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Optimization Analysis of Risk/Return of Large Equity Portfolios, Applying Option Strategies

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Problem Description

Considering portefolios consisting of different option contracts, we would like to examine how to minimize the risk and maximize the return of the portfolios. We will study the risk/return characteristics of options using various options and statistical simulation methods. The goal is to use Monte Carlo simulation to get a good understanding of the risk/return characteristics of various strategies on different horizons.

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This master thesis is to be carried out at the Department of Mathematical Sciences under guidance of Jacob Kooter Laading.

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PREFACE

"Optimization Analysis of Risk/Return of Large Equity Portfolios, Applying Option Strategies" is a 30 ECTS thesis leading to the degree of Master of Technology at the Department of Mathematical Sciences, Faculty of Information Technology, Mathematics and Electrical Engineering, Norwegian University of Science and Technology.

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Abstract

This is a study of the risk/return characteristics of large equity portfolios, consisting of different option contracts. In times when there is nervousness present in the financial market, and the future prospects of the market are highly uncertain, the importance of appropriate mathematical models is emphasized by traders. Tools to implement option strategies in different markets will be investigated, in order to minimize the risk and maximize the return of the portfolios. Using various options and statistical simulation methods, the risk/return characteristics of options will be studied. Monte Carlo simulation is used, in order to obtain a thorough understanding of the risk/return characteristics of various option strategies, applied on different indices. As the strategies investigated are meant to be applied by traders, the daily changes in portfolio value are the basis from which the risk of the strategies is estimated. There are various methods of estimating the risk available. In order to examine the risk characteristics, the probability of extreme outcomes is of interest. The Value-at-Risk and the expected shortfall are to be analyzed, when examining the risk of the strategies evaluated. Analyzing the properties of the expected return of the portfolios, the tails of the densities of the portfolio values, for the different strategies and indices, will be of interest.

Contents

Chapter 1

Introduction

1.1 Background

During 2007, the financial market went through a turbulent period characterized by nervousness and uncertainty. The main reason for this turbulence was the collapse in the subprime mortage industry. Subprime lending is the practice of making loans to borrowers who do not qualify for the best market interest rates because of their low credit rating. Beginning late in 2006, the U.S. subprime mortage industry entered what observers have refered to as a financial meltdown. A steep rise in the rate of subprime mortage foreclosures has caused more than 100 subprime mortage lenders to fail or file for bankruptcy, of which New Century Financial Corporation, the earlier second biggest subprime lender in the U.S., is the most prominent. The problems of these financial companies, causing the collapse in the $6.5 trillion mortage backed security market, have resulted in damaging effects on the U.S. housing market and the economy in general. In September 2007 Northern Rock, the fifth largest mortage provider in the UK, had to apply for emergency funding from the Bank of England, the UK's central bank, as a result of problems in international credit markets attributed to the subprime lending crisis. On August 15, 2007, concerns about the subprime mortage lending industry caused a sharp drop in stocks across the Nasdaq and Dow Jones, which affected almost all the stock markets worldwide. Record lows were observed in stock market prices across the Asian and European continents.

The subprime crisis also puts a downward pressure on the economic development, because fewer or more expensive loans inhibit investment in business and consumption, which run the economy. The threat to the extensive economy, caused by the financial market crisis and decline in the housing market were primary factors leading to the decisions by the U.S. Federal reserve to cut interest rates and present the economic stimulus package, passed by Congress and signed by President George W. Bush on February

13, 2008. Both actions were designed to stimulate economic growth and restore the confidence in the financial markets.

Mortage lenders and home builders suffered terribly, but losses also influenced other sectors, and some industries, such as metal and mining companies, were severly hit, even though they had minor connection with lending or mortages. Crisis has caused panic in financial markets and encouraged investors to take their money out of risky mortages bonds and shaky equities and put it into commodities as ”stores of value”. Most of the recent increases in global food prices have been the result of speculation and the collapse in the value of the US dollar. Top management has not escaped unscathed, as the CEOs of Merrill Lynch as well as Citigroup were forced to resign during one week. From July 2007 to March 2008 financial institutions fired more than 34,000 employees.

When the crisis first appeared, many analysts called it a domestic problem, that would only affect US housing markets. However, almost a year later, it can be seen that this is not the case. For instance, the Bank of China announced in August of 2007, that it holds $9.7 billion dollars of US Subprime debt. In January of 2008, Korean markets fell due to the ”selling spree” of shares of US mortages. Because of the global economy, and the huge Subprime ”pool” of mortages that was bought by investors world wide, the International Monetary Fund says that the worldwide losses caused by the US subprime mortage crisis could amount to $945 billion. It has yet to be seen if the US's stimulus plan will be sufficient to support the global economy.

The 2008 economic crisis which has generated fear of stagflation (stagnation and inflation), began in the United States in January 2008 when the oil price surpassed $100 a barrel, combined with the subprime mortage crisis and the crack in the US housing market. While the financial crisis was limited to developed countries, the inflation crisis has affected nearly every country in the world due to a variety of sustainability issues. It has been argued that the 2008 economic crisis is the first global crisis to include environmental in addition to economic issues, and it affected European countries with the Euro Zone Growth Outlook for 2008. April 2008 began with one of the worst periods in United States aviation history, with no fewer than four airlines all ceasing operations during a week. In the period of April and May 2008, a total of nine airlines ceased operations, due to rising fuel prices and reduced passenger numbers. Numerous factors have been cited for the crisis, including record oil prices and the historic weakness of the US dollar. Oil and gold hit historic highs while the US dollar declined to a historic low on Wednesday, February 27, 2008. Rising petroleum costs can also reverse globalization.

As the crisis is still going on with all its implications yet to be known, nervousness and uncertainty are characteristics which are present in the financial markets worldwide. In turbulent times in the financial markets, the importance of appropriate mathematical models is apparent. To hedge against the exposure for losses, it becomes important to utilize non-linear methods. Tools to implement different option strategies in different markets will be investigated, with reference to risk/return. Monte Carlo simulation will be used, in order to obtain a good understanding of the risk/return characteristics of various option strategies, applied on different indices. The Value-at-Risk and the expected shortfall will be studied, in order to compare the risk of the strategies and indices investigated. Analyzing the return of the strategies and indices evaluated, the densities of the portfolio values after having applied the strategies for three years, will be of interest.

1.2 Thesis Outline

This thesis consists of seven chapters. Initially, an introduction to the financial markets is presented, which enables the reader to understand the mathematics applied in the financial world. Secondly, the most common methods of measuring the risk, applied in finance are presented. The strategies and the data material are presented, before the results are to be analyzed.

- In chapter 2, the basics of financial markets and models for simulation of asset prices and pricing of options are presented.
- In chapter 3, the basics of risk management and the most common methods of measuring the risk, applied in the financial market are presented. These are the methods applied when evaluating the risk of the strategies and indices considered.
- In chapter 4, the motivation for this thesis and the option strategies applied are presented.
- In chapter 5, the data material used in this thesis and the implementation procedure is described.
- In chapter 6, the results for the strategies and indices considered are presented. The results are illustrated and analyzed, with reference to risk/reward.
- In chapter 7, a brief summary of our work is presented, along with future perspective in the extension to this thesis.

Chapter 2

Financial Models and Mathematical Models

2.1 Introduction to Options and Markets

2.1.1 European Options

A European call option is a contract with the following conditions:

- At a prescribed time in the future, known as the **expiry date** or **expiration date**, the **holder** of the option *may*
- purchase a prescribed asset, known as the **underlying asset** or, briefly, the **underlying** for a
- prescribed ammount, known as the **exercise price** or **strike price**.

This implies that the holder of the option has the right to buy the underlying asset, and not an obligation. On the other hand, the other part; the **writer** of the option, does have a potential obligation; he *must* sell the asset if the holder chooses to buy it. Moreover, the value of the option must be paid for at the time of opening the contract. Much of the work throughout this thesis will focus on the following problems:

- How much would one pay for this right, i.e. what is the value of an option?
- How can the writer minimize the risk associated with his obligation?

The option to *buy* an asset is known as a **call** option. The right to sell an asset is known as a **put** option and has payoff properties which are opposite to those of a call. A put option allows the holder to sell the asset on a certain date for a prescribed amount. The writer is then obliged to buy the asset.

2.1.2 Usage of Options

Options are primarily used by investors when using a hedging strategy or purely speculation. In finance you can not only buy shares or options in the traditional way, by hoping the asset will increase in value. You also have the opportunity to speculate in a decrease in value for a specific share. The concept of selling a share that you do not own is known as selling **short**. You will then make a profit from a fall in the shares. The opposite of a short position is a **long** position.

- Why would anyone write an option?

The motivation for writers of call options is that they expect the value of the underlying asset to fall, while writers of put options expect the value of the underlying asset to increase. The balance of different views on a particular asset makes the value of the asset stable. If everyone involved in the market expected the value of a specific asset to rise, its market price would be higher than it is. On the other hand, the value of an asset would be lower if everyone expected the value to decrease.

2.1.3 Hedging

The concept of **hedging** is a known strategy for investors who want to reduce the risk of their investments. The idea is to reduce the sensitivity of a portfolio to the movement of an underlying asset by taking opposite positions in different financial instruments. The central idea for this strategy is: *If a market can sell an option for more than it is worth and then hedge away all the risk for the rest of the option's life, he has locked in a guaranteed, risk-free profit.* The delta for a whole portfolio is the rate of change of its value with respect to changes in the underlying asset. Writing Π for the value of the portfolio,

$$\Delta = \frac{\partial \Pi}{\partial S}.$$

Thus, when delta hedging between an option and an asset, the position taken is called "delta-neutral", since the sensitivity of the hedged portfolio to asset price changes is instantaneously zero. For a general portfolio the maintenance of a delta-neutral position may require a short position in the underlying asset. In delta-hedging the largest random component of the portfolio is eliminated. One can hedge away effects due to the curvature of the portfolio value with respect to the underlying asset. This entails knowledge of the **gamma** of a portfolio, defined by

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}.$$

The decay of time value in a portfolio is represented by the **theta**, given by

$$\Theta = -\frac{\partial \Pi}{\partial t}.$$

Sensitivity to volatility is usually called the **vega** and is given by

$$\text{vega} = \frac{\partial \Pi}{\partial \sigma}.$$

Sensitivity to interest rate is called **rho**, where

$$\rho = \frac{\partial \Pi}{\partial r}.$$

The sensitivities mentioned are often referred to as the Greeks, although vega is not a greek letter.

2.1.4 Other Types of Options

The most used option nowadays is the American option. Note that the terms European/American used in this context do not refer to what kind of options are used on the respective continents, but simply refer to the technical specifications of the contract. The **American option** can be exercised at *any time* prior to expiry, while the **European option** only can be exercised *at* expiry. Additionally, so-called **Exotic options** exist, which depend on the history of an asset. Below are some examples of popular options and their properties:

- Barrier options (the option can either come into existence or become worthless if the underlying asset reaches some prescribed value before expiry)
- Asian options (the price depends on some form of average)
- Lookback options (the price depends on the asset price maximum or minimum)

2.1.5 Forward and Futures Contracts

A **forward contract** is an agreement between two parties whereby one contracts to buy a specified asset from the other for a specified price, known as the **forward price**, on a specified date in the future, known as the **delivery date** or **maturity date**. The main difference between a forward contract and an option, is that in a forward contract, the asset *has* to be delivered and paid for. An additional difference is that no money changes hands until delivery in a forward contract. A **futures contract** is essentially the same as a forward contract, but has some technical modifications. Futures are usually traded on an exchange with certain standard features of the contract, such as delivery date and contract size. When dealing with a forward contract, the profit or loss is only realized at the expiry date, while in a future contract, the value of the contract is evaluated every day. The change in value is paid to one party by the other, so that the net profit or loss is paid gradually over the lifetime of the contract.

2.2 A Simple Model for Asset Prices

The following main restrictive assumptions apply:

- The past history is fully reflected in the present price, which does not hold any further information.
- Markets respond immediately to any new information about the asset.

A simple model for generating asset prices, is the following **stochastic differential equation**

$$\frac{\mathrm{d}S}{S} = \sigma \mathrm{d}X + \mu \mathrm{d}t, \tag{2.1}$$

where $\frac{\mathrm{d}S}{S}$ represents the return of the asset, σ is the **volatility**, which measures the standard deviation of the returns. μ is a measure of the average rate of growth of the asset price, known as the drift. In this simple model μ is considered to be a constant, but can be regarded a function of S and t in more complex models. $\mathrm{d}t$ is a small subsequent interval. $\mathrm{d}X$ is the sample from a normal distribution, also known as a **Wiener process**, with the following properties:

- $\mathrm{d}X$ is a random variable, drawn from a normal distribution.
- The mean of $\mathrm{d}X$ is zero.
- The variance of $\mathrm{d}X$ is $\mathrm{d}t$.

2.3 Option Evaluation

2.3.1 Arbitrage and Risk Neutrality

One of the most fundamental concepts underlying the theory of derivative pricing and hedging is that of **arbitrage**. In financial terms, there are never any opportunities to make an instantaneous risk-free profit. The greatest risk-free return one can make on a portfolio of assets is the same as the return if the equivalent amount of cash were placed in a bank. A commonly used definition of risk of a portfolio is the variance of the return. A highly volatile stock with very uncertain return and thus a large variance is considered a risky asset. The **risk-neutral** approach stems from the observation that the growth rate μ does not appear in the Black-Scholes equation (2.12), which will be described in section (2.4). Although the value of an option depends on the standard deviation of the asset price, it does not depend on its rate of growth. The risk preferences of investors are irrelevant because all the risk inherent in an option can be hedged away i.e., there is no return to be made over and above the risk-free return. If a portfolio can be constructed with a derivative product and the underlying asset in such a way that the random component can be eliminated, the derivative product may generally be valued as if *all* the random walks involved are risk-neutral. This means that the drift term in the stochastic differential equation for the asset return is replaced by r wherever it appears. The random walk for the return on S is pretended to have drift r instead of μ. To calculate the probability density function of future values of S, the lognormal distribution (A.7) is used, with μ replaced by r. The derivation of this distribution can be found in (Appendix A). *It is utterly important to realize that the new probability density function is not that of S.* To calculate the expected value of the payoff $\Lambda(S)$ using this probability density function, multiply $\Lambda(S)$ by the risk-neutral probability density function and integrate over all possible future values of the asset, from zero to infinity. Finally, discounting to the present value of the option, obtaining

$$V(S,t) = \frac{\mathrm{e}^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \mathrm{e}^{-(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t))^2/2\sigma^2(T-t)} \Lambda(S') \frac{\mathrm{d}S'}{S'}. \tag{2.2}$$

2.3.2 Option Values, Payoffs and Strategies

When dealing with options

- V denotes the value of an option. To distinguish between the values of a call and a put option, we use $C(S,t)$ and $P(S,t)$ respectively. This value is a function of the current value of the underlying asset, S and time t: $V(S,t)$. Additionally, the value of an option depends on the following parameters:

- σ, the volatility of the underlying asset.
- E, the exercise price.
- T, the expiry.
- r, the interest rate.

If $S > E$ at expiry, it makes financial sense to exercise the call option, but in case $S < E$ at expiry, the option should not be exercised. Thus, the value of a call option at expiry can be written as

$$C(S,T) = \max(S - E, 0) \tag{2.3}$$

As a writer of a put option makes a profit if the underying asset decreases in value, the put option is worthless if $S > E$, but has the value $E - S$ for $S < E$. This leads to the payoff at expiry

$$P(S,T) = \max(E - S, 0) \tag{2.4}$$

Another special kind of option is the so-called **cash-or-nothing call**, which can be written as

$$V(S,T) = B\mathcal{H}(S - E) \tag{2.5}$$

Where $\mathcal{H}$ is the **Heaviside function**, which has value 0 when its argument is negative but is 1 otherwise. Options whith general payoffs are usually called **binaries** or **digitals**.

2.4 The Black-Scholes Model

2.4.1 Assumptions

- The asset price follows the lognormal random walk (2.1). Other models exist, and in many cases it is possible to perform the Black-Scholes analysis to derive a differential equation for the value of an option. Explicit formulae rarely exist for such models. However, this should not discourage their use, since an accurate numerical solution is usually quite straightforward.
- The risk-free interest rate r and the asset volatility σ are known functions of time over the life of an option.
- There are no transaction costs associated with hedging a portfolio.
- The underlying asset pays no dividends during the life of the option. This assumption can be dropped if the dividends are known beforehand. They can be paid either at discrete intervals or continuosly over the life of the option.

- There are no arbitrage possibilities. The absence of arbitrage opportunities means that all risk-free portfolios must earn the same return.
- Trading of the underlying asset can take place continuously. This is clearly an idealisation.
- Short selling is permitted and the assets are divisible. It is assumed that any number (not necessarily an integer) of the underlying asset can be bought and sold, and a trader may sell assets that he do not own.

2.4.2 Black-Scholes Partial Differential Equation

Suppose that the option value $V(S,t)$ depends only on S and t. Using Itô's lemma (A.5), writing

$$V(S,T) = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) \mathrm{d}t. \tag{2.6}$$

This gives the random walk followed by V. V is required to have at least one t derivative and two S derivatives. A portfolio consisting of one option and a number $-\Delta$ of the underlying asset can be constructed. The value of this portfolio is

$$\Pi = V - \Delta S. \tag{2.7}$$

The jump in the value of this portfolio in one time-step is

$$\mathrm{d}\Pi = \mathrm{d}V - \Delta \mathrm{d}S.$$

Here Δ is fixed during the time-step. Combining (2.1), (2.6) and (2.7), Π follows the random walk

$$\mathrm{d}\Pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) \mathrm{d}X + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) \mathrm{d}t. \tag{2.8}$$

The random component in this random walk can be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}. \tag{2.9}$$

Note that Δ is the value of $\partial V / \partial S$ at the *start* of the time-step dt. This results in a portfolio whose increment is wholly deterministic:

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \mathrm{d}t. \tag{2.10}$$

The concepts of arbitrage and supply and demand are appealed to, with the assumption of no transaction costs. The return on an amount Π invested in riskless assets would see a growth of $r\Pi dt$ in a time dt. If the right-hand side of (2.10) were greater than this amount, an arbitrager could make a guaranteed riskless profit by borrowing an amount Π to invest in the portfolio. The return for this risk-free strategy would be greater than the cost of borrowing. Conversely, if the right-hand side of (2.10) were less than $r\Pi dt$, the arbitrager would short the portfolio and invest Π in the bank. Thus,

$$r\Pi \mathrm{d}t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \mathrm{d}t. \tag{2.11}$$

Substituting (2.7) and (2.9) into (2.11) and dividing throughout by dt gives

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.12}$$

This is the **Black-Scholes partial differential equation**.

2.4.3 The Black-Scholes Formulae for European Options

When r and σ are constant, the exact, explicit solution for the European call is

$$C(S,t) = SN(d_1) - E\mathrm{e}^{-r(T-t)}N(d_2), \tag{2.13}$$

where $N(x)$ is the cumulative distribution function for a standardised normal random variable, given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}y^2} \mathrm{d}y.$$

In (2.13)

$$d_1 = \frac{log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2 = \frac{log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

To find the value of a put option, the **put-call parity** can be applied. The value of the portfolio is denoted by Π, thus;

$$\Pi = S + P - C$$

where P and C are the values of the put and call respectively. The payoff for this portfolio at expiry is

$$S + max(E - S, 0) - max(S - E, 0)$$

This can be rewritten as

$$S + (E - S) - 0 = E \quad \text{if } S \leq E$$

or

$$S + 0 - (S - E) = E \quad \text{if } S \geq E$$

Thus, the payoff is always the same whether S is greater or less than E at expiry. By discounting the value of the portfolio at expiry, the value of the portfolio at time t can be found to be $E\mathrm{e}^{-r(T-t)}$. Thus,

$$S + P - C = E\mathrm{e}^{-r(T-t)}.$$

Using the put-call parity, the solution for the European put option is found;

$$P(S,t) = E\mathrm{e}^{-r(T-t)}N(-d_2) - SN(-d_1). \tag{2.14}$$

2.5 Interest Rate Models and Derivative Products

One of the biggest assumptions made is that interest rates are constant. In reality this is not the case, but for simplicity, the interest rate is considered constant in this context, due to the relatively short duration of the option contracts. If the option contracts considered had a longer duration, it would have been more useful to include interest models. Some of the most common models used for this purpose are presented below. Models used for bond pricing will be presented.

2.5.1 Stochastic Interest Rates

The future course of an interest rate cannot realistically be forecasted, and it is therefore natural to model it as a random variable. The behavior of r, the interest rate received by the shortest possible deposit, which is commonly called the **spot interest rate** will be modelled. The spot interest rate r is supposed to be governed by a stochastic differential equation on the form

$$\mathrm{d}r = u(r,t)\mathrm{d}t + w(r,t)\mathrm{d}X. \tag{2.15}$$

The functional forms of $u(r,t)$ and $w(r,t)$ determine the behavior of the spot interest rate r.

The bond pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0. \tag{2.16}$$

The terms in the equation represent time decay, diffusion, drift and discounting, respectively. The drift term in the equation (2.16) is not the drift of the real spot rate u, but the drift of the rate called **risk-neutral spot rate**. This rate has a drift of $u - \lambda w$. When pricing interest rate derivatives it is important to model, and price, using the risk-neutral rate. This rate satisfies

$$\mathrm{d}r = (u - \lambda w)\mathrm{d}t + w\mathrm{d}X.$$

The risk-neutral drift can be written as $u - \lambda w$ and the volatility w as

$$u(r,t) - \lambda(r,t)w(r,t) = \eta(t) - \gamma(t)r \tag{2.17}$$

and

$$w(r,t) = \sqrt{\alpha(t)r + \beta(t)}. \tag{2.18}$$

The following assumptions are used when modelling interest rates;

- **Positive interest rates:** Interest rates are positive, except some extremely rare occurrences. The spot rate can be bounded below by a positive number if $\alpha(t) > 0$ and $\beta \leq 0$. The lower bound is $-\beta/\alpha$. r can go to infinity, with probability zero.

- **Mean reversion:** Examining the drift term, we see that for large r, the interest rate will tend to decrease towards the mean, which may be a function of time. When the rate is small, it will move up on average.

2.5.2 The Vasicek Model

The Vasicek model takes the form of (2.17) and (2.18) with $\alpha = 0$, $\beta > 0$ and all other parameters independent of time;

$$dr = (\eta - \gamma r)dt + \beta^{1/2}dX.$$

The value of a zero-coupon bond is given by

$$e^{A(t;T) - rB(t;T)}$$

where

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)}) \tag{2.19}$$

and

$$A = \frac{1}{\gamma^2}(B(t;T) - T + t)(\eta\gamma - \frac{1}{2}\beta) - \frac{\beta B(t;T)^2}{4\gamma}. \tag{2.20}$$

The model is mean-reverting to a constant level, but the drawback of the model is that the interest rates can easily turn negative.

2.5.3 Cox, Ingersoll & Ross (CIR)

The CIR model takes (2.17) and (2.18) as the interest rate model, with $\beta = 0$, and no time-dependence in the parameters;

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}dX.$$

The value of a zero-coupon bond is

$$e^{A(t;T)-rB(t;T)},$$

where A and B are given by (2.20) and (2.19).

Chapter 3

Risk Evaluation

3.1 Basic Concepts in Risk Management

A central issue in modern risk management is the measurement of risk. It is of interest to measure the risk exposure of a financial institution in order to determine the amount of capital that institution has to hold as a buffer against unexpected losses. The value of a portfolio at time s is denoted by $V(s)$. The portfolio may consist of a collection of stocks or bonds, a book of derivatives, a collection of risky loans or a financial institution's overall position in risky assets. The value of the portfolio is assumed to be observable at time s. For a given time horizon Δ, the loss of the portfolio over the period $[s, s+\Delta]$ is given by

$$L_{[s,s+\Delta]} = -(V(s+\Delta) - V(s)).$$

In risk management, the so-called profit-and-loss distribution is of concern. This is the distribution of the change in value, $V(s+\Delta) - V(s)$. However, in risk management the main concern is the probability of large losses and hence the upper tail of the loss distribution. A fixed horizon, Δ is considered. It will be convenient to measure time in units of Δ and to introduce a time series notation, moving from a generic process $Y(s)$ to the time series $(Y_t)_{t\in\mathbb{N}}$ with $Y_t = Y(t\Delta)$. Hence, the loss is written as

$$L_{t+1} = L_{[t\Delta,(t+1)\Delta]} = -(V_{t+1} - V_t).$$

The random variables V_t and V_{t+1} represent the portfolio values on days t to day $t+1$, respectively, and L_{t+1} is the loss from day t to day $t+1$.

3.1.1 Conditional and Unconditional Loss Distributions

In risk management, the decision has to be taken whether the conditional or the unconditional distribution of losses is of interest. The differences between conditional and unconditional loss distributions are strongly related to time series properties of the series of risk-factor changes $(X_t)_{t\in\mathbb{N}}$. Suppose that the risk-factor changes form a stationary time series with stationary distribution F_X on $\mathbb{R}^d$. This means that the distribution of $(X_t)_{t\in\mathbb{N}}$ is invariant under shifts of time and most time series models used in practice for the modelling of risk-factor changes satisfy this property. Denote the sigma field representing the publicly available information at time t by $\mathcal{F}_t$. Typically, $\mathcal{F}_t = \sigma(\{X_s : s \leq t\})$, the sigma field generated by past and present risk-factor changes, often called the history, up to and including time t. The conditional distribution of X_{t+1} given current information $\mathcal{F}_t$ is denoted by $F_{X_{t+1}|\mathcal{F}_t}$. In most stationary time series models relevant for risk management, $F_{X_{t+1}|\mathcal{F}_t}$ is *not* equal to the stationary distribution F_X. On the other hand, if $(X_t)_{t\in\mathbb{N}}$ is an independent and identically distributed series, obviously $F_{X_{t+1|\mathcal{F}_t}} = F_X$. The conditional loss distribution $F_{L_{t+1}|\mathcal{F}_t}$ is defined as the distribution of the loss operator $\ell_{[t]}(\cdot)$ under $F_{X_{t+1}|\mathcal{F}_t}$. For $\ell \in \mathbb{R}$,

$$F_{L_{t+1}|\mathcal{F}_t}(\ell) = P(\ell_{[t]}(X_{t+1}) \leq \ell|\mathcal{F}_t) = P(L_{t+1} \leq \ell|\mathcal{F}_t).$$

The conditional loss distribution gives the conditional distribution of the loss L_{t+1} in the next time period given current information $\mathcal{F}_t$. Conditional distributions are particularly relevant in market risk management. The unconditional loss-distribution $F_{L_{t+1}}$ on the other hand is defined as the distribution of $\ell_{[t]}(\cdot)$ under the stationary distribution F_X of risk-factor changes. The unconditional loss distribution is of particular interest if the time horizon over which the losses is measured is relatively large, as is frequently the case in credit risk management and insurance.

3.1.2 Risk Measurement

Risk measures are used for a variety of purposes. Among the most important ones are the following:

- **Determination of risk capital and capital adequacy.** One of the principal functions of risk management in the financial sector is to determine the amount of capital a financial institution needs to hold as a buffer against unexpected future losses on its portfolio in order to satisfy a regulator, who is concerned with the solvency of the institution. A related problem is the determination of appropriate margin requirements for investors trading at an organized exchange, which is typically done by the clearing house of the exchange.

- **Management tool.** Risk measures are often used by management as a tool for limiting the amount of risk a unit within a firm may take. For instance, traders in a bank are often constrained by the rule that the daily 95 % Value-at-Risk of their position should not exceed a given bound.

- **Insurance premiums.** Insurance premiums compensate an insurance company for bearing the risk of the insured claims. The size of this compensation can be viewed as a measure of the risk of these claims.

3.1.3 Risk Measures Based on Loss Distributions

Most modern measures of the risk in a portfolio are statistical quantities describing the conditional or unconditional loss distribution of the portfolio over some predetermined horizon Δ.

- Losses are the central object of interest in risk management and so it is natural to base a measure of risk on their distribution.

- The concept of a loss distribution makes sense on all levels of aggregation from a portfolio consisting of a single instrument to the overall position of a financial institution.

- If estimated properly, the loss distribution reflects netting and diversification effects.

- Loss distribution can be compared across portfolios

Working with loss distributions, there are two major problems. Firstly, any estimate of the loss distribution is based on past data. If the laws governing financial markets change, these past data are of limited use in predicting future risk. Secondly, there is a problem of estimating the loss distribution accurately in a stationary environment, particularly for large portfolios, and many apparently sophisticated risk-management systems are based on relatively crude statistical models for the loss distribution. However, this is not an argument against using loss distribution. Rather, it calls for improvement in the way loss distributions are estimated and for prudence in the practical application of risk-management models based on estimated loss distributions. In particular, risk measures based on the loss distribution should be complemented by information from hypothetical scenarios. Forward-looking information reflecting the expectations of market participants, such as implied volatilities, should be used in conjunction with statistical estimates in calibrating models of the loss distribution.

3.1.4 Heavy Tails

The normal distribution has shortcomings as a model of changes in market prices; in virtually all markets, the distribution of observed price changes displays a higher peak and heavier tails than can be captured with a normal distribution. This is in particular true over short time horizons. High peaks and heavy tails are characteristic of a market with small price changes in most periods, accompanied by occasional very large price changes. While other theoretical distributions may do a better job of fitting market data, the normal distribution offers many convenient properties for modelling and computation. Choosing a description of market data thus entails a compromise between realism and tractability. The qualitative property of having a high peak and heavy tails is often measured through kurtosis. The kurtosis of a random variable X with mean μ is given by

$$\frac{E[(X-\mu)^4]}{(E[(X-\mu)^2])^2},$$

assuming X has a finite fourth moment. Every normal random variable has a kurtosis of 3; distributions are sometimes compared on the basis of excess kurtosis; the difference between the kurtosis and 3. If two distributions have the same standard deviation, the one with higher kurtosis will ordinarily have a higher peak and heavier tails. Such a distribution is called leptokurtotic.

Kurtosis provides some information about the tails of a distribution, but it is far from a complete measure of the heaviness of the tails. Further information is provided by the rate of decay of a probability density or, equivalently, the number of finite moments or exponential moments. The following conditions define three important categories of distributions:

(i) $E[\exp(\theta X)] < \infty$ for all $\theta \in \mathcal{R}$
(ii) $E[\exp(\theta X)] < \infty$ for all $\theta < \theta^\star$ and $E[\exp(\theta X)] = \infty$ for all $\theta > \theta^\star$, for some $\theta^\star \in (0, \infty)$
(iii) $E[X^r] < \infty$ for all $r < \nu$ and $E[X^r] = \infty$ for all $r > \nu$, for some $\nu \in (0, \infty)$

The first category includes all normal random variables and all bounded random variables. The second category describes distributions with exponential tails and includes all gamma distributions and in particular the exponential density $\theta^\star \exp(-\theta^\star x)$. The third category (for which $E[\exp(\theta X)] = \infty$ for all $\theta > 0$) describes heavy-tailed distributions. (iii) includes the stable Paretian distribution, for which $\nu \leq 2$. The third category includes distributions whose tails decay like $x^{-\nu}$ and, more generally, regularly varying tails. These three categories are not exhaustive; the lognormal, for example, fits just between the second and third categories having $\theta^\star = 0$ and $\nu = \infty$.

Empirical data is necessarily finite, making it impossible to draw definite conclusions about the extremes of a distribution. The third category above tends to provide the best description of market data, with ν somewhere in the range of 3-7, depending on the market and the time horizon over which returns are measured. Although a mixture of two normal distributions can provide an arbitrarily large kurtosis, it lies within the first category above. The tail is ultimately determined by the larger of the two standard deviations, so it does not provide an entirely satisfactory model.

3.2 Value-at-Risk

Value-at-Risk is probably the most widely used risk measure in financial institutions and has also made its way into the Basel II capital-adequacy framework - hence it merits an extensive discussion. Some portfolio of risky assets and a fixed time horizon Δ is considered, and the distribution function of the corresponding loss distribution is denoted by $F_L(\ell) = P(L \leq \ell)$. L and L^Δ are not distinguished, nor are conditional and unconditional loss distributions; it is rather assumed that the choice has been made at the outset of the analysis and that F_L represents the distribution of interest. A statistic based on F_L which measures the severity of the risk of holding the portfolio over the time period Δ should be defined. An obvious candidate is the maximum possible loss, given by $\inf\{\ell \in \mathbb{R} : F_L(\ell) = 1\}$, a risk measure important in reinsurance. However, in most models of interest the support of F_L is unbounded so that the maximum loss is simply infinity. Moreover, by using the maximum loss, we neglect any probability information in F_L. Value-of-Risk is a straightforward extension of maximum loss, which takes these criticisms into account. The idea is simply to replace "maximum loss" by "maximum loss which is not exceeded with a given high probability", the so-called confidence level.

Given some confidence level $\alpha \in (0, 1)$. The Value-at-Risk of our portfolio at the confidence level α is given by the smallest number ℓ so that the probability of the loss L exceeding ℓ is no larger than $(1 - \alpha)$. Formally,

$$VaR_\alpha = \inf\{\ell \in \mathbb{R} : P(L > \ell) \leq 1 - \alpha\} = \inf\{\ell \in \mathbb{R} : F_L(\ell) \leq \alpha\},$$

Value-at-Risk is simply a quantile of the loss distribution. Typical values for α are $\alpha = 0.95$ or $\alpha = 0.99$. By definition, the Value-at-Risk at confidence level α does not give any information about the severity of losses which occur with probability less than $1 - \alpha$. This is clearly a drawback of Value-at-Risk as a risk measure.

The fact that Value-at-Risk numbers are given a literal intepretation in practice is misleading. The statement that the daily Value-at-Risk at confidence level $\alpha = 99\%$ of a particular portfolio is equal to l is understood as "with probability of 99% the loss of this position will be smaller than l". This interpretation is misleading for two reasons. The estimate of the loss distribution is subject to estimation error and the problem of model risk. Model risk can be defined as the risk that a financial institution incurs losses because its risk-management models are misspecified or because some of the assumptions underlying these models are not met in practice. For instance, a normal distribution might be applied to model losses whereas the real distribution is heavy-tailed, or the presence of volatility clustering or tail dependence might fail to be recognized in modelling the distribution of the risk-factor changes. Any financial model is by definition a simplified, and thus an imperfect representation of the economic world and the ways in which economic agents perform investment, trading or financing decisions. Thus, it is fair to say that any risk-management model is subject to model risk to some extent. These problems are most pronounced when trying to estimate Value-at-Risk at very high confidence levels.

Moreover, the above interpretation of Value-at-Risk neglects any problem related to market liquidity. A market for a security is termed liquid if investors can buy and sell large amounts of the security in a short time without affecting its price very much. Conversely, a market in which an attempt to trade has a large impact on price, or where trading is impossible since there is no counterparty willing to take the other side of the trade, is termed illiquid. Ideally we should try to factor the effects of market illiquidity into formal models, although this is difficult for a number of reasons. First, the price impact of trading a particular amount of a security at a given point in time is hard to measure; it depends on elusive factors such as market mood or the distribution of economic information among investors. Moreover, in illiquid markets, traders are forced to close their position gradually over time to minimize the price impact of their transactions. Obviously, this liquidation has to be done on a different timescale depending on the size of the position to be liquidated relative to the market. This in turn would lead to different time horizons Δ for different positions, rendering the aggregation of risk measures across portfolios impossible. In many practical situations the risk manager can therefore do no better than ignore the effect of market liquidity in computing Value-at-Risk numbers or related risk measures, and be aware of the ensuing problems in interpreting the results.

3.2.1 Choice of Value-at-Risk Parameters

Whenever working with risk measures based on the loss distribution, an appropriate horizon Δ has to be chosen. In the case of Value-at-Risk, the confidence level α has to be decided as well. There is no single optimal value for these parameters, but there are some considerations which might influence our choice. The risk-management horizon Δ should reflect the time period over which a financial institution is commited to hold its portfolio. This period is affected by contractual and legal constraints, and by liquidity considerations. Even in the absence of contractual constraints, a financial institution can be forced to hold a loss-making position in a risky asset if the market for that asset is not very liquid. For such positions, a relatively long horizon may be appropriate. Liquidity does vary across markets, and for overall risk management, an institution has to choose a horizon which best reflects its main exposures.

The choice of the confidence level α depends on the purpose. For instance, the Basel Committee purposes the use of Value-at-Risk at the 99% level and Δ equal to 10 days for market risk. In order to set limits for traders, a bank would typically take 95% and Δ equal to one day.

3.3 Expected Shortfall

Expected shortfall is closely related to Value-at-Risk. For a loss L with $E(|L|) < \infty$ and distribution function F_L, the expected shortfall at confidence level $\alpha \in (0,1)$ is defined by

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)\mathrm{d}u,$$

where $q_u(F_L) = F_L^{\leftarrow}(u)$ is the quantile function of F_L. Expected shortfall is thus related to Value-at-Risk by

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)\mathrm{d}u.$$

Instead of fixing a particular confidence level α, Value-at-Risk is averaged over all levels $u \geq \alpha$ and thus further consider the tail of the loss distribution. Obviously, ES_α depends only on the distribution of L and $\mathrm{ES}_\alpha \geq \mathrm{VaR}_\alpha$. For continous loss distributions, an even more intuitive expression can be derived, which shows that expected shortfall can be interpreted as the expected loss that is incurred in the event that Value-at-Risk is exceeded.

For an integrable loss L with continous distribution function F_L and any $\alpha \in (0,1)$,

$$\mathrm{ES}_\alpha = \frac{E(L; L \geq q_\alpha(L))}{1-\alpha} = E(L|L \geq \mathrm{VaR}_\alpha),$$

where the notation $E(X;A) := E(XI_A)$ is used for a generic integrable random variable X and a generic set $A \in \mathcal{F}$.

Expected shortfall for Gaussian loss distribution. Suppose that the loss distribution F_L is normal with mean μ and variance σ^2. $\alpha \in (0,1)$ is fixed. Then,

$$\mathrm{ES}_\alpha = \mu + \sigma \frac{\phi(\phi^{-1}(\alpha))}{1-\alpha},$$

where ϕ is the density of the standard normal distribution. Note that

$$\mathrm{ES}_\alpha = \mu + \sigma E\left(\frac{L-\mu}{\sigma}\Big|\frac{L-\mu}{\sigma} \geq q_\alpha\left(\frac{L-\mu}{\sigma}\right)\right);$$

hence it suffices to compute the expected shortfall for the standard normal random variable $\tilde{L} := (L-\mu)/\sigma$. We get

$$\mathrm{ES}_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\int_{\phi^{-1}(\alpha)}^{\infty} l\phi(l)\mathrm{d}l = \frac{1}{1-\alpha}[-\phi(l)]_{\phi^{-1}(\alpha)}^{\infty} = \frac{\phi(\phi^{-1}(\alpha))}{1-\alpha}.$$

Chapter 4

Risk Management Using Put Options in Equity Portfolios

4.1 Motivation

The goal of this thesis is to minimize the risk and maximize the return, having equity portfolios which consist of put options. Three different option strategies are applied, in order to analyze the risk and return of the strategies considered. Strategies which are of importance from a trader's point of view are particularly interesting. Beeing a trader with a relatively short horizon on his investments, the movement of the underlying index for a short period of time is of interest. Monte Carlo simulation is used, in order to obtain a good understanding of the risk/return characteristics of various strategies, applied on different indices. There are various methods of estimating the risk available. Value-at-Risk and the expected shortfall will be explored when examining the risk of the portfolios.

When evaluating an option's value, two obvious kinds of methods are available. Methods leading to closed solutions can be selected. A common method which leads to a closed solution is applying the Black-Scholes model. This is a widespread method, well known and applied by analysts and traders worldwide. This method is also applied in problems where it in fact is not valid. The main advantages of methods leading to closed solutions are their simplicity and speed. The major limitation of those methods is that they can not be used accurately to price options with an American-style exercise, as it only calculates the option price at one point of time, at expiry. As only one value is considered a solution when applying methods on this form, there is limited data available to do a proper statistical analysis. The methods mentioned can not be used to price exotic options which depend on the history of the asset's value either.

The other methods available are those based on simulations. A method based on simulation applies the lognormal random walk and leads to an accurate solution. Another advantage of this method is that simulation can be done independent of distributions. The idea of the simulation using lognormal random walk is that tomorrow's values of asset prices is unknown, and can not be predicted. Using the interest rate r as the drift term in the model instead of the expected value of the growth of the asset price, based on historical data, ensures that there are not any opportunities to make a guaranteed return of a risk-free investment. The concept of arbitrage is fundamental, and will be applied when evaluating the value of an option. The main advantage of this method is that the path for the movement of the underlying asset can be preserved. This is extremely valuable when pricing an American or Exotic option. In addition, the possibilities to measure the risk connected to the option are greater. Having a large set of possible outcomes, the tail of the distribution can be regarded as a risk measurement. This is appreciated by writers of options, who are interested in an estimated limit for how much they risk to lose with a certain probability. The drawback of this method is that it is more time-consuming than the Black-Scholes model. In this thesis, when time is not as vital as it may be in the real financial world, the method based on simulation will be prefered. This enables pricing of options which are more complicated than European options. In addition, risk measurement based on the distribution of the simulation results can be made.

4.2 Strategies

Portfolios consisting of a long position on the underlying asset and a put option will be considered, in addition to a portfolio consisting of two different positions on put options. The options chosen are three months European put options. In order to optimize the risk/return of the portfolio, different strategies have to be considered. To measure the risk, the Value-at-Risk and the expected shortfall will be utilized. These risk profiles should be as advantageous as possible for a trader who holds these particular portfolios.

Considering the strategies, there are several approaches which can be used. Some strategies which may be used by a trader are evaluated, while there of course are numerous strategies available to use. The strategies focused on are presented below.

4.2.1 Trading at Expiry

A long position is held on the underlying asset. To hedge against the exposure for losses, an at-the-money European three months put option is bought. This option is held for the whole period when it is valid, regardless of what happens during the period. When the option expires, the return is realized, in case the underlying asset has a value below the strike price. If the underlying asset has a value above the strike price, the option expires worthless. In any case, a new option is bought when the previous expires. The new option has the same properties as the previous, except from the strike, which is equal to the value of the underlying asset at the time when the new option is bought. This procedure is repeated for three years. During the period applying this strategy, the value of the portfolio is evaluated on a daily basis, where the values are discounted to the time when this strategy was started being applied. From the daily change in portfolio value, the expected shortfall and Value-at-Risk is estimated, in order to evaluate the risk exposure of the portfolio.

4.2.2 Trading on Predefined Levels

As for the strategy trading at expiry, the initial portfolio is identical, as it consists of a long position on the underlying asset and an at-the-money European put option. This option is held until the underlying asset has a value which differs more than 5% from the strike, or the option contract expires. If the underlying asset moves more than 5% above the strike, the option is sold at its market value, and a new option is bought, with properties equal to those for the old one, except from the strike, which is equal to the value of the underlying asset at the time when the new option was bought. In this case, the option has been sold for less than it was bought, but as the underlying asset has increased in value, the loss has been limited, and even a profit will be made in some cases. In case the underlying asset moves more than 5% below the strike, the option is sold for its market value. In this case, the value of the option will be greater when the option is sold than it was when it was bought. The portfolio value does not necessarily increase in this case, as a long position is held on the underlying asset, which must have decreased in value. If the option expires without having crossed the barriers 5% below or above the strike, the option is exercised if it expires when the underlying asset has a value below the strike. If the underlying asset has a value above the strike at expiry, the option expires worthless. In any case, a new option is bought with properties equal to the old one at expiry, except from the strike, which is equal to the current value of the underlying asset at the time. This strategy is practised for three years, i.e. at any time one long position on the underlying asset and one at-the-money three months European put option are held.

4.2.3 Put Spread

A strategy which is called put spread will be considered. This strategy consists in buying an at-the-money three months European put option, while simultaneously writing 10% out-of-the-money three months European put options. The concept of a self-financing portfolio is applied, as the number of options written corresponds to the price of the option which is bought. Thus, the amount used to buy the at-the-money option equals the amount received when writing the out-of-the-money options. If the value of the portfolio exceeds the initial value of the at-the-money option, the option held is sold, and the options written are bought. Simultaneously, the same strategy is applied, as a new at-the-money three months European put option is bought, while writing a number of 10% out-of-the-money three months European put options. By doing this, the amount received by the options written equals the amount used to buy the at-the-money option. The strike of the new at-the-money option is thus equal to the value of the underlying asset at the time when trading takes place. On the other hand, if the portfolio value exceeds the negative initial price of the at-the-money option, the option we are holding is sold, and the options written are bought. When selling and buying the options, a new at-the-money three months European put option is bought, and 10% out-of-the-money three months European put options are written, corresponding to the value for which the at-the-money option is bought. If the options expire without the portfolio value having crossed the barriers, the at-the-money option is exercised in case it has a value, while the out-of-the-money options have to be paid for in case they have a value at time of expiry. This strategy is repeated for three years.

Chapter 5

Data Material

5.1 Background for INDEX

5.1.1 S&P 500

Stock data from the index **S&P 500**, which is an index containing the stocks of 500 Large-Cap corporations, most of which are American, have been used. The index is the most notable of the many indices owned and maintained by **Standard & Poor's**. All the stocks in the index are those of large publicly held companies which trade in the two largest US stock markets, the **New York Stock Exchange** and **Nasdaq**. After the Dow Jones Industrial Average, the S&P 500 is the most widely watched index of large-cap US stocks. In addition, data for implied volatilities for options corresponding to the S&P 500 index have been used. The implied volatilities are calculated based on assumptions in the market. If the uncertainty in the market is high, it will lead to a relatively high implied volatility. On the other hand, if the market is stable, and no dramatic movement is expected in the near future, the implied volatility will be relatively low. There are index data available from 1998, illustrated in Figure 5.1.

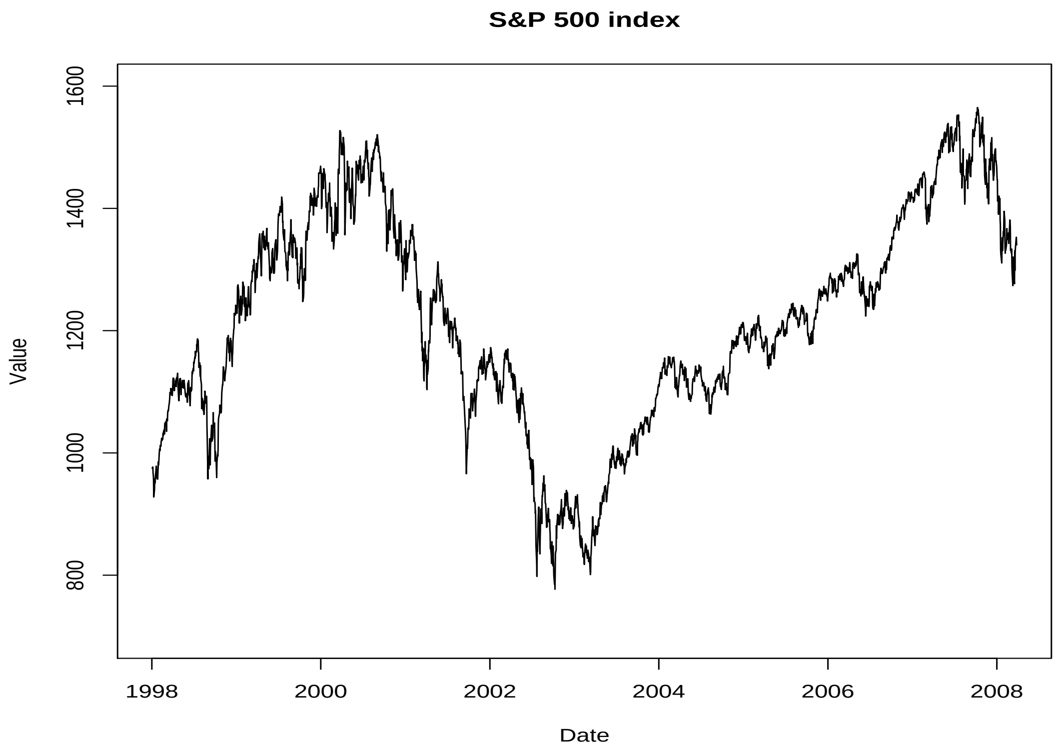

Figure 5.1: The S&P index from 1998.

The S&P 500 index increased in value until 2001, followed by a sharp drop in stock prices during 2001 and 2002 in stock exchanges across the United States, Canada, Asia and Europe. This downturn is often refered to as ”the Internet bubble bursting”. This downturn can be viewed as part of a larger **bear market** or correction, after a decade-long **bull market** had led to unusually high stock valuations. Some Internet companies (**Webvan**, **Exodus Communications** and **Pets.com**) went bankrupt. Others (**Amazon.com**, **eBay** and **Yahoo!**) went down dramatically in value, but remain in business today and have generally good long term growth prospects. An outbreak of accounting scandals (**Enron**, **Arthur Andersen**, **Adelphia** and **WorldCom**) was also a factor in the speed of the fall, as numerous large corporations were forced to restate earnings and investor confidence suffered. The **September 11 attacks** also contributed heavily to the stock market downturn, as investors became unsure about the prospects of **terrorism** affecting the United States economy. The stock exchanges recovered after this correction. There is an increase in the value of the S&P 500 index until the implications of the subprime lending crisis reached the surface during 2007. This led to nervousness in the stock markets worldwide, and a drop in the index at the end of the period when data are available, can be seen.

In Figure 5.2 the implied volatility for the S&P 500 index is illustrated. The volatility is high when the index has moved dramatically during a short period of time. The volatility reflects the views on the market, as a high volatility will occur when there is nervousness present in the market. In particular, there are high volatilities for the period following ”the Internet bubble bursting” during 2001 and 2002. There is a dramatic increase in volatility towards the end of 2007, which lasts for the rest of the period confined by data available. As the market has yet to know the implications of the collapse of the subprime lending market, the volatility remains high.

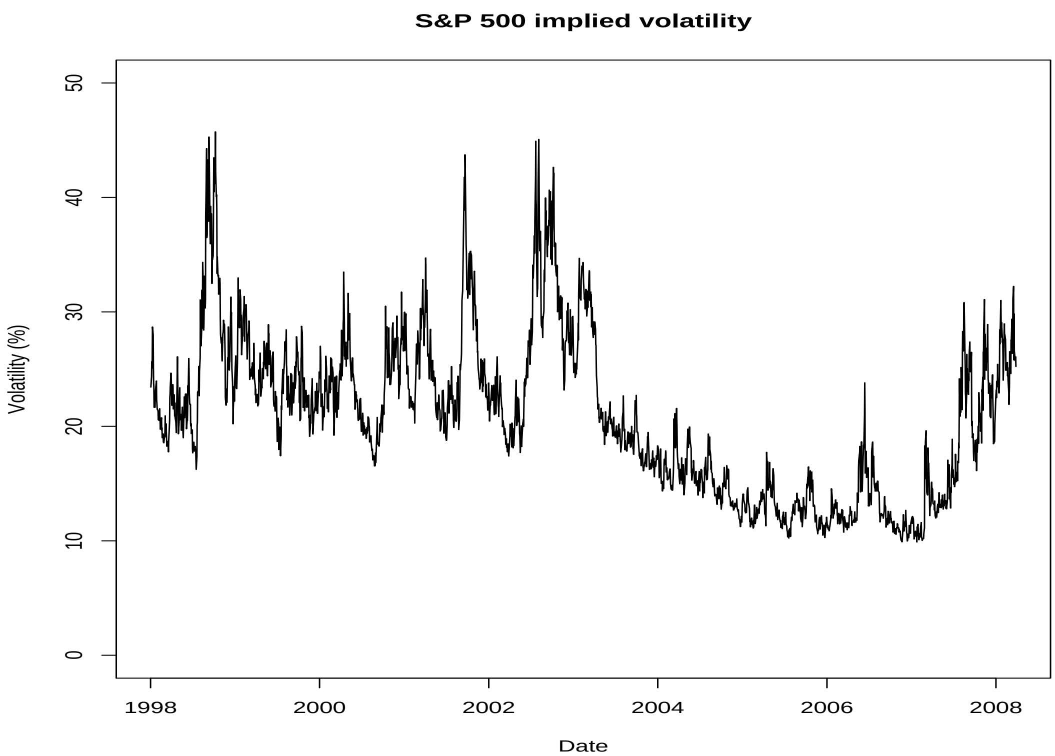

Figure 5.2: The implied volatility for the S&P index from 1998.

5.1.2 OSEBX

The **OSEBX** index is the Oslo Stock Exchange Benchmark Index, which is an investible index, comprimising the most traded shares listed on **Oslo Stock Exchange**, and contains 70 corporations. It is a semiannually revised free float adjusted index with changes implemented on January 1 and July 1 respectively. In the period between the review dates, the number of shares for each security is fixed with exception of adjustments for corporate actions with priority for existing shareholders. The OSEBX index is adjusted for dividend payments. As the Oslo Stock Exchange is a relatively small stock exchange, and the index contains only 70 corporations, this index is significantly more volatile than the S&P 500 index. There are index data available from 1998 until March 2008.

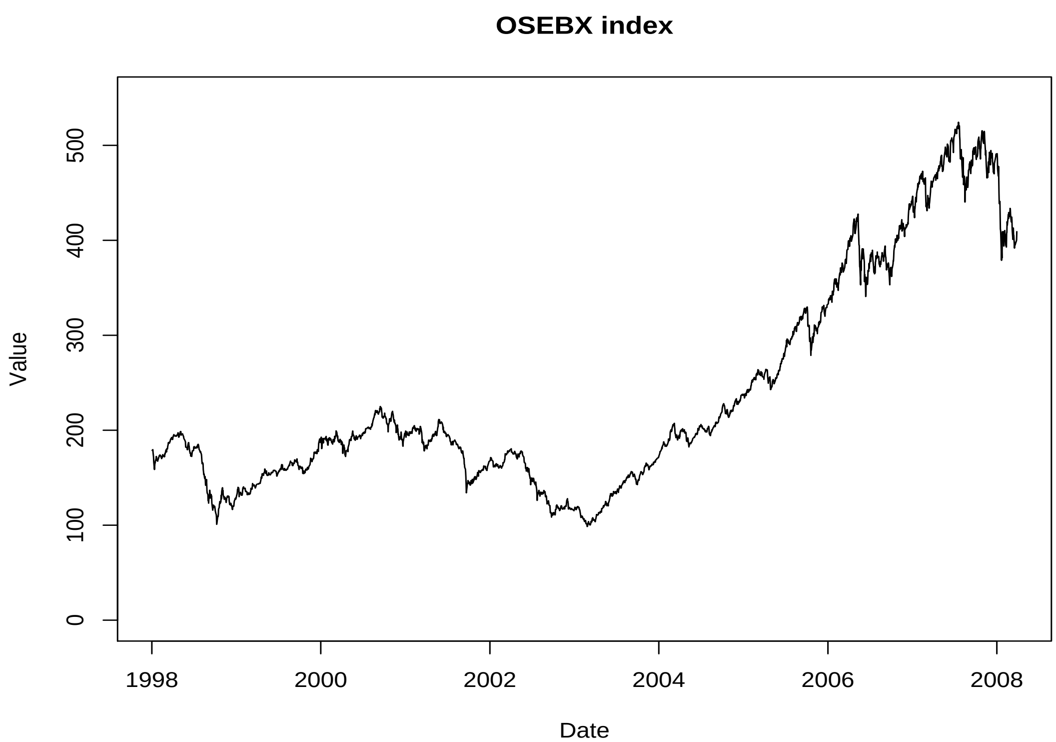

Figure 5.3: The OSEBX index from 1998.

The available data is illustrated in Figure 5.3. The OSEBX dropped dramatically in value, following ”the Internet bubble bursting” during 2001 and 2002. Starting in 2003, however, there has been an extreme increase in the value of the OSEBX index until the end of 2007, when the implications of the collapse of the subprime lending market affected most of the stock exchanges worldwide, including the Norwegian stock market. The reason for the extreme increase in the value of the OSEBX index is mainly the dramatic increase in the oil price. Many of the most traded shares listed on the Oslo Stock Exchange are companies in the oil industry, thus they are highly dependent of the oil price.

5.2 Implementation

The index data and implied volatilities for the S&P 500 index are given. For the other indices, the historical volatility is computed, based on the previous three years of the index. To approximate the volatility to those used in the market, the ratio between the implied and historical volatility for the S&P 500 index is used. This ratio is multiplied with the historical volatility of the particular index considered. This leads to a volatility which consider the history of the index, in addition to account for the expectations on the market. Even though the implied volatility of the S&P 500 index is used, and thereby the expectations on the largest companies on the United States Stock Exchange are considered, when calculating the implied/historical volatility ratio, the market worldwide is highly correlated with the United States market. Thus, this can be considered an applicable approximation.

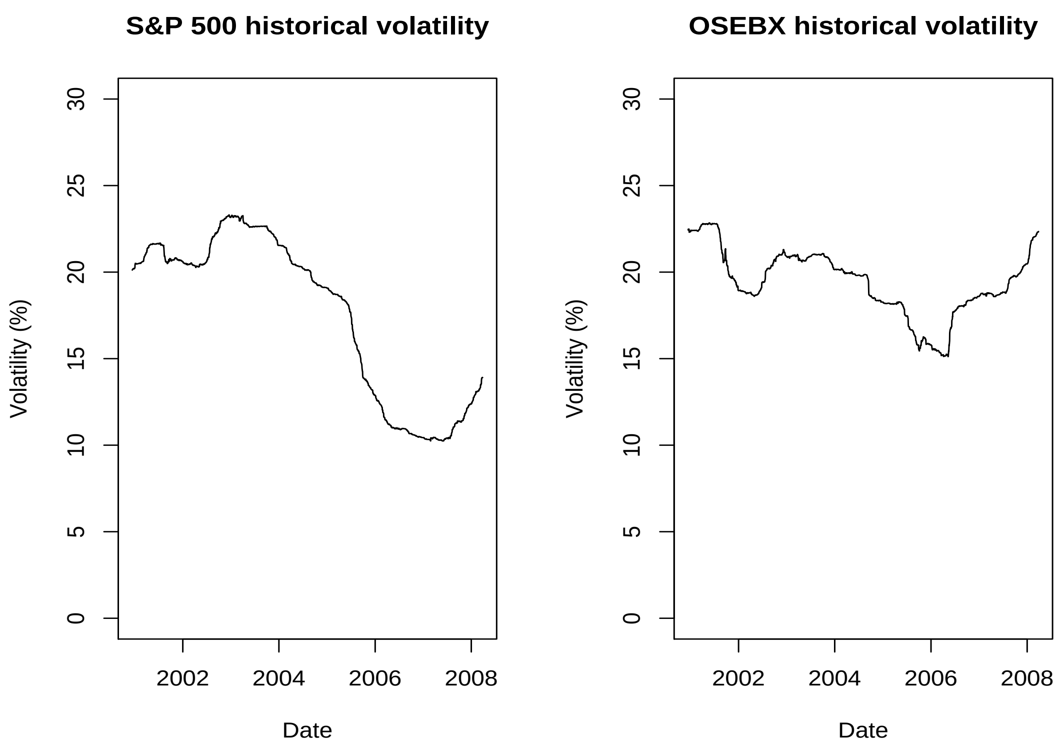

Figure 5.4: The historical volatility for the S&P 500 index and the OSEBX index.

In Figure 5.4, the historical volatility of the S&P 500 index and the OSEBX index are illustrated. The historical volatility is based on the movement of the index the past three years. The volatilities are relatively similar during the beginning of the period considered. Towards the end of 2005, the historical volatility of the S&P 500 index decreases significantly, while the historical volatility of the OSEBX index does not decrease as much during 2005. During 2006, the historical volatility of the OSEBX index is increasing towards the end of the period considered. The historical volatility of the S&P 500 index decreases during 2006 and the beginning of 2007. At the end of 2007, when the subprime lending crisis started to affect the stock markets worldwide, the historical volatility of the S&P 500 index increases. The historical volatility increases significantly towards the end of the period for both indices. The historical volatility for the OSEBX index is significantly higher than the historical volatility for the S&P 500 index during the last three years of the period considered. Examining the movements of the indices during the last five years, show that the movement of the OSEBX index is considerably more dramatic than the movement of the S&P 500 index.

In order to compute estimates for option prices, given specific volatilities, simulation has been used to predict paths for movements in the value of the underlying asset. $\mathrm{d}X$, a random variable drawn from a normal distribution is simulated, with mean zero and variance $\mathrm{d}t$. The results of the simulation are applied in the lognormal random walk. All simulation is done using the C programming language, applying the Mersenne Twister pseudorandom number generator, MT19937. The random number generator is developed by Makoto Matsumoto and Takuji Nishimura. The generator has a Mersenne prime period of $2^{19937} - 1$, and has passed the DIEHARD statistical test[1]. MT19937 generates 1370 k ints/sec and is among the fastest simulation quality generators available. Simulating the movement of an asset, results in values for the underlying asset at expiry. When applying the payoff function, we are able to calculate a mean value for the payoff, given different volatilities. The path of the movement of the asset is preserved, in order to be able to price more advanced options such as American or Exotic.

The underlying index is simulated using the lognormal random walk. In order to be able to analyze the results, 250000 possible paths for the movement of the index are simulated. The paths include the daily value of the underlying index for each day during the three years when the strategies are applied. The values of the options are evaluated each day, based on the value of the underlying index, the time to expiry and the strike of the particular option on the date and simulation path considered. The value of the underlying index, the value of the option, as well as the accumulated expenditures and earnings on buying and selling options are preserved for each date and each simulated path, so that the portfolio values are stored in a matrix. The portfolio values are discounted to the time when applying the strategies commenced.

Depending on the horizon the investor has on his investments, there are various approaches to estimate the risk. Having a long horizon on the investments, the change in portfolio value the last week or the last month might be of interest. Being a trader who evaluates the portfolio value on a daily basis, the daily change in portfolio value might be more appropriate to consider. As strategies which are meant to be applied by traders are applied, the change in portfolio value from one day to the next is of interest.

[1] A battery of statistical tests for measuring the quality of a set of random numbers

A new matrix is therefore created, containing the daily change in portfolio value. From the daily changes in portfolio values, the Value-at-Risk and expected shortfall of the portfolio can be estimated. The risk of the portfolio is estimated for each day during the period considered, applying the different methods of estimating the risk. Having stored the portfolio values for each simulated path, the density of the portfolio value can be examined at any given time during the period when the option strategies are applied.

Chapter 6

Results & Discussion

6.1 Trading at Expiry

In order to examine the properties of the risk and return of portfolios which consist of put options, three different strategies which may be of interest from a trader's point of view will be examined. All three strategies are applied on two different indices, which have significantly different volatilities. When estimating the risk involved in these strategies, the daily changes in portfolio value are to be considered. It could have been more appropriate to consider weekly or monthly changes in portfolio value, had the strategies in focus been meant to be applied by a financial institution with a longer horizon on its investments. Value-at-Risk and expected shortfall have been in focus, when examining the risk connected with the strategies. The first strategy considered is a strategy where trading takes place at expiry.

6.1.1 S&P 500

The properties of risk and return of the strategy is analyzed. A long position is being held on the underying asset, which in the first case is the S&P 500 index. At the time when beginning to use this strategy, an at-the-money three months European put option is bought. The option is held until expiry, when the return is realized, in case the option has a value at time of expiry. If the option expires worthless, the option has not led to any gain, but evidently as the underlying asset must have increased in value, this leads to a positive contribution to the overall portfolio value. When the option expires, a new at-the-money three months European put option is bought. The strike of the new option will thus be equal to the value of the underlying asset at the time when the option is bought. This strategy is repeated when the new option expires, and this strategy is exercised for three years. This is a hedging strategy, where a put option is constantly used as insurance for large drops in the value of the underlying asset.

All values presented in the following plots are discounted to the time when the strategy was commenced. The portfolio value consists of the value of the underlying asset, the value of the option at the particular time, and accumulated costs and profit of the options which have been bought and exercised. Figure 6.1 shows the portfolio values, change in portfolio values, Value-at-Risk and expected shortfall develop during the period when the strategy is applied. The Value-at-Risk and expected shortfall are based on the daily change in portfolio value. The Value-at-Risk represents the 95% quantile, which limit the daily loss by a certain value with 95 % probability, given the simulation results can be regarded as realistic values. The expected shortfall represents the mean value of the 5% largest daily losses, given the simulation results represent a realistic view on the behaviour of the underlying asset.

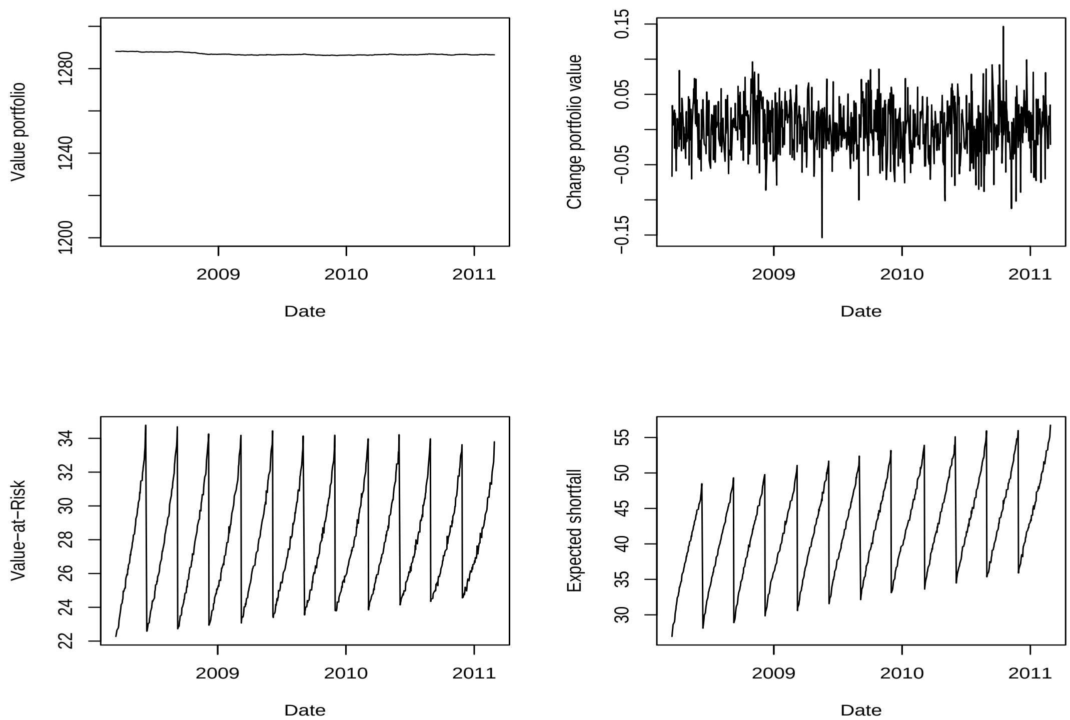

Figure 6.1: Portfolio values, daily change in portfolio values, Value-at-Risk and expected shortfall.

Of particular interest are the properties of the risk measurements; Value-at-Risk and expected shortfall, which vary periodically. They increase in value during the period when the option is valid, until the option expires, when a sudden drop in risk can be seen. The behaviour is repeated for the next option. For Value-at-Risk, the risk when a new option is bought increases slightly from the risk when the last option was bought. For the expected shortfall, the risk for the whole period when the option is valid increases slightly from one period to the next. When the behaviour of the Value-at-Risk and the expected shortfall is considered, the risk increases until the option held, expires. Value-at-Risk is based on the quantile which limits the maximum loss of the portfolio with 95% probability, given the simulated values are realistic. Thus, it is likely that this value is due to a change in the underlying asset, moving far above the strike while the expiry date of the option approaches. The reason why the exposure of risk seems to increase towards the expiry date of the option has to do with the simulated paths of the underlying asset which is far out-of-the-money. As time draws closer to expiry, the options unlikely to be exercised, will rapidly decrease in value. Thus, an increasing exposure of risk towards the expiry of the option is seen. When a new option is bought, the risk exposure of the portfolio will apparently drop, because of the sudden increase in time to expiry. Note that the risk again seems to increase towards expiry of the new option. This is due to the decreasing time value of the options which are out-of-the-money at the time considered.

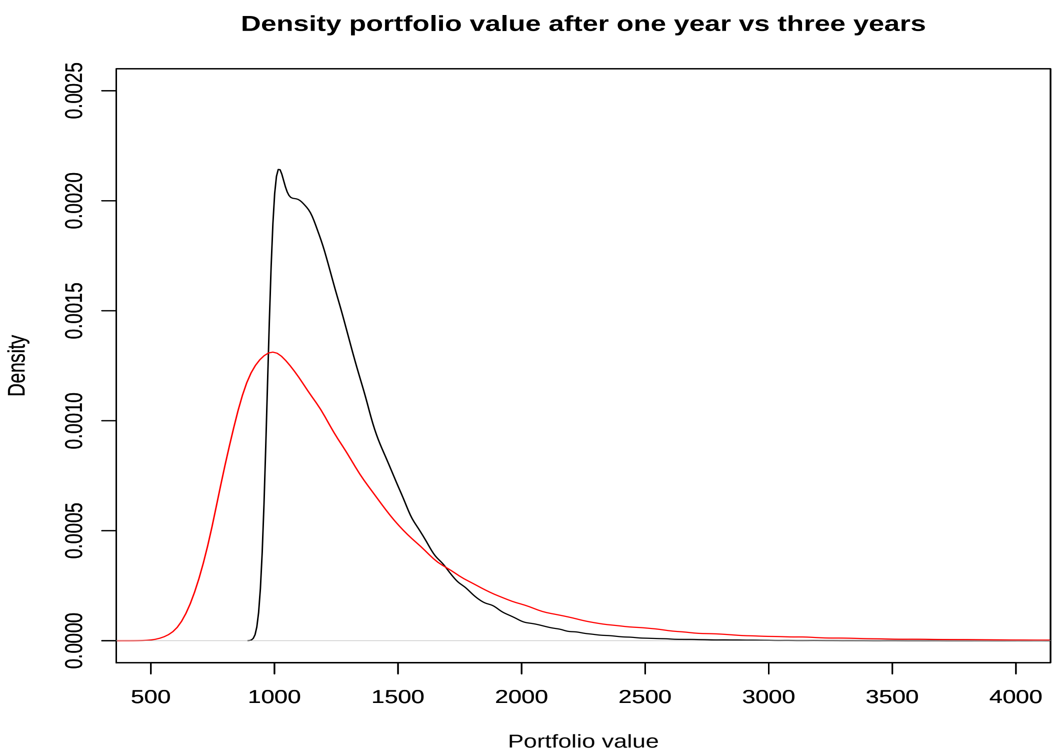

Figure 6.2: Density portfolio values, one year after having started using the option strategy, compared with the density of the portfolio values, three years after having started using the option strategy.

In Figure 6.2, the density of the portfolio values three years after having begun applying the option strategy is illustrated with a red line. The density of the portfolio values one year after having started applying the option strategy (black line) is included for comparison. The most probable portfolio values are greater for the portfolio after one year than after three years of applying the option strategy. Naturally, after having applied the strategy for three years, the possible outcomes will vary more. The probability for both large losses and to make a profit will be greater. As the expected portfolio value will not change significantly during the period considered, the probability for extreme outcomes will increase during the period.

6.1.2 OSEBX

The same strategy is considered for the Norwegian OSEBX index, holding a long position on the underlying asset for three years. At the same time an at-the-money three months European put option is bought. The option is held until expiry, when a new at-the-money three months European put option is bought. This strategy is repeated for three years.

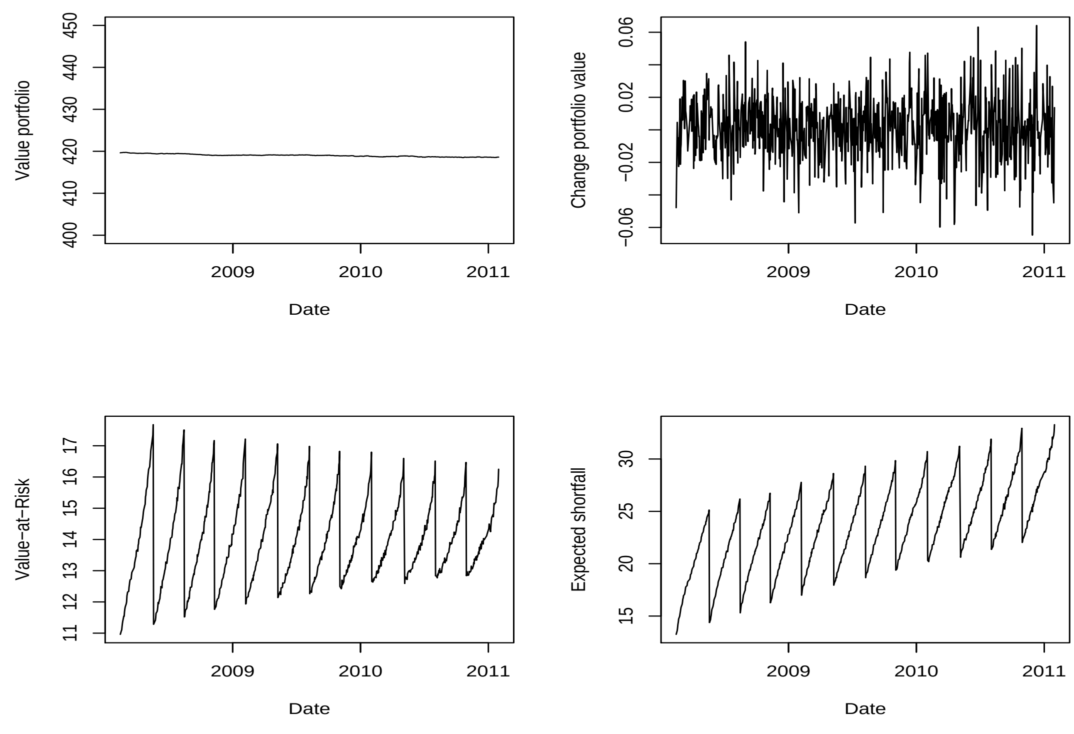

Figure 6.3: Portfolio values, daily change in portfolio values, Value-at-Risk and expected shortfall.

Figure 6.3 shows the resulting portfolio values, change in portfolio values, Value-at-Risk and expected shortfall for the period when the strategy is applied. From the Value-at-Risk and expected shortfall, the risk seems to behave periodically. The risk increases until expiry of the option, and drops suddenly when a new option is bought. The expected shortfall has similar properties to the expected shortfall for the S&P 500 index, and the risk during the life of the option increases from the lifetime of one option to the lifetime of the next which is bought. For the Value-at-Risk, the risk of the portfolio shortly before expiry of the option seems to decrease from one option to the following bought. At the same time, the risk shortly after having bought a new option seems to increase from an option to the next one bought.

Considering the Value-at-Risk and the expected shortfall, the risk increases towards expiry for all the periods when holding an option. This is due to the decrease of time value, so that an option which is out-of-the-money will decrease rapidly in value. This will occur to a greater extent when time draws closer to the date of expiry, thus the out-of-the-money options will be more likely to expire worthless. Apparently the risk for the S&P 500 index is greater than the risk for the OSEBX index. This is due to the initial value of the S&P 500 index when beginning to apply the strategy, which is significantly greater than the initial value of the OSEBX index.

In Figure 6.4, the density of the portfolio values one year after having started applying the option strategy is illustrated with a black line, and the density of the portfolio values after three years is illustrated with a red line. As noted for the S&P 500 index, the most probable outcomes lead to a greater portfolio value, after having applied the strategy for one year, than after three years. The possible outcomes will vary significantly more after having applied the strategy for three years than for one year. This is not difficult to understand, considering the properties of the strategy; as time goes by, the possible outcomes will vary more. The expected portfolio value will not vary much during the period considered. A particular skewness to the density of the portfolio values can be noted after having applied the strategy for three years. There will be more possible outcomes to the right of the peak of the density than to the left. As the expected values of the portfolio does not vary much, the most probable portfolio values will decrease during the period. There is a greater probability for large losses and large profits as time goes by.

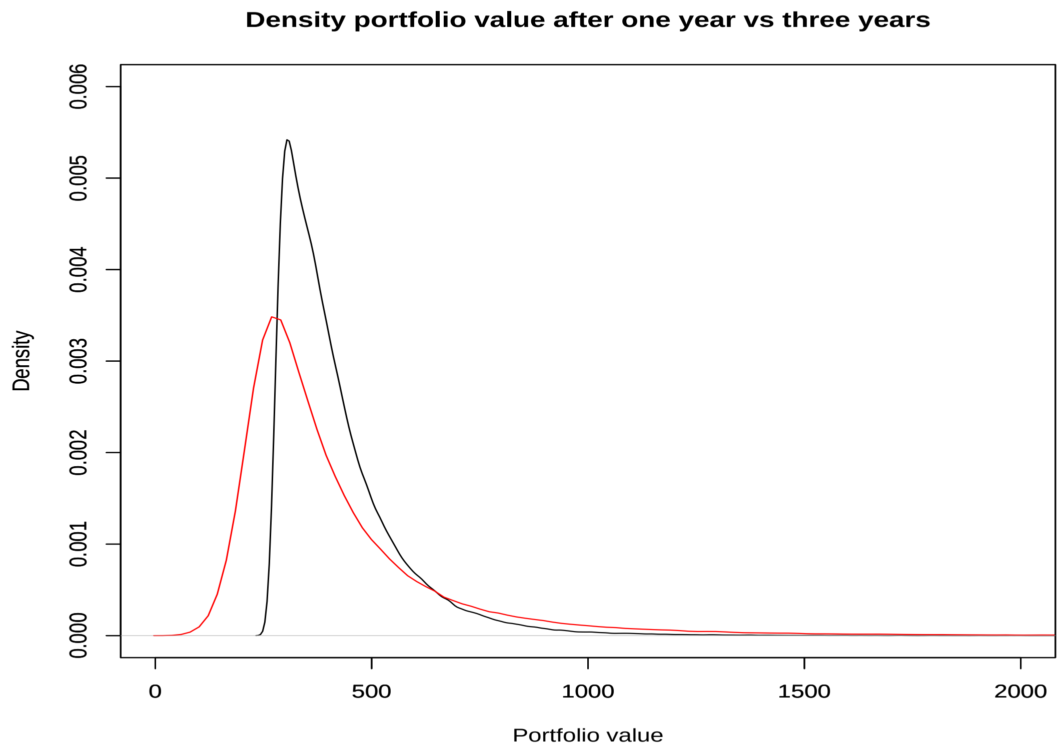

Figure 6.4: Density portfolio values, one year after having started using the option strategy, compared with the density of the portfolio values, three years after having started using the option strategy.

6.2 Trading at Predefined Levels

A strategy where trading takes place at predefined levels has been considered. The aim is to maximize the return and minimize the risk of the portfolio. In order to compare with the other strategies, the daily changes in portfolio value are estimated, from which estimates for Value-at-Risk and the expected shortfall are made.

6.2.1 S&P 500

Another kind of hedging strategy, used on the S&P 500 index is considered. This strategy consists in buying a long position on the underlying asset at the start of the period, while simultaneously buying an at-the-money three months European put option. The option is held until the underlying asset differs more than 5% from the strike price. If the underlying asset is more than 5% above the strike price, the option is sold at the market price, which evidently is less than for what the option was bought. Thus, the loss is realized, and a new at-the-money option is bought, at strike price equal to the value of the underlying asset at the particular time. In case the underlying asset has moved below 5% less than the strike price, the option is sold at market price. In this case, the option is worth more than it was at the time it was bought. Simultaneously as the option is sold, a new at-the-money three months European put option is bought, with strike price equal to the value of the underlying asset. If the underlying asset does not cross either barriers during the lifetime of the option, the option is exercised in case it has a value. If the underlying asset has a higher value than the strike price at expiry, the option expires worthless. A new option is bought when the previous option expires. This strategy is exercised for three years.

The portfolio value is evaluated every day during the period when the strategy is applied. The portfolio value consists of the value of the underlying asset and the value of the put option at the particular time, in addition to accumulated expenditures and profit of buying and selling options. All portfolio values are discounted to the time when this strategy was started being applied.

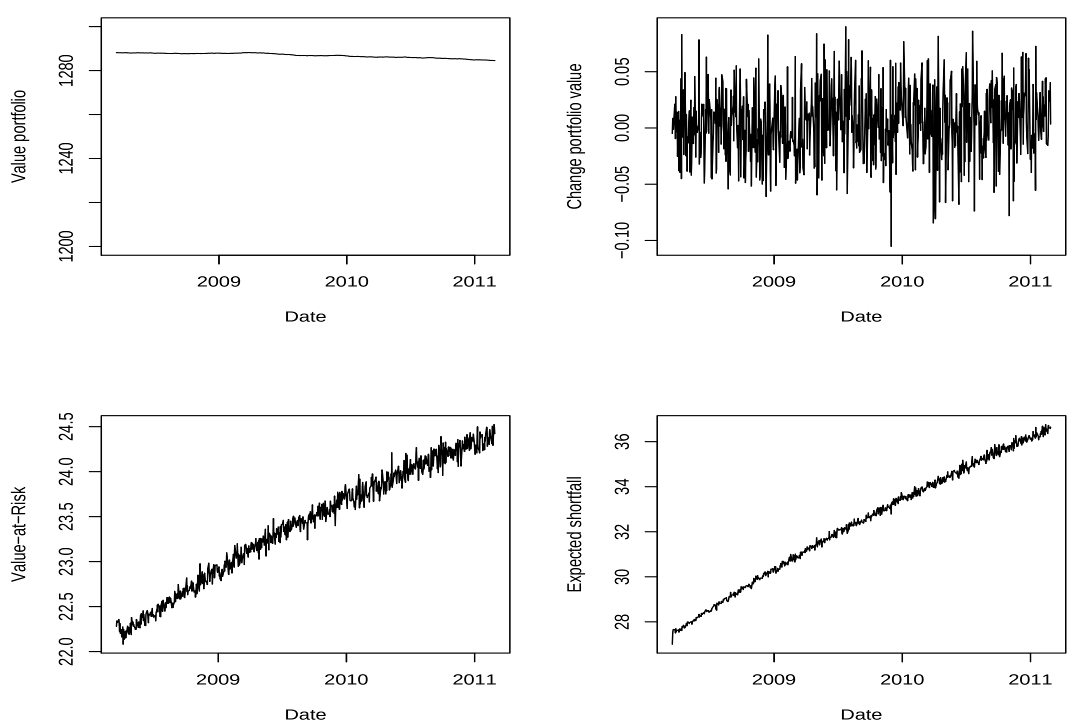

Figure 6.5: Portfolio values, daily change in portfolio values, Value-at-Risk and expected shortfall.

Figure 6.5 shows the portfolio values, change in portfolio values, Value-at-Risk and expected shortfall of the option strategy during the period when the strategy is used. The Value-at-Risk increases during the period. The variance of the Value-at-Risk does also increase during the period considered. The same tendency is seen for the expected shortfall. The expected shortfall increases during the period when the option strategy is applied, and so does its variance.

The reason why there is an apparent increase in risk during the period considered, could be explained by an increase in the number of options unlikeliy to be sold with a profit or expire with a value. When simulating the paths of the underlying index, a drift term equal to the interest rate obtained on a risk-free investment is used. This will evidently lead to more options out-of-the-money than in-the-money. Consequently, as the time goes by, there will be an increasing number of options unlikely to be exercised or sold with a profit. This leads to an increase in risk throughout the period when the option strategy is beeing applied.

The values of the Value-at-Risk and expected shortfall are smaller for the strategy which implies trading at predefined levels than for the strategy which implies trading at expiry. Apparently, the exposure for risk when the strategy trading at predefined levels is applied, is not as great as for the strategy trading at expiry. When trading at predefined levels, the options unlikely to expire with a value can be sold, while when trading at expiry, the options are held until they expire, regardless of what happens during the period when the strategy is applied. When trading at predefined levels, the options which have increased significantly in value may be sold, while when trading at expiry, the options might decrease in value towards expiry. When trading, depending on the movement of the underlying index, it seems as if the exposure to risk is not as great as for the strategy not affected by the movement of the underlying index.

Figure 6.6 shows the densities of the portfolio values for the S&P 500 index, applying the strategy trading at predefined levels. The density of the portfolio values after having used the strategy for one year is illustrated with a black line, while the density of the portfolio values after having applied the strategy for three years is illustrated with a red line. As noted for the strategy trading at expiry, the possible outcomes will vary more as time goes by. There will be a greater probability for large losses and large profits after having applied the strategy for three years than for one year.

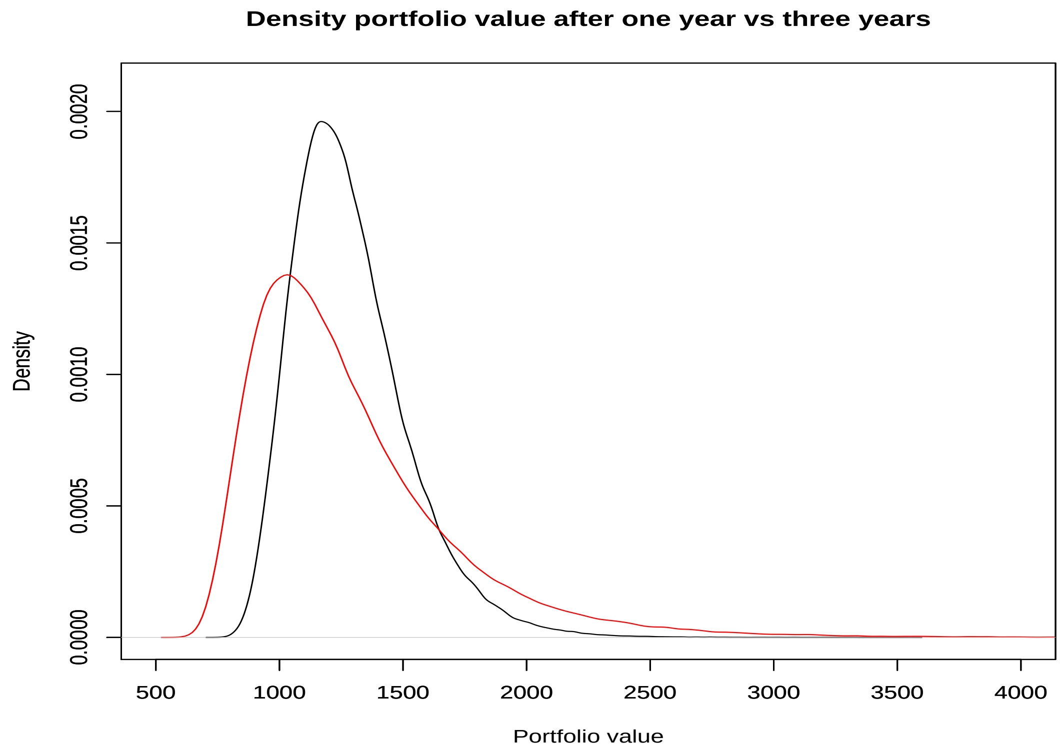

Figure 6.6: Density portfolio values, one year after having started using the option strategy, compared with the density of the portfolio values, three years after having started using the option strategy.

6.2.2 OSEBX

The strategy implemented on the S&P 500 index was also implemented on the Norwegian OSEBX index. Figure 6.7 shows the behaviour of the portfolio values, change in portfolio values, Value-at-Risk and expected shortfall during the period when the strategy is applied.

From Figure 6.7 it can be seen that the Value-at-Risk increases during the period considered, and so does its variance. The Value-at-Risk for the OSEBX index varies less than for the S&P 500 index. The expected shortfall increases during the period when the strategy is being used, but varies more towards the end of the period. The variance of the expected shortfall is less than the variance of the Value-at-Risk.

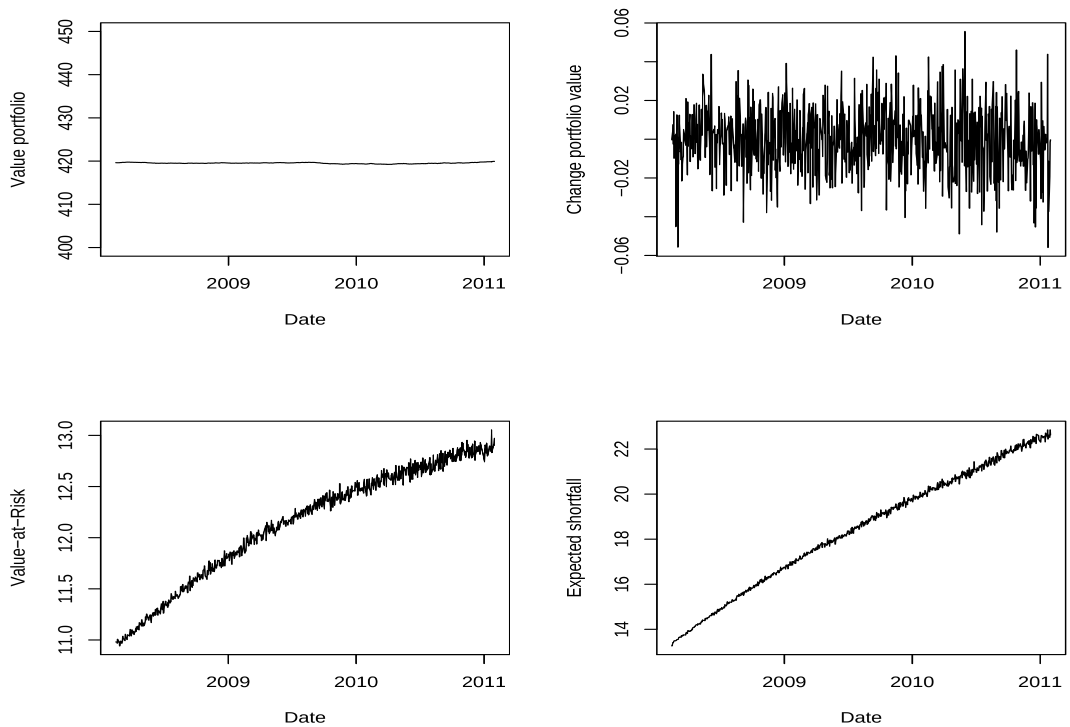

Figure 6.7: Portfolio values, daily change in portfolio values, Value-at-Risk and expected shortfall.

Note that for both the OSEBX index and the S&P 500 index, the risk apparently increases during the period considered. This is due to the increasing number of options which are close to the 5% barrier above the strike, leading to larger decreases in portfolio values, as the options are less likely to expire with a value.

As noted for the S&P 500 index, the values of Value-at-Risk and expected shortfall are smaller for the strategy which involves trading at predefined levels than for the strategy which involves trading at expiry. It is therefore obvious that the exposure for risk when applying the strategy trading at predefined levels is less, compared with the strategy which involves trading at expiry. It seems as if the exposure for risk can be reduced when trading based on the movement of the underlying index, compared with trading at expiry.

Figure 6.8 illustrates the densities of the portfolio values for the OSEBX index, when applying the strategy which involves trading at predefined levels. The density of the portfolio values after one year of using the strategy is illustrated with a black line, while the density of the portfolio values after three years is illustrated with a red line. As noted for the S&P 500 index, the possible outcomes vary more as time goes by. The probability of large losses and large profits will increase during the period considered. Note as well that the difference between the densities is not as great as for the S&P 500 index and for the strategy trading at expiry.

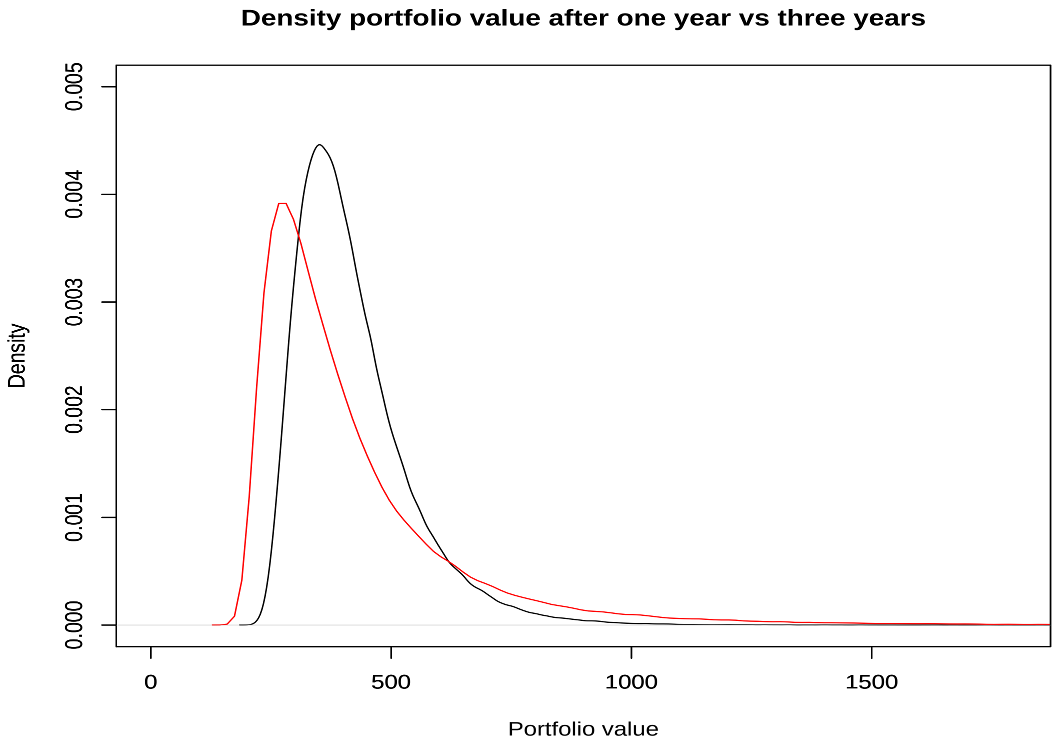

Figure 6.8: Density portfolio values, one year after having started using the option strategy, compared with the density of the portfolio values, three years after having started using the option strategy.

6.3 Put Spread

In order to compare the risk/return characteristics of portfolios consisting of put options, a strategy where the portfolio consists of put options exclusively is examined. Looking at the strategy from a trader's point of view, the short time movement in the portfolio value is of interest. Thus, the daily change in portfolio value is considered, when examining the risk of the strategy. From the daily change in portfolio value, estimates for Value-at-Risk and the expected shortfall can be made.

6.3.1 S&P 500

A strategy called put spread is considered. This strategy consists in buying an at-the-money three months European put option, while writing 10% out-of-the-money three months European put options. The number of options written depends on the price of the option bought, such that the amount spent on buying the option equals the amount received when writing the out-of-the-money options. This means the portfolio is self-financing, having no expenditures when creating the portfolio. Not having any position on the underlying asset; the portfolio consists only of the two different positions on the put options. The option held is sold, and the written options are bought if the value of the portfolio exceeds the initial price of the at-the-money put option. The option is sold and the out-of-the-money options are bought to their market value. A profit has been made by these transactions, and a new at-the-money three months European put option is bought, while writing a number of 10% out-of-the-money three months European put options, such that the sum of expenditures involved in these transactions equals zero. The strike of the new options will for the at-the-money option equal the value of the underlying asset at the time, while for the 10% out-of-the-money options, the strike will be 10% below the value of the underlying asset at the time.

On the other hand, if the value of the portfolio exceeds the negative of the initial price of the at-the-money option, the loss is realized, i.e. the option held is sold, and the options written are bought. A new at-the-money option is bought, with strike equal to the value of the underlying asset at the time, and a number of 10% out-of-the-money options are written, so that the amount for which the at-the-money option is bought equals the amount received when writing the out-of-the-money options. If the portfolio value does not cross either barriers during the lifetime of the options, the at-the-money option is exercised in case it has a value at expiry. If the out-of-the-money options have a value at expiry, the corresponding amount has to be paid. This strategy is repeated for three years, when the portfolio value is evaluated on a daily basis. First, the S&P 500 index is considered.

Figure 6.9 illustrates the portfolio values, the daily change in portfolio values and the corresponding Value-at-Risk and expected shortfall for the put spread strategy, applied on the S&P 500 index. The Value-at-Risk seems to fluctuate in a periodic manner, as the risk seems to increase immediately after having traded. The risk then seems to decrease, before a significant increase is noted towards the expiry of the options which have not been traded during their lifetime. The reason for the dramatical increase of the Value-at-Risk towards the date when most of the options expire, is that the value of the out-of-the-money options decreases rapidly when time draws closer to expiry, and the options are less likely to expire with a value. Recall that the Value-at-Risk is the quantile which limits the loss of the portfolio with a certain probability. In this case, the Value-at-Risk is the maximum loss one might have at the particular day, with 95% probability. The peaks of the Value-at-Risk seem to decrease during the period. This is due to the fact that as time passes, there will be an increasing number of paths where options have been traded before expiry. Thus, there will not be as many paths having options which expire simultaneously. The peaks occuring every three months will therefore not be as distinct as in the beginning of the period when the strategy is applied.

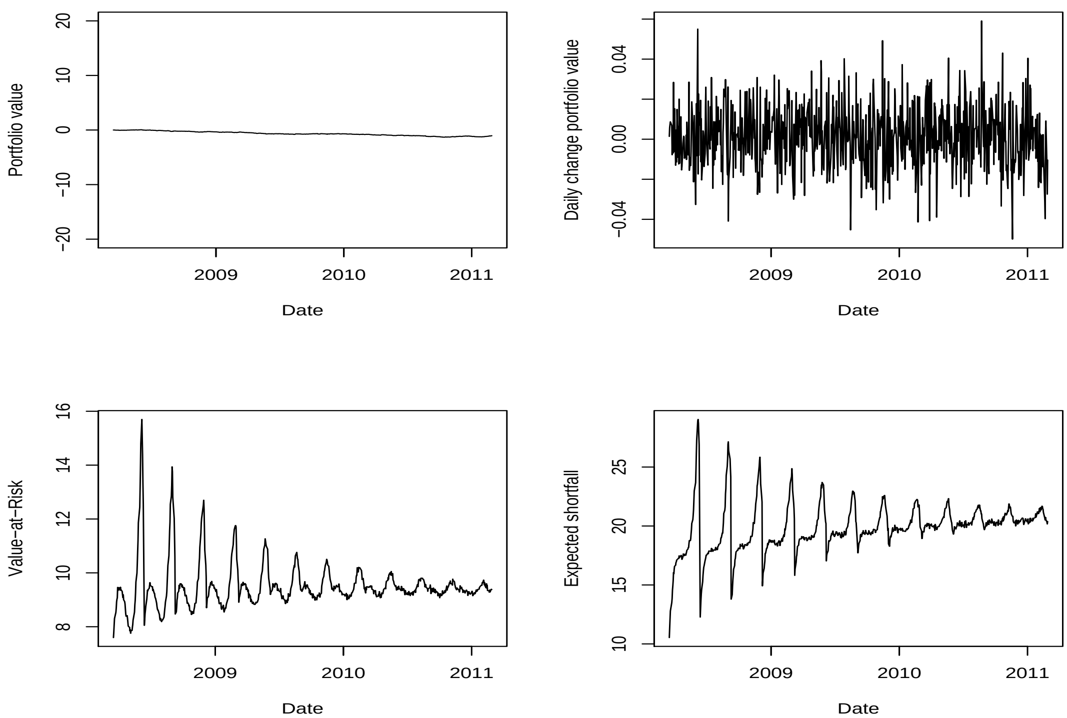

Figure 6.9: Portfolio value, daily change in portfolio values, Value-at-Risk and expected shortfall.

Examining the values of the expected shortfall, it seems to behave in a periodic manner, as noted for the Value-at-Risk. In the beginning of the period considered, the risk seems to increase immediately after having traded options. The risk then appears to stabilize, before it seems to increase towards the date of expiry of the options. The reason why the expected shortfall increases towards the date when most of the options expire, is that the options which are out-of-the-money a short time prior to expiry, decreases rapidly in value as time draws closer to expiry. As noted for the Value-at-Risk, the peaks which occur every three months seem to decrease in value as time goes by. As the expected shortfall is based on the paths which lead to the 5% largest losses, and not just one single path, the peaks do not decrease as rapidly as for the Value-at-Risk. The expected shortfall for the period immediately after having traded options seems to increase during the period. As a decreasing number of paths will consist of options expiring simultaneously, the expected shortfall will tend to stabilize.

In the beginning of the period, the risk profiles vary a lot more than the risk profiles for the strategy trading at predefined levels. When trading at expiry, the risk vary throughout the whole period considered. For the strategy trading at predefined levels, the risk is increasing during the period considered, while for the put spread strategy, the risk is fluctuating a lot in the beginning of the period, before it seems to stabilize towards the end of the period considered.

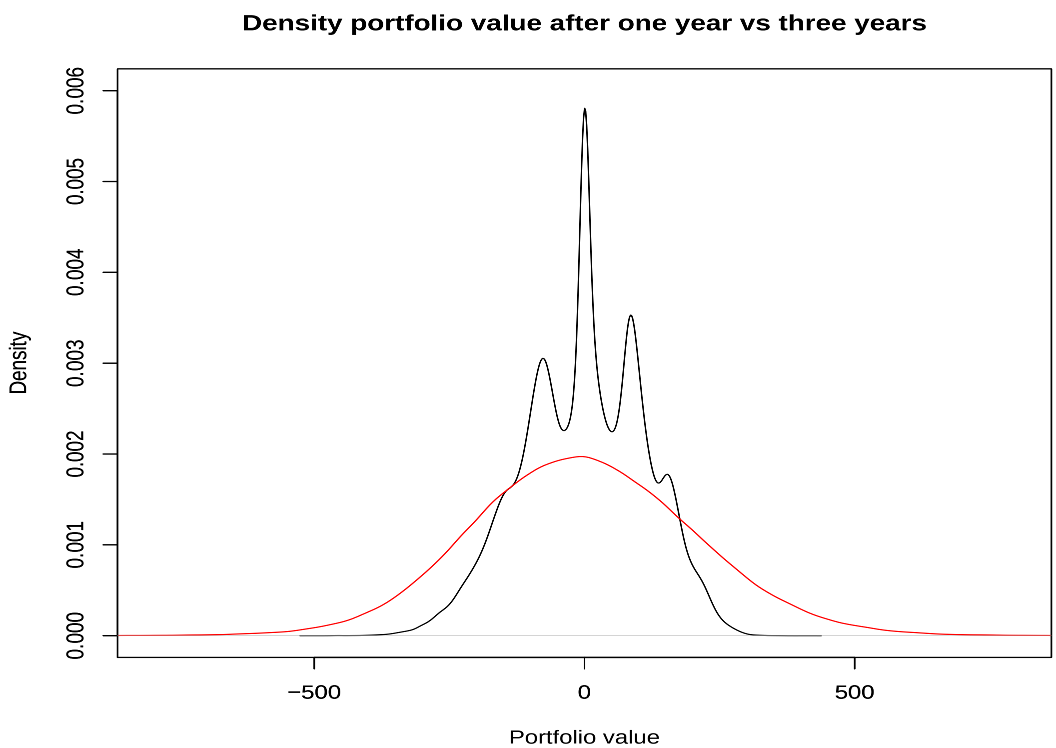

Figure 6.10: Density portfolio values, one year after having started using the option strategy, compared with the density of the portfolio values, three years after having started using the option strategy.

Figure 6.10 illustrates the densities of the portfolio values when applying the put spread strategy on the S&P 500 index. The density of the portfolio values one year after having started to apply the option strategy is illustrated with a black line, while the density of the portfolio values three years after having started applying the option strategy is illustrated with a red line. There is a significantly greater probability for the portfolio value to be close to zero after having applied the strategy for one year, than after three years. On the other hand, there will be a greater probability for large profits and losses after three years, than after one year of applying the strategy. The portfolio value has a peak at zero after one year of applying the strategy. There will be a number of paths where no trading have taken place prior to expiry, hence the peculiar shape of the density after one year of applying the strategy. The peaks on each side of the largest peak correspond to the positive and negative of the initial value for the at-the money options. The positive peak is slightly greater than the negative peak. Apparently there is a slightly greater probability for the portfolio value to cross the barrier corresponding to the positive of the initial value of the at-the-money option than the negative. The peaks seen for the density after one year of applying the strategy represent paths where no trading have been done prior to expiry, in the middle, and for the paths where trading have been done once prior to expiry on each side of the largest peak. The small peak seen on the right hand side, corresponds to paths where the portfolio value has crossed the barrier of the positive value of the at-the-money option twice prior to expiry. The distinct peaks seen in the density of the portfolio after one year of applying the strategy, disappear after three years.

Figure 6.11, illustrates how the estimated risk depends on the time horizon. The first plots illustrate the Value-at-Risk and expected shortfall, estimated based on the daily changes in portfolio value. Further, the Value-at-Risk and expected shortfall are estimated, based on the weekly and monthly changes in portfolio value. The Value-at-Risk increases in value as the time horizon increases. The greatest variance in Value-at-Risk is obtained when the estimates are based on weekly changes in portfolio value. The strategies applied are of interest from a trader's point of view, thus the daily changes in portfolio value will be of interest. Being a financial institution or a person with a longer horizon on his investments, it could have been more appropriate to utilize the weekly or monthly changes in portfolio value when exploring the risk. Similarly to the Value-at-Risk, the expected shortfall increases in value as the time horizon increases. It is also noted that the expected shortfall seems to vary the most when we base our estimates on weekly changes in portfolio value. Looking at the expected shortfall based on monthly changes in portfolio values, the expected shortfall increases during the period considered. The reason why the Value-at-Risk and the expected shortfall seem to increase in value when the estimates are besed on a longer time horizon is that there will be a greater probability for the portfolio value to change significantly, considering a longer time interval.

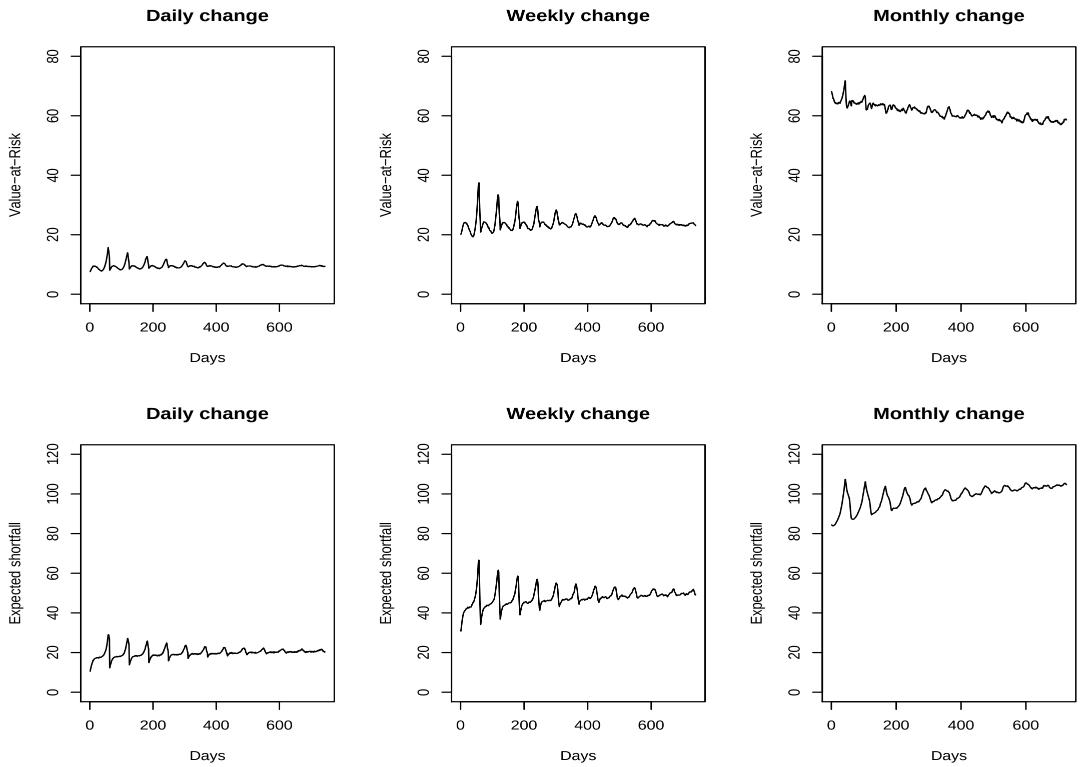

Figure 6.11: Estimated Value-at-Risk and expected shortfall, based on daily, weekly and monthly changes in portfolio value, respectively.

6.3.2 OSEBX

The put spread strategy is applied on the OSEBX index, and the value of the portfolio, daily change in portfolio value, and the corresponding Value-at-Risk and expected shortfall is illustrated in Figure 6.12.

The Value-at-Risk in Figure 6.12 shows a peak every three months, as was also noted for the S&P 500 index. The peaks tend to decrease in value as time goes by. As a decreasing number of paths will consist of options which expire simultaneously, the Value-at-Risk will decrease during the period considered. The reason why the Value-at-Risk seems to increase significantly in value towards the expiry of the options, is that the options which are out-of-the-money will decrease rapidly in value as the time draws closer to expiry, and the options are less likely to expire with a value. Contrary to the two other strategies considered, for the put spread strategy the Value-at-Risk for the OSEBX index is greater than the Value-at-Risk for the S&P 500 index. This is due to the volatility of the index, which is significantly greater than for the S&P 500 index.

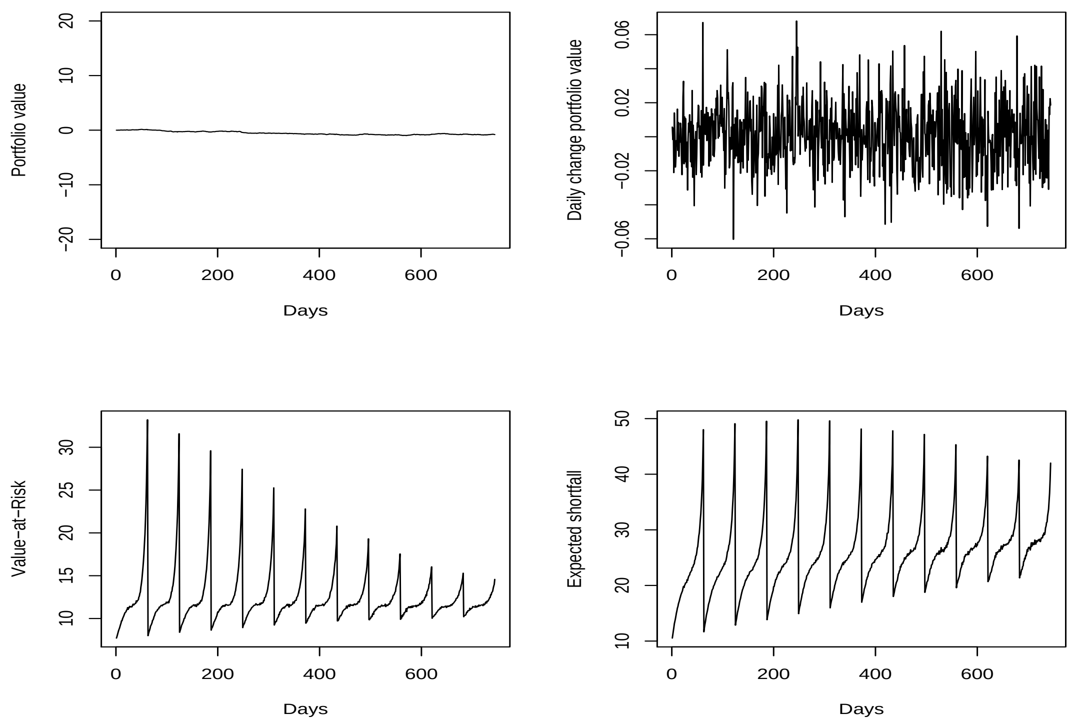

Figure 6.12: Portfolio value, daily change in portfolio value, Value-at-Risk and expected shortfall.

As the other strategies involve a long position on the underlying index, the S&P 500 index seems to give larger values of Value-at-Risk, even though the volatility of the OSEBX index is greater than for the S&P 500 index. The reason for this is that the initial value of the S&P 500 index is significantly greater than the initial value of the OSEBX index. Thus, the numeric change in the underlying index will be greater for the S&P 500 index than for the OSEBX index, even though the volatility is greater for the OSEBX index. Applying the put spread strategy, the portfolio value does not consist of any position on the underlying index, except from the positions on the options, which of course depend on the underlying index. The change in portfolio value will for the put spread strategy depend on the volatility of the index, hence the larger Value-at-Risk for the OSEBX index than for the S&P 500 index.

Similarly to the Value-at-Risk pattern, the expected shortfall has peaks every three months. For the OSEBX index, the peaks which occur every three months do not seem to change much in value during the period considered. Even though there will be less paths consisting of options which expire simultaneously as time passes, the peaks at time of expiry, every three months, will remain relatively unchanged. The expected shortfall immediately after the peaks, every three months tend to increase in value during the period considered.

As noted for the Value-at-Risk, the expected shortfall, when applying the put spread strategy, has greater values for the OSEBX index than for the S&P 500 index. Not surprisingly, a large volatility will lead to a large risk when applying the put spread strategy. The expected shortfall for the put spread strategy with the OSEBX as the underlying index is larger than the expected shortfall for the other strategies considered.

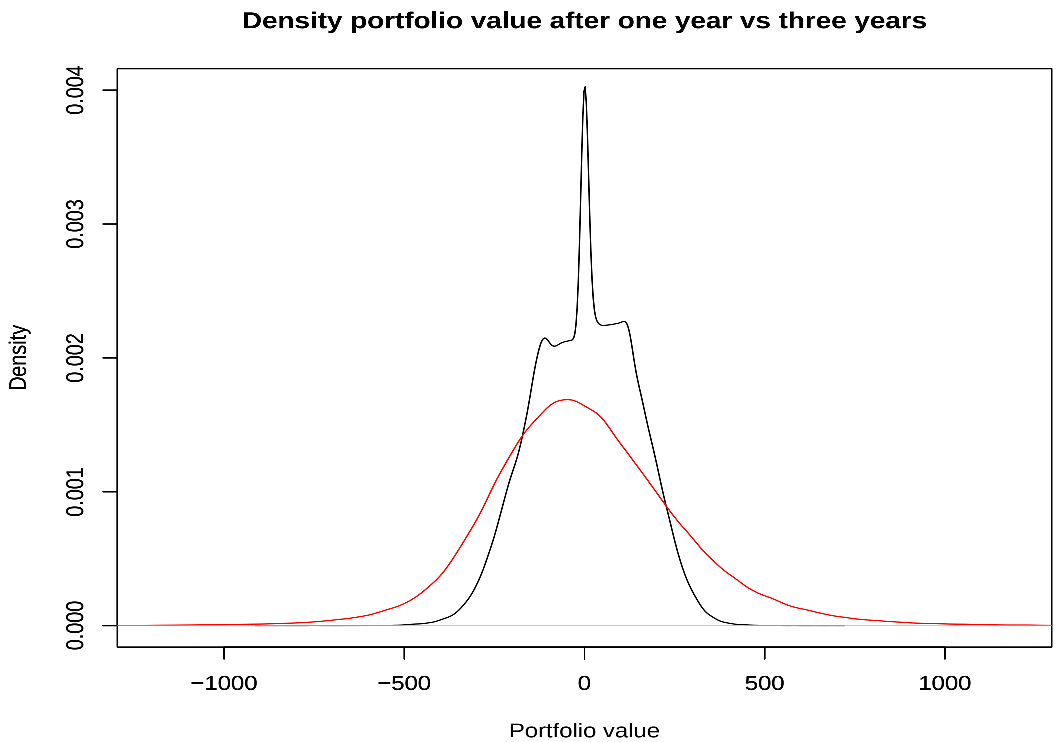

Figure 6.13: Density portfolio values, one year after having started applying the option strategy, compared with the density of the portfolio values three years after having started applying the option strategy.

In Figure 6.13, the density of the portfolio value after one year (black line) of using the option strategy is compared with the density after three years (red line). As was seen for the S&P 500 index, there is a greater probability for the portfolio value to be about zero after one year of exercising the strategy than after three years. After three years there will be a greater probability for the portfolio value to have decreased or increased significantly in value than after one year of applying the strategy. The peculiar shape of the density after having applied the strategy for one year is due to the fact that there will be a number of paths where there have been no trading prior to expiry. As the strategy consists of a self-financing portfolio, there will be a number of paths, where there have been no trading prior to expiry, and the options have expired worthless. The portfolio value will thus be zero for the paths where no trading have been done prior to expiry and the options have expired worthless. For the density after one year of applying the strategy, there are two smaller peaks on each side of the largest peak. The values of the peaks correspond to the negative and the positive of the initial value of the at-the-money option. As there will be a number of paths where there have been trading once prior to expiry, there are three distinct peaks on the density after one year of applying the strategy. After three years of trading, the portfolio values will be more evenly distributed, leading to a smoother density.

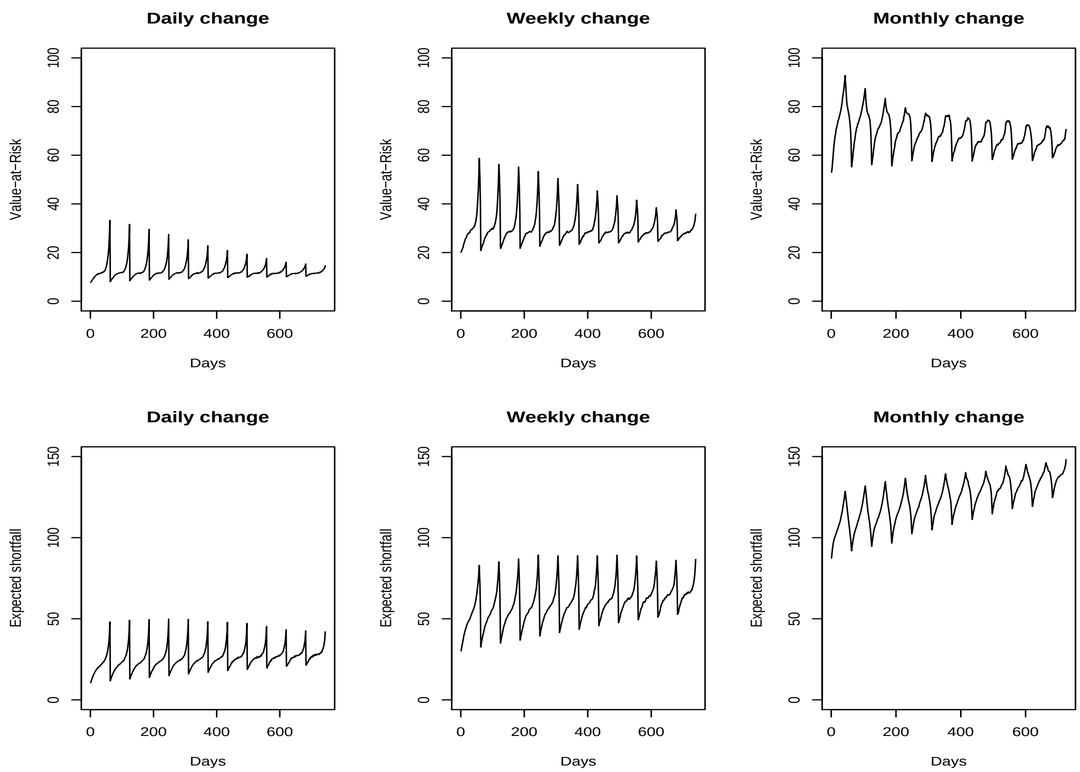

Figure 6.14: Value-at-Risk and expected shortfall based on daily change, weekly change and monthly change in portfolio value, respectively.

Figure 6.14 illustrates the Value-at-Risk and expected shortfall based on daily change, weekly change and monthly change in portfolio value, respectively. The Value-at-Risk increases when the estimates are based on a longer time interval. This is natural, as the portfolio have had more time to vary, thus the risk will be greater. For all horizons considered, the peaks which occur every three months tend to decrease towards the end of the period considered. Value-at-Risk varies the most when basing the estimates on weekly changes in portfolio value, and the least when basing on daily changes.

For the expected shortfall, the risk will increase when the estimates are besed on a longer period of time. It varies the most when basing the estimates on weekly changes in portfolio value, compared with the other horizons considered. For the expected shortfall based on monthly changes in portfolio value, the risk seems to increase throughout the period considered.

6.4 Analysis the Three Strategies

6.4.1 S&P 500

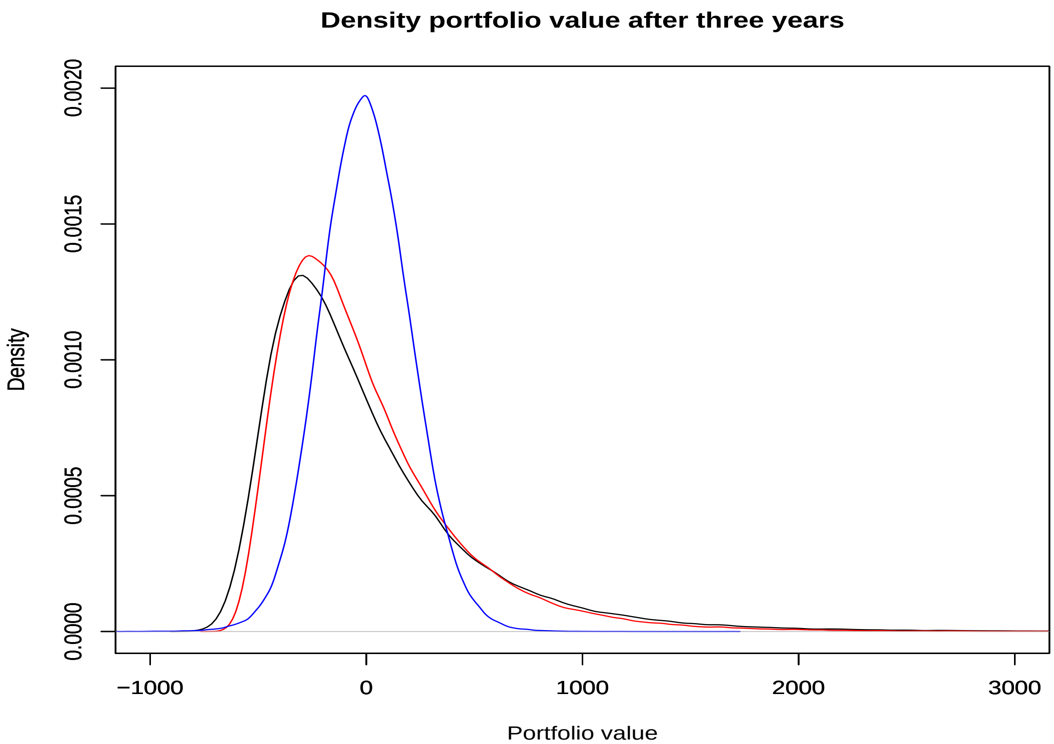

Figure 6.15: Density portfolio values of the strategy with trading at expiry in black, three years after having started applying the option strategy, compared with density of the portfolio values of the strategy with trading at predefined levels in red and the put spread strategy in blue at the same point of time.

Figure 6.15 shows the densities of the portfolio values for the three strategies considered, after three years of applying the strategies on the S&P 500 index. Trading at expiry, trading at predefined levels and the put spread strategy are represented by black, red and blue lines, respectively. In order to compare the densities, the initial value of the underlying index is subtracted at the time when the strategies, which imply trading at expiry and trading at predefined levels, were initiated. All values are discounted to the time when the strategies were initiated.

The density of the strategy implying trading at expiry is similar to the density of the strategy implying trading at predefined levels. The density of the put spread strategy is significantly different from the other strategies considered. Note the particular skewness of the densities for the strategies which imply trading at expiry and trading at predefined levels. The put spread strategy has a density more centered around zero. The strategies which imply trading at expiry and trading at predefined levels might lead to more extreme outcomes, with a greater probability than the put spread strategy, after three years of applying the strategies. From the densities, it seems like the put spread strategy has properties more appreciated by a trader than the other strategies considered. By applying the put spread strategy, the probability of large losses after three years will be significantly less than for the two other strategies. For the most probable positive portfolio values after three years, the probability will be greater for the put spread strategy than for the two other strategies. Applying the put spread strategy, the probability of large profits will not be as great as for the two other strategies.

6.4.2 OSEBX

In Figure 6.16, the densities of the three strategies considered, applied on the OSEBX index, are compared. Trading at expiry, trading at predefined levels and the put spread strategy are represented by black, red and blue lines, respectively. In order to compare the strategies, the value of the underlying index is subtracted at the time when the strategies which involve trading at expiry and trading at predefined levels were initiated. All values are discounted to the time when the strategies were initiated.

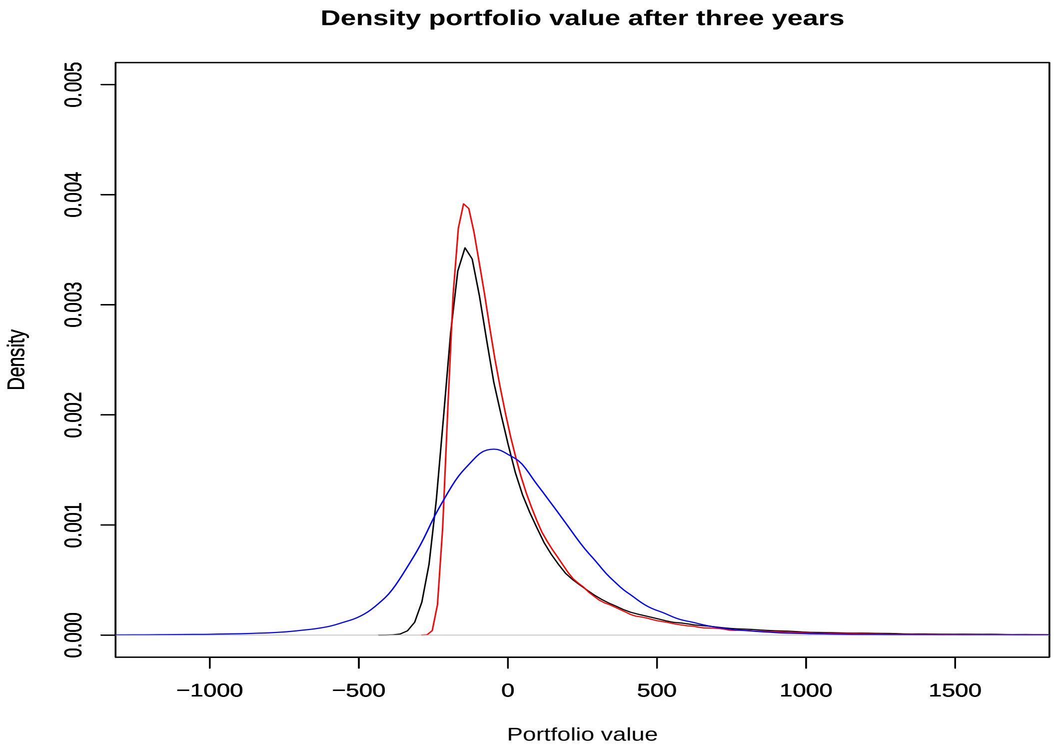

Figure 6.16: Density portfolio values of the strategy with trading at expiry in black, three years after having started applying the option strategy, compared with density of the portfolio values of the strategy which involves trading at predefined levels in red and the put spread strategy in blue, at the same point of time.

The densities of the strategies which involve trading at expiry and trading at predefined levels are similar after having applied the strategies for three years. The put spread strategy is significantly different from the other strategies considered. The densities of the OSEBX index differ significantly from the S&P 500 index. For the OSEBX index, there is a skewness of the densities of the strategies when trading at expiry and at predefined levels. Contrary to the densities of the S&P 500 index, the density for the put spread strategy, applied on the OSEBX index, has heavier tails than the two other strategies. The put spread strategy might lead to large losses with a greater probability than in the two other cases. The probability of making a profit is greater when applying the put spread strategy than for the two other strategies.

6.5 Summary of the Strategies Examined

Table 6.1 lists the expected portfolio value and the risk for the strategies and indices considered for comparison. Additionally, the ratio between expected portfolio value and risk for the respective strategies and indices have been considered. The portfolio value, the Value-at-Risk and the expected shortfall are the mean values of the period considered. When comparing the risk and the portfolio values, the value of the underlying index is subtracted from the portfolio value, for the strategies which involve trading at predefined levels and trading at expiry, at the time when the strategies were initiated. All values are discounted to the time when the strategies were initiated.

Table 6.1: Portfolio value, Value-at-Risk, expected shortfall and Portfolio value/Risk ratio for the strategies and indices considered.

	Trading at expiry	**Trading at predefined levels**	**Put spread**
PV S&P 500	−0.724	−1.27	−0.653
VaR S&P 500	27.7	23.4	9.51
ES S&P 500	42.0	32.3	19.8
PV/VaR S&P 500	−0.0261	−0.0543	−0.0686
PV/ES S&P 500	−0.0172	−0.0393	−0.0329
PV OSEBX	−0.773	0.00793	−0.540
VaR OSEBX	13.9	12.1	12.3
ES OSEBX	23.5	18.4	25.4
PV/VaR OSEBX	−0.0555	0.000653	−0.0440
PV/ES OSEBX	−0.0329	0.000430	−0.0213

For the S&P 500 index, the put spread strategy is the strategy where the exposure for risk is the smallest. As all the strategies considered have negative expected portfolio values, the ratios between the portfolio values and the risk are not completely comparable. The strategy which involves trading at expiry is the strategy exposed to the highest risk.

Considering the strategies for the OSEBX index, the strategy which involves trading at predefined levels leads to the lowest risk. For the OSEBX index, the strategy which involves trading at expiry and the put spread strategy seem to give similar results, as far as risk is concerned. The put spread strategy is the strategy with the largest expected shortfall, while the strategy trading at expiry gives the largest Value-at-Risk. Looking at the risk for the strategies and indices considered, it seems like the put spread strategy is the strategy where the risk is the smallest for the S&P 500 index, while the strategy trading at predefined levels is the strategy where the risk is the smallest for the OSEBX index.

Chapter 7

Summary

7.1 Conclusion

The aim of this thesis has been to establish tools to implement option strategies in different markets, with regard to risk/return. In times with turbulence in the financial market, the need for appropriate mathematical models is important. As the future movement of an asset or an index is not known, appropriate mathematical models will be helpful to make the right decision. When trading in a financial market with uncertain prospects, it is important for a trader to hedge his portfolios. Three different hedging strategies involving trading with European put options have been evaluated.

The risk profiles vary significantly, depending on the strategy applied. For the strategy trading at predefined levels, the risk seems to increase during the period investigated. By this strategy, the trading of the options will be equally distributed throughout the period when the strategy is applied, to a greater extent compared to the other strategies investigated. The strategy trading at expiry, leads to risk profiles, with a distinct increase in risk towards the expiry of the options and a great variation of the risk during the period investigated. In the beginning of the period, similar properties of the risk profiles are observed for the put spread strategy as for the trading at expiry. The probability of the barrier to be crossed for trading prior to expiry is less, compared to trading at predefined levels. The risk profile for the put spread strategy will have a cyclic behaviour in the beginning of the period. As more of the paths investigated will consist of portfolios traded prior to expiry, the Value-at-Risk will vary less towards the end of the period.

The discounted return of the portfolios for the strategies and indices investigated will all be close to zero, having applied the strategies for three years. Nevertheless, an analysis of the expected return can be made from the densities of the portfolio values. For the S&P 500 index, the strategies trading at expiry and at predefined levels show small differences. Still, the strategy trading at predefined levels appears to cause a slightly greater probability of minimal profit and losses, with concomitant avoidance of big losses compared to the strategy trading at expiry. The put spread strategy differs significantly from the two other strategies, as far as the density of the portfolio values is concerned. The probability of the portfolio value to have values closer to zero after having applied the strategy for three years is significantly greater, compared to the other strategies investigated. The difference is in particular great for relatively small positive portfolio values. There is a smaller probability of extreme outcomes when applying the put spread strategy, compared to the other evaluated strategies. The probability of relatively large losses and large profits will be smaller, when applying the put spread strategy. The risk for the put spread strategy, applied on the S&P 500 index is lower, compared to the two other strategies, and the probability of large losses is smaller. This strategy applied on this particular index appears to be the one to be prefered by traders who want to limit their risk exposure. Focusing exclusively at the positive tails of the densities, the strategies trading at predefined levels and at expiry are those leading to the greatest probability of large profits. Traders who do not make a point of the risk involved in their investments, but just look at the potential for large profits, may prefer these strategies. On the other hand, traders with this kind of view on their investments will probably prefer a strategy ”putting all their eggs in one basket”, rather than applying a hedging strategy.

The densities of the portfolio values using the put spread strategy differ significantly between the OSEBX index and the S&P 500 index. On the other hand, the densities by the trading at expiry and at predefined levels strategies are close to equal for the OSEBX index and the S&P 500 index. The strategy trading at predefined levels gives a slightly smaller probability of large losses compared to the strategy trading at expiry. The put spread strategy differs significantly from the other strategies investigated for the OSEBX index. The probability of large profits and losses is greater than for the other strategies. The strategy to be chosen by a trader is not as obvious for the OSEBX index as for the S&P 500 index. The strategy trading at predefined levels is the strategy leading to the lowest risk and the smallest probability of large losses. Traders concerned with the risk, wishing to avoid large losses in their portfolio values, may prefer the strategy trading at predefined levels, when the OSEBX is the underlying index. The Value-at-Risk and expected shortfall are similar for the put spread strategy and the strategy trading at expiry. However, the densities of the portfolio values after three years of applying the different strategies, indicate that the put spread strategy will give the greatest probability for both large profits and large losses.

The analysis shows that there are numerous choices available, regarding strategies. For the strategies, there are infinitely numbers of thresholds available. The prefered strategy depends on the character of the trader and to what degree he is willing to take risk. The aim has been to analyze the strategies, depending on the perspective of a trader, rather than finding an optimal strategy for trading.

7.2 Further Perspective

In this thesis, some strategies have been chosen at predefined conditions. The investigated strategies can be modified by changing the criteria for trading. Totally different strategies as straddle, strangle, butterfly or condor can be investigated. In this thesis, European put options have been investigated. The same strategies may be used upon call options. It might also be of interest analyzing strategies involving American or Exotic options. In this thesis, a stable market has been implied, what interest rate and volatility is concerned. The aim of this thesis has been to investigate tools for applying option strategies in markets with regard to risk/return. Focusing on this, the strategies have been evaluated in a stable market. Investigating a period of three years, interest rate and volatility will change. By defining these changes, new mathematical models can be evolved.

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Appendix A

Appendix

A.1 Itô's Lemma

Itô's lemma is the most important result about the manipulation of random variables required. The result is needed that, with probability 1;

$$\mathrm{d}X^2 \to \mathrm{d}t \text{ as } \mathrm{d}t \to 0 \tag{A.1}$$

Thus the smaller $\mathrm{d}t$ becomes, the more certainly $\mathrm{d}X^2$ is equal to $\mathrm{d}t$.

From Taylor series expansion it can be written

$$\mathrm{d}f = \frac{\mathrm{d}f}{\mathrm{d}S}\mathrm{d}S + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}S^2}\mathrm{d}S^2 + \cdots, \tag{A.2}$$

$\mathrm{d}S$ is given by (2.1). Here it is simply a number, so squaring it, finding that

$$\begin{aligned}\mathrm{d}S^2 &= (\sigma S \mathrm{d}X + \mu S\, \mathrm{d}t)^2 \\ &= \sigma^2 S^2 \mathrm{d}X^2 + 2\sigma\mu S^2 \mathrm{d}t\, \mathrm{d}X + \mu^2 S^2\, \mathrm{d}t^2\end{aligned}$$

Since $\mathrm{d}X^2 \to \mathrm{d}t$ to leading order

$$\mathrm{d}S^2 \to \sigma^2 S^2\, \mathrm{d}t$$

This is substituted into (A.2) and retain only those terms which are at least as large as $O(\mathrm{d}t)$. Using the definition of $\mathrm{d}S$ from (2.1), finding that

$$\mathrm{d}f = \frac{\mathrm{d}f}{\mathrm{d}S}(\sigma S\, \mathrm{d}X + \mu S\, \mathrm{d}t) + \frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 f}{\mathrm{d}S^2}\mathrm{d}t \tag{A.3}$$

$$= \sigma S \frac{\mathrm{d}f}{\mathrm{d}S}\mathrm{d}X + \left(\mu S \frac{\mathrm{d}f}{\mathrm{d}S} + \frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 f}{\mathrm{d}S^2}\right)\mathrm{d}t \tag{A.4}$$

The order of magnitude of $\mathrm{d}X$ is $O(\sqrt{\mathrm{d}t})$. Therefore the second derivative of f with respect to S appears in the expression for $\mathrm{d}f$ at order $\mathrm{d}t$. The order $\mathrm{d}t$ terms play a significant part in the later analysis, and any other choice for the order of $\mathrm{d}X$ would not lead to the interesting results discovered. It can be shown that any other order of magnitude for $\mathrm{d}X$ leads to unrealistic properties for the random walk in the limit $\mathrm{d}t \to 0$; if $\mathrm{d}X >> \sqrt{\mathrm{d}t}$ the random variable goes immediately to zero or infinity, and if $\mathrm{d}X << \sqrt{\mathrm{d}t}$ the random component of the walk vanishes in the limit $\mathrm{d}t \to 0$.

The result (A.4) can be generalized further by considering a function of the random variable S and of time $f(S,t)$. This entails the use of partial dervatives since there are now *two* independent variables, S and t. Expanding $f(S+\mathrm{d}S, t+\mathrm{d}t)$ in a Taylor series about (S,t), obtaining

$$\mathrm{d}f = \frac{\partial f}{\partial S}\mathrm{d}S + \frac{\partial f}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\mathrm{d}S^2 + \cdots ,$$

Using the expressions (2.1) for $\mathrm{d}S$ and A.1 for $\mathrm{d}X^2$, finding that the new expression for $\mathrm{d}f$ is

$$\mathrm{d}f = \sigma S\frac{\partial f}{\partial S}\mathrm{d}X + \left(\mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)\mathrm{d}t \qquad \text{(A.5)}$$

The function $f(S) = \log S$ will be considered. Differentiation of this function gives

$$\frac{\mathrm{d}f}{\mathrm{d}S} = \frac{1}{S} \text{ and } \frac{\mathrm{d}^2 f}{\mathrm{d}S^2} = -\frac{1}{S^2}$$

Using (A.4), arriving at

$$\mathrm{d}f = \sigma\mathrm{d}X + (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t$$

The jump $\mathrm{d}f$ of this constant coefficient stochastic differential equation is normally distributed, while f itself is the sum of jumps $\mathrm{d}f$. Since a sum of normal variables is also normal, $f - f_0$ has a normal distribution with mean $(\mu - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. $f_0 = \log S_0$ is the initial value of f. The probability density function of $f(S)$ is therefore

$$\frac{1}{\sigma\sqrt{2\pi t}}\mathrm{e}^{-(f-f_0-(\mu-\frac{1}{2}\sigma^2)t)^2/2\sigma^2 t} \qquad \text{(A.6)}$$

for $-\infty < f < \infty$. The probability density function of $f(S) = \log S$, which gives the probability density function of S itself

$$\frac{1}{\sigma S\sqrt{2\pi t}}\mathrm{e}^{-(\log(S/S_0)-(\mu-\frac{1}{2}\sigma^2)t)^2/2\sigma^2 t} \tag{A.7}$$

for $0 < S < \infty$. This is known as the **lognormal distribution**, and the random walk that gives rise to it is often called a lognormal random walk.

A.2 Black-Scholes for Barrier Options

In order to value European barrier options, some analytical Black-Scholes formulae exist. No dividend yield is assumed. The notation used is;

- S, the asset price;
- E, the exercise price;
- S_b, the barrier position;
- r, the interest rate;
- t, the time where the option is valuated;
- T, the expiry;
- σ, the volatility of the underlying asset.

$$a = \left(\frac{S_b}{S}\right)^{-1+\frac{2r}{\sigma^2}}$$

$$b = \left(\frac{S_b}{S}\right)^{1+\frac{2r}{\sigma^2}}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}y^2} \mathrm{d}y$$

$$d_1 = \frac{log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_3 = \frac{log(S/S_b) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_4 = \frac{log(S/S_b) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_5 = \frac{log(S/S_b) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_6 = \frac{log(S/S_b) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_7 = \frac{log(SE/S_b^2) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_8 = \frac{log(SE/S_b^2) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Up-and-out call

$$S(N(d_1) - N(d_3) - b(N(d_6) - N(d_8))) - Ee^{-r(T-t)}(N(d_2) - N(d_4) - a(N(d_5) - N(d_7)))$$

Up-and-in call

$$S(N(d_3) + b(N(d_6) - N(d_8))) - Ee^{-r(T-t)}(N(d_4) + a(N(d_5) - N(d_7)))$$

Down-and-out call

1. $E > S_b$:

$$S(N(d_1) - b(1 - N(d_8))) - Ee^{-r(T-t)}(N(d_2) - a(1 - N(d_7)))$$

2. $E < S_b$:

$$S(N(d_3) - b(1 - N(d_6))) - Ee^{-r(T-t)}(N(d_4) - a(1 - N(d_5)))$$

Down-and-in call

1. $E > S_b$:

$$Sb(1 - N(d_8)) - Ee^{-r(T-t)}a(1 - N(d_7))$$

2. $E < S_b$:

$$S(N(d_1) - N(d_3) + b(1 - N(d_6))) - Ee^{-r(T-t)}(N(d_2) - N(d_4) + a(1 - N(d_5)))$$

Down-and-out put

$$Ee^{-r(T-t)}(N(d_4) - N(d_2) - a(N(d_7) - N(d_5))) - S(N(d_3) - N(d_1) - b(N(d_8) - N(d_6))) \tag{A.8}$$

Down-and-in put

$$Ee^{-r(T-t)}(1 - N(d_4) + a(N(d_7) - N(d_5))) - S(1 - N(d_3) + b(N(d_8) - N(d_6)))$$

Up-and-in put
1. $E > S_b$:

$$Ee^{-r(T-t)}(1 - N(d_2) - a(N(d_7) - N(d_5))) - S(1 - N(d_1) - bN(d_8))$$

2. $E < S_b$:

$$Ee^{-r(T-t)}(1 - N(d_4) - aN(d_7)) - S(1 - N(d_3) - bN(d_6))$$

Up-and-in put
1. $E > S_b$:

$$Ee^{-r(T-t)}(N(d_4) - N(d_2) + aN(d_5)) - S(N(d_3) - N(d_1) + bN(d_6))$$

2. $E < S_b$

$$Ee^{-r(T-t)}(1 - N(d_4) - aN(d_5)) - S(1 - N(d_3) - bN(d_6))$$