_2 > -\pi$. That means $\alpha_2 - \beta_1 < 2\pi$ and $\alpha_1 > \beta_2$. This is fulfilled because of the way we chose the angles. Then by Lemma 4.4.1 there exists an analytic functional

Γ with support in $[e^{-iA}, e^{iA}]$ and $c_n(\Gamma) = G_m(n)$. Since $\operatorname{supp}\Gamma \subset [e^{-iA}, e^{iA}]$ we see that the limit as z approach -1 from both sides must be equal.

$$\lim_{x\to -1^+} \sum_{n\geq 0} c_n(\Gamma)x^n = \lim_{x\to -1^-} -\sum_{n<0} c_n(\Gamma)x^n$$

Then

$$\lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} c_n(\Gamma)r^{|n|}(-1)^n = 0$$

and since $c_n(\Gamma) = c_n(T_1)c_{m-n}(T_2)(-1)^{-m+n-1}$ we see that

$$\lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} r^{|n|}c_n(T_1)c_{m-n}(T_2)(-1)^{-m+n-1}(-1)^n = 0 \qquad (m \in \mathbb{Z}),$$

and then

$$\lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} r^{|n|}c_n(T_1)c_{m-n}(T_2) = 0 \qquad (m \in \mathbb{Z}).$$

4.4.3 The general case

Now we need to prove this without assuming that the support of T_1 and T_2 are contained in disjoint arcs. We still let the support of T_1 and T_2 be disjoint, but their support can now be anywhere on the circle.

We will show that T_1 and T_2 can be written as a sum of finitely many hyperfunctions T_{1i} and T_{2j}, each with support contained in an arc. Then we may repeat the above proof for all combinations of T_{1i} and T_{2j}.

Let $K_1 = \operatorname{supp} T_1$ and $K_2 = \operatorname{supp} T_2$. Since K_1 and K_2 are compact and disjoint we will show that there exist finite families of intervals $\{I_i\}$ and $\{J_j\}$ such that $K_1 \subset \cup_{i=1}^{n} I_i$ and $K_2 \subset \cup_{j=1}^{m} J_j$ and $I_i \cap J_j = \emptyset$ for all i and j. For each point of K_1 there is an interval with centre at that point that does not intersect K_2; take the interval with the same centre and half the length. These intervals will cover K_1, and similarly for K_2. Let $\{I_i\}$ and $\{J_j\}$ be these covers. We will see that $I_i \cap J_j = \emptyset$ for all i and j. We assume this is not true, so $I_i \cap J_j \neq \emptyset$ for some i and j. Assume J_j is the longest of the two intervals, and let x be the centre of I_i. If we double the length of J_j, we have that x will be in this interval. This is also true if I_i and J_j have equal length. Since $x \in K_1$, we have a contradiction because of the way we chose I_i and J_j. Then these covers will be disjoint. There exist finite subcovers by definition of compactness.

Let $f \in \mathscr{H}(\mathbb{C} \setminus \mathbb{T})$ be a function representing T_1. Let K_{11} be the part of $\operatorname{supp} T_1$ contained in I_1 and let $U_1 = \mathbb{C} \setminus K_{11}$. We also let $V_1 = (\mathbb{C} \setminus \operatorname{supp} T_1) \cup I_1$. We have that f is holomorphic in $U_1 \cap V_1 = \mathbb{C} \setminus \operatorname{supp} T_1$. By Theorem

4.3.1 there exist $f_1 \in \mathscr{H}(U_1)$ and $g_1 \in \mathscr{H}(V_1)$ such that $f = f_1 + g_1$ in $U_1 \cap V_1$. Then f_1 represents a hyperfunction T_{11} with support in K_{11} and g_1 represents a hyperfunction with support contained in $\mathbb{T} \setminus I_1$.

Now we define $U_2 = \mathbb{C} \setminus K_{12}$ and $V_2 = (\mathbb{C} \setminus \operatorname{supp} T_1) \cup I_1 \cup I_2$. Since g_1 is holomorphic in $U_2 \cap V_2 = (\mathbb{C} \setminus \operatorname{supp} T_1) \cup I_1$ we may use Theorem 4.3.1 again, and therefore there exist $f_2 \in \mathscr{H}(U_2)$ and $g_2 \in \mathscr{H}(V_2)$ such that $g_1 = f_2 + g_2$. Then $f = f_1 + f_2 + g_2$ where f_2 represents a hyperfunction T_{12} with support contained in I_2 and g_2 represents a hyperfunction with support contained in $\mathbb{T} \setminus (I_1 \cup I_2)$.

In the general case we have $U_i = \mathbb{C} \setminus K_{1i}$, $V_i = (\mathbb{C} \setminus \operatorname{supp} T_1) \cup_{k=1}^{i} I_k$ and $U_i \cap V_i = (\mathbb{C} \setminus \operatorname{supp} T_1) \cup_{k=1}^{i-1} I_k$. We continue like this until we have $f = f_1 + f_2 + ... + f_n + g_n$ where $g_n \in \mathscr{H}(\mathbb{T})$. Then we may write $T_1 = \sum_{i=1}^{n} T_{1i}$, and similarly $T_2 = \sum_{j=1}^{m} T_{2j}$.

Then we repeat the proof in section 4.4.2 for all combinations of T_{1i} and T_{2j}, and we are done.

4.5 Proof of Theorem 4.0.3, second statement

To prove the second part of theorem 4.0.3 we need a definition and a theorem from [10, Vol. I, page 382 and 404].

Definition 4.5.1. *Let $f(z)$ be analytic in a disc K with boundary L. The regular points of $f(z)$ are the points $\zeta \in L$ for which there can be found a neighbourhood $N(\zeta)$ and an analytic function $\varphi_\zeta(z)$ defined on $N(\zeta)$ such that $\varphi_\zeta(z) = f(z)$ for all z in $N(\zeta) \cap K$.*

Theorem 4.5.2. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1, such that

$$\lim_{n \to \infty} a_n = 0.$$

Then the series converges (in fact, uniformly) on every arc

$$z = e^{i\theta} (\alpha \leq \theta \leq \beta),$$

if all the points of the arc are regular points of $f(z)$.

We let $a_n = c_n(T_1)c_{m-n}(T_2) + c_{-n}(T_1)c_{m+n}(T_2)$. Since $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ extend eachother across $\mathbb{T} \setminus [e^{-iA}, e^{iA}]$ we see that -1 is a regular point.

Then the last part of theorem 4.0.3 follows: If $\lim_{n\to\infty} c_n(T_1)c_{m-n}(T_2) + c_{-n}(T_1)c_{m+n}(T_2) = 0$, then

$$\lim_{p\to\infty} \sum_{|n|\le p} c_n(T_1)c_{m-n}(T_2) = 0.$$

□

4.6 Comparison of the proofs

We will compare the proof of Theorem 4.0.3 to the proof of the existence of a nontrivial translation-invariant subspace of $l^p(w, \mathbb{Z})$.

We will see now that fact 3.7.2 can be turned into a statement about hyperfunctions. Let f be an entire function of order ≤ 1 that fulfils

$$\begin{aligned} |f(x)| &\le \tfrac{e^{k(x)}}{1+x^2}, & x \in \mathbb{R}, \\ |f(iy)| &\le e^{a|y|}, & y \in \mathbb{R}, \end{aligned}$$

with $a < \frac{1}{2}\pi$. Assume also that for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$f(z) \le C_\epsilon e^{ay+\epsilon|z|} \qquad \operatorname{Im} z \ge 0,$$

$$f(z) \le C_\epsilon e^{-ay+\epsilon|z|} \qquad \operatorname{Im} z < 0.$$

Then f corresponds to a hyperfunction T by Lemma 4.4.1 with support contained in $[e^{-ia}, e^{ia}]$ and we have $c_n(T) = f(n)$. If we rotate this hyperfunction by an angle π, like we did in subsection 4.4.2, its Fourier coefficients will be $(-1)^{-n-1}f(n)$, or equivalently $-(-1)^n f(n)$. We call this hyperfunction $\check{T}$. T and $\check{T}$ will have disjoint support since $a < \frac{1}{2}\pi$. Then we may interpret the sum

$$\sum_{n\in\mathbb{Z}} f(n)(-1)^n f(m-n) = 0$$

as the fact that the convolution of the Fourier coefficients of T with the Fourier coefficients of $\check{T}$, which is the same hyperfunction rotated by an angle π, is zero.

The Paley-Wiener theorem in Domar's proof corresponds to Theorem 4.1.2 and 4.1.3 in our proof. If K is a compact subset of $\mathbb{R}$ and we replace z by $-iz$ in the Fourier-Borel transform it will become the Fourier transform of an analytic functional on the real line:

$$FT(z) = \mathfrak{F}(T)(-iz) = \langle T_\zeta, e^{-iz\zeta} \rangle.$$

If we replace z by $-iz$ in Proposition 4.1.2 it becomes

$$|FT(z)| = |\mathfrak{F}(T)(-iz)| \leq C_\epsilon e^{H_K(-iz)+\epsilon|-iz|} = C_\epsilon e^{H_K(-iz)+\epsilon|z|}$$

and since $H_K(-iz) = \sup_{\zeta\in K} \operatorname{Re}(-iz\zeta) = \sup_{\zeta\in K} \operatorname{Re}(-i(x+iy)\zeta) = \sup_{\zeta\in K} y\zeta$, we get

$$|FT(z)| \leq C_\epsilon e^{\sup_{\zeta\in K}(y\zeta)+\epsilon|z|}.$$

We can do the same in Theorem 4.1.3. These theorems then give a correspondence between analytic functionals with compact support and entire functions that fulfil $|f(z)| \leq C_\epsilon e^{\sup_{\zeta\in K}(y\zeta)+\epsilon|z|}$ in the same way that the Paley-Wiener theorem gives a correspondence between distributions with compact support and entire functions that fulfil $|f(z)| \leq Ae^{2\pi M|z|}$ and $|f(x)| \leq \frac{C}{1+x^2}$.

In both proofs we have entire functions, g_m in Domar's proof and G_m in our proof. We know that $|g_m(z)| \leq C_m e^{2a|y|}$ and $|g_m(x)| \leq C_m \frac{1}{(1+x^2)^2}$. In our proof we have $|G_m(z)| \leq B_\epsilon e^{A|y|+\epsilon|z|}$ where $A < \pi$, but we do not have enough restrictions on the growth on $\mathbb{R}$ to be able to use the Paley-Wiener theorem. We use Lemma 4.4.1 instead.

The Poisson summation formula is what gives us

$$\sum_{n\in\mathbb{Z}} f(n)(-1)^n f(m-n) = 0$$

in Domar's article. In our proof we use Lemma 4.4.1 and then consider the hyperfunction in the point -1. This point is outside the support of the hyperfunction, so the limit as you approach -1 from both sides must be equal. Then we get

$$\lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} r^{|n|} c_n(T_1) c_{m-n}(T_2) = 0$$

instead. Our proof gives a meaning to the factor $r^{|n|}$.

Chapter 5

Hyperfunctions on $\mathbb{R}$

5.1 Introduction

We want to show that a formula similar to the one in Theorem 4.0.3 is valid for hyperfunctions on $\mathbb{R}$. Instead of a sum over the Fourier coefficients we will here get a convolution of the Fourier transforms. Our first attempt to prove this used parts of section 5.2, but it did not succeed. We have kept the section for its own interest. In section 5.3 we prove a formula for the convolution of Fourier transforms using similar methods as on $\mathbb{T}$.

First we need to define what we mean by a hyperfunction on $\mathbb{R}$.

Let Ω be an open subset of $\mathbb{R}$. We let $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ and $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Im} z < 0\}$. We also let $\mathbb{C}^\sharp = \mathbb{C} \setminus \mathbb{R}$ and $\tilde{\Omega} = \mathbb{C}^\sharp \cup \Omega$. We define $\mathcal{U}_\Omega$ to be the family of open sets V in $\mathbb{C}$ such that $V \cap \mathbb{R} = \Omega$. If V_1 and V_2 are open sets in $\mathbb{C}$ and $V_1 \subseteq V_2$, then $R_{V_1,V_2} : \mathscr{H}(V_2) \to \mathscr{H}(V_1)$ is the restriction map.

Definition 5.1.1. *The space of hyperfunctions is the complex vector space defined as the quotient*

$$\mathcal{B}(\Omega) := \mathscr{H}(\mathbb{C}^\sharp)/R_{\mathbb{C}^\sharp,\tilde{\Omega}}(\mathscr{H}(\tilde{\Omega}))$$

A hyperfunction in Ω is an element of $\mathcal{B}(\Omega)$.

This means that a hyperfunction is represented by pairs of holomorphic functions (f^+, f^-) in the upper and lower halfplanes. The next proposition shows that a hyperfunction can also be represented by a pair of holomorphic functions (f^+, f^-) in $V^+ = V \cap \mathbb{C}^+$ and $V^- = V \cap \mathbb{C}^-$, respectively.

Proposition 5.1.2. *For every $V \in \mathcal{U}_\Omega$, the natural map*

$$\mathcal{B}(\Omega) \xrightarrow{i_V} \mathscr{H}(V \setminus \Omega)/R_{V\setminus\Omega,V}(\mathscr{H}(V))$$

which assigns to the class $T \in \mathcal{B}(\Omega)$ of the function $f \in \mathscr{H}(\mathbb{C}^\sharp)$, the class T_1 of $f|(V \cap \mathbb{C}^\sharp)$ in the quotient space $\mathscr{H}(V \setminus \Omega)/R_{V\setminus\Omega,V}(\mathscr{H}(V))$, is an isomorphism of complex spaces. [2]

The product of a hyperfunction and an analytic function h is defined by

$$h(f^+, f^-) = (hf^+, hf^-).$$

The restriction and support of a hyperfunction on $\mathbb{R}$ is defined similarly as for $\mathbb{T}$:

Definition 5.1.3. *Let $L_1 \subset L_2$ be two nonempty subsets of $\mathbb{R}$. Let $T \in \mathcal{B}(L_2)$ be represented by $f \in \mathscr{H}(V_2 \setminus L_2)$, where $V_2 \in \mathcal{U}_{L_2}$. The restriction of T to L_1, denoted $T|L_1$, is the hyperfunction in $\mathcal{B}(L_1)$ associated to $f|((V_1 \cap V_2) \setminus L_1)$, for any open $V_1 \in \mathcal{U}_{L_1}$.*

Definition 5.1.4. *Let $T \in \mathcal{B}(\Omega)$. The support of T on $\mathbb{R}$, $\operatorname{supp} T$, is the complement on $\mathbb{R}$ of the largest open set $U \subseteq \mathbb{R}$ such that $T|U = 0$.*

This means that supp T is the complement on $\mathbb{T}$ of the largest open set $U \subseteq \mathbb{R}$ such that f^+ and f^- extend each other analytically across U. A hyperfunction has compact support if $\mathbb{R} \setminus U$ is compact.

We will now define the integral of a hyperfunction with compact support.

Definition 5.1.5. *Let $T \in \mathcal{B}(\Omega)$ be a hyperfunction with compact support. If $V \in \mathcal{U}(\Omega)$ and $f \in \mathscr{H}(V \setminus \operatorname{supp} T)$ represents T, we define the integral of T to be*

$$\int_\Omega T(x)\, dx = -\int_\gamma f\, dz$$

where γ is a union of disjoint Jordan curves in $V \setminus \operatorname{supp} T$ oriented so that the index of γ with respect to every point in supp T is 1.

We need the following proposition from [2].

Proposition 5.1.6. *Let Ω be a nonempty open subset of $\mathbb{R}$ and let $T \in \mathcal{B}(\Omega)$ have compact support. The function $\breve{T}$ defined in $\mathbb{C}^\sharp$ by*

$$\breve{T}(z) = \frac{1}{\pi} \int_\Omega \frac{T(x)}{z - x}\, \mathrm{d}x, \qquad z \in \mathbb{C}^\sharp,$$

is holomorphic in $\mathbb{C}^\sharp$ and has an analytic continuation to $\mathbb{C} \setminus \operatorname{supp} T$. Moreover, $\breve{T}(\infty) = \lim_{z\to\infty} \breve{T}(z) = 0$ and $\breve{T}$ represents $-2iT$. It is the only representative of $-2iT$ with these properties.

5.2 Fourier transform of a hyperfunction with compact support

Let T be a hyperfunction on $\mathbb{R}$ with compact support. We want to find the Fourier transform of T. We have

$$\int_{\mathbb{R}} T(x)e^{-ixw}\,\mathrm{d}x = -\int_{\gamma} -\frac{1}{2i}\breve{T}(z)e^{-izw}\,\mathrm{d}z = \int_{\gamma} \frac{1}{2i}\breve{T}(z)e^{-izw}\,\mathrm{d}z$$

where we let γ be a rectangle containing $\operatorname{supp} T$, see figure 5.2. Then

$$\int_{\mathbb{R}} T(x)e^{-ixw}\,\mathrm{d}x = \frac{1}{2i}\int_{\gamma} \breve{T}(z)e^{-izw}\,\mathrm{d}z =$$
$$=\frac{1}{2i}\int_{-x_0}^{x_0} -\breve{T}(x+iy_0)e^{-i(x+iy_0)w}\,\mathrm{d}x + \frac{1}{2i}\int_{-x_0}^{x_0} \breve{T}(x-iy_0)e^{-i(x-iy_0)w}\,\mathrm{d}x$$
$$+\frac{1}{2i}\int_{-y_0}^{y_0} \breve{T}(x_0+iy)e^{-i(x_0+iy)w}\,\mathrm{d}y + \frac{1}{2i}\int_{-y_0}^{y_0} -\breve{T}(-x_0+iy)e^{-i(-x_0+iy)w}\,\mathrm{d}y.$$

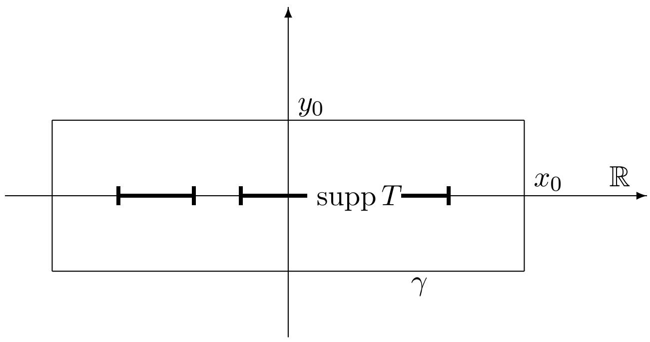

Figure 5.1: The rectangle of integration

We find an estimate of the integrals in y with $w \in \mathbb{R}$. We have

$$\begin{aligned} & \left|\int_{-y_0}^{y_0} \breve{T}(x_0+iy)e^{i(x_0+iy)w}\,\mathrm{d}y\right| \\ \leq\ & \int_{-y_0}^{y_0} e^{-yw}\,\mathrm{d}y\,|e^{ix_0w}| \sup_{y\in[-y_0,y_0]} \left|\breve{T}(x_0+iy)\right| \\ \leq\ & C\left|\breve{T}(x_0+iy*)\right| \end{aligned}$$

where $y*$ is the point where $\breve{T}$ attains its maximum. Similarly

$$\left| \int_{-y_0}^{y_0} \breve{T}(-x_0 + iy) e^{i(x_0+iy)w} \, \mathrm{d}y \right| \leq \tilde{C} \left| \breve{T}(-x_0 + i\tilde{y*}) \right|$$

for some $\tilde{y*}$. From Proposition 5.1.6 we see that when we let $x_0 \to \infty$ these integrals approach 0.

We now show that $\breve{T} \in L^2(\mathbb{R})$ for fixed y. We have

$$\begin{aligned} |\breve{T}(z)| &= \left| \frac{1}{\pi} \int_\Omega \frac{T(x)}{z - x} \, \mathrm{d}x \right| = \frac{1}{\pi} \left| \int_\gamma -\frac{T(\zeta)}{z - \zeta} \, \mathrm{d}\zeta \right| \\ &\leq \frac{1}{\pi} \left| \int_\gamma T(\zeta) \, \mathrm{d}\zeta \right| \sup_{\zeta \in \gamma} |\frac{1}{z - \zeta}| = C \frac{1}{|z - \zeta *|}, \end{aligned}$$

and then

$$|\breve{T}(x + iy)| \leq C \frac{1}{\sqrt{(x - x*)^2 + (y - y*)^2}} = C \frac{1}{\sqrt{(x - x*)^2 + K}},$$

so $\breve{T} \in L^2(\mathbb{R})$. Then we may write the Fourier transform of T as

$$\begin{aligned} FT(w) &= \lim_{x_0 \to \infty} \int_{-x_0}^{x_0} -\breve{T}(x + iy_0) e^{i(x+iy_0)w} \, \mathrm{d}x + \int_{-x_0}^{x_0} \breve{T}(x - iy_0) e^{i(x-iy_0)w} \, \mathrm{d}x \\ &= \lim_{x_0 \to \infty} \int_{-x_0}^{x_0} -\breve{T}(x + iy_0) e^{ixw} e^{-y_0 w} \, \mathrm{d}x + \int_{-x_0}^{x_0} \breve{T}(x - iy_0) e^{ixw} e^{y_0 w} \, \mathrm{d}x. \end{aligned}$$

5.3 Convolution of the Fourier transforms

We will now prove a formula similar to the one in Theorem 4.0.3 for hyperfunctions on $\mathbb{R}$. We will almost follow the scheme of the proof on $\mathbb{T}$. We need the following theorem from [9].

Theorem 5.3.1. Paley-Wiener-Ehrenpreis *The Fourier transform of a hyperfunction T with support contained in a compact set $K = [b, c]$ is an entire function, and for every $\epsilon > 0$ there is a constant C_ϵ such that*

$$|FT(z)| \leq \begin{cases} Ce^{yc + \epsilon|z|} & \operatorname{Im} z \geq 0 \\ Ce^{yb + \epsilon|z|} & \operatorname{Im} z < 0. \end{cases}$$

We will now prove the following:

Proposition 5.3.2. *If T and S are hyperfunctions with compact and disjoint support, then*

$$\lim_{a\to 0}\int_{-\infty}^{\infty} FT(x)FS(w-x)e^{-a|x|}\,\mathrm{d}x = 0.$$

Proof. Let T and S be two hyperfunctions with support contained in $[b,c]$ and $[d,e]$, respectively. We assume $c<d$, so the supports are disjoint. By Theorem 5.3.1 we have

$$|FT(z)| \leq \begin{cases} C_1 e^{yc+\epsilon|z|} & \operatorname{Im} z \geq 0 \\ C_1 e^{yb+\epsilon|z|} & \operatorname{Im} z < 0. \end{cases}$$

and

$$|FS(z)| \leq \begin{cases} C_2 e^{ye+\epsilon|z|} & \operatorname{Im} z \geq 0 \\ C_2 e^{yd+\epsilon|z|} & \operatorname{Im} z < 0. \end{cases}$$

We define a new function $G_w(z) = FT(z)FS(w-z)$, and then we have

$$|G_w(z)| = |FT(z)FS(w-z)| \leq \begin{cases} Ce^{y(c-d)+\epsilon|z|} & \operatorname{Im} z \geq 0 \\ Ce^{y(b-e)+\epsilon|z|} & \operatorname{Im} z < 0. \end{cases}$$

We define

$$B_L(u) = \int_L G_w(z)e^{izu}\,\mathrm{d}z$$

where L is a ray starting in 0. If $z = x+iy$ and $u = \alpha + i\beta$, we have $G_w(z)e^{izu} = G_w(z)e^{i(x+iy)(\alpha+i\beta)} = G_w(z)e^{-\alpha y - \beta x}e^{i(-\alpha x - \beta y)}$. For simplicity we let $-k = c-d$ and $-l = b-e$, and then

$$|G_w(z)e^{izu}| = |G_w(z)|e^{-\alpha y-\beta x} \leq \begin{cases} Ce^{-yk+\epsilon|z|}e^{-\alpha y-\beta x} & \operatorname{Im} z \geq 0 \\ Ce^{-yl+\epsilon|z|}e^{-\alpha y-\beta x} & \operatorname{Im} z < 0. \end{cases}$$

If u is given, we see that we need to choose the ray L carefully in order to make the integral convergent. When $u = i\beta$ where $\beta > 0$ for example, we can integrate along the positive part of the real line. We can sum it up as follows:

- If $\beta > 0$, we can choose $x > 0$ and $y = 0$.
- If $\beta < 0$, we can choose $x < 0$ and $y = 0$.
- If $\alpha > -k$, we can choose $x = 0$ and $y > 0$.
- If $\alpha < -l$, we can choose $x = 0$ and $y < 0$.

If we impose more restrictions, we can integrate along other rays as well. If we for example have both $\alpha > -k$ and $\beta > 0$, then we can integrate along any ray in the first quadrant.

Now we will show that B_L is independent of which ray we choose. We assume first that L_1 and L_2 are two rays in the upper half-plane, and let U consist of the points $u \in \mathbb{C}$ such that both $B_{L_1}(u)$ and $B_{L_2}(u)$ are finite. Let γ_r be the loop consisting of the parts of L_1 and L_2 that are contained in a ball with radius $r > 0$, together with a line c_r connecting the endpoints of these two parts of the rays. Let $z_1 \in L_1$ with $|z_1| = r$ and $z_2 \in L_2$ with $|z_2| = r$. The points of c_r are given by $z = \lambda z_1 + (1-\lambda)z_2$ where $\lambda \in [0,1]$. By assumption we know that for $u = \alpha + i\beta \in U$ we have that $e^{-y_1(k+\alpha)-\beta x_1+\epsilon|z_1|}$ and $e^{-y_2(k+\alpha)-\beta x_2+\epsilon|z_2|}$ will go to 0 when $r \to \infty$. If we put $z = \lambda z_1 + (1-\lambda)z_2$ in $e^{-y(k+\alpha)-\beta x+\epsilon|z|}$ we will see that this expression also will go to 0 when $r \to \infty$, and $|e^{-y(k+\alpha)-\beta x+\epsilon|z|}| \le e^{-\delta r}$ for some $\delta > 0$. Then

$$\left| \int_{c_r} G_w(z) e^{iuz} \, \mathrm{d}z \right| \le C\pi r e^{-\delta r} \to 0 \, \text{when} \, r \to 0.$$

Since $G_w e^{iuz}$ is entire, we have

$$\lim_{r \to \infty} \int_{\gamma_r} G_w(z) e^{iuz} \, \mathrm{d}z = 0$$

by Cauchy's theorem. We then see that $B_{L_1}(u) = B_{L_2}(u)$. Then we have shown that B_L is independent of L if L is in the upper half-plane. In the same way it can be shown that B_L is independent of L in all of $\mathbb{C}$. Then we define $B := B_L$. We see from the list we made that B will be defined for all $u \in \mathbb{C} \setminus [-k, -l]$, and it will be holomorphic there.

Now we let $a > 0$ and consider

$$B(ia) = \int_0^\infty G_w(x) e^{-ax} \, \mathrm{d}x$$

and

$$B(-ia) = \int_0^{-\infty} G_w(x) e^{ax} \, \mathrm{d}x = -\int_{-\infty}^0 G_w(x) e^{ax} \, \mathrm{d}x.$$

We see that these integrals converge by looking at the first two facts in the list on the previous page. Since B is holomorphic at 0, we must have $\lim_{a \to 0} B(ia) = \lim_{a \to 0} B(-ia)$, so

$$\lim_{a \to 0} \int_0^\infty G_w(x) e^{-ax} \, \mathrm{d}x = \lim_{a \to 0} -\int_{-\infty}^0 G_w(x) e^{ax} \, \mathrm{d}x,$$

and then

$$\lim_{a \to 0} \int_{-\infty}^\infty G_w(x) e^{-a|x|} \, \mathrm{d}x = 0.$$

Since $G_w(z) = FT(z)FS(w-z)$, we have

$$\lim_{a\to 0} \int_{-\infty}^{\infty} FT(x)FS(w-x)e^{-a|x|}\,\mathrm{d}x = 0.$$

The same result is true even if T and S are not assumed to have support in two intervals, but in arbitrary compact disjoint sets. We do as we did on $\mathbb{T}$, and write the hyperfunctions as a sum of finitely many hyperfunctions T_i and S_j with support in disjoint intervals. Then we repeat the above proof for all combinations of T_i and S_j, and we are done. □

Remark: The function B can be considered a hyperfunction with support in $[-k,-l]$, and it is the inverse Fourier transform of G_w up to a constant multiple, but we did not need that fact in our proof.

Chapter 6

Conclusion

We have looked at the problem of existence of entire functions under certain restrictions on the axes. The Beurling-Malliavin theorem gives some answers to this question. We saw that this theorem leads to one of the theorems in Domar's article.

In the end of his article Domar uses one of his theorems to show that there exists a nontrivial translation-invariant subspace of a certain weighted l^p-space. We have seen how this result can be interpretered as a statement about hyperfunctions. We showed that the generalization of this fact can be proved using extensions of Domar's tehnique. More precisely, we showed that if two hyperfunctions T_1 and T_2 on $\mathbb{T}$ have support contained in two disjoint arcs, then

$$\lim_{r \to 1^-} \sum_{n \in \mathbb{Z}} r^{|n|} c_n(T_1) c_{m-n}(T_2) = 0 \qquad (m \in \mathbb{Z}).$$

Our proof was based on the correspondence expressed by Lemma 4.4.1 between hyperfunctions and entire functions of order ≤ 1. To generalize the result further to any two hyperfunctions with compact support, we used a version of the Mittag-Leffler theorem. This gives a new proof to the statement by J. Esterle and R. Gay which uses complex analysis and gives a clear meaning of the factor $r^{|n|}$.

We also repeated the scheme for the hyperfunctions on the real line. We showed that if T and S are hyperfunctions on $\mathbb{R}$ with compact disjoint support, then

$$\lim_{a \to 0} \int_{-\infty}^{\infty} FT(z) FS(w - z) e^{-|x|} \, \mathrm{d}x = 0.$$

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