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Valuating Forward Contracts in the Electricity Market using Partial Integro-differential Equations

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Abstract

We will evaluate forward contracts in the electricity market. A thorough presentation of stochastic analysis for processes with discontinuous paths are provided, and some results concerning these from mathematical finance are stated.

Using a Feynman-Kac-type theorem by Pham [Pha98] we derive a partial integro-differential equation giving the forward price from the spot dynamics taken from Geman and Roncoroni [GR06]. This spot model is regime switching, so we get two equations.

These equations are then attempted solved numerically.

We suggest the following approach: When implementing boundary-conditions numerically we use values obtained from a Monte Carlo simulation of the spot dynamics to calibrate the boundary.

Forord

Gjennom matematikkstudiet har jeg forandret meg som menneske. En ny dimensjon har vokst frem og nye ferdigheter kommet. Jeg er sikker på at jeg kommer til å tenke tilbake på studietiden, hvertfall etter at alt det harde arbeidet er glemt, som en tid med undring, opplysning og moro.

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Bak enhver vellykket mann står det en kvinne sies det. Min vil vel neppe erkjenne seg som kvinne (kun 25 år men "vi begynner å bli gamle"), men hvertfall står hun bak meg. Takk Mariann, for tålmodighet og forståelse!

Trondheim, desember 2007.

Bjørn Waage Skogtrø

Contents

Introduction

Our mission is to derive the forward dynamics of a contract with the electricity spot as the underlying.

For some stochastic spot-models the forward-dynamics can not be found analytically, due to the complexity of the model. Hence numerical simulations, such as Monte Carlo simulations, are called for. Although accurate and straightforward to implement, Monte Carlo simulations are often computational inefficient, in that large number of random numbers must be generated.

We propose an alternative approach, by using a partial integro-differential equation.

Mean reverting jump diffusions have been used as a model for the electricity spot for some while. It is well-known that for such a jump-diffusion we can derive an associated partial integro-differential equation giving the conditional expectation of the spot value. However, solving such equations may pose a problem. In the case for partial differential equations we would argument for some boundary condition, and simply implement it using some discretization. Values outside the domain, such as those occuring from Neumann condition, could then be eliminated using "ghost" points for instance.

For integro-differential equations matters are worse. So, we will use the knowledge of the dynamics to find the missing values using Monte Carlo simulation.

The presentation is as follows: The first chapter is devoted to deriving the stochastic integral for semimartingales. Chapter two will present some relevant results from mathematical finance, and chapter three will give a short and heuristically introduction to means of solving equations occurring in finance.

The presentation is devoted to presenting ideas, and not proofs. These can be found in any of the references listed in the back.

Lastly, some notational issues, common in this field, is to be found in the appendix together with program code.

Part I

Theoretical Preliminaries

Chapter 1

Stochastic analysis

> "The theory of probability as a mathematical discipline can and should be developed from axioms in exactly the same way as geometry and algebra."
>
> A. N. Kolmogorov, Foundations of the Theory of Probability, 1933.

In this chapter our main concern is to develop the stochastic integral, but other results connected to our work in later chapters will be represented.

The main source is Protter [Pro05]. Since nearly every result in the sections to come can be found in [Pro05, Chapters I, II, III, IV] we make precise references only if results are found elsewhere.

1.1 Measure- and integration theory

L^p-spaces. We start with a measure space $(X, \mathcal{M}, \mu)$. For a measurable f on X and for $0 < p < \infty$ we define the norms and corresponding spaces

$$\|f\|_p = \Big(\int |f|^p \mathrm{d}\mu \Big)^{\frac{1}{p}}, \text{ and}$$

$$L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{R} : f \text{ measurable and } \|f\|_p < \infty\}.$$

For the case $p = \infty$ we define the norm

$$\begin{aligned} \|f\|_\infty &= \inf \Big\{ a \geq 0 : \mu(\{x : |f(x)| > a\}) > 0 \Big\} \\ &= \text{ess sup}_{x \in X} |f(x)|, \end{aligned}$$

and the space

$$L^\infty(X, \mathcal{M}, \mu) = \Big\{ \text{ measurable functions bounded } \mu\text{-a.e.} \Big\}.$$

Remark 1.1. Actually, for $1 \leq p \leq \infty$, $\|\cdot\|_p$ is only a seminorm unless we identify all the functions equal a.e. in equivalence classes. For the case $p < 1$ it is not a norm, indeed the triangle inequality is not valid.

It is a well-known fact that the spaces L^p with the associated norm $\|\cdot\|_p$ are Banach-spaces for the case $1 \leq p \leq \infty$.

Radon-Nikodym's Theorem. From integration theory we recall that for a signed measure ν and positive measure μ defined on some measure-space $(X, \mathcal{M})$, we say that ν is *absolutely continuous* with respect to μ, if $\mu(E) = 0$ implies $\nu(E) = 0$ for every $E \in \mathcal{M}$, and write $\nu \ll \mu$. If the opposite implication also hold, then the measures are *equivalent*. Also recall that two measures ν, μ are *mutually singular* if there exist $E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$, $E \cup F = X$ and $\nu(E) = \mu(F) = 0$.

Theorem 1.2 (Lebesgue-Radon-Nikodym [Fol99])**.** *Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. Then there exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that λ and ν are mutually singular, $\rho \ll \mu$ and $\nu = \lambda + \rho$. Also, there is a μ-integrable function $f : X \to \mathbb{R}$ such that $\mathrm{d}\rho = f\mathrm{d}\mu$, and such functions are equal up to a set of measure zero.*

The function f is referred to as the Radon-Nikodym *derivative*, and we write

$$\frac{\mathrm{d}\rho}{\mathrm{d}\mu} = f.$$

As a matter of fact, the Radon-Nikodym theorem can be use to define the conditional expectation and, as we shall see later, there are other applications of this theorem.

1.2 Stochastic analysis

Throughout this section assume that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1.2.1 Basic definitions

Stochastic processes. We know the stochastic processes well from the discrete-time case. In the continuous time setting the construction is highly technical, and we shall not pursue it. We just state the usual definition:

Definition 1.3. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a *stochastic process* is a parametrized collection of stochastic variables $\{X_t(\omega) : t \in \mathbb{T}\}$, where $\mathbb{T}$ is the parameter set.

Fixing an $\omega \in \Omega$, we refer to $X_t(\omega)$ as the *path*, and interpret this as one possible outcome, out of many, from our experiment.

The parameter set $\mathbb{T}$ can be $\mathbb{N}$, $\mathbb{R}_+$ or subsets thereof. Later we concentrate on the case $\mathbb{T} = [0, T_{max}]$, with T_{max} bounded, but for now, we take $\mathbb{T} = \mathbb{R}_+$. No generality is lost in this assumption, since we can embed

$[0, T_{max}]$ in $\mathbb{R}_+$ by setting the stochastic process in question to zero outside the set of interest in a suitable fashion. Recall that all stochastic variables are by definition $(\Omega, \mathcal{F})$-measurable, and so we will refer to *measurable* processes as the ones that are measurable with respect to the sigma-algebra

$$\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F},$$

where, as usual, $\mathcal{B}(\mathbb{R}_+)$ is the Borel sigma-algebra on $\mathbb{R}_+$.

The usual hypothesis. To our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we add a *filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, i.e. an increasing family of sigma-algebras such that

$$\mathcal{F}_s \subset \mathcal{F}_t \text{ if } s \leq t.$$

We usually omit the parenthesis, and just write $\mathcal{F}_t$ when we refer to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Having introduced this concept, we say that a process is *adapted* to a filtration if it is $\mathcal{F}_t$-measurable for all times.

Filtrations play an important part in mathematical finance since they keep track of information: intuitively this means we can decide if an event has occurred by the time t by looking at $\mathcal{F}_t$.

Any stochastic process X generates the *natural filtration* $\mathcal{F}_t^X = \sigma(X_s | s \leq t)$ by its pre-images. The process is obviously adapted to this filtration, and furthermore this is the smallest filtration making X adapted. Hence to say that a process is $\mathcal{F}_t$-adapted is equivalent with saying that we have the inclusion $\mathcal{F}_t^X \subset \mathcal{F}_t$. Interpreting this as a statement on the information available to the market, then the information available is not give solely by the process, i.e. the price history, but also other information is available.

Augmenting the natural filtration of a process with the $\mathbb{P}$-null sets of $\mathcal{F}$ thus completing $\mathcal{F}_t$ we can state

Assumption 1 (The ususal hypothesis)**.** *Assume we are given a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$*, where* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$*. The filtration* $\mathcal{F}_t$ *satisfy the* usual hypotheses *if*

- $\mathcal{F}_0$ *contains all the* $\mathbb{P}$*- null sets of* $\mathcal{F}$ *and*
- $\cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$.

Again interpreting this from an financial perspective, the first property in Assumption 1 states that already at time zero, we know which events that will not happen and the second property states that an observer can "see" into the immediate future.

Really, the filtrations play an essential part in much of this theory, as can be seen from the fact that they occur many definitions, e.g. martingales: recall from basic stochastic analysis that a *martingale* is an $\mathcal{F}_t$-adapted stochastic process $X \in L^1(\mathbb{P})$ with the property $\mathbf{E}[X_s|\mathcal{F}_t] = X_t$ for $t \leq s$.

Stopping times. We say that the map $T : \Omega \to \mathbb{R}_+$ is a *stopping time* if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

Stopping times are very useful in stochastic analysis, for instance to "tame the continuum of time" as Chung so nicely put it, in order to make the transition from uniform to local properties. They play an essential part for instance in defining the stochastic integral, but are also useful to determine when some process reach a specific level, for instance in path-dependent options, e.g. Asian options.

Levy-processes. For our purposes, we shall be working with the following class of stochastic processes:

Definition 1.4. A *Levy-process* is a stochastic process X_t adapted to the filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$ with $X_0 = 0$, satisfying the following:

(i) It has stationary increments, $X_s - X_t \sim X_{s-t}$ for $0 \leq s \leq t < \infty$

(ii) It has increments independent of the past, $X_s - X_t$ independent of $\mathcal{F}_t$ for $0 \leq s \leq t < \infty$

(iii) It is continuous with probability one

An important result is that for a Levy-process X we can always find an unique modification[1] which is càdlàg and also a Levy-process. We take this modification by default.

The Levy-processes posses some nice mathematical properties, as we will see, and allows for the introduction of jumps in the paths of the financial asset. On the other side, it can be challenging to valuate assets with Levy driving processes, and finding the precise jump-properties can be difficult.

These processes include relatively familiar ones from basic courses in stochastic processes, as the following short example show:

Example 1.5 (Levy processes)**.** Perhaps the two most famous Levy-processes are the *Brownian motion* and the *Poisson process.* ★

The third property in Definition 1.4 is the one to take special notice of, and the following French acronyms are often used to describe processes that are continuous a.s.: A *càdlàg* function is right-continuous functions with left limits, and a *càglàd* function is left-continuous functions with right limits. We put $\mathbb{D}$ and $\mathbb{L}$ for the for the space of càdlàg and càglàd processes respectively. It can readily be shown that these processes may have at most a countable number of jumps by property (iii) of Definition 1.4.

[1] Recall the stochastic processes X and Y are *modifications* if $X_t = Y_t$ a.s. for each $t \geq 0$.

Characterization of Levy-processes. For any càdlàg stochastic process we can use the left limit $X_{t-} = \lim_{s\uparrow t} X_s$ to define the *jump*

$$\Delta X_t = X_t - X_{t-}$$

at time t and say that the process has *bounded jumps* if $\sup_t |\Delta X_s| < M$ for some constant $M \in \mathbb{R}$. Now let X be Levy, and take $\Lambda \in \mathcal{B}(\mathbb{R})$ with a closure not intersecting zero. Then we can define the sum of all jumps of size belonging to Λ as

$$N(t, \Lambda) = \sum_{0<s\leq t} \mathbf{1}_\Lambda(\Delta X_s)$$

It is clear that $N(t, \cdot)$ is a counting process keeping track of the number of jumps of size in a given interval, and with some reflection we can convince ourselves that it has inherited the stationary independent increments from X; thus $N(t, \cdot)$ must be a Poisson process. Also, from the definition it is obvious that $\Lambda \mapsto N(\cdot, \Lambda)$ is countable additive for disjoint sets, and so it is a measure, or more precisely a *Poisson random measure*. Now, for a Borel-measurable function f we can put

$$\int_\Lambda f(x)N(t, \mathrm{d}x) = \sum_{0<s\leq t} f(\Delta X_s)\mathbf{1}_\Lambda(\Delta X_s),$$

and the integral is well-defined since N is a counting measure.

From the Poisson random measure we can define the expected number of jumps of any given size in a time-interval of length one:

Definition 1.6. For a Levy-process, and Borel set Λ whose closure do not intersect zero, we define the *Levy-measure*

$$\nu(\Lambda) = \mathbf{E}\left[N(1, \Lambda)\right].$$

The Levy-measure is a Radon-measure describing the nature the jumps completely. To simplify notation we make the choice $\nu(\{0\}) = 0$. It also hold that

$$\int_{\mathbb{R}} \min(1, x^2)\mathrm{d}\nu(x) < \infty.$$

It turns out that the Levy-processes can be described completely by the triplet (α, σ^2, ν):

Theorem 1.7 (Levy-Khintchine formula)**.** *Let X be a Levy-process with Levy-measure ν. Then*

$$\mathbf{E}\left[e^{iuX_t}\right] = e^{-t\psi(u)},$$

where we have the characteristic exponent

$$\psi(u) = -i\alpha u + \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \Big(1 - e^{iux} + iux\mathbf{1}_{|x|<1}(x)\Big)\mathrm{d}\nu(x),$$

for $\alpha, \sigma \in \mathbb{R}$. We call (α, σ^2, ν) the Levy-triplet*, and given any such triplet the corresponding process is unique in distribution.*

If the Levy-measure is finite, we put $\nu(\mathbb{R}) = \lambda$ for the expected number of jumps and define the probability measure $f(x) = \nu(x)/\lambda$. Assuming that $\alpha = \sigma = 0$, it can be shown that X must be a compound Poisson process:

Example 1.8 (Example 1.5 revisited)**.** We now have the following examples of Levy-processes:

(i) The *drift* has characteristic function $\Phi(u) = \exp(-i\alpha u)$

(ii) The *Brownian motion* has characteristic function $\Phi(u) = \exp(-\sigma^2 u^2 t/2)$

(iii) The *compound Poisson process*: Let N be a Poisson process with intensity λ, $(J_i)_{i\geq 1}$ a sequence of i.i.d. stochastic variables with distribution $f(x) = \nu(x)/\lambda$, for ν a finite Levy-measure. We then put

$$M_t = \sum_{t=0}^{N_t} J_i,$$

and we have the characteristic function

$$\begin{aligned}\Phi(u) &= \exp\Big(t\int_{\mathbb{R}}(e^{iux}-1)\mathrm{d}\nu(x)\Big)\\ &= \exp\Big(\lambda t\int_{\mathbb{R}}(e^{iux}-1)\mathrm{d}f(x)\Big).\end{aligned}$$

Adding all these to one we get a *Levy jump-diffusion*. Figure 1.1 illustrates the components, for $\alpha = 1.2, \lambda = 10$ and the jumps distributed according to $\mathcal{N}(0, 5)$. ★

For a compound Poisson process, we could *compensate* it by subtracting the Levy-measure in the following manner:

$$\widetilde{M_t} = \int_{[0,t]\times\mathbb{R}} z\Big(N(\mathrm{d}s, \mathrm{d}z) - \mathrm{d}\nu(z)\mathrm{d}s\Big),$$

thus making $\widetilde{M}$ a martingale.

One result on Levy-processes will prove to be useful:

Theorem 1.9. *If X is a Levy-process we can write*

$$X_t = Y_t + Z_t,$$

where Y and Z are Levy-processes such that $Y \in L^p(\mathbb{P})$ for $p \geq 1$ is a martingale with bounded jumps and Z has paths of finite variation on compact sets.

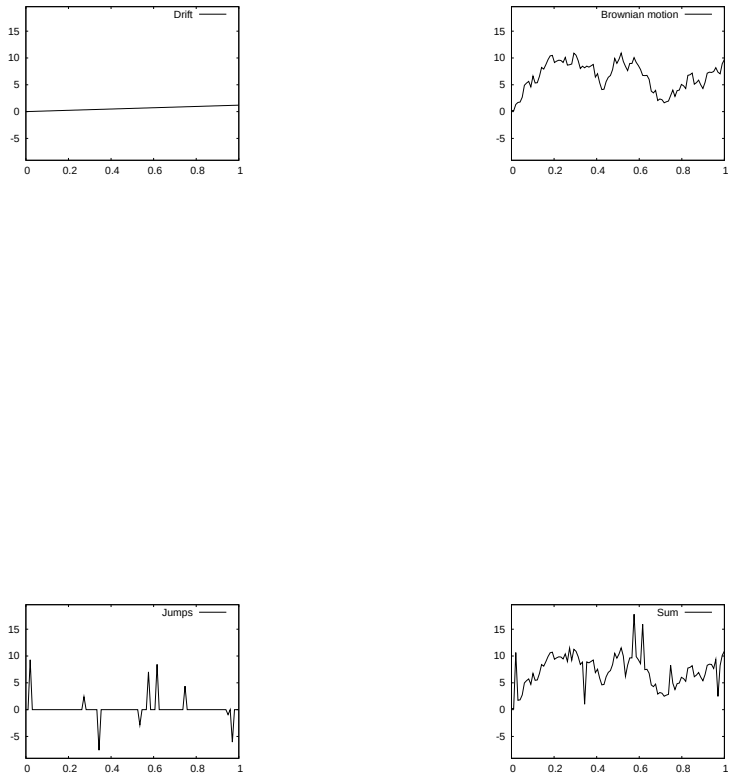

Figure 1.1: The components of a jump-diffusion.

1.2.2 Stochastic integration.

In the sections to come we will derive the stochastic integral. Protter [Pro05] gives a readable account on stochastic integration-theory with respect to semimartingales, and defines the semimartingales as "good integrators". This approach can be justified by the fact that the semimartingales are the largest collection of processes for which a stochastic integral can be derived from

simple processes to general ones by continuous extension. We follow Protter's approach since it has obvious analogies to the theory of Lebesgue integration and allow us to present the theory compactly avoiding most of the functional-theoretic arguments.

Local martingales. In defining the stochastic integral for semimartingales a great deal of generality has been obtained. Historically, the development has been from Brownian motion, martingales then local martingales and lastly semimartingales.

As we will see in the next chapter, the class of semimartingales are the needed degree of generality concerning mathematical finance.

We know from the general theory on the Itô-integral that for a general integrand the integral need not be a martingale, but just a local martingale. We make this notion more precise.

By *localization*, i.e. finding a sequence of stopping times converging to infinity and using the *stopped process*

$$X^T = (X^T)_t = X_t \mathbf{1}_{t \leq T} + X_T \mathbf{1}_{t > T},$$

we can generalize the martingale-property to a local one:

Definition 1.10. Let $(T_n)_{n \geq 1}$ be a sequence of stopping times increasing to infinity a.s., and X an adapted and càdlàg process such that $X^{T_n} \mathbf{1}_{T_n > 0}$ is a martingale for each n. Then X is a *local martingale*.

Taking the sequence $T_n = n$ we see that any càdlàg martingale is a local martingale. The local martingales forms a vector space, and obviously the martingales are included in this set by the preceding sentence. Local martingales satisfying $\mathbf{E}\left[\sup_{s \leq t} X_s\right] < \infty$ are martingales; however, it is important to note that the space of local martingales include many examples that are not martingales.

The stochastic integral for simple predictable processes. Quite analogous to the simple functions from measure theory, we state:

Definition 1.11. A process H is *simple predictable* if it can be written as

$$H_t = H_0 + \sum_{i=1}^{n} H_i \mathbf{1}_{(T_i, T_{i+1}]}(t)$$

for $0 = T_1 \leq \cdots \leq T_{n+1} < \infty$ a finite sequence of stopping times and $H_i \in \mathcal{F}_{T_i}$ such that $|H_i| < \infty$ a.s. for $0 \leq i \leq n$. The collection of all such processes we denote by $\mathbf{S}$.

We put $\mathbf{L}^0$ for the space of all finite-valued stochastic variables, and equip this space with the topology of convergence in probability. We also

need a topology on $\mathbf{S}$ and choose the uniform topology, writing $\mathbf{S}_u$ for this space. Now we can define the linear map $I_X : \mathbf{S}_u \to \mathbf{L}^0$, acting on processes and yielding stochastic variables, by

$$I_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i(X_{T_{i+1}} - X_{T_i}).$$

Isolate the collection of all the càdlàg adapted processes X that make the map continuous; then pick all stopped processes X^t for $t \in [0, \infty)$ who belong to this class and name them *semimartingales*.

We see that the semimartingales are the processes X that give meaning to the map I_X defined on $\mathbf{S}$ for arbitrary finite integration limits.

Theorem 1.12. *Properties of semimartingales include:*

(i) The semimartingales form a vector space

(ii) They do no depend on the null-sets: for probability-measures $\mathbb{Q} \sim \mathbb{P}$ then a $\mathbb{P}$-semimartingale is a $\mathbb{Q}$-semimartingale

Example 1.13. Elementary examples include adapted càdlàg processes of finite variation on compact sets, L^2-martingales with càdlàg paths and Brownian motion.

The next step is to extend the integral from $\mathbf{S}$ to a more general class of integrands. It can be shown that the space $\mathbf{S}$ is dense in the set of all càglàd functions $\mathbb{L}$ where convergence is *uniformly on compacts in probability* (ucp), that is for each $t \geq 0$ and $\epsilon > 0$

$$\mathbb{P}(\sup_{0 \leq s \leq t} |H_s^n - H_s| \geq \epsilon) \to 0 \text{ as } n \to 0.$$

Now we are ready for

Definition 1.14 (Stochastic integral for càglàd adapted processes)**.** For a semimartingale X we extend $I_{X^t} : \mathbf{S} \to \mathbb{D}$ in a continuous fashion using the density of $\mathbf{S}$ in $\mathbb{L}$, so that we now have the mapping $I_{X^t} : \mathbb{L}_{ucp} \to \mathbb{D}_{ucp}$, and define the *stochastic integral*

$$J_{X^t}(H) = \int H_s \mathrm{d}X_s. \tag{1.1}$$

Thus we have defined the stochastic integral for processes with some smoothness restriction on the paths, that is the number of jumps are at most countable.

For many applications like Itô 's formula and Girsanov's Theorem this is sufficient, but for stochastic differential equations and some applications fundamental for mathematical finance, we need a more general integral.

Example 1.15 (The Itô-integral)**.** Taking $X = B$, for B a standard Brownian motion, we see that we get the Itô-integral, since the integrand is evaluated at the left side of the interval.

Quadratic variation. Recall the fundamental problem Itô faced [Itô51] when deriving the stochastic integral for the Brownian motion was the unbounded variation of the paths on compact time intervals. His solution was to restrict the analysis to processes adapted to the filtration generated by the Brownian motion and to continue in a functional-theoretic manner using isometries of Hilbert-spaces.

When deriving the stochastic integral for more general processes, the quadratic variation acts like an inner-product, thus playing an important role in this theory. We continue our discussion in the spirit of Protter.

Definition 1.16. Take X and Y to be semimartingales. Then the *quadratic co-variation* is given by

$$[X,Y] = XY - \int X_- \mathrm{d}Y - \int Y_- \mathrm{d}X. \tag{1.2}$$

If $Y = X$ we get the *quadratic variation*

$$[X,X] = X^2 - 2\int X_- \mathrm{d}X.$$

The quadratic variation is càdlàg , increasing and of finite variation.

Example 1.17. For the standard Brownian motion we have the quadratic variation $[B,B]_t = t^2$. More generally, for a Levy-process

$$L_t = \mu t + \sigma B_t + \int_{[0,t]\times\mathbb{R}} zN(\mathrm{d}s, \mathrm{d}z)$$

where $\mu, \sigma \in \mathbb{R}$, B is a standard Brownian motion and N a Poisson random measure we have

$$[L,L]_t = \sigma^2 t + \int_{[0,t]\times\mathbb{R}} z^2 N(\mathrm{d}s, \mathrm{d}z).$$

★

If we consider the map $(X,Y) \mapsto [X,Y]$ it is clear that it is symmetric and bi-linear, so we have the following suggestive *polarization* identity

$$[X,Y] = \frac{1}{2}[X+Y, X-Y] - [X,X] - [Y,Y],$$

reflecting the Hilbert-space structure of the inner product $[\cdot,\cdot]$.

Example 1.18. A well-known result connected to the quadratic variation is the following: Recall that we have the integral

$$\int_0^t B_s \mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t,$$

which equals to

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s \tag{1.3}$$

where $f(x) = x^2/2$. This differs from the ordinary Lebesgue-Stieltjes change of variables formula for $f \in C^1$ and a process A with continuous paths of finite variation

$$f(A_t) - f(A_0) = \int_0^t f'(A_s)\mathrm{d}A_s. \tag{1.4}$$

This difference is due to the quadratic variation, which introduces the term

$$\frac{1}{2}\int_0^t f''(B_s)\mathrm{d}[B,B]_s,$$

to the change of variables formula, and looking to Example 1.17 we see that $\mathrm{d}[B,B]_s = \mathrm{d}s$ and (1.4) yield (1.3). ★

We also note that the quadratic variation of a process of finite variation is zero, and that by a trivial rearrangement of (1.2) we can find the *integration by parts* formula

$$XY = \int X_-\mathrm{d}Y + \int Y_-\mathrm{d}X + [X,Y].$$

The stochastic integral for predictable processes. We start with a definition:

Definition 1.19. We define the *predictable* sigma-algebra $\mathcal{P}$ on $\mathbb{R}_+ \times \Omega$ as the smallest sigma-algebra that make all the processes in $\mathbb{L}$ measurable, i.e.

$$\mathcal{P} = \sigma(H | H \in \mathbb{L}).$$

For short we say that X is *predictable* when X is measurable with respect to the predictable sigma-algebra. We write $\mathrm{b}\mathcal{P}$ for the *bounded* $\mathcal{P}$-measurable processes.

To relate this definition to the previous measurability-condition stated in the first paragraph of Section 1.2.1, we see that we have the containment

$$\mathcal{P} \subset \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}.$$

It is a fact that this sigma-algebra is also generated by the continuous adapted, càglàd adapted and left-continuous processes [RY05, Proposition 5.1]. Our mission in this Section is to define the stochastic integral for such predictable processes.

With this in mind, we must first state a result describing the semimartingales completely:

Theorem 1.20 (Bichteler-Dellacherie)**.** *A process X is a semimartingale if and only if it can be decomposed as the sum*

$$X_t = X_0 + M_t + A_t$$

where M is a local martingale, a process A of finite variation and $A_0 = M_0 = 0$.

For applications we think of this as a deterministic signal A plus a noise-term M.

This decomposition is not unique as can readily be seen by adding and subtracting a compensated Poisson process to the two parts above respectively, but if, in addition, A is predictable then this decomposition is unique, and we say that X is a *special* semimartingale. As we have noted earlier, the semimartingales do not depend on the null-sets; however special semimartingales do.

Remark 1.21. By unique in this setting, we of course mean unique a.s. This is valid for all such statements to come also.

Example 1.22. All Levy-processes are semimartingales: By Theorem 1.9 they can be decomposed into the sum of an L^p-martingale, $p \geq 1$, with bounded jumps and a process of finite variation on compact sets. Hence by Theorem 1.20 they are semimartingales, and even special semimartingales by the comments preceding this example. ★

Now we have positioned ourselves so that we can make some progress. We assume that X is a special semimartingale with decomposition

$$X_t = M_t + A_t,$$

i.e. M a local martingale and A a predictable process of finite variation.

Remark 1.23. This we can do without loss of generality, since if X is a general semimartingale such that $X_t = X_0 + M_t + A_t$ we can put $X^* = X_t - X_0$ and continue our discussion with X^* instead of X.

Next we define the space of semimartingales $\mathcal{H}^2$ consisting of all semimartingales such that the norm

$$\|X\|_{\mathcal{H}^2} = \left\| [M,M]_\infty^{1/2} \right\|_2 + \left\| \int_0^\infty |\mathrm{d}A_s| \right\|_2$$

is finite.

Theorem 1.24. *The space $\mathcal{H}^2$ is a Banach space.*

We observe that for $H \in \mathrm{b}\mathbb{L}$ and $X \in \mathcal{H}^2$ then $\int H_s \mathrm{d}X_s \in \mathcal{H}^2$, and given the decomposition $X = M + A$ then the integral can be decomposed as

$$\int H_s \mathrm{d}X_s = \int H_s \mathrm{d}M_s + \int H_s \mathrm{d}A_s.$$

So we have

$$\left\| \int_0^\infty H_s \mathrm{d}X_s \right\|_{\mathcal{H}^2} = \left\| \left(\int_0^\infty H_s^2 \mathrm{d}[M,M]_s \right)^{1/2} \right\|_2 + \left\| \int_0^\infty |H_s| |\mathrm{d}A_s| \right\|_2 .$$

This allows us to define a metric on the space of bounded predictable processes: for any $X \in \mathcal{H}^2$ and $H, J \in \mathrm{b}\mathcal{P}$ we put

$$\mathrm{d}_X(J,H) = \left\| \left(\int_0^\infty (H_s - J_s)^2 \mathrm{d}[M,M]_s^{1/2} \right) \right\|_2 + \left\| \int_0^\infty |H_s - J_s| |\mathrm{d}A_s| \right\|_2 .$$

What remains at this point is to prove the density of $\mathrm{b}\mathbb{L}$ in $\mathrm{b}\mathcal{P}$ under d_X, and apply this to approximate the integral for bounded predictable integrands: For $X \in \mathcal{H}^2$ and $H^n \in \mathrm{b}\mathbb{L}$ Cauchy with respect to d_X, then

$$\int H_s^n \mathrm{d}X_s$$

is Cauchy in $\mathcal{H}^2$. Finally, after showing that this limit is well-defined, we are finally here:

Definition 1.25 (Stochastic integral for bounded predictable processes)**.** For a semimartingale $X \in \mathcal{H}^2$ and $H \in \mathrm{b}\mathcal{P}$ and sequence $H^n \in \mathrm{b}\mathbb{L}$ such that $\mathrm{d}_X(H^n, H) \to 0$ we define the unique *stochastic integral* as

$$Y_t = \lim_{n\to\infty} \int_0^t H_s^n \mathrm{d}X_s.$$

Thus the integral for bounded predictable processes is defined as the limit of stochastic integrals or simple predictable processes.

We note that one could take the point of view that, for a given semimartingale X, the stochastic integral acts like a linear operator $\int : \mathrm{b}\mathcal{P} \to \mathbf{L}^0$, and use duality to construct the integral.

The stochastic integral for arbitrary semimartingales. As we have done before in the definition of local martingales, we could relax our hypothesis that the integrands are bounded predictable and that the semimartingales belongs to $\mathcal{H}^2$ by finding some local property for which we could still define the integral. We complete our discussion on the stochastic integral in this section, having reached the necessary generality.

Definition 1.26. For $X = M + A$ in $\mathcal{H}^2$ we say that $H \in \mathcal{P}$ is $(\mathcal{H}^2, X)$-*integrable* if

$$\mathbf{E}\left[\int_0^\infty H_s^2 \mathrm{d}[M,M]_s \right] + \mathbf{E}\left[(\int_0^\infty |H_s| |\mathrm{d}A_s|)^2 \right] < \infty$$

For X a general semimartingale and H predictable and $(\mathcal{H}^2, X)$-integrable we put $H^n = H\mathbf{1}_{|H|\leq n}$. Then H^n is bounded and predictable for each n, and

$$\int H_s^n \mathrm{d}X_s$$

is Cauchy in $\mathcal{H}^2$.

Definition 1.27. Let X be a semimartingale in $\mathcal{H}^2$ and let $H \in \mathcal{P}$ be $(\mathcal{H}^2, X)$-integrable and finally $H^n = H\mathbf{1}_{|H|\leq n}$. We define the *stochastic integral* as

$$\int H_s \mathrm{d}X_s = \lim_{n\to\infty} \int H_s^n \mathrm{d}X_s.$$

So the stochastic integral for predictable processes is defined as the limit of stochastic integrals for bounded predictable processes.

Now we can finish off the discussion with this last definition, using the following version of a stopped processes:

$$X^{T-} = X_t \mathbf{1}_{0\leq t<T} + X_T \mathbf{1}_{t\geq T}.$$

Definition 1.28 (The stochastic integral for semimartingales)**.** Assume that X is a semimartingale and $H \in \mathcal{P}$. The stochastic integral *exist* if there is a sequence of stopping-times $T_n \uparrow \infty$ a.s. such that $X^{T_n-} \in \mathcal{H}^2$ for each $n \geq 1$ and such that H is $(\mathcal{H}^2, X^{T_n-})$-integrable for all n.

Supposing that the integral exist for such X and H we define the *stochastic integral*

$$\int H_s \mathrm{d}X_s = \int H_s \mathrm{d}X_s^{T_n-}$$

on $[0, T_n)$, and say that H is X-integrable and write this as $H \in L(X)$.

A necessary condition for $H \in L(X)$ is that the integral

$$\int_0^t H_s^2 \mathrm{d}[X, X]_s$$

exist and is finite for all $t \geq 0$.

Theorem 1.29. *We have the following properties of the stochastic integral, where X is a semimartingale:*

(i) Let $H \in L(X)$. Then

$$\int H_s \mathrm{d}X_s$$

is a semimartingale

(ii) If X in addition is of finite variation, and $H \in L(X)$ such that

$$\int_0^t |H_s||\mathrm{d}X_s|,$$

exist a.s. for each $t \geq 0$, then the integral agrees with the Lebesgue-Stieltjes integral $\omega - by - \omega$

(iii) If $a, b \in \mathbb{R}$ and $H, J \in L(X)$ then $aH + bJ \in L(X)$ and

$$\int (aH_s + bJ_s)\mathrm{d}X_s = a \int H_s \mathrm{d}X_S + b \int J_s \mathrm{d}X_s$$

(iv) If Y is another semimartingale, and $H \in L(X) \cap L(Y)$ then $H \in L(X + Y)$ and

$$\int H_s \mathrm{d}(X_s + Y_s) = \int H_s \mathrm{d}X_s + \int H_s \mathrm{d}Y_s$$

(v) Let $H \in L(X)$ and suppose that $\mathbb{Q}$ is a probability-measure such that $\mathbb{Q} \ll \mathbb{P}$. Then the stochastic integral under $\mathbb{Q}$ is indistinguishable from the stochastic integral under $\mathbb{P}$, given that the latter integral exist.

1.2.3 Stochastic analysis

Itô 's formula. We have already seen the change of variable formula for processes of finite variation. Itô's formula generalize this to semimartingales, and tells us that a semimartingale is still a semimartingale under composition with a C^2-function.

Theorem 1.30 (Itô 's formula)**.** *For X a semimartingale and $f \in C^2(\mathbb{R})$ then $f(X)$ is a semimartingale and we have the following formula*

$$\begin{aligned} &f(X_t) - f(X_0) = \\ &\int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}[X,X]_s \\ &+ \int_{[0,t]\times\mathbb{R}} f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2}f''(X_{s-})(\Delta X_s^2)N(\mathrm{d}s, \mathrm{d}z) \end{aligned}$$

The stochastic exponential. The stochastic exponential can be used, amongst other things, to solve stochastic differential equations. Here is

Theorem 1.31. *For a semimartingale X with $X_0 = 0$ there is a unique solution of the stochastic differential equation*

$$\mathrm{d}Z_t = Z_{t-}\mathrm{d}X_t, \tag{1.5}$$

and the solution is given by the stochastic exponential

$$Z_t = \exp\Big(X_t - \frac{1}{2}[X,X]_t\Big) \prod_{0<s\leq t} (1+\Delta X_s)\exp\Big(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\Big) \quad (1.6)$$

We sometimes write $\mathcal{E}(X)$ for this unique solution.

Looking to (1.6) we see that if we want to model any financial asset by a stochastic differential equation like (1.5), then for the jumps must we must have $\Delta X_s \geq -1$, else X could take negative values.

Girsanov's theorem. If A is a process of finite variation, $A_0 = 0$ and such that it has locally integrable total variation, then one could subtract a predictable process $\widetilde{A}$ of finite variation such that

$$A - \widetilde{A}$$

is a local martingale. The process $\widetilde{A}$ is unique, and called the *compensator* of A.

Example 1.32. For the Poisson process N with parameter λ we have the compensator $\widetilde{N} = \lambda t$, which is continuous hence predictable, and so

$$N_t - \lambda t$$

is a martingale. ★

For a semimartingale X such that $[X,X]_{T_n} \in L^1(\mathbb{P})$ for the sequence of stopping-times $T_n \uparrow \infty$, we define the *conditional quadratic variation* $\langle X,X\rangle$ as the compensator of the quadratic variation $[X,X]$. For X a continuous semimartingale we have that $[X,X]_t = \langle X,X\rangle_t$, and for a general semimartingale we have $[X,X]_t = \langle X,X\rangle_t + \sum_{s\leq t}(\Delta X_s m)^2$.

Girsanov's theorem is of fundamental importance in mathematical finance. It states the existence of a measure transformation for absolutely continuous measures, and we state several versions:

Theorem 1.33 (Girsanov-Meyer Theorem)**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent, and let X be a semimartingale under $\mathbb{P}$, with decomposition*

$$X_t = M_t + A_t.$$

Then X is also a semimartingale under $\mathbb{Q}$ and has decomposition $X = L+C$, where

$$L_t = M_t - \int_0^t \frac{1}{Z_u}\mathrm{d}[Z,M]_u$$

is a $\mathbb{Q}$-local martingale and $C = X - L$ is a $\mathbb{Q}$-FV-process.

Theorem 1.34 (Girsanov-Meyer for predictable processes)**.** *Let X be a $\mathbb{P}$-local martingale with $X_0 = 0$. Let $\mathbb{Q}$ be a probability-measure such that $\mathbb{P} \sim \mathbb{Q}$, and let*

$$Z_t = \mathbf{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\,\middle|\,\mathcal{F}_t\right].$$

If $\langle X, Z\rangle$ exist for $\mathbb{P}$, then under $\mathbb{Q}$ we have the decomposition

$$X_t = \left(X_t - \int_0^t \frac{1}{Z_{s-}}\mathrm{d}\langle X, Z\rangle_s\right) + \int_0^t \frac{1}{Z_{s-}}\mathrm{d}\langle X, Z\rangle_s$$

It is possible to derive similar results for the case when we merely have $\mathbb{Q} \ll \mathbb{P}$ also.

We finish off with a result by Chan [Cha99], more suitable for our purposes. Suppose that we have the dynamics

$$\mathrm{d}S_t = \mu_t S_{t-}\mathrm{d}t + \sigma_t S_{t-}\mathrm{d}B_t + \mathrm{d}N_t$$

where B is a standard Brownian motion, N a compound Poisson process with intensity λ and i.i.d. jumps J_i. μ_t, σ_t are continuous and deterministic. We incorporate the "drift" originating from the compound Poisson process into μ by putting

$$\mathrm{d}S_t = \Big(\mu_t + \int_{\mathbb{R}} z\mathrm{d}\nu(z)\Big)S_{t-}\mathrm{d}t + \sigma_t S_{t-}\mathrm{d}B_t + \mathrm{d}M_t,$$

where M is the compensated N. Note that $\int_{\mathbb{R}} z\mathrm{d}\nu(z) = \lambda\mathbf{E}[J]$.

Proposition 1.35. *Assume θ_t is predictable and*

$$\mathbf{E}\left[\int_0^t \theta_s^2\mathrm{d}s\right] < \infty,$$

Then

$$Z_t = \exp\Big(-\int_0^t \theta_s\mathrm{d}B_s - \frac{1}{2}\int_0^t \theta_s^2\mathrm{d}s\Big) \tag{1.7}$$

is a non-negative local martingale with $Z_0 = 1$, and Z is positive.

Remark 1.36. In Chan's paper we have chosen the $H = 1$ and $h = 0$, since this is sufficient for our application of this result.

Theorem 1.37 ([Cha99])**.** *Let $\mathbb{Q}$ and $\mathbb{P}$ be measures such that $\mathbb{Q} \ll \mathbb{P}$. Then we have*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z_T,$$

for Z given by (1.7), for some θ for which $\mathbf{E}[Z_t] = 1$.

Under $\mathbb{Q}$

$$W_t = B_t + \int_0^t \theta_s\mathrm{d}s$$

is a Brownian motion, and the Levy-measure is unchanged. θ is given by the relation

$$\sigma_t \theta_t = \mu_t + \int_{\mathbb{R}} z \mathrm{d}\nu(z) - r_t,$$

for r a continuous deterministic function.

1.3 The connection between stochastic- and differential equations

The connection between stochastic differential equations and partial differential equations have been known for a long time. Since an Itô diffusion is a strong Markov process, we can identify its transition semi-group, and study the semi-group to gain insight in the process itself. To study the semi-group one can look at its infinitesimal generator, which is a partial integro-differential operator.

1.3.1 The infinitesemal generator.

Assume we have a stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t + \int_{\mathbb{R}} \gamma(X_{t-}, z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \tag{1.8}$$

i.e. *jump-diffusion* or *Levy-diffusion*, with $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

(i) There is a constant $C < \infty$ such that

$$|\mu(x)|^2 + |\sigma(x)|^2 + \int_{\mathbb{R}} |\gamma(x, z)|^2 \mathrm{d}\nu(z) \leq C(1 + |x|^2),$$

for all $x \in \mathbb{R}$.

(ii) There is a constant $K < \infty$ such that

$$|\mu(x) - \mu(y)|^2 + |\sigma(x) - \sigma(y)| + \int_{\mathbb{R}} |\gamma(x, z) - \gamma(y, z)| \mathrm{d}\nu(z) \leq K|x - y|^2.$$

Then we define the *infinitesemal generator* of (1.8), for $u \in C_0^2(\mathbb{R})$, as

$$\begin{aligned} \mathcal{A}u &= \mu(x)u_x(t, x) + \frac{1}{2}\sigma^2(x)u_{xx}(t, x) \\ &+ \int_{\mathbb{R}} \Big(u(t, x + \gamma(t, x, z)) - u(t, x) - \mathbf{1}_{|z| \leq 1}\gamma(t, x, z)u_x(t, x) \Big) \mathrm{d}\nu(z). \end{aligned} \tag{1.9}$$

1.3.2 The Feynman-Kac formula

While developing his path integral Feynman "discovered" what we now know as the Feynman-Kac formula, later proven by Kac. It states that for a given partial differential equation, we can find a solution as the expectation of an Itô-diffusion like

$$X_s^{(t,x)} = x + \int_t^s b(u, X_u^{(t,x)})\mathrm{d}u + \int_t^s \sigma(u, X_u^{(t,x)})\mathrm{d}W_u \tag{1.10}$$

for $t \leq s < \infty$. We state not the Theorem itself, but a analogue result, to be found in [KS98]. First we must make some assumptions:

Assumption 2. *Assume that $b, \sigma : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ are continuous and satisfies the linear growth condition*

$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \leq K^2(1 + \|x\|^2)$$

for every $(t,x) \in [0,\infty) \times \mathbb{R}^n$ and $K > 0$. For fixed $T > 0$ and constants $L > 0, \lambda \geq 1$ consider the continuous functions $g(x) : \mathbb{R}^n \to \mathbb{R}$, $f(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $k(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$. Assume that they satisfy

$$|g(x)| \leq L(1 + \|x\|^{2\lambda}) \text{ or } g(x) \geq 0 \quad \forall \quad x \in \mathbb{R}^n$$

and

$$|f(t,x)| \leq L(1 + \|x\|^{2\lambda}) \text{ or } f(t,x) \geq 0 \quad \forall \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Theorem 1.38 (Feynman-Kac formula [KS98])**.** *Assume that all assumptions listed in Assumption 2 hold. Let $\mathcal{A}$ be the infinitesimal generator of the Itô-diffusion (1.10). Suppose that $u(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ belongs to $C^{1,2}([0,T) \times \mathbb{R}^n)$, and that u solves the partial differential equation*

$$\begin{cases} -u_t + ku &= \mathcal{A}u + f \quad \text{in } [0,T) \times \mathbb{R}^n \\ u(T,x) &= g(x) \quad x \in \mathbb{R}^n. \end{cases}$$

In addition assume that

$$\max_{0 \leq t \leq T} |u(x,t)| \leq M(1 + \|x\|^{2\kappa}) \quad x \in \mathbb{R}^n$$

for some constants $M > 0$ and $\kappa \geq 1$. Then $u(x,t)$ has the unique solution

$$u(x,t) = \mathbf{E}^{t,x}\Big[g(X_T)\exp\Big(-\int_0^T k(u, X_u)\mathrm{d}u\Big) + \int_0^T f(u, X_u)\exp\Big(-\int_t^s k(u, X_u)\mathrm{d}u\Big)\mathrm{d}s\Big] \text{ on } \mathbb{R}^n \times [0,T].$$

Proof. See [KS98, Theorem 5.7.6, p. 366]. □

The Feynman-Kac formula is frequently used, for instance when we want to valuate exotic options, in which case finding a solution by analytical methods can be quite difficult.

For the more general case of jump-diffusions Pham [Pha98] have given a similar result. Again there are several assumptions that must hold, but first we make some general remarks on the setting. Assumption 1 holds. We are given a Brownian motion B and a homogeneous Poisson random measure N with intensity measure $\mathrm{d}\nu(z)\mathrm{d}t$, where ν is the Levy measure, and the compensated Poisson random measure is given by $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}\nu(z)\mathrm{d}t$. The jump-diffusion has dynamics given by

$$\mathrm{d}X_t = b(t, X_{t-})\mathrm{d}t + \sigma(t, X_{t-})\mathrm{d}B_t + \int_{\mathbb{R}^n} \gamma(t, X_{t-}, z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z). \tag{1.11}$$

We have the infinitesemal generator $\mathcal{A}$ associated with (1.11). As earlier we are looking for a classical solution $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ to the Cauchy problem

$$\begin{cases} -u_t + ku &= \mathcal{A}u + f \quad \text{for all} \quad (t, x) \in [0, T) \times \mathbb{R}^n \\ u(T, x) &= g(x) \qquad \text{for all} \quad x \in \mathbb{R}^n. \end{cases} \tag{1.12}$$

We now state the conditions that must hold.

Assumption 3. *First, assume that the functions b, σ, γ, c, f and g are continuous. In addition the following must hold:*

(P1) There exists $\beta > 0$ such that for all $t \in [0, T]$ and $x, \zeta \in \mathbb{R}^n$ then

$$\zeta^{\mathsf{T}}\sigma\sigma^{\mathsf{T}}(t, x)\zeta \geq \beta|\zeta|^2.$$

(P2) b and σ are bounded and locally Lipschitz in (x, t).

(P3) There exists a function $\rho : \mathbb{R}^n \to \mathbb{R}$ with

$$\int_{\mathbb{R}^n} \rho^2(z)\mathrm{d}\nu(z) < \infty,$$

such that

$$\begin{aligned} |\gamma(t, x, z)| &\leq \rho(z) \qquad \text{for all} \quad (t, x) \in [0, T] \times \mathbb{R}^n \\ |\gamma(t, x, z) - \gamma(t, y, x)| &\leq \rho(z)|x - y| \quad \text{for all } (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n. \end{aligned}$$

Also, we demand that

$$(t, x) \to \int_{\mathbb{R}^n} \gamma(t, x, z)\mathrm{d}\nu(z)$$

is locally Lipschitz in (t, x).

(P4) *There exists a constant $K > 0$ such that for all $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$*

$$|b(t,x) - b(t,y)| - |\sigma(t,x) - \sigma(t,y)| \leq K|x - y|.$$

(P5) *There exists a constant $K > 0$ such that for all $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$*

$$|f(t,x) - f(s,y)| - |g(x) - g(y)| \leq K\Big(|t - s| + |x - y|\Big).$$

(P6) *k is bounded and locally Hölder-continuous in (t, x).*

(P7) *The measure ν is bounded*

We can finally state Pham's result:

Proposition 1.39 ([Pha98])**.** *If the conditions in Assumption 3 holds, then there exists a unique solution*

$$u \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$$

of the Cauchy problem (1.12). The solution satisfies

$$|u(t,x)| \leq C(1 + |x|^q) \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^n,$$

for some $q \in [0, 2]$. The solution is given by

$$u(t,x) = \mathbf{E}^{t,x}\Big[\exp\Big(-\int_t^T k(u, X_u)\mathrm{d}u\Big)g(X_T) + \int_t^T \exp\Big(-\int_t^s k(u, X_u)\mathrm{d}u\Big)f(s, X_s)\mathrm{d}s\Big]. \quad (1.13)$$

Proof. The proof is given in Pham [Pha98]. □

Chapter 2

Mathematical finance

> A finance professor and a mathematics student are walking down a street. Suddenly the student spots a 100 $ bill lying on the sidewalk. As the student bends down to pick up the bill, the professor says: "Don't pick it up. It is quite impossible that a 100 $ bill is lying on the sidewalk – indeed if it were, somebody else would already have picked it up."
>
> Ad.lib. after Schachermeyer [Sch04].

This chapter will be devoted to a brief introduction to mathematical finance. We will give some definitions and examples, and state some important result. From there we shall continue with a description of the electricity market, and it's fundamental characteristics. Lastly we will look at some of the existing models of the electricity spot.

2.1 Historical background and some concepts

The early history. The history of mathematical finance can be traced back to the year 1900 when Bachelier defended his doctoral thesis, in where the Brownian motion was treated rigorously and applied in a financial context for the first time. He proposed to use the Brownian motion as the driving term in the stock dynamics

$$\mathrm{d}S_t = \sigma S_0 \mathrm{d}B_t.$$

Some fifty years later Samuelson was the next to make progress, and suggested that one should use the exponential of Bachelier's model, since negative-valued stocks had yet to be observed:

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}B_t.$$

Meanwhile mathematicians like Kolmogorov, Itô, Meyer and Levy had placed the theory of probability on a solid foundation and invented the stochastic

calculus. Several attempts had been made to solve the question on the pricing of options during this time.

The pricing of options. An option is an example of a derivative security, something we will define later, but for now we just describe it as a financial contract whose value is derived from some underlying security, for instance stocks, commodities or bonds. The option gives the holder the right to sell the underlying security at a specific time and price, regardless of the actual value the security may have at the time.

In the mid-sixties, Black and Scholes were working on the problem of pricing an option, considering a portfolio consisting of a stock, bond and an option, applying

$$\mathrm{d}S_t = \alpha S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t$$

as the stock-dynamics [BS73]. After some time they arrived at the paramount insight that it was the risk that was the key to the problem. A couple of years later Merton joined them, and brought with him tools from stochastic analysis and the understanding of continuous time stochastic processes. Merton realized that by continuously adjusting the portfolio consisting of one bond, stock and option he could make the risk vanish.

In 1997 Merton and Scholes earned the Nobel Price in economics due to their result. Black had passed away in 1995, so he was ineligible for the prize.

Empirical objections. The application of stochastic analysis by Black & Scholes is often thought of as the pinnacle of mathematical finance. However, the Black & Scholes model has flaws, and can not capture the empirical facts observed. Observed skewness and leptokurtic properties are not replicated, but also path-wise properties like abrupt changes (jumps) in the prices are hard to replicate without using large volatilities in this continuous model. Trying to remedy these flaws, Merton was the first to propose a model including jumps [Mer76]:

$$\mathrm{d}S_t = \alpha S_{t-} \mathrm{d}t + \sigma S_{t-} \mathrm{d}B_t + S_{t-} \mathrm{d}N_t.$$

Here we have the additional process N, representing the jumps. Depending on the application, N could for instance be a Poisson or compound Poisson process. Merton choose Gaussian distributed jumps.

Remark 2.1. Black & Scholes ran into problems when they wanted to publish their result, presumably because of Black's non-academic line of work as a consultant. They had to try three times to get the article now so famous published. Merton had on his own hand written a more general version, but graciously delayed his publication until Black & Scholes article appeared.

2.2 Arbitrage-free pricing

For semimartingales, the stochastic integral has been developed. As we will see later, semimartingales are the natural degree of generalization.

Price process. Assuming the usual conditions described in Assumption 1, we will in the following sections use the following notation: We let $S_t = (S_t^0, S_t^1, \ldots, S_t^n)$ be a $n+1$-dimensional semimartingale, with

$$S_t^i : [0,T] \times \Omega \to \mathbb{R}, \text{ for } i \in \{0,1,\ldots,n\},$$

denoting the *price process* of our securities. Following the usual practise, we let S_t^0 be our riskless investment, and we assume that $S_t^0 > 0$ for all $t \in [0, T_{max}]$, and $S_0^0 = 1$. We assume that there are no cash flows associated with this security, such as dividends and that the market is frictionless. Recalling that $L(S)$ denote the collection of all predictable processes integrable with respect to S, we shall refer to $(S, L(S), \mathbb{P})$ as a *model*.

Remark 2.2. To have some boundedness, we assume that S is locally bounded, i.e. there is some sequence of stopping times T_n converging to infinity almost surely, such that $S_{t \wedge T_n} < \infty$ uniformly for all $n \in \mathbb{N}$. For instance, we could take a Levy-process with bounded jumps.

We have already mentioned options, which is one example of the following class:

Definition 2.3. A *derivative security*, or equivalently a *contingent claim*, is a $\mathcal{F}_T$-adapted function $F(\omega) \in L^2(\mathbb{P})$ where T is the *strike time*.

Let us start with a simple example, illustrating the idea:

Example 2.4 (An European option)**.** An *European call* option written on a stock S is a contingent claim giving its owner the right to buy stocks at the time T for the price K. Is has the *pay-off function*

$$H(\omega) = \max\{S(T) - K, 0\},$$

with K the *strike price*.

Let the initial value of the stock be 100 €. We assume the simple case $\Omega = \{\omega_1, \omega_2\}$ with corresponding probabilities $\mathbb{P} = (\frac{1}{2}, \frac{1}{2})$, as illustrated in Figure 2.1. Also, let there be an option written on this stock with strike price $K = 100$ € at time T. Reasoning according to classical probability analysis we find the value of the option as

$$V_{cl} = \mathbf{E}_{\mathbb{P}}[H] = (150 - 100)p = 25\,€.$$

Assuming that there is is no price connected to borrowing, consider the following actions by a trader at time zero:

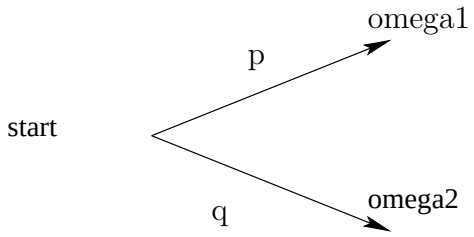

Figure 2.1: The possible values of the stock

1. She borrows 41.67€ from the bank, at no cost

2. Then sell one option valued $V_{cl} = 25$€ with strike time T and strike price 100€

3. Buy $\frac{2}{3}$ of a stock

To recapitulate, she commits herself to supply one stock at the price 100€ at time T to the buyer of the option, and she has bought two thirds of a stock.

Case	Up (ω_1)	Down (ω_2)
Loan	−41.67€	−41.67€
Option	−50€	0€
Stock	$\frac{2}{3}$150€	$\frac{2}{3}$75€
Difference	8.33€	8.33€

Table 2.1: Summary of the investors profit at time $t = T$.

At the strike time one out of two things happen. If the stock goes up she has stock holdings worth 100€(= $\frac{2}{3}$ 150€) and must supply one stock to the buyer of the option, at the cost 50€. After paying her debt to the bank she is left with a profit of 8.33€. If the stock is down she is left with 50€(= $\frac{2}{3}$75€) in stocks, and the buyer of the option does not exercise his right. She makes a profit of 8.33€ after repaying the bank. Thus she is guaranteed to make a profit regardless of the value the stock takes at the time T. A summary of the investors profit is shown in Table 2.1.

However, according to Black and Scholes' reasoning we find another value for the option. This involves replacing the probability $\mathbb{P}$ with another probability $\mathbb{Q}$ so that S is a martingale under this new probability, or more intuitively, transforming the probability measure so that we have a fair game. Since the stock initially is worth 100€ we want to have q such that

$$\mathbf{E}_{\mathbb{Q}}[S_T] = 150q + 75(1-q) = 100,$$

under this new probability measure. Solving for q gives

$$q = \frac{1}{3},$$

and so

$$V_{bs} = \mathbf{E}_{\mathbb{Q}}[H] = (150 - 100)q = 16.67 €$$

is the true value of the option at time zero. ★

This example introduced two fundamental concepts in pricing derivatives: Pricing a claim by transforming the measure so that the price process becomes a martingale under this new measure, and the concept of *arbitrage*. As we saw, if the option was incorrectly priced then a clever arbitrageur could make an arbitrary large profit without any risk.

It is common to refer to $\mathbb{P}$ as the *physical* probability measure, and $\mathbb{Q}$ as the *financial* probability measure.

To make the notion of arbitrage more precise, we must introduce new terminology:

Definition 2.5. A *trading strategy* is a real-valued $n+1$-dimensional process $\phi \in L(S)$.

This is a natural definition since any trader should only be allowed to adjust his portfolio by the information given up to time t reflected by the fact that ϕ must be adapted to the filtration $\mathcal{F}_t$.

Remark 2.6. To allow for some diversity in our language, we use *portfolio* interchangeably with strategy.

We interpret the value of ϕ_i at any time as our holdings of security i in our portfolio at the time T. The requirement that our strategies should be predictable is equivalent with the fact that the content of this portfolio is determined on the basis of our knowledge up to the time t.

Example 2.7. A buy-and-hold strategy could be represented as

$$\phi_t = f\mathbf{1}_{(T_1,T_2]}$$

for stopping-times T_1, T_2 such that $T_1 \leq T_2$ and f is $\mathcal{F}_{T_1}$-measurable. ★

Straightforwardly we can define the *gain process* generated by ϕ and the corresponding price process S as the integral

$$G(t) = \int_0^t \phi_u \, \mathrm{d}S_u.$$

To illustrate the need of having predictable strategies, we look to the following example:

Example 2.8. Let the price process be be given as the negative of a compensated Poisson process, i.e.

$$S_t = \lambda t - N_t.$$

Denoting the first time N makes a jump by T_1, we consider the strategy of buying one unit of the security at $t = 0$ and selling it just before the price falls, i.e. we have $\phi_t = 1_{[0,T_1)}(t)$. Since ϕ is not left-continuous it is not càglàd and hence not predictable either. From this strategy we have the gain

$$\int_0^t \phi_u \, \mathrm{d}S_u = \begin{cases} \lambda t, & \text{if } t < T_1 \\ \lambda T_1, & \text{if } t \geq T_1, \end{cases}$$

and thus we have an arbitrage opportunity. We see that we must restrict the possible strategies to predictable ones. ★

One wonder why not simple predictable processes, cf. Definition 1.11 with the obvious interpretation as linear combination of buy-and-hold strategies, are used to represent the actions taken by an investor. The answer is partly connected to the use of replicating portfolios and the continuous updating used in the Black & Scholes analysis, and the more mathematically argument that we can approximate the integrand by simple ones.

Definition 2.9. A trading strategy ϕ is called *self-financing* if at time t it can be written as

$$\mathrm{d}(\phi_t S_t) = \phi_t \mathrm{d}S_t$$

in the differential notation.

This translates to the investor placing an initial investment at time zero, and with any increased value of the portfolio at any time $t > 0$ is due solely to the gain, and no additional infusion of cash. We are now able to be mathematical precise in the characterization of arbitrage:

Definition 2.10. A model is *arbitrage free* if there are no self-financing trading strategies a such that

- $\phi_0 S_0 = 0$
- $\phi_T S_T \geq 0$ a.s, with
- $\mathbb{P}(\phi_T S_T > 0) > 0$.

Going back to Example 2.4, the we recall the fundamental idea of switching between measures to find the correct value of the claim. This relationship between $\mathbb{P}$ and $\mathbb{Q}$ is such a fundamental one in mathematical finance, that it has its own name:

Definition 2.11. A probability measure $\mathbb{Q}$ on $\mathcal{F}$ equivalent to $\mathbb{P}$ is called an *equivalent local martingale measure* (EMM) if S is a local martingale under $\mathbb{Q}$.

Remark 2.12. The probability measure $\mathbb{Q}$ is also known as the *risk-neutral* measure.

Now the question is on the existence of an equivalent martingale measure. From Radon-Nikodym's theorem we know that given equivalent measures $\mathbb{P}$ and $\mathbb{Q}$ then we can find the Radon-Nikodym derivative, i.e. random variable in this case, $Z \in L^1(\mathbb{P})$ such that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z \quad \Longrightarrow \quad \mathbb{Q}(\Lambda) = \mathbf{E}\left[1_\Lambda Z\right],$$

and put

$$Z_t = \mathbf{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \,\middle|\, \mathcal{F}_t\right].$$

To show that Z is a martingale, we could use the Novikov condition

$$\mathbf{E}\left[\exp\left(\frac{1}{2}[Z,Z]_\infty\right)\right] < \infty.$$

By equivalence of $\mathbb{Q}$ and $\mathbb{P}$ we can apply Girsanov's Theorem to the price-process. Anticipating the events somewhat, recall the technique for the continuous case:

Example 2.13. Let us look at the model proposed by Black & Scholes. We are given a probability-space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and the following dynamics:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \tag{2.1}$$

We have the constants $\mu, \sigma, r \in \mathbb{R}$, and standard Brownian motion B. Now we postulate the existence of $\mathbb{Q}$, and must somehow find a measure-transform so to make S a $\mathbb{Q}$-martingale. According to the discussion above, the Radon-Nikodym derivative is then given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z_T,$$

and our task is to find this Z. From Girsanov's Theorem we know that the semimartingale $S = A + M$ there is a decomposition $S = L + C$ for S a semimartingale under $\mathbb{Q}$ with L a local martingale and C a finite variation process. We seek to find such a decomposition such that $C = 0$. We have that

$$\sigma \int_0^t S_u \mathrm{d}B_u - \int_0^t \frac{1}{Z_u} \mathrm{d}[Z, \sigma \int_0^\cdot S_u \mathrm{d}B_u]$$

is a $\mathbb{Q}$-local martingale. Writing

$$Z_t = 1 - \int_0^t \frac{\mu}{\sigma} Z_u \mathrm{d}B_u, Z_0 = 1$$

we find that

$$\int_0^t \frac{1}{Z_u} \mathrm{d}[Z, \sigma \int_0^{\cdot} S_u \mathrm{d}B_u] = -\mu \int_0^t S_u \mathrm{d}u$$

since $[B, B]_t = t$ and hence that

$$\sigma \int_0^t S_u \mathrm{d}B_u + \mu \int_0^t S_u \mathrm{d}u \tag{2.2}$$

is a $\mathbb{Q}$-local martingale and

$$\mathrm{d}S_t = \theta S_t \mathrm{d}L_t$$

with $L_t = B_t + \mu t/\sigma$ from (2.2), and $\theta = \mu/\sigma$. ★

Remark 2.14. The constant θ in Example 2.13 is frequently referred to as the *market price of risk*.

As noted earlier the question on which strategies to consider is an important one, and we have seen mathematical and economical arguments to choose predictable processes. However, not all predictable processes should be considered and strong economical arguments imply that we should make some restrictions on the strategies.

To disallow trading strategies like the *doubling strategy*, where a profit can be guaranteed if only enough money can be borrowed, we must place restrictions on the trading strategies, and so we say that a predictable trading strategy is *admissible* if there is a lower bound $K \in \mathbb{R}$ such that

$$\int_0^t \phi_u \mathrm{d}S_u \geq -K$$

a.s. for all $t \geq 0$. That is to say that there is a finite line of credit for an investor.

Before we proceed, we note that it is of great importance to make the right choice when defining strategies. As we have pointed out, one easily could rationalize the simple predictable processes as our choice of strategies. This choice, however, make the mathematics quite unappealing from an economical viewpoint, for instance in the definition of no-arbitrage conditions (no free lunch), so in search for a more economically pleasing theory predictable strategies are the proper choice.

Remark 2.15. Some more trivia: The French phrase "la martingale" is used to refer to this doubling strategy. An example illustrating the doubling strategy is presented in [KS98, Example 2.3, p.8].

The question of the connection between arbitrage and equivalent martingale measures for semimartingales have been resolved by Delbaen and Schachermeyer, who give a disposition in [DS06]. The theory is quite technical for the general case, i.e. for unbounded semimartingales, and beyond our scope. We are quite happy to restrict ourselves to the case for bounded semimartingales, and will only try to make some general remarks, since we now are on the edges of our knowledge considering the powerful interplay of analysis, topology and functional analysis.

With this in mind, let us define the following:

$$K = \Big\{ \int_0^\infty \phi_u \, \mathrm{d}S_u \,|\, \phi \text{ admissible and } \lim_{t\to\infty} \int_0^t \phi_u \, \mathrm{d}S_u \text{ exists a.s.} \Big\}, \tag{2.3}$$

$$C = \{ f \in L^\infty(\mathbb{P}) \,|\, \text{ there is } g \in K \text{ such that } f \leq g \}. \tag{2.4}$$

We interpret K as all the claims available through admissible predictable strategies at the prize zero.

These spaces let us define a more general arbitrage condition:

Definition 2.16. We say that a semimartingale S satisfies the *no free lunch with vanishing risk* (NFLVR) condition if $\overline{C} \cap L^\infty_+ = \{0\}$.

The closure of C is taken with respect to the topology induced by $\|\cdot\|_\infty$, i.e. the norm associated with L^∞.

Theorem 2.17 (The fundamental theorem of asset pricing [DS06])**.** *For a locally bounded semimartingale S there exists a equivalent local martingale measure $\mathbb{Q}$ if and only if S satisfies the NFLVR criterion.*

The NFLVR-property will be used in the following manner: We hypothesize that the market arbitrage-free, which imply the NFVLR-property and thus the existence of an equivalent local martingale measure. Then we can apply Girsanov's theorem to find such a measure.

Remark 2.18. The most general case has yet to be achieved in our discussion. For the case where the semimartingale is not necessarily bounded, there is an equivalent probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that S is a sigma-martingale under $\mathbb{Q}$. A sigma martingale is, with some degree of simplification, a process that can be written as the integral of a strictly positive predictable process, with respect to a local martingale.

Delbaen and Schachermeyer [DS06] are able to show that for a locally bounded process S satisfying the NFLVR-property, then S must be a semimartingale, thus our *a priori* assumption that the price process must be a semimartingale is justified.

Contrary to the continuous case, we do not have uniqueness in the equivalent martingale measure, so given any semimartingale there are several choices of equivalent martingale measures.

Example 2.19. As in the previous example, we take real coefficients: let $\mu, \sigma_i \in \mathbb{R}$ for $i = 1, 2, 3$. Assumption 1 applies. Suppose we have the following price process

$$\mathrm{d}S_t = \mu S_{t-}\mathrm{d}t + \sigma_1 S_{t-}\mathrm{d}W_t + \sigma_2 S_{t-}\mathrm{d}B_t + \sigma_3 S_{t-}\mathrm{d}M_t,$$

where W and B are independent Brownian motions, and M is a compensated Poisson process, i.e. $M_t = N_t - \lambda t$ for N a Poisson process with intensity λ independent of both W and B. We let $\widetilde{M}$ be the sum of the three martingales. To find a equivalent martingale measure, all we have to do is to eliminate the drift, and we show four possibilities:

a) We can proceed as shown in Example 2.13, and choose one of the Brownian motions to be our starting point:

$$Z_t^1 = 1 - \int_0^t \mu Z_u^1 S_u \mathrm{d}B_u \text{ with } Z_0^1 = 1.$$

Then by the following

$$\int_0^t \frac{1}{Z_u^1}\mathrm{d}\langle Z^1, \widetilde{M}\rangle_u = -\mu \int_0^t \frac{1}{Z_u^1} Z_u^1 S_u \mathrm{d}[B, B]_u = -\mu \int_0^t S_u \mathrm{d}u$$

and Girsanov's Theorem we see that S has no drift under $\mathbb{Q}$, and change of measure given by $\mathrm{d}\mathbb{Q}_1 = Z_T^1 \mathrm{d}\mathbb{P}$.

b) Similarly, we can take use W to find a change of measure $\mathrm{d}\mathbb{Q}_2 = Z_T^2 \mathrm{d}\mathbb{P}$ precisely as above, with W in place of B.

c) If we put

$$Z_t^3 = 1 - \mu \int_0^t Z_u^3 \frac{1}{\lambda} S_u \mathrm{d}M_u, \text{ with } Z_0^3 = 1$$

we find

$$\int_0^t \frac{1}{Z_u^3}\mathrm{d}\langle Z^3, \widetilde{M}\rangle_u = -\mu \int_0^t \frac{1}{Z_u^3} Z_u^3 \frac{1}{\lambda} S_u \mathrm{d}\langle M, M\rangle_u = -\mu \int_0^t \frac{1}{\lambda} S_u \lambda \mathrm{d}u.$$

Again, we see that S has no drift under $\mathbb{Q}$, and we have the relationship $\mathrm{d}\mathbb{Q}_3 = Z_T^3 \mathrm{d}\mathbb{P}$.

d) Lastly, any linear combination of the above:

$$Z_t^4 = 1 - \mu \int_0^t Z_u^4 \Big(\alpha S_u \mathrm{d}B_u + \beta S_u \mathrm{d}W_u + \gamma \frac{1}{\lambda} S_u \mathrm{d}M_u\Big) \text{ with } Z_0^4 = 1$$

where α, β and γ are non-negative and such that $\alpha + \beta + \gamma = 1$.

We have still not exhausted the possible choices. In part d) we can replace the deterministic combinations with random ones, and obtain several other possibilities. ★

2.3 The electricity market

In the beginning of the nineties, a world-wide process to deregulate the national power started. Since then the electricity system has been open for competition, and this in turn have led to the introduction of power markets, where power is traded; both the spot itself and derived products such as futures and forwards. Examples of such markets include the European Energy Exchange (EEX), The New York Mercantile Exchange (NYMEX) and the Nordic Power Exchange (NordPool). In addition there is a significant over-the-counter (OTC) market. We shall be concerned with the forward contract only:

Definition 2.20. A *forward contract* is an agreement between a seller, taking the *short* position, and a buyer, taking the *long* position, that a transaction of an asset is to take place at some future time T for a certain price $F(t,T)$. This agreed upon price is called the *forward price*.

In mathematical terms we can write the arbitrage-free forward price from Definition 2.20 as

$$\mathbf{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)\mathrm{d}u}\Big(S_T - F(t,T)\Big)\,\Big|\,\mathcal{F}_t\right] = 0, \text{ for } t \leq T \leq T_{max},$$

where $\mathbb{Q}$ is an equivalent martingale measure. Being a contingent claim $F(t,T)$ must be adapted, so this relation simplifies to

$$F(t,T) = \mathbf{E}_{\mathbb{Q}}\left[S_T\,\Big|\,\mathcal{F}_t\right].$$

In contrast to other commodities, the electricity stands out in several important ways. The electricity can not be stored once produced, at least in some financial satisfying way, so the spot is not tradeable. Also, it is produced for immediate consumption, and only useful for any practical purposes if delivered during a period of time. In addition the lack of any practical means to transport the electricity over any greater distance, make the demand inelastic and also dependent on local conditions, such as supply and demand or for given weather conditions. For instance the spot-price increase after weather-forecasts predicting colder weather. This is the origin of a powerful mean-reverting behavior.

A common way to hedge the risk is to use a simple buy-and-hold strategy, illustrated in the following example.

Example 2.21. Assume we have the spot price $S(0) = 100\,€$, at the initial time, and that there risk free interest rate is 10% *per annum*, continuously compounded. Suppose that the one-year forward price written on this commodity is $F(0,1) = 130\,€$. The following strategy would then yield a profit:

1. Borrow $100\,€$

2. Buy one item of the commodity

3. Enter into a forward contract to sell the commodity at the time $t = T$ for a amount of 130€

At the time T the trader would sell the commodity and receive 130€, repay the loan (with interest) of 110€ and make a profit of 20€. ★

However, this simple buy-and-hold strategy does not apply, due to the non-storability of the electricity. So, in contrast to other commodities markets where participants could store the commodity to hedge the risk from price-variations, the non-storability of electricity make the price closely connected to the supply and demand at any given time.

For a further discussion of the electricity market, the articles by Lucia and Schwartz [LS02] and Benth and Koekebakker [BK05] give a thorough discussion - in particular they discuss the NordPool. Information of a more pragmatic nature is also readily available on the respective web-sites.

2.3.1 Modelling the electricity spot

There are generally two approaches one could take when valuating financial derivatives: One could derive a model of the underlying security and from this valuate the claim, or one could alternatively try to model the dynamics for the claim directly. This latter approach shall be of no interest to us in this thesis; we shall only consider the former method. We note that the former method tend to give more analytical tractable expressions though, so it is not without interest.

Any sound model must try to capture as much of the properties of the spot as possible. This is to say that both path-wise and statistical properties should be taken into account. For the electricity spot observed path-wise properties include seasonality at three levels: annually, monthly and intra-day, and in addition the spot demonstrates large jumps, and a strong reversion to a mean level. Obviously we can not say something specific about the statistics of the spot model without an empirical study of time series.

We shall discuss two of the proposed spot models in some detail in the following sections.

2.3.2 The Schwartz model

Schwartz proposed the model holding his name for general commodities in [Sch97], and later he applied it to the electricity spot in cooperation with Lucia [LS02]. Originally the model did not include jumps, but we have chosen to include them to capture the influence of jumps in the spot price and statistics. Due tho the mean-reverting nature of the electricity spot, the dynamics are given as an Ornstein-Uhlenbeck process:

$$\mathrm{d}\widetilde{S}_t = \left(\mu - \alpha \widetilde{S}_{t-}\right)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}N_t. \tag{2.5}$$

We use the notation $\widetilde{S} = \log S$. In the stochastic differential equation (2.5) $\mu \in \mathbb{R}$ denotes a mean level, $\alpha \in \mathbb{R}$ the speed of which we have mean reversion and $\sigma \in \mathbb{R}$ the volatility. Further, we have that B is a standard Brownian motion and the representation

$$N_t = \int_{[0,t]\times\mathbb{R}} zN(\mathrm{d}s, \mathrm{d}z),$$

for the Poisson random measure N.

2.3.3 Geman & Roncoronis model

The spot model proposed by H. Geman and A. Roncoroni are presented in [GR06], and really only differs from (2.5) in the last term:

$$\mathrm{d}\widetilde{S}_t = \left(\mu - \alpha\widetilde{S}_{t-}\right)\mathrm{d}t + \sigma\mathrm{d}B_t + h(S_{t-})\mathrm{d}N_t. \tag{2.6}$$

The function h represent some threshold $c \in \mathbb{R}$ where the jumps are allowed take on negative values, i.e. the jumps can now also be negative to inflict an abrupt fall in the spot price

$$h(S_{t-}) = \begin{cases} 1, & \text{if } S_{t-} \leq c, \\ -1, & \text{if } S_{t-} > c. \end{cases}$$

This is an example of a *regime switching* model.

The jump size distribution is given as

$$p(x) = \frac{\gamma \exp(-\gamma x)}{1 - \exp(-\gamma\Delta_{max})},$$

for $0 \leq x \leq \psi$, where $\gamma, \Delta_{max}, \psi \in \mathbb{R}$.

From [GR06] we find the parameters of (2.6) we shall use for the three different regions.

All relevant constants are listed in Table 2.2.

Parameters							
Model	μ	α	σ	Δ_{max}	γ	c	λ
ECAR	3.0923	38.8938	1.8355	3.3835	0.3129	5.5923	9.0
COB	2.8928	42.8844	1.3631	1.0169	1.0038	3.8928	2.0
PJM	3.2002	13.3815	1.4453	1.6864	0.5016	4.7002	9.667

Table 2.2: Listing of parameters for the spot model (2.6), taken from [GR06].

2.3.4 Pricing in the electricity market

We saw in Example 2.19 that there are several ways one can find an equivalent martingale measure. The approach taken by Merton, who first proposed jump diffusions, are shown in the next example.

Example 2.22. For the classical jump-diffusion price process

$$\begin{cases} \mathrm{d}R_t = rR_t\mathrm{d}t,\ R_0 = r \\ \mathrm{d}S_t = S_{t-}\Big[\mu\mathrm{d}t + \sigma\mathrm{d}B_t + \mathrm{d}N_t\Big] \end{cases} \tag{2.7}$$

proposed by Merton [Mer76], we have the constants μ and σ, B a standard Brownian motion and N a compound Poisson process with intensity λ. Merton did not price the risk premium associated with jumps:

$$\theta = \frac{\mu + \int_{\mathbb{R}} z\mathrm{d}\nu(z) - r}{\sigma}.$$

The risk-neutral dynamics are then given by

$$\mathrm{d}S_t = S_t\Big[r\mathrm{d}t + \sigma\mathrm{d}W_t + \mathrm{d}M_t\Big],$$

where M is the compensated N and W is a $\mathbb{Q}$-Brownian motion, as follows from Theorem 1.37.

This is to say that the risk can be diversified away, and to let the price of jumps be propagated to the price of the claim in question. ★

Several authors have adopted Merton's approach and applied it to the electricity market: For our purposes Cartea and Figueroa [CF05] and Benth *et al.* [BEHN03] are relevant applications of Merton's choice in the electricity-market.

Chapter 3

Numerical methods in finance

> "At that time, the notion of partial differential equations was very, very strange on Wall Street"
>
> R. C. Merton, Derivative Strategies, 1998.

In this chapter we review some of theory of numerical solution of partial differential equations, extend this theory to the integro-differential case, and finish off with a discussion of Monte Carlo simulations.

3.1 Introduction

The complexity of the models occurring in mathematical finance have increased steadily over the last century and, in many cases, to the point where purely analytical insight is hard to obtain. For the two stochastic models we are concerned with in this thesis, only one of them are tractable by pen and paper solely when we want to valuate derivatives. And so, we have no option but to resort to numerical simulations.

In mathematical finance two numerical approaches occur frequently: One exploits the connection between partial differential equations and stochastic differential equations, and the other exploits the ability to generate large amounts of random (actually, pseudo-random) numbers by computers.

Remark 3.1. In this chapter we will use the notation $u(x,t)$ as opposed to $u(t,x)$ used in Chapter 1, so to keep in line with current notation used in textbooks. We are confident that this small change of notation will present no challenge to our readers.

3.2 Partial differential equations

In this section we will motivate the study of integro-differential equations in finance. The theory on this subject is quite extensive, so we will not go into details.

3.2.1 Parabolic differential equations

A linear operator on $\mathbb{R}^n$ is said to be *semi-elliptic* if it is given by

$$L = \sum_{i,j=1}^{n} a_{ij}(x,t)\partial_{x_i x_j} + \sum_{j=1}^{n} b_j(x,t)\partial_{x_j} + c(x,t), \tag{3.1}$$

where the coefficients a_{ij}, b_j and c are all bounded and real, L satisfy the ellipticity condition $y^{\mathsf{T}} a(x,t) y \geq \xi |y|^2$ for some non-negative real ξ and $y \in \mathbb{R}^n$. Operators of the form

$$\partial_t + L \tag{3.2}$$

are then *parabolic*. This brings us to *parabolic* partial differential equations

$$\begin{cases} u_t + Lu &= 0 \text{ on } U \times (0,T] \\ u(x,0) &= h(x) \text{ for } x \in \mathbb{R}^n, \end{cases} \tag{3.3}$$

for $U \subset \mathbb{R}^n$, not necessarily bounded. We often restrict h to, say, the class of continuous functions with at most polynomial growth.

Examples of parabolic equations include diffusion equations like the heat equation and convection-diffusion equations like the Black-Scholes equation.

For our purposes the typical example of a parabolic equation is the Black-Scholes equation:

Example 3.2 (Black-Scholes PDE)**.** We recall the dynamics used by Black & Scholes in their derivation of the option price, illustrated in Example 2.13:

$$\begin{cases} \mathrm{d}R_t = rR_t \mathrm{d}t, R_0 = r \\ \mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t. \end{cases} \tag{2.1}$$

We let the interest-rate $r \in \mathbb{R}$ be constant, i.e. we assume that the large investor do not get any benefits of being large. First we write the risk-neutral dynamics of this equation, which we can find from Girsanov's Theorem: Under the risk-neutral measure $\mathbb{Q}$

$$\mathrm{d}W_t = \mathrm{d}B_t + \theta \mathrm{d}t$$

is a Brownian motion, and upon inserting we find the risk-neutral dynamics

$$\mathrm{d}S_t = (\mu - \sigma\theta) S_t \mathrm{d}t + \sigma \mathrm{d}W_t = rS_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

and $\theta = (\mu - r)/\sigma$. Next, we find the infinitesimal generator

$$\mathcal{A}u = rxu_x + \frac{1}{2}\sigma^2 x^2 u_{xx}.$$

for $u \in C^{2,1}(\mathbb{R} \times [0,T])$. Assuming that h is continuous and such that

$$|h(x)| \leq L(1 + |x|^2) \text{ or } h(x) \geq 0,$$

for $L \geq 0$ we can apply the Feynman-Kac formula. Now it is straightforward to see that the degenerate backward parabolic equation

$$\begin{cases} u_t + rxu_x + \frac{1}{2}\sigma^2 x^2 u_{xx} - ru = 0, \text{ on } \mathbb{R} \times [0,T) \\ u(T,x) = h(x), \qquad\qquad\qquad\qquad x \in \mathbb{R} \end{cases} \tag{3.4}$$

known as the *Black-Scholes* partial differential equation, has the unique solution

$$u(x,t) = \mathbf{E}^{t,x}\Big[e^{-rt}h(S_t)\Big].$$

★

We note that (3.4) is *backwards* but can easily be turned into a forward equation by the transformation $\tau = T - t$.

Initial- and boundary conditions. From Example 3.2 we learned that Feynman-Kac's formula only provide us with one condition, and since the equation is of second order in x we need additional conditions.

Consider for instance an European call in Example 3.2: Then

$$h(x) = \max\{x - K, 0\}.$$

We see that if S_t is zero in (2.1) then $\mathrm{d}S_t$ is zero, and so S must be constant. This is a deterministic, and the only such, solution of (2.1). If S is zero at the terminal time T then from the pay-off function we see that there is zero payoff, and so the call option is worthless on $S = 0$. Hence we could put

$$u(0,t) = 0.$$

Considering the other extreme, when the asset price increases without bounds, it will become more likely that the option will be exercised, regardless of the actual value of the option. Hence, as x approaches infinity, we have

$$u(x,t) = x.$$

Localization. Neither from a financial nor numerical perspective is an unbounded domain, as we have in Example 3.2, satisfying. The limitations on storage in computers imply that we must localize the problem to a bounded domain before we can solve it. Later we will introduce an integral term into the parabolic equation (3.2) and form a parabolic partial integro-differential equation, but we leave this for now and return to the subject in a moment.

3.2.2 Parabolic integro-differential equations

Now, what is the case for price-processes with jumps?

Example 3.3. Let us take a closer look at Merton's model, briefly encountered in Section 2.1 and Example 2.22:

$$\begin{cases} \mathrm{d}R_t = rR_t\mathrm{d}t,\ R_0 = r \\ \mathrm{d}S_t = \Big(\mu + \int_{\mathbb{R}} z\mathrm{d}\nu(z)\Big)S_{t-}\mathrm{d}t + \sigma S_{t-}\mathrm{d}W_t + S_{t-}\mathrm{d}M_t, \end{cases}$$

with M_t the compensated Poisson process. We can appeal to Theorem 1.37 by Chan, and find a change of measure such that

$$\mathrm{d}W_t = \mathrm{d}B_t + \theta\mathrm{d}t$$

is a $\mathbb{Q}$-martingale with the price of risk

$$\theta = \frac{\mu + \int_{\mathbb{R}} z\mathrm{d}\nu(z) - r}{\sigma}.$$

Inserting we find the risk-neutral dynamics

$$\mathrm{d}S_t = rS_{t-}\mathrm{d}t + \sigma S_{t-}\mathrm{d}W_t + S_{t-}\mathrm{d}M_t,$$

and the infinitesemal generator

$$\mathcal{A}u = rxu_x + \frac{1}{2}\sigma^2x^2u_{xx} + \int_{\mathbb{R}}\Big(u(x+zx) - u(x) - zxu_x\Big)\mathrm{d}\nu(z)$$

defined for $C^{2,1}$ functions on $\mathbb{R}\times[0,T)$. Now we can use the result by Pham, Proposition 1.39, and find that the solution of the partial integro-differential problem

$$\begin{cases} u_t + rxu_x + \frac{1}{2}\sigma^2x^2u_{xx} + \int_{\mathbb{R}}\Big(u(x+zx,t) - u(x,t) - zxu_x(x,t)\Big)\mathrm{d}\nu(z), \\ u(x,T) = h(x) \end{cases}$$

is given as

$$u(x,t) = \mathbf{E}^{t,x}[e^{-r(T-t)}h(S_T)],$$

with Lipschitz pay-off function h. For example, we could take $h(x) = \max\{x-K,0\}$ for the European call option. ★

Hence, by introducing jumps into our price process, we get a *partial integro-differential* equation

$$\begin{cases} u_t + Lu + Iu &= 0 \text{ on } U\times(0,T] \\ u(x,0) &= h(x) \text{ for } x\in\mathbb{R}^n, \end{cases} \tag{3.5}$$

for I some integral operator. Typically, I is of the type

$$Iu = \int_{\mathbb{R}^n} \Big(u(x + \gamma(t,x,z),t) - u(x,t) - \mathbf{1}_{|z|\leq 1}\gamma(t,x,z)u_x(x,t) \Big) \mathrm{d}\nu(z),$$

with ν the Levy-measure of the associated price process. Often this measure is finite and compactly supported, for instance for all jump-diffusions with bounded jumps, and so the problem simplifies to

$$Iu = \int_E \Big(u(x + \gamma(t,x,z),t) - u(x,t) \Big) \mathrm{d}\nu(z)$$

for E compact, after moving the u_x-term from the integral- to the differential operator. In many cases the simplification can be taken even further, so we could take for instance $\gamma(t,x,z) = z$ and we are left with the integral

$$\begin{aligned} Iu &= \int_E \Big(u(x+z,t) - u(x,t) \Big) \mathrm{d}\nu(z) \\ &= \lambda \int_E u(x+z,t)\mathrm{d}p(z) - \lambda u(x,t), \end{aligned}$$

after using the relations $p(z) = \nu(z)/\lambda$ and $\int \mathrm{d}p(z) = 1$. So, for the simplest non-trivial scenario we are left with the integral operator

$$Iu = \int_E u(x+z,t)\mathrm{d}p(z). \tag{3.6}$$

Localization and truncation. Suppose that the domain $U = \mathbb{R}$ in the parabolic integro-differential equation (3.5). To have a tractable problem, we must localize the problem to a bounded subset of the real numbers, and so we define

$$\begin{cases} \bar{u}_t + L\bar{u} + I\bar{u} & = 0 \text{ on } [-A,A] \times (0,T] \\ \bar{u}(x,0) & = h(x) \text{ for } x \in (-A,A) \end{cases}$$

as the solution of this localized problem.

Needless to say, this introduce some error, and analysis of this matter should be performed.

Boundary conditions. Another issue is due to the non-locality of the integral term. To find the integral for some inner point $x \in (-A,A)$ we must know u for the values $x + y$ with $y \in \mathrm{supp}\{\nu\}$. Obviously, difficulties arise when we try to compute u outside $[-A,A]$. The remedy is to introduce some auxiliary numerical boundary condition

$$\bar{u}(x,t) = g(x,t) \text{ for } x \notin (-A,A)$$

to the localized problem. Possible choices for could be $g = 0$ or $g = h$, for h the pay-off function of the given problem.

3.3 Finite difference methods

Definitions and notation. Assume that we have the two-dimensional case and the domain $\Omega = [-A, A] \times [0, T]$. We partition the domain with N points in the space coordinate, and M in the time coordinate. Let the step sizes in t and x-direction be given as $\Delta t = T/M$ and $\Delta x = 2A/N$. we put

$$t_m = m\Delta t, \text{ for } m = 0, \ldots, M \text{ and}$$
$$x_j = -A + j\Delta x \text{ for } j = 0, \ldots, N.$$

We approximate u at the points listed above, and write

$$v_j^m = u(x_j, t_m).$$

From the definition of the derivative we have

$$u_t(x,t) = \lim_{\Delta t \to 0} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{u(x,t) - u(x, t-\Delta t)}{\Delta t}.$$

This motivates the forward- and backward difference approximations

$$u_t \approx \frac{v_j^{m+1} - v_j^m}{\Delta t}, \quad u_t \approx \frac{v_j^m - v_j^{m-1}}{\Delta t},$$

and the central difference

$$u_t \approx \frac{v_j^{m+1} - v_j^{m-1}}{2\Delta t}.$$

Analogously we define the differences for the x-partial derivative, and in addition the second derivative central difference

$$u_{xx} \approx \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\Delta x^2}.$$

3.3.1 The differential case

For the purely differential case, we look at the parabolic equation given by (3.3), for some elliptic differential operator L given by (3.1). Applying the difference approximations stated above we can derive the following discrete operator:

$$L_d v_j^m = a_j^m \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\Delta x^2} + b_j^m \frac{v_{j+1}^m - v_j^m}{\Delta x} + c_j^m v_j^m, \tag{3.7}$$

where $a_j^m = a(x_j, t_m)$ and likewise for b_j^m and c_j^m for $0 < j < N$ and $0 \leq m \leq M$. Trivial rewriting yields

$$L_d v_j^m = \frac{1}{\Delta x^2} a_j^m v_{j-1}^m + \frac{1}{\Delta x^2}\Big(\Delta x^2 c_j^m - \Delta x b_j^m - 2a_j^m\Big)v_j^m + \frac{1}{\Delta x^2}\Big(a_j^m + \Delta x b_j^m\Big)v_{j+1}^m. \tag{3.8}$$

From this we see that we get a tri-diagonal sparse matrix.

Now we define the θ-scheme[1]:

$$-\frac{v_j^{m+1} - v_j^m}{\Delta t} = \theta L_d v_j^m + (1-\theta) L_d v_j^{m+1}$$

for $\theta \in [0,1]$. This enables us to define the following schemes:

- For $\theta = 1$ we have the *explicit* Euler scheme: $v_j^{m+1} = (I - \Delta t L_d) v_j^m$
- For $\theta = 0$ we have the *implicit* Euler scheme: $(I + \Delta t L_d) v_j^{m+1} = v_j^m$

The explicit scheme is potentially unstable so stringent conditions must be met to ensure stability, but on a positive note the scheme is computationally efficient since L_d is the origin of a tri-diagonal matrix. For the implicit scheme there are no stringent stability condition, however the scheme is computational inefficient in that for each iteration a system of equations must be solved.

Other methods exist, like the Crank-Nicolson scheme where $\theta = 1/2$.

3.3.2 The integro-differential case

Numerical evaluation. To evaluate the integral (3.6) we could apply any numerical method for integration. For instance, we could use the trapezoid rule: We partition $[B_l, B_u]$ into intervals of size Δx as we have used in the finite difference approximation earlier, and choose integers K_l, K_u such that

$$[B_l, B_u] \subset [(K_l - 1/2)\Delta x, (K_u + 1/2)\Delta x].$$

The integral is then given by

$$\int_{B_l}^{B_u} u(x_j + z, t_m) \mathrm{d}p(z) \approx \sum_{i=K_l}^{K_u} p_i v_{i+j}^m$$

for

$$p_i = \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} \mathrm{d}p(z)$$

and $v_{i+j}^m = u(x_j + z_i, t_m)$. Hence we have the discrete integral operator

$$I_d v_j^m = \sum_{i=K_l}^{K_u} p_i v_{i+j}^m, \tag{3.9}$$

[1] This θ is not to be mistaken with the price of risk

so for each v_j^m we get a sum of $K_u - K_l$ terms. Clearly, this operator is the origin of a dense matrix: For $j \in \{0, \ldots, N\}$ we get

$$\begin{aligned} I_d v_j^m &= \sum_{i=K_l}^{K_u} v_{j+i}^m p_i \\ &= v_{K_l+j}^m \, p_{K_l} + \cdots + v_{K_u+j}^m \, p_{K_u}. \end{aligned}$$

Since j varies in $0, \ldots, N$ we need the values $v_{K_l}, \ldots, v_{N+K_u}$, hence we get the $(N + K_u - K_l) \times (N + K_u - K_l)$ band-matrix

$$\begin{pmatrix} p_{K_l} & \cdots & & p_{K_u} & & & & \\ & \ddots & & & \ddots & & & \\ & & & & & & & \\ & p_{K_l+j} & \cdots & & & p_{K_u+j} & & \\ & & \ddots & & & & \ddots & \\ & & & & & & & \\ & & & p_{K_l+N} & \cdots & & & p_{K_u+N} \end{pmatrix}.$$

Finite difference solution. We now extend the methods outlined in Section 3.3.1, and find the following θ-scheme for the partial integro-differential equation:

$$\begin{aligned} -\frac{v_j^{m+1} - v_j^m}{\Delta t} =& \theta_1 L_d v_j^m + (1-\theta_1) L_d v_j^{m+1} \\ &+ \theta_2 I_d v_j^m + (1-\theta_2) I_d v_j^{m+1}, \end{aligned}$$

again for $\theta_1, \theta_2 \in [0, 1]$. We define explicit and implicit solutions as before.

A natural choice would be to use an implicit scheme for the differential term, since a sparse matrix arise from L_d. From the operator I_d, however, a dense arise so an explicit scheme is preferred for this term since solving a large system of linear equations at each time step is computational demanding. Hence we choose $\theta_1 = 0$ and $\theta_2 = 1$, and get an *implicit-explicit* (IMEX) scheme

$$(I + \Delta t L_d) v_j^{m+1} = (I - \Delta t I_d) v_j^m,$$

after trivial rewriting. Assuming we have the initial auxiliary boundary condition $u(x,t) = g(x,t)$ for $x \notin (-A, A)$ we get:

3.4 Monte Carlo simulations

The idea. The Monte Carlo method exploits the fact that we can take volume as an analogy for probability. If we think of the integral

$$\int_0^1 f(x) \mathrm{d}x$$

Algorithm 1 Finite difference solution

if $i = 1, \ldots, N$ **then**
  $v_j^0 \leftarrow x_j$
else if $i \neq 1, \ldots, N$ **then**
  $v_j^0 \leftarrow g(x_j, 0)$
end if
for $m = 0, \ldots, M-1$ **do**
  if $i = 1, \ldots, N$ **then**
    $v_j^{m+1} \leftarrow (I + \Delta t L_d)^{-1}(I - \Delta t I_d)v_j^m$
  else if $i \neq 1, \ldots, N$ **then**
    $v_j^{m+1} \leftarrow g(x_j, t_{m+1})$
  end if
end for
Apply boundary conditions

as an expectation with x some random variable, then the strong low of large numbers imply

$$\frac{1}{n}\sum_{i=1}^{n} f(U_i) \longrightarrow \int_0^1 f(x)\mathrm{d}x$$

as $n \to \infty$ a.s. if $U_i \sim \mathrm{Unif}(0,1)$. One of the advantages of Monte Carlo simulations is immediate: we can relax the regularity restrictions on f.

Generation of random numbers. To generate random numbers from any given distribution there are several alternatives. For instance, the *inverse method* is useful: If F is the cumulative distribution function of the probability distribution we want to draw numbers from, we set

$$X = F^{-1}(U), \text{ for } U \sim \mathrm{Unif}(0,1).$$

Obviously, the requirement is that it is possible to obtain the inverse of F. This method is often applied when it comes to generate exponentially distributed numbers.

Generation of sample paths. The solution of stochastic differential equations are indeed a large subject in it's own right. We shall not give any detailed treatment, just mention the bare minimum so that our application are justified.

The problem is to approximate processes such as

$$X_T = x_t + \int_t^T \mu(s, X_{s-})\mathrm{d}S_t + \int_t^T \sigma(s, X_{s-})\mathrm{d}B_s + \int_t^T \gamma(s, X_{s-})\mathrm{d}N_s,$$

where $x_t \in \mathbb{R}$, for B a Brownian motion and N a compound Poisson process with intensity λ. The coefficients

$$\sigma : \mathbb{R}^2 \to \mathbb{R},\ \gamma : \mathbb{R}^2 \to \mathbb{R}$$

are taken to be continuous, for instance, but other alternatives exist.

Given a partition $t = \tau_0 < \tau_1 < \cdots < \tau_M = T$ of the time interval $[t, T]$, we define the iterative *Euler approximation*, for the case $\gamma = 0$ initially:

$$X_{n+1} = X_n + \mu(\tau_n, X_n)(\tau_{n+1} - \tau_n) + \sigma(\tau_n, X_n)(B_{\tau_{n+1}} - B_{\tau_n}),$$

for $n = 0, \ldots, M-1$. We have the initial value X_0. We have

$$\begin{aligned} \mathbf{E}\left[B_{\tau_{n+1}} - B_{\tau_n}\right] &= 0, \text{ and} \\ \mathbf{E}\left[B_{(\tau_{n+1}} - B_{\tau_n})^2\right] &= (\tau_{n+1} - \tau_n), \end{aligned}$$

from the usual properties of the Brownian motion. A thorough treatment of this case is given in Kloeden and Platen [KP92].

Turning to the case $\gamma \neq 0$, we must incorporate the jumps. Recall that the compound Poisson process is given by

$$\sum_{j=0}^{N(t)} Y_j,$$

where the Y's are i.i.d. and N a Poisson process with intensity λ.

The discrete version reads

$$X_{n+1} = X_n + \mu(\tau_n, X_n)(\tau_{n+1} - \tau_n) + \sigma(\tau_n, X_n)(B_{\tau_{n+1}} - B_{\tau_n}) + \sum_{j=N(\tau_i)+1}^{N(\tau_{i+1})} Y_j. \tag{3.10}$$

The implementation is mostly straightforward, but some issues are connected with the simulation of the jumps. Especially, some authors suggest that one should simulate the jump times and sizes upfront, and then insert them into the partition $t = \tau_0 < \cdots < \tau_n = T$. This is not convenient, for instance, when doing Monte Carlo simulations, since we want to take the sum of vectors with the same size.

We shall therefore apply another approach, taken from Glasserman [Gla04]

As a final comment, we note that the independent increments of the Poisson process have been used, and that

$$N_{\tau_{i+1}} - N_{\tau_i} \sim Poisson$$

with mean

$$\mathbf{E}\left[N_{\tau_{i+1}} - N_{\tau_i}\right] = \lambda(\tau_{i+1} - \tau_i)$$

.

Algorithm 2 Solution of (3.10)

1: Initialize
2: Generate $B \sim \mathcal{N}(0,1)$
3: Generate $N \sim Poisson(\lambda(\tau_{n+1} - \tau_n))$
4: **if** $N = 0$ **then**
5: Set $Y = 0$
6: **else if** $M \geq 1$ **then**
7: Generate $M = Y_1, \ldots, Y_M$
8: **end if**
9: Set $X_{\tau_{n+1}} = X_{\tau_n} + \mu(\tau_n, X_n)(\tau_{n+1} - \tau_n) + \sigma(\tau_n, X_n)\sqrt{\tau_{i+1} - \tau_i}\, B + M$

Part II

Results and discussion

Chapter 4

Results

4.1 Analytical results

This chapter will present our results. The methods we use are outlined earlier, so the chapter contain calculations mainly.

4.1.1 The Forward Price w.r.t. Schwartz' model

Let the usual hypothesis listed in Assumption 1 be satisfied. Assuming that the electricity market satisfy the no free lunch with vanishing risk criteria, we know that there exist an equivalent local martingale measure $\mathbb{Q}$ by the fundamental theorem of asset pricing and can apply the arbitrage-free pricing method illustrated in a previous chapter.

Recall the modified Schwartz model (2.5)

$$\mathrm{d}\widetilde{S}_t = \left(\mu - \alpha\widetilde{S}_{t-}\right)\mathrm{d}t + \sigma\mathrm{d}B_t + \mathrm{d}N_t,$$

with the short hand notation $\widetilde{S} = \log S$. For a discussion of the model, look to Section 2.3.2.

We shall employ the ideas outlined in Section 2.3. We know from Theorem 1.37 that there is a $\theta \in \mathbb{R}$ such that

$$\mathrm{d}W_t = \mathrm{d}B_t + \theta\mathrm{d}t,$$

is a $\mathbb{Q}$-martingale, and use the Girsanov theorem to find the measure-transformation. As as illustrated in Example 2.22 we get the risk-neutral dynamics

$$\mathrm{d}\widetilde{S}_t = \left(\mu + \int_{\mathbb{R}} z\mathrm{d}\nu(z) - \sigma\theta - \alpha\widetilde{S}_{t-}\right)\mathrm{d}t + \sigma\mathrm{d}W_t + \mathrm{d}M_t, \tag{4.1}$$

where we use the compensated compound Poisson process. Now we put $\widehat{\mu} = \mu + \int_{\mathbb{R}} z\mathrm{d}\nu(z) - \sigma\theta$.

Result 1. *The forward price with respect to Lucia and Schwartz' model, is given by*

$$F(t,T) = S_t^{e^{-\alpha(T-t)}} \exp\Big(\frac{\widehat{\mu}}{\alpha}(1-e^{-\alpha(T-t)})\Big) \exp\Big(\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T-t)})\Big) \exp\Big(\int_t^T \psi(e^{-\alpha(T-s)})\mathrm{d}s\Big). \quad (4.2)$$

We have the following:

(i) The price of risk is given by the relation $\theta = \frac{\mu+\int_{\mathbb{R}} z\mathrm{d}\nu(z)}{\sigma} = \frac{\mu+\lambda\int_{\mathbb{R}}\mathrm{d}p(z)}{\sigma}$

(ii) $F(T,T) = S_T$ *as should be expected*

Proof. To solve the stochastic differential equation (4.1) we multiply by a suitable integrating factor and integrate by parts:

$$\begin{aligned}\mathrm{d}(e^{\alpha t}\widetilde{S}_t) &= e^{\alpha t}\mathrm{d}\widetilde{S}_t + \widetilde{S}_t\mathrm{d}(e^{\alpha t}) + \mathrm{d}[e^{\alpha t},\widetilde{S}]_t \\ &= e^{\alpha t}\widehat{\mu}\mathrm{d}t + e^{\alpha t}\sigma\mathrm{d}W_t + e^{\alpha t}\mathrm{d}M_t.\end{aligned}$$

Now we integrate over $[t,T]$ and multiply by $e^{-\alpha T}$ to arrive at

$$\begin{aligned}\widetilde{S}_T =& e^{-\alpha T}\left(\widetilde{S}_t e^{\alpha t} + \int_t^T e^{\alpha s}\widehat{\mu}\mathrm{d}s + \int_t^T e^{\alpha s}\sigma\mathrm{d}W_s + \int_t^T e^{\alpha s}\mathrm{d}M_s\right) \\ =& e^{-\alpha(T-t)}\widetilde{S}_t + \int_t^T e^{-\alpha(T-s)}\widehat{\mu}\mathrm{d}s \\ &+ \int_t^T e^{-\alpha(T-s)}\sigma\mathrm{d}W_s + \int_t^T e^{-\alpha(T-s)}\mathrm{d}M_s.\end{aligned}$$

Taking the exponential leads to

$$\begin{aligned}S_T &= \exp\Big(e^{-\alpha(T-t)}\log S_t + \int_t^T e^{-\alpha(T-s)}\widehat{\mu}\mathrm{d}s \\ &\quad + \int_t^T e^{-\alpha(T-s)}\sigma\mathrm{d}W_s + \int_t^T e^{-\alpha(T-s)}\mathrm{d}M_s\Big) \\ &= S_t^{e^{-\alpha(T-t)}} \exp\Big(\int_t^T e^{-\alpha(T-s)}\widehat{\mu}\mathrm{d}s\Big) \\ &\quad \exp\Big(\int_t^T e^{-\alpha(T-s)}\sigma\mathrm{d}W_s\Big) \exp\Big(\int_t^T e^{-\alpha(T-s)}\mathrm{d}M_s\Big).\end{aligned}$$

At this point we take expectations with respect to the risk-neutral measure.

$$\begin{aligned}
F(t,T) =& \mathbf{E}_{\mathbb{Q}}\Big[S_T \,\Big|\, \mathcal{F}_t\Big] \\
=& \mathbf{E}_{\mathbb{Q}}\Big[S_t^{e^{-\alpha(T-t)}} \exp\Big(\int_t^T e^{-\alpha(T-s)}\widehat{\mu}\mathrm{d}s\Big) \\
& \exp\Big(\int_t^T e^{-\alpha(T-s)}\sigma\mathrm{d}W_s\Big)\exp\Big(\int_t^T e^{-\alpha(T-s)}\mathrm{d}M_s\Big)\,\Big|\,\mathcal{F}_t\Big] \\
=& S_t^{e^{-\alpha(T-t)}} \exp\Big(\int_t^T e^{-\alpha(T-s)}\widehat{\mu}\mathrm{d}s\Big) \\
& \mathbf{E}_{\mathbb{Q}}\Big[\exp\Big(\int_t^T e^{-\alpha(T-s)}\sigma\mathrm{d}W_s\Big)\exp\Big(\int_t^T e^{-\alpha(T-s)}\mathrm{d}M_s\Big)\,\Big|\,\mathcal{F}_t\Big].
\end{aligned}$$

Next we appeal to the independence of the W- and N-integrals, and use the following two results: Since W is a Brownian motion under $\mathbb{Q}$ we have that

$$\mathbf{E}_{\mathbb{Q}}\Big[\exp\Big(\int_t^T e^{-\alpha(T-s)}\sigma\mathrm{d}W_s\Big)\Big] = \exp\Big(\frac{1}{2}\int_t^T e^{-2\alpha(T-s)}\sigma^2\mathrm{d}s\Big).$$

We also have the the following relation, justified in Section A.3 in the Appendix:

$$\mathbf{E}_{\mathbb{Q}}\Big[\exp\Big(\int_t^T e^{-\alpha(T-s)}\mathrm{d}M_s\Big)\Big] = \exp\Big(\int_t^T \psi(e^{-\alpha(T-s)})\mathrm{d}s\Big).$$

Continuing our proof, we find

$$\begin{aligned}
F(t,T) = S_t^{e^{-\alpha(T-t)}} & \exp\Big(\int_t^T e^{-\alpha(T-s)}\widehat{\mu}\mathrm{d}s\Big) \\
& \exp\Big(\frac{1}{2}\int_t^T e^{-2\alpha(T-s)}\sigma^2\mathrm{d}s\Big)\exp\Big(\int_t^T \psi(e^{-\alpha(T-s)})\mathrm{d}s\Big),
\end{aligned}$$

and after calculating the integrals we are done. □

4.1.2 An PIDE for the forward price

In this section we shall find a partial integro-differential equation giving the forward curve. By the Markovian property for Levy-processes we know that

$$\mathbf{E}\left[X_t|\mathcal{F}_t\right] = \mathbf{E}\left[X_t|X_t = x\right],$$

supposing the process is adapapted to a suitable filtered probability-space and such that the integral exist. We will use this fact implicitly.

The partial integro-differential equation. We recall the model proposed by Geman and Roncoroni, which we restate for convenience:

$$\mathrm{d}\widetilde{S}_t = \Big(\mu - \alpha\widetilde{S}_{t-}\Big)\,\mathrm{d}t + \sigma\mathrm{d}B_t + h(\widetilde{S}(t-))\mathrm{d}N_t, \tag{2.6}$$

with the notation $\widetilde{S}$ as before and

$$h(\widetilde{S}_{t-}) = \begin{cases} 1 & \text{if } \widetilde{S}_{t-} \leq c, \\ -1 & \text{if } \widetilde{S}_{t-} > c. \end{cases}$$

We will make use of Proposition 1.39 due to Pham. Setting aside questions of regularity for the moment, we shall continue with our derivation of the partial integro-differential equation, and arguing as in the previous section we rewrite (2.6) on the risk neutral form

$$\mathrm{d}\widetilde{S}_t = \Big(\widehat{\mu} - \alpha\widetilde{S}_{t-}\Big)\,\mathrm{d}t + \sigma\mathrm{d}W_t + h(\widetilde{S}_{t-})\mathrm{d}M_t.$$

Here we have used the same price of risk as in the previous section, so $\widehat{\mu} = \mu + \int_{\mathbb{R}} z\mathrm{d}\nu(z) - \sigma\theta$. Then we have the corresponding infinitesemal generator with $u \in C_0^2$:

$$\begin{aligned}\mathcal{A}u(x,t) = \Big(\widehat{\mu} - \alpha\, x\Big)u_x + \frac{\sigma^2}{2}u_{xx} \\ + \int_{\mathbb{R}} \Big(u(x + zh(x), t) - u(x,t) - zh(x)u_x\Big)\mathrm{d}\nu(z),\end{aligned} \tag{4.3}$$

defined on $[0,T] \times \mathbb{R}$.

Before we present out second result, we let $\mathcal{A}^*$ denote the generator we find after we have applied Itô on (2.6). Since we never actually need this generator, we do not write it out. Equivalently, we could have made the appropriate transformations on the integro-differential equation, but the calculations are messy and not informative so we leave them out.

Result 2. *The price $F = F(t,T,s) \in C_0^2[\mathbb{R} \times [0,T]]$ of a forward contract with (2.6) as the underlying spot-model, is given as the solution of the Cauchy problem*

$$\begin{cases} F_\tau &= \mathcal{A}^*F \quad \text{on } \mathbb{R} \times (0,T] \\ F(0,T,s) &= \exp(s) \quad \text{for } s \in \mathbb{R}. \end{cases} \tag{4.4}$$

We have the unique solution

$$F(t,T,s) = \mathbf{E}_{\mathbb{Q}}^{t,s}[S_T] \tag{4.5}$$

Here the integro-differential operator $\mathcal{A}^*$ is the one shortly described in the text before the result. It is defined on $[0,T]\times\mathbb{R}$, and τ is the time to maturity, that is $\tau = T - t$.

Proof. We look at (2.6). First, note that since all the conditions in Assumption 3 hold for constant and linear coefficients, we find by a straightforward application of Proposition 1.39 that the solution to the problem

$$\begin{cases} -u_t &= \mathcal{A}u \quad \text{on } \mathbb{R} \times [0,T) \\ u(x,T) &= g(x) \quad \text{for } x \in \mathbb{R} \end{cases} \tag{4.6}$$

is given by

$$u(x,t) = \mathbf{E}_{\mathbb{Q}}^{t,x}[g(\widetilde{S}_T)]$$

after making the choices $f = k = 0$ and $g(x) = \exp(x)$. This gives the price of the forward contract with initial values given by $\widetilde{S}$. The integro-differential operator $\mathcal{A}$ is defined by (4.3) on the domain $(x,t) \in \mathbb{R} \times [0,T)$.

Next, we apply the transformation $\tau = T - t$ to transform (4.6) into a forward equation. Since it is more convenient to have the solution in terms of S_t instead of $\widetilde{S}_t$, we change variables

$$\widetilde{S}_t = \log S_t = x \Leftrightarrow S(t) = e^x$$

we can rewrite (4.6) to a partial integro-differential equation giving the forward price by putting $F(t,T,y) = u(\log x, t)$:

$$F(t,T,y) = \mathbf{E}_{\mathbb{Q}}\left[g(S_T)|S_t = y\right],$$

and we see that u gives the price of the forward contract after the transformation. □

The equation we will be solving is (4.6), since it is easier to solve problems with constant coefficients numerically. We will then transform it back to our problem afterwords. It is therefore our object of study in the following.

Since ν is a compactly supported on $E = [0, \Delta_{max}]$, and a bounded measure, we can rewrite the integral term of $\mathcal{A}$ in the fashion outlined in Section 3.3.2, and find

$$\begin{aligned} &\int_E \Big(u(x + z\,h(x), t) - u(x,t) - zh(x)u_x(x,t)\Big)\mathrm{d}\nu(z) \\ &= \int_E u(x + z\,h(x), t)\mathrm{d}\nu(z) - \lambda u(x,t) - h(x)u_x(x,t)\int_{\mathbb{R}} x\mathrm{d}\nu(z). \end{aligned}$$

Also, we put $\widehat{\mu}_{\mp} := \widehat{\mu} \mp \int_{\mathbb{R}} z\mathrm{d}\nu(z)$ depending on $h(\widetilde{S}_{t-})$, and insert back into (4.3) to find that

$$\mathcal{A}u = \frac{\sigma^2}{2}u_{xx} + \Big(\widehat{\mu}_{\mp} - \alpha x\Big)u_x - \lambda u(x,t) + \lambda \int_E u(x \pm z, t)\mathrm{d}p(z).$$

Boundary conditions. Since we have a parabolic problem, we must apply boundary conditions on both boundaries. Keep in mind that the original problem is not stated on a bounded domain; this only occurs after we have truncated the domain for numerical solution. Hence the introduction of boundary conditions are motivated by the numerical implementation.

One possibility would be to assume that the asymptotics follow some real function f so that

$$u_x(x,\cdot) \approx f(x) \text{ as } x \to \pm\infty,$$

and possibly taking f as the payoff-function, so to "extend" our solution [BNR07]. Some take the function f equal to zero, thus specifying a Neumann type boundary condition.

Another approach would be to apply convexity-restrictions, and use the second-derivative of the forward price with respect to the underlying, $F_{SS} = 0$, which after the usual transformation yields

$$u_{xx} + u_x = 0 \text{ as } x \to \pm\infty.$$

In effect, we are assuming that the solution is linear for large values of S. This is the boundary condition we will use.

One last condition, on $x = 0$, is $F_\tau = 0$, representing an absorbing state. We get $u_\tau = 0$.

4.2 Numerical implementation

The forward price with respect to Schwartz model. Implementation of the analytical forward price with respect to the dynamics by Schwartz is relatively straightforward. The main issue is the integral

$$\int_{\mathbb{R}} \Psi(e^{-\alpha(T-s)})\mathrm{d}s,$$

which has to be solved numerically due to complexity. Actually, this is a frequently occurring problem in exp-Levy models. Several approaches can be applied to this end; we choose a relatively simple method found in [DPS00] known as the tanh-rule. The method is first order accurate $\mathcal{O}(h)$, and has the nice property that it is robust with respect to singularities in the integrand. The precise formula is given as

$$\int_a^b f(x)\mathrm{d}x \approx 2\tanh\Big[\frac{\Delta x}{2} f\Big(\frac{a+b}{2}\Big)\Big]. \tag{4.7}$$

4.2.1 Solution by finite differences

As we have noted earlier, it is beneficial to solve the transformed problem (4.6) instead of the problem (4.4) yielding the forward-price for reasons of stability.

Recall that we are working on the convection-diffusion problem including an integral-term

$$\begin{cases} u_\tau + \Big(\widehat{\mu}_\pm + \alpha x\Big)u_x = \frac{\sigma^2}{2}u_{xx} - \lambda u + \lambda \int_E u(x \pm z, t)\mathrm{d}p(z), \\ u(x,0) = \exp(x) \end{cases} \tag{4.6}$$

given on $\mathbb{R} \times [0, T)$. Even for small values of x the advection term dominates the diffusion term.

We recall that $\widehat{\mu}_\pm = \pm \int_{\mathbb{R}} z\mathrm{d}\nu(z) = \pm\lambda \mathbf{E}[z]$, where the expectation is with respect to the jump-probability density.

Differential operators. The differential operators are implemented in the same manner for both cases, i.e. small and big x-values, so we conclude the discussion on these matters at this point.

For the discrete differential operators, there are a variety of choices. Since both a first- and second derivative are present, we must use a second-order central scheme to approximate the first-order term in order to have $\mathcal{O}(\Delta x^2)$.

Implicit-explicit formulation. Convection-diffusion problems are often difficult to approximate using finite differences. So, to solve (4.10) we apply an implicit-explicit method in combination with time stepping.

Based on these remarks, and those earlier, we make the choices

- $G(u, Du, D^2u) = (\widehat{\mu}_\mp - \alpha x)u_x + \frac{\sigma^2}{2}u_{xx} - \lambda u$, and
- $H(Du, Iu) = \lambda \int_E u(x \pm z, t)\mathrm{d}p(z)$.

G will be solved implicitly, and H explicitly.

In addition using time splitting, we arrive at the implicit-explicit scheme

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = G_d v_j^{m+1} + H_d v_j^m.$$

Rearranging we find that we have to solve the problem

$$v_j^{m+1} = (I - \Delta t G_d)^{-1}(I + \Delta t H_d)v_j^m, \tag{4.8}$$

and apply Algorithm 3.

Let us proceed and discuss the cases for $0 \leq x \leq c$ and $c \leq x \leq A$ separately, i.e. the cases for small- and large initial $\widetilde{S}$ values.

Algorithm 3 Solution of (4.10) by finite differences

if $m = 0$ **then**
  for $j = 0, \ldots, N-1$ **do**
    $v_j^0 \leftarrow x_j$
  end for
  for $j = N, \ldots, N+K$ **do**
    $v_j^{m+1} \leftarrow exp(x_j)$
  end for
else if $m = 1, \ldots, M$ **then**
  for $j = 0, \ldots, N-1$ **do**
    $v_j^{m+1} \leftarrow (I + \Delta t I_d) v_j^m$
    $v_j^{m+1} \leftarrow (I - \Delta t D_d)^{-1} v_j^m$
  end for
  for $j = N, \ldots, N+K$ **do**
    $v_j^{m+1} \leftarrow exp(x_j)$
  end for
  Insert boundary conditions using Monte Carlo simulation
end if

4.2.2 Small x-values.

Starting with the case $0 \leq x \leq c$, we find that the we must solve the problem

$$\begin{cases} u_\tau + \Big(\widehat{\mu}_+ + \alpha x\Big) u_x = \frac{\sigma^2}{2} u_{xx} - \lambda u + \lambda \int_E u(x+z,t) \mathrm{d}p(z) \\ u(x,0) = \exp(x) \end{cases} \tag{4.9}$$

on the x-interval $[0, c]$, and, by the convolutional effect of the integral we need values for u on the strictly larger domain $[0, c + \Delta_{max}]$.

Next we partition the domain $[0, c]$ into N parts, and put $\Delta x = c/N$. We also choose $K \in \mathbb{N}$ such that $K = \lfloor \Delta_{max}/\Delta x \rfloor$, and hence have $(N+K)\Delta x \approx c + \Delta_{max}$.

Numerical evaluation of the integral. The integral deserves some special attention. As noted earlier, we make the choice $\Delta x = \Delta z$ so as to simply things somewhat. For any $j \in \{1, \ldots, N-1\}$ we have to find the integral

$$\begin{aligned} I_d v_j^m &= \int_0^{\Delta_{max}} v(x_j + z) \mathrm{d}p(z) \\ &= \int_{x_j}^{x_j + \Delta_{max}} v(y) p(y - x_j) \mathrm{d}y, \end{aligned}$$

where we have left out the time dependency. We may also do this in the following without any warning.

Since we have taken $K \in \mathbb{N}$ such that $K\Delta x \approx \Delta_{max}$, applying the Trapezoidal rule, we find the numerical approximation

$$I_d v_j^m = \sum_{i=0}^{K-1} \int_{x_j+y_i}^{x_j+y_{i+1}} v(y)p(x_j - y)\mathrm{d}y,$$

and since $\Delta y = \Delta x$ we get $x_j + y_i = x_{i+j}$ and thus, putting $p_i = p(x_i)$, we have

$$\begin{aligned} I_d v_j^m &= \frac{1}{2}\Delta x \sum_{i=0}^{K-1} \Big(v_{i+j}^m p_i + v_{i+j+1}^m p_{i+1} \Big) \\ &= \frac{1}{2}\Delta x \Big(v_j^m p_0 + v_{j+K}^m p_K + 2\sum_{i=1}^{K-1} v_{i+j}^m p_i \Big), \text{ for } j = 1, \ldots, N-1. \end{aligned}$$

Writing out the sum and collecting terms, we can identify the linear system

$$I_d v_j^m = P v_j^m,$$

for all $j \in 1, \ldots, N-1$ where

$$P = \begin{pmatrix} p_0 & 2p_1 & \cdots & 2p_{K-1} & p_K & 0 & \ldots & 0 \\ & \ddots & & & & \ddots & & \\ & & & p_0 & 2p_1 & \cdots & 2p_{K-1} & p_K \end{pmatrix} \in \mathbb{R}^{N-1, N+K-2},$$

acting on all the points of the domain $[0, c + \Delta_{mac}]$ yielding results in $[0, c]$.

Boundary conditions. To approximate the boundary-conditions we use

$$u_\tau(0) = \frac{v_0^{m+1} - v_0^m}{\Delta t} \Leftrightarrow v_0^{m+1} = v_0^m,$$

and the approximation

$$u_{xx}(c, \tau) + u_x(c, \tau) = 0 \Leftrightarrow v_N^m = \frac{1}{2}(1 + \frac{\Delta x}{2})v_{N-1}^m + (1 - \frac{\Delta x}{2})v_{N+1}^m,$$

which is $\mathcal{O}(\Delta x^2)$ since we have used the second order central difference scheme to approximate the first derivative.

The unknown values, for instance v_{N+1}^m, will be found using a Euler discretization and Monte Carlo simulation of the stochastic differential equation (2.6). This is accomplished using the methods described in Section 3.4.

4.2.3 Large x-values

For large values the discussion is mostly as in the previous Section. Some exceptions, however are present.

This time the problem we seek to solve is

$$\begin{cases} u_\tau + \Big(\widehat{\mu}_- + \alpha x\Big)u_x = \frac{\sigma^2}{2}u_{xx} - \lambda u + \lambda \int_E u(x-z,t)\mathrm{d}p(z) \\ u(x,0) = \exp(x) \end{cases} \tag{4.10}$$

We take the interval $[c, A]$ and partition it in N intervals. As before, we choose $K = \lfloor \Delta_{\max}/\Delta \rfloor$. We notice the slight difference in the coefficient for the convection term; for large values of x it is to be regarded as unimportant.

Numerical evaluation of the integral. In precise manner as before we find

$$\begin{aligned} I_d v_j^m &= \int_0^{\Delta_{max}} v(x_j - z)\mathrm{d}p(z) \\ &= \int_{x_j - \Delta_{max}}^{x_j} v(y)p(x_j - y)p(y)\mathrm{d}y, \end{aligned}$$

by changing variables. This we evaluate using the Trapezoidal rule and find

$$I_d v_j^m = \sum_{i=0}^{K-1} \int_{y_i}^{y_{i+1}} v(y)p(x_j - y)\mathrm{d}y,$$

where $y_i = x_j - \Delta_{max} + i\Delta x$. Calculating the inner integral yields

$$\begin{aligned} I_d v_j^m &= \frac{1}{2}\Delta x \sum_{i=0}^{K-1} \Big(v_{j+i-K}p_{K-i} + v_{j+i-K+1}p_{K-i+1}\Big) \\ &= \frac{1}{2}\Delta x\Big(v_{j-K}p_K + v_j p_0 + 2\sum_{i=1}^{K-1} v_{j+i-K}p_{K-i}\Big), \text{ for } j = 1, \ldots, N-1. \end{aligned}$$

For instance, taking $j = 1$, we find

$$v_{1-K}p_K + 2v_{2-K}p_{K-1} + \cdots + 2v_0 p_1 + v_1 p_0,$$

and we see that the order of the p's are reversed in comparison to the previous case.

The linear system can be derived in the same manner as in the previous section.

Boundary conditions. Fort this case, we apply the boundary condition

$$u_{xx}(c,\tau) + u_x(c,\tau) = u_{xx}(A,\tau) + u_x(A,\tau) = 0,$$

and as before we arrive at

$$v_0^m = \frac{1}{2}(1 + \frac{\Delta x}{2})v_{-1}^m + (1 - \frac{\Delta x}{2})v_1^m,$$
$$v_N^m = \frac{1}{2}(1 + \frac{\Delta x}{2})v_{N-1}^m + (1 - \frac{\Delta x}{2})v_{N+1}^m.$$

4.3 Numerical results

Using the values listed in Table 2.2 find the forward prices with respect to Schwartz' model, shown in Figure 4.1 for the initial spot-value of \$ 0.5.

Plots of the forward curves with respect to the model proposed by Geman and Roncoroni are shown in Figure 4.2, and Figure 4.3 respectively, for the COB model.

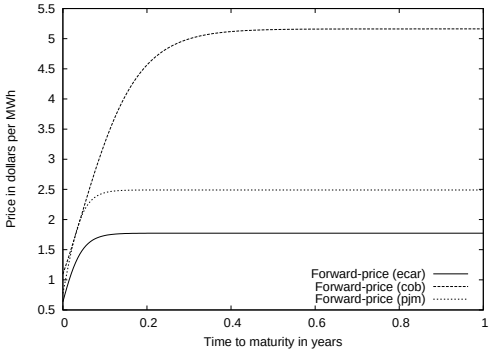

Figure 4.1: Forward price in $/MWh versus the time to maturity in years.

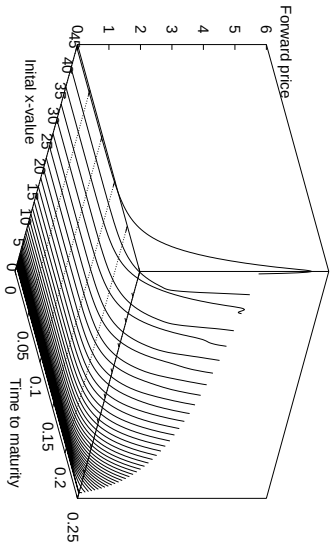

Figure 4.2: Forward price in $/MWh versus time in years and the value of S small.

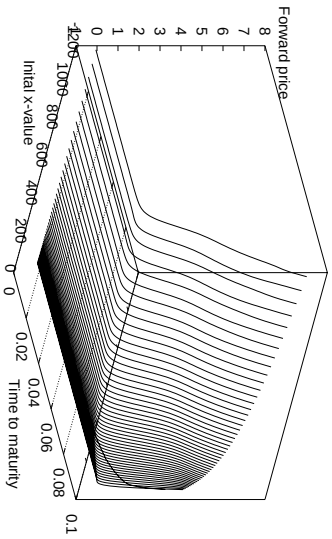

Figure 4.3: Forward price in $/MWh versus time in years and the value of S large.

Chapter 5

Discussion

In this Chapter we will discuss the results, and point to possible errors and improvements.

5.1 Results

The forward-price with respect to Schwartz' model. Looking to Figure 4.1 we see that as the time to maturity τ increases the forward curve flattens out. This is in line with what one should expect, since there is no seasonality present. The fact that $F(T,T) = S$ validates for the model, is another indication that the calculated results are valid.

The forward-price with respect to Geman and Roncoroni's model. Figures 4.2 and 4.3 illustrate the forward curves for small- and large initial values respectively. We would expect the same kind of behaviour as observed in the results from the Schwartz case.

Given that we have an convection-diffusion including an integral term, we should also be able to observe some convection. For small values of x, i.e. for small spot-values the convection term is not very dominating, so for this case diffusion should be the major force. Looking to Figure 4.2 this is indeed the case; we have strong diffusion and the value decreases rapidly to just above zero.

For the case of larger spot-price, the convection should be more prominent. This is not the case. The suspicion is that this part of the implementation is not correct; be it due to boundary conditions or some error in the dicretization and the question has not been resolved for this case. We therefore focus on the case for small spot-values in the following.

Comparison of the forward prices. A comparison of the forward-prices calculated by the two model studied are shown in Figure 5.1, for a initial value of $\widetilde{S} = 1$. From the figure we see that there is a large discrepancy

between the two prices. As the time-to-maturity increases, the price become

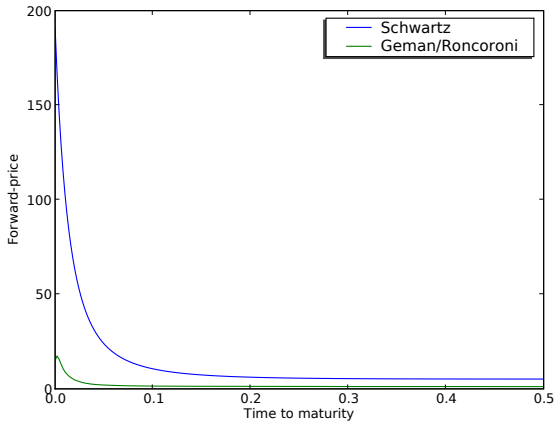

Figure 5.1: Forward prices in $\$/MWh$ versus time for the COB model and for the Schwartz and Geman/Roncoroni models respectively.

more similar, but still there is a significant difference.

The large discrepancy as $\tau \to 0$ is surely due to the singularity of the integral occurring in the Schwartz model. This singularity, as we briefly have mentioned, cases problems for implementation and make the forward-price inaccurate at $\tau = 0$.

5.2 Numerical implementation

The implementation of the finite differences for the differential operators are straightforward. Some care must be taken so to avoid oscillations, a frequent problem in when approximating convection-diffusion equations with finite differences, but this is resolved by taking small enough increments.

The integral operator. We recall that Merton chose log-normal distributed jumps in his original proposal (2.7), and was able to derive a closed-form expression for the option price. This is due to his choice of jump-size distribution. This is to be kept in mind when choosing jump-size distribution.

Alternatively, one could approximate the probability-distribution p defined on a compact interval, with another distribution with matching moments up to a given order on the whole of $\mathbb{R}$. From a numerical point of view this is beneficial: when truncating the domain on which the integro-differential is defined, the integral then is to be taken over this same interval. Thus a dense $n \times n$-matrix, instead of a $n \times k$-matrix, for $k \neq n$, arise.

Of course other numerical methods to evaluate the integral could be applied. This may lead to simpler numerical algorithms, or, more likely, to higher complexity of the implementation. Also, accuracy could be increased.

Monte Carlo simulations. The first thing to take notice of, is that we simulate the paths on intervals, say $[0, 0.5]$ and use the same grid-size as the partition in the finite difference implementation. Obviously this is for simplicity. However, to have convergence we must perform a great number iterations since, typically, we have $N = 100$ intervals and so we would need to repeat this process perhaps one million times ideally.

The importance of convergence is another issue. The calibration of the boundary inevitably leads to the introduction of error.

5.3 Unresolved issues

Large spot-values: As we have already mentioned, the issue for large spot-values have not been completely resolved, and so further work on this case are advised

Boundary conditions: Some investigations on the importance of convergence in the Monte Carlo simulations

Generalize: The next natural step would be to introduce seasonality, and inhomogeneous Poisson processes

Appendix A

Appendix

A.1 Notation

We use the notation $\partial_x u = u_x$ for the first partial derivative in x, and analogous for higher order derivatives and time-derivatives.

The expectation operator is defined as

$$\mathbf{E}[X] = \int_\Omega X \mathrm{d}P,$$

and occasionally we will write

$$\mathbf{E}^{t,x}[X] = \mathbf{E}[X|X_t = x].$$

We define the floor operator $\lfloor \cdot \rfloor$ as the integer part of any real number.

A.2 Derivation of Schwartz model

Let us assume that the logarithm of the spot price can be written as a sum of some deterministic function f of time, which represent the seasonal variation, and a stochastic process X. Regarding the choice of X we want to allow for the event of jumps, so we shall take X to be a Levy process. Also, from the discussion in Section 2.3, we know that there is a strong mean reversion present in the electricity market, and thus X should be a Ornstein-Uhlenbeck Levy process:

$$\mathrm{d}X_t = -\alpha(t)X_t\mathrm{d}t + \sigma_t \mathrm{d}B_t + \mathrm{d}N_t.$$

α and σ are continuous functions of time, and represent the speed of mean reversion and volatility respectively. B is a Brownian motion, and N is a compound Poisson process. Recall from Levy-Khintchine that we can represent N as the integral $N = \int_{\mathbb{R}} zN(\mathrm{d}t, \mathrm{d}z)$, where, with some horrific

abuse of notation, N is the random Poisson measure associated with the process N.

Since the log-price of the spot is given by the relation

$$\log S_t = f(t) + X_t, \tag{$*$}$$

we have that

$$S_t = \exp\left(f(t) + X_t\right),$$

and an application of Itô on this relation yields

$$\begin{aligned}
\mathrm{d}S_t =& \dot{f}(t)S_t\mathrm{d}t + \Big(- \alpha(t)X_t\mathrm{d}t + \sigma(t)\mathrm{d}B_t\Big)S_t \\
&+ \frac{1}{2}\sigma^2(t)S_t\mathrm{d}t + \int_{\mathbb{R}} \Big(S_{t-}e^z - S_{t-}\Big)N(\mathrm{d}t, \mathrm{d}z) \\
=& \left(\dot{f}(t) + \frac{1}{2}\sigma^2(t) - \alpha(t)X(t)\right) S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}B_t \\
&+ \int_{\mathbb{R}} \left(e^z - 1\right) S_{t-}N(\mathrm{d}t, \mathrm{d}z),
\end{aligned}$$

where $\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$ denotes derivation with respect to time. Inserting for $X_t = \log S_t - f(t)$ from $(*)$ we find

$$\begin{aligned}
\mathrm{d}S_t =& \left(\dot{f}(t) + \frac{1}{2}\sigma^2(t) - \alpha(t)(\log S_t - f(t))\right) S_t\mathrm{d}t \\
&+ \sigma(t)S_t\mathrm{d}B_t + \int_{\mathbb{R}} S_{t-}\left(e^z - 1\right) N(\mathrm{d}t, \mathrm{d}z) \\
=& \left(\mu(t) + \frac{1}{2}\sigma^2(t) - \alpha(t)\log S_t\right) S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}B_t \\
&+ \int_{\mathbb{R}} S_{t-}\left(e^z - 1\right) N(\mathrm{d}t, \mathrm{d}z),
\end{aligned}$$

where we have written $\mu(t) = \dot{f}(t) - \alpha(t)f(t)$. A second application of Itô, but now on $\widetilde{S} = \log S_t$, gives

$$\begin{aligned}
\mathrm{d}(\log S_t) =& \frac{1}{S_t}\left((\mu(t) + \frac{1}{2} - \alpha(t)\log S_t)S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}B_t\right) - \frac{1}{2}\sigma^2(t)S_t^2\frac{1}{S_t^2}\mathrm{d}t \\
&+ \int_{\mathbb{R}} \log\left(S_{t-} + S_{t-}(e^z - 1)\right) - \log(S_{t-})N(\mathrm{d}t, \mathrm{d}z) \\
=& \Big(\mu(t) - \alpha(t)\log S_t\Big)\mathrm{d}t + \sigma(t)\mathrm{d}B_t + \int_{\mathbb{R}} \log\left(\frac{e^z S_{t-}}{S_{t-}}\right) N(\mathrm{d}t, \mathrm{d}z) \\
=& \Big(\mu(t) - \alpha(t)\log S_t\Big)\mathrm{d}t + \sigma(t)\mathrm{d}B_t + \int_{\mathbb{R}} zN(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

Putting $\widetilde{S} = \log S$, we have arrived at the dynamics

$$\mathrm{d}\widetilde{S}_t = \left(\mu(t) - \alpha(t)\widetilde{S}_t\right)\mathrm{d}t + \sigma(t)\mathrm{d}B_t + \mathrm{d}N_t. \tag{A.1}$$

A.3 A messy integral

In the derivation of the forward-price with respect to Lucia and Schwartz' model, we run into

$$\mathbf{E}_{\mathbb{Q}}[\exp\Big(\int_t^T e^{-\alpha(T-s)}\mathrm{d}N_s\Big)] = \exp\Big(\int_t^T \psi(e^{-\alpha(T-s)})\mathrm{d}s\Big).$$

We complete the calculations now, but first some justification is appropriate. From Lemma 15.1 in [CT04] we have:

Lemma A.1. *Let $f : [0,T] \to \mathbb{R}$ be left continuous, and Z a Levy-process. Then*

$$\mathbf{E}\left[\exp\Big(i\int_0^T f(s)\mathrm{d}Z_s\Big)\right] = \exp\Big(\int_0^T \psi(f(s))\mathrm{d}s\Big) \tag{A.2}$$

where ψ is the characteristic exponent of Z.

Proof. Consider first the case for f a simple predictable integrand:

$$\int f(s)\mathrm{d}Z_s = \sum_{j=1}^N f_i(Z_j - Z_{j-1}).$$

Then we have

$$\mathbf{E}\left[\exp\Big(i\int f(s)\mathrm{d}Z_s\Big)\right] = \prod_{j=1}^N \mathbf{E}\left[\exp\Big(if_j(Z_j - Z_{j-1})\Big)\right]$$
$$= \prod_{j=1}^N \mathbf{E}\left[\exp\Big(i(s_j - s_{j-1})\psi(f_j)\Big)\right] = \exp\Big(\int \psi(f(t))\mathrm{d}s\Big).$$

The result follows now since any left-continuous integrand can be approximated by simple integrands. □

Recalling the given jump size distribution from Section 2.3.3, we have the Levy-measure

$$\nu(\mathrm{d}z) = \lambda\frac{\gamma\exp(-\gamma z)}{1-\exp(-\gamma\Delta_{max})}\mathrm{d}z, \quad 0 \le z \le \Delta_{max},$$

and we also have the characteristic function for a compound Poisson process

$$\psi(u) = \int_{\mathbb{R}}(e^{iuz} - 1)\nu(\mathrm{d}z).$$

Now we have that $f(s) = -i\exp(-\alpha(T-s))$ in Lemma A.1, and hence by substituting this into (A.2) we must find

$$\frac{\lambda\gamma}{1-\exp(-\gamma\Delta_{max})}\int_t^T\int_0^{\Delta_{max}}\Big(\exp(e^{-\alpha(T-s)}z) - 1\Big)\exp(-\gamma z)\mathrm{d}z\mathrm{d}s$$
$$= \frac{\lambda\gamma}{1-\exp(-\gamma\Delta_{max})}\int_t^T\int_0^{\Delta_{max}}\exp(e^{-\alpha(T-s)} - \gamma)z - \exp(-\gamma z)\mathrm{d}z\mathrm{d}s.$$

Straightforward calculations yields:

$$\int_t^T \frac{1}{e^{-\alpha(T-s)} - \gamma} \Big[\exp\Big((e^{-\alpha(T-s)} - \gamma)\Delta_{max} \Big) - 1 \Big] + \frac{1}{\gamma} \Big(e^{-\gamma \Delta_{max}} - 1 \Big) \mathrm{d}s.$$

Now the first part of the integral has a singularity when s is such that $\exp(-\alpha(T-s)) - \gamma = 0$. This causes problems for integration, so we must use some sort of numerical integration to evaluate the integral, and we could choose from a range of methods that are able to cope with such singularities.

Appendix B

Fortran and Python code

Calculation of the Schwartz forward price

```
!
! Calculation of the forward-price
! Author: Bjørn Waage Skogtrø
!
module ecar
  ! parameters for the ECAR model
  implicit none
  save
  real :: alpha=38.8938
  real :: gamma=0.3129
  real :: delta_max=3.3835
  !real :: mu=3.0923
  real :: mu=12.58806
  real :: sigma=1.8355
  real :: lambda=9.0
end module ecar

module pjm
  ! parameters for the PJM model
 implicit none
 save
 real :: alpha=42.8844
 real :: gamma=0.5016
 real :: delta_max=1.6864
 !real :: mu=3.2002
 real :: mu=7.01529
 real :: sigma=1.3631
 real :: lambda=9.6667
end module pjm

module cob
  ! parameters for the COB model
  implicit none
  save
  real :: alpha=13.3815
  real :: gamma=1.0038
  real :: delta_max=1.0169
  !real :: mu=2.8928
  real :: mu=0.84683
  real :: sigma=1.3631
  real :: lambda=2.0
end module cob
```

```
module forw_ecar
  use ecar
  implicit none
  ! Other variables
  integer :: n=10E3

  ! Functions
contains

  ! The integrand from the jump-term
  real function f(x)
    implicit none
    real, intent(in) :: x
    f=(exp((exp(-alpha*x)-gamma)*delta_max)-1)/(exp(-alpha*x)-gamma)
  end function f

  ! Numerical integration
  subroutine tanh_rule(x0,x1,n,res)
    implicit none
    !
    real, intent(in) :: x0,x1
    integer, intent(in) :: n
    real, intent(out) :: res
    !
    real, dimension(n) :: x
    real :: dx=0.0, mid=0.0
    integer :: i=0

    dx=(x1-x0)/(n-1)
    do i=0,n-1
       x(i+1)=i*dx
    end do
    do i=1,n-1
       mid=(x(i+1)+x(i))/2.0
       res=res+2*tanh(0.5*dx*f(mid))
    end do
  end subroutine tanh_rule

  ! The part of the forward associated with jumps
  real function jmppris(t0,t1)
    implicit none
    real :: integral
    real, intent(in) :: t0,t1
```

```
    integer :: n=1E3
    call tanh_rule(t0,t1,n,integral)
    jmppris=gamma*(integral + (t1-t0)*(exp(-gamma*delta_max)-1)/gamma)&
         /(1-exp(-gamma*delta_max))
  end function jmppris

  ! The part of the forward associated with BM
  real function bmpris(t0,t1)
    implicit none
    real, intent(in) :: t0,t1
    bmpris=(sigma**2)*(1-exp(-2*alpha*(t1-t0)))/(4*alpha)
  end function bmpris

  ! The part of the forward associated with the drift
  real function driftpris(t0,t1)
    implicit none
    real, intent(in) :: t0,t1
    driftpris=mu*(1-exp(-alpha*(t1-t0)))/alpha
  end function driftpris

  ! The forward itself
  subroutine forward(pris,x,t0,t1)
    implicit none
    real, intent(out) :: pris
    real, intent(in) :: x,t0,t1
    real :: res
    res=x**exp(-alpha*(t1-t0))*exp(jmppris(t0,t1) + bmpris(t0,t1))
    pris=res
  end subroutine forward
end module forw_ecar

module forw_cob
  use cob
  implicit none
  ! Other variables
  integer :: n=10E3

  ! Functions
contains

  ! The integrand from the jump-term
  real function f(x)
    implicit none
    real, intent(in) :: x
```

```
  f=(exp((exp(-alpha*x)-gamma)*delta_max)-1)/(exp(-alpha*x)-gamma)
end function f

! Numerical integration
subroutine tanh_rule(x0,x1,n,res)
  implicit none
  !
  real, intent(in) :: x0,x1
  integer, intent(in) :: n
  real, intent(out) :: res
  !
  real, dimension(n) :: x
  real :: dx=0.0, mid=0.0
  integer :: i=0

  dx=(x1-x0)/(n-1)
  do i=0,n-1
     x(i+1)=i*dx
  end do
  do i=1,n-1
     mid=(x(i+1)+x(i))/2.0
     res=res+2*tanh(0.5*dx*f(mid))
  end do
end subroutine tanh_rule

! The part of the forward associated with the jumps
real function jmppris(t0,t1)
  implicit none
  real :: integral
  real, intent(in) :: t0,t1
  integer :: n=1E3
  call tanh_rule(t0,t1,n,integral)
  jmppris=gamma*(integral + (t1-t0)*(exp(-gamma*delta_max)-1)/gamma)&
       /(1-exp(-gamma*delta_max))
end function jmppris

! The part of the forward associated with the jumps
real function bmpris(t0,t1)
  implicit none
  real, intent(in) :: t0,t1
  bmpris=(sigma**2)*(1-exp(-2*alpha*(t1-t0)))/(4*alpha)
end function bmpris

! The part of the forward associated with the jumps
```

```
  real function driftpris(t0,t1)
    implicit none
    real, intent(in) :: t0,t1
    driftpris=mu*(1-exp(-alpha*(t1-t0)))/alpha
  end function driftpris

  ! The forward itself
  subroutine forward(pris,x,t0,t1)
    implicit none
    real, intent(out) :: pris
    real, intent(in) :: x,t0,t1
    real :: res
    res=x**exp(-alpha*(t1-t0))*exp(jmppris(t0,t1) &
         + bmpris(t0,t1))! + driftpris(t0,t1))
    pris=res
  end subroutine forward
end module forw_cob

module forw_pjm
  use pjm
  implicit none
  ! Other variables
  integer :: n=10E3

  ! Functions
contains
  ! The integrand from the jump-term
  real function f(x)
    implicit none
    real, intent(in) :: x
    f=(exp((exp(-alpha*x)-gamma)*delta_max)-1)/(exp(-alpha*x)-gamma)
  end function f

  ! Numerical integration
  subroutine tanh_rule(x0,x1,n,res)
    implicit none
    !
    real, intent(in) :: x0,x1
    integer, intent(in) :: n
    real, intent(out) :: res
    !
    real, dimension(n) :: x
    real :: dx=0.0, mid=0.0
    integer :: i=0
```

```
  dx=(x1-x0)/(n-1)
  do i=0,n-1
     x(i+1)=i*dx
  end do
  do i=1,n-1
     mid=(x(i+1)+x(i))/2.0
     res=res+2*tanh(0.5*dx*f(mid))
  end do
end subroutine tanh_rule

! The part of the forward associated with the jumps
real function jmppris(t0,t1)
  implicit none
  real :: integral
  real, intent(in) :: t0,t1
  integer :: n=1E3
  call tanh_rule(t0,t1,n,integral)
  jmppris=gamma*(integral + (t1-t0)*(exp(-gamma*delta_max)-1)/gamma)&
       /(1-exp(-gamma*delta_max))
end function jmppris

! The part of the forward associated with the jumps
real function bmpris(t0,t1)
  implicit none
  real, intent(in) :: t0,t1
  bmpris=(sigma**2)*(1-exp(-2*alpha*(t1-t0)))/(4*alpha)
end function bmpris

! The part of the forward associated with the jumps
real function driftpris(t0,t1)
  implicit none
  real, intent(in) :: t0,t1
  driftpris=mu*(1-exp(-alpha*(t1-t0)))/alpha
end function driftpris

! The forward itself
subroutine forward(pris,x,t0,t1)
  implicit none
  real, intent(out) :: pris
  real, intent(in) :: x,t0,t1
  real :: res
  res=x**exp(-alpha*(t1-t0))*exp(jmppris(t0,t1) &
       + bmpris(t0,t1))! + driftpris(t0,t1))
```

```
    pris=res
  end subroutine forward
end module forw_pjm
```

Solving the PIDE for small values

```
#!/usr/bin/python
"""
Solving the pide
"""
## Modules
from numpy import *
import scipy.linalg

from pylab import plot,show

## Self-implemented modules
import viz
from cob import *
from funcs import *

# Want 2 significant digits, and tiny numbers supressed
set_printoptions(precision=3, suppress=True, linewidth=220)

## Partition-size
N=40
NX=N-2
NT=250

## Spacedomain
x_start=0.0; x_stop=2.71

## Increments
dt=(t_stop-t_start)/float(NT)
dx=(x_stop-x_start)/float(N)
print "dx=",dx,"\tdt=",dt

## Arrays
t=linspace(t_start,t_stop,NT)
x_inner=linspace(x_start+dx,x_stop-dx,NX+1)
x=hstack((x_start,x_inner,x_stop))

# The overlap caused by the integro-operator
```

```
K=int(floor(delta_max/dx))
x_add=x_stop+dx*array(range(1,K+1))
x_full=hstack((x,x_add))
print "N=",N,"\t\tK=",K

# Coefficients
a=-lam*mu - alpha*x[1:N-1]
b=sigma**2/float(2)

## Initial condition
u=zeros((NT,N+K+1))
u[0,:]=x_full

## generate the bondries:
u1=sde(NT,x_full[N+1])
u[0,N]=0.5*(1-0.5*dx)*u[0,N-1]+0.5*(1+0.5*dx)*u1[0]

for i in range(1,NT):
    ## Solving
    u[i,1:N-1]=u[i,1:N-1]+dt*J_small(dx,u[i-1,1:N+K-1],N,NX,K)
    M=eye(NX)-dt*H(a,dx,N,K,NX)-dt*G(b,dx,NX)
    u[i,2:N]=linalg.solve(M,u[i-1,2:N])
    ## Outside the domain
    #u[i,N:]=forward(log(x_full[N:]),i*dt)
    u[i,N:]=x_full[N:]
    ## Bondary conditions
    u[i,0]=u[i-1,0]
    u[i,N]=0.5*(1-0.5*dx)*u[i,N-1]+0.5*(1+0.5*dx)*u1[i]
```

Solving the PIDE for large values

```
#!/usr/bin/python
"""
Solving the pide
"""
## Modules
from numpy import *
import scipy.linalg

## Self-implemented modules
import viz
from cob import *
from funcs import *
```

```
from pylab import plot,show,figure,subplot

# Want 2 significant digits, and tiny numbers supressed
set_printoptions(precision=3, suppress=True, linewidth=200)

## Partition-size
N=60
NX=N-2
NT=250

## Timedomain
t_start=0.0; t_stop=0.1
## Spacedomain
x_start=c; x_stop=2*c

## Increments
dt=(t_stop-t_start)/float(NT)
dx=(x_stop-x_start)/float(N)
print "dx=",dx,"\tdt=",dt

## Arrays
t=linspace(t_start,t_stop,NT)
x_inner=linspace(x_start+dx,x_stop-dx,NX+1)
x=hstack((x_start,x_inner,x_stop))

# The overlap caused by the integro-operator
K=int(floor(delta_max/dx))
x_add=x_start-K*dx+dx*array(range(K))
x_full=hstack((x_add,x))
print "N=",N,"\t\tK=",K

# Coefficients
a=lam*mu - alpha*x[1:N]
b=sigma**2/float(2)

## Initial condition
u=zeros((NT,N+K+1))
u1=sde(NT,x_full[K-1])
u2=sde(NT,(N+K+1)*dx)
u[0,:]=x_full

for i in range(1,NT):
    ## Solving
```

```
    u[i,K+1:-1]=u[i,K+1:-1]+dt*J_big(dx,u[i-1,1:-1],N,NX,K)
    M=eye(NX+1)-dt*H_big(a,dx,N,K,NX)-dt*G_big(b,dx,NX)
    u[i,K+1:-1]=linalg.solve(M,u[i-1,K+1:-1])

    ## Outside the domain
    u[i,:K]=x_full[:K]
    ## Bondary conditions
    u[i,K]=0.5*(1-0.5*dx)*u1[i]+0.5*(1+0.5*dx)*u[i,K+1]
    u[i,N+K]=0.5*(1-0.5*dx)*u[i,N+K-1]+0.5*(1+0.5*dx)*u2[i]
```

Functions used in calculation

```
#!/usr/bin/python
from numpy import *
import scipy.linalg
import numpy.random as ran

from cob import *
import forw

## The Schwartz forward price (Fortran module)
def forward(x,t):
    return forw.forw_cob.forward(x,0.0,t)

## The probability-distribution
def p(x):
    if x<0 or x>delta_max:
        return 0
    else:
        return gamma*exp(-gamma*x)/(1-exp(-gamma*delta_max))

## Second order
def G(b,dx,NX):
    # Diagonals
    ld=ones(NX-1)
    d=-2*ones(NX)
    ud=ones(NX-1)

    # Second-order differential term:
    D=b*(diag(d,k=0)+diag(ld,k=-1)+diag(ud,k=1))/dx**2 - lam*eye(NX)
    # assembling the matrix
    return D

def G_big(b,dx,NX):
```

```
    # Diagonals
    ld=ones(NX)
    d=-2*ones(NX+1)
    ud=ones(NX)
    # Second-order differential term:
    D=b*(diag(d,k=0)+diag(ld,k=-1)+diag(ud,k=1))/dx**2 - lam*eye(NX+1)
    # assembling the matrix
    return D

##  First order
def H(a,dx,N,K,NX):
    # Diagonals
    ld=ones(NX-1)
    d=ones(NX)
    ud=ones(NX-1)
    # First-order differential term
    D=a*(diag(ud,k=1)-diag(ld,k=-1))/dx
    return D

## First order
def H_big(a,dx,N,K,NX):
    # Diagonals
    ld=ones(NX)
    d=ones(NX+1)
    ud=ones(NX)
    # First-order differential term
    D=a*(diag(ud,k=1)-diag(ld,k=-1))/dx
    return D


def J_big(dx,u,N,NX,K):
    # Integral-operator
    res=[]
    for i in range(K+1):
        if i==0 or i==K:
            res.append(p(dx*i))
        else:
            res.append(2*p(dx*i))
    res.reverse()
    prob=hstack((res,zeros(NX)))
    I=zeros((NX+1,N+K-1))
    for i in range(0,NX+1):
        I[i,:]=roll(prob,i)
    # return the product
```

```
    #print I.shape,u.shape
    return 0.5*dx*lam*dot(I,u)


def J_small(dx,u,N,NX,K):
    # Integral-operator
    res=[]
    for i in range(K+1):
        if i==0 or i==K:
            res.append(p(dx*i))
        else:
            res.append(2*p(dx*i))

    prob=hstack((res,zeros(NX-1)))
    I=zeros((NX,N+K-2))
    for i in range(0,NX):
        I[i,:]=roll(prob,i)
    # return the product
    return 0.5*dx*lam*dot(I,u)


def sde(NT,x0):
    # number of bathces
    n=NT
    m=100

    t=linspace(t_start,t_stop,n)
    dt=(t_stop-t_start)/float(n+1)

    ## generate p-distributed numbes
    jmp=zeros(n)

    y=zeros((m,n),float)
    gjsnitt=zeros(n,float)
    for i in range(m):
         y[:,0]=x0
         jmp=ran.poisson(lam*dt,n)
         dx=ran.normal(0,1,n)*sqrt(dt)
         for j in range(0,n-1):
             y[i,j+1]=y[i,j]*exp((mu_mc - alpha*y[i,j])*dt + sigma*dx[j]\
                                 +jmp[j]*jmpprob(1))

    for i in range(n):
        _gjn=sum(y[:,i])/float(m)
```

```
        gjsnitt[i]=_gjn
    return gjsnitt

def jmpprob(n):
    r=[]
    for i in range(n):
        r.append(-log(1-ran.uniform(0,1)*(1-exp(-gamma*delta_max)))/gamma)
    return array(r)
```

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