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On Fourier Series in Convex Domains

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Abstract

We consider systems of complex exponential functions in spaces of square integrable functions. Some classical one-dimensional theory is reviewed, in particular, we emphasize the duality between the Riesz bases of complex exponential functions in L^2-spaces and complete interpolating sequences in PW^2-spaces of entire functions of exponential type. Basis properties for L^2-spaces over planar convex domains are then studied in detail. The convex domain in question is shown to be crucial for what basis properties the corresponding L^2-space possesses. We explain some results related to Fuglede's conjecture about existence of orthonormal bases and then a result by Lyubarskii and Rashkovskii regarding Riesz bases for L^2-spaces over convex polygons, symmetric with respect to the origin. Finally, we make a modest attempt to apply the techniques by Lyubarskii and Rashkovskii combined with approximation of plurisubharmonic functions using logarithms of moduli of entire functions, to construct a complete system of exponential functions in the space of square integrable functions over a disk. This work is not completed yet.

Preface

This text is submitted as the concluding part of my Master of Science degree in mathematics. The work has been done during the academic year 2006/2007 under the supervision of Professor Yurii Lyubarskii at the Department of Mathematical Sciences, NTNU.

The purpose of this work has been to study systems of complex exponential systems in spaces of square integrable functions over planar domains. The subject has been very interesting as it combines two of my favourite fields, complex analysis and functional analysis, in a very elegant way.

Acknowledgements

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Chapter 1

Introduction

We start with an informal introduction to some of the problems we will work on in this text. Consider the space of square integrable functions on the interval $[-\pi, \pi]$, that is $L^2(-\pi, \pi)$. It is an elementary fact from Fourier analysis that the set $\mathcal{E}(\mathbf{Z}) = \{e^{in\cdot} : n \in \mathbf{Z}\}$ constitutes an orthogonal basis for $L^2(-\pi, \pi)$. With a suitable normalization factor it is an orthonormal basis for the same space. There is nothing special about the interval $[-\pi, \pi]$. Any interval has the same basis property if we change the exponent set in the system of exponential functions accordingly.

Let us change the exponent set, that is, replace the integers $\mathbf{Z}$ with some other discrete set Λ. We can then study how properties of the system of exponential functions $\mathcal{E}(\Lambda) = \{e^{i\lambda\cdot} : \lambda \in \Lambda\}$ in $L^2(-\pi, \pi)$ changes as we change Λ. For instance, we can try to find out what kind of properties Λ must possess if we want $\mathcal{E}(\Lambda)$ to be an orthonormal basis for $L^2(-\pi, \pi)$. Alternatively, we can lower the requirements on $\mathcal{E}(\Lambda)$ and restrict ourselves to Riesz bases, frames or complete systems. A natural question is then: What kind of properties of Λ determines how "good" the system $\mathcal{E}(\Lambda)$ is in $L^2(-\pi, \pi)$? This question is intimately connected to sampling and interpolation properties of Λ in the space of entire functions of exponential type π and in this way methods from function theory come into use.

Another direction to explore, is similar problems in several dimensions. Let Ω be some bounded domain in $\mathbf{R}^n$ and consider the corresponding Lebesgue space, $L^2(\Omega)$. It is natural to ask for which sets Ω the space $L^2(\Omega)$ has an orthonormal basis of exponential functions. This is closely related to a conjecture by Bent Fuglede [Fug74], saying that $L^2(\Omega)$ has an orthonormal basis of exponential functions if and only if it is possible to tile $\mathbf{R}^n$ with non-overlapping translates of Ω. This has been proved wrong in dimensions 3 and higher, but it is true for convex sets in the plane. This means that the only convex sets Ω in the plane for which $L^2(\Omega)$ has an orthonormal basis of exponential functions are rectangles and hexagons. One might then wonder what kind of properties systems of exponential functions have $L^2(\Omega)$ when Ω is a convex set, which does not tile the plane by translations. Lyubarskii and Rashkovskii answered this question in [LR00] for convex polygons M symmetric with respect to the origin. $L^2(M)$ has Riesz bases of complex exponential functions. For another natural domain, namely the disk $D \subset \mathbf{R}^2$, not much is known. However, it is conjectured that no Riesz basis of exponential functions for $L^2(D)$ exists. If this is true, what kind of systems of exponential functions are there in $L^2(D)$?

Outline of the text

In Chapter 2 we give necessary results from real, complex and functional analysis in order to make the text self-contained. Most of the material in the chapter is assumed to be known in advance.

In Chapter 3 we introduce Paley-Wiener spaces of entire functions and their relations to Lebesgue spaces. The duality between complete interpolating sequences for Paley-Wiener spaces and Riesz bases of complex exponential functions for L^2-spaces is especially emphasized.

Chapter 4 is about Fuglede's conjecture. It starts with an informal introduction to unbounded operators in Hilbert spaces, because this topic is not covered in the course in functional analysis given at NTNU. The chapter then explains the background for Fuglede's conjecture and some simple cases are proved. A summary of the latest results about the conjecture is given in the end.

Chapter 5 is concerned with the article [LR00] about Riesz bases for L^2-spaces over convex symmetric polygons. Quite a lot of details are given, because the technique will be used in Chapter 6.

In Chapter 6 we try to use the methods developed by Lyubarskii and Rashkovskii in [LR00] combined with an approximation technique to get a uniqueness result for a space of entire functions of two complex variables. This work is not completed yet.

Chapter 2

Preliminaries

We start with some preliminary material from real and complex analysis, functional analysis and Fourier analysis. Much of the material is assumed to be known from before and hardly no proofs are given. Other parts of this chapter are not assumed to be known in advance. For these parts we sometimes illustrate the theory with examples, but we still give no proofs. References are of course given in any case. It should be noted that the exposition is not by any means complete. Only definitions and results needed in this text are stated. The references should be consulted if one would like to get the full stories.

2.1 Notation

Some words about notation should be said in order to avoid misunderstandings. The basic notation is given here, more will be introduced when needed.

C will in general denote a positive constant. If a constant changes throughout an estimate we will sometimes use an index to stress that it is not the same constant. If f and g are some real valued functions and $f \leq Cg$, we will sometimes write $f \prec g$. If $f \prec g$ and $f \succ g$ we write $f \asymp g$. When f and g has the same asymptotic behaviour, that is

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1,$$

we write $f \sim g$.

Let A be a subset of a set X. The characteristic function of A, $\chi_A : X \to \{0,1\}$, is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

$\#A$ is the cardinality of the set A.

The zero set of a complex valued function $f : X \to \mathbf{C}$, where X is some set, is $Z(f) = \{x \in X : f(x) = 0\}$.

The end of a proof will be marked with a black rectangle ∎, and the end of an example will be marked with a black triangle ▲.

2.2 Entire functions

We will be particularily concerned with a special class of analytic functions, namely those who are analytic in the whole complex plane. A reference for this material is [Lev96].

Definition 2.1 If f analytic in all of $\mathbf{C}$, then f is called *entire*.

The notation $f \in \mathrm{Hol}(\mathbf{C})$ will sometimes be used.

Example 2.2 Some examples of entire functions are polynomials and exponential functions and sums and products of such. ▲

Theorem 2.3 (Liouville) *If f is entire and bounded, then f is constant.*

Since the only bounded entire functions are the constants, it is of interest to consider entire functions with different growth properties. We start with a couple of definitions. In the following we define the maximum modulus of a function f as $M_f(r) = \max_{|z|=r} |f(z)|$.

Definition 2.4 The *order* of an entire function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

The *type* of an entire function f with respect to the order ρ is defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^\rho}.$$

If f is an entire function of order 1 and of normal type, i.e. $0 < \sigma < \infty$, then f is said to be an *entire function of exponential type σ*.

Example 2.5 Let $f(z) = \sin \pi z = \frac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right)$ and compute the maximum modulus on a circle of radius r

$$M_f(r) = \max_{|z|=r} |f(z)| = Ce^{\pi|\mathrm{Im}\, z|} = Ce^{\pi r},$$

where C is some positive constant. We may then calculate the order

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} = \limsup_{r\to\infty} \frac{\log(\log C + \pi r)}{\log r} = 1$$

and the type

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r} = \limsup_{r\to\infty} \frac{\log Ce^{\pi r}}{r} = \pi.$$

Thus we see that $\sin \pi z$ is an entire function of exponential type π. ▲

An infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is said to converge if for some M the limit

$$\lim_{N\to\infty} \prod_{n=M}^{N} (1+a_n)$$

is different from 0 or infinity.

Theorem 2.6 *The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ with $1+a_n \neq 0$ converges simultaneously with the series $\sum_{n=1}^{\infty}\log(1+a_n)$ whose terms represent the values of the principal branch of the logarithm.*

Define the functions

$$G(u,p) = \begin{cases} 1-u, & p=0 \\ (1-u)\exp\left(u+\frac{u^2}{2}+\ldots+\frac{u^p}{p}\right), & p>0 \end{cases}.$$

The functions $G(\cdot,\cdot)$ are called the Weierstrass primary factors. There is a useful representation for entire functions in terms of infinite products.

Theorem 2.7 *Each entire function f can be represented in the form*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}, p_n\right),$$

where g is an entire function, $a_1, a_2, \ldots$ are the non-zero roots of f, and m is the multiplicity of the root of f at the origin.

If we restrict ourselves to entire functions of finite order, we get what is called the Hadamard factorization.

Theorem 2.8 *An entire function f of finite order ρ may be represented in the form*

$$f(z) = z^m e^{P_q(z)} \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}, p\right), \tag{2.1}$$

where $a_1, a_2, \ldots$ are all non-zero roots of the function f, $p \le \rho$, P_q is a polynomial in z of degree $q \le \rho$, and m is the multiplicity of the root of f at the origin.

Example 2.9 Consider $f(z) = \sin \pi z$, with zero set $Z(f) = \mathbf{Z}$ and order $\rho = 1$. $\sin \pi z$ has the canonical expansion

$$\sin \pi z = z e^{P_1(z)} \prod_{n=-\infty}^{\infty}{}' \left(1-\frac{z}{n}\right) e^{z/n}. \tag{2.2}$$

The prime means that we should omit the factor corresponding to $n = 0$. The polynomial P_1 is to be found. Differentiate both sides of (2.2) logarithmically

$$\frac{\pi \cos \pi z}{\sin \pi z} = \frac{1}{z} + P_1'(z) + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z-n} + \frac{z}{n} \right].$$

It can be shown that the partial fractions decomposition of $\pi \frac{\cos \pi z}{\sin \pi z}$ is

$$\pi \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z-n} + \frac{1}{n} \right],$$

hence P_1 must be constant. Since

$$\lim_{z \to 0} \frac{\sin \pi z}{z} = \pi,$$

we get $e^{P_1(z)} = \pi$. We have obtained the following product representation for $\sin \pi z$

$$\sin \pi z = \pi z \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{z/n}. \tag{2.3}$$

The convergence is absolute for every $z \in \mathbf{C}$, so we can simplify the right-hand side of (2.3) by writing

$$\pi z \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) \left(1 + \frac{z}{n}\right) e^{z/n} e^{-z/n} = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

We now have a nice product expansion for $\sin \pi z$

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

▲

Several complex variables

Quite often we will be working with sets in $\mathbf{C}^2$ and functions defined in $\mathbf{C}^2$, so we need some general definitions and theory. This will we be given for $\mathbf{C}^n$. Some references for the theory of several complex variables are [Sha92] and [Hör73]. We identify the complex n-space as $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$ and a point $z \in \mathbf{C}^n$ is given as $z = (z_1, ..., z_n)$, where $z_j = x_j + iy_j \in \mathbf{C}$, $x_j, y_j \in \mathbf{R}$, for $j = 1, ..., n$. The notation $z = x + iy \in \mathbf{C}^n$, where $x, y \in \mathbf{R}^n$ will also be used. The norm of $z \in \mathbf{C}^n$ is

$$|z| = \left(|z_1|^2 + ... + |z_n|^2\right)^{1/2}.$$

$\mathbf{C}^n$ is equipped with an inner product $\langle \cdot, \cdot \rangle : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}$, defined by $\langle z, \zeta \rangle = \sum_{j=1}^n z_j \bar{\zeta}_j$ for $z, \zeta \in \mathbf{C}^n$. The differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

are fundamental when we define analyticity.

Definition 2.10 A function $f \in C^1(\Omega)$, $\Omega \subset \mathbf{C}^n$ is open, is *analytic* (or holomorphic) in Ω if it satisfies the system of homogenous Cauchy-Riemann equations

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, ..., n,$$

for all $z \in \Omega$. We often write $f \in \mathrm{Hol}(\Omega)$.

We will be working with zero sets of analytic functions of several variables and need to give some definitions related to manifolds. The definitions are very brief and one should perhaps consult a book, e.g. [Sha92].

Definition 2.11 Let X be a topological space. X is called a *topological manifold* of real dimension n if X is a Hausdorff space and for every point $x \in X$ there exists an open set $U \ni x$ homeomorphic to an open set in $\mathbf{R}^n$. If the real dimension is an even number $2n$, we sometimes call X a topological manifold of complex dimension n.

Definition 2.12 Let X be a topological manifold of complex dimension n. X is called an *analytic manifold* of complex dimension n if there is a family $\mathcal{F}$ of homeomorphisms φ, mapping open sets $U_\varphi \subset X$ to open sets $V_\varphi \subset \mathbf{C}^n$ such that

- If $\varphi, \varphi' \in \mathcal{F}$, then the mapping

  $$\varphi' \circ \varphi^{-1} : \varphi(U_\varphi \cap U_{\varphi'}) \to \varphi'(V_\varphi \cap V_{\varphi'})$$

  is analytic.

- $$\bigcup_{\varphi \in \mathcal{F}} U_\varphi = X.$$

- If φ_0 is a homeomorphism between the open set $U_0 \subset X$ and an open set in $\mathbf{C}^n$ and the mapping

  $$\varphi \circ \varphi_0^{-1} : \varphi_0(U_0 \cap U_\varphi) \to \varphi(U_0 \cap U_\varphi)$$

  as well as its inverse is analytic for all $\varphi \in \mathcal{F}$, it follows that $\varphi_0 \in \mathcal{F}$.

Definition 2.13 A non-empty set of points $z \in \mathbf{C}^n$ that satisfies the system of equations

$$\sum_{i=1}^n a_{ij} z_i = b_j, \qquad j = 1, ..., k, \qquad a_{ij}, b_j \in \mathbf{C},$$

is called an *analytic plane*. The complex dimension of an analytic plane is $n-k$, where

$$\operatorname{rank}(a_{ij})_{i,j=1}^{n} = k < n.$$

Liouville's theorem has an n-dimensional generalization.

Theorem 2.14 (Liouville) *If the entire function $f : \mathbf{C}^n \to \mathbf{C}$ is bounded in all of $\mathbf{C}^n$, then it is constant.*

Consider a function $f \in \operatorname{Hol}(\mathbf{C}^n)$ and define

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

We may now define order and type analogously to the one-dimensional case.

Definition 2.15 The *order* of a function $f \in \operatorname{Hol}(\mathbf{C}^n)$ is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

The *type* of f with respect to the order ρ is defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^\rho}.$$

2.3 Subharmonic functions

The following material may be found in [HK76].

Definition 2.16 Let Ω be a domain in the complex plane. A function $u : \Omega \to [-\infty, \infty)$ is said to be *subharmonic* in Ω if it at each point $z_0 \in \Omega$ satisfies the following two conditions:

1. upper semicontinuity:

$$u(z_0) = \lim_{\delta\to 0} \sup_{|z-z_0|<\delta} u(z)$$

2. the mean value inequality:

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) \mathrm{d}\theta$$

   with $0 < r < \varepsilon$ for some $\varepsilon > 0$.

The next example shows that the logarithm of the modulus of an analytic function is subharmonic.

Example 2.17 If $f \in \mathrm{Hol}(\Omega)$, then $u = \log|f|$ is subharmonic in Ω. Condition 1 is trivially fulfilled. If $f(z_0) = 0$, then $u(z_0) = -\infty$ and condition 2 is ok. If $f(z_0) \neq 0$ then some branch of $\log f$ is analytic in some neighborhood of z_0. Re $(\log f) = u$ is harmonic and therefore subharmonic as well. ▲

Example 2.18 Let u be subharmonic in a domain Ω and φ be an increasing convex function defined on the range of u. We then have

$$\varphi(u(z)) \leq \varphi\left(\frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) \mathrm{d}\theta\right) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(u\left(z_0 + re^{i\theta}\right)\right) \mathrm{d}\theta,$$

where the last inequality is justified by Jensen's inequality. Upper semicontinuity follows trivially, thus $\varphi \circ u$ is a subharmonic function in Ω. ▲

The function in the next example will be used a lot in the following chapters.

Example 2.19 Let $f(z) = \pi|\mathrm{Im}\, z|$. The imaginary part of a complex number is a harmonic function, and therefore also subharmonic. The modulus function is convex, so we may use the same argument as in Example 2.18 to prove that f is subharmonic.

Definition 2.20 Let $M_u(r) = \sup_{|z|=r} u^+(z)$, where $u^+ = \max\{0, u\}$. The *order* of the subharmonic function u is defined as

$$\rho_u = \limsup_{r\to\infty} \frac{\log M_u(r)}{\log r}.$$

The definition above is analogous to the definition of order of an entire function f. $\log|f|$ is a subharmonic function, and this is why there is one less "log" in the definition of the order of a subharmonic function.

F. Riesz' theorem on the local representation of a subharmonic function in terms of an integral is essential in the theory of subharmonic functions.

Theorem 2.21 *If u is subharmonic in a domain $D \subset \mathbf{C}$, then u has the following representation in every compact set $G \subset D$*

$$u(z) = \iint_G \log|z-\zeta|\,\mathrm{d}\mu_\zeta + h(z), \tag{2.4}$$

where h is harmonic in G and μ is a positive Borel measure. The integral on the right hand side is called the logarithmic potential of u and μ is called the Riesz measure of u. If $u \in C^2(G)$, then the Riesz measure has the form $\mathrm{d}\mu_\zeta = \frac{1}{2\pi}\triangle u(\zeta)\mathrm{d}m_\zeta$, where $\mathrm{d}m$ is the usual Lebesgue measure in the plane and $\triangle$ is the Laplace operator.

Plurisubharmonic functions

Definition 2.22 Let $\Omega \subset \mathbf{C}^n$ be a domain and $u : \Omega \to [-\infty, \infty)$ be upper semicontinuous. We say that u is *plurisubharmonic* if, for each complex line $\ell = \{a + bz\} \subset \mathbf{C}^n$,

$a, b \in \mathbf{C}^n, z \in \mathbf{C}$, the function

$$z \mapsto u(a + bz), \quad a, b \in \mathbf{C}^n, z \in \mathbf{C},$$

is subharmonic on $\Omega \cap \ell$. We sometimes write $u \in \mathrm{PSH}(\Omega)$.

Definition 2.23 A complex-valued function $f \in C^2(\Omega)$, $\Omega \subset \mathbf{C}^n$, is said to be *pluriharmonic* if for every complex line $\ell = \{a + bz\} \subset \mathbf{C}^n$, $a, b \in \mathbf{C}^n, z \in \mathbf{C}$, the function $\zeta \mapsto f(a + b\zeta)$ is harmonic on the set $\Omega \cap \ell$.

The class of plurisubharmonic functions is properly contained in the class of subharmonic functions when $n > 1$. When $n = 1$ the classes coincide. Nevertheless, many of the properties of subharmonic functions are carried over to plurisubharmonic functions.

Example 2.24 If $f \in \mathrm{Hol}(\Omega)$, then $\log|f| \in \mathrm{PSH}(\Omega)$.

Example 2.25 Let $u \in \mathrm{PSH}(\Omega)$ and $\varphi : \mathbf{R} \to \mathbf{R}$ be a convex monotone non-decreasing function. Then $\varphi \circ u \in \mathrm{PSH}(\Omega)$.

A two-dimensional analog of Example 2.19 will be important in Chapter 6.

Example 2.26 $\pi|\mathrm{Im}\, z|$, $z \in \mathbf{C}^2$. $\mathrm{Im}\, z$ is a pluriharmonic function and the norm is convex, so $\pi|\mathrm{Im}\, z|$ is plurisubharmonic in $\mathbf{C}^2$.

2.4 Functional analysis

We first state two of the cornerstones of functional analysis, the Hahn-Banach theorem and the closed graph theorem. After that, some theory for sequences in Banach spaces and Hilbert spaces are given. In the end the Lebesgue and Hardy spaces are introduced. A reference for the basics of functional analysis is [DS58]. Some more specialized references are presented on the way.

We will denote a general Banach space by X and its dual by X^*, while a general Hilbert space will be denoted by H.

Theorem 2.27 (Hahn-Banach) *Let f be a bounded linear functional on a subspace Z of a Banach space X. Then there exists a bounded linear functional $\tilde{f}$ which is an extension of f to X, that is $\tilde{f}|_Z = f$, and has the same norm, $\|\tilde{f}\|_X = \|f\|_Z$.*

An important consequence of the Hahn-Banach theorem is the following:

Theorem 2.28 *Let Z be a closed subspace of a Banach space X and let $\delta(x) = \mathrm{dist}(x, Z)$. Then there exists a non-trivial functional $f \in X^*$ such that $f|_Z = 0$, $\|f\| = 1$ and $f(x) = \delta(x)$.*

Definition 2.29 Let $T : \mathcal{D}(T) \to Y$ be a linear operator between two Banach spaces X and Y, where $\mathcal{D}(T) \subset X$. T is a *closed* operator if its graph

$$\mathcal{G}(T) = \{(x, y) : \ x \in \mathcal{D}(T),\ y = Tx\}$$

is closed in $X \times Y$.

Theorem 2.30 (Closed Graph Theorem) *Let X and Y be Banach spaces and $T : \mathcal{D}(T) \to Y$ a closed linear operator, where $\mathcal{D}(T) \subset X$. If $\mathcal{D}(T)$ is closed in X, the operator T is bounded.*

A useful criterion for determining whether a linear operator is closed or not is the following:

Theorem 2.31 *Let $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and X and Y are Banach spaces. Then T is closed if and only if it has the following property. If $x_k \to x$, where $x_k \in \mathcal{D}(T)$, and $Tx_k \to y$, then $x \in \mathcal{D}(T)$ and $Tx = y$.*

Sequences in Banach and Hilbert spaces

We will be working mostly with Hilbert spaces, but first we define some concepts in Banach spaces and state some general results. After that we will look more carefully at sequences in Hilbert spaces. Most of the material is taken from [You01]. We start with some definitions.

Definition 2.32 Given a Banach space X and a sequence $\{x_j\} = \{x_j\}_{j=1}^{\infty}$ in X. $\{x_j\}$ is a *Schauder basis* if for each $x \in X$ there exist unique scalars $\{c_j\} = \{c_j\}_{j=1}^{\infty}$ such that $x = \sum_{j=1}^{\infty} c_j x_j$. The convergence is to be understood in the norm topology, that is

$$\left\| x - \sum_{j=1}^{n} c_j x_j \right\| \to 0 \quad \text{as } n \to \infty.$$

In the sequel we will write basis and mean Schauder basis. Not every separable Banach space has a basis, as shown by Enflo in 1973. Every Hilbert space has one though. In fact, every Hilbert space has an orthonormal basis (see Definition 2.38). We will need the notion of equivalent bases, so here is a definition.

Definition 2.33 Two bases $\{x_j\}$ and $\{y_j\}$ for a Banach space X are called *equivalent* if

$$\sum_{j=1}^{\infty} c_j x_j < \infty \quad \Leftrightarrow \quad \sum_{j=1}^{\infty} c_j y_j < \infty.$$

A criterion for equivalence of bases is given below.

Theorem 2.34 *Two bases $\{x_j\}$ and $\{y_j\}$ for a Banach space X are equivalent if and only if there exists a bounded invertible operator $T : X \to X$ such that $Tx_j = y_j$ for all j.*

Definition 2.35 Let $\{x_j\}$ be a sequence in a Banach space X. $\{x_j\}$ is *complete*[1] in X if the linear span of $\{x_j\}$ is dense in X.

[1]The word *total* is also quite common.

This means that if $\{x_j\}$ is complete in X, then for any $x \in X$ and any $\varepsilon > 0$ there exists $y \in \operatorname{span}\{x_j\}$ such that $\|x - y\| < \varepsilon$. We repeat a couple of elementary definitions.

Definition 2.36 A sequence $\{x_j\}$ in a Banach space X is called *minimal* if for each j the element x_j lies outside the closed linear span of the other elements.

Definition 2.37 Let $\{x_j\}$ be a sequence in a Hilbert space H. If

$$\langle x_k, x_l \rangle = C\delta_{kl} = C \cdot \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$$

then $\{x_j\}$ is an *orthogonal sequence* in H. If $C = 1$, $\{x_j\}$ is an *orthonormal sequence* in H.

Definition 2.38 A complete orthonormal sequence in a Hilbert space H is called an *orthonormal basis* for H.

A famous criterion for an orthonormal sequence to be complete, is Parseval's identity.

Theorem 2.39 *Let $\{x_j\}$ be an orthonormal sequence in a Hilbert space H. The sequence $\{x_j\}$ is complete in H if and only if Parseval's identity holds for all $x \in H$, that is*

$$\|x\|^2 = \sum_j |\langle x, x_j \rangle|^2 ,$$

for all $x \in H$.

It will be useful to know what biorthogonal sequences are.

Definition 2.40 Let $\{x_j\}$ and $\{y_j\}$ be sequences in a Hilbert space H. They constitute a *biorthogonal* system if $\langle x_k, y_l \rangle = \delta_{kl}$.

For a given sequence $\{x_j\}$ there exists by the Hahn-Banach theorem a system $\{y_j\}$ biorthogonal to $\{x_j\}$ if and only if $\{x_j\}$ is minimal. The biorthogonal system is uniquely determined if and only if the minimal system is complete.

We will be working extensively with sequences of the form $\mathcal{E}(\Lambda) = \{e^{i\langle \lambda, \cdot \rangle} : \lambda \in \Lambda\}$, where Λ is some discrete subset of $\mathbf{R}^n$, in spaces of square integrable functions (more about these spaces soon). Such spaces do not always possess orthonormal bases of the form $\mathcal{E}(\Lambda)$ and we must loosen up the requirements on the basis-like sequence we are seeking. One such type of sequence is a Riesz basis.

Definition 2.41 A basis for a Hilbert space H is called a *Riesz basis* if it is equivalent to an orthonormal basis, that is, if it is the image of an orthonormal basis under a bounded invertible operator.

It can be proved that $\{x_j\}$ is a Riesz basis for a Hilbert space H if and only if the sequence $\{x_j\}$ is complete in H, and there exist positive constants A and B such that for an arbitrary positive integer n and arbitrary scalars $c_1, ..., c_n$ one has

$$A \sum_{j=1}^{n} |c_j|^2 \le \left\| \sum_{j=1}^{n} c_j x_j \right\|^2 \le B \sum_{j=1}^{n} |c_j|^2 . \tag{2.5}$$

This characterization is sometimes taken as the definition of a Riesz basis. Condition (2.5) corresponds to Parseval's identity for orthonormal bases.

L^p-spaces

Given a σ-finite measure space $(X, \mathcal{M}, \mu)$. We define the quantities

$$\|u\|_p = \left(\int_X |u(x)|^p \, d\mu \right)^{1/p} \qquad 1 \le p < \infty \tag{2.6}$$

and

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x \in X} |u(x)| \tag{2.7}$$

and say that $u \in L^p(X)$ if u is measurable with respect to the σ-algebra $\mathcal{M}$ and $\|u\|_p < \infty$ for $1 \le p \le \infty$. $\|\cdot\|_p$ is a norm if we consider functions with norm equal to zero to be equivalent. With the norm (2.6) and $1 \le p < \infty$ or with the norm (2.7) and $p = \infty$, $L^p(X)$ is a Banach space. For $p = 2$ it is a Hilbert space. For us, X is usually a subset of $\mathbf{R}^n$ or $\mathbf{C}^n$, $\mathcal{M}$ is the Lebesgue σ-algebra and μ is the Lebesgue measure. When we deal with several variables, we will use the notation "$\mathrm{d}m_x$" for the Lebesgue measure. The subscript x indicates the variable we consider. In one dimension we will stick to the usual "$\mathrm{d}x$", where x is the variable in question.

We get a special type of L^p-spaces if we let μ be the counting measure. Given a sequence $a = \{a_j\}_{j=1}^{\infty}$, where $a_j \in \mathbf{C}$ for all $j \in \mathbf{N}$. We say that $a \in \ell^p = \ell^p(\mathbf{N})$ for $1 \le p < \infty$ if

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} < \infty. \tag{2.8}$$

The space of bounded sequences ℓ^∞ consists of all sequences $a = \{a_j\}_{j=1}^{\infty}$ with

$$\|a\|_\infty = \sup_j |a_j| < \infty, \quad \forall j \in \mathbf{N}. \tag{2.9}$$

More about L^p-spaces can be found in [Fol99].

Hardy spaces

We will only need Hardy spaces over the upper half-plane $\mathbf{C}_+$, so we skip the alternative version of Hardy spaces over the unit disk. The Hardy space over the upper half-plane H_+^2 is defined as the set of complex-valued functions f which are analytic in the upper half-plane and satisfies

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx \, < \infty.$$

The definition can be generalized to other domains and other p than 2. If we give H_+^2 the norm

$$\|f\|^2 = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, \mathrm{d}x,$$

then H_+^2 is a Banach space. The Hardy space over the lower half-plane H_-^2 is defined analogously. Some properties of functions $f \in H_+^2$, and similarly for $f \in H_-^2$, are:

1. H_+^2 is a Hilbert space with the inner product

   $$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \, \mathrm{d}x$$

   for $f, g \in H_+^2$.
2. The boundary function

   $$f(x) = \lim_{y \to 0} f(x+iy)$$

   exists almost everywhere and $\|f\|_{H_+^2}^2 = \|f\|_{L^2(\mathbf{R})}$.
3. f can be reconstructed from its boundary values using the Cauchy integral:

   $$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} \, \mathrm{d}t, \quad \operatorname{Im} z > 0.$$

4. If a sequence $\{\lambda_j\} \subset \mathbf{C}_+$ is located in a horizontal strip $0 < m \leq \operatorname{Im} z \leq M < \infty$ and separated $|\lambda_k - \lambda_l| > \varepsilon > 0$ for $k \neq l$, then

   $$\sum_j |f(\lambda_j)|^2 \leq C \|f\|_{H_+^2}^2$$

   for some constant C independent of f.
5. The dual space of H_+^2 can be identified with H_-^2 and vice versa. The functional $F_\psi \in \left(H_+^2\right)^*$ which corresponds to the function $\psi \in H_-^2$ is of the form

   $$F_\psi(\varphi) = \int_{-\infty}^{\infty} \varphi(t)\psi(t) \, \mathrm{d}t,$$

   and

   $$\|F_\psi\|_{\left(H_+^2\right)^*} = \|\psi\|_{H_-^2}.$$

We now link Hardy spaces to the Fourier transform.

Theorem 2.42 (Paley-Wiener) *A function f is from H^2_+ if and only if it admits the representation*

$$f(z) = \frac{1}{2\pi}\int_0^\infty \varphi(t)e^{itz}\,\mathrm{d}t,$$

for some $\varphi \in L^2(0,\infty)$. Then

$$\|f\|_{H^2_+} = 2\pi\|\varphi\|_{L^2(0,\infty)}.$$

More on H^p-spaces in the upper half-plane can be found in [Dur02, Chapter 11] or in [Lev96, Chapter 19].

2.5 Real analysis and Fourier analysis

Some basic results from real analysis are needed, namely Fubini's theorem and the dominated convergence theorem.

Theorem 2.43 (The Fubini Theorem) *Suppose that $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ are σ-finite measure spaces. If $f \in L^1(\mu\times\nu)$, then $f \in L^1(\nu)$ for a.e. $x \in X$, $f \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f(x,y)\,\mathrm{d}\nu$ and $h(y) = \int f(x,y)\,\mathrm{d}\mu$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively and*

$$\iint f\,\mathrm{d}(\mu\times\nu) = \int\left[\int f(x,y)\,\mathrm{d}\nu(y)\right]\mathrm{d}\mu(x) = \int\left[\int f(x,y)\,\mathrm{d}\mu(x)\right]\mathrm{d}\nu(y)$$

holds.

Theorem 2.44 (The Dominated Convergence Theorem) *Let $(X,\mathcal{M},\mu)$ be a measure space. Let $\{f_n\}$ be a sequence in $L^1(X)$ such that $f_n \to f$ almost everywhere, and there exists a nonnegative $g \in L^1(X)$ such that $|f_n| \le g$ almost everywhere for all n. Then $f \in L^1(X)$ and*

$$\int f\,\mathrm{d}\mu = \lim_{n\to\infty}\int f_n\,\mathrm{d}\mu.$$

We will not work very much with distributions, but in Chapter 4 we will need to know what the distributional derivative means.

Definition 2.45 Let $\Omega \subset \mathbf{R}^n$ and let u and v be locally integrable in Ω, that is $u, v \in L^1_{\mathrm{loc}}(\Omega)$. The function v is said to be the *distributional derivative* of u in the direction x_j if

$$\int_\Omega u\frac{\partial\varphi}{\partial x_j}\,\mathrm{d}m_x = -\int_\Omega v\varphi\,\mathrm{d}m_x$$

for all test functions $\varphi \in C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ denotes the set of all smooth functions defined on Ω with compact support properly contained in Ω.

The Cauchy principal value needs to be defined.

Definition 2.46 If $f \in L^1_{\text{loc}}(\mathbf{R}^n)$, $n \in \mathbf{N}$, we define the principal value of the integral

$$\int_{\mathbf{R}^n} f(x)\, \mathrm{d}m_x$$

to be

$$\text{V.P.} \int_{\mathbf{R}^n} f(x)\, \mathrm{d}m_x = \lim_{R \to \infty} \int_{|x|<R} f(x)\, \mathrm{d}m_x.$$

Much of this text will be concerned with consequences of properties of the Fourier transform. To be consistent with existing research literature, as well as to avoid some rather non-standard normalizations (at least from a mathematical point of view), we will use two different definitions of the Fourier transform. In Chapters 3, 5 and 6 we will use the following:

Definition 2.47 For $f \in L^1(\mathbf{R}^n)$ we define its *Fourier transform* $\hat{f}$ as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-i\langle \xi, x\rangle}\, \mathrm{d}m_x.$$

The *inverse Fourier transform* is defined as

$$\mathcal{F}^{-1}f(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} f(x) e^{i\langle \xi, x\rangle}\, \mathrm{d}m_x.$$

In Chapter 4 we use:

Definition 2.48 For $f \in L^1(\mathbf{R}^n)$ we define its *Fourier transform* $\hat{f}$ as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i\langle \xi, x\rangle}\, \mathrm{d}m_x.$$

The *inverse Fourier transform* is defined as

$$\mathcal{F}^{-1}f(\xi) = \int_{\mathbf{R}^n} f(x) e^{2\pi i\langle \xi, x\rangle}\, \mathrm{d}m_x.$$

The Riemann-Lebesgue lemma holds no matter which definition we use.

Theorem 2.49 (The Riemann-Lebesgue lemma) *If $f \in L^1(\mathbf{R}^n)$ the following holds:*

$$\lim_{|\xi| \to +\infty} |\hat{f}(\xi)| = 0.$$

The Fourier transform can be extended to $L^2(\mathbf{R}^n)$.

Theorem 2.50 *The Fourier transform defined on $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ extends uniquely to an operator on $L^2(\mathbf{R}^n)$. The Fourier transform defined on $L^1(\mathbf{R}^n)$ and the operator obtained by extension to $L^2(\mathbf{R}^n)$ coincides on $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. If $f \in L^2(\mathbf{R}^n)$, then $\hat{f}$ is the limit in $L^2(\mathbf{R}^n)$ of the sequence g_n defined by*

$$g_n(\xi) = \int_{-n}^{n} \dots \int_{-n}^{n} f(x) e^{-i\xi x} \, \mathrm{d}m_x$$

if we use Definition 2.47 and

$$g_n(\xi) = \int_{-n}^{n} \dots \int_{-n}^{n} f(x) e^{-2\pi i\xi x} \, \mathrm{d}m_x$$

if we use Definition 2.48.

Plancherel's theorem needs to be adjusted according to which definition we use.

Theorem 2.51 (Plancherel) *If $f, g \in L^2(\mathbf{R}^n)$, then $(2\pi)^{n/2}\|f\| = \|\hat{f}\|$ if we use Definition 2.47 and $\|f\| = \|\hat{f}\|$ if we use Definition 2.48.*

All results in this section may be found in [Fol99].

Chapter 3

Paley-Wiener spaces

In this chapter we will introduce a space of entire functions of exponential type which are square integrable on the real axis, called the Paley-Wiener space. In the first section we look at the definition, as well as some fundamental theorems. After that we will describe a certain duality between interpolation problems in the Paley-Wiener space and systems of exponential functions in $L^2(-\pi, \pi)$, with an example illustrating the matter. In Section 3.5 we look at Riesz bases formed by subspaces for $L^2(-\pi, \pi)$. Using this, we are able to relax a separation condition used in the previous sections. In the end we consider Paley-Wiener spaces of entire functions of several complex variables and introduce the problems we will work with in the remaining chapters.

3.1 Paley-Wiener spaces in one variable

A function $f \in L^2(\mathbf{R})$ is called bandlimited if its Fourier transform $\hat{f}$ has compact support. For simplicity, we will assume that the support of $\hat{f}$ is contained in the interval $[-\pi, \pi]$. Bandlimited functions are of fundamental importance in signal analysis. In signal analysis a function is called a signal and the integrability condition ($f \in L^2(\mathbf{R})$) means that the signal has finite energy. That a signal is bandlimited means that its frequencies are bounded. These signals seem to make sense physically, but there is one drawback. In Theorem 3.1 we will see that bandlimited functions are entire functions. This implies that they cannot have compact support. In the language of signal analysis, this means that they are signals which last for an infinitely long period of time. Nevertheless, this class of functions is used extensively in practial applications.

The celebrated Paley-Wiener theorem [PW34] says that a bandlimited function may be extended to an entire function of exponential type.

Theorem 3.1 (Paley and Wiener, 1934) *A function f has the representation*

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t) e^{ixt} \, \mathrm{d}t, \tag{3.1}$$

where $\psi \in L^2(-\pi,\pi)$*, if and only if it is possible to extend* f *to an entire function of exponential type* π *and* $f \in L^2(\mathbf{R})$*.*

A small proposition is needed before we can prove Theorem 3.1.

Proposition 3.2 *If* f *is an entire function such that* $|f(z)| \leq Ae^{B|z|}$ *for all* $z \in \mathbf{C}$ *and* $|f(x)| \leq M$ *for all* $x \in \mathbf{R}$*. Then*

$$|f(x+iy)| \leq Me^{B|y|}.$$

Proof: Assume that $y > 0$, let $\varepsilon > 0$ and let

$$g(z) = e^{i(B+\varepsilon)z} f(z).$$

We have $|g(x)| = |f(x)| \leq M$ for all $x \in \mathbf{R}$ and $g(iy) \to 0$ as $y \to \infty$. Let N denote the maximum value of $|g(z)|$ on the positive part of the imaginary axis. The Phragmèn-Lindelöf theorem then implies that $|g(z)| \leq \max(N,M)$ for every z in the first quadrant. By the same argument we get the same estimate in the second quadrant. Applying the maximum principle to a large rectangle with lower edge on the real axis, then gives that $M \geq N$. Now, $|g(z)| \leq M$ and

$$|f(z)| = \left| e^{-i(B+\varepsilon)z} \right| |g(z)| \leq Me^{(B+\varepsilon)y}$$

for $\operatorname{Im} z > 0$. Let $\varepsilon \to 0$ and the desired result follows. If $y < 0$, the same result is obtained by considering $f(-z)$. ∎

We follow the proof by Boas given in [You01].

Proof of Theorem 3.1: ($\Rightarrow$) The function f is entire since the integrand is an entire function of $z \in \mathbf{C}$. Plancherel's theorem yields

$$\|f\|_{L^2(\mathbf{R})} = \frac{1}{\sqrt{2\pi}} \|\psi\|_{L^2(-\pi,\pi)},$$

hence f is square integrable along the real axis. Moreover,

$$|f(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|e^{izt}\right| |\psi(t)| \,\mathrm{d}t \leq \frac{1}{2\pi} e^{\pi|\operatorname{Im} z|} \int_{-\pi}^{\pi} |\psi(t)| \,\mathrm{d}t \leq Ce^{\pi|\operatorname{Im} z|}$$

for all $z \in \mathbf{C}$, so f is of exponential type.

($\Leftarrow$) Assume that $f \in L^2(\mathbf{R})$ and that f is an entire function of exponential type π. Let ψ be the Fourier transform of f,

$$\psi(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt} \,\mathrm{d}x.$$

Then $\psi \in L^2(\mathbf{R})$ and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t)e^{ixt} \,\mathrm{d}t.$$

We need to show that $\psi(t) = 0$ for almost every $t \in \mathbf{R} \setminus [-\pi, \pi]$. Let $R > 0$ and consider the contour $\gamma_R = [R, R+iR] \cup [R+iR, -R+iR] \cup [-R+iR, -R]$. Define

$$I = \int_{\gamma_R} f(z)e^{-izt} dz.$$

Cauchy's theorem implies that $I = -\int_{-R}^{R} f(x)e^{-ixt}\,\mathrm{d}x$. We need to show that $I \to 0$ as $R \to \infty$ when $|t| > \pi$. Since

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)e^{-ixt}\,\mathrm{d}x = \psi(t)$$

we will obtain $\psi(t) = 0$ for almost every $t \in \mathbf{R} \setminus [-\pi, \pi]$. Assume that $t < -\pi$, then

$$|I| \leq \int_0^R |f(R+iy)|\, e^{ty}\,\mathrm{d}y + e^{tR}\int_{-R}^{R} |f(x+iR)|\,\mathrm{d}x + \int_0^R |f(-R+iy)|\, e^{ty}\,\mathrm{d}y. \quad (3.2)$$

Denote the three integrals on the right-hand side of (3.2) by I_1, I_2 and I_3 respectively. The function f is square integrable on the real axis and entire, thus $|f(x)| \leq M$ for some finite M and all $x \in \mathbf{R}$. By Proposition 3.2 $|f(x+iR)| \leq Me^{\pi R}$ for all $x \in \mathbf{R}$. This means that $I_2 \leq 2RMe^{(t+\pi)R}$, but $t < -\pi$, so $I_2 \to 0$ as $R \to \infty$.

Write

$$I_1 = \int_0^{R_1} |f(R+iy)|\, e^{ty}\,\mathrm{d}y + \int_{R_1}^{R} |f(R+iy)|\, e^{ty}\,\mathrm{d}y,$$

where $0 < R_1 < R$. By Proposition 3.2, the function f is uniformly bounded in each horizontal strip and $f(x) \to 0$ as $|x| \to \infty$ since $f \in L^2(\mathbf{R})$, thus for each fixed R_1, $|f(R+iy)|$ tends to zero uniformly as R tends to infinity for each $0 < y < R_1$. Consider $\int_{R_1}^{R} |f(R+iy)|\, e^{ty}\,\mathrm{d}y$. Again by Proposition 3.2

$$\int_{R_1}^{R} |f(R+iy)|\, e^{ty}\,\mathrm{d}y \leq M\int_{R_1}^{R} e^{(t+\pi)y}\,\mathrm{d}y = \frac{M}{t+\pi}\left(e^{(t+\pi)R} - e^{(t+\pi)R_1}\right) \to 0$$

as R and R_1 tends to infinity. The integral I_3 is treated in the same way. If $t > \pi$, we may use the same arguments with a similar contour in the lower half-plane to obtain the same conclusion. ∎

The space of entire functions of exponential type π which are square integrable on the real axis is called a Paley-Wiener space, and is denoted PW_π^2. In view of the Paley-Wiener theorem we have the characterization

$$PW_\pi^2 = \left\{ f : \mathbf{C} \to \mathbf{C} :\ f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(t)e^{izt}\,\mathrm{d}t,\ \psi \in L^2(-\pi,\pi) \right\}.$$

We equip PW_π^2 with the inner product $\langle f, g\rangle = \int_{\mathbf{R}} f(x)\overline{g(x)}\,\mathrm{d}x$. This inner product induces a norm on PW_π^2, thus $\|f\|_{PW_\pi^2} = \|f\|_{L^2(\mathbf{R})}$. With this norm PW_π^2 becomes a Hilbert space, (see e.g. [Sei04, Chapter 6]).

The Whittaker-Kotel'nikov-Shannon sampling theorem says that every function from PW_π^2 can be reconstructed from its values at integers.

Theorem 3.3 (Whittaker-Kotel'nikov-Shannon) *If $f \in PW_\pi^2$, then*

$$f(z) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(z-k)}{\pi(z-k)} \tag{3.3}$$

and

$$\|f\|_{PW_\pi^2} = \|\{f(k)\}\|_{\ell^2(\mathbf{Z})}. \tag{3.4}$$

The series in (3.3) converges both in PW_π^2 and uniformly on compact subsets of $\mathbf{C}$. Conversely, given $\{c_k\} \in \ell^2(\mathbf{Z})$, then (3.3) defines a function $f \in PW_\pi^2$, which solves the interpolation problem $f(k) = c_k$.

Proof: From the Paley-Wiener theorem we know that each function $f \in PW_\pi^2$ has the representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t) e^{itz} \, \mathrm{d}t \tag{3.5}$$

where $\psi \in L^2(-\pi, \pi)$. ψ has a Fourier expansion

$$\psi(t) = \sum_{k=-\infty}^{\infty} c_k e^{-ikt}. \tag{3.6}$$

The coefficients c_k are

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t) e^{ikt} \, \mathrm{d}t = f(k). \tag{3.7}$$

Plugging (3.6) and (3.7) into (3.5) gives

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} f(k) e^{i(z-k)t} \, \mathrm{d}t = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(z-k)}{\pi(z-k)}.$$

Plancherel's theorem and Parseval's identity imply that

$$\|f\|_{PW_\pi^2} = \|f\|_{L^2(\mathbf{R})} = \frac{1}{\sqrt{2\pi}} \|\psi\|_{L^2(-\pi,\pi)} = \|\{c_k\}\|_{\ell^2(\mathbf{Z})} = \|\{f(k)\}\|_{\ell^2(\mathbf{Z})},$$

hence the series on the right-hand side of (3.3) converges in the norm of PW_π^2. For any $f \in PW_\pi^2$ we may derive the following estimate from the Paley-Wiener theorem and Plancherel's theorem

$$|f(x+iy)| \leq e^{\pi|y|} \|f\|_{PW_\pi^2}.$$

A consequence of this inequality is that convergence in PW_π^2 implies convergence in horizontal strips, which again implies convergence on compact subsets of $\mathbf{C}$.

Given a sequence $\{c_k\} \in \ell^2(\mathbf{Z})$. The function defined in (3.3) is easily seen to solve the interpolation problem. The function defined in (3.6) is from $L^2(-\pi, \pi)$, thus the function defined in (3.5) is from PW_π^2. ∎

Theorem 3.3 is also known as the sampling theorem. It is fundamental in signal analysis as it provides the foundation for conversion between digital and analog signals.

One can generalize Paley-Wiener spaces to $p \neq 2$. PW_π^p is the space of entire functions of exponential type π whose restriction to the real line is in $L^p(\mathbf{R})$. When $1 \leq p \leq \infty$, PW_π^p is a Banach space with the $L^p(\mathbf{R})$-norm. There is an analog of Theorem 3.3 for these spaces, called the Plancherel-Polya theorem. In this case (3.4) is weakened a bit, instead of equality one has $\|f\|_{PW_\pi^p} \asymp \|\{f(k)\}\|_{\ell^p(\mathbf{Z})}$. More about this matter is written in [Lev96, Chapter 20].

3.2 Duality

In this section we will explore the duality between systems of complex exponential functions in $L^2(-\pi,\pi)$ and interpolation problems for PW_π^2. We will let $\Lambda = \{\lambda_k\}$ denote a sequence of complex numbers located in a horizontal strip and $\mathcal{E}(\Lambda) = \{e^{i\lambda_k t}\}$ the corresponding sequence of complex exponential functions.

Definition 3.4 Λ is a *complete interpolating sequence* for PW_π^2 if for each $\{a_k\} \in \ell^2$ there exists a unique $f \in PW_\pi^2$ solving the interpolation problem

$$f(\lambda_k) = a_k,$$

for all k.

The main goal is to prove the following.

Theorem 3.5 *Λ is a complete interpolating sequence for PW_π^2 if and only if $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(-\pi,\pi)$.*

Λ is called a set of uniqueness for PW_π^2 if $f|_\Lambda = 0$ implies that $f \equiv 0$, for any $f \in PW_\pi^2$.

Proposition 3.6 *$\mathcal{E}(\Lambda)$ is complete in $L^2(-\pi,\pi)$ if and only if Λ is a set of uniqueness for PW_π^2.*

Proof: It is a consequence of the Hahn-Banach theorem that $\mathcal{E}(\Lambda)$ is complete in $L^2(-\pi,\pi)$ if and only if there does not exist a non-trivial functional F such that $F|_{\mathcal{E}(\Lambda)} = 0$. $L^2(-\pi,\pi)$ is a Hilbert space and by Riesz' representation theorem any functional F applied to the function $e^{i\lambda_k t}$ is of the form

$$F(e^{i\lambda_k t}) = \int_{-\pi}^{\pi} \psi(t) e^{i\lambda_k t}\, \mathrm{d}t \tag{3.8}$$

for some $\psi \in L^2(-\pi,\pi)$. According to the Paley-Wiener theorem the right-hand side of (3.8) defines a function $f \in PW_\pi^2$. The function f vanishes at Λ if and only if F vanishes at $\mathcal{E}(\Lambda)$ and f is non-trivial if and only if F is non-trivial. ∎

Let $\{x_k\}$ be a sequence in a Hilbert space H. The sequence $\{\langle x, x_k\rangle\}$ is called the *moment sequence* of $x \in H$. The set of all moment sequences with respect to the sequence $\{x_k\}$ is called the *moment space* of $\{x_k\}$. In our case $x_k(t) = e^{i\lambda_k t}$, where λ_k is some complex number, and $H = L^2(-\pi, \pi)$. The moment sequence of a function $\psi \in L^2(-\pi, \pi)$ is of the form

$$\left\{\left\langle \psi, e^{i\lambda_k \cdot} \right\rangle\right\} = \left\{\int_{-\pi}^{\pi} \psi(t) e^{i\lambda_k t}\,\mathrm{d}t\right\}. \tag{3.9}$$

According to the Paley-Wiener theorem the right-hand side of (3.9) defines a sequence $\{f(\lambda_k)\}$, where f is from PW_π^2. To solve the interpolation problem stated in Definition 3.4 is therefore equivalent to show that the moment space of a complete sequence $\{e^{i\lambda_k \cdot}\} \subset L^2(-\pi, \pi)$ is equal to ℓ^2.

Let us now see that the moment space characterizes complete sequences up to equivalence. Recall that two sequences are equivalent if one can pass between them by the means of a bounded invertible operator.

Theorem 3.7 *Two complete sequences belonging to a separable Hilbert space are equivalent if and only if they have the same moment space.*

Proof: $(\Rightarrow)$ Let $\{x_k\}$ and $\{y_k\}$ be complete sequences in H. Suppose that there exists a bounded invertible operator $T : H \to H$ such that $Tx_k = y_k$ for all $k \in \mathbf{N}$. The two sequences $\{x_k\}$ and $\{y_k\}$ have the same moment space if the system

$$\langle x, x_k \rangle = \langle y, y_k \rangle\,, \quad k \in \mathbf{N}, \tag{3.10}$$

has a unique solution x given y, and a unique solution y whenever x is given. The solution is clearly $x = T^* y$, where T^* is the adjoint operator of T.

$(\Leftarrow)$ Assume that $\{x_k\}$ and $\{y_k\}$ have the same moment space. For a given $y \in H$ the system (3.10) then has a unique solution $x \in H$. We need to show that this implies that $\{x_k\}$ and $\{y_k\}$ are equivalent, that is, to show that there exists a bounded invertible operator $T : H \to H$ such that $Tx_k = y_k$ for all k.

Let X be the linear span of $\{x_k\}$ and $x = \sum_{k=1}^N c_k x_k$. Define an operator $T : H \to H$ by the relation

$$Tx = \sum_{k=1}^{N} c_k y_k.$$

The operator is clearly linear and maps $\{x_k\}$ to $\{y_k\}$. We need to show that T is well-defined, bounded and invertible. The space X is dense in H, so if the properties above can be shown, the operator extends uniquely to a bounded invertible operator defined on the closure of X, i.e. H.

Assume that T is not well-defined. Then there exist scalars $\{c_k\}$ such that

$$\sum_{k=1}^{N} c_k x_k = 0 \quad \text{and} \quad \sum_{k=1}^{N} c_k y_k \neq 0.$$

Choose $y = \sum_{k=1}^{N} c_k y_k$ and let x be the solution of (3.10), then

$$0 = \left\langle x, \sum_{k=1}^{N} c_k x_k \right\rangle = \sum_{k=1}^{N} \bar{c}_k \langle x, x_k \rangle = \sum_{k=1}^{N} \bar{c}_k \langle y, y_k \rangle = \left\langle y, \sum_{k=1}^{N} c_k y_k \right\rangle = \|y\|^2,$$

which is a contradiction. T is well-defined.

To show that T is bounded, we introduce a new operator $A : H \to H$ defined by the relation

$$Ay = x,$$

where x is the unique solution to (3.10). We prove that A is linear and bounded. Let $h_1, h_2 \in H$ and a_1, a_2 be scalars, then

$$\begin{aligned} \langle A(a_1 h_1 + a_2 h_2), x_k \rangle &= \langle a_1 h_1 + a_2 h_2, y_k \rangle \\ &= a_1 \langle h_1, y_k \rangle + a_2 \langle h_2, y_k \rangle \\ &= a_1 \langle Ah_1, x_k \rangle + a_2 \langle Ah_2, x_k \rangle \\ &= \langle a_1 Ah_1 + a_2 Ah_2, x_k \rangle . \end{aligned}$$

$\{x_k\}$ is complete, thus $A(a_1 h_1 + a_2 h_2) = a_1 A h_1 + a_2 A h_2$ and A is linear. Let $\{h_n\}$ be a sequence in H such that $h_n \to h$ and $Ah_n \to g$, then

$$\langle h, y_k \rangle = \lim_{n \to \infty} \langle h_n, y_k \rangle = \lim_{n \to \infty} \langle Ah_n, x_k \rangle = \langle g, x_k \rangle$$

for all k. By the definition of A, $Ah = g$, and by the closed graph theorem A is bounded. Now, let $x \in X$. For every $y \in H$ we have

$$|\langle Tx, y \rangle| = |\langle x, Ay \rangle| \leq \|x\| \|A\| \|y\|$$

and

$$\|Tx\| \leq \|x\| \|A\|,$$

thus T is bounded.

The same argument can be used if one would like to prove the existence of a bounded linear operator $S : H \to H$, such that $Sy_n = x_n$. It then follows that $ST = I = TS$ and T is therefore invertible. ∎

Proposition 3.8 *The moment space M of a Riesz basis in a Hilbert space H is equal to ℓ^2.*

Proof: If $\{x_k\}$ is a Riesz basis for H then there exists another Riesz basis $\{y_k\}$ for H, which is biorthogonal to $\{x_k\}$. Assume that $\{c_k\} \in \ell^2$, then the series $\sum_{k=1}^{\infty} c_k y_k$ converges to some element x in H, which satisfies (3.10), thus $\ell^2 \subset M$. To prove the opposite inclusion we consider the expansion $x = \sum_{k=1}^{\infty} \langle x, x_k \rangle y_k$. It is valid for every element of H, so $\{\langle x, x_k \rangle\} \in \ell^2$, thus $M \subset \ell^2$. We conclude that $M = \ell^2$. ∎

Proposition 3.8 says that the moment space of a Riesz basis is ℓ^2 and in particular that the moment space of an orthonormal basis is ℓ^2. Applying Theorem 3.7 to this situation, we see that a complete sequence in a Hilbert space is a Riesz basis if and only if its moment space is ℓ^2.

We summarize the results of this section in a chain of equivalences:

$$\Lambda \text{ is a complete interpolating sequence for } PW_\pi^2.$$

$$\Updownarrow$$

$$\text{The moment space of the complete sequence } \mathcal{E}(\Lambda) \subset L^2(-\pi,\pi) \text{ equals } \ell^2.$$

$$\Updownarrow$$

$$\text{The complete sequence } \mathcal{E}(\Lambda) \subset L^2(-\pi,\pi) \text{ is a Riesz basis.}$$

Theorem 3.5 is proved.

3.3 A complete interpolating sequence for PW_π^2

Given a sequence $\{c_k\} \in \ell^2$. The Whittaker-Kotel'nikov-Shannon theorem gives us the unique solution of the interpolation problem $f(k) = c_k$ for all $k \in \mathbf{Z}$, where f should be from the space PW_π^2. $\mathbf{Z}$ is precisely the zero set of $\sin \pi z$. In this section we prove a theorem similar to the Whittaker-Kotel'nikov-Shannon theorem, where the interpolation nodes are zeros of a function which is very much like sine.

Definition 3.9 Let f be an entire function with zero set $\Lambda = \{\lambda_k\}_{k\in\mathbf{Z}}$. f is a *sine-type function* if

1. all zeros lie in a horizontal strip, $|\text{Im } \lambda_k| \leq H$, where $H > 0$,
2. all zeros are uniformly separated, $|\lambda_k - \lambda_l| > \varepsilon > 0$, $k \neq l$, $\varepsilon > 0$,
3. for any $\delta > 0$ there exist constants depending only on δ, such that

$$|f(z)| \asymp e^{\pi|\text{Im } z|}$$

whenever $\text{dist}(z, \Lambda) > \delta$.

We need the following lemma on the derivative of a sine-type function.

Lemma 3.10 *Let S be a sine-type function and $\Lambda = \{\lambda_k\}_{k\in\mathbf{Z}}$ be its zero set. There exist constants N_1 and N_2 such that*

$$0 < N_1 < |S'(\lambda_k)| < N_2 < \infty \quad \textit{for every } k \in \mathbf{Z}.$$

Proof: See [Lev96, Chapter 22].

The next result illustrates the power of the duality between solving interpolation problems in Paley-Wiener spaces and proving properties of systems of complex exponential functions in Lebesgue spaces.

Theorem 3.11 (Levin) *The zero set $\Lambda = \{\lambda_k\}_{k\in\mathbf{Z}}$ of a sine-type function S is a complete interpolating sequence for PW_π^2.*

We need to show that for each given $\{a_k\} \in \ell^2(\mathbf{Z})$ there exists a unique function $f \in PW_\pi^2$ such that $f(\lambda_k) = a_k$ for all $k \in \mathbf{Z}$. The proof is split into two parts, that is an existence part and a uniqueness part.

Proof: (Existence.) Consider a generalized Lagrange interpolating series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)}, \tag{3.11}$$

where S is the sine-type function. This is a solution when the sum is finite, we must therefore show that the corresponding infinite sum is convergent. We define

$$f_N(z) = \sum_{k=-N}^{N} a_k \frac{S(z)}{S'(\lambda_k)(z-\lambda_k)}.$$

By the estimate for the derivative of a sine-type function from Lemma 3.10.

$$0 < N_1 < |S'(\lambda_k)| < N_2 < \infty, \tag{3.12}$$

we observe that the only possibly unbounded part of f_N is $g_N(z) = \sum_{k=-N}^{N} \frac{a_k}{z-\lambda_k}$. g_N has possible singularities in both the upper and lower half-plane $\mathbf{C}_+$ and $\mathbf{C}_-$. We make a vertical shift, $\varphi_N(z) = g_N(z - 2iH)$, where H is the constant in condition 1 in Definition 3.9, to collect all singularities in the upper half-plane. Now, the function $\varphi_N \in H_-^2$ and we can estimate its H_-^2-norm. Recall that the dual space of H_-^2 may be identified with H_+^2, and that the norm of the function φ_N is

$$\|\varphi_N\|_{H_-^2} = \sup_{\substack{\psi\in H_+^2 \\ \|\psi\|=1}} \left| \int_{-\infty}^{\infty} \varphi_N(x)\overline{\psi(x)}\,\mathrm{d}x \right|.$$

Fix some $\psi \in H_+^2$ with norm equal to one, then

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \varphi_N(x)\overline{\psi(x)}\,\mathrm{d}x \right| &= \left| \int_{-\infty}^{\infty} \sum_{k=-N}^{N} \frac{a_k}{x-2iH-\lambda_k}\overline{\psi(x)}\,\mathrm{d}x \right| \\ &\le \sum_{k=-N}^{N} \left| \int_{-\infty}^{\infty} \frac{a_k}{x-2iH-\lambda_k}\overline{\psi(x)}\,\mathrm{d}x \right|, \end{aligned}$$

where the interchange of summation and integration is justified by the dominated convergence theorem. Further, we may write

$$\sum_{k=-N}^{N}\left|\int_{-\infty}^{\infty}\frac{a_k}{x-2iH-\lambda_k}\overline{\psi(x)}\,\mathrm{d}x\right| = \sum_{k=-N}^{N}|a_k|\left|\int_{-\infty}^{\infty}\frac{\overline{\psi(x)}}{x-2iH-\lambda_k}\,\mathrm{d}x\right|$$
$$= 2\pi\sum_{k=-N}^{N}|a_k||\overline{\psi(2iH+\lambda_k)}|,$$

because a function in H^2_+ can be reconstructed from its boundary values, using the Cauchy integral. An application of the Cauchy-Schwartz inequality gives us

$$2\pi\sum_{k=-N}^{N}|a_k||\overline{\psi(2iH+\lambda_k)}| \le 2\pi\left(\sum_{k=-N}^{N}|a_k|^2\right)^{1/2}\left(\sum_{k=-N}^{N}|\overline{\psi(2iH+\lambda_k)}|^2\right)^{1/2}. \tag{3.13}$$

For the last sum on the right-hand side (3.13), we use another property of H^2_+, namely that the sum of the squared modulus of a function evaluated at a separated sequence is less than some constant times the H^2_+-norm of the function. We have

$$\sum_{k=-N}^{N}|\overline{\psi(2iH+\lambda_k)}|^2 \le C\,\|\psi\|^2_{H^2_+},$$

thus

$$2\pi\sum_{k=-N}^{N}|a_k||\overline{\psi(2iH+\lambda_k)}| \le C_1\left(\sum_{k=-N}^{N}|a_k|^2\right)^{1/2},$$

where the constant is independent of N. Since $\{a_k\}\in\ell^2(\mathbf{Z})$ the sum on the left-hand side of the equality above is convergent for any N. Let N tend to infinity, then $f_N\to f$ and $f\in PW^2_\pi$.

(Uniqueness.) We have proved that a solution exists. Assume that the solution is not unique. This means that there exist functions $f_1, f_2\in PW^2_\pi$ such that $f_1(\lambda_k)=a_k$ and $f_2(\lambda_k)=a_k$. Let us consider the function $h=f_1-f_2$, then $h(\lambda_k)=0$ for all $k\in\mathbf{Z}$. If we can show that this implies that $h\equiv 0$, we are done. Let S be a sine-type function with zero set equal to Λ. Then the function $\Phi=\frac{h}{S}$ is entire, since h is entire and $Z(S)\subset Z(h)$. We want to show that $\Phi(z)=0$ for every $z\in\mathbf{C}$. From the definition of sine-type functions we have

$$0<c_\delta e^{\pi|\operatorname{Im} z|}\le|S(z)|\le C_\delta e^{\pi|\operatorname{Im} z|}<\infty$$

for $|z-\lambda_k|>\delta>0$. Since $h\in PW^2_\pi$ we have the following estimate

$$|h(z)|\le\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{|\operatorname{Im} z|t}|\psi(t)|\,\mathrm{d}t\le Ke^{|\operatorname{Im} z|\pi},$$

hence an estimate for $|\Phi(z)|$ is

$$|\Phi(z)| = \frac{|h(z)|}{|S(z)|} \leq K/c_\delta,$$

when $|z-\lambda_k| > \delta > 0$. The function Φ is analytic in the whole complex plane and particularly inside the disks $\{z \in \mathbf{C} : |z-\lambda_k| < \delta, \lambda_k \in \Lambda\}$. By the maximum principle Φ is bounded in the set $\{z \in \mathbf{C} : |z-\lambda_k| < \delta, \lambda_k \in \Lambda\}$ as well. Liouville's theorem then implies that Φ must be constant throughout all of $\mathbf{C}$. It remains to prove that Φ is zero somewhere in $\mathbf{C}$. Since $h \in PW_\pi^2$, h must be square integrable along horizontal lines, i.e. it vanishes somewhere where S does not, implying that Φ must be identically 0. ∎

In view of Theorem 3.5, the result above can be read as follows.

Corollary 3.12 *If $\Lambda = \{\lambda_k\}$ is the zero set of a sine-type function, then $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(-\pi,\pi)$.*

3.4 More about complex sequences and relations to PW_π^2

Let us for a moment look at the situation through our signal analysis glasses. Theorem 3.3 told us that it is possible to reconstruct a bandlimited signal using regularly spaced samples of the signal. In practice, a signal is very often disturbed by some noise and it might be desirable to sampled sparsely at the noisy parts of the signal and more densely at the parts where the noise is not that present. One is led to irregular sampling and an immediate question is how dense/sparse one should sample in order to have a stable reconstruction? Let us introduce two other types of sequences, sampling sequences and interpolating sequences.

Definition 3.13 $\Lambda = (\lambda_k) \subset \mathbf{C}$ is called a *sampling sequence* for PW_π^2 if there exist constants A and B, $0 < A \leq B < \infty$, such that

$$A\,\|f\|^2 \leq \sum_{\lambda_k \in \Lambda} |f(\lambda_k)|^2 \leq B\,\|f\|^2, \quad \forall f \in PW_\pi^2.$$

If we do not know whether the solution to the interpolation problem stated in Definition 3.4 is unique or not, then Λ is called an *interpolating sequence.*

From a signal analytic point of view, sampling sequences are such that a signal can be reconstructed from its values at the points of the sequence in a stable way. The interpolation property implies that Λ is non-redundant.

It is clear that the density of the sequence Λ is important for the sampling or interpolating properties of Λ. If the sequence is too dense, it is hard for a function to fit the points, while still being from the space PW_π^2. If the sequence is too sparse it will be hard to reconstruct the function in a stable way. The Beurling densities make this reasoning precise. Let Λ be a real and separated sequence and let $n^+(r)$ and $n^-(r)$ be the maximal and minimal number of points to be found in an interval of length r. The upper and lower uniform densities of Beurling are then defined as

$$D^+(\Lambda) = \lim_{r\to\infty} \frac{n^+(r)}{r}, \quad D^-(\Lambda) = \lim_{r\to\infty} \frac{n^-(r)}{r}.$$

It can then be proved that (see [Sei04])

$$\begin{aligned} D^-(\Lambda) > 1 &\Rightarrow \Lambda \text{ sampling sequence} &&\Rightarrow D^-(\Lambda) \geq 1 \\ D^+(\Lambda) < 1 &\Rightarrow \Lambda \text{ interpolating sequence} &&\Rightarrow D^+(\Lambda) \leq 1, \end{aligned}$$

and since a complete interpolating sequence is both sampling and interpolating we see that the density of such a sequence must be uniform, i.e.

$$D^-(\Lambda) = D^+(\Lambda) = 1.$$

Ortega-Cerdà and Seip completely characterized the sampling sequences for PW_π^2 in [OCS02], while a complete characterization of the interpolating sequences is not known.

Full description of the complete interpolating sequences for PW_π^2

Consider a sequence Λ of complex numbers. Assuming that Λ was contained in a horizontal strip, Pavlov was in [Pav79] able to describe the complete interpolating sequences for PW_π^2. Nikol'skii [Nik80] improved Pavlov's result, getting rid of the upper bound on the imaginary part of points in Λ. In 1991, Minkin [Min91] described the complete interpolating sequences for PW_π^2 without any boundedness assumptions on Λ. This result was again generalized by Lyubarskii and Seip [LS97] to PW_π^p, where $1 < p < \infty$ and their description is the one stated here.

Since the sequence Λ may have elements with unbounded imaginary part, it is necessary to modify the setting of the interpolation problem a bit. For which sequences $\Lambda = \{\lambda_k\} = \{\xi_k + i\eta_k\}$ does the interpolation problem

$$f(\lambda_k) = a_k, \quad \text{for all } k$$

have a unique solution $f \in PW_\pi^p$, when $\{a_k\}$ is assumed to satisfy

$$\sum_k |a_k|^p e^{-p\pi|\eta_k|}(1 + |\eta_k|) < \infty.$$

We need some preliminaries in order to state the Lyubarskii-Seip theorem.

Definition 3.14 A sequence $\Lambda \subset \mathbf{C}$ is said to satisfy the *Carleson condition* if

$$\inf_k \prod_{k\neq l} \left|\frac{\lambda_k - \lambda_l}{\lambda_k - \bar{\lambda}_l}\right| > 0.$$

If Λ is contained in a horizontal strip, then Carleson's condition is equivalent to the separation condition, $\inf_{k\neq l}|\lambda_k-\lambda_l|>0$.

Definition 3.15 Let $Q(x,r)$ be a square centered in $x\in\mathbf{R}$ with side length $2r$ and sides parallell to the real and imaginary axis. A sequence $\Lambda\subset\mathbf{C}$ is *relatively dense* if there exists some $r_0>0$ such that $\Lambda\cap Q(x,r_0)\neq\emptyset$ for each $x\in\mathbf{R}$.

Pavlov's characterization involves boundedness of the Hilbert transform $\mathcal{H}$

$$\mathcal{H}:f\mapsto\frac{1}{\pi i}\int\frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau$$

in the weighted space

$$L^p_w(\mathbf{R})=\left\{f:\int_{\mathbf{R}}|f(x)|^p w(x)\,\mathrm{d}x<\infty\right\}.$$

There are several criterions for the Hilbert transform to be bounded in $L^p_w(\mathbf{R})$ and one of them can be formulated in terms of the Muckenhoupt (A_p) condition for the weight w:

$$\sup_I\left(\frac{1}{|I|}\int_I w(x)\,\mathrm{d}x\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1}<\infty,$$

where the supremum is over all intervals $I\subset\mathbf{R}$, see [HMW73]. Lyubarskii and Seip uses a discrete version of the Hilbert transform in their characterization, as well as a discrete version of the (A_p) condition:

$$\sup_{\substack{k\in\mathbf{Z}\\ n>0}}\left(\frac{1}{n}\sum_{j=k+1}^{k+n}w_j\right)\left(\frac{1}{n}\sum_{j=k+1}^{k+n}w_j^{-1/(p-1)}\right)^{p-1}<\infty,$$

where $\{w_k\}$ is the discrete analog of w. We will not dig into the details on how this is used in the proof, but refer to the article [LS97].

Theorem 3.16 (Lyubarskii and Seip, 1997) *Let $\Lambda=\{\lambda_k\}\subset\mathbf{C}$ and if $0\in\Lambda$, let $\lambda_0=0$. Λ is a complete interpolating sequence for PW_π^p, $1<p<\infty$, if and only if the following three conditions hold.*

1. *The sequences $\Lambda\cap\mathbf{C}_+$ and $\Lambda\cap\mathbf{C}_-$ satisfy the Carleson condition in $\mathbf{C}_+$ and $\mathbf{C}_-$ respectively, and also $\inf_{k\neq l}|\lambda_k-\lambda_l|>0$.*
2. *The limit*
$$S(z)=(z-\lambda_0)\lim_{R\to\infty}\prod_{|\lambda_k|<R,\,k\neq0}\left(1-\frac{z}{\lambda_k}\right)$$
exists and represents an entire function of exponential type π.
3. *There exists a relatively dense subsequence $\Gamma=\{\gamma_j\}\subset\Lambda$ such that the sequence $\{|S'(\gamma_j)|^p\}$ satisfies the discrete (A_p) condition.*

Defining $F(x)=|S(x)|/\mathrm{dist}(x,\Lambda)$, $x\in\mathbf{R}$, we may replace statement 3. by the following:

3'. $F^p(x)$, $x \in \mathbf{R}$, satisfies the continuous (A_p) condition.

Let us check that this agrees with our observations in Section 3.3.

Example 3.17 We use Theorem 3.16 to check that the zero set Λ of a sine-type function forms a complete interpolating sequence for PW_π^2. Λ is located in a horizontal strip and separated, thus the first condition is ok. The second condition is fulfilled since S is an entire function of exponential type π by definition and therefore it has a representation in terms of an infinite product according to the factorization theorem of Hadamard. We may shift Λ vertically, such that $0 < h \leq \operatorname{Im}\lambda_k \leq H < \infty$, h and H are some constants, without affecting the interpolation property. Fix some interval $I \subset \mathbf{R}$, then

$$\frac{1}{|I|^2}\int_I \frac{|S(x)|^2}{\operatorname{dist}(x,\Lambda)^2}\,\mathrm{d}x \int_I \frac{\operatorname{dist}(x,\Lambda)^2}{|S(x)|^2}\,\mathrm{d}x \leq C\int_I |S(x)|^2\,\mathrm{d}x \int_I |S(x)|^{-2}\,\mathrm{d}x. \tag{3.14}$$

S has no zeros on the real line, hence $|S(x)|^{-2}$ is bounded on for all $x \in \mathbf{R}$. Moreover, S is bounded on the real line, so the right-hand side of (3.14) is bounded. Λ satisfies the conditions in Thorem 3.16 and is indeed a complete interpolating sequence for PW_π^2. ▲

3.5 Block interpolation in one variable

The material in this section is based on the last section of the article [LR00] and notes by Yurii Lyubarskii. A similar result was obtained in [AI01] with a different approach.

Let $S : \mathbf{C} \to \mathbf{C}$ be an entire function and $\Lambda = \{\lambda\}$ its zero set. We assume that all zeros are simple and that S satisfies the following conditions:

1. $0 < 4h \leq |\operatorname{Im}\lambda| \leq H < \infty$
2. For each $\varepsilon > 0$, $|S(z)| \asymp e^{\pi|\operatorname{Im} z|}$ when $\operatorname{dist}(z,\Lambda) > \varepsilon$.

The zeros are for simplicity assumed to lie in a strip in the upper half-plane, because we will work with the $L^2(\mathbf{R})$-norm of rational functions with singularities at Λ. This assumption does not affect the generality of the problem, since functions from PW_π^2 are square integrable on any horizontal line. The number 4 is included for further convenience. Observe that S satisfies two out of three conditions for being a sine-type function. If S satisfies the separation condition

$$|\lambda_i - \lambda_j| > \varepsilon, \quad \varepsilon > 0,$$

for all $\lambda_i, \lambda_j \in \Lambda$, then Theorem 3.11 tells us that Λ is a complete interpolating sequence for PW_π^2. In this section we assume that S does not satisfy the separation condition and we will show that it is still possible to extract a Riesz basis for $L^2(-\pi,\pi)$ from Λ. The proof will be more involved than the proof of Theorem 3.11. Let us start

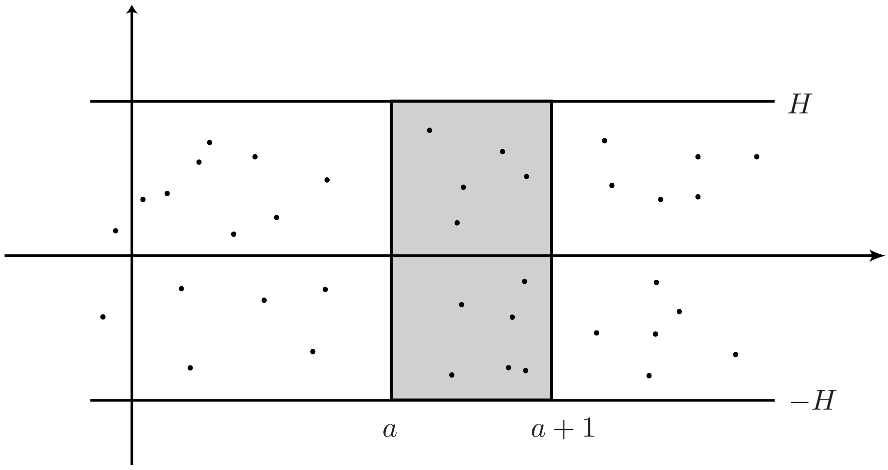

Figure 3.1: There are at most N zeros in each such rectangle, taking their multiplicity into account.

with some preliminary lemmas. We divide Λ into smaller pieces, called blocks and estimate the size of the blocks and the number of zeros in each block.

Lemma 3.18 *For all $a \in \mathbf{R}$ there exists a number $N > 0$ such that*

$$\sup_{a\in\mathbf{R}} \#\{\lambda \in \Lambda : \ a \le \operatorname{Re}\lambda \le a+1\} < N.$$

The lemma is illustrated in Figure 3.1 and a proof can be found in [Lev96].

Corollary 3.19 *There exists a partition $\{\Lambda_k\}$ of Λ such that $\Lambda = \cup_k \Lambda_k$ and*

1. $\sup_k M_k = M < \infty$, *where* $M_k = \#\Lambda_k$,
2. *for each k, there exists some $\varepsilon > 0$ such that*

$$4\varepsilon + \max_{\lambda\in\Lambda_k} \operatorname{Re}\lambda \le \min_{\lambda\in\Lambda_{k+1}} \operatorname{Re}\lambda,$$

3. $\sup_k (\operatorname{diam}\Lambda_k) = \sup_k \left(\sup_{\lambda_i,\lambda_j\in\Lambda_k} |\lambda_i - \lambda_j|\right) = D < \infty.$

To each $f \in PW_\pi^2$ we associate a Lagrangian interpolation series

$$f(z) \sim \sum_{\lambda\in\Lambda} f(\lambda) \frac{S(z)}{S'(\lambda)(z-\lambda)}.$$

We write $\sim$ since we do not know anything about convergence yet. Using the partition into blocks, we may write

$$f(z) \sim \sum_k \sum_{\lambda\in\Lambda_k} f(\lambda) \frac{S(z)}{S'(\lambda)(z-\lambda)} = \sum_k G_k(z).$$

The inner sums converge, since they are finite sums and none of their terms blow up.

The two main theorems of this section are the following.

Theorem 3.20 *Let G_k, Λ, Λ_k and f be as above, then*

$$\sum_k \|G_k\|^2_{L^2(\mathbf{R})} \asymp \|f\|^2_{PW^2_\pi}$$

and $f(z) = \sum_k G_k(z)$, with convergence both in the sense of PW^2_π and uniformly on compact subsets of $\mathbf{C}$.

Theorem 3.21 *Let S, Λ and Λ_k be as above and*

$$P_k(z) = \prod_{\lambda\in\Lambda_k} (z-\lambda),$$

where $\deg P_k = M_k = \#\Lambda_k$. Moreover, let $\{Q_k(z)\}$ be a sequence of polynomials, where $\deg Q_k < M_k$, such that

$$\sum_k \left\| \frac{Q_k}{P_k} \right\|^2_{L^2(\mathbf{R})} < \infty.$$

Let $r_k = Q_k/P_k$. For some $f \in PW^2_\pi$ we then have

$$f(z) = \sum_k S(z) r_k(z)$$

in the sense of PW^2_π and

$$\|f\|^2_{PW^2_\pi} \asymp \sum_k \|r_k\|^2_{L^2(\mathbf{R})}.$$

The constants in the double inequality are independent of the choice of the polynomials Q_k as long as $\deg Q_k < M_k$.

Define a sequence of subspaces X_k by

$$X_k = \left\{ f \in \mathrm{Hol}(\mathbf{C}) : \; f(z) = S(z)\frac{Q_k(z)}{P_k(z)} \right\}. \tag{3.15}$$

It is clear that if Theorem 3.21 holds, then the sequence X_k defined by (3.15) forms a Riesz basis of subspaces for PW^2_π, which in turn can be mapped to a Riesz basis of subspaces for $L^2(-\pi,\pi)$ using the Fourier transform.

We need quite a lot of preparations in order to the proofs. First, we state some important theorems without proofs.

Theorem 3.22 (Minkowski) *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and let f be an $\mathcal{M}\times\mathcal{N}$-measurable function on $X \times Y$.*

1. *If $f \geq 0$ and $1 \leq p < \infty$, then*

$$\left[\int \left(\int f(x,y)\,\mathrm{d}\nu(y)\right)^p \mathrm{d}\mu(x)\right]^{1/p} \leq \int \left[\int f(x,y)^p\,\mathrm{d}\mu(x)\right]^{1/p} \mathrm{d}\nu(y).$$

2. *If $1 \leq p \leq \infty$, $f(\cdot,y) \in L^p(\mu)$ for almost every y, and the function $y \mapsto \|f(\cdot,y)\|_p$ is in $L^1(\nu)$, then $f(x,\cdot) \in L^1(\nu)$ for almost every x, then function $x \mapsto \int f(x,y)\,\mathrm{d}\nu(y)$ is in $L^p(\mu)$, and*

$$\left\|\int f(\cdot,y)\,\mathrm{d}\nu(y)\right\|_p \leq \int \|f(\cdot,y)\|_p\,\mathrm{d}\nu(y).$$

A proof can be found in [Fol99].

Definition 3.23 A nonnegative Borel measure μ is a *Carleson measure* on the upper half-plane if for all $x \in \mathbf{R}$ and all $h > 0$

$$\mu\left((x, x+h) \times (0,h)\right) \leq Ch,$$

where C is independent of x and h. The smallest such C is called the Carleson norm of μ.

Theorem 3.24 (Carleson embedding theorem) *A nonnegative Borel measure μ is a Carleson measure if and only if there exists a constant C such that*

$$\left\{\int_{\mathbf{C}_+} |f(z)|^2\,\mathrm{d}\mu(z)\right\}^{1/2} \leq C\|f\|_{H^2_+},$$

for all $f \in H^2_+$.

A proof of this theorem is given in [Koo80]. A simpler version of this result was obtained by Gabriel in [Gab35] and the case we will consider is actually simple enough to be proved by hands. However, we will refer to Carleson's embedding theorem in order to be short.

The following results about rational functions will be crucial in the proof of Theorem 3.21.

Lemma 3.25 *Let $r : \mathbf{C} \to \mathbf{C}$ be a rational function with the property that*

$$\lim_{|z|\to\infty} r(z) = 0.$$

Let γ be a simple closed curve such that all the poles of r lies in the interior of γ. Moreover, let Γ^+ and Γ^- be the bounded and unbounded parts of $\mathbf{C} \setminus \gamma$. Then

$$r(z) = \frac{1}{2\pi i}\int_\gamma \frac{r(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \quad z \in \Gamma^-.$$

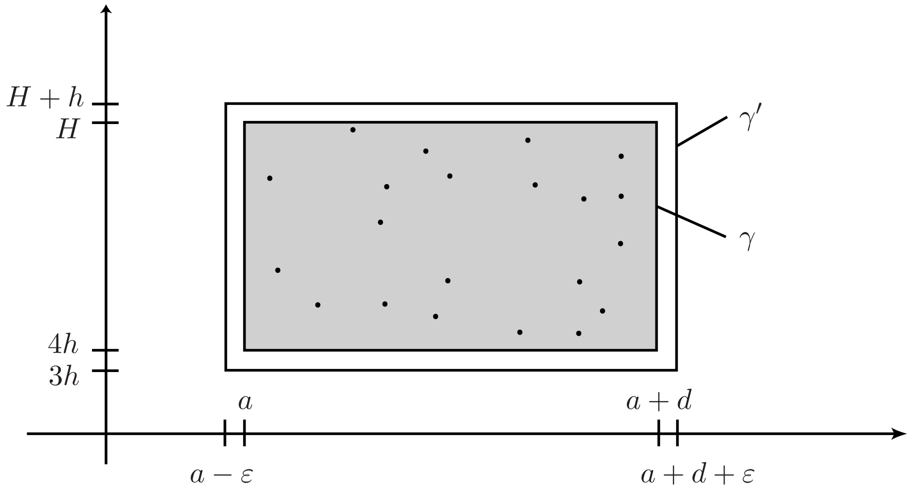

Figure 3.2: Illustration of the construction in Lemma 3.26.

Proof: This is just a special version of the residue theorem. ∎

Recall that the degree of a rational function is the maximum of the degree of the numerator and the degree of the denominator.

Lemma 3.26 *Let r be a rational function such that $\lim_{|z|\to\infty} r(z) = 0$ and $\|r\|_{L^2(\mathbf{R})} = K$. Let $\deg r \leq M$. Since $\lim_{|z|\to\infty} r(z) = 0$, the degree of r is the degree of the denominator. Assume that all poles of r are located inside the rectangle*

$$\gamma = [a+4ih, a+d+4ih] \cup [a+d+4ih, a+d+iH] \cup [a+4ih, a+iH] \\ \cup [a+iH, a+d+iH],$$

where h and H are fixed, $0 < d < D = \sup_k \operatorname{diam} \Lambda_k$ and $a \in \mathbf{R}$. Also, let $\varepsilon > 0$ and

$$\gamma' = [a-\varepsilon+3ih, a+d+\varepsilon+3ih] \cup [a+d+\varepsilon+3ih, a+d+\varepsilon+i(H+h)] \\ \cup [a-\varepsilon+3ih, a-\varepsilon+i(H+h)] \\ \cup [a-\varepsilon+i(H+h), a+d+\varepsilon+i(H+h)].$$

We then have

$$\|r\|_{L^2(\gamma')} = \left(\int_{\gamma'} |r(z)|^2 \, |\mathrm{d}z| \right)^{1/2} \leq AK,$$

where A depends on M, ε, h and D, but is independent of r, a and d.

The construction in the lemma is shown in Figure 3.2.

Proof: Let $m = \deg r$ and $\lambda_1, \lambda_2, ..., \lambda_m$ be the poles of r. Define $p(z) = \prod_{j=1}^m (z - \lambda_j)$, such that $r(z) = q(z)/p(z)$ for some polynomial q of degree less than or equal to $m-1$.

Let $\bar{p}(z) = \overline{p(\bar{z})} = \prod_{j=1}^{m}(z - \bar{\lambda}_j)$ and $r_1(z) = r(z)p(z)/\bar{p}(z) = q(z)/\bar{p}(z)$. We see that

$$\left|\frac{p(x)}{\bar{p}(x)}\right| = 1 \quad \Rightarrow \quad \|r_1\|_{L^2(\mathbf{R})} = K.$$

The function $r_1 \in H^2_+$, since $\bar{\lambda}_j \in \mathbf{C}_-$, for $j = 1, ..., m$. Now, for any Lebesgue measurable $A \subset \mathbf{C}_+$ define a measure μ by

$$\mu(A) = \text{length}(A \cap \gamma').$$

Observe that μ is a Carleson measure whose Carleson norm is bounded by some constant C_1, which only depends on ε, d, h and H. We then have

$$\|r_1\|^2_{L^2(\gamma')} = \int_{\gamma'} |r_1(z)|^2 \, |\mathrm{d}z| = \int_{\gamma'} |r_1(z)|^2 \, \mathrm{d}\mu(z) \leq BC_1 \|r_1\|^2_{H^2_+}$$

where the last inequality is justified by Carleson's embedding theorem (Theorem 3.24) and B is a positive constant. Moreover,

$$\|r_1\|_{H^2_+} = \|r_1\|_{L^2(\mathbf{R})} = K,$$

hence $\|r_1\|^2_{L^2(\gamma')} \leq BC_1\sqrt{K}$. An estimate for $\|r_1\|_{L^2(\gamma')}$ is obtained, but what we really want is an estimate for $\|r\|_{L^2(\gamma')}$. We have

$$\|r\|_{L^2(\gamma')} = \left\|r_1 \frac{\bar{p}}{p}\right\|_{L^2(\gamma')}$$

and since

$$\min_{z\in\gamma'} |p(z)| \geq (\min(\varepsilon, h))^m \quad \text{and}$$
$$\max_{z\in\gamma'} |\bar{p}(z)| \leq (2H + 2h + 2\varepsilon + D)^m,$$

we get that $|\bar{p}(z)/p(z)| \leq C_2$ on γ'. Hence

$$\|r\|_{L^2(\gamma')} \leq C_2 \|r_1\|_{L^2(\gamma')} \leq BC_3 K,$$

where C_3 is depending on H, h, ε, D and M. ∎

We are now ready to prove Theorem 3.21.

Proof of Theorem 3.21: From the assumptions in the theorem we have $r_k = Q_k/P_k$ and $\sum_{k=-\infty}^{\infty} \|r_k\|^2_{L^2(\mathbf{R})} < \infty$. Now, choose the sequence of rectangles

$$\gamma_k = [a_k + 4ih, a_k + d_k + 4ih] \cup [a_k + d_k + 4ih, a_k + d_k + iH] \cup [a_k + 4ih, a_k + iH]$$
$$\cup [a_k + iH, a_k + d_k + iH]$$

such that $\Lambda_k \subset \operatorname{int}(\gamma_k)$, $a_k+d_k+3\varepsilon < a_{k+1}$ and $d_k \le D+\varepsilon/2$. Let γ_k' be obtained from γ_k as it was done in Lemma 3.26 and write $R_k(z) = S(z)r_k(z)$. Then $\sum_{k=m}^n R_k(x) = S(x)\sum_{k=m}^n r_k(x)$ and by the growth assumption on S we have $|S(x)| \asymp 1$ for all $x \in \mathbf{R}$. We would like to prove that

$$\left\| \sum_{k=m}^n R_k \right\|_{L^2(\mathbf{R})}^2 \le C \sum_{k=m}^n \|r_k\|_{L^2(\mathbf{R})}^2 . \tag{3.16}$$

Let us start estimating $h_{m,n}(x) = \sum_{k=m}^n r_k(x)$. The function $h_{m,n}$ has poles in $\mathbf{C}_+$ and

$$\lim_{|z|\to\infty} h(z) = 0,$$

so $h_{m,n} \in H^2_-$. The dual space of H^2_- may be identified with H^2_+, hence $h_{m,n}$ may be considered as a bounded linear functional on H^2_+. We use this to estimate the norm of $h_{m,n}$

$$\|h_{m,n}\|_{L^2(\mathbf{R})} = \|h_{m,n}\|_{H^2_-} = \sup_{\substack{\varphi\in H^2_+\\ \|\varphi\|=1}} \left| \int_{-\infty}^{\infty} h_{m,n}(x)\varphi(x)\,\mathrm{d}x \right| .$$

Let φ be a function from H^2_+, then

$$\left| \int_{-\infty}^{\infty} h_{m,n}(x)\varphi(x)\,\mathrm{d}x \right| \le \sum_{k=m}^n \left| \int_{-\infty}^{\infty} r_k(x)\varphi(x)\,\mathrm{d}x \right| .$$

By Lemma 3.25 we get

$$\sum_{k=m}^n \left| \int_{-\infty}^{\infty} r_k(x)\varphi(x)\,\mathrm{d}x \right| \le \sum_{k=m}^n \left| \int_{-\infty}^{\infty} \frac{1}{2\pi i} \int_{\gamma_k'} \frac{r_k(\zeta)}{\zeta - x}\,\mathrm{d}\zeta\, \varphi(x)\,\mathrm{d}x \right|$$

and Fubini's theorem justifies the change of the order of integration

$$\sum_{k=m}^n \left| \int_{-\infty}^{\infty} \frac{1}{2\pi i} \int_{\gamma_k'} \frac{r_k(\zeta)}{\zeta - x}\,\mathrm{d}\zeta\, \varphi(x)\,\mathrm{d}x \right| = \sum_{k=m}^n \left| \int_{\gamma_k'} r_k(\zeta) \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(x)}{\zeta - x}\,\mathrm{d}x\,\mathrm{d}\zeta \right| .$$

Recall that a function $\varphi \in H^2_+$ can be reconstructed from its boundary values using the Cauchy integral, thus

$$\begin{aligned} \sum_{k=m}^n \left| \int_{\gamma_k'} r_k(\zeta) \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(x)}{\zeta - x}\,\mathrm{d}x\,\mathrm{d}\zeta \right| &= \sum_{k=m}^n \left| \int_{\gamma_k'} r_k(\zeta)\varphi(\zeta)\,\mathrm{d}\zeta \right| \\ &\le \sum_{k=m}^n \|r_k\|_{L^2(\gamma_k')} \|\varphi\|_{L^2(\gamma_k')}, \end{aligned}$$

where the last step is justified by the Cauchy-Schwarz inequality. Using Lemma 3.26 and then applying Cauchy-Schwarz once again we get

$$\begin{aligned}\sum_{k=m}^{n} \|r_k\|_{L^2(\gamma_k')}\|\varphi\|_{L^2(\gamma_k')} &\le C \sum_{k=m}^{n} \|r_k\|_{L^2(\mathbf{R})}\|\varphi\|_{L^2(\gamma_k')} \\ &\le C \left(\sum_{k=m}^{n} \|r_k\|^2_{L^2(\mathbf{R})}\right)^{1/2} \left(\sum_{k=m}^{n} \|\varphi\|^2_{L^2(\gamma_k')}\right)^{1/2}.\end{aligned}$$

We need to estimate the last factor

$$\sum_{k=m}^{n} \|\varphi\|^2_{L^2(\gamma_k')} = \sum_{k=m}^{n} \int_{\gamma_k'} |\varphi(\zeta)|^2 \,|\mathrm{d}\zeta| \le \int_{\bigcup_{k=-\infty}^{\infty}} |\varphi(\zeta)|^2 \,|\mathrm{d}\zeta|.$$

The measure μ generated by the system of curves $\bigcup_{k=-\infty}^{\infty} \gamma_k'$, such that for any $A \subset \mathbf{C}_+$

$$\mu(A) = \text{length}\left(A \cap \left(\bigcup_{k=-\infty}^{\infty} \gamma_k'\right)\right),$$

is a Carleson measure. If now $\|\varphi\|_{H^2_+} = 1$, Carleson's embedding theorem implies that

$$\int_{\bigcup_{k=-\infty}^{\infty}} |\varphi(\zeta)|^2 \,|\mathrm{d}\zeta| \le C_1.$$

We have arrived at

$$\left\|\sum_{k=m}^{n} R_k(x)\right\|^2_{L^2(\mathbf{R})} \le C_2 \sum_{k=m}^{n} \|r_k\|^2_{L^2(\mathbf{R})},$$

where C_2 does not depend upon m and n. Let $m < n$ and let $m, n \to \infty$ and then $m, n \to -\infty$ to get convergence in $L^2(\mathbf{R})$. To get the desired estimate (3.16) we let $m \to -\infty$ and $n \to \infty$. ∎

Proof of Theorem 3.20: Let γ_k' be as above. The residue theorem gives us

$$\begin{aligned}G_k(z) &= S(z) \sum_{\lambda \in \Lambda_k} \frac{f(\lambda)}{S'(\lambda)(z-\lambda)} \\ &= S(z) \sum_{\lambda \in \Lambda_k} \mathrm{Res}\left[\frac{f(\zeta)}{S(\zeta)(z-\zeta)}, \lambda\right] \\ &= \frac{S(z)}{2\pi i} \int_{\gamma_k'} \frac{f(\zeta)}{S(\zeta)(z-\zeta)}\,\mathrm{d}\zeta\end{aligned}$$

for (at least) $z \in \mathbf{C} \setminus (\gamma_k' \bigcup \mathrm{int}(\gamma_k'))$ and in particular for $x \in \mathbf{R}$, such that

$$G_k(x) = \frac{S(x)}{2\pi i} \int_{\gamma_k'} \frac{f(\zeta)}{S(\zeta)(x-\zeta)}\,\mathrm{d}\zeta. \tag{3.17}$$

Denote the right hand side of (3.17) by f_k. We would like to prove that $f_k(z) = G_k(z)$ for all $z \in \mathbf{C}$. To do that we prove the following

- f_k is entire
- $f_k(\lambda) = f(\lambda), \quad \lambda \in \Lambda_k$
- $f_k(\lambda) = 0, \quad \lambda \in \Lambda \setminus \Lambda_k$.

The function f_k is of exponential type less than or equal to π and square integrable on the real line. If all the three points above are proved, then $f_k|_\Lambda = G_k|_\Lambda$ and by the uniqueness part of Theorem 3.11, they are equal in the whole complex plane.

We start by proving that f_k is entire. It is possible to choose $\varepsilon > 0$ such that $\{|\zeta - \lambda| < \varepsilon\}_{\lambda \in \Lambda_k}$ is a disjoint family of disks, containing Λ_k. The singularities of f_k are the points $\lambda \in \Lambda_k$, thus we may write

$$f_k(z) = \frac{S(z)}{2\pi i} \sum_{\lambda \in \Lambda_k} \int_{|\zeta - \lambda| = \varepsilon} \frac{f(\zeta)}{S(\zeta)(z - \zeta)} \, \mathrm{d}\zeta. \tag{3.18}$$

Since Λ_k is finite, the singularities are isolated. If we can show that f_k is bounded near each $\lambda \in \Lambda_k$, we can use Riemann's theorem on removable singularities. Fix $\lambda \in \Lambda_k$ and consider z close to λ. Let $\varepsilon = \frac{1}{2}|z - \lambda|$.

$$\frac{S(z)}{2\pi i} \int_{|\zeta - \lambda| = \varepsilon} \frac{f(\zeta)}{S(\zeta)(z - \zeta)} \, \mathrm{d}\zeta$$

is the only term of (3.18) which contributes to the growth of f_k near λ. We have

$$|f_k(z)| = \frac{|S(z)|}{2\pi} \left| \int_{|\zeta - \lambda| = \varepsilon} \frac{f(\zeta)}{S(\zeta)(z - \zeta)} \, \mathrm{d}\zeta \right|.$$

Observe that $|\zeta - z| \asymp |z - \lambda|$ and that the length of the path of integration is approximately $|z - \lambda|$, thus

$$\frac{|S(z)|}{2\pi} \left| \int_{|\zeta - \lambda| = \varepsilon} \frac{f(\zeta)}{S(\zeta)(z - \zeta)} \, \mathrm{d}\zeta \right| \leq C_1 |S(z)| \max_{|\zeta - \lambda| = \varepsilon} \frac{|f(\zeta)|}{|S(\zeta)|} \leq C.$$

f_k is bounded near λ and by Riemann's theorem on removable singularites, f_k is entire.

The two other points above is ok because of Cauchy's integral theorem. The integral representation in (3.17) is therefore valid for any $z \in \mathbf{C}$ and we may write G_k instead of f_k.

We will now prove the following:

$$\|G_k\|_{L^2(\mathbf{R})} \leq C \|f\|_{L^2(\gamma_k')}. \tag{3.19}$$

S is bounded on the real line, so we must prove

$$\left\| \int_{\gamma_k'} \frac{f(\zeta)}{S(\zeta)(\cdot - \zeta)} \, \mathrm{d}\zeta \right\|_{L^2(\mathbf{R})} \leq C \|f\|_{L^2(\gamma_k')}.$$

When $\zeta \in \gamma_k'$, then

$$\left\| \frac{1}{\cdot - \zeta} \right\|_{L^2(\mathbf{R})} \leq \text{constant}.$$

From Minkowskii's inequality for integrals (Theorem 3.22) we get

$$\begin{aligned}
\left\| \int_{\gamma_k'} \frac{f(\zeta)}{S(\zeta)} \frac{\mathrm{d}\zeta}{\cdot - \zeta} \right\|_{L^2(\mathbf{R})} &\leq \left(\int_{\mathbf{R}} \left[\int_{\gamma_k'} \left| \frac{f(\zeta)}{S(\zeta)} \right| \frac{|\mathrm{d}\zeta|}{|x - \zeta|} \right]^2 \mathrm{d}x \right)^{1/2} \\
&\leq \int_{\gamma_k'} \left[\int_{\mathbf{R}} \left| \frac{f(\zeta)}{S(\zeta)} \right|^2 \frac{\mathrm{d}x}{|x - \zeta|^2} \right]^{1/2} |\mathrm{d}\zeta| \\
&= \int_{\gamma_k'} \left| \frac{f(\zeta)}{S(\zeta)} \right| \left\| \frac{1}{\cdot - \zeta} \right\|_{L^2(\mathbf{R})} |\mathrm{d}\zeta|,
\end{aligned}$$

and from the Cauchy-Schwarz inequality we get

$$\int_{\gamma_k'} \left| \frac{f(\zeta)}{S(\zeta)} \right| \left\| \frac{1}{\cdot - \zeta} \right\|_{L^2(\mathbf{R})} |\mathrm{d}\zeta| \leq C \left\| \frac{f(\zeta)}{S(\zeta)} \right\|_{L^2(\gamma_k')} \left(\int_{\gamma_k'} |\mathrm{d}\zeta| \right)^{1/2} \leq C_1 \left\| \frac{f(\zeta)}{S(\zeta)} \right\|_{L^2(\gamma_k')},$$

since the length of γ_k' is bounded. S is uniformly bounded from below on $\bigcup_k \gamma_k'$, thus the estimate (3.19) is obtained. We are left to show that

$$\sum_k \|f\|^2_{L^2(\gamma_k')} \leq \|f\|^2_{PW^2_\pi}.$$

$e^{i\pi z}$ is bounded from above and below on $\bigcup_k \gamma_k'$, so $\sum_k \|f\|^2_{L^2(\gamma_k')}$ is bounded if $\sum_k \|e^{i\pi z} f\|^2_{L^2(\gamma_k')}$ is bounded. We define $\Phi(z) = e^{i\pi z} f(z)$. The function Φ is from H^2_+, since $e^{i\pi z} PW^2_\pi$ is a closed subspace of H^2_+, more precisely $e^{i\pi z} PW^2_\pi = H^2_+ \ominus e^{2i\pi z} H^2_+$ (see [Sei04, chapter 5]). The H^2_+-norm of Φ is $\|\Phi\|_{H^2_+} = \|f\|_{L^2(\mathbf{R})}$. Now,

$$\sum_k \|f\|^2_{L^2(\gamma_k')} \leq C \sum_k \|\Phi\|^2_{L^2(\gamma_k')} = \sum_k \int_{\gamma_k'} |\Phi(z)|^2 \, |\mathrm{d}z| \leq \int_{\cup_k \gamma_k'} |\Phi(z)|^2 \, |\mathrm{d}z|.$$

For any Lebesgue measurable $A \subset \mathbf{C}_+$, define a measure μ by

$$\mu(A) = \text{length} \left(A \cap \left(\bigcup_k \gamma_k' \right) \right).$$

The measure μ is clearly a Carleson measure in the upper half-plane, and we may apply Carleson's embedding theorem. We have

$$\int_{\cup_k \gamma_k'} |\Phi(z)|^2 \, |\mathrm{d}z| = \int_{\mathbf{C}_+} |\Phi(z)|^2 \, \mathrm{d}\mu(z) \leq C \, \|\Phi\|_{H^2_+}^2$$

and conclude that

$$\sum_k \|f\|_{L^2(\gamma_k')}^2 \leq C \|\Phi\|_{H^2(\mathbf{C}^+)}^2 = C \|f\|_{L^2(\mathbf{R})}^2.$$

Uniform convergence on compact sets follows by the same argument as in the proof of Theorem 3.3. ∎

3.6 Paley-Wiener spaces in several variables

In this section we introduce Paley-Wiener spaces of entire functions of several complex variables and results analogous to the some of the results obtained in Sections 3.1 and 3.2.

Definition 3.27 Given a bounded domain $\Omega \subset \mathbf{R}^n$, we define the corresponding Paley-Wiener space by the relation

$$PW_\Omega^2 = \left\{ f : \mathbf{C}^n \to \mathbf{C};\ f(z) = \frac{1}{(2\pi)^n} \int_\Omega e^{i\langle z,w\rangle} \varphi(w) \, \mathrm{d}m_w,\ \varphi \in L^2(\Omega) \right\}.$$

We would like to present a several-dimensional version of the Paley-Wiener theorem called the Plancherel-Polya theorem. To do this we need some definitions.

Definition 3.28 Let Ω be a convex set in $\mathbf{R}^n$. Its *supporting function* is defined as

$$H_\Omega(y) = \sup_{\xi \in \Omega} \langle y, \xi \rangle .$$

A geometrical interpretation of the supporting function when $|y| = 1$ is as follows. $\langle y, \xi \rangle$ is the projection of ξ onto the line spanned by y. $H_\Omega(y)$ is therefore the distance to the closest hyperplane P perpendicular to y such that the half-space $\{\xi \in \mathbf{R}^n : \langle y, \xi \rangle > H_\Omega(y)\}$, whose boundary is P, contains no points of Ω. An illustration of the idea is shown in Figure 3.3. Plancherel and Polya introduced a function called the P-indicator of a function f. It describes the growth of f in the imaginary part of $\mathbf{C}^n$.

Definition 3.29 Let α be a point on the unit sphere of $\mathbf{R}^n$ and let

$$h_f(\alpha; x) = \limsup_{R \to \infty} \frac{1}{R} \log |f(x_1 + i\alpha_1 R, ..., x_n + i\alpha_n R)| .$$

The *P-indicator* of f is defined as

$$h_f(\alpha) = \sup_{x \in \mathbf{R}^n} h_f(\alpha; x).$$

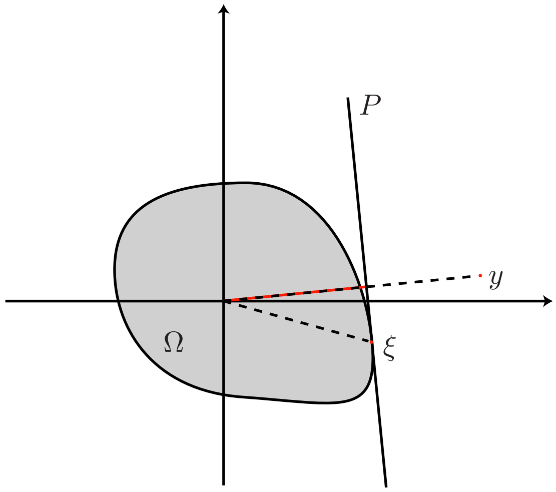

Figure 3.3: The length of the red and black dotted line is $H_\Omega(y)$.

Theorem 3.30 (Plancherel-Polya) *A function $f : \mathbf{C}^n \to \mathbf{C}$ is an entire function of exponential type and in $L^2(\mathbf{R}^n)$ if and only if there is a bounded domain $\Omega_\varphi \subset \mathbf{R}^n$ and a function $\varphi \in L^2(\Omega_\varphi)$ such that*

$$f(z) = \frac{1}{(2\pi)^n} \int_{\Omega_\varphi} e^{i\langle z,w\rangle} \varphi(w)\, \mathrm{d}m_w. \tag{3.20}$$

When this representation is valid, the supporting function of the convex hull of Ω_φ coincides with the P-indicator of the function f.

We present the proof given in [Ron74], but first we need a lemma.

Lemma 3.31 *Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function of exponential type satisfying the following conditions:*

1. $\limsup_{y\to\infty} y^{-1} \log |f(iy)| = \sigma$
2. $f \in L^1(\mathbf{R})$.

Then the Fourier transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-itx}\, \mathrm{d}x$$

vanishes for $t > \sigma$.

Proof: The function f must be less than some constant M on the real line. Then $|f|^2 \leq M\,|f|$ on the real line, which implies that f is from $L^2(\mathbf{R})$. The Paley-Wiener theorem applies and $\hat{f}$ vanishes for $t > \sigma$. ∎

Proof of Theorem 3.30: ($\Leftarrow$) Let $\varphi \in L^2(\Omega)$ and $\varphi \equiv 0$ in $\mathbf{R}^n \setminus \Omega$, where Ω is some bounded convex domain. Then φ is also from $L^1(\Omega)$ and the integral

$$\int_\Omega e^{i\langle z,w\rangle}\varphi(w)\,\mathrm{d}m_w$$

makes sense. Suppose that

$$f(z) = \frac{1}{(2\pi)^n}\int_\Omega e^{i\langle z,w\rangle}\varphi(w)\,\mathrm{d}m_w. \tag{3.21}$$

We would like to prove that f is an entire function of exponential type and that f is from $L^2(\mathbf{R}^n)$. That $f \in L^2(\mathbf{R}^n)$ follows from Plancherel's theorem. Moreover, the function f is entire since the integral in (3.21) is absolutely and uniformly convergent in any bounded subset of $\mathbf{C}^n$. To prove the growth property, we estimate the function f. Let α be a point on the unit sphere of $\mathbf{R}^n$ and let $R \in \mathbf{R}$, then

$$\begin{aligned}
|f(x+i\alpha R)| &= \left|\frac{1}{(2\pi)^n}\int_\Omega e^{i\langle x,w\rangle}e^{-R\langle y,w\rangle}\varphi(w)\,\mathrm{d}m_w\right| \\
&\le \frac{1}{(2\pi)^n}\int_\Omega e^{-R\langle y,w\rangle}|\varphi(w)|\,\mathrm{d}m_w \\
&\le \frac{1}{(2\pi)^n}\exp\left(R\sup_{w\in\Omega}\langle\alpha,w\rangle\right)\int_\Omega|\varphi(w)|\,\mathrm{d}m_w \\
&\le C\exp\left(RH_\Omega(\alpha)\right).
\end{aligned} \tag{3.22}$$

For any $z \in \mathbf{C}^n$ we then have

$$|f(z)| \le Ce^{a|y|} \le Ce^{a|z|},$$

where $a = \sup_{|\alpha|=1} H_\Omega(\alpha)$, thus f is of exponential type.

From (3.22) we see that $h_f(\alpha;x) \le H_\Omega(\alpha)$, then

$$h_f(\alpha) \le H_\Omega(\alpha). \tag{3.23}$$

We did not use any other properties of Ω, other than that φ vanishes identically on the complement, thus (3.23) holds for the convex hull $ch\Omega_\varphi$ of Ω_φ, i.e.

$$h_f(\alpha) \le H_{ch\Omega_\varphi}(\alpha). \tag{3.24}$$

($\Rightarrow$) Suppose $f : \mathbf{C}^n \to \mathbf{C}$ is an entire function of exponential type and from $L^2(\mathbf{R}^n)$. Assume first that $f \in L^1(\mathbf{R}^n)$, then its Fourier transform

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-i\langle\xi,x\rangle}\,\mathrm{d}m_x \tag{3.25}$$

exists as an ordinary Lebesgue integral. Consider a straight line through the origin and let this line be the x'_1-axis in the new coordinate system $x'_1, ..., x'_n$. Also, let $\gamma_{p,q}$ be the cosine of the angle between the x'_q-axis and the x_p-axis. Then

$$x_p = \sum_{q=1}^{n} x'_q \gamma_{p,q}.$$

Define a new function

$$g(x') = f(a_1 + \gamma_{1,1} x'_1, ..., a_n + \gamma_{n,1} x'_n),$$

where

$$a_p = \sum_{q=2}^{n} x'_q \gamma_{p,q}, \quad p = 1, ..., n.$$

Consider the integral

$$\int_{\mathbf{R}^n} |g(x')| \, \mathrm{d}m_x.$$

It is finite since $f \in L^1(\mathbf{R}^n)$ and Fubini's theorem tells us that

$$\int_{-\infty}^{\infty} |g(x')| \, \mathrm{d}x'_1$$

exists for almost every $(x'_2, ..., x'_n) \in \mathbf{R}^{n-1}$. Apply the same change of coordinates to the integral in (3.25), then

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbf{R}^n} f(x) e^{-i\langle \xi, x\rangle} \, \mathrm{d}m_x \\ &= \int_{\mathbf{R}^n} g(x') \exp\left(-i \sum_{p=1}^{n} \xi_p \sum_{q=1}^{n} x'_q \gamma_{p,q}\right) \mathrm{d}m_x. \end{aligned}$$

Simple computations show that

$$\sum_{p=1}^{n} \xi_p \sum_{q=1}^{n} x'_q \gamma_{p,q} = x'_1 \sum_{p=1}^{n} \xi_p \lambda_{p,1} + \sum_{p=1}^{n} \xi_p \sum_{q=2}^{n} x'_q \lambda_{p,q} = x'_1 \sum_{p=1}^{n} \xi_p \lambda_{p,1} + \sum_{p=1}^{n} \xi_p a_p,$$

hence

$$\hat{f}(\xi) = \int_{\mathbf{R}^{n-1}} \exp\left(-i \sum_{p=1}^{n} \xi_p a_p\right) \left\{ \int_{-\infty}^{\infty} g(x') \exp\left(-i x'_1 \sum_{p=1}^{n} \xi_p \gamma_{p,q}\right) \mathrm{d}x'_1 \right\} \mathrm{d}x'_2 ... \mathrm{d}x'_n.$$

The inner integral exists for almost every $(x'_2, ..., x'_n) \in \mathbf{R}^{n-1}$. Also, we may write

$$g(x') = f(a_1 + z'_1 \gamma_{1,1}, ..., a_n + z'_1 \gamma_{n,1})\big|_{y'_1 = 0},$$

where $z_1' = x_1' + iy_1'$, and g, as a function of x_1', may be extended to an entire function of exponential type in the complex z_1'-plane. Moreover,

$$\begin{aligned}\limsup_{R\to\infty} \frac{1}{R} \log |f(a_1 + iR\gamma_{1,1}, ..., a_n + iR\gamma_{n,1})| &= h_f(\gamma_{1,1}, ..., \gamma_{n,1}; a_1, ..., a_n) \\ &\leq h_f(\gamma_{1,1}, ..., \gamma_{n,1}).\end{aligned}$$

Using Lemma 3.31 we see that

$$\int_{-\infty}^{\infty} g(x') \exp\left(-ix_1' \sum_{p=1}^{n} \xi_p \gamma_{p,q}\right) \mathrm{d}x_1' = 0$$

for almost every $(x_2', ..., x_n')$ whenever

$$\sum_{p=1}^{n} \xi_p \gamma_{p,1} \geq h_f(\gamma_{1,1}, ..., \gamma_{n,1}).$$

Then $\hat{f}(\xi) = 0$ in the half-space $\sum_{p=1}^{n} \xi_p \gamma_{p,1} \geq h_f(\gamma_{1,1}, ..., \gamma_{n,1})$. The x_1'-axis was arbitrary, so $\hat{f}(\xi) = 0$ for all ξ outside the intersection of all the halfspaces $\langle \xi, \alpha \rangle \leq h_f(\alpha)$, $\alpha \in \mathbf{R}^n$, i.e. the convex hull of Ω. Consequently,

$$H_{\mathrm{ch}\,\Omega}(\alpha) \leq h_f(\alpha).$$

Let us now examine the case when $f \in L^2(\mathbf{R}^n)$, but not necessarily from $L^1(\mathbf{R}^n)$. Let

$$f_\varepsilon(z) = f(z) \prod_{j=1}^{n} \frac{\sin \varepsilon z_j}{\varepsilon z_j}.$$

Using the Cauchy-Schwarz inequality we get

$$\int_{\mathbf{R}^n} |f_\varepsilon(x)| \,\mathrm{d}m_x \leq \left(\int_{\mathbf{R}^n} |f(x)|^2 \,\mathrm{d}m_x\right)^{1/2} \left(\int_{\mathbf{R}^n} \prod_{j=1}^{n} \frac{\sin^2 \varepsilon x_j}{\varepsilon^2 x_j^2} \,\mathrm{d}m_x\right)^{1/2}.$$

Both integrals on the right-hand side are finite, hence $f_\varepsilon \in L^1(\mathbf{R}^n)$. We have proved that the Fourier transform of f_ε vanishes outside some convex set Ω_ε, such that

$$H_{\Omega_\varepsilon}(\alpha) \leq h_{f_\varepsilon}(\alpha) = h_f(\alpha) + \varepsilon$$

for any $\alpha \in \mathbf{R}^n$. We calculate the Fourier transform of f_ε

$$\begin{aligned}\hat{f}_\varepsilon(\xi) &= \int_{\mathbf{R}^n} f_\varepsilon(x) e^{-i\langle \xi, x\rangle} \,\mathrm{d}m_x \\ &= \int_{\mathbf{R}^n} f(x) \prod_{j=1}^{n} \frac{\sin \varepsilon x_j}{\varepsilon x_j} e^{-i\langle \xi, x\rangle} \,\mathrm{d}m_x\end{aligned}$$

$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{f}(\eta)e^{i\langle x,\eta\rangle}\,\mathrm{d}m_\eta\, e^{-i\langle \xi,x\rangle}\prod_{j=1}^{n}\frac{\sin\varepsilon x_j}{\varepsilon x_j}\,\mathrm{d}m_x$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \prod_{j=1}^{n}\frac{\sin\varepsilon x_j}{\varepsilon x_j}e^{-i\langle x,\xi-\eta\rangle}\,\mathrm{d}m_x\,\hat{f}(\eta)\,\mathrm{d}m_\eta$$
$$= \frac{1}{(2\varepsilon)^n}\int_{\xi_1-\varepsilon}^{\xi_1+\varepsilon}\dots\int_{\xi_n-\varepsilon}^{\xi_n+\varepsilon}\hat{f}(\eta)\,\mathrm{d}\eta_n\dots\mathrm{d}\eta_1,$$

and see that

$$\limsup_{\varepsilon\to 0}\hat{f}_\varepsilon(\xi) = \hat{f}(\xi)$$

for almost every $\xi \in \mathbf{R}^n$. For every positive ε, we have $\hat{f}(\xi) = 0$ for almost every ξ outside the convex set with supporting function $h_f + \varepsilon$. This means that the smallest convex set $\Omega_{\hat{f}}$ for which $\hat{f}$ vanishes on the complement has the property

$$H_{\Omega_{\hat{f}}}(\alpha) \leq h_f(\alpha). \tag{3.26}$$

The function f is of exponential type, so its P-indicator is bounded. Then $H_{\Omega_{\hat{f}}}$ is bounded as well, and this implies that $\Omega_{\hat{f}}$ is bounded. Let $\varphi = \hat{f}$ and we obtain the representation (3.20).

Finally, compare (3.24) and (3.26) to see that

$$H_{\Omega_\varphi}(\alpha) = h_f(\alpha)$$

for all $\alpha \in \mathbf{R}^n$. ∎

One of the other basic results in one dimension was the sampling theorem by Whittaker, Kotel'nikov and Shannon. This result is extendable to several dimensions.

Theorem 3.32 *Let $S = [-\pi,\pi]\times\cdots\times[-\pi,\pi] \subset \mathbf{R}^n$ and let f be a function from PW_S^2. Then*

$$f(z) = \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} f(k_1,...,k_n)\prod_{j=k_1}^{k_n}\frac{\sin\pi(z-j)}{\pi(z-j)} \tag{3.27}$$

and

$$\|f\|_{PW_S^2} = \|f(k_1,...,k_n)\|_{\ell^2(\mathbf{Z}^n)}\,.$$

The series in (3.27) converges in the norm of PW_S^2 and uniformly on compact subsets of $\mathbf{C}^n$.

The proof is similar to the proof of Theorem 3.3 and is omitted.

Our main motivation for introducing Paley-Wiener spaces of several variables is to be able to say something about complex exponential systems in L^2-spaces over domains in several dimensions. The arguments in the proof of the duality theorem we proved in Section 3.2 works equally well for several variables, so the duality idea can be transfered to several variables. We are again looking for complete interpolating sequences for Paley-Wiener spaces in order to get Riesz bases of families of complex exponential functions in the corresponding L^2-spaces.

Not much is known about complete interpolating sequences in several variables. For most domains $\Omega \subset \mathbf{R}^n$ we do not know whether they exist or not. A complete description is even further away. One reason for this might be that the methods from the one-variable case does not transfer to several variables. In one dimension, we described complete interpolating sequences as the zero set of some generating function. In several variables this does not work, because the zero set of an analytic function of n-variables is loosely speaking a complex manifold of dimension $n-1$, which is certainly not a discrete set when $n > 1$.

However, there are some domains Ω which we know things about. In Chapter 4 we discuss the domains Ω with the property that $L^2(\Omega)$ has an orthonormal basis of complex exponential functions. This corresponds to precise interpolation in the corresponding Paley-Wiener space PW^2_Ω, with Parseval's identity. This problem has gained a lot of attention the past years because of its relation to a conjecture by Fuglede. Fuglede conjectured [Fug74] that it is possible to fill up $\mathbf{R}^n$ with non-overlapping translates of Ω if and only if $L^2(\Omega)$ possesses an orthonormal basis of complex exponential functions. The conjecture has been disproved in a number of cases, but in the case of convex sets in the plane, Iosevich, Katz and Tao proved in [IKT03] that the conjecture holds. This means that if $\Omega \subset \mathbf{R}^2$ is convex, then $L^2(\Omega)$ has an orthonormal basis of complex exponential functions if and only if Ω is a rectangle or a hexagon, since they are the only convex sets which tile $\mathbf{R}^2$ by translations. This leads us to another question: What if $\Omega \subset \mathbf{R}^2$ is convex, but not a rectangle or a hexagon? What is the 'best' possible system of exponential functions for $L^2(\Omega)$? What about about polygons with more vertices? And what about the disk? These questions will be fundamental in Chapter 5 and Chapter 6.

Chapter 5 will be concerned with convex polygons Ω in the plane, which are symmetric with respect to the origin. Lyubarskii and Rashkovskii showed in [LR00] that there exist complete interpolating sequences for PW^2_Ω and by duality that there exist Riesz bases of complex exponential functions for $L^2(\Omega)$. Their technique is based on one-variable methods, using a generating function whose zero set is a union of hyperplanes. The pairwise intersections of the hyperplanes is a discrete set, having the desired interpolating properties.

The next set we look at is the plane disk D. In Chapter 4 we show that there does not exist orthonormal bases in terms of complex exponential functions for $L^2(D)$, so the next best thing to look for is Riesz bases. One might expect a negative result about this, mainly because of the following unpublished result by Ortega-Cerdà [OC06].

Theorem 3.33 *For any convex smooth bounded domain $\Omega \subset \mathbf{R}^n$ there are no complete* *interpolating sequences for PW_Ω^p when $p \neq 2$, $1 < p < \infty$.*

The proof relies on Fourier multipliers and Fefferman's theorem and will not be reproduced here. It is however of common belief that the following is true.

Conjecture 3.34 *Let $\Omega \subset \mathbf{R}^n$ be a convex set. If at least one point on $\partial\Omega$ has a non-vanishing Gaussian curvature, then there exist no complete interpolating sequences for PW_Ω^2.*

These observations indicate that we should look for even weaker sequences than the complete interpolating ones. In Chapter 6 we try to apply the method from the case of a convex symmetric polygon to the disk to get a set of uniqueness for PW_Ω^2, which again will lead to a complete system of exponential functions in $L^2(D)$. This work is not completed yet.

Finally we mention Landau's generalization of the Beurling densities to several variables.

Theorem 3.35 (Landau's density theorem) *Let Λ be a uniformly separated sequence in $\mathbf{R}^n$, U the unit cube in $\mathbf{R}^n$ and Ω a subset of $\mathbf{R}^n$. Define $n^+(r)$ and $n^-(r)$ as the largest and smallest number of points from Λ which can be found in a translate of rU. The Landau densities are then*

$$D^+(\Lambda) = \limsup_{r\to\infty} \frac{n^+(r)}{r} \text{ and } D^-(\Lambda) = \liminf_{r\to\infty} \frac{n^-(r)}{r}.$$

If Λ is an interpolating sequence for PW_Ω^2, then $D^+(\Lambda) \leq \frac{|\Omega|}{(2\pi)^n}$. If Λ is a sampling sequence for PW_Ω^2, then $D^-(\Lambda) \geq \frac{|\Omega|}{(2\pi)^n}$.

A complete interpolating sequence for PW_Ω^2 must therefore have density

$$D^+(\Lambda) = D^-(\Lambda) = \frac{|\Omega|}{(2\pi)^n}.$$

See the original article [Lan67] for a proof.

Chapter 4

Fuglede's conjecture

To be able to state the problem of this chapter we need some definitions.

Definition 4.1 Let Ω be a Lebesgue measurable subset of $\mathbf{R}^n$. We say that Ω is a *spectral set* if there exists a discrete set $\Lambda \subset \mathbf{R}^n$ such that $\mathcal{E}(\Lambda) = \left\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\right\} = \{e_\lambda : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\Omega)$. Λ is then said to be a *spectrum* for Ω. The pair (Ω, Λ) is sometimes called a *spectral pair*.

Example 4.2 Let $\Omega = [0,1]^n$ be the unit cube in $\mathbf{R}^n$. Then Ω is a spectral set, and a spectrum for Ω is $\Lambda = \mathbf{Z}^n$. The orthonormal basis is in this case the usual "Fourier basis" $\mathcal{E}(\mathbf{Z}^n)$. ▲

We need the concept of translational tilings.

Definition 4.3 Let Ω be a Lebesgue measurable subset of $\mathbf{R}^n$. We say that Ω *tiles* $\mathbf{R}^n$ *by translations* if there is a discrete subset $T \subset \mathbf{R}^n$ such that

$$\sum_{t\in T} \chi_\Omega(x+t) = 1,$$

for almost every $x \in \mathbf{R}^n$.

We are now ready to state Fuglede's conjecture, also known as the spectral set conjecture.

Conjecture 4.4 (Fuglede, 1974) *Let Ω be a Lebesgue measurable subset of $\mathbf{R}^n$. Ω is a spectral set if and only if it tiles $\mathbf{R}^n$ by translations.*

In this chapter we will discuss Fuglede's conjecture for various cases. The first section contains some of Fuglede's motivation for stating the conjecture and one of his results. In the second section we prove Fuglede's conjecture for the case when the tiling set or the spectrum is assumed to be a lattice. After that we give an overview over what is known in two dimensions and some two-dimensional examples. Finally we have a brief look at known results in other dimensions and some open questions.

4.1 Origin of the problem and a result by Fuglede

Most of the research papers nowadays only state the conjecture and refer to the original paper [Fug74] for the functional analytic origin of the conjecture. In this section we will try to explain the origin of the problem and a result obtained by Fuglede himself. To be able to understand Fuglede's result we need some theory for unbounded linear operators in Hilbert spaces. We start with some general theory for unbounded linear operators, using the formal expression for the momentum operator from quantum mechanics defined on various domains to display some of the challenges related to this class of operators. After a quick look at their spectral theory, we consider a system of unbounded operators, and this will lead us to a nice result by Fuglede. It should be noted that the exposition of the material on unbounded operators is of a quite heuristic nature.

A linear operator between two Banach spaces (we will only work with Hilbert spaces) for which there is no constant C such that $\|D\| \leq C$ is called an unbounded linear operator. Unbounded operators are a bit different to work with than bounded operators, in particular the concept of adjoint operators needs to be generalized. Much of the material which now follows is taken from [DS63, chapter XII].

Recall that for a bounded linear operator $T : H \to H$, where H is a Hilbert space, the adjoint operator T^* is the operator $T^* : H \to H$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$. It should be noticed that such an operator always exists, is unique and is bounded and linear. If $T = T^*$, then T is called self-adjoint, so for bounded self-adjoint linear operators we have

$$\langle Tx, y\rangle = \langle x, Ty\rangle \,, \tag{4.1}$$

for all $x, y \in H$. For unbounded operators things are not as simple. The Hellinger-Toeplitz theorem is of basic importance and says that a linear operator T defined on all of a Hilbert space H satisfying (4.1) for all $x, y \in H$, must be bounded. This means that an unbounded operator cannot be defined on all of H and immediately leads to the problem of determining suitable domains for unbounded operators. Later in this section we will see that important properties (especially symmetry and self-adjointness) of unbounded operators depend heavily on the choice of domains. First we will show that in order to have a resonable definition of the adjoint operator, the domain should be some dense subset of the Hilbert space in question.

Definition 4.5 Let $T : \mathcal{D}(T) \to H$ be a linear operator, where $\mathcal{D}(T)$ is a dense subset of H. The *adjoint* operator $T^* : \mathcal{D}(T^*) \to H$ is defined as follows. The domain of T^*, $\mathcal{D}(T^*)$, is all $y \in H$ such that there exists a $y^* \in H$ satisfying $\langle Tx, y\rangle = \langle x, y^*\rangle$ for all $x \in \mathcal{D}(T)$ and for each $y \in \mathcal{D}(T^*)$ the adjoint operator is defined by $y^* = T^*y$.

Observe that for bounded linear operators this definition coincides with the definition in the text above. Let us see what happens if we define T on a set $\mathcal{D}(T)$ which is not dense in H. The orthogonal complement $\mathcal{D}(T)^\perp$ of the closure of $\mathcal{D}(T)$, $\overline{\mathcal{D}(T)}$, contains a non-zero element y_1. Then $\langle x, y^* \rangle = \langle x, y^* \rangle + \underline{\langle x, y_1 \rangle} = \langle x, y^* + y_1 \rangle$ and y^* is not unique. On the contrary, if $\mathcal{D}(T)$ is dense in H, then $\overline{\mathcal{D}(T)} = H$ and $\mathcal{D}(T)^\perp = \{0\}$. This implies that if $\langle x, y_1 \rangle = 0$ for all $x \in \mathcal{D}(T)$, then $y_1 = 0$ and y^* is unique.

An operator T satisfying (4.1) for all $x, y \in \mathcal{D}(T)$ is called symmetric. For bounded linear operators symmetry and self-adjointness is the same. For unbounded linear operators it is not. It can be shown that a densely defined linear operator T is symmetric if and only if T^* is an extension of T, that is $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $T^*\big|_{\mathcal{D}(T)} = T$. If $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $T = T^*$, then T is self-adjoint. A self-adjoint operator is therefore always symmetric, but not vice versa. Notice that for bounded linear operators, also this definition coincides with the one given earlier. We are now ready for an example, illustrating the theory just outlined.

Example 4.6 Consider the differential operator $D = -i\frac{\mathrm{d}}{\mathrm{d}x}$ defined on[1]

$$\mathcal{D}(D) = \left\{ u \in AC[0,1] : \ \frac{\mathrm{d}u}{\mathrm{d}x} \in L^2(0,1) \right\}.$$

The operator D is easily seen to been unbounded. It can be shown that $\mathcal{D}(D)$ is dense in $L^2(0,1)$, see e.g. [Yos80]. For $u \in \mathcal{D}(D)$ and $v \in L^2(0,1)$, we have

$$\begin{aligned} \langle Du, v \rangle &= -i \int_0^1 \frac{\mathrm{d}u}{\mathrm{d}x} \overline{v(x)} \, \mathrm{d}x \\ &= -iu(1)v(1) + iu(0)v(0) + \int_0^1 u(x) \overline{-i\frac{\mathrm{d}v}{\mathrm{d}x}} \, \mathrm{d}x \\ &= -iu(1)v(1) + iu(0)v(0) + \langle u, Dv \rangle \, . \end{aligned}$$

As seen from the calculation, with this domain D is neither symmetric nor self-adjoint. It is easy to find a symmetric restriction D_1 of D. If we let $\mathcal{D}(D_1) = \{u \in \mathcal{D}(D) : u(0) = u(1) = 0\}$, then

$$\langle D_1 u, v \rangle = -iu(1)v(1) + iu(0)v(0) + \langle u, D_1 v \rangle = \langle u, D_1 v \rangle \, ,$$

for all $u \in \mathcal{D}_1(D)$, $v \in L^2(0,1)$, hence D_1 is a symmetric restriction of D. However, D_1 is not self-adjoint, because $\mathcal{D}(D_1) \neq \mathcal{D}(D_1^*)$. In fact,

$$\mathcal{D}(D_1^*) = \left\{ u \in L^2(0,1) : \ D_1 u \in L^2(0,1) \right\},$$

so $\mathcal{D}(D_1) \subset \mathcal{D}(D_1^*)$ and D_1^* is an extension of D_1. Consider the operator D_2 having the same formal expression as D, but defined on

$$\mathcal{D}(D_2) = \{ u \in \mathcal{D}(D) : \ u(0) = u(1) \} \, .$$

[1] If $u \in AC[0,1]$, the set of absolutely continuous functions on the interval $[0,1]$, then for every $\varepsilon > 0$ there exist some $\delta > 0$ such that for any set of disjoint intervals $(a_1, b_1), ..., (a_N, b_N)$, $\sum_{j=1}^{N} (b_j - a_j) < \delta$ implies that $\sum_{j=1}^{N} |u(b_j) - u(a_j)| < \varepsilon$. An absolutely continuous function is differentiable almost everywhere, see e.g. [Fol99].

Now

$$\langle D_2 u, v\rangle = iu(0)\,(v(0) - v(1)) + \langle u, D_2 v\rangle \,,$$

which means that $\langle D_2 u, v\rangle = \langle u, D_2 v\rangle$ if $\mathcal{D}(D_2^*) = \mathcal{D}(D_2)$ and D_2 is self-adjoint. ▲

The example above raises some questions. When can we find some self-adjoint restriction or extension of an unbounded operator? How many self-adjoint extensions are there? This problem is handled by a theorem of von Neumann [vN29]. We need two definitions.

Definition 4.7 Let $T : \mathcal{D}(T) \to H$ be a linear operator, where $\mathcal{D}(T)$ is a dense subset of a Hilbert space H. If T has a closed linear extension it is called *closable*. If $\overline{T}$ is a closed linear extension of T such that every other linear extension of T is an extension of $\overline{T}$, then $\overline{T}$ is called the *closure* of T.

Definition 4.8 Let T be a closed symmetric operator defined on a dense subset of a Hilbert space H. n_+ is the dimension of the linear subspace $\mathcal{N}_+$ and n_- is the dimension of the linear subspace $\mathcal{N}_-$, where

$$\mathcal{N}_+ = \{x \in \mathcal{D}(T^*) : \; T^* x = ix\}$$
$$\mathcal{N}_- = \{x \in \mathcal{D}(T^*) : \; T^* x = -ix\}\,.$$

The numbers n_+ and n_- are called the *deficiency indices* of T.

Theorem 4.9 (von Neumann, 1929) *Let T be a closed symmetric operator defined on a dense subset of a Hilbert space H with deficiency indices n_+ and n_-. T has a self-adjoint extension if and only if $n_+ = n_-$.*

A proof can be found in [DS63].

Example 4.10 Consider the symmetric operator D_1 as defined above. To calculate its deficiency indices we need to find the dimension of the subspaces $\mathcal{N}_+$ and $\mathcal{N}_-$ of $L^2(0,1)$. By simple calculations we see that

$$\mathcal{N}_+ = \big\{u \in \mathcal{D}(D_1) : \; -u' = u\big\} = \big\{u = Ce^{-x}\big\}$$
$$\mathcal{N}_- = \big\{u \in \mathcal{D}(D_1) : \; u' = u\big\} = \{u = Ce^{x}\}\,,$$

where C is some constant. The dimension of each of the subspaces is clearly 1 and D_1 has by the von Neumann theorem self-adjoint extensions. This is indeed in agreement with our example above, where we found one such extension. ▲

It can be shown that $D = -i\frac{\mathrm{d}}{\mathrm{d}x}$ has different extension properties when the domain is a dense subset of the square integrable functions on the real line, the half-line and an interval. If D_R is the operator $-i\frac{\mathrm{d}}{\mathrm{d}x}$ defined on the subset

$$\mathcal{D}(D_R) = \left\{u \in AC(\mathbf{R}) : \; \frac{\mathrm{d}u}{\mathrm{d}x} \in L^2(\mathbf{R})\right\}$$

of $L^2(\mathbf{R})$, then D_R is self-adjoint. If we move to the half-line $\mathbf{R}_+ = (0, \infty)$, then the operator D_+ defined on

$$\mathcal{D}(D_+) = \left\{ u \in AC(\mathbf{R}_+) : \frac{\mathrm{d}u}{\mathrm{d}x} \in L^2(\mathbf{R}_+),\, u(0) = 0 \right\}$$

has no self-adjoint extension and as we saw above, when considered on an interval, the operator D_1 has self-adjoint extensions.

We know move on to the spectral representation of self-adjoint operators.

Definition 4.11 Let $T : \mathcal{D}(T) \to H$ be a self-adjoint linear operator, where $\mathcal{D}(T)$ is a dense subset of a Hilbert space H. The operator

$$U = (T - iI)(T + iI)^{-1},$$

where I is the identity operator, is called the *Cayley transform* of T.

Definition 4.12 Let U be a bounded linear operator. If U is bijective and $U^* = U^{-1}$, then U is called *unitary.*

Proposition 4.13 *Let T be as in Definition 4.11. Its Cayley transform U is a unitary operator.*

A proof of this proposition is given in [Lax02].

Definition 4.14 Let M be a closed subspace of the Hilbert space H. The *orthogonal projection* from H to M is the operator $P : H \to M$ defined by $Px = y$ if $x = y + z$, where $y \in M$ and $z \in M^{\perp}$.

For orthogonal projections we have the following equivalence:

Proposition 4.15 *A bounded linear operator $P : H \to H$, where H is a Hilbert space, is an orthogonal projection if and only if P is self-adjoint and $P^2 = P$.*

Theorem 4.16 *Let T be a self-adjoint operator in a Hilbert space H, with domain $\mathcal{D}(T)$. There is a spectral resolution for T, that is an orthogonal projectionvalued measure E defined for all Borel measurable subsets of $\mathbf{R}$, with the following properties:*

1. *$E(\emptyset) = 0$, $E(\mathbf{R}) = I$.*
2. *For any pair of measurable sets A and B, $E(A \cap B) = E(A)E(B)$.*
3. *For every measurable set A, $E^*(A) = E(A)$.*
4. *E commutes with T, that is, for any measurable set A, $E(A)$ maps the domain $\mathcal{D}(T)$ of T into $\mathcal{D}(T)$, and for all $x \in \mathcal{D}(T)$, $TE(A)x = E(A)Tx$.*
5. *The domain $\mathcal{D}(T)$ of T consists of all elements x for which*

$$\int_{\mathbf{R}} t^2 \,\mathrm{d}\left\langle E\left(t\right)x, x\right\rangle < \infty$$

and

$$Tx = \int_{\mathbf{R}} t \,\mathrm{d}E\,(t)\,x. \tag{4.2}$$

The integral in (4.2) should be explained. Consider the interval $I = [m, M] \subset \mathbf{R}$ and any decomposition $I = \cup_j [t_{j-1}, t_j]$ into disjoint subsets. The integral $\int_m^M t\,\mathrm{d}E(t)x$ is defined as the limit of the Riemann sums $\sum_j t^* E\left((t_{j-1}, t_j]\right)x$, where $t^* \in (t_{j-1}, t_j]$, as $\max_j |t_j - t_{j-1}|$ tends to zero. The expression $\int_{\mathbf{R}} t\,\mathrm{d}E\,(t)\,x$ is then defined as

$$\int_{\mathbf{R}} t\,\mathrm{d}E\,(t)\,x = \lim_{\substack{M\to\infty \\ m\to-\infty}} \int_m^M t\,\mathrm{d}E\,(t)\,x.$$

The original proof of the spectral representation for self-adjoint operators given by von Neumann uses the Cayley transform. The Cayley transform U is a unitary operator and a spectral representation for this is known. Unitary operators have spectrum on the unit circle, so their spectral measures are supported by the unit circle. The idea is to pull back the spectral representation of U with a conformal mapping to get a spectral representation for the unbounded operator T. More about this proof, as well as other proofs, may be found in [Lax02].

The support of the measure E is the spectrum of the operator T. Recall that the resolvent set $\rho(T)$ of an operator T is the set of points $\lambda \in \mathbf{C}$ for which there exists a bounded linear operator A, such that $A(T - \lambda I) = (T - \lambda I)A = I$, in other words $A = (T - \lambda I)^{-1}$. The spectrum of T is $\sigma(T) = \mathbf{C} \setminus \rho(T)$. One can prove (see e.g. [DS63]) that the spectrum of a symmetric operator, and in particular a self-adjoint operator, is real. This is the reason why the integration in Theorem 4.16 is done on the real line and not in the complex plane.

We need a way to define commutation for self-adjoint operators, based on spectral resolutions.

Definition 4.17 Let T_1 and T_2 be self-adjoint operators in a Hilbert space H, with domains $\mathcal{D}(T_1)$ and $\mathcal{D}(T_2)$. *Commutation in the sense of spectral measures* means that

$$E_1(A)E_2(B) = E_2(B)E_1(A)$$

for all Borel sets $A, B \subset \mathbf{R}$, where E_1 and E_2 are the spectral measures associated with T_1 and T_2.

Finally, we are ready to the state the problem posed to Bent Fuglede by Irving Segal in 1958:

Problem: Given the differential operators $D_j = -i\frac{\partial}{\partial x_j}$, $j = 1, ..., n$, acting on $L^2(\Omega)$ in the sense of distributions, where $\Omega \subset \mathbf{R}^n$ is an open connected set. Since each D_j is acting in the sense of distributions, we consider

$$\mathcal{D}(D_j) = \left\{u \in L^2(\Omega) :\ D_j u \in L^2(\Omega)\right\}$$

as its domain. The question is then, for which domains Ω does there exist self-adjoint restrictions $T_1, ..., T_n$ of $D_1, ..., D_n$, which are commuting in the sense of spectral measures?

Let now $T = (T_1, ..., T_n)$ be a family of self-adjoint operators in a Hilbert space H, commuting in the sense of spectral measures. We follow [Fug74] to give some properties of T. The domain of T is defined as $\mathcal{D}(T) = \cap_{j=1}^{n} \mathcal{D}(T_j)$. There is a spectral representation of T, based on the spectral representations for the individual operators T_j, if and only if the operators are commuting in the sense of spectral measures,

$$T = \int_{\mathbf{R}^n} t \, \mathrm{d}E(t),$$

meaning that

$$T_j = \int_{\mathbf{R}} t \, \mathrm{d}E_j(t),$$

for $j = 1, ..., n$, where E is the spectral measure on $\mathbf{R}^n$ associated with T. The support of E is the spectrum of T, $\sigma(T)$. By the spectrum of T we mean a point $\lambda \in \mathbf{R}^n$ such that λ_j is in the spectrum of T_j for all $j = 1, ..., n$. The point spectrum $\sigma_p(T)$ is the set of eigenvalues for T. An eigenvalue for T is a point $\lambda \in \mathbf{R}^n$ such that

$$\mathbf{E}(\{\lambda\}) = \{u \in \mathcal{D}(T) : \; Tu = \lambda u\} \neq \{0\} .$$

$\mathbf{E}(\{\lambda\})$ is called the eigenspace associated with the eigenvalue λ. It should also be noticed that the image of the eigenvalue λ under the measure E is the orthogonal projection operator of H onto $\mathbf{E}(\{\lambda\})$. If $E(\sigma_p(T)) = I$, then T is said to have a pure point spectrum. In this case, the union of all the eigenspaces is dense in H and $\overline{\sigma_p(T)} = \sigma(T)$. For T to have a pure point spectrum it is sufficient that the spectrum is discrete in $\mathbf{R}^n$.

The goal of this section is to state and prove Fuglede's result related to the problem stated above. To do this we need two propositions which we state without proofs, and a definition. The proofs of the propositions can be found in [Fug74].

Proposition 4.18 *Let T and H be as above and let P be a finite dimensional orthogonal projection on H. Denote a point in the spectrum $\sigma(T)$ by $\lambda = (\lambda_1, ..., \lambda_n)$. Assume there exists a finite constant C such that*

$$\|u - Pu\|^2 \leq C \sum_{j=1}^{n} \|T_j u - \lambda_j u\|^2$$

for all $u \in \mathcal{D}(T)$. Then[2] *$0 \neq E(\{\lambda\}) \leq P$, that is, λ is an eigenvalue for T with eigenspace $\mathbf{E}(\{\lambda\})$ contained in the range of P. If $E(\{\lambda\}) = P$, then λ is an isolated point of $\sigma(T)$ and the distance between λ and $\sigma(T) \setminus \{\lambda\}$ is larger than or equal to $C^{-1/2}$.*

[2] Here we mean the partial ordering obtained by saying that $E(\lambda) \leq P$ if $\langle E(\lambda)x, x\rangle \leq \langle Px, x\rangle$ for all $x \in H$. For any self-adjoint operator T, the expression $\langle Tx, x\rangle$ is real, so the ordering does certainly make sense.

Definition 4.19 A non-empty open set $\Omega \subset \mathbf{R}^n$ is called a *Nikodym set* if every distribution u on Ω such that all $D_j u$ are in $L^2(\Omega)$, $j = 1, ..., n$, is itself in $L^2(\Omega)$. A connected Nikodym set is called a *Nikodym region.*

Proposition 4.20 *An open connected set $\Omega \subset \mathbf{R}^n$ is a Nikodym region if and only if the Lebesgue measure of Ω is finite and there is some finite constant C, depending on Ω, such that the following two conditions are satisfied for any u in the space $\left\{u \in L^2(\Omega) : \ D_j u \in L^2(\Omega),\, j = 1, ..., n\right\}$:*

(i) $\|u\|^2 \le \left|\int_\Omega u(x)\,\mathrm{d}m_x\right|^2 + C\sum_{j=1}^n \|D_j u\|^2$,

(ii) $\|u - \langle u, e_\lambda\rangle\, e_\lambda\|^2 \le C\sum_{j=1}^n \|D_j u - \lambda_j u\|^2$

for any $\lambda \in \mathbf{R}^n$. Recall from the beginning of the chapter that $e_\lambda = e^{2\pi i\langle\lambda,\cdot\rangle}$.

Theorem 4.21 (Fuglede,1974) *Let $\Omega \subset \mathbf{R}^n$ be a Nikodym region with Lebesgue measure equal to 1.*

a) *Let $T = (T_1, ..., T_n)$ denote a commuting (in the sense of spectral measures) family of self-adjoint restrictions T_j of D_j on $L^2(\Omega)$, $j = 1, ..., n$. Then T has a discrete spectrum, each point $\lambda \in \sigma(T)$ being a simple eigenvalue for T with the eigenspace $\mathbf{C}e_\lambda$, and hence $\mathcal{E}(\sigma(T)) = \{e_\lambda : \ \lambda \in \sigma(T)\}$ is an orthonormal basis for $L^2(\Omega)$. Moreover,*

$$\sigma(T) = \sigma_p(T) = \{\lambda \in \mathbf{R}^n : \ e_\lambda \in \mathcal{D}(T)\}.$$

b) *Conversely, let Λ denote a subset of $\mathbf{R}^n$ such that $\mathcal{E}(\Lambda)$ is an orthonormal basis for $L^2(\Omega)$. Then there exists a unique commuting (in the sense of spectral measures) family $T = (T_1, ..., T_n)$ of self-adjoint restrictions T_j of D_j on $L^2(\Omega)$ with the property that $\mathcal{E}(\Lambda) \subset \mathcal{D}(T)$, or equivalently that $\Lambda = \sigma(T)$.*

Proof:

a) For any $\lambda \in \sigma(T)$ let P_λ be the one-dimensional projection of $L^2(\Omega)$ on $\mathbf{C}e_\lambda$, where $P_\lambda u = \langle u, e_\lambda\rangle\, e_\lambda$ for any $u \in L^2(\Omega)$. Ω is a Nikodym region, so by Proposition 4.20

$$\|u - P_\lambda u\|^2 = \|u - \langle u, e_\lambda\rangle\, e_\lambda\|^2 \le C\sum_{j=1}^n \|T_j u - \lambda_j u\|^2$$

and we may apply Proposition 4.18. λ is an eigenvalue for T with eigenspace $\mathbf{C}e_\lambda$, which is equal to the range of P_λ. By the last assertion of Proposition 4.18 λ must be isolated. λ was an arbitrary element of $\sigma(T)$, so $\sigma(T)$ must necessarily be discrete. T has then a pure point spectrum and $\sigma_p(T) = \sigma(T)$ (no closure on $\sigma_p(T)$ is necessary, since $\sigma_p(T)$ is a discrete subset of $\mathbf{R}^n$ and is consequently closed). To prove the last claim of part **a)**, we observe that we have the following circle of implications

$$\lambda \in \sigma(T) \Rightarrow e_\lambda \in \mathcal{D}(T) \Rightarrow Te_\lambda = \lambda e_\lambda \Rightarrow \lambda \in \sigma(T). \tag{4.3}$$

This means that

$$\sigma(T) = \{\lambda \in \mathbf{R}^n : \ e_\lambda \in \mathcal{D}(T)\} = \{\lambda \in \mathbf{R}^n : \ Te_\lambda = \lambda e_\lambda\}.$$

b) Assume that $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(\Omega)$. From (4.3) we have

$$\Lambda \subset \sigma(T) \Leftrightarrow \mathcal{E}(\Lambda) \subset \mathcal{D}(T)$$

if the operator T exists. The system $\mathcal{E}(\sigma(T))$ is an orthonormal extension of $\mathcal{E}(\Lambda)$, but $\mathcal{E}(\Lambda)$ is dense in $L^2(\Omega)$, so Λ must be equal to the spectrum of T.

To prove uniqueness, we notice that we must have $T_j e_\lambda = \lambda_j e_\lambda$, $j = 1, ..., n$, for all $\lambda \in \mathbf{R}^n$ such that $e_\lambda \in \mathcal{D}(T)$, since each T_j should be a restriction of D_j on $L^2(\Omega)$. Especially, we need to have $T_j e_\lambda = \lambda_j e_\lambda$ for $\lambda \in \Lambda$, when we would like that $\mathcal{E}(\Lambda) \subset \mathcal{D}(T)$. $\mathcal{E}(\Lambda)$ was assumed to be an orthonormal basis for $L^2(\Omega)$, meaning that T_j must be the closure of $T_j^0 = T|_M$, where M is the subspace of $L^2(\Omega)$ spanned by $\mathcal{E}(\Lambda)$. When a closure exists it is unique.

To prove existence, we define T_j to be the closure of $T_j^0 = D_j|_M$. T_j is self-adjoint since $T_j e_\lambda = \lambda_j e_\lambda$ for all $\lambda \in \Lambda$. $\mathcal{D}(T_j) \subset \mathcal{D}(D_j)$ since D_j is closed and the operators $T_1, ..., T_n$ commutes. We have $\mathcal{E}(\Lambda) \subset M \subset \mathcal{D}(T)$. ∎

4.2 When the tiling set or the spectrum is a lattice

In this section we prove Fuglede's conjecture with the assumption that the tiling set or the spectrum is a lattice. We first give the definition of a lattice, its dual lattice and the fundamental domain of a lattice.

Definition 4.22 Given an invertible $n \times n$ matrix A. A set Λ is called a *lattice* if $\Lambda = A\mathbf{Z}^n$.

$$\Lambda^* = \{\mu : \ \langle \mu, \lambda \rangle \in \mathbf{Z}, \ \forall \lambda \in \Lambda\}$$

is called the *dual lattice* of Λ. $Q_\Lambda := A[0,1]^n$ is called the *fundamental domain* of the lattice Λ.

Recall the notation $\mathcal{E}(\Lambda) = \left\{e^{2\pi i \langle \lambda, \cdot \rangle} : \ \lambda \in \Lambda\right\} = \{e_\lambda : \ \lambda \in \Lambda\}$, where Λ is some discrete subset of $\mathbf{R}^n$. We first prove that $\mathcal{E}(\mathbf{Z}^n)$ is an orthonormal basis for $L^2([0,1]^n)$ and then that $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(Q_{\Lambda^*})$ if Λ is a lattice with dual lattice Λ^*. We need two auxillary results.

Theorem 4.23 (Stone-Weierstrass) *Let X be a closed and bounded subset of $\mathbf{R}^n$ and let $\mathcal{A}$ be an algebra of complex-valued continuous functions that separates points and which is closed under complex conjugation. If no element of $\mathcal{A}$ vanishes at any point of X then $\mathcal{A}$ is dense in $C(X)$.*

Proposition 4.24 *The set of continuous functions on $\mathbf{R}^n$ is dense in $L^2(\mathbf{R}^n)$.*

Proofs of both results can be found in [Fol99].

Proposition 4.25 *$\mathcal{E}(\mathbf{Z}^n)$ is an orthonormal basis for $L^2([0,1]^n)$.*

Proof: Orthonormality is easily checked by applying Fubini's theorem. Denote the linear span of $\mathcal{E}(\mathbf{Z}^n)$ by $\mathcal{A}$. We need to check that $\mathcal{A}$ is dense in $L^2([0,1]^n)$. $\mathcal{A}$ is clearly an algebra, since $e_\kappa e_{\kappa'} = e_{\kappa+\kappa'}$, $\kappa, \kappa' \in \mathbf{Z}^n$. $\mathcal{A}$ separates points, since we for any $x, y \in [0,1]^n$, $x \neq y$, can find $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Also $\overline{e_\kappa} = e_{-\kappa}$, so $\mathcal{A}$ is closed under complex conjugation. $[0,1]^n$ is closed and bounded and no element of $\mathcal{A}$ vanishes at any point of $[0,1]^n$, thus the Stone-Weierstrass theorem applies. The span of $\mathcal{E}(\mathbf{Z}^n)$ is dense in $C([0,1]^n)$ in the supremum norm and therefore also in the L^2-norm. Proposition 4.24 then implies that span of $\mathcal{E}(\mathbf{Z}^n)$ is dense in $L^2([0,1]^n)$. ∎

Proposition 4.26 *Let Λ be a lattice. $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(Q_{\Lambda^*})$, where Q_{Λ^*} is the fundamental domain of the dual lattice Λ^*.*

Proof: Let $\lambda, \lambda' \in \Lambda$, then

$$\langle e_\lambda, e_{\lambda'} \rangle = \int_{Q_{\lambda^*}} e^{2\pi i \langle \lambda - \lambda', x \rangle} \, \mathrm{d}m_x = 0,$$

since $\langle \lambda - \lambda', x \rangle \in \mathbf{Z}$ when $x \in \Lambda^*$, hence $\mathcal{E}(\Lambda)$ is an orthogonal system. The density argument in the proof of Proposition 4.25 is also valid for $\mathcal{E}(\Lambda)$, so $\mathcal{E}(\Lambda)$ is a basis. ∎

A useful working criterion for a set to be spectral, is the following:

Proposition 4.27 *Let $\Omega \subset \mathbf{R}^n$ be of unit Lebesgue measure and $\Lambda \subset \mathbf{R}^n$ be a discrete set. Λ is a spectrum for Ω if and only if*

$$\sum_{\lambda \in \Lambda} |\hat{\chi}_\Omega(x - \lambda)|^2 = 1 \tag{4.4}$$

for almost every $x \in \mathbf{R}^n$ and $\lambda - \lambda' \in Z(\hat{\chi}_\Omega)$, $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Lambda$.

Condition (4.4) is fundamental in the construction of a multiresolution analysis in the theory of wavelets. More about this can be found in [Dau92, chapter 5].

Proof: The criterion for $\mathcal{E}(\Lambda)$ to be an orthonormal system in $L^2(\Omega)$ is that $\langle e_\lambda, e_{\lambda'} \rangle = 0$, whenever $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$ and that $\langle e_\lambda, e_\lambda \rangle = 1$. We have

$$\langle e_\lambda, e_\lambda \rangle = \int_\Omega \mathrm{d}m_x = |\Omega| = 1$$

and

$$\langle e_\lambda, e_{\lambda'} \rangle = \int_\Omega e^{2\pi i \langle \lambda, x \rangle} e^{-2\pi i \langle \lambda', x \rangle} \, \mathrm{d}m_x = \hat{\chi}_\Omega(\lambda - \lambda'),$$

hence $\mathcal{E}(\Lambda)$ is orthonormal if and only if $\lambda - \lambda' \in Z(\hat{\chi}_\Omega)$, whenever $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$. We want $\mathcal{E}(\Lambda)$ to be an orthonormal basis, so we also need completeness of

$\mathcal{E}(\Lambda)$ in $L^2(\Omega)$. Completeness of an orthonormal system in $L^2(\Omega)$ is equivalent to the validity of Parseval's identity

$$\sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle|^2 = \|f\|^2_{L^2(\Omega)}\,, \tag{4.5}$$

for any function $f \in L^2(\Omega)$. It is enough to have (4.5) fulfilled for a dense subset of $L^2(\Omega)$, because we will then have it for the closed linear span of the dense subset, i.e. all of $L^2(\Omega)$. The trigonometric polynomials are dense in $L^2(\Omega)$, so we only need to consider $f(\cdot) = e^{2\pi i\langle x,\cdot\rangle}$ for almost every $x \in \mathbf{R}^n$. Then (4.5) takes the form

$$\sum_{\lambda\in\Lambda} |\hat{\chi}_\Omega(x-\lambda)|^2 = 1.$$

■

Theorem 4.28 (Fuglede, 1974) *Let $\Omega \subset \mathbf{R}^n$ and suppose that Λ is a lattice with dual lattice Λ^*. Then Ω tiles $\mathbf{R}^n$ with translation set Λ if and only if Λ^* is a spectrum for Ω.*

This result was first proved by Bent Fuglede in [Fug74]. The proof we give here is taken from the short note [Ios07] by Alex Iosevich.

Proof: Assume for simplicity that $|\Omega| = 1$. Taking Definition 4.3 and Proposition 4.27 into account, we must prove that

$$\sum_{\lambda\in\Lambda} \chi_\Omega(x+\lambda) = 1 \text{ for a.e. } x \in \mathbf{R}^n \;\Leftrightarrow\; \sum_{\mu\in\Lambda^*} |\hat{\chi}_\Omega(\xi+\mu)|^2 = 1 \text{ for a.e. } \xi \in \mathbf{R}^n.$$

Let

$$f(x) = \sum_{\lambda\in\Lambda} \chi_\Omega(x+\lambda) \quad \text{and} \quad F(\xi) = \sum_{\mu\in\Lambda^*} |\hat{\chi}_\Omega(\xi+\mu)|^2\,.$$

($\Rightarrow$) We assume that $f(x) = 1$ for almost every $x \in \mathbf{R}^n$. F is periodic with respect to Λ^*, so we denote the fundamental domain of Λ^* by Q_{Λ^*}. Q_{Λ^*} obviously tiles $\mathbf{R}^n$ with Λ^*. From Proposition 4.26 we know that $\mathcal{E}(\Lambda) = \left\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\right\}$ is an orthogonal basis for $L^2(Q_{\Lambda^*})$. F is clearly in $L^2(Q_{\Lambda^*})$, so we may compute its Fourier coefficients.

$$\begin{aligned}
\hat{F}(\lambda) &= \frac{1}{|Q_{\Lambda^*}|}\int_{Q_{\Lambda^*}} e^{-2\pi i\langle\xi,\lambda\rangle} \sum_{\mu\in\Lambda^*} |\hat{\chi}_\Omega(\xi+\mu)|^2 \, dm_\xi \\
&= \frac{1}{|Q_{\Lambda^*}|}\sum_{\mu\in\Lambda^*}\int_{Q_{\lambda^*}} e^{-2\pi i\langle\xi,\lambda\rangle} |\hat{\chi}_\Omega(\xi+\mu)|^2 \, dm_\xi. \qquad (4.6)
\end{aligned}$$

A change of variables gives

$$\text{r.h.s. of (4.6)} = \frac{1}{|Q_{\Lambda^*}|}\sum_{\mu\in\Lambda^*}\int_{Q_{\lambda^*}+\mu} e^{-2\pi i\langle\xi-\mu,\lambda\rangle} |\hat{\chi}_\Omega(\xi)|^2 \, dm_\xi$$

$$= \frac{1}{|Q_{\Lambda^*}|} \int_{\mathbf{R}^n} e^{-2\pi i \langle \xi, \lambda \rangle} \left| \hat{\chi}_\Omega(\xi) \right|^2 \, \mathrm{d}m_\xi,$$

since $\langle \lambda, \mu \rangle \in \mathbf{Z}$ and Q_{Λ^*} tiles $\mathbf{R}^n$ with Λ^*. The Fourier transform of $\left|\hat{\chi}_\Omega(\xi)\right|^2$ is $\chi_\Omega * \overline{\chi}_\Omega$, thus

$$\begin{aligned} \frac{1}{|Q_{\Lambda^*}|} \int_{\mathbf{R}^n} e^{-2\pi i \langle \xi, \lambda \rangle} \left| \hat{\chi}_\Omega(\xi) \right|^2 \, \mathrm{d}m_\xi &= \frac{1}{|Q_{\Lambda^*}|} \int_{\mathbf{R}^n} \chi_\Omega(\lambda - x) \overline{\chi}_\Omega(x) \, \mathrm{d}m_x \\ &= \frac{|\Omega \cap (\Omega + \lambda)|}{|Q_{\Lambda^*}|}. \end{aligned}$$

Λ tiles Ω, so if $\lambda \neq \{0, ..., 0\}$, then

$$\frac{|\Omega \cap (\Omega + \lambda)|}{|Q_{\Lambda^*}|} = 0$$

and if $\lambda = \{0, ..., 0\}$, then

$$\frac{|\Omega \cap (\Omega + \lambda)|}{|Q_{\Lambda^*}|} = \frac{|\Omega|}{|Q_{\Lambda^*}|} = \frac{1}{|Q_{\Lambda^*}|}. \tag{4.7}$$

From the definition of the fundamental domain of a lattice, we see that

$$|Q_\Lambda| \, |Q_{\Lambda^*}| = 1. \tag{4.8}$$

We now compute the Fourier coefficients of f. The function f is in $L^2(Q_\Lambda)$ and it is Λ-periodic. Let $\mu \in \Lambda^*$, then

$$\begin{aligned} \hat{f}(\mu) &= \frac{1}{|Q_\Lambda|} \int_{Q_\Lambda} e^{-2\pi i \langle x, \mu \rangle} \sum_{\lambda \in \Lambda} \chi_\Omega(x + \lambda) \, \mathrm{d}m_x \\ &= \frac{1}{|Q_\Lambda|} \sum_{\lambda \in \Lambda} \int_{Q_\Lambda} e^{-2\pi i \langle x, \mu \rangle} \chi_\Omega(x + \lambda) \, \mathrm{d}m_x \\ &= \frac{1}{|Q_\Lambda|} \int_{\mathbf{R}^n} e^{-2\pi i \langle x, \mu \rangle} \chi_\Omega(x) \, \mathrm{d}m_x \\ &= \frac{\hat{\chi}_\Omega(\mu)}{|Q_\Lambda|}. \end{aligned}$$

We have assumed that $f(x) = 1$ for almost every x, thus $\hat{f}(\mu) = 1$ if $\mu = \{0, ..., 0\}$ and $\hat{f}(\mu) = 0$ if $\mu \neq \{0, ..., 0\}$. This means that $\hat{f}(0) = |\Omega| \, |Q_\Lambda|^{-1} = |Q_\Lambda|^{-1} = 1$, which implies that $|Q_\Lambda| = 1$. Because of (4.7) and (4.8) we get that

$$\hat{F}(0, ..., 0) = \frac{1}{|Q_{\Lambda^*}|} = |Q_\Lambda| = 1.$$

We saw above that $\hat{F}(\lambda) = 0$ if $\lambda \neq \{0, ..., 0\}$. It now follows that $F(\xi) = 1$ for almost every $\xi \in \mathbf{R}^n$.

($\Leftarrow$) From the above we know that $\hat{f}(\mu) = |Q_\Lambda|^{-1} \hat{\chi}_\Omega(\mu)$. We assume that $F(\xi) = 1$ and this implies that $\hat{\chi}_\Omega(\mu) = 0$ if $\mu \neq \{0, ..., 0\}$, which in turn means that

$$\hat{f}(\mu) = 0, \tag{4.9}$$

if $\mu \neq \{0, ..., 0\}$. If $\mu = \{0, ..., 0\}$, then by direct computation

$$\hat{f}(0, ..., 0) = \frac{\hat{\chi}_\Omega(0, ..., 0)}{|Q_\Lambda|} = \frac{1}{|Q_\Lambda|} = |Q_{\Lambda^*}|\,.$$

Equation (4.7) and our assumption gives that

$$\hat{F}(0, ..., 0) = \frac{1}{|Q_{\Lambda^*}|} = 1,$$

which implies that $|Q_{\Lambda^*}| = 1$ and then that $\hat{f}(0, ..., 0) = 1$. Together with (4.9) we get that $f(x) = 1$ for almost every $x \in \mathbf{R}^n$. ∎

4.3 Results in the plane

In this section we start with some examples. We prove explicitly that neither the triangle nor the disk are spectral sets. Later in the section we give a much more general result which comprises the two examples. Nevertheless, the explicit proofs are interesting in their own right. In the first case we have an example of infinite systems of orthogonal exponential functions which are not complete. The second example deals with a case where every system of orthogonal exponential functions is finite. Our examples consequently illustrate two different things that can prevent a set from being spectral. Both examples originate from [Fug74].

Example 4.29 (A triangle is not a spectral set) Let Ω be the triangle

$$\Omega = \left\{(x, y) \in \mathbf{R}^2 : \ x, y > 0, x + y < 1\right\}.$$

We want to show that there exist infinite families of orthonormal exponential functions in $L^2(\Omega)$ and that none of them are complete in $L^2(\Omega)$.

Recall the notation $\mathcal{E}(\Lambda) = \{e_\lambda : \ \ \lambda \in \Lambda\} = \{\sqrt{2}e^{2\pi i\langle\lambda,\cdot\rangle} : \ \ \lambda \in \Lambda\}$. Notice the normalization factor $\sqrt{2}$, which is included because we want to work with a simple triangle Ω and a simple set Λ.

Assume that $(0, 0) \in \Lambda$ and let $\lambda = (\alpha, \beta) \in \mathbf{R}^2$. The following relation must be fulfilled in order to have orthogonality

$$\langle e_\lambda, e_0\rangle = 2\int_\Omega e^{2\pi i(\alpha x+\beta y)}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{4.10}$$

Equation (4.10) means that

$$\frac{1}{\alpha\beta(\alpha-\beta)}\left(\beta-\alpha-\beta e^{2\pi i\alpha}+\alpha e^{2\pi i\beta}\right)=0. \tag{4.11}$$

Consequently,

$$\alpha,\beta\neq 0,\quad \alpha\neq\beta \quad\text{and}\quad \beta-\alpha-\beta e^{2\pi i\alpha}+\alpha e^{2\pi i\beta}.$$

In order to fulfill the last equality above we need α and β to satisfy $e^{2\pi i\alpha}=e^{2\pi i\beta}=1$, hence

$$\lambda=(p,q),\quad \text{with}\quad p,q\in\mathbf{Z}\setminus\{0\},\quad p\neq q. \tag{4.12}$$

In other words, $\mathcal{E}(\Lambda)$ is orthonormal in $L^2(\Omega)$ if and only if every $\lambda\in\Lambda-\Lambda$ with $\lambda\neq(0,0)$ satisfies (4.12). This means in particular that the mapping

$$\Lambda\ni(\alpha,\beta)\mapsto\alpha\in\mathbf{R} \tag{4.13}$$

must be injective. An example of a set with this property is the set $\Lambda=\{(-p,p):p\in\mathbf{Z}\}$.

Let us now consider an arbitrary set Λ with the properties above, and prove that it cannot be complete in $L^2(\Omega)$. Let

$$S=\left\{(x,y)\in\mathbf{R}^2:\ 0<x,y<1/2\right\}$$

and define $f=\chi_S e_{\lambda_0}$ where $\lambda_0=(p_0,q_0)\in\mathbf{Z}\times\mathbf{Z}$ is given. Clearly $f\in L^2(\Omega)$ and we can easily calculate its norm

$$\|f\|^2=2\int_0^{1/2}\int_0^{1/2}\mathrm{d}x\,\mathrm{d}y=1/2.$$

Recall that an orthonormal system $\mathcal{E}(\Lambda)\subset L^2(\Omega)$ is complete in $L^2(\Omega)$ if and only if Parseval's identity

$$\|f\|^2_{L^2(\Omega)}=\sum_{\lambda\in\Lambda}|\langle f,e_\lambda\rangle|^2$$

holds for every $f\in L^2(\Omega)$. After some calculations we find that $|\langle f,e_\lambda\rangle|^2\neq 0$ if and only if p_0-p and q_0-q are odd numbers. If none of the five points

$$(p_0,q_0),\ (p_0+1,q_0),\ (p_0-1,q_0),\ (p_0,q_0+1),\ (p_0,q_0-1)$$

belongs to Λ, then

$$\sum_{\lambda\in\Lambda}|\langle f,e_\lambda\rangle|^2\leq\frac{4}{\pi^2}\sum_{k=3,5,\dots}\left(\frac{1}{k^2}+\frac{4}{\pi^2k^4}\right)<1/2=\|f\|^2_{L^2(\Omega)},$$

which clearly contradicts Parseval's identity. Next, by the injectivity of the mapping (4.13) it is possible to choose five integers $q_1,\dots,q_5$ such that none of the points $(0,q_j)$ is in Λ. If we apply the result above to $\lambda_0=(0,q_j)$ for $j=2,3,4$, we see that for each of the j's one of the points $(1,q_j)$ or $(-1,q_j)$ belongs to Λ. It is clear that this violates the injectivity of the mapping (4.13), thus no orthonormal system of the form $\mathcal{E}(\Lambda)$ can be complete in $L^2(\Omega)$. ▲

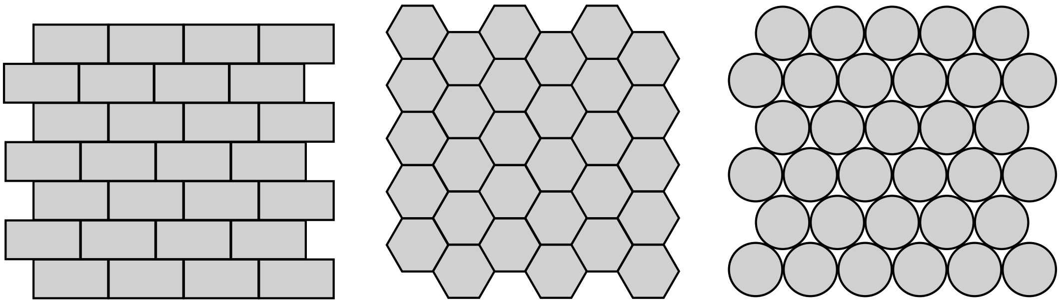

Figure 4.1: A rectangle tiles $\mathbf{R}^2$ in the obvious way. A hexagon tiles $\mathbf{R}^2$ in a honeycomb looking way, while a disk does not tile $\mathbf{R}^2$ by translation.

In the next example we prove that the plane disk is not a spectral set. This was first shown in [Fug74]. A more general result, also by Fuglede, was proved in [Fug01]: There does not exist an infinite set Λ such that any two elements in $\mathcal{E}(\Lambda)$ are pairwise orthogonal in $L^2(B)$, where B is the unit ball in $\mathbf{R}^n$.

We give two proofs, both based on the same idea, but the argumentation differs a bit. The first one is from [Kol04], while the second is Fuglede's original proof.

Definition 4.30 Let Ω be a Lebesgue measurable subset of $\mathbf{R}^n$ and let $\Lambda \subset \mathbf{R}^n$ be a discrete set. We say that Ω *packs* $\mathbf{R}^n$ *with* Λ *at level* ℓ if

$$\sum_{\lambda \in \Lambda} \chi_\Omega(x+\lambda) \leq \ell,$$

for almost every $x \in \mathbf{R}^n$. We say that $\Omega + \Lambda$ is a *packing*.

We will need a circle packing theorem by the Norwegian mathematician Axel Thue:

Theorem 4.31 (Thue, 1890) *The best possible packing of non-overlapping identical disks is the one obtained by a hexagonal tiling, as shown in Figure 4.1. The density of this packing is $\pi/\sqrt{12}$, i.e. the packing level is $\pi/\sqrt{12}$.*

See [PA95] for a proof.

Recall that $\Lambda \subset \mathbf{R}^n$ has asymptotic density, $\operatorname{dens}(\Lambda) = \rho$, if

$$\lim_{R\to\infty} \frac{\#\left(\Lambda \cap B_R(x)\right)}{|B_R(x)|} = \rho$$

for all $x \in \mathbf{R}^n$, where $B_R(x)$ is the ball of radius R centered in x.

Proposition 4.32 *Let $\Omega \subset \mathbf{R}^n$ tile $\mathbf{R}^n$ with Λ. Then* $\operatorname{dens}(\Lambda) = |\Omega|^{-1}$.

Proof: Fix $x \in \mathbf{R}^n$ and let R be some positive number. Assume that Ω tiles $\mathbf{R}^n$ with Λ. This means that $\sum_{\lambda\in\Lambda} \chi_\Omega(\xi - \lambda) = 1$ for almost every $\xi \in \mathbf{R}^n$. Then

$$|B_R(x)| = \int_{B_R(x)} \sum_{\lambda \in \Lambda} \chi_\Omega(\xi - \lambda)\, \mathrm{d}m_\xi$$

$$
\begin{aligned}
&= \sum_{|\lambda - x| < R} \int_{B_R(x)} \chi_\Omega(\xi - \lambda)\,\mathrm{d}m_\xi \\
&= \#\left(\Lambda \cap B_R(x)\right) \int_{B_R(x)} \chi_\Omega(\xi - \lambda)\,\mathrm{d}m_\xi .
\end{aligned}
$$

Let $R \to \infty$, then $\mathrm{dens}\,(\Lambda) = |\Omega|^{-1}$. ∎

Proposition 4.33 *If $\Omega + \Lambda$ is a packing at level ℓ, then Λ has density uniformly bounded by $\ell\,|\Omega|^{-1}$.*

The proof of this proposition is almost identical to the proof above.

Proof: Fix $x \in \mathbf{R}^n$ and let R be some positive number. By assumption, the set Ω packs $\mathbf{R}^n$ at level ℓ, i.e. $\sum_{\lambda\in\Lambda} \chi_\Omega(\xi - \lambda) \le \ell$ for almost every $\xi \in \mathbf{R}^n$.

$$
\begin{aligned}
|B_R(x)| &\ge \frac{1}{\ell} \int_{B_R(x)} \sum_{\lambda\in\Lambda} \chi_\Omega(\xi - \lambda)\,\mathrm{d}m_\xi \\
&= \frac{1}{\ell} \sum_{|\lambda - x| < R} \int_{B_R(x)} \chi_\Omega(\xi - \lambda)\,\mathrm{d}m_\xi \\
&= \frac{1}{\ell} \#\left(\Lambda \cap B_R(x)\right) \int_{B_R(x)} \chi_\Omega(\xi - \lambda)\,\mathrm{d}m_\xi .
\end{aligned}
$$

If we let $R \to \infty$, we get $\mathrm{dens}\,(\Lambda) \le \ell\,|\Omega|^{-1}$. ∎

Example 4.34 (The disk is not a spectral set) Let $D = \{x \in \mathbf{R}^2 : \ |x| < \frac{1}{\sqrt{\pi}}\}$, then $|D| = 1$. Also, let $\Lambda \subset \mathbf{R}^2$, $\lambda, \lambda' \in \Lambda$ and $|\lambda - \lambda'| = \rho$. We would like to find a relation between Λ and the Bessel function of order 1. Choose a coordinate system such that $\lambda - \lambda' = (\rho, 0)$, then

$$
\langle e_\lambda, e_{\lambda'} \rangle = \int_D e^{2\pi i \langle \lambda - \lambda', x\rangle}\,\mathrm{d}m_x = \int_D e^{2\pi i \rho x_1}\,\mathrm{d}m_x .
$$

We change to polar coordinates

$$
\int_D e^{2\pi i \rho x_1}\,\mathrm{d}m_x = \int_0^{1/\sqrt{\pi}} r \int_{-\pi}^{\pi} e^{2\pi i \rho r \cos\theta}\,\mathrm{d}\theta\,\mathrm{d}r
$$

and work with the inner integral

$$
\int_{-\pi}^{\pi} e^{2\pi i \rho r \cos\theta}\,\mathrm{d}\theta = \int_{-\pi}^{\pi} \left[\cos\left(2\pi\rho r\cos\theta\right) + i \sin\left(2\pi\rho r \cos\theta\right)\right]\,\mathrm{d}\theta .
$$

Both integrands are even, hence

$$
\int_{-\pi}^{\pi} \left[\cos\left(2\pi\rho r\cos\theta\right) + i \sin\left(2\pi\rho r \cos\theta\right)\right]\,\mathrm{d}\theta = 2\int_0^{\pi} \cos\left(2\pi\rho r\cos\theta\right)\,\mathrm{d}\theta
$$

$$+ 2i \int_0^\pi \sin(2\pi\rho r \cos\theta)\, \mathrm{d}\theta. \quad (4.14)$$

The last integral on the right-hand side of (4.14) is zero, since

$$\int_0^\pi \sin(2\pi\rho r \cos\theta)\, \mathrm{d}\theta = \int_{-\pi/2}^{\pi/2} \sin\left(2\pi\rho r \cos\left(\alpha + \frac{\pi}{2}\right)\right) \mathrm{d}\alpha = 0.$$

According to [SL98], the first integral on the right-hand side of (4.14) is

$$2\int_0^\pi \cos(2\pi\rho r \cos\theta)\, \mathrm{d}\theta = 2\pi J_0(2\pi\rho r),$$

where J_0 is the Bessel function of order 0. Using the formula $\int x J_0(x)\,\mathrm{d}x = xJ_1(x)$ from [SL98], where J_1 is the Bessel function of first order, we have

$$\langle e_\lambda, e_{\lambda'}\rangle = \int_0^{1/\sqrt{\pi}} r \int_{-\pi}^{\pi} e^{2\pi i \rho r \cos\theta}\,\mathrm{d}\theta\,\mathrm{d}r = \int_0^{1/\sqrt{\pi}} 2\pi r J_0(2\pi\rho r)\,\mathrm{d}r = 2\sqrt{\pi} J_1(2\pi\rho). \tag{4.15}$$

From (4.15) we see that $\mathcal{E}(\Lambda)$ is orthonormal in $L^2(D)$ if and only if the distance between any two points in Λ is a zero of the Bessel function J_1. Assume that D is a spectral set. The first zero of J_1 is approximately 3,832, thus the smallest distance between points in Λ is approximately $r_0 = 1,081$. Now, center a disk D_1 of radius $r_0/2$ in each point of Λ. This is a packing of the plane at level ℓ. We would like to find this ℓ and use Thue's theorem to obtain a contradiction. From Plancherel's theorem we have

$$\int_{\mathbf{R}^2} |\hat{\chi}_D(\xi)|^2\, \mathrm{d}m_\xi = 1$$

and since Λ is assumed to be a spectrum for D we have from Proposition 4.27 that

$$\sum_{\lambda\in\Lambda} |\hat{\chi}_D(\xi+\lambda)|^2 = 1.$$

Proposition 4.32 now gives that $\mathrm{dens}\,(\Lambda) = 1$. Taking Proposition 4.33 into account, we get

$$1 = \mathrm{dens}\,(\Lambda) \le \ell \left(\frac{\pi r_0^2}{4}\right)^{-1}.$$

Thue's theorem gives that

$$\frac{\pi r_0^2}{4} \le \ell \le \frac{\pi}{\sqrt{12}},$$

which implies that $r_0 \le 2/12^{1/4} \approx 1,075 < 1,081$. We have got a contradiction and D is not a spectral set.

Alternatively, we could have argued in the more geometric way Fuglede did. If Λ consists of 3 elements constituting an equilateral triangle, then $\mathcal{E}(\Lambda)$ is an orthonormal system if the sidelength of the triangle is a zero of the Bessel function J_1. If Λ consists of 4 elements, then the 6 different lines joining the elements can not have the same length. If there exists an orthonormal family $\mathcal{E}(\Lambda)$, then there must be a non-trivial algebraic relation between the elements of Λ. This again implies that there must be a non-trivial algebraic relation between the positive zeros of J_1. Such a relation does presumably not exist. The pattern is clear; when we increase the cardinality of Λ, we need more and more involved algebraic relations between the positive zeros of the Bessel function J_1. An orthonormal system in $L^2(D)$ can therefore have three elements, but probably not more. ▲

The examples above described some planar domains which are not spectral sets. It would be nice to have some generalizations of these results. In [Kol00] Kolountzakis proved a generalization of the triangle-case.

Theorem 4.35 (Kolountzakis) *Let Ω be a convex, non-symmetric, bounded open set in $\mathbf{R}^n$ of measure 1. Then Ω is not a spectral set.*

In [IKP99] it was proved that the ball in $\mathbf{R}^n$, $n > 1$, does not have a spectrum.

Theorem 4.36 (Iosevich, Katz and Pedersen) *An affine image of the unit ball in $\mathbf{R}^n$, $n > 1$, is not a spectral set.*

The arguments from this article was generalized to symmetric convex bodies with smooth boundary in $\mathbf{R}^n$, $n \geq 2$ and to symmetric convex plane bodies with piecewise smooth boundary with at least one point of non-vanishing curvature in [IKT01].

Theorem 4.37 (Iosevich, Katz and Tao) *Suppose that Ω is a symmetric convex body in $\mathbf{R}^n$, $n \geq 2$. If the boundary of Ω is smooth, then Ω does not admit a spectrum. The same conclusion holds in $\mathbf{R}^2$ if the boundary of Ω is piecewise smooth, and has at least one point of non-vanishing Gaussian curvature.*

Finally, it was proved by Iosevich, Katz and Tao [IKT03] that the only convex planar sets which are spectral are quadrilaterals and hexagons. These are also the only convex sets which tiles $\mathbf{R}^2$ by translations, see Figure 4.1, meaning that Fuglede's conjecture is true for convex sets in the plane.

Theorem 4.38 (Iosevich, Katz and Tao) *Let Ω be a convex compact set in the plane. The Fuglede conjecture holds. More precisely, Ω is a spectral set if and only if it tiles $\mathbf{R}^2$ by translations.*

The proofs of these results rest upon facts about the Fourier transform of the characteristic function of the convex set in question, as well as density properties of Λ. We will not go into details about this.

It should be noted that the conjecture is still open for non-convex sets in the plane.

4.4 Other dimensions

In this last section, we will try to give a brief overview of the latest research regarding Fuglede's conjecture, as well as some references. Even though the conjecture was formulated in 1974, the biggest progress has been made during the last ten years. The two-dimensional case was treated thoroughly earlier in this chapter, so the intention of this section is to tell what is known and what is unknown for $\mathbf{R}^n$, when $n \neq 2$.

For $n \geq 3$ the conjecture has recently been disproved. A breakthrough was made by Terence Tao in 2004. In [Tao04] Tao gave an example of a spectral set in $\mathbf{R}^n$, $n \geq 5$, which does not tile $\mathbf{R}^n$ by translations, proving that one of the directions in Fuglede's conjecture does not hold. The opposite direction was disproved for $n \geq 5$ by Kolountzakis and Matolcsi in [KM06b], where a non-spectral tiling set was constructed. In dimension 4 Matolcsi, [Mat05], modified Tao's example to disprove the "spectral set $\Rightarrow$ tiling set" part and Farkas and Révész, [FR06], gave an example of a non-spectral tile. The three-dimensional case was solved by Kolountzakis and Matolcsi, [KM06a], in the direction "spectral set $\Rightarrow$ tiling set" and by Farkas, Matolcsi and Móra, [FMM06], in the direction "tiling set $\Rightarrow$ spectral set". Even though the conjecture fails in general in $\mathbf{R}^n$, when $n \geq 3$, it might still be true for some special classes of sets, e.g. for convex sets, as in the case when $n = 2$.

The conjecture is still open in dimensions 1 and 2. The planar case was treated earlier. On the real line there are a couple of results supporting the conjecture. Łaba proved that the conjecture holds for a union of two intervals in [Łab01]. Also, it is true that if $\Omega \subset \mathbf{R}$ has measure 1 and is contained in an interval of length strictly less than $3/2$, then Ω is a spectral set if and only if Ω tiles the real line by translations, see [KŁ04].

Finally, it should be mentioned that Iosevich and Kolountzakis are planning a book on Fuglede's conjecture.

Chapter 5

Complete interpolating sequences for PW_M^2, when M is a polygon

We follow the article [LR00] by Lyubarskii and Rashkovskii. Let M be a convex polygon in $\mathbf{R}^2$, which is symmetric with respect to the origin. We repeat the definition of the Paley-Wiener space PW_M^2 of entire functions of two variables and some properties. The space is defined by

$$PW_M^2 = \left\{ f : \mathbf{C}^2 \to \mathbf{C} : \; f(z) = \frac{1}{4\pi^2} \int_M e^{i\langle z,w\rangle} \varphi(w)\,\mathrm{d}m_w, \; \varphi \in L^2(M) \right\}.$$

When tt is equipped with the $L^2(\mathbf{R}^2)$-norm, it is a Hilbert space. We identify $\mathbf{R}^2$ as the real plane in $\mathbf{C}^2$, dm is the usual Lebesgue measure in the plane and $\langle \cdot, \cdot \rangle$ is the $\mathbf{C}^2$ inner product, i.e. $\langle z, \zeta \rangle = z_1\bar{\zeta_1} + z_2\bar{\zeta_2}$, where $z = (z_1, z_2), \zeta = (\zeta_1, \zeta_2) \in \mathbf{C}^2$.

The supporting function of a convex set $M \subset \mathbf{R}^2$ was defined as

$$H_M(x) = \sup_{\xi \in M} \langle x, \xi \rangle .$$

According to Theorem 3.30, any $f \in PW_M^2$ satisfies the estimate

$$|f(x+iy)| \leq Ce^{H_M(y)}. \tag{5.1}$$

This fact will be used quite a lot throughout the chapter.

Our main objective is to construct some discrete sequence $\Omega \subset \mathbf{C}^2$ (our sequence will turn out to be real), such that for any sequence $\{a_\omega\}_{\omega\in\Omega} \in \ell^2(\Omega)$ the interpolation problem

$$f(\omega) = a_\omega, \quad \forall \omega \in \Omega \tag{5.2}$$

has a unique solution $f \in PW_M^2$. Recall that such a sequence Ω is called a complete interpolating sequence for PW_M^2. A full description of such sequences is (as far as we know) not known. To obtain complete interpolating sequences for PW_M^2 is by the duality reasoning in Section 3.6 equivalent to obtain Riesz bases of complex exponential

functions for the corresponding Lebesgue space $L^2(M)$. When M is a rectangle or a hexagon, we know by Theorem 4.38 that $L^2(M)$ has an orthonormal basis of complex exponential functions. From the main theorem in this chapter we will therefore get basis properties for domains that we do not get from Theorem 4.38.
In the one-dimensional case, complete interpolating sequences were proven to be the zero set of some entire function. Trying to lift this idea to several dimensions will immediately lead to problems. The main reason for this is, loosely speaking, that the zero set of an entire function of n complex variables is not a discrete set, but rather a $(n-1)$-dimensional complex analytic manifold.

The strategy for solving the specific interpolation problem encountered in this chapter, is to construct an entire function where the zero set is a union of hyperplanes. The interpolating sequence is then chosen to be the union of all pairwise intersections of the hyperplanes.

The outline of the chapter is as follows. First we give a representation for the polygon M and use this representation to write the supporting function for M explicitly. We move on by constructing the generating function S and proving some of its properties. By reducing the problem to one complex variable we are then able to prove a uniqueness theorem for PW_M^2. Finally, we state a result about the existence of a solution of the interpolation problem (5.2).

5.1 Contructing a sequence

Let M be a convex polygon with $2N$ vertices, symmetric with respect to the origin. We want to obtain a nice representation for M, in order to express the supporting function for M in an applicable way. Denote the vertices by $a^{(k)}$, $k = 1, ..., 2N$ according to the counterclockwise order they appear in, as shown in Figure 5.1 for $N = 5$. By symmetry we have $a^{(k)} = -a^{(k+N)}$. Now, define vectors $b^{(k)} \in \mathbf{R}^2$ by

$$b^{(k)} = \frac{1}{2\pi}\left(a^{(k+1)} - a^{(k)}\right), \quad k = 1, ..., N.$$

The vectors $b^{(k)}$ are also shown in Figure 5.1. Using the $b^{(k)}$'s we able to represent M explicitly

$$M = \sum_{k=1}^{N}\left[-\pi b^{(k)}, \pi b^{(k)}\right] = \left\{\sum_{k=1}^{N} t_k b^{(k)};\ |t_k| \le \pi,\ 1 \le k \le N\right\}. \tag{5.3}$$

We easily see that if we use the right-hand side of (5.3), the supporting function of M can be given explicitly

$$H_M(x) = \sup_{\xi\in M}\langle x, \xi\rangle = \sup_{-\pi\le t_k\le\pi}\left\{\sum_{k=1}^{N} t_k\left\langle x, b^{(k)}\right\rangle\right\} = \pi\sum_{k=1}^{N}\left|\left\langle x, b^{(k)}\right\rangle\right|.$$

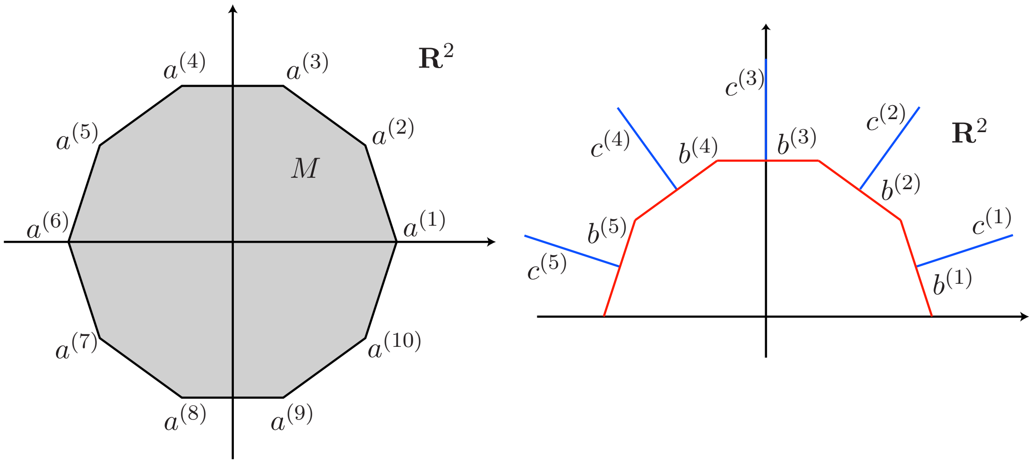

Figure 5.1: A convex polygon M, symmetric with respect to the origin, with $N = 5$. The vectors $a^{(k)}$, $b^{(k)}$ and $c^{(k)}$ are indicated.

Based on the supporting function of M we define a function $S : \mathbf{C}^2 \to \mathbf{C}$ by

$$S(z) = \prod_{k=1}^{N} \sin\left(\pi\left(\left\langle z, b^{(k)}\right\rangle - \alpha_k\right)\right),$$

where $\alpha_k \in \mathbf{R}$ and will be specified later. Soon we will see that S plays the same role as the sine-type functions did in the one-dimensional case in Section 3.3. S will generate a complete interpolating sequence for PW_M^2. The rest of this section consists of a number of properties of S and its zero set. Notice first that S is an entire function. Moreover, the growth of S is controlled by the supporting function of M, as shown in the next proposition.

Proposition 5.1 *Given $\delta > 0$, then*

$$|S(z)| \asymp \exp\left(H_M(y)\right)$$

for z such that

$$\operatorname{dist}\left(\left\langle z, b^{(k)}\right\rangle - \alpha_k, \mathbf{Z}\right) > \delta$$

when $k = 1, ..., N$.

In the sequel we will write $E_\delta = \left\{z \in \mathbf{C}^2 : \operatorname{dist}\left(\left\langle z, b^{(k)}\right\rangle - \alpha_k, \mathbf{Z}\right) \leq \delta\right\}$.

Proof: Let $\zeta = \xi + i\eta \in \mathbf{C}$. If $\operatorname{Im}\zeta > 1$, then

$$|\sin \pi\zeta| = \frac{1}{2}\left|e^{i\pi\zeta} - e^{-i\pi\zeta}\right| \leq \frac{1}{2}\left(\left|e^{i\pi\zeta}\right| + \left|e^{-i\pi\zeta}\right|\right) \leq Ce^{\pi\eta} = Ce^{\pi \operatorname{Im}\zeta}$$

and similarly if $\operatorname{Im}\zeta < -1$

$$|\sin \pi\zeta| = \frac{1}{2}\left|e^{i\pi\zeta} - e^{-i\pi\zeta}\right| \leq \frac{1}{2}\left(\left|e^{i\pi\zeta}\right| + \left|e^{-i\pi\zeta}\right|\right) \leq Ce^{-\pi\eta} = Ce^{\pi|\operatorname{Im}\zeta|},$$

so for $|\mathrm{Im}\,\zeta| > 1$ we have
$$|\sin \pi\zeta| \leq Ce^{\pi|\mathrm{Im}\,\zeta|}.$$

When $|\mathrm{Im}\,\zeta| \leq 1$ we use periodicity. The function $\sin \pi\zeta$ has period 2, thus we need only consider one such period. The zeros are located at the integers. It is possible to find some constant $c = c(\delta)$ such that $|\sin \pi\zeta| \geq ce^{\pi|\mathrm{Im}\,\zeta|}$ when $\mathrm{dist}(\zeta, \mathbf{Z}) > \delta$. For the rest of the period we may use the maximum principle to argue that $|\sin \pi\zeta|$ is bounded from above. We have obtained

$$|\sin \pi\zeta| \asymp e^{\pi|\mathrm{Im}\,\zeta|},$$

for $\zeta \in \mathbf{C}$ such that $\mathrm{dist}(\zeta, \mathbf{Z}) > \delta$.

If we substitute ζ with $\left\langle z, b^{(k)} \right\rangle - \alpha_k$ we have

$$\left|\sin \pi \left(\left\langle z, b^{(k)} \right\rangle - \alpha_k\right)\right| \asymp e^{\pi\left|\mathrm{Im}\left(\left\langle z,b^{(k)}\right\rangle - \alpha_k\right)\right|} = e^{\pi\left|\left\langle y,b^{(k)}\right\rangle\right|},$$

for $z \notin E_\delta$. We now have

$$\begin{aligned} |S(z)| &= \prod_{k=1}^{N} \left|\sin \pi \left(\left\langle z, b^{(k)} \right\rangle - \alpha_k\right)\right| \asymp \prod_{k=1}^{N} e^{\pi\left|\left\langle y,b^{(k)}\right\rangle\right|} \\ &= \exp\left(\sum_{k=1}^{N} \pi \left|\left\langle y, b^{(k)} \right\rangle\right|\right) = \exp\left(H_M(y)\right) \end{aligned}$$

for $z \notin E_\delta$. ∎

In the next proposition we prove that the zero set of S is a union of hyperplanes.

Proposition 5.2 *Let $Z(S)$ denote the zero set of S. The set $Z(S)$ is the union of the hyperplanes*

$$P^{(k,n)} = \left\{z \in \mathbf{C}^2 : \left\langle z, b^{(k)} \right\rangle = n + \alpha_k\right\}, \quad n \in \mathbf{Z}, k = 1, ..., N.$$

Proof: The zero set of the factor $\sin\left(\pi\left(\left\langle z, b^{(k)}\right\rangle - \alpha_k\right)\right)$ is the union of the disjoint hyperplanes
$$P^{(k,n)} = \left\{z \in \mathbf{C}^2 : \left\langle z, b^{(k)} \right\rangle = n + \alpha_k\right\}, \quad n \in \mathbf{Z}.$$

The zero set of S is then
$$Z(S) = \bigcup_{\substack{k=1,\dots,N \\ n\in\mathbf{Z}}} P^{(k,n)}.$$

∎

Each hyperplane $P^{(k,n)}$ may be represented in the following way

$$P^{(k,n)} = \left\{z = b^{(k,n)} + c^{(k)}\zeta;\ \zeta \in \mathbf{C}\right\}, \tag{5.4}$$

where $b^{(k,n)} = (n+\alpha_k)\frac{b^{(k)}}{|b^{(k)}|^2}$ and $c^{(k)}$ is the unit normal to $b^{(k)}$ in $\mathbf{R}^2$ which is turned into $\frac{b^{(k)}}{|b^{(k)}|}$ if rotated $\pi/2$ in the clockwise direction. The vectors $c^{(k)}$ are shown in Figure 5.1 for $N=5$. Using this representation of the hyperplanes we are able to prove that each pair of hyperplanes intersect in a point and that all such points are located in $\mathbf{R}^2$. The union of all such points will form a complete interpolating sequence for PW_M^2. It will also be shown that we can collect all points being in the intersection of three or more planes in an exceptional set, having Lebesgue measure zero in the space of parameters $\mathbf{R}^N_{(\alpha)}$.

Proposition 5.3 *Let $\omega^{(k_1,n_1)(k_2,n_2)}$ be the points in the intersection*

$$P^{(k_1,n_1)} \cap P^{(k_2,n_2)}$$

and let Ω be the set of all such points, when $k_1 \neq k_2$. Then Ω is a subset of $\mathbf{R}^2$.

Proof: Letting $k_1 \neq k_2$ and using the representation (5.4) for the hyperplanes, we see that a point in the intersection $P^{(k_1,n_1)} \cap P^{(k_2,n_2)}$ must satisfy

$$\omega^{(k_1,n_1)(k_2,n_2)} = b^{(k_1,n_1)} + c^{(k_1)}\zeta_1 = b^{(k_2,n_2)} + c^{(k_2)}\zeta_2, \quad \zeta_1,\zeta_2 \in \mathbf{C}. \tag{5.5}$$

The last equality in (5.5) is a system of two equations with unique solutions

$$\begin{aligned}
\zeta_1 &= \frac{c_2^{(k_2)}}{c_1^{(k_1)}c_2^{(k_2)} - c_1^{(k_2)}c_2^{(k_1)}}\left(b_1^{(k_2,n_2)} - b_1^{(k_1,n_1)} - \frac{c_1^{(k_2)}}{c_2^{(k_2)}}\left(b_2^{(k_2,n_2)} - b_2^{(k_1,n_1)}\right)\right)\\
\zeta_2 &= \frac{c_2^{(k_1)}}{c_1^{(k_1)}c_2^{(k_2)} - c_1^{(k_2)}c_2^{(k_1)}}\left(b_1^{(k_2,n_2)} - b_1^{(k_1,n_1)} - \frac{c_1^{(k_1)}}{c_2^{(k_1)}}\left(b_2^{(k_2,n_2)} - b_2^{(k_1,n_1)}\right)\right).
\end{aligned}$$

The numbers ζ_1 and ζ_2 are real. Both $b^{(k_1,n_1)}$ and $c^{(k_1)}$ are real as well, hence a point $\omega^{(k_1,n_1)(k_2,n_2)}$ in the intersection $P^{(k_1,n_1)} \cap P^{(k_2,n_2)}$ must necessarily be unique and real.

We introduce some notation

$$\zeta_1 = x_{(k_2,n_2)}^{(k_1,n_1)} \quad \zeta_2 = x_{(k_1,n_1)}^{(k_2,n_2)}, \tag{5.6}$$

and

$$\begin{aligned}
\Omega_k^{(k_1,n_1)} &= \left\{\omega^{(k_1,n_1)(k,n)} : n \in \mathbf{Z}\right\}, & X_k^{(k_1,n_1)} &= \left\{x_{(k,n)}^{(k_1,n_1)} : n \in \mathbf{Z}\right\}\\
\Omega^{(k_1,n_1)} &= \bigcup_{k\neq k_1} \Omega_k^{(k_1,n_1)}, & X^{(k_1,n_1)} &= \bigcup_{k\neq k_1} X_k^{(k_1,n_1)}.
\end{aligned}$$

Then

$$\Omega = \bigcup_{\substack{k=1,..,N\\ n_1\in\mathbf{Z}}} \Omega^{(k_1,n_1)} \subset \mathbf{R}^2.$$

∎

Proposition 5.4 *Let Ω be as in Proposition 5.3. Then Ω contains no points lying in the intersection of three or more hyperplanes for all*

$$\alpha = (\alpha_1, ..., \alpha_N) \in \mathbf{R}^N \setminus E_M,$$

where E_M is a countable collection of hyperplanes in the parameter space $\mathbf{R}^N_{(\alpha)}$ and is of zero Lebesgue measure.

Proof: We must determine the exceptional set E_M and show that its N-dimensional Lebesgue measure is zero. We do not want to consider points lying in the intersection of three or more hyperplanes and would therefore like to exclude points satisfying

$$\omega^{(k_1,n_1)(k_2,n_2)} = \omega^{(k_1,n_1)(k_3,n_3)}, \quad (k_2, n_2) \neq (k_3, n_3). \tag{5.7}$$

Such a point lies in the intersection $P^{(k_1,n_1)} \cap P^{(k_2,n_2)} \cap P^{(k_3,n_3)}$. Condition (5.7) implies a linear relation between $n_1 + \alpha_1$, $n_2 + \alpha_2$ and $n_3 + \alpha_3$ and defines a hyperplane in $\mathbf{R}^N_{(\alpha)}$. A hyperplane in $\mathbf{R}^N_{(\alpha)}$ is of N-dimensional Lebesgue measure zero. The exceptional set consists of the union of such hyperplanes for all triples (k_1, n_1), (k_2, n_2) and (k_3, n_3). The union is countable, hence the N-dimensional Lebesgue measure of the set E_M is zero. ∎

It is important to know when Ω is uniformly separated, and the next proposition gives us a condition for this.

Proposition 5.5 *Let*

$$s_j^{(k)} = \frac{1}{\left|\left\langle b^{(j)}, c^{(k)} \right\rangle\right|}, \quad j \neq k$$

and eliminate multiple points. Ω is uniformly separated if and only if $s_l^{(k)}/s_m^{(k)} \in \mathbf{Q}$ for all $k, l, m \in \{1, 2, ..., N\}$.

Proof: Consider the sequence $X_k^{(k_1,n_1)}$ and calculate the distance between the points. Using the explicit formulas for the points $x_{(k,n+1)}^{(k_1,n_1)}$ and $x_{(k,n)}^{(k_1,n_1)}$ we see that

$$x_{(k,n+1)}^{(k_1,n_1)} - x_{(k,n)}^{(k_1,n_1)} = \frac{c_1^{(k)} b_2^{(k)} - c_2^{(k)} b_1^{(k)}}{c_2^{(k_1)} c_1^{(k)} - c_2^{(k)} c_1^{(k_1)}} \frac{1}{\left|b^{(k)}\right|^2}.$$

Further, using the fact that $\left\langle b^{(k)}, c^{(k)} \right\rangle = 0$ we obtain

$$\left| x_{(k,n+1)}^{(k_1,n_1)} - x_{(k,n)}^{(k_1,n_1)} \right| = \frac{1}{\left|\left\langle b^{(k)}, c^{(k_1)} \right\rangle\right|} = s_k^{(k_1)}.$$

Thus we see that the points in the sequence $X_l^{(k_1,n_1)} \cup X_m^{(k_1,n_1)}$, $l \neq m$, are separated if and only if $s_l^{(k_1)}/s_m^{(k_1)}$ is a rational number. ∎

The example below illustrates that even very simple polygons may not have the separation property.

Example 5.6 Let M be the polygon, symmetric with respect to the origin, where $N = 3$, $a^{(1)} = (1,0)$, $a^{(2)} = (1/2,1)$ and $a^{(3)} = (-1/2,1)$. We calculate $b^{(k)}$ and $c^{(k)}$, for $k = 1$ and $k = 2$

$$\begin{aligned} b^{(1)} &= \left(\frac{-1}{4\pi}, \frac{1}{2\pi}\right), \quad b^{(2)} = \left(\frac{-1}{2\pi}, 0\right) \\ c^{(1)} &= \left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right), \quad c^{(2)} = (0,-1)\,. \end{aligned}$$

Then $s_1^{(2)} = 2\pi$ and $s_2^{(1)} = \sqrt{5}\pi$ and $s_2^{(1)}/s_1^{(2)} = \sqrt{5}/2$, which is irrational. The polygon M does not have the separation property. ▲

We say that a polygon M is approximated by polygons with the separation property, if for a given $\varepsilon > 0$ there exists a polygon N with the separation property, such that for any $x \in M$ we have $\operatorname{dist}(x, N) < \varepsilon$. For possible applications we include a result without proof, saying that any convex polygon symmetric with respect the origin can be approximated by polygons with the separation property.

Proposition 5.7 *Any convex polygon in $\mathbf{R}^2$ symmetric with respect to the origin can be approximated, from the inside and from the outside, by polygons having the separation property.*

A proof can be found in [LR00].

5.2 A uniqueness theorem

Recall from Section 3.2 that if Ω is a set of uniqueness for PW_M^2, then the system $\mathcal{E}(\Omega) = \left\{e^{i\langle\omega,\cdot\rangle} : \omega \in \Omega\right\}$ is complete in $L^2(M)$. This should motivate the following theorem.

Theorem 5.8 *Let Ω be as in Section 5.1, $\alpha \notin E_M$ and $f \in PW_M^2$. If $f\big|_\Omega = 0$, then $f \equiv 0$.*

The method of this proof is close to the method used in the uniqueness proof of Theorem 3.11. The main difference is of course that we have two complex variables instead of one. However, we will see that this problem can be reduced one dimension. Before we give the proof, we need some preliminaries about zero sets of entire functions in $\mathbf{C}^2$. References for this material are [Ron74] and [Sha92].

Definition 5.9 Let $G \subset \mathbf{C}^2$ be a domain. A set $A \subset G$ is called an *analytic set* if for any point $z_0 \in A$, there is a neighborhood V of z_0 and a system of functions $\{f_i\}_{i=1}^n \subset \operatorname{Hol}(V)$ such that

$$A \cap V = \{z \in V : \; f_i(z) = 0, i = 1, ..., n\}\,. \tag{5.8}$$

It follows from the definition that if $f \in \mathrm{Hol}(\mathbf{C}^2)$, then

$$Z(f) = \left\{z \in \mathbf{C}^2 : \; f(z) = 0\right\}$$

is an analytic set. The next property of the zero set of the entire function S from the previous section is stated informally as a fact, because the formal theorem is too involved to write here.

Fact 5.10 *The zero set of the entire function $S : \mathbf{C}^2 \to \mathbf{C}$ is locally a one-dimensional analytic manifold, except for a set of discrete points.*

Definition 5.11 Let A be an analytic set in some domain in $\mathbf{C}^2$. If the functions in (5.8) can be chosen in some neighborhood V of z_0 such that the rank of the Jacobi matrix

$$\left(\frac{\partial f_i}{\partial z_j}\right)_{\substack{i=1,\ldots,n \\ j=1,2}}$$

is equal to $0 \leq k \leq 2$, then z_0 is called a *regular point* of A. The complex dimension of A at z_0 is $n - k = m$. The set of regular points in A is denoted A^*. $A \setminus A^*$ are the *singular points* of A.

Definition 5.12 Let z_0 be a regular point of $Z(f)$, when $f \in \mathrm{Hol}(\mathbf{C}^2)$. f has a power series expansion around z_0

$$f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha (z - z_0)^\alpha,$$

where α is a two-dimensional multi-index. The *multiplicity* of z_0 is defined as

$$\gamma_f(z_0) = \min_{a_\alpha \neq 0} |\alpha| \, .$$

Instead of the definition above, we could have defined the multiplicity of a zero z_0 of an entire function f in a more geometric way. Let z_0 be a regular point of $Z(f)$ and consider an analytic plane P containing z_0, such that P is not a tangent plane of $Z(f)$ at z_0. When restricted to P, f is an analytic function of one variable, and z_0 is an (isolated) zero. Multiplicity of z_0 may then be used as in one-variable theory. It can be proved that the multiplicity is independent of the choice of P, as long as P is not tangent to $Z(f)$ at z_0.

Moreover, the multiplicity is an integer-valued continuous function on the regular points of $Z(f)$, hence it is constant on the connected parts of the set of regular points.

Theorem 5.13 *Let $f, g \in \mathrm{Hol}(\mathbf{C}^2)$ and $Z(g) \subset Z(f)$. If $\gamma_f(z_0) \geq \gamma_g(z_0)$ for all $z_0 \in Z(g)$, then $f/g \in \mathrm{Hol}(\mathbf{C}^2)$.*

See [Sha92, p. 129] for a proof.

We will now introduce a generating function for $X^{(k_1,n_1)}$. $X^{(k_1,n_1)}$ may be considered as a subset of a plane parametrized by $\zeta \in \mathbf{C}$, so we consider the one-variable function

$$L_k^{(k_1,n_1)}(\zeta) = \sin \pi \left(\left\langle b^{(k)}, c^{(k_1)} \right\rangle \zeta - \beta_k^{k_1,n_1} \right), \quad \zeta \in \mathbf{C},$$

where the numbers $\beta_k^{k_1,n_1}$ are chosen such that

$$L_k^{(k_1,n_1)} \Big|_{X_k^{(k_1,n_1)}} = 0.$$

The function

$$L^{(k_1,n_1)}(\zeta) = \prod_{k \neq k_1} L_k^{(k_1,n_1)}(\zeta)$$

is then a generating function for $X^{(k_1,n_1)}$. In the next propositions we will need the following notation. Let

$$\sigma^{(k_1)} = \pi \sum_{k=1}^{N} \left| \left\langle b^{(k)}, c^{(k_1)} \right\rangle \right|.$$

Proposition 5.14 *The function $L^{(k_1,n_1)}$ is an entire function of one variable and satisfies*

$$\left| L^{(k_1,n_1)}(\zeta) \right| \asymp \exp \left(\sigma^{(k_1)} \left| \operatorname{Im} \zeta \right| \right)$$

when $\operatorname{dist}\left(\zeta, X^{(k_1,n_1)}\right) > \delta$ *for some* $\delta > 0$.

Proof: The proof is contained in the proof of Proposition 5.1. ∎

The proposition below is a one-dimensional uniqueness result, which is needed in the proof of Theorem 5.8.

Proposition 5.15 *If $f \in PW^2_{\sigma^{(k_1)}}$ and $f\big|_{X^{(k_1,n_1)}} = 0$, then $f \equiv 0$. $X^{(k_1,n_1)}$ is a complete interpolating sequence for $PW^2_{\sigma^{(k_1)}}$ if it is uniformly separated.*

Notice that $PW^2_{\sigma^{(k_1)}}$ is a space of entire functions of *one* complex variable.

Proof: Recall the proof of Theorem 3.11. The uniqueness part of the proof did not demand that the sequence was uniformly separated, so the proof applies to the first part of this proposition as well.

If $X^{(k_1,n_1)}$ is uniformly separated, then $L^{(k_1,n_1)}$ is a sine-type function and $X^{(k_1,n_1)}$ is a complete interpolating sequence for $PW^2_{\sigma^{(k_1)}}$ by Theorem 3.11. ∎

Proof of Theorem 5.8: For all $k_1 = 1, .., N$ and $n_1 \in \mathbf{Z}$ we consider the trace of the function f on the hyperplane $P^{(k_1,n_1)}$ and define

$$f^{(k_1,n_1)}(\zeta) = f\left(b^{(k_1,n_1)} + c^{(k_1)} \zeta \right), \quad \zeta \in \mathbf{C}.$$

We make a growth estimate for $f^{(k_1,n_1)}$,

$$\begin{aligned}\left|f^{(k_1,n_1)}(\zeta)\right| &= \left|f\left(b^{(k_1,n_1)} + c^{(k_1)}\zeta\right)\right| \\ &\asymp \exp\left(H_M\left(\operatorname{Im}\left(b^{(k_1,n_1)} + c^{(k_1)}\zeta\right)\right)\right) \\ &= \exp\left(H_M\left(\operatorname{Im}\left(c^{(k_1)}\zeta\right)\right)\right) \\ &= \exp\left(\pi\left|\operatorname{Im}\zeta\right|\sum_{k=1}^{N}\left|\left\langle b^{(k)}, c^{(k_1)}\right\rangle\right|\right) \\ &= e^{\sigma^{(k_1)}|\operatorname{Im}\zeta|},\end{aligned}$$

when

$$\operatorname{dist}\left(\zeta, X^{(k_1,n_1)}\right) > \delta > 0.$$

The estimate implies that $f^{(k_1,n_1)} \in PW^2_{\sigma^{(k_1)}}$. By assumption

$$f^{(k_1,n_1)}\big|_{X^{(k_1,n_1)}} = 0,$$

and Proposition 5.15 then implies that $f^{(k_1,n_1)} \equiv 0$. In other words, $f\big|_{P^{(k_1,n_1)}} = 0$. Define a function

$$\Phi(z) = \frac{f(z)}{S(z)}, \quad z \in \mathbf{C}^2.$$

The multiplicity of the zeros of each of the factors of S is 1, and the regular parts of the zero set of the factors do not coincide, so the multiplicity of the zeros of S is 1. Moreover, since $Z(S) \subset Z(f)$, we may apply Theorem 5.13 to see that Φ is an entire function. Since $f \in PW_M^2$ and because of the growth estimate for S from Proposition 5.1, we see that

$$|\Phi(z)| \leq \frac{C_f e^{H_M(z)}}{c_S e^{H_M(z)}} \leq C_\delta, \quad z \notin E_\delta.$$

To extend this estimate to any $z \in \mathbf{C}^2$ we use plurisubharmonicity properties of $\log|\Phi|$. Let z_0 be a point on a plane $P_1 \subset Z(S)$. Take any analytic plane P_2 containing z_0, such that the tangent spaces of P_1 and P_2 does not coincide at z_0. P_1 and P_2 intersect in the point z_0. We may now apply the mean value inequality to $\log|\Phi|$ at z_0 to see that $\log|\Phi|$ is bounded in all of $\mathbf{C}^2$. This means that Φ is bounded in $\mathbf{C}^2$ as well. Liouville's theorem now tells us that $\Phi(z)$ is equal to some constant for every $z \in \mathbf{C}^2$. We would like to show that this constant must be equal to zero. The Riemann-Lebesgue lemma gives

$$|f(x + (i,i))| \to 0, \quad \text{as } |x| \to \infty,\ x \in \mathbf{R}^2, \tag{5.9}$$

and by the growth estimate for S, we get

$$|S(x + (i,i))| \asymp 1, \quad \text{as } |x| \to \infty,\ x \in \mathbf{R}^2. \tag{5.10}$$

(5.9) and (5.10) then implies that $\Phi(x + (i,i)) \to 0$ as $|x| \to \infty$, for $x \in \mathbf{R}^2$. This means that $\Phi \equiv 0$, which again implies that $f \equiv 0$. ∎

5.3 An existence theorem

In this section we assume that $\alpha \notin E_M$ and that Ω is uniformly separated. We would like to state a result about the existence of a function, $f \in PW_M^2$, solving the interpolation problem

$$f(\omega) = a_\omega, \quad \forall \omega \in \Omega,$$

for each given $\{a_\omega\}_{\omega\in\Omega} \in \ell^2(\Omega)$. If such a function exists, it is unique by Theorem 5.8. The set Ω is then a complete interpolating sequence for the space PW_M^2.

Before we state the existence theorem, we need some notational preparation. For each point $\omega = \omega^{(k_1,n_1)(k_2,n_2)} \in \Omega$, we define a function $\varphi_\omega : \mathbf{C}^2 \to \mathbf{C}$ by

$$\varphi_\omega(z) = \prod_{k\neq k_1,k_2} \sin\left(\pi\left(\left\langle z, b^{(k)}\right\rangle - \alpha_k\right)\right) r_{k_1,n_1}(z) r_{k_2,n_2}(z), \tag{5.11}$$

where

$$r_{k,n}(z) = \frac{\sin\left(\pi\left(\left\langle z, b^{(k)}\right\rangle - \alpha_k\right)\right)}{\left\langle z, b^{(k)}\right\rangle - n - \alpha_k}.$$

The zero set of φ_ω is $\Omega \setminus \omega$ and since the sequence Ω is real and uniformly separated, we have the estimate

$$|\varphi_\omega(\omega)| \asymp 1.$$

For simplicity we will denote $|\varphi_\omega(\omega)|$ by c_ω.

Theorem 5.16 *Let Ω be uniformly separated. Given a sequence $\{a_\omega\}_{\omega\in\Omega} \in \ell^2(\Omega)$, then the interpolation problem*

$$f(\omega) = a_\omega, \quad \forall \omega \in \Omega,$$

has a solution $f = f_a \in PW_M^2$ of the form

$$f_a(z) = \sum_{\omega\in\Omega} \frac{a_\omega}{c_\omega} \varphi_\omega(z), \quad z \in \mathbf{C}^2. \tag{5.12}$$

The series converges both in the norm of PW_M^2 and uniformly on compact sets in $\mathbf{C}^2$.

Using the duality between Paley-Wiener spaces and Lebesgue spaces, we may write Theorem 5.16 in a different way.

Theorem 5.17 *Let Ω be uniformly separated and $\alpha \notin E_M$. Then the system $\mathcal{E}(\Omega) = \left\{e^{i\langle\omega,\cdot\rangle} : \omega \in \Omega\right\}$ is a Riesz basis for the space $L^2(M)$. This means that each function $\varphi \in L^2(M)$, can be expanded in a non-harmonic Fourier series*

$$\varphi(x) = \sum_{\omega\in\Omega} c_\omega(\varphi) e^{i\langle x,\omega\rangle},$$

which converges in $L^2(M)$-norm, and

$$\|\varphi\|_{L^2(M)}^2 \asymp \sum_{\omega\in\Omega} |c_\omega(\varphi)|^2 .$$

5.4 What if Ω is not separated?

We saw in Example 5.6 that even very simple polygons may not have the separation property. There are some possible ways to deal with this problem.

In Section 5.1 we saw that it is possible to approximate any convex polygon symmetric with respect to the origin with a polygon having the separation property. Another alternative which was proposed by Lyubarskii and Rashkovskii, is to introduce block interpolation. We did this for the one-dimensional problem in Section 3.5. Lyubarskii and Rashkovskii used similar ideas to obtain a block interpolation procedure for the case encountered in this chapter. This was done in [LR00]. A last possible approach might be to modify the generating function S. This can be done by changing the factors from sines to sine-type functions with real zeros. To our knowledge it is not known whether it is possible to choose the sine-type functions such that the set Ω will be uniformly separated.

Chapter 6

Towards a uniqueness theorem for PW_D^2, when D is a disk

The subject of study in this chapter is the space PW_D^2, where $D \subset \mathbf{R}^2$ is the closed disk of radius π, centered at the origin. It is defined, in the same way as PW_M^2 was defined in Chapter 5, by

$$PW_D^2 = \left\{ f : \mathbf{C}^2 \to \mathbf{C} : \; f(z) = \frac{1}{4\pi^2} \int_D e^{i\langle z,w\rangle} \varphi(w) \, \mathrm{d}m_w, \varphi \in L^2(D) \right\}.$$

Just as PW_M^2, the space PW_D^2 is a Hilbert space when it is equipped with the $L^2(\mathbf{R}^2)$-norm. We use the same notation as earlier, a point $z \in \mathbf{C}^2$ may be represented in the form $z = (z_1, z_2)$, where $z_1, z_2 \in \mathbf{C}$ or the form $z = x + iy$, where $x, y \in \mathbf{R}^2$. The inner product on $\mathbf{C}^2$ is defined as $\langle z, \zeta \rangle = z_1\bar{\zeta}_1 + z_2\bar{\zeta}_2$, where $z, \zeta \in \mathbf{C}^2$. The usual Lebesgue measure in the plane will be denoted $\mathrm{d}m$.

Recall the definition of the supporting function of a convex set $\Omega \subset \mathbf{R}^2$

$$H_\Omega(y) = \sup_{\xi \in \Omega} \langle y, \xi \rangle \,, \quad y \in \mathbf{R}^2.$$

We see that

$$H_D(y) = \pi \, |y| \,, \quad y \in \mathbf{R}^2.$$

By the Plancherel-Polya theorem (Theorem 3.30) we know that every $f \in PW_D^2$ satisfies the estimate

$$|f(z)| \le Ce^{H_D(y)} = Ce^{\pi|y|}.$$

The idea of this chapter is to use the techniques developed in Chapter 5 to see if we can extract some information about PW_D^2. To do this we need a series representation for the supporting function of D. This can be done using the plurisubharmonicity of the function $u(z) = \pi \, |\mathrm{Im}\, z|$. Let us first consider a one-dimensional analog of u, namely $\pi \, |\mathrm{Im}\, z|$, for $z \in \mathbf{C}$, and see how a series approximation of such a subharmonic function can be obtained. We will then try to lift the idea to two complex variables.

The problem of approximating a subharmonic function with a series can be stated in much greater generality: Given a subharmonic function u in $\mathbf{C}$. Can we find an entire function f such that the difference $u(z) - \log|f(z)|$ in some sense is small? How good can such an approximation be? If we can find such an entire function f, then $\log|f|$ can be written as a series due to the canonical representation of entire functions as infinite products.

Approximation of subharmonic functions by logarithms of moduli of entire functions was used as a tool by Beurling and Malliavin in [BM62] and by Mergeljan in [Mer62]. The general problem of approximating a subharmonic function of finite order was first considered by Azarin. In [Aza69] it was proved that for any subharmonic function u of finite order ρ, there exists an entire function f such that

$$u(z) - \log|f(z)| = o\left(|z|^\rho\right), \quad z \to \infty,\ z \notin E,$$

where E is some small exceptional set. In [Yul85] Yulmukhametov made a breakthrough by proving the following: Given a subharmonic function u of finite order, there exists an entire function f such that

$$u(z) - \log|f(z)| = O\left(\log|z|\right), \quad z \to \infty,\ z \notin E,$$

where E is an exceptional set satisfying some natural smallness estimates, depending on the order of u. In [LM01] Lyubarskii and Malinnikova was able to improve Yulmukhametov's result.

Theorem 6.1 (Lyubarskii and Malinnikova, 2001) *Let u be a subharmonic function in $\mathbf{C}$, with Riesz measure μ. Suppose*

$$\mu\left(\{z:\ R < |z| \le qR\}\right) > 1,\ R > R_0,$$

for some $R_0 > 0$ and $q > 1$. Then there exists an entire function f satifying

$$\sup_{R>0} \frac{1}{\pi R^2} \int_{|z|<R} |u(z) - \log|f(z)||\, \mathrm{d}m_z < \infty,$$

and for each $\varepsilon > 0$ there exists a set $E_\varepsilon \subset \mathbf{C}$ with

$$\limsup_{R\to\infty} \frac{m\left(\{z \in E_\varepsilon :\ |z| < R\}\right)}{\pi R^2} < \varepsilon$$

such that

$$u(z) - \log|f(z)| = O(1), \quad z \to \infty,\ z \notin E_\varepsilon.$$

Some applications, as well as a proof of Yulmukhametov's result, are given in [Dra01].

Let us now say some words about a technique which is often used in these kinds of approximations. Recall that, according to Theorem 2.21, each subharmonic function u in $\Omega \subset \mathbf{C}$ has in every compact set $G \subset \Omega$ the representation

$$u(z) = \iint_G \log|z - \zeta|\, \mathrm{d}\mu_\zeta + h(z),$$

where h is harmonic in G. The measure μ was called the Riesz measure of u. Approximations of the type mentioned above are usually done by the means of this representation. One way to approximate u is to construct the entire function f using an atomization procedure. It is carried out by splitting the Riesz measure μ into measures μ_k such that $\mu_k(\mathbf{C}) = n_k$, where n_k is some integer. Then n_k points are put in the support of μ_k for each $k \in \mathbf{Z}$. These points correspond to the zeros of the entire function f. We will not describe this method further in its generality. In Section 6.1 we will instead describe it for a simple case, namely when $u(z) = \pi\,|\mathrm{Im}\,z|$ and $z \in \mathbf{C}$. This case shows some of the ideas of the general method, but one should be aware that the general case is much more complicated. More about this technique can be found in [Aza69], [Yul85] and [LM01].

In Section 6.2 we consider the case when $u(z) = \pi\,|\mathrm{Im}\,z|$, $z \in \mathbf{C}^2$. The function u is as mentioned earlier, plurisubharmonic in $\mathbf{C}^2$ and the questions are the same. How to find an entire function f such that the difference $u(z) - \log|f(z)|$ in some sense is small? How good can such an approximation be?

In Section 6.1 we will see that the entire function used in the approximation of a subharmonic function u will be constructed almost directly from the logarithmic kernel of its Riesz representation. In several complex variables the logarithmic kernel disappears from the Riesz representation: Plurisubharmonic functions are subharmonic, thus a plurisubharmonic function u in $\Omega \subset \mathbf{C}^n$ can in each compact set $G \subset \Omega$ be represented as

$$u(z) = -\int_G |\zeta - z|^{2-2n}\,\mathrm{d}\mu_\zeta + h(z),$$

where μ is the Riesz measure associated with u and h is harmonic in G. It is therefore of interest to be able to represent plurisubharmonic functions as an integral with a logarithmic kernel. This problem has been considered by Sekerin in a series of papers, see [Sek84], [Sek86b], [Sek86a] and [Sek92].

In order to state the most relevant representation theorem by Sekerin we need some notational preparations. Recall that

$$B_r(w) = \{z \in \mathbf{C}^n : \ |z - w| < r\}$$

and

$$\mathbf{S}^{2n-1} = \{z \in \mathbf{C}^n : \ |z| = 1\}\,.$$

Definition 6.2 A convex compact set $K \subset \mathbf{C}^2$, symmetric with respect to the origin is called a *Steiner set* if there exists an even Borel measure $\nu \geq 0$ on the sphere $\mathbf{S}^3$, such that the supporting function H_K of K can be represented in the form

$$H_K(z) = \int_{\mathbf{S}^3} |\mathrm{Re}\,\langle z, w\rangle|\,\mathrm{d}\nu_w.$$

Definition 6.3 A plurisubharmonic function u in $\mathbf{C}^2$ is called a *logarithmic potential* if there exists a Borel measure $\mu \geq 0$ on $[0, \infty) \times \mathbf{S}^3$, such that for every $0 < R < \infty$ we have

$$u(z) = \int_{[0,R]\times \mathbf{S}^3} \log |t - \langle z, w\rangle| \, \mathrm{d}\mu(t, w) + H_R(z),$$

where H_R is pluriharmonic in $B_R(0)$. The measure μ is called the *logarithmic measure* of u.

Theorem 6.4 (Sekerin, 1992) *An even supporting function is a logarithmic potential if and only if it is the supporting function of a Steiner compact set.*

The proof is given in [Sek92]. In the same article it is also noted that a closed disk D with center at the origin is a Steiner compact set, thus we know that the supporting function of D, that is $u(z) = \pi |\mathrm{Im}\, z|$, is a logaritmic potential.

One more definition is needed in order to state a result on approximation of plurisubharmonic functions with logarithms of moduli of entire functions.

Definition 6.5 A set $\Omega \subset \mathbf{C}^n$, $n \geq 2$, is called a *C_0-set* if it can be covered by a union of balls of the form $B_{\delta_j}(z_j)$ for which the condition

$$\limsup_{R\to\infty} \frac{1}{R^{2n-2}} \sum_{|z_j|<R} \delta_j^{2n-2} = 0$$

holds.

Theorem 6.6 (Sekerin, 1989) *Let $u \in \mathrm{PSH}(\mathbf{C}^n)$ be a logarithmic potential and assume that $u(z) \leq C|z|^\rho + d$, where C, ρ and d are constants. Then there exists an entire function f, whose zeros set is a union of hyperplanes, such that the estimate*

$$|u(z) - \log |f(z)|| \leq C \left(\log^3 |z|\right) |z|^{\rho(1-1/2n)}$$

holds outside a C_0-set.

In Section 6.2 we try to approximate a particular plurisubharmonic function, namely $u(z) = \pi |\mathrm{Im}\, z|$, $z \in \mathbf{C}^2$, with the logarithm of the modulus of an entire function. We expect that our approximation is better than the one in Theorem 6.6, but we are unfortunately not able to give a full proof yet.

6.1 Approximation of $\pi |\mathrm{Im}\, z|$, when $z \in \mathbf{C}$

We are considering the subharmonic function $u(z) = \pi |\mathrm{Im}\, z|$, when $z \in \mathbf{C}$ and would like to construct an entire function f such that $u(z) - \log |f(z)|$ is small. Let us first represent u as an integral with a logarithmic kernel.

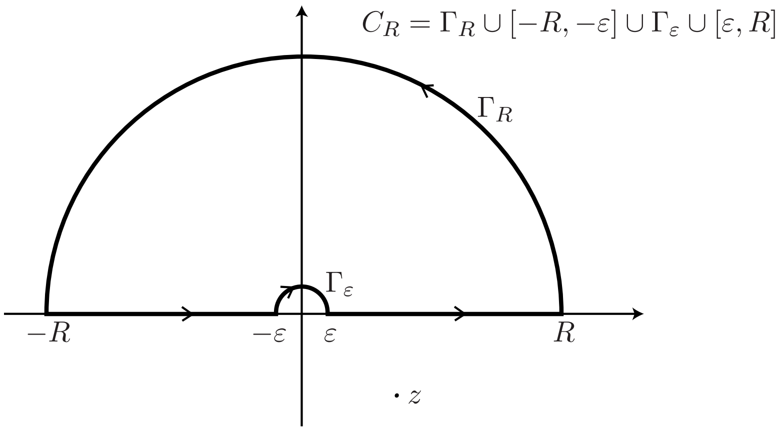

Figure 6.1: The contour used in the proof of Proposition 6.7.

Proposition 6.7 *Let $z = x + iy \in \mathbf{C}$, then the following holds*

$$\pi|\mathrm{Im}\,z| = \text{V.P.}\int_{-\infty}^{\infty} \log\left|1 - \frac{z}{t}\right| \,\mathrm{d}t. \tag{6.1}$$

Proof: Assume that $\mathrm{Im}\,z < 0$. From the definition of the principal value of an integral we have

$$\text{V.P.}\int_{-\infty}^{\infty} \log\left|1 - \frac{z}{t}\right| \,\mathrm{d}t = \mathrm{Re}\lim_{R\to\infty}\int_{-R}^{R} \log\left(1 - \frac{z}{t}\right) \,\mathrm{d}t.$$

To evaluate this integral we use contour integration. Consider the integral

$$\int_{C_R} \log\left(1 - \frac{z}{\zeta}\right) \,\mathrm{d}\zeta,$$

where C_R is the contour illustrated in Figure 6.1. The integrand, $\log(1 - z/\zeta)$ with $\zeta \in \mathbf{C}$, is analytic with respect to ζ in the interior of C_R and Cauchy's theorem gives

$$\int_{C_R} \log\left(1 - \frac{z}{\zeta}\right) \,\mathrm{d}\zeta = 0.$$

We split the contour, C_R, into four pieces as indicated in Figure 6.1. By a small estimate we get

$$\left|\int_{\Gamma_\varepsilon} \log\left(1 - \frac{z}{\zeta}\right) \,\mathrm{d}\zeta\right| \to 0 \quad \text{as } \varepsilon \to 0,$$

hence

$$\int_{-R}^{R} \log\left(1 - \frac{z}{t}\right) \,\mathrm{d}t = -\int_{\Gamma_R} \log\left(1 - \frac{z}{\zeta}\right) \,\mathrm{d}\zeta.$$

Since $|\zeta| \gg |z|$, $\log(1 - z/\zeta) \sim -z/\zeta$, and we may write

$$\int_{-R}^{R} \log\left(1 - \frac{z}{t}\right) \,\mathrm{d}t = \int_{\Gamma_R} \frac{z}{\zeta} \,\mathrm{d}\zeta + O\left(R^{-1}\right).$$

Let $\zeta = Re^{i\theta}$, then

$$\int_{-R}^{R} \log\left(1 - \frac{z}{t}\right) \mathrm{d}t = iz \int_0^{\pi} \mathrm{d}\theta + O\left(R^{-1}\right) = i\pi z + O\left(R^{-1}\right).$$

Letting R tend to infinity and taking real parts, we get

$$\text{V.P.} \int_{-\infty}^{\infty} \log\left|1 - \frac{z}{t}\right| \mathrm{d}t = \pi\,|y|\,.$$

A similar calculation gives the same answer if the imaginary part of z is assumed to be positive. ∎

Proposition 6.7 shows that u has a very simple Riesz representation, namely

$$u(z) = \text{V.P.} \int_{-\infty}^{\infty} \log\left|1 - \frac{z}{t}\right| \mathrm{d}t,$$

where the Riesz measure of u is simply the Lebesgue measure on the real line. The support of the Riesz measure of u is the real line, so we split the real line into intervals $I_k = [k - 1/2, k + 1/2)$ for $k \in \mathbf{Z}$. Each I_k has measure one. Observe that the center of mass of each I_k is k, since

$$\int_{I_k} (t - k)\,\mathrm{d}t = 0.$$

We may now write

$$\text{V.P.} \int_{-\infty}^{\infty} \log\left|1 - \frac{z}{t}\right| \mathrm{d}t = \text{V.P.} \sum_{k=-\infty}^{\infty} \int_{I_k} \log\left|1 - \frac{z}{t}\right| \mathrm{d}t. \tag{6.2}$$

The atomization procedure leads us to replace the right-hand side of (6.2) by a sum without integrals, which corresponds to the function $\log|\sin\pi z|$.

Theorem 6.8 *Let $u(z) = \pi\,|\mathrm{Im}\, z|$ and $f(z) = \sin\pi z$, then the estimate*

$$|u(z) - \log|f(z)|| = O(1),$$

holds as $z \to \infty$, outside the set $E_\varepsilon = \bigcup_{k\in\mathbf{Z}} \{z : \ |z - k| < \varepsilon\}$.

This was already proved in the the proof of Proposition 5.1. We will now give another proof demonstrating a general technique.

Proof: From Example 2.9 we know that

$$\sin\pi z = \pi z \prod_{k=-\infty}^{\infty}{}' \left(1 - \frac{z}{k}\right) e^{z/k},$$

thus

$$\log|\sin\pi z| = \log\pi\,|z| + \sum_{k=-\infty}^{\infty}{}' \left|1 - \frac{z}{k}\right| + \mathrm{Re}\, z \sum_{k=-\infty}^{\infty}{}' \frac{1}{k},$$

where the sums should be taken in the sense of principal value, so the last sum on the right-hand side vanishes. Using the splitting of the real line into intervals I_k we may write

$$\begin{aligned} u(z) &= \sum_{k=-\infty}^{\infty}{}' \int_{I_k} \log\left|1-\frac{z}{t}\right| \,\mathrm{d}t + \int_{-1/2}^{1/2} \log\left|1-\frac{z}{t}\right| \,\mathrm{d}t \\ &= \sum_{k=-\infty}^{\infty}{}' \int_{I_k} \log\left|1-\frac{z}{t}\right| \,\mathrm{d}t + O\left(\log|z|\right). \end{aligned}$$

The interval I_k has length one for all k, so

$$|u(z) - \log|f(z)|| = \left|\sum_{k=-\infty}^{\infty}{}' \int_{I_k} \left[\log\left|1-\frac{z}{t}\right| - \log\left|1-\frac{z}{k}\right|\right] \mathrm{d}t\right| + O(1). \tag{6.3}$$

The right-hand side of (6.3) is what we are going to estimate. Fix $z \in \mathbf{C}$ outside the exceptional set and split the complex plane into three regions:

$$\begin{aligned} B_1 &:= \{\zeta \in \mathbf{C} :\ |\zeta| < |z| - 10\} \\ B_2 &:= \{\zeta \in \mathbf{C} :\ |z| - 10 \le |\zeta| < |z| + 10\} \\ B_3 &:= \{\zeta \in \mathbf{C} :\ |\zeta| \ge |z| + 10\}. \end{aligned}$$

Also, define

$$\begin{aligned} K_i &:= \{k :\ I_k \cap B_i \neq \emptyset\} \qquad i = 1,2,3 \\ w_i &:= \sum_{k \in K_i} \int_{I_k} \left[\log\left|1-\frac{z}{t}\right| - \log\left|1-\frac{z}{k}\right|\right] \mathrm{d}t. \end{aligned}$$

We will estimate the modulus of each of the w_i's separately.

Estimate of w_1:

Observe that

$$\begin{aligned} \log\left|1-\frac{z}{t}\right| - \log\left|1-\frac{z}{k}\right| &= \log\left|1-\frac{t}{z}\right| - \log\left|1-\frac{k}{z}\right| - \log\left|\frac{t}{k}\right| \\ &= \mathrm{Re}\left\{\log\left(1-\frac{t}{z}\right) - \log\left(1-\frac{k}{z}\right) - \log\frac{t}{k}\right\}. \end{aligned}$$

Then

$$\begin{aligned} |w_1(z)| &= \left|\sum_{k \in K_1} \int_{I_k} \mathrm{Re}\left\{\log\left(1-\frac{t}{z}\right) - \log\left(1-\frac{k}{z}\right) - \log\frac{t}{k}\right\} \mathrm{d}t\right| \\ &\le \sum_{k \in K_1} \left|\int_{I_k} \left\{\log\left(1-\frac{t}{z}\right) - \log\left(1-\frac{k}{z}\right) - \log\left(1+\frac{t-k}{k}\right)\right\} \mathrm{d}t\right|. \end{aligned}$$

We have $|t-k| \le |k|$ and $|t| < |z|$, thus

$$\int_{I_k} \left[\log\left(1-\frac{t}{z}\right) - \log\left(1-\frac{k}{z}\right)\right] \mathrm{d}t \sim \int_{I_k} \left[\frac{k}{z} - \frac{t}{z}\right] \mathrm{d}t = 0$$

and

$$\int_{I_k} \log\left(1+\frac{t-k}{k}\right) \mathrm{d}t \sim \int_{I_k} \frac{t-k}{k} \mathrm{d}t = 0,$$

since k is the center of mass of I_k. We have obtained $|w_1(z)| = 0$.

Estimate of w_2*:*

There are a finite number of intervals I_k in the region B_2 and the number does not depend on z. When z is outside the exceptional set, the integrand is bounded, hence $|w_2(z)| = O(1)$.

Estimate of w_3*:*

We introduce the function $L(\zeta) = \log(1 - z/\zeta)$. It is analytic in $\mathbf{C} \setminus E_\varepsilon$ and Taylor's formula gives

$$L(t) - L(k) - L'(k)(t-k) = \int_k^t L''(\tau)(\tau - t)\,\mathrm{d}\tau.$$

Then

$$\begin{aligned}\int_{I_k} \left[\log\left|1-\frac{z}{t}\right| - \log\left|1-\frac{z}{k}\right|\right] \mathrm{d}t &= \int_{I_k} \mathrm{Re}\left\{L(t) - L(k) - L'(k)(t-k)\right\} \mathrm{d}t \\ &= \int_{I_k} \mathrm{Re}\left\{\int_k^t L''(\tau)(\tau-t)\,\mathrm{d}\tau\right\} \mathrm{d}t,\end{aligned}$$

due to cancellation. Now,

$$\begin{aligned}\left|\int_{I_k} \mathrm{Re}\left\{\int_k^t L''(\tau)(\tau-t)\,\mathrm{d}\tau\right\} \mathrm{d}t\right| &\le \sup_{t\in I_k}\left\{|t-k| \sup_{\tau\in[k,t]} \left|L''(\tau)\right| |\tau - t|\right\} \\ &\prec \sup_{\tau\in I_k} \frac{|z||z-2\tau|}{|\tau|^2|\tau-z|^2}.\end{aligned}$$

We may now use the fact that $\tau \in B_3$, $|\tau| \ge |z| + 10$, hence

$$\sup_{\tau\in I_k} \frac{|z||z-2\tau|}{|\tau|^2|\tau-z|^2} \prec \sup_{\tau\in I_k} \frac{1}{|\tau-z|^2} \sim \frac{1}{|k-z|^2}.$$

Adding up all the contributions from B_3 gives

$$|w_3(z)| \prec \sum_{|k|\ge|z|+10} \frac{1}{|k-z|^2} = \sum_{|j|\succ 1} \frac{1}{|j|^2} = O(1).$$

Conclusion of the argument:

Put the estimates for w_1, w_2 and w_3 into the right-hand side of (6.3), and we get the desired estimate

$$|u(z) - \log|f(z)|| = O(1),$$

as $z \to \infty$, when $z \notin E_\varepsilon$. ∎

6.2 Representation of $\pi\,|\mathrm{Im}\, z|$, when $z \in \mathbf{C}^2$

Let us now consider $u(z) = \pi\,|\mathrm{Im}\, z|$, when $z \in \mathbf{C}^2$. By Theorem 6.4 we know that u is a logarithmic potential. Using Proposition 6.7 we are able to a concrete expression for the logarithmic measure of u.

Proposition 6.9 *Let $z = x + iy \in \mathbf{C}^2$, where $x, y \in \mathbf{R}^2$, then*

$$\pi|y| = \frac{1}{2}\mathrm{V.P.} \iint_{\mathbf{R}^2} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \frac{\mathrm{d}m_\xi}{|\xi|}. \tag{6.4}$$

Proof: From the definition of the principal value of an integral we have

$$\mathrm{V.P.} \iint_{\mathbf{R}^2} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \frac{\mathrm{d}m_\xi}{|\xi|} = \lim_{R\to\infty} \iint_{|x|<R} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \frac{\mathrm{d}m_\xi}{|\xi|}.$$

We may change to polar coordinates. Let $\xi = r(\cos\theta, \sin\theta)$, then

$$\lim_{R\to\infty} \iint_{|x|<R} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \frac{\mathrm{d}m_\xi}{|\xi|} = \lim_{R\to\infty} \int_{-\pi}^{\pi}\int_0^R \log\left|1 - \frac{\langle z, (\cos\theta, \sin\theta)\rangle}{r}\right| \mathrm{d}r\,\mathrm{d}\theta.$$

Let

$$k(r, \theta, z) = \log\left|1 - \frac{\langle z, (\cos\theta, \sin\theta)\rangle}{r}\right|.$$

To be able to apply Proposition 6.7 we need to change the limits of integration. We have

$$\lim_{R\to\infty} \int_{-\pi}^{\pi}\int_0^R k(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta = \lim_{R\to\infty} \int_0^{\pi}\int_{-R}^R k(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta.$$

Using the dominated convergence theorem we may move the limit inside the outer integral

$$\lim_{R\to\infty} \int_0^{\pi}\int_{-R}^R k(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi} \lim_{R\to\infty} \int_{-R}^R k(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta.$$

Proposition 6.7 then yields that

$$\begin{aligned}\int_0^{\pi} \lim_{R\to\infty} \int_{-R}^R k(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{\pi} \lim_{R\to\infty} \int_{-R}^R \log\left|1 - \frac{\langle z, (\cos\theta, \sin\theta)\rangle}{r}\right| \mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{\pi} \pi|y_1\cos\theta + y_2\sin\theta|\,\mathrm{d}\theta.\end{aligned}$$

Write $y = |y|(\cos\alpha, \sin\alpha)$, then

$$\int_0^\pi \pi|y_1 \cos\theta + y_2 \sin\theta|\,\mathrm{d}\theta = \pi|y| \int_0^\pi |\cos(\theta-\alpha)|\,\mathrm{d}\theta = 2\pi|y|.$$

We conclude that

$$\text{V.P} \iint_{\mathbf{R}^2} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \frac{\mathrm{d}m_\xi}{|\xi|} = 2\pi|y|.$$

∎

Now that we have established one integral representation, we quite easily obtain one more. We introduce a new measure. For any Lebesgue measurable set $E \subset \mathbf{R}^2$ we define a measure μ by the relation

$$\mu(E) = \iint_E \mathrm{d}\mu = \iint_E \frac{\mathrm{d}m_\xi}{2|\xi|}.$$

The measure μ is by Proposition 6.9 the logarithmic measure of $u(z) = \pi\,|\mathrm{Im}\, z|$.

Proposition 6.10 *Let $z = x + iy \in \mathbf{C}^2$ and $\mathbf{R}^2_+ = \{(x_1, x_2) \in \mathbf{R}^2 :\ x_2 > 0\}$, then*

$$\pi|y| = \iint_{\mathbf{R}^2_+} \log\left|1 - \frac{\langle z, \xi\rangle^2}{|\xi|^4}\right| \mathrm{d}\mu_\xi. \tag{6.5}$$

Proof: We use the representation for $\pi|y|$ obtained in Proposition 6.9 and switch to polar coordinates, where $\xi = r(\cos\theta, \sin\theta)$,

$$\begin{aligned}\pi|y| &= \text{V.P.} \iint_{\mathbf{R}^2} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \mathrm{d}\mu_\xi \\ &= \frac{1}{2} \lim_{R\to\infty} \int_{-\pi}^{\pi} \int_0^R \log\left|1 - \frac{\langle z, (\cos\theta, \sin\theta)\rangle}{r}\right| \mathrm{d}r\,\mathrm{d}\theta.\end{aligned}$$

We may change the limits of integration

$$\lim_{R\to\infty} \int_{-\pi}^{\pi} \int_0^R \log\left|1 - \frac{\langle z, (\cos\theta, \sin\theta)\rangle}{r}\right| \mathrm{d}r\,\mathrm{d}\theta = \lim_{R\to\infty} \int_0^\pi \int_0^R l(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta,$$

where

$$l(r, \theta, z) = \log\left|1 - \frac{\langle z, (\cos\theta, \sin\theta)\rangle^2}{r^2}\right|$$

By applying the dominated convergence theorem we get

$$\lim_{R\to\infty} \int_0^\pi \int_0^R l(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^\pi \lim_{R\to\infty} \int_0^R l(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta.$$

The inner integral can be understood as an ordinary integral, hence

$$\int_0^\pi \lim_{R\to\infty} \int_0^R l(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^\pi \int_0^\infty l(r, \theta, z)\,\mathrm{d}r\,\mathrm{d}\theta.$$

If we go back to Cartesian coordinates we get what we need

$$\frac{1}{2}\int_0^{\pi}\int_0^{\infty} l(r,\theta,z)\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\pi}\int_0^{\infty} \log\left|1 - \frac{\langle z,(\cos\theta,\sin\theta)\rangle^2}{r^2}\right|\mathrm{d}r\,\mathrm{d}\theta$$
$$= \iint_{\mathbf{R}^2_+} \log\left|1 - \frac{\langle z,\xi\rangle^2}{|\xi|^4}\right|\mathrm{d}\mu_\xi. \tag{6.6}$$

∎

Remark: Notice that the right-hand side of (6.6) is not taken in the sense of principal value. This is not unexpected, since the difference between the representations (6.4) and (6.5) is similar to the difference between the two product representations

$$\pi z \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{z/n} \quad \text{and} \quad \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

of the function $\sin \pi z$, when $z \in \mathbf{C}$.

6.3 Towards approximation of $\pi\,|\mathrm{Im}\,z|$, $z \in \mathbf{C}^2$

As mentioned earlier, we would like a series representation for $u(z) = \pi\,|\mathrm{Im}\,z|$. The support of the logarithmic measure μ of u is $\mathbf{R}^2$, so the next step is to split $\mathbf{R}^2$ into cells of unit μ-measure. This will enable us to write the integral from Proposition 6.9 as a sum of integrals where the integration is performed over sets with μ-measure equal to one. Split $\mathbf{R}^2$ into disjoint sets

$$A_k = \left\{\xi \in \mathbf{R}^2 : \; R_k \le |\xi| < R_{k+1}\right\},$$

where $R_k = \frac{1}{\pi}k^2$ and $k = 0, 1, 2, ...,$ then

$$\mu(A_k) = \int_{A_k} \mathrm{d}\mu = \int_{R_k}^{R_{k+1}}\int_0^{2\pi} \mathrm{d}\theta\,\mathrm{d}r = 4k + 2.$$

Split each set A_k into $4k + 2$ cells

$$Q_{k,l} = \{\xi \in A_k : \; \psi_{k,l} \le \arg\xi < \psi_{k,l+1}\},$$

where $l = 0, 1, ..., 4k + 1$, $\psi_{k,l} = \frac{2\pi l}{4k+2}$ and $\psi_{k,4k+1} = \psi_{k,0}$, then

$$\mu(Q_{k,l}) = \int_{Q_{k,l}} \mathrm{d}\mu = \int_{R_k}^{R_{k+1}}\int_{\psi_{k,l}}^{\psi_{k,l+1}} \mathrm{d}\theta\,\mathrm{d}r = 1.$$

The splitting is sketched in Figure 6.2. We have

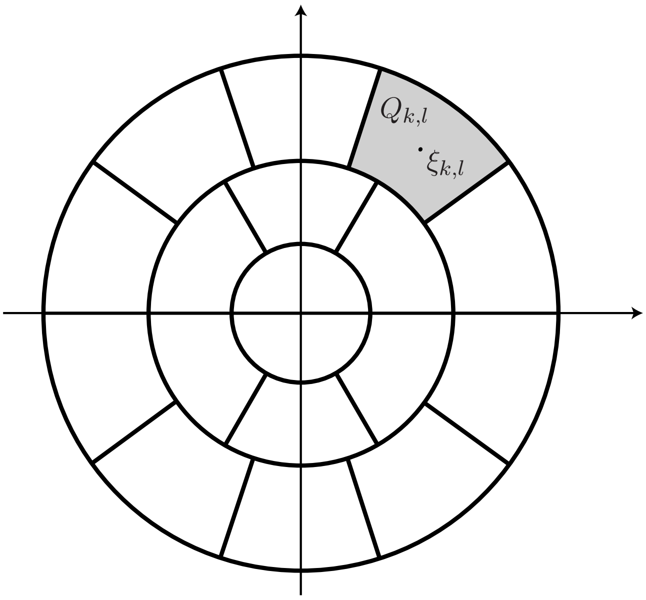

Figure 6.2: The splitting of the plane into cells $Q_{k,l}$ of unit μ-measure.

$$\bigcup_{\substack{k=0,1,2\dots\\ l=0,1,\dots,4k+1}} Q_{k,l}=\mathbf{R}^2 \quad \text{and} \quad \bigcup_{\substack{k=0,1,2\dots\\ l=0,1,\dots,2k}} Q_{k,l}=\mathbf{R}^2_+\cup\left([0,\infty)\times\{0\}\right).$$

Moreover, $Q_{k,l}\cap Q_{k',l'}=\emptyset$ when $(k,l)\neq(k',l')$. Since the μ-measure of each cell $Q_{k,l}$ is one, we may now write

$$\pi|y|=\text{V.P.}\iint_{\mathbf{R}^2}\log\left|1-\frac{\langle z,\xi\rangle}{|\xi|^2}\right|\,\mathrm{d}\mu_\xi=\sum_{k=0}^{\infty}\sum_{l=0}^{4k+1}\iint_{Q_{k,l}}\log\left|1-\frac{\langle z,\xi\rangle}{|\xi|^2}\right|\,\mathrm{d}\mu_\xi$$

and

$$\pi|y|=\iint_{\mathbf{R}^2_+}\log\left|1-\frac{\langle z,\xi\rangle^2}{|\xi|^4}\right|\,\mathrm{d}\mu_\xi=\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\iint_{Q_{k,l}}\log\left|1-\frac{\langle z,\xi\rangle^2}{|\xi|^4}\right|\,\mathrm{d}\mu_\xi$$

for the plane and half-plane representation respectively.

Based on our representation of $u(z)=\pi\,|\mathrm{Im}\,z|$ as an integral with a logarithmic kernel and the atomization above, we are able to construct an entire function with zero set equal to the union of a set of hyperplanes.

Proposition 6.11 *Assume that each $\xi_{k,l}$ is the μ-center of mass of $Q_{k,l}$, where the measure μ and the sets $Q_{k,l}$ are defined as above. Define a function $S:\mathbf{C}^2\to\mathbf{C}$ by the relation*

$$S(z)=\prod_{k=0}^{\infty}\prod_{l=0}^{4k+1}\left(1-\frac{\langle z,\xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right).$$

Then S is an entire function whose zero set is a union of hyperplanes.

Proof: The points $\xi_{k,l}$ are distributed symmetrically with respect to the origin and in particular, $\xi_{k,l} = -\xi_{k,l+2k+1}$, thus

$$\prod_{k=0}^{\infty}\prod_{l=0}^{4k+1}\left(1-\frac{\langle z,\xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right) = \prod_{k=0}^{\infty}\left[\prod_{l=0}^{2k}\left(1-\frac{\langle z,\xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right)\prod_{l=2k+1}^{4k+1}\left(1-\frac{\langle z,\xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right)\right]$$
$$= \prod_{k=0}^{\infty}\left[\prod_{l=0}^{2k}\left(1-\frac{\langle z,\xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right)\prod_{l=2k+1}^{4k+1}\left(1+\frac{\langle z,\xi_{k,l+2k+1}\rangle}{|\xi_{k,l+2k+1}|^2}\right)\right].$$

If we change the indices in the last product inside the brackets, we get

$$\prod_{k=0}^{\infty}\prod_{l=0}^{4k+1}\left(1-\frac{\langle z,\xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right) = \prod_{k=0}^{\infty}\prod_{l=0}^{2k}\left(1-\frac{\langle z,\xi_{k,l}\rangle^2}{|\xi_{k,l}|^4}\right). \tag{6.7}$$

Convergence of the product on the right-hand side of (6.7) is determined by the convergence of the series

$$\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\frac{|\langle z,\xi_{k,l}\rangle|^2}{|\xi_{k,l}|^4}.$$

The Cauchy-Schwarz inequality implies that

$$\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\frac{|\langle z,\xi_{k,l}\rangle|^2}{|\xi_{k,l}|^4} \le |z|^2\sum_{k=0}^{\infty}\sum_{l=0}^{2k}|\xi_{k,l}|^{-2}.$$

The length of each $\xi_{k,l}$ is approximately k^2 and $|\xi_{k,l}| \neq 0$ for any pair (k,l), hence

$$\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\frac{|\langle z,\xi_{k,l}\rangle|^2}{|\xi_{k,l}|^4} \sim |z|^2\sum_{k=1}^{\infty}\sum_{l=0}^{2k}k^{-4} \sim |z|^2\sum_{k=1}^{\infty}k^{-3}. \tag{6.8}$$

The Weierstrass M-test now tells us that the right-hand side of (6.8) converges uniformly for any compact subset of $\mathbf{C}^2$. The infinite product

$$\prod_{k=0}^{\infty}\prod_{l=0}^{4k+1}\left(1-\frac{\langle z,\xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right)$$

then converges uniformly on compact subsets of $\mathbf{C}^2$ and S is an entire function.

The zero set of S is

$$Z(S) = \bigcup_{k=0}^{\infty}\bigcup_{l=0}^{4k+1}\left\{z \in \mathbf{C}^2 :\ \langle z,\xi_{k,l}\rangle = |\xi_{k,l}|^2\right\},$$

which is clearly a union of hyperplanes. ∎

We are unfortunately not able to give a full proof of the next claim.

Claim 6.12 *Let S, $\xi_{k,l}$, $Q_{k,l}$ and μ be as above and let $z = x + iy \in \mathbf{C}^2$. Then*

$$|\pi\,|y| - \log|S(z)|| = O(1) \tag{6.9}$$

as $|z| \to \infty$, outside the set

$$E_\varepsilon = \bigcup_{k=0}^{\infty} \bigcup_{l=0}^{4k+1} \left\{ z \in \mathbf{C}^2 : \left| 1 - \frac{\langle z, \xi_{k,l}\rangle}{|\xi_{k,l}|^2} \right| < \varepsilon \right\},$$

for some positive ε.

Let us outline the idea of how we believe that this can be proved and where the troubles are.

We first rewrite (6.9) using atomization. Observe that

$$\log|S(z)| = \log\left| \prod_{k=0}^{\infty} \prod_{l=0}^{4k+1} \left(1 - \frac{\langle z, \xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right)\right| = \sum_{k=0}^{\infty}\sum_{l=0}^{4k+1} \log\left|1 - \frac{\langle z, \xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right|$$

and

$$\log|S(z)| = \log\left| \prod_{k=0}^{\infty} \prod_{l=0}^{2k} \left(1 - \frac{\langle z, \xi_{k,l}\rangle^2}{|\xi_{k,l}|^4}\right)\right| = \sum_{k=0}^{\infty}\sum_{l=0}^{2k} \log\left|1 - \frac{\langle z, \xi_{k,l}\rangle^2}{|\xi_{k,l}|^4}\right|. \tag{6.10}$$

From Proposition 6.9 we have

$$\pi|y| = \text{V.P.} \iint_{\mathbf{R}^2} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \,\mathrm{d}\mu_\xi.$$

Taking the atomization into account

$$\pi|y| = \sum_{k=0}^{\infty}\sum_{l=0}^{4k+1} \iint_{Q_{k,l}} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \,\mathrm{d}\mu_\xi,$$

then

$$|\pi|y| - \log|S(z)|| = \left| \sum_{k=0}^{\infty}\sum_{l=0}^{4k+1} \iint_{Q_{k,l}} \log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| \,\mathrm{d}\mu_\xi - \sum_{k=0}^{\infty}\sum_{l=0}^{4k+1} \log\left|1 - \frac{\langle z, \xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right| \right|.$$

The μ-measure of each cell $Q_{k,l}$ is one, thus

$$|\pi|y| - \log|S(z)|| = \left| \sum_{k=0}^{\infty}\sum_{l=0}^{4k+1} \iint_{Q_{k,l}} \left[\log\left|1 - \frac{\langle z, \xi\rangle}{|\xi|^2}\right| - \log\left|1 - \frac{\langle z, \xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right| \right] \mathrm{d}\mu_\xi \right|. \tag{6.11}$$

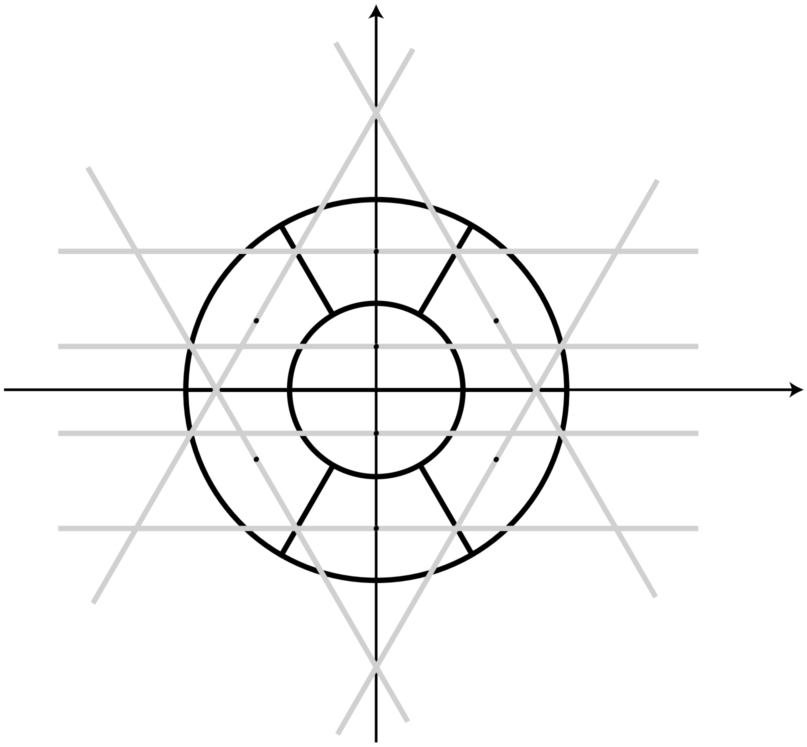

Figure 6.3: The union of all the strips is a piece of the intersection of the exceptional set and the real plane.

We may do the same steps using the half-plane representation from Proposition 6.10 and (6.10), then

$$|\pi|y| - \log|S(z)|| = \left|\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\iint_{Q_{k,l}}\left[\log\left|1-\frac{\langle z,\xi\rangle^2}{|\xi|^4}\right| - \log\left|1-\frac{\langle z,\xi_{k,l}\rangle^2}{|\xi_{k,l}|^4}\right|\right]\,\mathrm{d}\mu_\xi\right|. \tag{6.12}$$

Let us now split $\mathbf{R}^2$ into three regions to estimate contributions from each of the regions seperately. The exceptional set

$$E_\varepsilon = \bigcup_{k=0}^{\infty}\bigcup_{l=0}^{4k+1}\left\{z \in \mathbf{C}^2 :\ \left|1-\frac{\langle z,\xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right| < \varepsilon\right\}$$

is an open set containing the zero set of S. A piece of the intersection of the exceptional set and the real plane is illustrated in Figure 6.3. When z is in the exceptional set, the quantity $|\log|S(z)||$ is very large and it is clear that an estimate will not hold in this set. Fix some $z \in \mathbf{C}^2 \setminus E_\varepsilon$ and split $\mathbf{R}^2$ into three regions

$$D_1 = \left\{\xi \in \mathbf{R}^2 :\ |\xi|^2 \le \frac{1}{10}\,|\langle z,\xi\rangle|\right\}$$

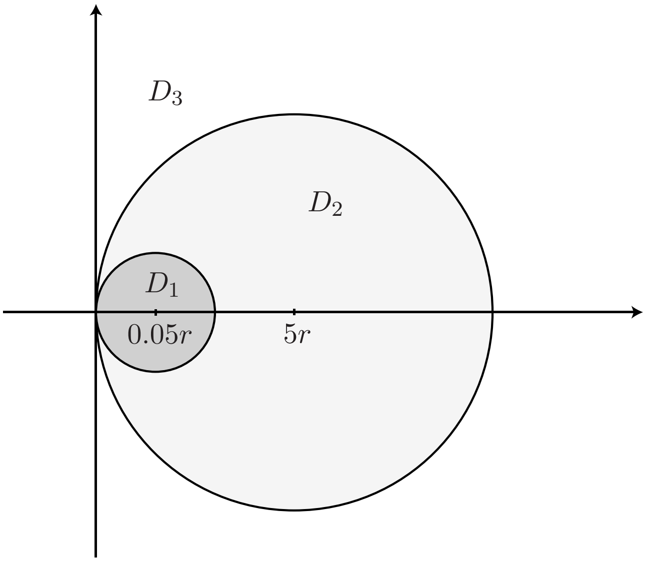

Figure 6.4: The splitting of $\mathbf{R}^2$ when $z = (r, 0) + i(0, 0)$. Notice that the scale is wrong. The small circle should be much smaller compared to the big one.

$$
\begin{aligned}
D_2 &= \left\{ \xi \in \mathbf{R}^2 : \ \frac{1}{10} \left| \langle z, \xi \rangle \right| < |\xi|^2 \leq 10 \left| \langle z, \xi \rangle \right| \right\} \\
D_3 &= \left\{ \xi \in \mathbf{R}^2 : \ |\xi|^2 > 10 \left| \langle z, \xi \rangle \right| \right\}.
\end{aligned}
$$

In Figure 6.4 we have sketched the splitting of $\mathbf{R}^2$ into the sets D_1, D_2 and D_3, when z is real and located on the ξ_1-axis.

To simplify the notation we let

$$
\begin{aligned}
\Lambda_1 &= \{k, l : \ Q_{k,l} \cap D_1 \neq \emptyset\} \\
\Lambda_2 &= \{k, l : \ Q_{k,l} \cap D_2 \neq \emptyset\} \\
\Lambda_3 &= \{k, l : \ Q_{k,l} \cap D_3 \neq \emptyset\} \\
\Lambda_i^+ &= \Lambda_i \cap \mathbf{R}^2_+, \quad i = 1, 2, 3.
\end{aligned}
$$

Contribution from remote summands

We now give an estimate in the region D_3, using the half-plane representation for $\pi |y|$. Then

$$
\begin{aligned}
\left| \pi |y| - \log |S(z)| \right| &= \left| \sum_{k,l \in \Lambda_3^+} \left[\iint_{Q_{k,l}} \log \left| 1 - \frac{\langle z, \xi \rangle^2}{|\xi|^4} \right| \mathrm{d}\mu_\xi - \log \left| 1 - \frac{\langle z, \xi_{k,l} \rangle^2}{|\xi_{k,l}|^4} \right| \right] \right| \\
&\leq \left| \sum_{k,l \in \Lambda_3^+} \left[\sup_{\xi \in Q_{k,l}} \left\{ \log \left| 1 - \frac{\langle z, \xi \rangle^2}{|\xi|^4} \right| \right\} - \log \left| 1 - \frac{\langle z, \xi_{k,l} \rangle^2}{|\xi_{k,l}|^4} \right| \right] \right|, \quad (6.13)
\end{aligned}
$$

since $\mu(Q_{k,l}) = 1$, for all $k, l \in \Lambda_3^+$. Because $\xi \in D_3$ we have $|\xi|^2 > 10\,|\langle z, \xi\rangle|$. Also,

$$\log\left|1 - \frac{\langle z, \xi\rangle^2}{|\xi|^4}\right| = \mathrm{Re}\,\log\left(1 - \frac{\langle z, \xi\rangle^2}{|\xi|^4}\right),$$

thus we have

$$\log\left|1 - \frac{\langle z, \xi\rangle^2}{|\xi|^4}\right| \sim -\mathrm{Re}\,\frac{\langle z, \xi\rangle^2}{|\xi|^4}.$$

Now,

$$\begin{aligned}
\text{r.h.s. of (6.13)} &\sim \left|\sum_{k,l\in\Lambda_3^+}\left[\mathrm{Re}\,\frac{\langle z, \xi_{k,l}\rangle^2}{|\xi_{k,l}|^4} - \sup_{\xi\in Q_{k,l}}\mathrm{Re}\,\frac{\langle z, \xi\rangle^2}{|\xi|^4}\right]\right| \\
&\le \sum_{k,l\in\Lambda_3^+}\left[\left|\mathrm{Re}\,\frac{\langle z, \xi_{k,l}\rangle^2}{|\xi_{k,l}|^4}\right| + \sup_{\xi\in Q_{k,l}}\left|\mathrm{Re}\,\frac{\langle z, \xi\rangle^2}{|\xi|^4}\right|\right] \\
&\le \sum_{k,l\in\Lambda_3^+}\left[\frac{|\langle z, \xi_{k,l}\rangle|^2}{|\xi_{k,l}|^4} + \sup_{\xi\in Q_{k,l}}\frac{|\langle z, \xi\rangle|^2}{|\xi|^4}\right].
\end{aligned}$$

For $\xi \in Q_{k,l}$ we have $\max|\xi| = (k+1)^2/\pi$ and $\min|\xi| = k^2/\pi$. Together with the Cauchy-Schwarz inequality this implies that

$$\sup_{\xi\in Q_{k,l}}\frac{|\langle z, \xi\rangle|^2}{|\xi|^4} \le \sup_{\xi\in Q_{k,l}}\frac{|z|^2\,|\xi|^2}{|\xi|^4} \prec |z|^2\,k^{-4}.$$

Moreover, the length of each $\xi_{k,l}$ is approximately k^2, thus

$$\frac{|\langle z, \xi_{k,l}\rangle|^2}{|\xi_{k,l}|^4} \prec |z|^2\,k^{-4}.$$

This means that

$$\sum_{k,l\in\Lambda_3^+}\left[\frac{|\langle z, \xi_{k,l}\rangle|^2}{|\xi_{k,l}|^4} + \sup_{\xi\in Q_{k,l}}\frac{|\langle z, \xi\rangle|^2}{|\xi|^4}\right] \prec \sum_{k,l\in\Lambda_3^+}\frac{|z|^2}{k^4}.$$

Each annulus A_k has approximately k cells, so

$$\sum_{k,l\in\Lambda_3^+}\frac{|z|^2}{k^4} \prec \sum_{k\ge|z|}\frac{|z|^2}{k^3} \prec \frac{|z|^2}{|z|^2} = O(1)$$

as $|z| \to \infty$. The estimate is proved for the region D_3.

Towards the final estimate

We are not able to estimate the contributions from the regions D_1 and D_2 yet. The region D_2 has not been considered at all, but for the region D_1 we have made some progress.

Consider the region D_1 and use the plane representation

$$|\pi |y| - \log |S(z)|| = \left| \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} \left[\log \left| 1 - \frac{\langle z, \xi \rangle}{|\xi|^2} \right| - \log \left| 1 - \frac{\langle z, \xi_{k,l} \rangle}{|\xi_{k,l}|^2} \right| \right] \mathrm{d}\mu_\xi \right|.$$

Let us rewrite the equality above to a form where we are able to take advantage of the fact that $\xi \in D_1$, that is $10\,|\xi|^2 \le |\langle z, \xi\rangle|$. We have

$$\log \left| 1 - \frac{\langle z, \xi \rangle}{|\xi|^2} \right| - \log \left| 1 - \frac{\langle z, \xi_{k,l} \rangle}{|\xi_{k,l}|^2} \right| = \\ \log \left| 1 - \frac{|\xi|^2}{\langle z, \xi \rangle} \right| - \log \left| 1 - \frac{|\xi_{k,l}|^2}{\langle z, \xi_{k,l} \rangle} \right| + 2 \log \frac{|\xi_{k,l}|}{|\xi|} - \log \frac{|\langle z, \xi_{k,l} \rangle|}{|\langle z, \xi \rangle|},$$

thus

$$|\pi |y| - \log |S(z)|| \le \left| \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} \left[\log \left| 1 - \frac{|\xi|^2}{\langle z, \xi \rangle} \right| - \log \left| 1 - \frac{|\xi_{k,l}|^2}{\langle z, \xi_{k,l} \rangle} \right| \right] \mathrm{d}\mu_\xi \right| \\ + \left| \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} 2 \log \frac{|\xi_{k,l}|}{|\xi|} \,\mathrm{d}\mu_\xi \right| + \left| \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} \log \frac{|\langle z, \xi_{k,l} \rangle|}{|\langle z, \xi \rangle|} \,\mathrm{d}\mu_\xi \right|.$$

Now define

$$S_1(z) = \left| \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} \left[\log \left| 1 - \frac{|\xi|^2}{\langle z, \xi \rangle} \right| - \log \left| 1 - \frac{|\xi_{k,l}|^2}{\langle z, \xi_{k,l} \rangle} \right| \right] \mathrm{d}\mu_\xi \right|$$

$$S_2(z) = \left| \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} 2 \log \frac{|\xi_{k,l}|}{|\xi|} \,\mathrm{d}\mu_\xi \right|$$

$$S_3(z) = \left| \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} \log \frac{|\langle z, \xi_{k,l} \rangle|}{|\langle z, \xi \rangle|} \,\mathrm{d}\mu_\xi \right|.$$

We will estimate S_1, S_2 and S_3 separately and start with S_2.

$$S_2(z) = 2 \left| \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} \log \frac{|\xi_{k,l}|}{|\xi|} \,\mathrm{d}\mu_\xi \right| = 2 \left| \mathrm{Re} \sum_{k,l\in\Lambda_1} \iint_{Q_{k,l}} \log \left(1 + \frac{\xi - \xi_{k,l}}{\xi_{k,l}} \right) \mathrm{d}\mu_\xi \right|.$$

We have $|\xi - \xi_{k,l}| \leq |\xi_{k,l}|$, thus

$$\log\left(1 + \frac{\xi - \xi_{k,l}}{\xi_{k,l}}\right) \sim \frac{\xi - \xi_{k,l}}{\xi_{k,l}}.$$

Now,

$$S_2(z) \sim \left|\operatorname{Re} \sum_{k,l \in Q_{k,l}} \frac{1}{\xi_{k,l}} \iint_{Q_{k,l}} (\xi - \xi_{k,l})\, \mathrm{d}\mu_\xi\right| = 0,$$

since $\xi_{k,l}$ is the μ-center of mass of $Q_{k,l}$ for all $k, l \in \Lambda_1$.

The estimate for S_3 goes almost like the estimate for S_2. We have

$$S_3(z) = \left|\operatorname{Re} \sum_{k,l \in \Lambda_1} \iint_{Q_{k,l}} \log\left(1 + \frac{\langle z, \xi - \xi_{k,l}\rangle}{\langle z, \xi_{k,l}\rangle}\right) \mathrm{d}\mu_\xi\right|.$$

Again we may use $\frac{\langle z, \xi - \xi_{k,l}\rangle}{\langle z, \xi_{k,l}\rangle}$ instead of $\log\left(1 + \frac{\langle z, \xi - \xi_{k,l}\rangle}{\langle z, \xi_{k,l}\rangle}\right)$, hence

$$\begin{aligned} S_3(z) &\sim \left|\operatorname{Re} \sum_{k,l \in \Lambda_1} \frac{1}{\langle z, \xi_{k,l}\rangle} \iint_{Q_{k,l}} \langle z, \xi - \xi_{k,l}\rangle\, \mathrm{d}\mu_\xi\right| \\ &= \left|\operatorname{Re} \sum_{k,l \in \Lambda_1} \frac{1}{\langle z, \xi_{k,l}\rangle} \left\langle z, \iint_{Q_{k,l}} (\xi - \xi_{k,l})\, \mathrm{d}\mu_\xi \right\rangle\right| = 0. \end{aligned}$$

Let us now look at S_1:

$$S_1(z) = \left|\operatorname{Re} \sum_{k,l \in \Lambda_1} \iint_{Q_{k,l}} \left[\log\left(1 - \frac{|\xi|^2}{\langle z, \xi\rangle}\right) - \log\left(1 - \frac{|\xi_{k,l}|^2}{\langle z, \xi_{k,l}\rangle}\right)\right] \mathrm{d}\mu_\xi\right|.$$

The point ξ is in D_1, so we may use that

$$\log\left(1 - \frac{|\xi|^2}{\langle z, \xi\rangle}\right) \sim -\frac{|\xi|^2}{\langle z, \xi\rangle}.$$

Then

$$S_1(z) \sim \left|\operatorname{Re} \sum_{k,l \in \Lambda_1} \iint_{Q_{k,l}} \left[\frac{|\xi_{k,l}|^2}{\langle z, \xi_{k,l}\rangle} - \frac{|\xi|^2}{\langle z, \xi\rangle}\right] \mathrm{d}\mu_\xi\right|.$$

If we add and subtract $|\xi_{k,l}|^2 / \langle z, \xi\rangle$ in the integrand we get

$$\left|\operatorname{Re} \sum_{k,l \in \Lambda_1} \iint_{Q_{k,l}} \left[\frac{|\xi_{k,l}|^2}{\langle z, \xi_{k,l}\rangle} - \frac{|\xi|^2}{\langle z, \xi\rangle}\right] \mathrm{d}\mu_\xi\right| \leq \left|\operatorname{Re} \sum_{k,l \in \Lambda_1} \iint_{Q_{k,l}} \left[\frac{|\xi_{k,l}|^2}{\langle z, \xi_{k,l}\rangle} - \frac{|\xi_{k,l}|^2}{\langle z, \xi\rangle}\right] \mathrm{d}\mu_\xi\right|$$

$$+\left|\operatorname{Re}\sum_{k,l\in\Lambda_1}\iint_{Q_{k,l}}\left[\frac{|\xi_{k,l}|^2}{\langle z,\xi\rangle}-\frac{|\xi|^2}{\langle z,\xi\rangle}\right]\mathrm{d}\mu_\xi\right|.$$

Denote the two expressions on the right-hand side above by $S_{1,1}$ and $S_{1,2}$, respectively.

For $S_{1,2}$ we have

$$S_{1,2}(z)=\left|\operatorname{Re}\sum_{k,l\in\Lambda_1}\iint_{Q_{k,l}}\frac{|\xi_{k,l}|^2-|\xi|^2}{\langle z,\xi\rangle}\mathrm{d}\mu_\xi\right|\le\sum_{k,l\in\Lambda_1}\sup_{\xi\in Q_{k,l}}\frac{\left||\xi_{k,l}|^2-|\xi|^2\right|}{|\langle z,\xi\rangle|}.$$

$\xi\in D_1$, so $|\langle z,\xi\rangle|\ge 10\,|\xi|^2\sim|\xi_{k,l}|^2$, also $\left||\xi_{k,l}|^2-|\xi|^2\right|\sim|\xi_{k,l}|^{1/2}$, hence

$$\sum_{k,l\in\Lambda_1}\sup_{\xi\in Q_{k,l}}\frac{\left||\xi_{k,l}|^2-|\xi|^2\right|}{|\langle z,\xi\rangle|}\prec\sum_{k,l\in\Lambda_1}|\xi_{k,l}|^{-3/2}.$$

This sum can be estimated quite easily. There are about k cells in each annulus A_k, the length of $\xi_{k,l}$ is approximately k^2 and k runs from 1 to $C\,|z|^{1/2}$, where C some positive constant. Then

$$\sum_{k,l\in\Lambda_1}|\xi_{k,l}|^{-3/2}\prec\sum_{k=1}^{|z|^{1/2}}k^{-2}=O(1)$$

as $|z|\to\infty$.

Next, consider $S_{1,1}$ and write the integrand with a common denominator

$$S_{1,1}(z)=\left|\operatorname{Re}\sum_{k,l\in\Lambda_1}\frac{|\xi_{k,l}|^2}{\langle z,\xi_{k,l}\rangle}\iint_{Q_{k,l}}\frac{\langle z,\xi-\xi_{k,l}\rangle}{\langle z,\xi\rangle}\mathrm{d}\mu_\xi\right|.$$

Using the fact that $\xi\in D_1$ and that $\mu(Q_{k,l})=1$ we obtain

$$\left|\operatorname{Re}\sum_{k,l\in\Lambda_1}\frac{|\xi_{k,l}|^2}{\langle z,\xi_{k,l}\rangle}\iint_{Q_{k,l}}\frac{\langle z,\xi-\xi_{k,l}\rangle}{\langle z,\xi\rangle}\mathrm{d}\mu_\xi\right|\prec\sum_{k,l\in\Lambda_1}\sup_{\xi\in Q_{k,l}}\frac{|\langle z,\xi-\xi_{k,l}\rangle|}{|\langle z,\xi\rangle|}.$$

The Cauchy-Schwarz inequality and again the fact that $\xi\in Q_{k,l}$, then

$$\sum_{k,l\in\Lambda_1}\sup_{\xi\in Q_{k,l}}\frac{|\langle z,\xi-\xi_{k,l}\rangle|}{|\langle z,\xi\rangle|}\prec|z|\sum_{k,l\in\Lambda_1}|\xi_{k,l}|^{-3/2}=O(|z|)$$

as $|z|\to\infty$. This is not the desired estimate. Comparing with Sekerin's result (Theorem 6.6) we have

$$\lim_{|z|\to\infty}\frac{\left(\log^3|z|\right)|z|^{3/4}}{|z|}=0,$$

and see that our result is in fact worse than Sekerin's. We expect that more elaborate methods are needed to obtain a better estimate than $O(|z|)$. Some attempts were made without being able to improve the estimate. This is a subject for further work.

When estimates for all the three regions are obtained, we add them up to get an estimate for the difference

$$|\pi\,|y| - \log|S(z)||$$

as $|z| \to \infty$ for z outside the exceptional set E_ε.

What if Claim 6.12 holds?

Assume for a moment that the approximation of the plurisubharmonic function $u(z) = \pi\,|\mathrm{Im}\,z|$ with $\log|S(z)|$, where S is defined by

$$S(z) = \prod_{k=0}^{\infty} \prod_{l=0}^{4k+1} \left(1 - \frac{\langle z, \xi_{k,l}\rangle}{|\xi_{k,l}|^2}\right),$$

has been proved. The zero set of S is a union of hyperplanes, so it should be possible to proceed as it was done in [LR00]. One should prove that each pair of hyperplanes intersect in a point and consider the union of all such points. This point set will then be a candidate for a set of uniqueness for PW^2_D. If we are able to prove such a uniqueness result we will by duality obtain a complete system of complex exponential functions in the corresponding space $L^2(\Omega)$. No attempts were made in this direction due to the lack of proof of Claim 6.12.

Chapter 7

Conclusions and further work

We have seen that properties of families of complex exponential functions in the space $L^2(\Omega)$ depend highly on the domain $\Omega \subset \mathbf{R}^n$ in question. For planar sets, we have observed that L^2-spaces over rectangles and hexagons possess orthonormal bases of complex exponential functions, while L^2-spaces over convex polygons, which are symmetric with respect to the origin, in general only admit Riesz bases. For the disk, we know that no orthonormal basis exists. It is still an open question whether there exist Riesz bases for the disk, but one might expect an answer in the negative. However, in the last chapter we tried to apply the methods of Lyubarskii and Rashkovskii to the disk, hoping to construct a complete system of complex exponential functions for the L^2-space over a disk. This work is unfortunately not completed yet.

Some further work related to this text is first of all to complete the work started in Chapter 6. One can then investigate the resulting system of exponential functions and hopefully get some kind of series representation for functions from $L^2(D)$.

A convex triangle A is another simple and interesting domain. It is convex, so by Theorem 4.35 there does not exist an orthonormal basis of complex exponential functions for $L^2(A)$. Being non-symmetric, it is not covered by the Lyubarskii-Rashkovskii theorem and we do not know whether there are any results at all about Riesz bases for $L^2(A)$.

More generally we may ask for which bounded convex sets $\Omega \subset \mathbf{R}^2$ the space $L^2(\Omega)$ possesses a Riesz basis of complex exponential functions. This problem is related to a similar problem for bounded convex subsets G of the complex plane. The L^2-space above is then replaced by a generalized Hardy space, called a Smirnov space. The Smirnov space $E^2(G)$ is the closure of the set of all polynomials in z with respect to the norm

$$\|f\|^2_{E^2(G)} = \int_{\partial G} |f(z)|^2 \, |\mathrm{d}z| \, .$$

More about Smirnov spaces can be found in [Dur02]. It was shown by Levin and Lyubarskii in [LL75] that if G is a convex polygon, then there are Riesz bases of complex exponential functions for $E^2(G)$. Lyubarskii and Seip [LS94] and Lutsenko and Yulmukhametov [Lut92] showed independently that there do not exist Riesz bases of exponential functions for $E^2(G)$ when some part of the boundary of G is curved. In view of these results one might expect something similar for bounded convex sets in $\mathbf{R}^2$. This is a topic for further research.

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