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Mathematical Model of the Geomagnetic Field

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Problem Description

One of the famous problems of geophysics is the reconstruction of the magnetic field of the earth using surface data. The aim of the project is to study the mathematical model of this problem, analyze the non-uniqueness phenomena known as the Backus effect and look at the modern theoretical approaches to resolving the Backus problem.

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Abstract

We begin with a description of a mathematical model of the geomagnetic field and some discussion of the classical non-uniqueness results of Backus. Then we look at more recent results concerning reconstruction of the geomagnetic field from intensity and the normal component of the field. New stability estimate for this reconstruction is obtained.

The most beautiful things in life are meaningless if they are not shared

Preface

This thesis marks the end of a five year programme I have attended since August 2001 at the Norwegian University of Science and Technology culminating in the degree Master of Science. This degree is in physics and mathematics, with specialization in mathematical analysis. In this thesis I have worked with the geomagnetic field, and thus had the opportunity to combine physics and mathematics, this I found delightful. The thesis it is written in LaTeXand represents 20 weeks worth of work (+1 week Easter holiday).

I would like to use this opportunity to thank Hans Michael Øvergaard and Kristian Stormark, which have "lived" in the office next door during this work. Their their support and friendship have made this a great time. I would also like to thank Dr. Geir Arne Hjelle for proofreading this text and learning me some tricks in LaTeX. It has made this thesis come out better.

Last and most I want to thank my Amazing Academic Advisor Associate Professor Eugenia Malinnikova for having the time and effort to answer my many questions, and for guiding me through this thesis in an elegant way.

Trondheim, 16 June 2006.

Kjetil Thorsen

Contents

Introduction

The aim of this thesis is to look at a mathematical model for the geomagnetic field and the challenges it offers.

The geomagnetic field is a complicated phenomenon. Both its origin and behavior are not completely understood. We are only going to look at the time independent field in the atmosphere, and it turns out that in this case there is, at least from a mathematical point of view, a very nice model that describes it. Using this model, we will consider different empirical data and see if they are sufficient to reconstruct the geomagnetic field.

Our starting point is Maxwell's equations, which are the classical tool to describe electromagnetic fields. Motivated by [Gri99] and [AE92], we take a closer look at this system for electromagnetic fields. Then following [BPC96] we analyze Maxwell's equations and see, that under some reasonable assumptions, the geomagnetic field in the atmosphere can be approximated by the gradient of a harmonic function. Measurements show that the main contribution to the geomagnetic field is due to sources inside the earth. So the gradients of functions harmonic outside the earth provide a good mathematical model of the geomagnetic field.

One of the famous problems in geophysics is the reconstruction of the geomagnetic field using surface data. A natural question is then: *can more than one magnetic field satisfy these data?* We have collected some answers to this question and will present them in this text.

From [BPC96] we get an explicit formula for reconstructing the magnetic field from vector data on a sphere, and thus answering no to our question. The most famous answer is due to Backus in [Bac70] for the situation when intensity on a spherical surface is known. Here he constructs a simple example of two fields with the same intensity on a sphere. This example is reproduced in the text along with two others using the same technique. A different approach to this problem is due to [AOP04] where Clebsch-Gordan coefficients are used to obtain general uniqueness criteria. They use this to show that if two fields have a finite expansion then they are equal provided they have the same intensity on a sphere. Both this approach and the result are included in the text. The criterion is also used to study an example.

A natural follow up question would then be: if we know the intensity on two spheres, are there still more than one magnetic field which fit these data? Attempts of answering this has been made using the techniques and examples for the one sphere case. But it seems that there is no simple counter example.

The next question is if we know the intensity everywhere outside a sphere are there more than one magnetic field that fit these data? The answer is no and we present the proof given by Backus in [Bac68].

A more modern approach to the reconstruction problem is when in addition to the intensity the sign of the normal component is assumed known. It is shown in [KHLM97] that this uniquely determines a magnetic field. Also in a follow up paper [KHLM99] a stability estimate for this uniqueness is given. Both of these results are presented in this text. We use the latter to find a more explicit stability estimates for this situation.

The most part of the thesis consists of the review of some known results from books and articles, mainely [BPC96], [AOP04], [KHLM97] and [KHLM99]. We tried to unify the approach and write a consistent treatment of the subject. Some of the proofs are modified and/or simplified. The two sphere problem is known in geomagnetism. Discussions of the examples for two sphere in section 3.5 is new as well as those for the one sphere in sections 3.3 and 3.5.

The main new result of the thesis is an explicit stability estimate in section 4.2.5. Step 1 of the proof is contained in [KHLM99], where general domain instead of $\mathbb{R}^3 \setminus B$ is considered. A stability estimate in [KHLM99] for the uniform norm of the field on the sphere on which the data is known is obtained by a normal family argument. Our aim was to give an explicit proof and get a quantitative estimate.

Overview of the Thesis

Chapter 1: Geomagnetic and Mathematical Preliminaries

We start with a short introduction of the geomagnetic field and some motivation to study it. Then general notation and background about harmonic functions used in this text are summarized. Some units and constants are listed in the end of the chapter.

Chapter 2: The Magnetic Field in the Atmosphere

We analyze Maxwell's equations using real data. It turns out that the gradient of functions harmonic in the atmosphere approximate the geomagnetic field well. Also we see that we can divide the geomagnetic field into two parts, one due to internal sources and one due to external. Expanding harmonic functions in spherical harmonics, we find an explicit formula for the reconstruction of the geomagnetic field in the atmosphere from vector data on a sphere.

Chapter 3: Reconstruction from Intensity

In this chapter we consider the problem: can two different magnetic fields have the same intensity on a sphere? We also look at if this holds for two spheres. Expanding harmonic functions in spherical harmonics, we get examples of different fields with same intensity on a sphere. Moreover, we show that these examples does not hold for the two-sphere case. Further we use Clebsch-Gordan coefficients in expanding the product of two spherical harmonics. From this we get a general formula for constructing fields with same intensity on spheres. Then we use the formula to show that two fields corresponding to functions having finite expansion are equal if they have the same intensity on a sphere. We also use this formula to get another example of two fields with same intensity on a sphere. However none of these has the same intensity on two spheres. We finish the chapter by showing uniqueness from intensity everywhere outside a sphere.

Chapter 4: Reconstruction from Intensity and Dip Equator

Here we consider the problem: can two different magnetic fields have the same intensity and same sign of the normal component on a sphere? First we show that this cannot happen, and then we look at two magnetic fields $\mathbf{B}_1$, $\mathbf{B}_2$, for which the difference of the intensities is small on a sphere and the signs of the normal components are the same (at least where the normal components are not very small). We find that on this sphere we have under some conditions that

$$|\mathbf{B}_1 - \mathbf{B}_2| < C\varepsilon^{1/6},$$

where ε depend on how close the data are and C is some specific constant.

Chapter 1

Geomagnetic and Mathematical Preliminaries

1.1 The geomagnetic field

There is a magnetic field everywhere. From the outer reaches of the galaxy to, where you (probably) are now, the surface of the earth. In interstellar space, the strength of the magnetic field is about 1nT. It is created from magnetically oriented dust particles and observed by how it polarizes light. Within the solar system, the sun's magnetic field dominates interplanetary space. Solar winds are continuously streaming from the sun carrying along charged particles. The magnetic field they create is about 5nT and is directed away the sun, or towards it, in huge sectors. Immediately surrounding the earth and up to 10-20 earth radii we have the magnetosphere. We call the magnetic field here for the geomagnetic field. Sometimes we also divide this into three parts, depending on where it originates from. The main geomagnetic field is created inside the earth. This field has a strength of 30-60 μT on the earth's surface, and it has a slow and steady variation of a few nT a year. This field can be approximated by that of a dipole, or a bar magnet, at the center of the earth, as we can see in Figure 1.1[1].

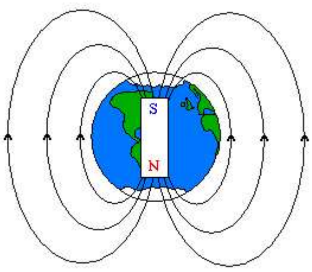

Figure 1.1: The earth's magnetic field look like a dipole.

[1]From http://www.sunblock99.org.uk/sb99/people/DMackay/magearth.html, 15.06.2006

Magnetic rock on the earth's surface also creates a magnetic field, and its strength is about some hundreds of nT.
The geomagnetic field also has a contribution from outside the earth, this creates current systems in the ionosphere and magnetosphere, and this imposes a magnetic field. This field is mostly due to the solar winds and has a strength of up to 80nT. It also varies rapidly dependent on the solar activity.

Magnetic forces was probably first discovered in China about 4000 years ago, where one observed that certain iron ores, such as lodestone, have the tendency to attract small pieces of iron. The geomagnetic field was at least known in China in 1088 when it was used for navigational proposes. Still today, many people use it to navigate. However, the most important thing about the geomagnetic field is that it protects the earth's surface against charged particles of the solar wind. The geomagnetic field becomes a shield which the particles collide in, as illustration in Figure 1.2[2].

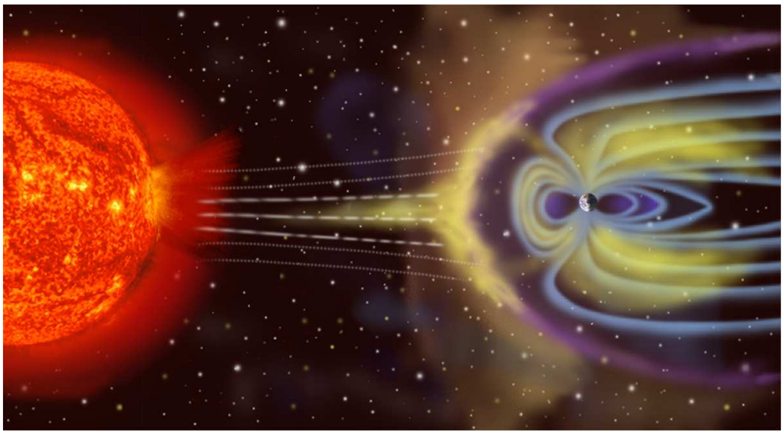

Figure 1.2: The earth's magnetic field look like a dipole.

These particles could be fatal to the living organisms on earth. Even now, the collisions can cause great damage. These collisions are known as magnetic storms and they can cause significantly damages to power grids, telecommunications system and oil pipelines. This happened in 1989 when a magnetic storm shut down a substantial part of the Canadian electrical power grid.

Therefore, it is important to study the geomagnetic field and how it evolves. Studying the geomagnetic field is also an important source to gain information about how the earth is build up inside. Also in connection with sending up space shuttles, knowledge of the geomagnetic field is important.

[2]From http://en.wikipedia.org/wiki/Geomagnetic_field, 15.06.2006

1.2 Notation and Terminology

In this section some general notation and conventions used in the text are summarized, mostly adapted from [BPC96].

We will work in the Euclidean space $\mathbb{R}^3$ where we denote a point by $\mathbf{r} = (x, y, z)$. This space will be equipped with the usual Euclidean norm $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, where we write r for short. For the standard axial unit vectors we write $\hat{\mathbf{x}} = (1, 0, 0)$, $\hat{\mathbf{y}} = (0, 1, 0)$ and $\hat{\mathbf{z}} = (0, 0, 1)$. Also a point $\mathbf{r} \in \mathbb{R}^3$ can be written as $\mathbf{r} = r\hat{\mathbf{r}}$, where $|\hat{\mathbf{r}}| = 1$. We will also use spherical coordinates where we have the following relations

$$\begin{aligned} x &= r \cos \lambda \sin \theta \\ y &= r \sin \lambda \sin \theta \\ z &= r \cos \theta \end{aligned}$$

For a subset $E \subset \mathbb{R}^3$ we will write $\overline{E}$, $\text{int}(E)$ and ∂E for the closure, interior and boundary of E. We define the boundary of E to be $\partial E = \overline{E} \setminus \text{int}(E)$, and for the boundary of E in $\mathbb{R}^3 \cup \{\infty\}$ we write $\partial^\infty E$.

An open ball of center $\mathbf{x}$ and radius R is denoted by $B(\mathbf{x}, R)$ and its boundary points by $S(\mathbf{x}, R)$. When the center $\mathbf{x} = 0$ we just write B_R and S_R. If in addition the radius is one we simply write B and S. For an annulus we write $A(r_1, r_2) = \{\mathbf{r} \in \mathbb{R}^3 : r_1 < |\mathbf{r}| < r_2\}$.

For $n \in \mathbb{N}$ and a subset $E \subset \mathbb{R}^3$ we define the L^n norm of a function f on E by

$$\|f\|_{L^n(E)} = \left(\int_E |f|^n d\mathbf{r} \right)^{1/n}.$$

By L^2-estimate we will mean the estimate in L^2 norm. We let $L^n(E) = \{f \colon E \to \mathbb{C} : \|f\|_{L^n(E)} < \infty\}$.
For a complex number z we will denote the complex conjugate by $\overline{z}$.

For $f \in L^1(S)$ we define

$$< f >= \int_S f \, d\hat{\mathbf{s}}.$$

Saying that a function $f(x) = o(x)$ when $|x| \to a$ means $\lim_{|x| \to a} \frac{f(x)}{|x|} = 0$.

We define the gradient of a function f by $\nabla f = (\partial_x f, \partial_y f, \partial_z f)$ and Laplacian operator Δ by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. In spherical coordinates this becomes

$$\begin{aligned} \nabla &= \hat{\mathbf{r}} \partial_r + \hat{\lambda} \frac{1}{r \sin \theta} \partial_\lambda + \hat{\theta} \frac{1}{r} \partial_\theta \\ \Delta &= \frac{1}{r} \partial_r (r \partial_r + 1) + \frac{1}{r^2} \left(\frac{1}{\sin^2 \theta} \partial_\lambda^2 + \partial_\theta^2 + \frac{1}{\tan \theta} \partial_\theta \right). \end{aligned}$$

Thus by defining

$$\begin{array}{lll} \nabla_r = \mathbf{r}\partial_r & \text{and} & \nabla_1 = \hat{\lambda}\frac{1}{\sin\theta}\partial_\lambda + \hat{\theta}\partial_\theta \\ \Delta_r = r\partial_r(r\partial_r + 1) & \text{and} & \Delta_1 = \frac{1}{\sin^2\theta}\partial_\lambda^2 + \partial_\theta^2 + \frac{1}{\tan\theta}\partial_\theta, \end{array}$$

we have the relations $r\nabla = \nabla_r + \nabla_1$ and $r^2\Delta = \Delta_r + \Delta_1$.

1.3 Harmonic functions

Let E be a domain in $\mathbb{R}^3$, then a function $f\colon E \to \mathbb{R}$ is called *harmonic* on E if $f \in C^2(E)$ and $\Delta f \equiv 0$. The set of all harmonic functions on E is denoted by $\mathcal{H}(E)$. And the set of all non-negative harmonic functions on E is denoted by $\mathcal{H}_+(E)$. If E is unbounded we write $\mathcal{H}_0(E)$ for the harmonic functions in E which vanishes at infinity.

Fact 1 (Extremum principle). *Let $f \in \mathcal{H}(E)$, then f achieve its extremum on $\partial^\infty E$.*

Proof. See [AG01, Theorem 1.2.4]. □

Fact 2. *The spherical harmonics $\{Y_l^m\}_{l,m}$, for $0 \le |m| \le l \le \infty$, is an orthonormal basis for $L^2(S)$, where*

$$Y_l^m(\lambda,\theta) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\lambda},$$

and P_l^m are Legendre functions given by

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\partial_x^{l+m}(x^2-1)^l, \quad \text{for} \quad -l \le m \le l.$$

And $P_l^m = 0$ for $l < |m|$. By orthonormality we mean $< Y_{l'}^{m'}\overline{Y_l^m} >= \delta_{ll'}\delta_{mm'}$. Also we have $\overline{Y_l^m} = (-1)^m Y_l^{-m}$.

Proof. See [AW95, Equation (12.167)]. □

Fact 3. $|Y_l^m|^2 \le \frac{2l+1}{4\pi}$ *for every l and m.*

Proof. This follows from [Jac75, Equation (3.69)]. □

If p is a polynomial of degree l then we say that p is homogenous if $p(\mathbf{r}) = r^l p(\hat{\mathbf{r}})$. We then denote p by p_l. For the set of all homogenous polynomials of degree l we write $\mathcal{P}_l$. Specially we write $\mathcal{H}_l$ for the set of homogenous harmonic polynomials of degree l.

For a homogenous harmonic polynomial h_l we have

$$\begin{aligned} 0 = r^2\Delta h_l(\mathbf{r}) &= h_l(\hat{\mathbf{r}})\Delta_r r^l + r^l\Delta_1 h_l(\hat{\mathbf{r}}) \\ &= [l(l+1)h_l(\hat{\mathbf{r}}) + \Delta_1 h_l(\hat{\mathbf{r}})]r^l, \end{aligned}$$

so

$$\Delta_1 h_l(\hat{\mathbf{r}}) = -l(l+1)h_l(\hat{\mathbf{r}}). \tag{1.1}$$

Fact 4. *For any smooth functions f and g we have*

$$< \nabla_1 f \cdot \nabla_1 g >= - < f \Delta_1 g > .$$

Proof. This follows by direct computations. □

Using Fact 4 we see that

$$\begin{aligned} < \nabla_1 \overline{Y_{l'}^{m'}} \cdot \nabla_1 Y_l^m > &= - < \overline{Y_{l'}^{m'}} \Delta_1 Y_l^m > \\ &= l(l+1) < \overline{Y_{l'}^{m'}} Y_l^m > \\ &= l(l+1)\delta_{ll'}\delta_{mm'} . \end{aligned} \tag{1.2}$$

Fact 5. *Let $f \in L^2(S)$, then there are unique scalars f_l^m such that*

$$f(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m(\hat{\mathbf{r}})$$

where $f_l^m =< \overline{Y_l^m} f >$.

Proof. See Theorem 3 on p.51 in [BPC96]. □

1.4 Measurements and units

We are going to use the units in Table 1.1.

Quantity	Symbol	SI unit	Name
Area	A	m^2	
Current	$\mathbf{I}$	A	ampere
Current density	$\mathbf{J}$	A/m^2	
Electric charge	q	C	coulomb
Electric charge density	ρ	C/m^2	
Electric displacement	$\mathbf{D}$	C/m^2	
Electric field	$\mathbf{E}$	V/m	volt/meter
Electric polarization	$\mathbf{P}$	C/m^2	
Length	L	m	meter
Magnetic displacement	$\mathbf{H}$	A/m	
Magnetic field	$\mathbf{B}$	T	tesla
Magnetization	$\mathbf{M}$	A/m	
Time	t	s	second

Table 1.1: Some quantities used, their symbol, unit and name.

We will use the constants in Table 1.2.

Quantity	Symbol	Value
Earth radii	R_{-1}	$6.371 \times 10^6 m$
micro	μ	10^{-6}
nano	n	10^{-9}
Speed of light	c	$299,792,458 \frac{m}{s}$
Vacuum permeability	μ_0	$4\pi \times 10^{-7} \frac{kg\ m}{s^2\ A^2}$
Vacuum permittivity	ϵ_0	$\frac{1}{\mu_0 c^2}$

Table 1.2: Some physical constants.

Chapter 2

The Magnetic Field in the Atmosphere

The goal of this chapter is to build a mathematical model that describes the magnetic field in the atmosphere.

2.1 Maxwell's equations

If we have two electrically charged particles we will experience a force between them. This is called an electrical force. A number of experiments have resulted in Coulomb's law that connects the force between them with their charge and position. From this law we can define an electric field. We also experience a force when we have two charged particles in relative motion. This is another force, and Lorentz's law connects it with the magnetic field. Moreover, Biot-Savart's law connect the magnetic field with the motion of the charges. These laws are justified through many experiments. In short we can say that the electric field is given by the positions of the charged particles and the magnetic field by their velocities. In nature there are lot of charged particles close together, and their mutual forces make them move, so here the electric and magnetic field always co-exists. So naturally we will often work with both fields at the same time and then we use the term electromagnetic field to refer to them both. Also we call the charges creating a field the sources of the field. From Coulomb's and Biot-Savart's law we can deduce four famous relations between the flux and circulation to an electromagnetic field and its sources. These are called Maxwell's equations and are standard in physics, so for a derivation see any respectable book on electromagnetism, for example [Gri99].

Maxwell's equations. *For any closed surface A and closed path L we have the following macroscopic relations between an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.*

$$\begin{aligned}
\oint_A \mathbf{E} \cdot d\mathbf{A} &= q/\epsilon_0 \\
\oint_A \mathbf{B} \cdot d\mathbf{A} &= 0 \\
\oint_L \mathbf{E} \cdot d\mathbf{L} &= -\frac{d}{dt}\int_{A_L} \mathbf{B} \cdot d\mathbf{A} \\
\oint_L \mathbf{B} \cdot d\mathbf{L} &= \mu_0(\mathbf{I} + \epsilon_0 \frac{d}{dt}\int_{A_L} \mathbf{E} \cdot d\mathbf{A}).
\end{aligned}$$

Here q is the total charge inside A, ϵ_0 is the vacuum permittivity, A_L is the surface bounded by L, μ_0 is the vacuum permeability, and $\mathbf{I}$ is the total current through A_L.

The term macroscopic relations is used because there are some microscopic electromagnetic effects that are not correctly explained by these equations. Thus on such scales we really need to use quantum mechanics. Moreover a true microscopic field would be very complex and varies wildly on small intervals in time and space. For example the magnetic field in an atom is averaged to about 1T over the size of the atom (10^{-10}m). So the intensity fluctuates wildly between values that are millions of times larger than the typical geomagnetic intensity. We will also have these wild fluctuations in time due to the high speed of the electrons. Therefore most practical implementation of these equations on microscopic scales would be very difficult. These fluctuations can also cause problems when applying Maxwell's equations on macroscopic scales, but then we can average over time and space. And in most cases if this is done in a satisfactory way, we get nice smooth fields which does not vary so much, but looks the same as the "true" field on a macroscopic scale. See [Ros65] for a discussion of the averaging process.

We are interested in the local nature of the electromagnetic field, so we want to write the integral equations into differential form. If we assume that the surface A is independent of time we can take the differentiations inside the integrals in the last two equations. Then we can use Green's and Stokes' theorem to rewrite the left hand sides,

$$\begin{aligned}
\int_V \nabla \cdot \mathbf{E} dV &= \int_V \rho/\epsilon_0 dV \\
\int_V \nabla \cdot \mathbf{B} dV &= 0 \\
\int_A (\nabla \times \mathbf{E}) \cdot d\mathbf{A} &= \int_A -\frac{\partial}{\partial t}\mathbf{B} \cdot d\mathbf{A} \\
\int_A (\nabla \times \mathbf{B}) \cdot d\mathbf{A} &= \mu_0 \int_A (\mathbf{J} + \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}) \cdot d\mathbf{A}.
\end{aligned}$$

Here ρ is charge density and $\mathbf{J}$ is current density. We have smoothed the sources, so therefore are ρ, $\mathbf{J}$, $\mathbf{E}$ and $\mathbf{B}$ continuous. Since these equations hold for every volume and surface the integrands must therefore be equal. Then we get Maxwell's equations in differential form,

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{B} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}.
\end{aligned}$$

2.2 Electromagnetic field in a medium

We are going to look at how an electrical field affects a neutral media. A neutral media is a media with no sources. Usually it consist of neutral atoms/molecules or randomly oriented dipoles. A neutral atom or molecule consists of a positive charged core and a negative charged electron cloud surrounding it. The electrical charges just balance, so there is no electromagnetic field outside the atom/molecule. An electric dipole is a particle that consists of an equal amount of positive and negative charges, but the positive and negative charge centers are shifted a bit. This creates an electric field outside the particle, even if the total charge is zero. A magnetic dipole is orbiting current. For example electrons orbiting around their core and electrons spinning around their axis are magnetic dipoles, and they will create electromagnetic fields. In nature electrical and magnetic dipoles almost always co-exists, so we will just talk about dipoles.

If we have a neutral media consisting of neutral atoms/molecules and randomly oriented dipoles, then by definition there will be no sources due to this media. Suppose we expose it to an electromagnetic field. Then the dipoles will orient in accordance to the field, so the fields from the dipoles will no longer cancel each other. Also the core of the neutral atoms/molecules will be pushed in the direction of the electric field and the electron cloud in the opposite way. If the exterior field is strong enough it could pull the atom apart. Then clearly this will create an electric field. If not, then the stretching of the atom will reach a balance with the positive charge shifted one way and the negative the other. Thus we will get an electrical dipole. In this way we get induced sources in the media, so the media is no longer neutral. We call these sources *polarization sources* and the field they create *polarization field*.

How can we get induced sources when we have the law of charge conservation? This may seem somewhat odd, but it accrues when we average Maxwell's equations to write them on differential form. When we average over (macroscopically speaking) neutral media, we forget that there really are sources there. And because of this, in our model, the fields are no longer linearly dependent on the sources.

We divide the total electromagnetic field, $\mathbf{F}$, into two parts.

$$\mathbf{F} = \mathbf{F}^{(F)} + \mathbf{F}^{(P)},$$

where $\mathbf{F}^{(P)}$ is the polarized field due to the polarization sources and $\mathbf{F}^{(F)}$ is the "free" field due to all the other sources. Let σ be the sources of $\mathbf{F}$, then we also can divide σ,

$$\sigma = \sigma^{(F)} + \sigma^{(P)}.$$

If we look at the electromagnetic field $\mathbf{F}^{(F)}$, created from the "free" sources, then induced sources would have no effect on the field. Thus in a way we have conservation of charge in a macroscopic sense. This means that the field is linearly dependent on its sources. Therefore we define the following fields,

$$\begin{array}{lrcl} \text{The polarization field} & \mathbf{P} & = & -\epsilon_0 \mathbf{E}^{(P)} \\ \text{Electric displacement} & \mathbf{D} & = & \epsilon_0 \mathbf{E}^{(F)} \\ \text{The magnetization field} & \mathbf{M} & = & \mathbf{B}^{(P)}/\mu_0 \\ \text{Magnetic displacement} & \mathbf{H} & = & \mathbf{B}^{(F)}/\mu_0. \end{array}$$

Although there are no common accepted names on these fields in literature their letters and meanings are very often the same. The minus sign in the definition of the polarization field is due to convention since $\mathbf{E}^{(P)}$ is oppositely directed of $\mathbf{E}^{(F)}$. These definitions lead to the following relations

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} &= \mathbf{B}/\mu_0 - \mathbf{M}. \end{aligned}$$

Using these relations we can rewrite Maxwell's equations,

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho^{(F)} \\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J}^{(F)} + \partial_t \mathbf{D}. \end{aligned}$$

By understanding Maxwell's equations in this way we can say that electromagnetic fields are (locally) linearly dependent on their sources.

2.3 The geomagnetic field in the atmosphere

By measuring or somehow predicting the sizes of terms, we can get a notion of how each term contributes in a relation. In this way we will analyze one of Maxwell's equations. We will look at them in the atmosphere, and hope that this analysis will lead to some simplification. We will look at the following equation

$$\nabla \times \mathbf{H} = \mathbf{J}^{(F)} + \partial_t \mathbf{D}. \tag{2.1}$$

We want to get a better feeling of how it behaves in the atmosphere. From observations in the atmosphere we have the following data.

$$\begin{aligned} |\mathbf{H}| &\approx 30 - 60 \cdot 10^{-6} A/m \\ |\mathbf{J}^{(F)}| &\approx 10^{-11} A/m^2 \\ |\mathbf{D}| &\approx 10^{-9} C/m^2. \end{aligned}$$

We are looking on length scales of order R_{-1} = the radius of the earth $\approx 6.25 \cdot 10^6$m and time scales of order t. Then the terms in equation (2.1) have the following magnitude,

$$\begin{aligned} |\nabla \times \mathbf{H}| &\approx 4 \cdot 10^{-6} A/m^2 \\ |\mathbf{J}^{(F)}| &\approx 10^{-11} A/m^2 \\ |\partial_t \mathbf{D}| &\approx \frac{10^{-9}}{t} As/m^2. \end{aligned}$$

If we look at time scales $t \gg 2.5 \cdot 10^{-4}$s, we see that $|\nabla \times \mathbf{H}|$ is dominating the equation totally. Therefore we will neglect the other terms. The atmosphere is very slightly magnetically polarizable, so we set $\mathbf{M} = 0$ or equivalently $\mu_0 \mathbf{H} = \mathbf{B}$. Then we have

$$\nabla \times \mathbf{B} = 0. \tag{2.2}$$

Although (2.1) and (2.2) are in a way close in the atmosphere, this do not in general imply that their solutions are close. To verify that we have to compare the solution of the approximated equation with the solution of the real equation or with observed data. We are not going to do this, but it turns out that the solutions are close.

The atmosphere is simply connected so we can use the following fact.

Fact 6. *Let* $\mathbf{F}$ *be a continuous differentiable vector field in a simply connected region* E*, then*

$$\nabla \times \mathbf{F} = 0 \Leftrightarrow \mathbf{F} = \nabla V,$$

for some function V*.*

Proof. See [Kre99, Theorem 3 p. 475]. □

Thus there exists a function ψ such that

$$\mathbf{B} = -\nabla \psi. \tag{2.3}$$

We will say that ψ is the potential to $\mathbf{B}$. Now it follows from the third of Maxwell's equations that

$$\nabla \cdot \mathbf{B} = -\Delta \psi = 0.$$

So ψ is harmonic. We state this in a result.

Result 1. *The geomagnetic field is in the atmosphere well approximated by the gradient of a harmonic function.*

2.4 Internal and external fields

In the last section we concluded that we can neglect $\mathbf{J}^{(F)}$ in the atmosphere. This means that we look at magnetic fields without any sources there. In Gauss' theory of the earth's magnetic field sources are located above the atmosphere or within the earth. As announced we now want to split the magnetic field dependent on its sources. We will split the field in a internal and an exterior field, $\mathbf{B}^{(I)}$ and $\mathbf{B}^{(E)}$ respectively. Then for the total magnetic field B we have

$$\mathbf{B} = \mathbf{B}^{(I)} + \mathbf{B}^{(E)}.$$

Let R_{-1} be the radius of the earth and R_{+1} the radius of the ionosphere, we have

$$\begin{array}{rcll} \nabla \cdot \mathbf{B}^{(I)} & = & 0 & \text{everywhere} \\ \nabla \cdot \mathbf{B}^{(E)} & = & 0 & \text{everywhere} \\ \nabla \times \mathbf{B}^{(I)} & = & 0 & \text{when } r > R_{-1} \\ \nabla \times \mathbf{B}^{(E)} & = & 0 & \text{when } r < R_{+1}. \end{array}$$

Now it follows from Fact 6 that there exists potentials $\psi^{(I)}$ and $\psi^{(E)}$ such that

$$\begin{array}{lcl} \mathbf{B}^{(I)} = -\nabla\psi^{(I)} & \text{and} & \Delta\psi^{(I)} = 0 \text{ for } r > R_{-1} \\ \mathbf{B}^{(E)} = -\nabla\psi^{(E)} & \text{and} & \Delta\psi^{(E)} = 0 \text{ for } r < R_{+1}. \end{array}$$

2.5 Reconstruction from vector data

It is always a question of accuracy when describing a real physical phenomena. One can never describe it exact, so one is forced to look at idealized situations, approximations or probability models. This is also the case for the geomagnetic field. We will therefore only look at the "well approximated" magnetic field in Result 1. This result says, that for the geomagnetic field, $\mathbf{B}$, in the atmosphere there exists a harmonic function ψ such that

$$\mathbf{B} = -\nabla\psi.$$

Let the annulus $A(r_1, r_2)$, for $0 \leq r_1 < r_2 \leq \infty$, be the atmosphere. Then ψ will be harmonic there. Fix $r \in (r_1, r_2)$ and define $\psi_r(\hat{\mathbf{r}}) = \psi(r\hat{\mathbf{r}})$. Clearly $\psi_r \in L^2(S)$, so by Fact 5,

$$\psi(\mathbf{r}) = \psi_r(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_l^m(r) Y_l^m(\hat{\mathbf{r}}).$$

Here the coefficients $\psi_l^m(r)$ depend on r. Since ψ is harmonic we need

$$\begin{aligned} 0 = r^2 \Delta\psi &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [Y_l^m(\hat{\mathbf{r}}) \Delta_r \psi_l^m(r) + \psi_l^m(r) \Delta_1 Y_l^m(\hat{\mathbf{r}})] \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [\Delta_r \psi_l^m(r) - l(l+1)\psi_l^m(r)] Y_l^m(\hat{\mathbf{r}}). \end{aligned}$$

We used $r^2\Delta = \Delta_r + \Delta_1$ and (1.1). Now $\{Y_l^m\}_{l,m}$ form a basis for $L^2(S)$, thus the above equation holds only if

$$\Delta_r \psi_l^m(r) - l(l+1)\psi_l^m(r) = r^2\partial_r^2\psi_l^m(r) + 2r\partial_r\psi_l^m(r) - l(l+1)\psi_l^m(r) = 0, \tag{2.4}$$

for every l and m. This is an Euler-Cauchy equation which has solutions of the form r^n. Putting $\psi_l^m(r) = r^n$ into (2.4) we get

$$[n(n-1) + 2n - l(l+1)]r^n = 0,$$

thus $n = -l - 1$ or $n = l$. The general solution of (2.4) is then

$$\psi_l^m(r) = g_l^m r^{-(l+1)} + k_l^m r^l. \tag{2.5}$$

Where g_l^m and k_l^m are some constants which we will call the *Gauss' coefficients* of the function ψ. Putting this expression into the formula for ψ we get

$$\psi(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[g_l^m r^{-(l+1)} + k_l^m r^l]Y_l^m(\hat{\mathbf{r}}). \tag{2.6}$$

Applying $-r^{-1}\nabla_r$ and $-r^{-1}\nabla_1$ on (2.6) we get the radial and the spherical part of the magnetic field.

$$B_r(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[(l+1)g_l^m r^{-(l+2)} - lk_l^m r^{l-1}]Y_l^m(\hat{\mathbf{r}}) \tag{2.7}$$

$$\mathbf{B}_s(\mathbf{r}) = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}[g_l^m r^{-(l+2)} + k_l^m r^{l-1}]\nabla_1 Y_l^m(\hat{\mathbf{r}}). \tag{2.8}$$

Fix $r = R$, then knowing $\psi_l^m(R)$ corresponds to knowing ψ on a sphere since $\psi_l^m(R) =< \overline{Y_l^m(\hat{\mathbf{s}})}\psi(R\hat{\mathbf{s}}) >$. If we only know $\psi_l^m(R)$, there are infinitely many choices of Gauss' coefficients which satisfy (2.5). Thus there are infinitely many potentials which are equal on a given sphere. However, if we know $\psi_l^m(r)$ for two values of r, there is just one choice of Gauss' coefficients that satisfy (2.5), so this is then sufficient to reconstruct the magnetic field. Since it is $\mathbf{B}$ and not ψ that is observable we will show how to reconstruct the field form its values on a sphere. We use (2.7) and the orthonormality to the basis vectors to project B_r onto $\overline{Y_l^m}$,

$$< \overline{Y_l^m(\hat{\mathbf{s}})}B_r(r\hat{\mathbf{s}}) >= (l+1)g_l^m r^{-(l+2)} - lk_l^m r^{l-1}. \tag{2.9}$$

Also we have $< \nabla_1\overline{Y_{l'}^{m'}} \cdot \nabla_1 Y_l^m >= l(l+1)\delta_{ll'}\delta_{mm'}$ from (1.2), thus using (2.8) we get

$$< \nabla_1\overline{Y_l^m(\hat{\mathbf{s}})} \cdot \mathbf{B}_s(r\hat{\mathbf{s}}) >= -l(l+1)(g_l^m r^{-(l+2)} + k_l^m r^{l-1}). \tag{2.10}$$

Solving (2.9) and (2.10) for g_l^m and k_l^m gives

$$(2l+1)r^{-(l+2)}g_l^m =< \overline{Y_l^m(\hat{\mathbf{s}})}B_r(r\hat{\mathbf{s}}) > - \frac{< \nabla_1 \overline{Y_l^m(\hat{\mathbf{s}})} \cdot \mathbf{B}_s(r\hat{\mathbf{s}}) >}{l+1} \tag{2.11}$$

$$-(2l+1)r^{l-1}k_l^m =< \overline{Y_l^m(\hat{\mathbf{s}})}B_r(r\hat{\mathbf{s}}) > + \frac{< \nabla_1 \overline{Y_l^m(\hat{\mathbf{s}})} \cdot \mathbf{B}_s(r\hat{\mathbf{s}}) >}{l}. \tag{2.12}$$

Now (2.12) does not hold for $l = 0$. However, since $Y_0^0 = 1$, the term $k_0^0 r^0 Y_0^0$ is constant in the expansion (2.6) for ψ, thus $\mathbf{B}$ is independent of it and we can choose it arbitrarily.

In the last section we saw that ψ can be divided into two functions $\psi^{(I)}$ and $\psi^{(E)}$, where $\psi^{(I)}$ is harmonic for $r > R_{-1}$ and $\psi^{(E)}$ is harmonic for $r < R_{+1}$. Now assume we only have internal sources, thus $\psi = \psi^{(I)}$. The gauss coefficients is still given by (2.11) and (2.12). Because we only have internal sources it seems reasonable to assume that $\mathbf{B}$ vanishes at infinity. Let $r \to \infty$ in (2.12). We assumed that $\mathbf{B}$ vanish at infinity, so the right hand side goes to zero. In order for the equation to hold we therefore need $k_l^m = 0$ for $l, m \geq 1$. For convenience we assume ψ vanishes at infinity, so $k_0^0 = 0$. Then we have

$$\psi^{(I)}(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_l^m r^{-(l+1)} Y_l^m(\hat{\mathbf{r}}).$$

Not that this is a general formula for harmonic functions. We state it in a lemma.

Lemma 7. *Let $0 \leq r_1 < r_2 \leq \infty$ and assume $f \in \mathcal{H}_0(A(r_1, r_2))$, then*

$$f(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m r^{-(l+1)} Y_l^m(\hat{\mathbf{r}}),$$

where f_l^m are the Gauss coefficients to f, given by $(l+1)f_l^m =< \overline{Y_l^m}\partial_r f >$.

We can proceed in a similar way to find $\psi^{(E)}$. Assume we have no internal sources and that $|\mathbf{B}|$ is bounded at zero. Let $r \to \infty$ in (2.11). Then the left hand side goes to infinity while the right hand side is bounded. This imply $g_l^m = 0$ for every l and m. Thus

$$\psi^{(E)}(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} k_l^m r^l Y_l^m(\hat{\mathbf{r}}).$$

We conclude that in a magnetic field in the atmosphere the g_l^m coefficients are due to internal sources and k_l^m to external. Note that if we only have an internal or external field, then due to (2.9) and (2.10), we only need to know the spherical component, $\mathbf{B}_s$, or the normal component, B_r, on a sphere, in order to reconstruct it in the whole atmosphere.

Also in the general case we set $k_0^0 = 0$ for convenience. Further by (2.11) we have

$$r^{-2} g_0^0 = \overline{Y_0^0} < B_r(r\hat{\mathbf{s}}) >,$$

because Y_0^0 is constant. In addition we have

$$< B_r(r\hat{\mathbf{s}}) >= \frac{1}{4\pi}\int_S B_r(r\hat{\mathbf{s}})d\hat{\mathbf{s}} = \frac{1}{4\pi r^2}\int_{S(r)} B_r(\mathbf{s})d\mathbf{s}$$
$$= \frac{1}{4\pi r^2}\int_{\partial B(r)} \mathbf{B}(\mathbf{s})\cdot d\mathbf{s} = \frac{1}{4\pi r^2}\int_{B(r)} (\nabla\cdot\mathbf{B})d\mathbf{s} = 0.$$

Here we used Stokes' theorem in the next last, and Maxwell equation $\nabla\cdot\mathbf{B} = 0$ in the last step. This implies that $g_0^0 = 0$. We end this section by summing up our results.

Result 2. *Suppose* $\psi \in \mathcal{H}_0(A(r_1, r_2))$ *for* $0 \le r_1 < r_2 \le \infty$ *and let* $\mathbf{B} = -\nabla\psi$ *be its corresponding magnetic field, then*

$$\begin{aligned}
\psi(\mathbf{r}) &= \sum_{l=1}^{\infty}\sum_{m=-l}^{l}[g_l^m r^{-(l+1)} + k_l^m r^l]Y_l^m(\hat{\mathbf{r}}) \\
B_r(\mathbf{r}) &= \sum_{l=1}^{\infty}\sum_{m=-l}^{l}[(l+1)g_l^m r^{-(l+2)} - l k_l^m r^{l-1}]Y_l^m(\hat{\mathbf{r}}) \\
\mathbf{B}_s(\mathbf{r}) &= -\sum_{l=1}^{\infty}\sum_{m=-l}^{l}[g_l^m r^{-(l+2)} + k_l^m r^{l-1}]\nabla_1 Y_l^m(\hat{\mathbf{r}}),
\end{aligned}$$

where g_l^m *and* k_l^m *are given by (2.11) and (2.12). Also if* $\mathbf{B}$ *is created only by internal sources and vanish at infinity, then* $k_l^m = 0$. *If* $\mathbf{B}$ *is created only by external sources, then* $g_l^m = 0$. *Moreover, if we have any of those cases, then we only need* $\mathbf{B}_s$ *or* B_r *to reconstruct the field, and its Gauss coefficients are given by (2.9) or (2.10).*

Chapter 3

Reconstruction from Intensity

In the previous chapter we found out that it is enough to know the magnetic field on a sphere in order to reconstruct it. In practice it is difficult to collect such data. We can measure the magnetic field on the earth's surface, but since the magnetic observatories are few (< 200) and unevenly distributed it is hard to get measurements that corresponds to that of a sphere (see [Lan87] for a table of magnetic observatories).
If we use satellites for these measurements we avoid these problems. A low orbiting satellite would take about 22 days to achieve a 1° spacing between the measurements. However, to measure the magnetic field accurate, we have to know the orientation of the satellite correspondingly accurate. This is very difficult and expensive, and inaccurate orientation data affect the end result significantly. The intensity of the geomagnetic field can however be measured very precise, so we want to determined if we can use this to reconstruct the field. We ask the question: can two different magnetic fields have the same intensity on a sphere? And we also look at if this holds for two spheres.

To answer this question we will look at some simple examples and special cases. These will not immediately imply something about the geomagnetic field, but will give us some notion of what to expect. In practice some geomagnetic models of intensity data differs from measurements of the magnetic field with up to 1000nT near the equator (see [BPC96, Subsection 4.4.1]). From this we would expect non-uniqueness, and it had been nice with some example giving this a theoretical foundation.

The main geomagnetic field is the field created from sources inside the earth. Measurements indicate that the annual mean magnetic field is 99.9% internally produced. We will therefore ignore the field created from external sources in the following and concentrate on the main geomagnetic field. It now follow from Result 2 that our potential, ψ, is on the form

$$\psi(\mathbf{r}) = \sum_{l=1}^{\infty} r^{-(l+1)} \sum_{m=-l}^{l} \psi_l^m Y_l^m(\hat{\mathbf{r}}), \tag{3.1}$$

Let $\mathbf{B}_u$ and $\mathbf{B}_v$ be internally produced magnetic fields such that $|\mathbf{B}_u| = |\mathbf{B}_v|$ on a sphere containing the sources. We want to know if this imply that $\mathbf{B}_u = \pm\mathbf{B}_v$. A follow up question would then be if we can say $\mathbf{B}_u = \pm\mathbf{B}_v$ when $|\mathbf{B}_u| = |\mathbf{B}_v|$ on two spheres. This is a very interesting situation and is still an open problem.

Now $\mathbf{B}_u = -\nabla u$ and $\mathbf{B}_v = -\nabla v$ for some harmonic functions u and v. Also we have

$$|\nabla u| = |\nabla v| \;\Leftrightarrow\; \nabla u \cdot \nabla u = \nabla v \cdot \nabla v \Leftrightarrow \nabla(u+v)\cdot\nabla(u-v) = 0.$$

This imply that finding two different magnetic field with the same intensity is equivalent to finding two non-zero orthogonal fields. We will try to do this by looking at some examples. Define $\phi = u + v$, $\psi = u - v$, $\mathbf{B}^+ = -\nabla\phi$, $\mathbf{B}^- = -\nabla\psi$, then our plan is first to let ϕ be an explicit harmonic potential, and then try to find ψ such that $\nabla\phi\cdot\nabla\psi = \mathbf{B}^+\cdot\mathbf{B}^- = 0$, on one or two spheres.

3.1 An easy example, $\phi = r^{-1}Y_0^0$

We start with the easy example $\phi = r^{-1}Y_0^0$, where $Y_0^0 = \frac{1}{\sqrt{4\pi}}$. Then

$$\begin{aligned}\nabla\phi &= < \frac{-1}{\sqrt{4\pi}r^2}, 0, 0 > \\ \nabla\psi &= < \partial_r\psi, (r\sin\theta)^{-1}\partial_\lambda\psi, r^{-1}\partial_\theta\psi > .\end{aligned}$$

Then imposing the orthogonality condition we have

$$\nabla\phi\cdot\nabla\psi = \frac{-1}{\sqrt{4\pi}r^2}\partial_r\psi = 0.$$

Using the expression (3.1) for ψ, the above equation implies

$$\partial_r\psi(\mathbf{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} -(l+1)r^{-(l+2)}\psi_l^m Y_l^m(\hat{\mathbf{r}}) = 0.$$

Then since $\{Y_l^m\}_{l,m}$ form a basis for $L^2(S)$ the above equation holds if and only if

$$-(l+1)r^{-(l+2)}\psi_l^m = 0,$$

for every l and m. Thus every $\psi_l^m = 0$, so $\psi \equiv 0$. This means that there are no non-zero fields orthogonal to $-\nabla r^{-1}Y_0^0$. This means that if the sum of two fields are $r^{-1}Y_0^0$ and they have the same intensity on a sphere, then they are equal.

Note also that the implication $\partial_r\psi = 0 \Rightarrow \psi \equiv 0$ follows directly from the fact that ψ vanishes at infinity. And also that in this example the Gauss coefficient $g_0^0 \neq 0$, so in accordance to what we found out in the previous chapter, $-\nabla\psi$, and thus $-\nabla u$ and $-\nabla v$, does not correspond to actual magnetic fields. However, the next example we are going to look at does.

3.2 Backus example, $\phi = r^{-2}Y_1^0$

Now we are going to look at a famous example constructed by Backus in [Bac70]. This will show non-uniqueness from intensity on a sphere. We will also try to extend this non-uniqueness to two spheres. Let $\phi = r^{-2}Y_1^0$, where $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$. Then setting $\mu = \cos\theta$ we get

$$\begin{aligned}\nabla\phi &= \sqrt{\frac{3}{4\pi}} < -2\mu r^{-3}, 0, -r^{-3}\sin\theta > \\ \nabla\psi &= < \partial_r\psi, \frac{1}{r\sin\theta}\partial_\lambda\psi, r^{-1}\partial_\theta\psi > .\end{aligned}$$

Thus

$$\nabla\phi\cdot\nabla\psi = \sqrt{\frac{3}{4\pi}}(-2\mu r^{-3}\partial_r\psi - r^{-4}\sin\theta\partial_\theta\psi).$$

Using $\partial_\theta = -\sin\theta\partial_\mu$, then $\nabla\phi\cdot\nabla\psi = 0$ if and only if

$$(1-\mu^2)\partial_\mu\psi - 2r\mu\partial_r\psi = 0.$$

Using the expression (3.1) for ψ, we get

$$\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\psi_l^m r^{-(l+2)}[(1-\mu^2)\partial_\mu Y_l^m + 2\mu(l+1)Y_l^m] = 0. \tag{3.2}$$

Define $g_l^m = 0$ for $l < |m|$ and

$$g_l^m = \left[\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right]^{1/2} \qquad \text{for } 0 \le |m| \le l.$$

Also let $Y_l^m = 0$ for $l < |m|$. We have the following relations from [EWK28]

$$\begin{aligned}(1-\mu^2)\partial_\mu Y_l^m = &\quad (l+1)g_l^m Y_{l-1}^m - l g_{l+1}^m Y_{l+1}^m \\ \mu Y_l^m = &\quad g_l^m Y_{l-1}^m + g_{l+1}^m Y_{l+1}^m.\end{aligned} \tag{3.3}$$

Substitute these into (3.2)

$$\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\psi_l^m r^{-(l+2)}[3(l+1)g_l^m Y_{l-1}^m + (l+2)g_{l+1}^m Y_{l+1}^m] = 0,$$

and define $\psi_l^m = 0$ for $l < |m|$ and $\psi_0^0 = 0$. Then we can rewrite this last equation as

$$\sum_{m=-\infty}^{\infty}\sum_{l=|m|}^{\infty}[3(l+2)g_{l+1}^m\psi_{l+1}^m + (l+1)g_l^m\psi_{l-1}^m r^2]r^{-(l+3)}Y_l^m = 0.$$

The spherical harmonics, Y_l^m, form a basis for $L^2(S)$, so the above equation holds if and only if

$$3(l+2)g_{l+1}^m\psi_{l+1}^m + (l+1)g_l^m\psi_{l-1}^m r^2 = 0,$$

for $0 \le |m| \le l$. Since $g_{l+1}^m \neq 0$ we can rewrite it as

$$\psi_{l+1}^m = -\frac{1}{3}\frac{(l+1)g_l^m}{(l+2)g_{l+1}^m}\psi_{l-1}^m r^2. \tag{3.4}$$

If this equation shall hold for two different r-values, we must have $\psi_l^m = 0$ for all l and m. Thus we have uniqueness from intensity on two spheres. For the one sphere case we fix $r = R$. Then by definition we have $\psi_{|m|-1}^m = 0$, so (3.4) implies that for any positive integer n, $\psi_{|m|+2n+1}^m = 0$. Also we can choose $\psi_{|m|}^m$ arbitrarily, and (3.4) will determine $\psi_{|m|+2n}^m$. If this series converge we have an example of two different fields with same intensity on a sphere.

One sphere: convergence

We are going to look at convergence of

$$\sum_{m=-\infty}^{\infty}\sum_{l=|m|}^{\infty}\psi_l^m r^{-(l+1)}Y_l^m(\hat{\mathbf{r}}).$$

By writing out the expression (3.4) for ψ_l^m, we see after some calculations that $|\psi_{l+1}^m| \le \frac{R^2}{3}|\psi_{l-1}^m|$. Then set $\alpha_l^m(\mathbf{r}) = \psi_l^m r^{-(l+1)}Y_l^m(\hat{\mathbf{r}})$. Now α_l^m is only non-zero for $l = |m| + 2n$, where n is a non-zero integer. Using $|Y_l^m|^2 \le \frac{2l+1}{4\pi}$ from Fact 3, we have

$$|\alpha_{|m|+2n}^m(\mathbf{r})| \le |\psi_{|m|}^m| r^{-(|m|+1)}\left(\frac{R}{\sqrt{3}r}\right)^{2n}\left(\frac{2|m|+4n+1}{4\pi}\right)^{1/2}.$$

Thus

$$\sum_{m=-\infty}^{\infty}\sum_{l=|m|}^{\infty}|\alpha_l^m(\mathbf{r})| \le \sum_{m=-\infty}^{\infty}|\psi_{|m|}^m| r^{-(|m|+1)}\sum_{n=0}^{\infty}\left(\frac{R}{\sqrt{3}r}\right)^{2n}\left(\frac{2|m|+4n+1}{4\pi}\right)^{1/2},$$

where $\alpha_0^0 = \psi_0^0 = 0$. The series $f_n = \sum_{n=0}^{\infty}\left(\frac{R}{\sqrt{3}r}\right)^{2n}\left(\frac{2|m|+4n+1}{4\pi}\right)^{1/2}$ will converge for every m and $r > \frac{R}{\sqrt{3}}$ due to the ratio test. If we choose

$$\psi_{|m|}^m \le C\beta^{|m|}$$

for a constant C and $\beta < 1$ and then apply the ratio test once again, we see that the double series converge absolutely for $r > \max\{\frac{R}{\sqrt{3}}, \beta\}$.However, since $Y_l^{-m} = (-1)^m\overline{Y_l^m}$ by Fact 2, we have to choose $\psi_l^{-m} = (-1)^m\overline{\psi_l^m}$ in order for ψ to be real. Now the series converges to a real harmonic function *psi*. Then we can easily find u, v, such that $|\nabla u| = |\nabla v|$ on a sphere and $u \neq \pm v$. Therefore we have the following result.

Result 3 (Non-uniqueness)**.** *A magnetic field is not uniquely determined up to sign by its intensity on a sphere.*

3.3 Example, $\phi = c_1r^{-2}Y_1^0 + c_2r^{-3}Y_2^0$

Now we let $\phi = c_1r^{-2}Y_1^0 + c_2r^{-3}Y_2^0$, where $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$ and $Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$. Writing μ for $\cos\theta$ and s for $\sin\theta$, and also let $b = \sqrt{\frac{5}{16\pi}}c_2$ and $a = \sqrt{\frac{3}{4\pi}}c_1b^{-1}$, such that $b^{-1}\phi = a\mu r^{-2} + (3\mu^2 - 1)r^{-3}$. Then

$$\begin{aligned}\nabla\phi &= b < -2a\mu r^{-3} - 3(3\mu^2-1)r^{-4}, 0, -asr^{-3} - 6\mu sr^{-4} > \\ \nabla\psi &= < \partial_r\psi, (rs)^{-1}\partial_\lambda\psi, r^{-1}\partial_\theta\psi >,\end{aligned}$$

and we have

$$\nabla\phi\cdot\nabla\psi = -br^{-4}\{[2a\mu r + 3(3\mu^2-1)]\partial_r\psi + [asr + 6\mu s]r^{-1}\partial_\theta\psi\}$$

We set this equal to zero and set in the expansion (3.1) for ψ. If we use the relations (3.3) and drop the, for the moment, superfluous superscript m's, we get after some calculations

$$\begin{aligned}15(l+3)g_{l+2}g_{l+l}\psi_{l+2} + 3a(l+2)g_{l+1}\psi_{l+1}r^2 + 3[5(l+1)g_l^2 + (l+3)g_{l+1}^2 - (l+1)]\psi_l r^2 + \\ +a(l+1)g_l\psi_{l-1}r^4 + 3(l+1)g_{l-1}g_l\psi_{l-2}r^4 = 0.\end{aligned}$$

Clearly $g_l \to \frac{1}{2}$ when $l \to \infty$. So dividing the above equation by l and letting $l \to \infty$ we get the corresponding asymptotic relation,

$$15\psi_{l+2} + 6a\psi_{l+1}r^2 + 6\psi_l r^2 + 2a\psi_{l-1}r^4 + 3\psi_{l-2}r^4 = 0.$$

We will look on the one sphere case and set therefore $r = 1$. The characteristic equation for this difference equation is

$$15x^4 + 6ax^3 + 6x^2 + 2ax + 3 = 0. \tag{3.5}$$

If we are going to have convergent solutions, the characteristic equation needs roots with modulo less than one. Therefore we write this as a function of a,

$$a(x) = -\frac{15x^4 + 6x^2 + 3}{6x^3 + 2x}.$$

Then if (3.5) shall have roots with modulo less than one, a must be in the image of $(-1, 1)$ under $a(x)$. We can see $a(x)$ in Figure 3.1.

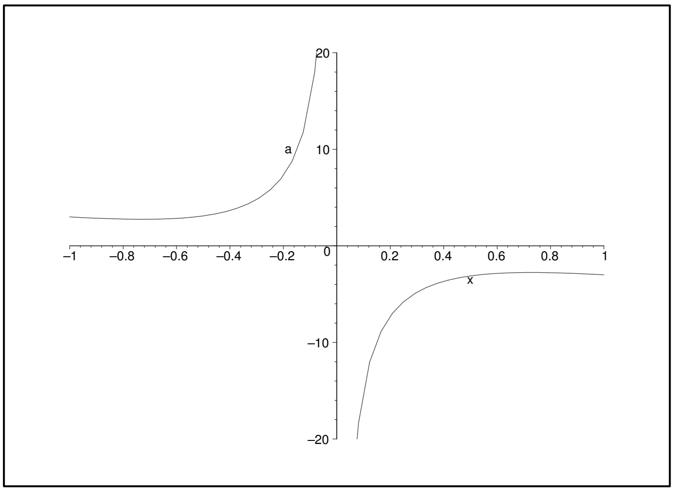

Figure 3.1: The function $a(x)$ between -1 and 1.

To find the values $a(x)$ takes in $(-1, 1)$, we find its extremum points. Differentiating $a(x)$ we get

$$a'(x) = -\frac{60\,x^3 + 12\,x}{6\,x^3 + 2\,x} + \frac{\left(15\,x^4 + 6\,x^2 + 3\right)\left(18\,x^2 + 2\right)}{\left(6\,x^3 + 2\,x\right)^2}.$$

Setting this equal to zero and solving for x, we get two real roots,

$$s_+ = \sqrt{\frac{3 + 2\sqrt{6}}{15}} \quad \text{and} \quad s_- = -\sqrt{\frac{3 + 2\sqrt{6}}{15}}.$$

We have $a(\pm 1) = \mp 3$ and $a(s_\pm) = \frac{2}{s_\mp}$, so from figure 3.1 we see that

$\lvert a\rvert < \frac{2}{s_+}$	no convergent solution
$\frac{2}{s_+} < \lvert a\rvert < 3$	two convergent solutions
$3 < \lvert a\rvert$	one convergent solution

Analyzing this case for the two sphere case becomes very difficult and computational demanding. And for more involved examples it gets even harder. We will therefore look at another technique to get examples like this.

3.4 General formula

We want a general formula for constructing orthogonal fields. The technique we used in the three previous examples can be used to look at any field, but as we saw it becomes very difficult and computational demanding for involved examples. Therefore we have little hope to use this in order to find a general formula. We will follow [AOP04] to get a general expression for $\mathbf{B}^+ \cdot \mathbf{B}^-$. First we have the following key relation,

$$\Delta(uv) = v\Delta u + u\Delta v + 2(\nabla u \cdot \nabla v).$$

This follow by direct calculation. Since ϕ and ψ are harmonic we only need to find $\Delta(\phi\psi)$. Expanding ϕ and ψ in spherical harmonics we get

$$\begin{aligned}\phi &= \sum_{l^+=1}^{\infty} \sum_{m^+=-l^+}^{l^+} \phi_{l^+}^{m^+} r^{-(l^++1)} Y_{l^+}^{m^+} \\ \psi &= \sum_{l^-=1}^{\infty} \sum_{m^-=-l^-}^{l^-} \psi_{l^-}^{m^-} r^{-(l^-+1)} Y_{l^-}^{m^-}.\end{aligned}$$

Thus

$$\phi\psi = \sum_{l^+,m^+} \sum_{l^-,m^-} \psi_{l^+}^{m^+} \phi_{l^-}^{m^-} r^{-(l^++l^-+2)} Y_{l^+}^{m^+} Y_{l^-}^{m^-}.$$

Now we can use the following fact.

Fact 8.

$$Y_{l_1}^{m_1} Y_{l_2}^{m_2} = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m=-l}^{l} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} < l_1 0; l_2 0 | l 0 >< l_1 m_1; l_2 m_2 | l m > Y_l^m,$$

where $< l_1 m_1; l_2 m_2 | l m >$ are the Clebsch-Gordan coefficients. For further details see [Edm57, Chapter 3], specially see (3.6.10) or (3.6.11) in that chapter for an explicit expression. However, the main properties of the Clebsch-Gordan coefficients that we are going to use are that they are only nonzero when $m = m_1 + m_2$ and $|l_1 - l_2| \le l \le l_1 + l_2$, and that $< l_1 0; l_2 0 | l 0 >$ is nonzero only when $l_1 + l_2 + l$ is even.

By this fact we have

$$\phi\psi = \sum_{l^+,m^+} \sum_{l^-,m^-} \sum_{k=|l^+-l^-|}^{l^++l^-} \sum_{m_k=-k}^{k} \sqrt{\frac{(2l^++1)(2l^-+1)}{4\pi(2k+1)}} < l^+ 0; l^- 0 | k 0 >< l^+ m^+; l^- m^- | k m_k > \\ \times \phi_{l^+}^{m^+} \psi_{l^-}^{m^-} r^{-(l^++l^-+2)} Y_k^{m_k}.$$

We have $\Delta = r^{-2}(\Delta_r + \Delta_1)$, and also

$$\begin{aligned}\Delta_r r^{-(l^++l^-+2)} &= (r^2\partial_r^2 + 2r\partial_r) r^{-(l^++l^-+2)} \\ &= (l^++l^-+2)(l^++l^-+3) r^{-(l^++l^-+2)} - 2(l^++l^-+2) r^{-(l^++l^-+2)} \\ &= (l^++l^-+1)(l^++l^-+2) r^{-(l^++l^-+2)}.\end{aligned}$$

By (1.1), $\Delta_1 Y_k^{m_k} = -k(k+1)Y_k^{m_k}$, so

$$\Delta r^{-(l^+ + l^- + 2)} Y_k^{m_k} = [(l^+ + l^- + 1)(l^+ + l^- + 2) - k(k+1)] r^{-(l^+ + l^- + 4)} Y_k^{m_k}.$$

We have the simple relation

$$a(a+1) - k(k+1) = a^2 + a - k^2 - k = a^2 - k^2 + a - k = (a+k)(a-k) - (a-k) = (a+k+1)(a-k).$$

Using this with $a = l^+ + l^- + 1$, we can write

$$\Delta r^{-(l^+ + l^- + 2)} Y_k^{m_k} = (l^+ + l^- + 2 + k)(l^+ + l^- + 1 - k) r^{-(l^+ + l^- + 4)} Y_k^{m_k}.$$

Finally we have

$$\mathbf{B}^+ \cdot \mathbf{B}^- = \frac{1}{2} \sum_{l^+=1}^{\infty} \sum_{m^+=-l^+}^{l^+} \sum_{l^-=1}^{\infty} \sum_{m^-=-l^-}^{l^-} \sum_{k=|l^+-l^-|}^{l^++l^-} \sum_{m_k=-k}^{k} \sqrt{\frac{(2l^+ + 1)(2l^- + 1)}{4\pi(2k+1)}} < l^+0; l^-0|k0 >$$
$$\times < l^+m^+; l^-m^-|km_k > (l^+ + l^- + 2 + k)(l^+ + l^- + 1 - k) \phi_{l^+}^{m^+} \psi_{l^-}^{m^-} r^{-(l^+ + l^- + 4)} Y_k^{m_k}.$$

Since $\{Y_l^m\}$ are orthogonal, it follows that $\mathbf{B}^+ \cdot \mathbf{B}^- = 0$ if and only if

$$\sum_{l^+,m^+} \sum_{l^-,m^-} \sqrt{(2l^+ + 1)(2l^- + 1)} (l^+ + l^- + 2 + k)(l^+ + l^- + 1 - k)$$
$$\times < l^+0; l^-0|k0 >< l^+m^+; l^-m^-|km_k > \phi_{l^+}^{m^+} \psi_{l^-}^{m^-} r^{-(l^+ + l^-)} = 0, \tag{3.6}$$

for every pair (k, m_k) such that $0 \le |m_k| \le k$.

This was the relation we sought. We will use this to get a general result about fields with finite expansion.

Result 4 (Finite expansion). *If two internally produced magnetic fields have the same intensity on a sphere and their sum and difference only have a finite number of non-zero Gauss coefficients, then they differ at most by a sign.*

Proof. Along with the previous discussion in this section, we have to show that if $\mathbf{B}^+$ and $\mathbf{B}^-$ have finite expansions and are orthogonal on a sphere, then one of the fields is identically zero. Assume that none of the fields are identically zero, then since $\mathbf{B}^+$ has only finitely many non-zero Gauss' coefficients there exists L^+, M^+ such that $\phi_{L^+}^{M^+} \neq 0$ and $\phi_{l^+}^{m^+} = 0$ when $l^+ > L^+$ or $l^+ = L^+$ and $m^+ > M^+$. Similarly there are constants L^-, M^- for ψ.
Now we look at (3.6) when $k = L^+ + L^-$ and $m_k = M^+ + M^-$. If the Clebsch-Gordan coefficients shall be nonzero m^+, m^-, l^+, l^- has to fulfill some relations listed in Fact 8. Namely $m_k = M^+ + M^- = m^+ + m^-$, thus $m^+ = M^+$ and $m^- = M^-$. Ana also $k = L^+ + L^- \le l^+ + l^-$ so $l^+ = L^+$ and $l^- = L^-$. These are the only options so we are left with

$$< L^+0; L^-0|L^+ + L^-0 >< L^+M^+; L^-M^-|L^+ + L^-M^+ + M^- > \phi_{L^+}^{M^+} \psi_{L^-}^{M^-} = 0.$$

If we let $J = j_1 + j_2$ and $M = m_1 + m_2$, then we have from [Edm57, Equation (3.6.12)],

$$< j_1m_1; j_2m_2|JM >= \left[\frac{(2j_1)!(2j_2)!(J+M)!(J-M)!}{(2J)!(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!}\right]^{1/2}.$$

Clearly $< L^+0; L^-0|L^+ + L^-0 >< L^+M^+; L^-M^-|L^+ + L^-M^+ + M^- >\neq 0$, thus $\phi_{L^+}^{M^+}\psi_{L^-}^{M^-} = 0$. This contradict the assumption that both fields are not identical zero. □

3.5 Another example, $\phi = r^{-3}Y_2^1$

We will use (3.6) to look at $\phi = r^{-3}Y_2^1$, this means that all Gauss' coefficients of ϕ are zero except ϕ_2^1, thus

$$\sum_{l^-,m^-}\sqrt{(2l^-+1)}(l^-+4+k)(l^-+3-k) < 20; l^-0|k0 >< 21; l^-m^-|km_k > \psi_{l^-}^{m^-}r^{-l^-} = 0, \tag{3.7}$$

for every pair (k, m_k) such that $0 \le |m_k| \le k$.

Also $< 20; l^-0|k0 >\neq 0$ only when $2 + l^- + k$ is even and $< 21; l^-m^-|km_k >\neq 0$ when $m_k = m^- + 1$ and $|l^- - 2| \le k \le l^- + 2$. Therefore to get a non-trivial equation we must have

$$m^- = m_k - 1 \quad \text{and} \quad |k-2| \le l^- \le k+2 \quad \text{and} \quad l^- + k \text{ even},$$

where $0 \le |m_k| \le k$. We have $\psi_l^m = 0$, when $l \le 0$ or $l < |m|$, so we can let $k - 2 \le l^- \le k + 2$. Taking into account that $l^- + k$ must be even we have $l^- \in \{k-2, k, k+2\}$. Now (3.7) reduces to

$$\begin{aligned}\sqrt{(2k-3)}(k+1) < 20; k-2\ 0|k0 >< 21; k-2\ m_k-1|km_k > \psi_{k-2}^{m_k-1}r^4 \quad &+ \\ +3\sqrt{(2k+1)}(k+2) < 20; k0|k0 >< 21; k\ m_k-1|km_k > \psi_k^{m_k-1}r^2 \quad &+ \\ +5\sqrt{(2k+5)}(k+3) < 20; k+2\ 0|k0 >< 21; k+2\ m_k-1|km_k > \psi_{k+2}^{m_k-1} \quad &= \quad 0.\end{aligned}$$

If we replace the Clebsch-Gordan coefficients by their values using [AS72, Table 27.9.4], we get

$$\begin{aligned}\frac{k+1}{2k-1}\sqrt{\frac{(k+m_k-1)(k+m_k-2)(k^2-m_k^2)}{2k-3}}\psi_{k-2}^{m_k-1}r^4+ \\ +\frac{3(k+2)(2m_k-1)}{(2k-1)(2k+3)}\sqrt{(k-m_k+1)(k+m_k)(2k+1)}\psi_k^{m_k-1}r^2- \\ -\frac{5(k+3)}{2k+3}\sqrt{\frac{(k-m_k+3)(k-m_k+2)(k-m_k+1)(k+m_k+1)}{2k+5}}\psi_{k+2}^{m_k-1} = 0.\end{aligned} \tag{3.8}$$

This yields the three terms recurrence relation

$$\psi_{k+2}^{m_k-1} = -\frac{3(k+2)(2m_k-1)}{5(k+3)(2k-1)}\sqrt{\frac{(2k+5)(k+m_k)(2k+1)}{(k-m_k+3)(k-m_k+2)(k+m_k+1)}}\psi_k^{m_k-1}r^2 -$$
$$-\frac{(2k+3)(k+1)}{5(k+3)(2k-1)}\sqrt{\frac{(2k+5)(k+m_k-1)(k+m_k-2)(k^2-m_k^2)}{(k-m_k+3)(k-m_k+2)(k-m_k+1)(k+m_k+1)(2k-3)}}\psi_{k-2}^{m_k-1}r^4. \tag{3.9}$$

One sphere: convergence

Fix $r = R$ and then define p, q by rewriting (3.9) as

$$\psi_{k+2}^{m_k-1} = p(k, m_k)\psi_k^{m_k-1}R^2 + q(k, m_k)\psi_{k-2}^{m_k-1}R^4. \tag{3.10}$$

We want to choose $\{\psi_l^m\}$ that fulfill (3.10) and such that

$$\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\psi_l^m r^{-(l+1)}Y_l^m(\hat{\mathbf{r}})$$

converges. Let $\alpha_l^m(\mathbf{r}) = \psi_l^m r^{-(l+1)}Y_l^m(\hat{\mathbf{r}})$, then it is clearly sufficient to show that $\sum_{l=N}^{\infty}\sum_{m=-l}^{l}|\alpha_l^m| < \infty$ for some integer N.
If we let $k \to \infty$, then $p(k, m_k) \sim -\frac{3}{5k}$ and $q(k, m_k) = -\frac{1}{5}$, so it follows that there exist an integer N such that for $k \geq N$ we have

$$|\psi_{k+2}^{m_k-1}| \leq \frac{1}{5}R^2|\psi_k^{m_k-1}| + \frac{2}{5}R^4|\psi_{k-2}^{m_k-1}|.$$

If we choose $\{\psi_{|m|}^m, \psi_{|m|+1}^m\}$ such that they are bounded for all m, then ψ_l^m will at least be bounded for finite l. This means that there exists a constant A such that for $\alpha > 0$ we have $|\psi_{N-2}^{m_k-1}| \leq (\alpha R)^{N-2}A$ and $|\psi_N^{m_k-1}| \leq (\alpha R)^N A$, and it follows that

$$|\psi_{k+2}^{m_k-1}| \leq \left(\frac{1}{5\alpha^2} + \frac{2}{5\alpha}\right)(\alpha R)^{k+2}A.$$

Letting $\alpha = \left(\frac{\sqrt{21}+1}{10}\right)^{1/2}$ we get $|\psi_{N+2}^{m_k-1}| \leq (\alpha R)^{k+2}A$. It then follows by induction that $|\psi_l^m| \leq (\alpha R)^l A$ for $l \geq N$. Using $|Y_l^m|^2 \leq \frac{2l+1}{4\pi}$ from Fact 3 we have

$$|\alpha_l^m(\mathbf{r})| \leq \sqrt{\frac{2l+1}{4\pi}}r^{-(l+1)}|\psi_l^m| \leq \sqrt{\frac{2l+1}{4\pi}}r^{-(l+1)}(\alpha R)^l A, \qquad \text{for } l \geq N.$$

Then by the ratio test we have

$$\sum_{l=N}^{\infty}\sum_{m=-l}^{l}|\alpha_l^m(\mathbf{r})| \leq \sum_{l=N}^{\infty}2l\sqrt{\frac{2l+1}{4\pi}}r^{-1}\left(\frac{\alpha R}{r}\right)^l A < \infty,$$

for $r > \alpha R$.

Two spheres: uniqueness

Fix m_k and define the a, b, c by rewriting (3.8) as

$$a(k)\psi_{k-2}r^4 + b(k)\psi_k r^2 + c(k)\psi_{k+2} = 0. \tag{3.11}$$

Here we drop the superscript m_k, since they are fixed. Also fix $k \geq |m_k| + 3$ such that $abc \neq 0$. Assume $\psi_{k-2} \neq 0$, then we can divide by $a(k)\psi_{k-2}$, and (3.11) becomes

$$r^4 + \frac{b(k)\psi_k}{a(k)\psi_{k-2}}r^2 + \frac{c(k)\psi_{k+2}}{a(k)\psi_{k-2}} = 0.$$

We can think of the left hand side of this equations as a real polynomial in r, and rewrite the equation as

$$(r^2 - r_1^2)(r^2 - r_2^2) = r^4 - (r_1^2 + r_2^2)r + r_1^2 r_2^2 = 0,$$

where

$$-(r_1^2 + r_2^2) = \frac{b(k)\psi_k}{a(k)\psi_{k-2}}$$
$$r_1^2 r_2^2 = \frac{c(k)\psi_{k+2}}{a(k)\psi_{k-2}}.$$

This we can again rewrite as

$$\frac{\psi_k}{\psi_{k-2}} = \frac{-a(k)(r_1^2 + r_2^2)}{b(k)} \tag{3.12}$$
$$\frac{\psi_{k+2}}{\psi_{k-2}} = \frac{a(k)r_1^2 r_2^2}{c(k)}. \tag{3.13}$$

Now using (3.12) twice, we have

$$\frac{\psi_{k+2}}{\psi_{k-2}} = \frac{\psi_{k+2}}{\psi_k}\frac{\psi_k}{\psi_{k-2}} = \frac{a(k+2)a(k)(r_1^2 + r_2^2)^2}{b(k+2)b(k)}. \tag{3.14}$$

These equations holds for every k, and if we let k be big, we see from (3.8) that

$$a(k) \sim \frac{k^{3/2}}{2\sqrt{2}} \quad , \quad b(k) \sim \frac{3k^{1/2}}{2\sqrt{2}} \quad , \quad c(k) \sim -\frac{5k^{3/2}}{2\sqrt{2}}.$$

Now if $k \to \infty$, then using (3.13) we get

$$\lim_{k\to\infty} \frac{\psi_{k-2}}{\psi_{k+2}} = \lim_{k\to\infty} \frac{c(k)}{a(k)r_1^2 r_2^2} = -\frac{5}{r_1^2 r_2^2} \neq 0.$$

However (3.14) gives us

$$\lim_{k\to\infty}\frac{\psi_{k-2}}{\psi_{k+2}}=\lim_{k\to\infty}\frac{b(k+2)b(k)}{a(k+2)a(k)(r_1^2+r_2^2)^2}=0.$$

This is a contradiction, so if (3.11) holds for two different r-values we must have $\psi_{k-2}=0$, and (3.11) reduces to

$$\psi_{k+2}=-\frac{b(k)}{c(k)}\psi_k r^2.$$

Then for this to hold for two different r-values we must have $\psi_{k+2}\psi_k=0$, so it follows by an easy induction argument that $\psi_l=0$ for all l, and thus $\mathbf{B}^-=0$.

3.6 Intensity outside a sphere

We have seen several examples of different magnetic fields with the same intensity on a sphere. However, if we know the intensity of the geomagnetic field everywhere outside the earth then it can reconstructed.

Result 5 (Intensity outside a sphere)**.** *A magnetic field is uniquely determined up to sign by its intensity outside a sphere containing its sources.*

Proof. It follows from the discussion at the start of this chapter that it is enough to show that $\mathbf{B}^+\cdot\mathbf{B}^-=0$ everywhere outside a sphere imply that one of the fields is identically zero.

Assume $\mathbf{B}^+,\mathbf{B}^-\neq 0$, then due to (3.1) we can write

$$\phi(\mathbf{r})=\sum_{l=M}^{\infty}\frac{h_l^+(\hat{\mathbf{r}})}{r^{l+1}}$$
$$\psi(\mathbf{r})=\sum_{l=N}^{\infty}\frac{h_l^-(\hat{\mathbf{r}})}{r^{l+1}},$$

where $h_M^+,h_N^-\neq 0$. Using $r\nabla=\nabla_r+\nabla_1$ we have

$$\nabla\phi(\mathbf{r})=\sum_{l=M}^{\infty}[-(l+1)h_l^+\hat{\mathbf{r}}+\nabla_1 h_l^+]r^{-(l+2)}$$
$$\nabla\psi(\mathbf{r})=\sum_{l=N}^{\infty}[-(l+1)h_l^-\hat{\mathbf{r}}+\nabla_1 h_l^-]r^{-(l+2)}.$$

The orthogonality conditions gives

$$\begin{aligned}\nabla\phi\cdot\nabla\psi &= \sum_{l^+=M}^{\infty}\sum_{l^-=N}^{\infty}[(l^++1)(l^-+1)h_{l^+}^+h_{l^-}^-+\nabla_1 h_{l^+}^+\cdot\nabla_1 h_{l^-}^-]r^{-(l^++l^-+4)}\\ &= [(M+1)(N+1)h_M^+h_N^-+\nabla_1 h_M^+\cdot\nabla_1 h_N^-]r^{-(M+N+4)}+\mathcal{O}(r^{-(M+N+5)})=0.\end{aligned}$$

Now by multiplying the above equation with r^{M+N+4} and let $r \to \infty$ we see that

$$(M+1)(N+1)h_M^+ h_N^- + \nabla_1 h_M^+ \cdot \nabla_1 h_N^- = 0.$$

It then follow that

$$\left(\nabla \frac{h_M^+}{r^{M+1}}\right) \cdot \left(\nabla \frac{h_N^-}{r^{N+1}}\right) = [(M+1)(N+1)h_M^+ h_N^- + \nabla_1 h_M^+ \cdot \nabla_1 h_N^-] r^{M+N+4} = 0.$$

Then by Result 4 er have $h_M^+ = 0$ or $h_N^- = 0$. This contradicts our assumption and it follows that at least one of $\mathbf{B}^+$ or $\mathbf{B}^-$ is identically zero. □

Chapter 4

Reconstruction from Intensity and Dip Equator

So far we have found a way to reconstruct a magnetic field from its values on a sphere, and we have found several examples of different magnetic fields with the same intensity on a sphere. Good empirical data for the geomagnetic field on a sphere are not available, so our reconstruction formula will not be used in practice. Good empirical data for the intensity of the geomagnetic field on a sphere are available, but these are not sufficient for reconstruction. This is often referred to as *the Backus effect* due to his counter-example in [Bac70]. The urge of reconstructing the geomagnetic field is still there, so we will try to add some new information to good intensity data in order to reach our goal.

When reconstructing the geomagnetic field in practice one gets some large errors. These errors specially arise in the neighborhood of the earth's equator. The errors seem to be too big to be caused only by discretization errors, so one suspects that the Backus effect has a major influence here. One reconstruction technique often used in order to avoid these errors is to add a small set of vectorial data near the equator. Then even low-quality vectorial data together with good intensity data greatly reduces the Backus effect.

What is so special about the geomagnetic field near the equator? One thing is that here the vector field is almost entirely horizontal, meaning that the vertical component is almost zero. We will call the set of points where the radial component of a magnetic field is zero for the *dip equator*. These empirical results motivate us to ask if we can get uniqueness if we in addition to intensity data on a sphere know the dip equator. In this chapter we will first show that the intensity and sign of the normal component of a magnetic field on a sphere uniquely determines the field. Then we will look at how imperfect reconstruction data affect this uniqueness and come up with a stability estimate.

4.1 Uniqueness

We want to show that if two magnetic fields have the same intensity and their normal component have the same sign on a sphere, then they are the same. We start by formally defining the dip equator, D_0^0, of a function u on S_{R_0},

$$D_0^0 = \{\mathbf{x} \in S_{R_0} : \hat{\mathbf{r}} \cdot \nabla u(\mathbf{x}) = \partial_r u(\mathbf{x}) = 0\}.$$

Note that the dip equator divides S_{R_0} into regions where $\partial_r u$ has a determined sign. We will denote these sets by

$$D_0^+ = \{\mathbf{x} \in S_{R_0} : \partial_r u(\mathbf{x}) > 0\}$$
$$D_0^- = \{\mathbf{x} \in S_{R_0} : \partial_r u(\mathbf{x}) < 0\}.$$

We will write D_0 for all these three sets, and call them the *signed dip equator*. Note that $S_{R_0} = D_0^+ \cup D_0^0 \cup D_0^-$. Let $\Omega = \mathbb{R}^3 \setminus \overline{B_{R_0}}$. We will look at potentials $u, v \in \mathcal{H}_0(\Omega) \cap C^1(\overline{\Omega})$ which are connected through the following equivalence relation,

$$u \sim v \quad \Leftrightarrow \quad |\nabla u| = |\nabla v| \text{ and } \operatorname{sign}(\partial_r u) = \operatorname{sign}(\partial_r v) \quad \text{on } S_{R_0}.$$

We want to show that $u \sim v \Rightarrow u \equiv v$ when $u, v \in \mathcal{H}_0(\Omega) \cap C^1(\overline{\Omega})$. First we have

$$|\nabla u|^2 - |\nabla v|^2 = \nabla u \cdot \nabla u - \nabla v \cdot \nabla v = \nabla(u+v) \cdot \nabla(u-v) = 0. \tag{4.1}$$

Define $h = u - v$. Now h is harmonic and it follows from the extremum principle (Fact 1) that it achieves its extremum at $\partial^\infty \Omega$. If h is non-zero it has at least one extremum on S_{R_0} since it vanishes at infinity. Assume h is non-zero and thus has an extremum $\mathbf{x}_0 \in S_{R_0}$. Then we will use the following key lemma from [BJS64, pp. 151-152, Theorem III].

Lemma 9. *Let E be a domain with differentiable boundary. If $f \in \mathcal{H}(E) \cap C^1(\overline{E})$ is not constant and f has extremum that lies on the boundary, then at this point we have $\nabla f = \alpha \hat{\mathbf{n}}$ for some $\alpha \neq 0$, where $\hat{\mathbf{n}}$ denotes the normal to ∂E.*

By Lemma 9, $\hat{\mathbf{r}} \cdot \nabla h(\mathbf{x}_0) = \hat{\mathbf{r}} \cdot \nabla u(\mathbf{x}_0) - \hat{\mathbf{r}} \cdot \nabla v(\mathbf{x}_0) = \alpha \neq 0$, thus $\mathbf{x}_0 \notin D_0^0$. However, from (4.1) we have

$$\nabla h(\mathbf{x}_0) \cdot \nabla(u+v)(\mathbf{x}_0) = \alpha \hat{\mathbf{r}} \cdot \nabla(u+v)(\mathbf{x}_0) = \alpha[\hat{\mathbf{r}} \cdot \nabla u(\mathbf{x}_0) + \hat{\mathbf{r}} \cdot \nabla v(\mathbf{x}_0)] = 0.$$

This is a contradiction since $\mathbf{x}_0 \in D_0^+ \cup D_0^-$ implies that $\hat{\mathbf{r}} \cdot \nabla u(\mathbf{x}_0)$ and $\hat{\mathbf{r}} \cdot \nabla v(\mathbf{x}_0)$ are non-zero and have the same sign.

Result 6. *A magnetic field is uniquely determined by its intensity and the sign of its normal component on a sphere surrounding its sources.*

This result does not immediately solve our problems since B_r is hard to measure for the same reasons as $\mathbf{B}$ is. However, we only need to know where B_r is small and what sign it has to determine the signed dip equator, and it turns out that this can be done with a satisfactory accuracy. In the next section we are going to look at how good reconstruction we are guaranteed from some uncertain reconstruction data.

4.2 Convergence

In practice we only measure the intensity and the signed dip equator of a magnetic field up to some accuracy. Therefore we will assume some uncertainty in our collected data and try to find out how this affects the accuracy of the reconstructed field.
We are concerned with the main geomagnetic field outside the earth and will assume that its strength and signed dip equator will be measured on the sphere S_{R_0} in the atmosphere. Let R_{-1} = earth radii and $\Omega' = \mathbb{R}^3 \setminus \overline{B_{R_{-1}}}$. We will consider potentials of the following class

$$M = \{f \in \mathcal{H}(\Omega') \cap C(\overline{\Omega'}) : |\nabla f| < B_{\max} \text{ on } S_{R_{-1}} \text{ and } |f(\mathbf{x})| < \left(\frac{K}{|\mathbf{x}|}\right)^2 \text{ for } \mathbf{x} \in \Omega'\}.$$

Empirical data will reduce the set of potentials that can correspond to the geomagnetic field, and we will assume that the range of several measurements will cover the values of the true field. Using this we can predict a constant δ, such that if some measurements $|\nabla u|$, $|\nabla v|$ are taken of the same field, we will have

$$\Big||\nabla u(\mathbf{x})| - |\nabla v(\mathbf{x})|\Big| < \delta \quad \text{for every } \mathbf{x} \in S_{R_0}.$$

To express the uncertainty in the signed dip equator we define the following sets for a harmonic function f,

$$\begin{aligned} D^-_\epsilon(f) &= \{\mathbf{x} \in S_{R_0} : \partial_r f(\mathbf{x}) < -\epsilon\} \\ D^0_\epsilon(f) &= \{\mathbf{x} \in S_{R_0} : |\partial_r f(\mathbf{x})| \le \epsilon\} \\ D^+_\epsilon(f) &= \{\mathbf{x} \in S_{R_0} : \partial_r f(\mathbf{x}) > \epsilon\}. \end{aligned}$$

Assume that the sphere is divided into the three sets D^+, D^- and D^0, then we say that these sets form *a signed ϵ-dip equator* for u if

$$D^+ \subset D^+_\epsilon(u) \quad \text{and} \quad D^- \subset D^-_\epsilon(u) \quad \text{and} \quad D^0 \subset D^0_{2\epsilon}(u). \tag{4.2}$$

Then specially for two functions u, v to have the same signed ϵ-dip equator we need

$$D^+ \subset D^+_\epsilon(u) \cap D^+_\epsilon(v) \quad \text{and} \quad D^- \subset D^-_\epsilon(u) \cap D^-_\epsilon(v) \quad \text{and} \quad D^0 \subset D^0_{2\epsilon}(u) \cap D^0_{2\epsilon}(v).$$

Assume $u, v \in M$ have the same signed ϵ-dip equator and $||\nabla u| - |\nabla v|| < \delta$ on S_{R_0} and let $h = u - v$. We want a point estimate for ∇h in S_{R_0}, or equivalently we want to estimate $\|\nabla h\|_{L^\infty(S_{R_0})}$. To do this we will proceed in four steps and we will operate on the different spheres in Figure 4.1.

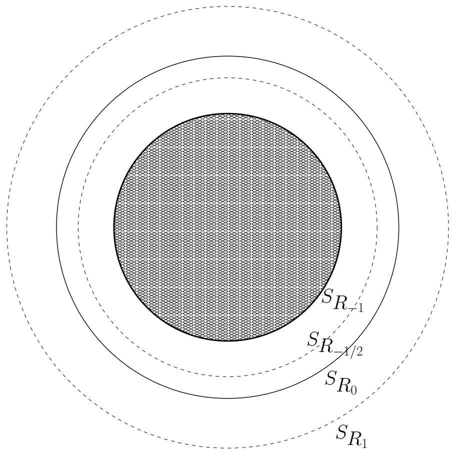

Figure 4.1: Different spheres in the atmosphere.

Here $S_{R_{-1}}$ will be the surface of the earth, and S_{R_0} will be the sphere where our data is collected. These data are usually collected by a low orbiting satellite, so we assume that R_{-1} and R_0 are relative close. The spheres $S_{R_{-1/2}}$ and S_{R_1} are some auxiliary spheres that we will choose in the end.

4.2.1 Step 1: $\|h\|_{L^2(S_{R_0})}$

In this first step we will get an L^2-estimate for ∇h on S_{R_0}. To do this we will follow [KHLM99], but to get the best explicit estimate in Lemma 13 we combine the proof of [HL97, Lemma 1.34] and [KHLM99, Lemma 3], and we also use Harnack's inequality from [AG01].

Auxiliary results

Theorem 10. *(Harnack's inequalities) If $f \in \mathcal{H}_+(B(\mathbf{x}_0, r))$, then*

$$\frac{(r - |\mathbf{x} - \mathbf{x}_0|)r}{(r + \|\mathbf{x} - \mathbf{x}_0\|)^2} f(\mathbf{x}_0) \leq f(\mathbf{x}) \leq \frac{(r + |\mathbf{x} - \mathbf{x}_0|)r}{(r - \|\mathbf{x} - \mathbf{x}_0\|)^2} f(\mathbf{x}_0)$$

for each $\mathbf{x} \in B(\mathbf{x}_0, r)$.

Proof. See [AG01, Theorem 1.4.1] □

Lemma 11. *Suppose $f \in \mathcal{H}(B) \cap C(\overline{B})$. If $f(\mathbf{x}) < f(\mathbf{x}_0)$ for any $\mathbf{x} \in B$ and some $\mathbf{x}_0 \in \partial B$, then*

$$\frac{\partial f}{\partial r}(\mathbf{x}_0) \geq \frac{f(\mathbf{x}_0) - f(0)}{4}.$$

Proof. Define $g(\mathbf{x}) = \frac{1}{|\mathbf{x}|} - 1$, then $g(\mathbf{x}) \in \mathcal{H}_+(B \setminus \{0\})$. For $\epsilon > 0$ we define

$$h_\epsilon(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}_0) + \epsilon g(\mathbf{x}).$$

Clearly $h_\epsilon \in \mathcal{H}(A(R,1)) \cap C(\overline{A(R,1)})$ for any $R \in (0,1)$, thus by the extremum principle, (Fact 1), h_ϵ achieves its maximum on $S_R \cup S$. Clearly $h_\epsilon \leq 0$ on S and $h_\epsilon(\mathbf{x}_0) = 0$, and since $f(\mathbf{x}) < f(\mathbf{x}_0)$ for $|\mathbf{x}| = R$ we may take $\epsilon > 0$ so small such that $h_\epsilon \leq 0$ on S_R. Then it follows that $h_\epsilon \leq 0$ in $A(R,1)$, thus

$$\frac{\partial h_\epsilon}{\partial r}(\mathbf{x}_0) \geq 0 \quad \text{or} \quad \frac{\partial f}{\partial r}(\mathbf{x}_0) \geq -\epsilon \frac{\partial g}{\partial r}(\mathbf{x}_0) = \epsilon > 0.$$

Now we estimate ϵ. We want $\epsilon > 0$ such that

$$\epsilon \leq \frac{f(\mathbf{x}_0) - f(\mathbf{x})}{g(\mathbf{x})} \quad \text{for every } \mathbf{x} \in S_R.$$

Define $w(\mathbf{x}) = f(\mathbf{x}_0) - f(\mathbf{x})$ in B. Then $w \in \mathcal{H}_+(B)$, so by Harnack's inequalities (Theorem 10) there holds

$$\frac{1-R}{(1+R)^2} w(0) < w(\mathbf{x}).$$

Thus it is sufficient that

$$\begin{aligned} \epsilon &\leq \frac{1-R}{(1+R)^2} \frac{w(0)}{g(\mathbf{x})} \qquad \text{for } \mathbf{x} \in S_R \\ &= \frac{R}{(1+R)^2} [f(\mathbf{x}_0) - f(0)]. \end{aligned}$$

This holds for every $R \in (0,1)$. Also $0 < \frac{R}{(1+R)^2} < \frac{1}{4}$ when $R \in (0,1)$, so we can choose $\epsilon = \alpha[f(\mathbf{x}_0) - f(0)]$ for $\alpha \in (0, \frac{1}{4})$. Then

$$\frac{\partial f}{\partial r}(\mathbf{x}_0) \geq \alpha[f(\mathbf{x}_0) - f(\mathbf{x})],$$

and this holds also for $\alpha = \frac{1}{4}$. □

Corollary 12. *Suppose* $f \in \mathcal{H}(B(\mathbf{y},R)) \cap C(\overline{B(\mathbf{y},R)})$. *If* $f(\mathbf{x}) < f(\mathbf{x}_0)$ *for any* $\mathbf{x} \in B(\mathbf{y},R)$ *and some* $\mathbf{x} \in S(\mathbf{y},R)$, *then*

$$\frac{\partial f}{\partial \hat{\mathbf{n}}}(\mathbf{x}_0) \geq \frac{f(\mathbf{x}_0) - f(\mathbf{y})}{4R},$$

where $\hat{\mathbf{n}}$ *denotes the unit normal to* $S(\mathbf{y},R)$.

Proof. The statement follows immediately from Lemma 11 by using that $g(\mathbf{x}) = f(R\mathbf{x} + \mathbf{y}) \in \mathcal{H}(B)$. □

Lemma 13. *Suppose* $f \in \mathcal{H}(\Omega) \cap C^1(\overline{\Omega})$ *and* $|f(\mathbf{x})| < \left(\frac{K}{|\mathbf{x}|}\right)^2$. *If* f *achieves maximum at* $\mathbf{x}_0 \in S_{R_0}$, *then*

$$f(\mathbf{x}_0) < 3(2K|\nabla f(\mathbf{x}_0)|)^{2/3}.$$

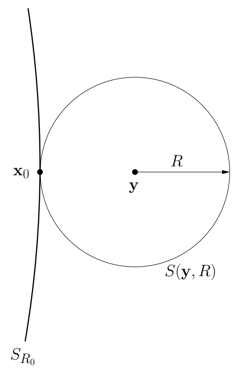

Figure 4.2: Sketch for the proof in lemma 13.

Proof. We let $B(\mathbf{y}, R) \subset \Omega \cup \{\mathbf{x}_0\}$ such that $\mathbf{x}_0 \in S(\mathbf{y}, R)$, as in Figure 4.2.
Now $\mathbf{x}_0$ is a maximum point for f on $B(\mathbf{y}, R)$, so by Lemma 9 the gradient of f at $\mathbf{x}_0$ is normal to $S(\mathbf{y}, R)$. Applying Corollary 12 we get

$$|\nabla f(\mathbf{x}_0)| = \frac{\partial f}{\partial \hat{\mathbf{n}}}(\mathbf{x}_0) \geq \frac{f(\mathbf{x}_0) - f(\mathbf{y})}{4R}. \tag{4.3}$$

Thus

$$\begin{aligned} f(\mathbf{x}_0) &\leq f(\mathbf{y}) + 4R|\nabla f(\mathbf{x}_0)| \\ &< \left(\frac{K}{R_0 + R}\right)^2 + 4R|\nabla f(\mathbf{x}_0)| \\ &< \left(\frac{K}{R}\right)^2 + 4R|\nabla f(\mathbf{x}_0)|. \end{aligned}$$

This holds for every $R > 0$. We want to find the R that minimize the right hand side therefore we differentiate it with respect to R and set this equal to zero. Solving this for R we get

$$R = \left(\frac{K^2}{2|\nabla f(\mathbf{x}_0)|}\right)^{1/3}.$$

This is clearly the minimum since when $R \to 0, \infty$ the expression goes to infinity. Now using this value for R in (4.3) the result follows. □

Estimate

Since h is harmonic in Ω it achieves its extremum on $\partial^\infty\Omega$. If $h \equiv 0$ our final result will hold, so we assume $h \not\equiv 0$. Since h vanish at infinity it will have at least one extremum on S_{R_0}. Let x_0 be a maximum point of $|h|$. If $\mathbf{x}_0 \in D^0_{2\epsilon}$, then by definition $|\nabla h(\mathbf{x}_0)| \leq |\nabla u(\mathbf{x}_0)| + |\nabla v(\mathbf{x}_0)| \leq 4\epsilon$. If $\mathbf{x}_0 \in D^+_\epsilon \cup D^-_\epsilon$, then the situation will be as in Figure 4.3, and we will use elementary geometric considerations to get an estimate for $|\nabla h(\mathbf{x}_0)|$. In the figure O stands for $\mathbf{x}_0$.

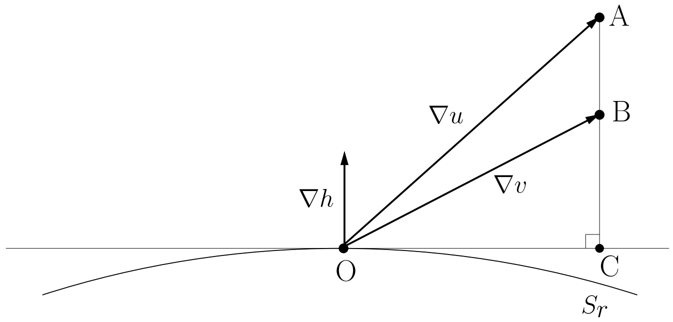

Figure 4.3: Sketch of when $\mathbf{x}_0 \in D_\epsilon^+ \cup D_\epsilon^-$, where $\mathbf{x}_0$=O in the figure.

Now ∇u and ∇v are on the same side of OC in the figure since $\partial_r u(\mathbf{x}_0)$ and $\partial_r v(\mathbf{x}_0)$ are of equal sign. Since $\mathbf{x}_0$ is an extreme point it follows from Lemma 9 that $\nabla h(\mathbf{x}_0)$ is a nonzero radial vector. Because of this we have two right-angled triangles, OCA and OCB. Using the Pythagoras' theorem on these we get

$$\begin{aligned} |OA|^2 &= |OC|^2 + |CA|^2 \\ |OB|^2 &= |OC|^2 + |CB|^2. \end{aligned}$$

Taking the difference of these two equations and use that $|CA| = |CB| + |BA|$, we get

$$(|OA| + |OB|)(|OA| - |OB|) = |OA|^2 - |OB|^2 = |CA|^2 - |CB|^2 = |BA|^2 - 2|CB||BA|.$$

This is a second order polynomial equation in $|BA|$ and its positive root is

$$\begin{aligned} |BA| &= \sqrt{|CB|^2 + (|OA| + |OB|)(|OA| - |OB|)} - |CB| \\ &< \sqrt{|CB|^2 + 2|OA|(|OA| - |OB|)} - |CB| \\ &< \sqrt{\epsilon^2 + 2\delta B_{\max}} - \epsilon. \end{aligned}$$

We used that $\sqrt{x^2 + A} - x$ is decreasing when A is positive, and that $|\nabla u| < B_{\max}$ on $S_{R_{-1}}$ and therefore also on S_{R_0}. Now since $|\nabla h(\mathbf{x}_0)| = |BA|$ we have

$$|\nabla h(\mathbf{x}_0)| < \sqrt{\epsilon^2 + 2\delta B_{\max}} - \epsilon.$$

Let $R(\epsilon, \delta) = \max\{\sqrt{\epsilon^2 + 2\delta B_{\max}} - \epsilon, 4\epsilon\}$ such that $|\nabla h(\mathbf{x}_0)| \leq R(\epsilon, \delta)$.
Next we have $|h(\mathbf{x})| \leq |u(\mathbf{x})| + |v(\mathbf{x})| \leq \left(\frac{\sqrt{2}K}{|\mathbf{x}|}\right)^2$, so we can apply Lemma 13 and get a bound for $|h|$

$$\|h\|_{L^\infty(S_{R_0})} = |h(\mathbf{x}_0)| \leq 6(K|\nabla h(\mathbf{x}_0)|)^{2/3}.$$

Then it follows that

$$\|h\|_{L^2(S_{R_0})} = \left(\int_{S_{R_0}} |h|^2 d\mathbf{r}\right)^{1/2} \leq 2\sqrt{\pi} R_0 \|h\|_{L^\infty(S_{R_0})}.$$

4.2.2 Step 2: $\|\nabla h\|_{L^2(S_{R_1})}$

In this step we will get an L^2-estimate for ∇h on a sphere outside S_{R_0}. Lemma 3 in [KHLM99] connects harmonic functions with their gradients. The lemma is taken from [Mit94, Corollary 5.18] and stated without proof. The lemma is stated for any harmonic function u which tends to zero at infinity and any convex smooth and closed surface Σ. The lemma fails for the simple case when $u = \frac{1}{r}$ and $\Sigma = S$. Also the strict inequality for the relation $\sim$ in the lemma clearly does not hold when $u = 0$. A correct version of this lemma for the special case when Σ is a sphere is stated and proved in this subsection (Lemma 16), and this will be the key point to get the estimate.

Auxiliary results

Proposition 14. *If $0 \le r_1 < R < r_2 \le \infty$ and $f \in \mathcal{H}_0(A(r_1, r_2))$, then*

$$\int_{S_R} |\nabla f|^2 d\mathbf{s} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (2l+1)(l+1)|f_l^m|^2 R^{-(2l+2)},$$

where f_l^m are the Gauss coefficients of f.

Proof. By Lemma 7 we have

$$f(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m r^{-(l+1)} Y_l^m(\hat{\mathbf{r}}). \tag{4.4}$$

Let $\mathbf{F} = -\nabla f$, then $|\nabla f|^2 = |\mathbf{F}|^2 = \mathbf{F} \cdot \overline{\mathbf{F}} = F_r \overline{F}_r + \mathbf{F}_s \cdot \overline{\mathbf{F}}_s$, where $F_r = -r^{-1} \nabla_r f$ is the radial part and $\mathbf{F}_s = -r^{-1} \nabla_1 f$ the spherical part of $\mathbf{F}$. It then follows from (4.4) by direct calculations that

$$\begin{aligned} F_r(\mathbf{r}) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l+1) f_l^m r^{-(l+2)} Y_l^m(\hat{\mathbf{r}}) \\ \mathbf{F}_s(\mathbf{r}) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} -f_l^m r^{-(l+2)} \nabla_1 Y_l^m(\hat{\mathbf{r}}). \end{aligned}$$

Using this we get

$$\int_{S_R} F_r \overline{F}_r d\mathbf{s} = \sum_{l,m} \sum_{l',m'} (l+1)(l'+1) f_l^m \overline{f_{l'}^{m'}} R^{-(l+l'+4)} \int_{S_R} Y_l^m \overline{Y_{l'}^{m'}} d\mathbf{s}.$$

Now $\{Y_l^m\}_{l,m}$ are orthonormal and independent on r so letting $\mathbf{s} = R\omega$ we have

$$\int_{S_R} Y_l^m \overline{Y_{l'}^{m'}} d\mathbf{s} = R^2 \int_S Y_l^m \overline{Y_{l'}^{m'}} d\omega = R^2 \delta_{ll'} \delta_{mm'}.$$

Thus

$$\int_{S_R} F_r \overline{F}_r d\mathbf{s} = \sum_{l,m} (l+1)^2 |f_l^m|^2 R^{-(2l+2)}. \tag{4.5}$$

We have from equation (1.2) that $< \nabla_1 \overline{Y_{l'}^{m'}} \cdot \nabla_1 Y_l^m >= l(l+1)\delta_{ll'}\delta_{mm'}$. So with a similar change of variable as above will give

$$\int_{S_R} \nabla_1 Y_l^m \cdot \nabla_1 \overline{Y_{l'}^{m'}} d\mathbf{s} = R^2 l(l+1)\delta_{ll'}\delta_{mm'}.$$

Thus

$$\int_{S_R} \mathbf{F}_s \cdot \overline{\mathbf{F}}_s d\mathbf{s} = \sum_{l,m} \sum_{l',m'} f_l^m \overline{f_{l'}^{m'}} R^{-(l+l'+4)} \int_{S_R} \nabla_1 Y_l^m \cdot \nabla_1 \overline{Y_{l'}^{m'}} d\mathbf{s} = \sum_{l,m} l(l+1)|f_l^m|^2 R^{-(2l+2)}. \tag{4.6}$$

Now the results follows by combining (4.5) and (4.6). □

Corollary 15. *If* $0 \le r_1 < r < R < r_2 \le \infty$ *and* $f \in \mathcal{H}_0(A(r_1, r_2))$*, then*

$$\|\nabla f\|^2_{L^2(S_R)} = \int_{S_R} |\nabla f|^2 d\mathbf{s} \le \int_{S_r} |\nabla f|^2 d\mathbf{s} = \|\nabla f\|^2_{L^2(S_r)}.$$

Proof. This follows immediately from Proposition 14. □

Lemma 16. *If* $f \in \mathcal{H}_0(A(R,\infty)) \cap C(\overline{A}(R,\infty))$ *for* $R > 0$ *and* $f(\mathbf{x}) = o\left(\frac{1}{|\mathbf{x}|}\right)$ *when* $|\mathbf{x}| \to \infty$*, then*

$$\int_{\mathbb{R}^3 \setminus B_R} |\nabla f|^2 (r-R) d\mathbf{s} \le \int_{S_R} |f|^2 d\mathbf{s}. \tag{4.7}$$

Proof. By Lemma 7 we have

$$f(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m r^{-(l+1)} Y_l^m(\hat{\mathbf{r}}). \tag{4.8}$$

Since $f = o\left(\frac{1}{|\mathbf{x}|}\right)$ we have $f_0^0 = 0$. Also $|f|^2 = f\overline{f}$, so using (4.8) and the orthonormality of $\{Y_l^m\}_{l,m}$ it follows immediately that

$$\int_{S_R} |f|^2 d\hat{\mathbf{s}} = \sum_{l,m} \sum_{l',m'} f_l^m \overline{f_{l'}^{m'}} R^{-(l+l'+2)} \int_{S_R} Y_l^m \overline{Y_{l'}^{m'}} d\mathbf{s} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} |f_l^m|^2 R^{-2l}. \tag{4.9}$$

Now for the left hand side of (4.7) we have

$$\int_{\mathbb{R}^3 \setminus B_R} |\nabla f|^2 (r-R) d\mathbf{s} = \int_R^{\infty} (r-R) \int_{S_r} |\nabla f(\mathbf{r})|^2 d\mathbf{s} dr.$$

We have $f_0^0 = 0$ so it follows from Proposition 14 that

$$\int_{\mathbb{R}^3\setminus B_R} |\nabla f|^2 (r-R) d\mathbf{s} = \sum_{l,m} (l+1)(2l+1)|f_l^m|^2 \int_R^\infty r^{-(2l+1)} - Rr^{-(2l+2)} dr \tag{4.10}$$

$$= \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{l+1}{2l} |f_l^m|^2 R^{-2l}. \tag{4.11}$$

Now the result follows by comparing (4.9) and (4.11). □

Estimate

By Corollary 15 we have

$$\begin{aligned} \int_{A(R_0,R_1)} (r-R_0)|\nabla h|^2 d\mathbf{r} &= \int_{R_0}^{R_1} r - R_0 \int_{S_r} |\nabla h|^2 d\hat{\mathbf{s}} dr \\ &\geq \|\nabla h\|^2_{L^2(S_{R_1})} \int_{R_0}^{R_1} r - R_0 dr \\ &= \frac{1}{2}(R_1 - R_0)^2 \|\nabla h\|^2_{L^2(S_{R_1})}. \end{aligned}$$

Then from Lemma 16 it follows that

$$\int_{A(R_0,R_1)} (r-R_0)|\nabla h|^2 d\mathbf{r} \leq \int_{A(R_0,\infty)} (r-R_0)|\nabla h|^2 d\mathbf{r} \leq \int_{S_{R_0}} |h|^2 d\mathbf{s}.$$

Thus we have the following estimate

$$\|\nabla h\|_{L^2(S_{R_1})} \leq \frac{\sqrt{2}}{R_1 - R_0} \|h\|_{L^2(S_{R_0})}.$$

This gives us an L^2-estimate for ∇h on any bigger sphere then S_{R_0}, but we see that when we get close to S_{R_0} this bound grows to infinity. Thus we have no control of what happens on S_{R_0}. We will in the next two steps use that h is bounded on $S_{R_{-1}}$ and that it vanishes at infinity in order to get a uniform bound for ∇h on S_{R_0}.

4.2.3 Step 3: $\|\nabla h\|_{L^2(S_{R_{-1/2}})}$

In this step we find an estimate for ∇h on a smaller sphere than the one where we collected the data. The fact we use to prove this follows from the proof of Lemma 2.1 in [KM94] by doing some minor changes.

Auxiliary results

Fact 17. *If* $f \in \mathcal{H}_0(\mathbb{R}^3 \setminus \overline{B_R}) \cap C(\mathbb{R}^3 \setminus B_R)$ *and* $R \geq r_1 < r_2 < r_3 < \infty$ *is such that* $r_2 = r_1^\alpha r_3^{1-\alpha}$, *then*

$$\|\nabla f\|_{L^2(S_{r_2})} \leq \|\nabla f\|^\alpha_{L^2(S_{r_1})} \|\nabla f\|^{1-\alpha}_{L^2(S_{r_3})}.$$

Proof. It follows from the proof of Lemma 2.1 in [KM94]. □

Estimate

By Fact 17

$$\|\nabla h\|_{L^2(S_{R_{-1/2}})} \leq \|\nabla h\|^\alpha_{L^2(S_{R_{-1}})} \|\nabla h\|^{1-\alpha}_{L^2(S_{R_1})}.$$

Since $\nabla h < B_{\max}$ on S_{-1} we have

$$\|\nabla h\|_{L^2(R_{-1})} = \left(\int_{S_{R_{-1}}} |\nabla h|^2 d\mathbf{s} \right)^{1/2} \leq 2\sqrt{\pi} B_{\max} R_{-1}.$$

Thus

$$\|\nabla h\|_{L^2(S_{R_{-1/2}})} \leq (2\sqrt{\pi} B_{\max} R_{-1})^\alpha \|\nabla h\|^{1-\alpha}_{L^2(S_{R_1})} \qquad \text{where } R_{-1/2} = R_{-1}^\alpha R_1^{1-\alpha}.$$

4.2.4 Step 4: $\|\nabla h\|_{L^\infty(S_{R_0})}$

In this last step we use Poisson's integral formula and the Kelvin transform in order to get our uniform estimate of ∇h on the sphere where we collected the data.

Auxiliary results

Definition 18. *The Poisson kernel of* $B(\tilde{\mathbf{x}}, R)$ *is the function*

$$P(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi R} \frac{R^2 - |\mathbf{x} - \tilde{\mathbf{x}}|^2}{|\mathbf{x} - \mathbf{y}|^3} \quad (\mathbf{y} \in S(\tilde{\mathbf{x}}, R); \mathbf{x} \in \mathbf{R}^3 \setminus \{\mathbf{y}\}).$$

Theorem 19 (Poisson's integral)**.** *If* $h \in \mathcal{H}(B_R) \cap C(\overline{B}_R)$ *then*

$$h(\mathbf{x}) = \int_{S_R} h(\mathbf{y}) P(\mathbf{x}, \mathbf{y}) d\mathbf{y}$$

Proof. See [AG01, Corollary 1.3.4]. □

Definition 20 (Kelvin transform)**.** *If* $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{y}\}$, *then the* inverse *of* $\mathbf{x}$ *with respect to* $S(\mathbf{y}, R)$ *is the point*

$$\mathbf{x}^* = \frac{R^2}{|\mathbf{x} - \mathbf{y}|^2} (\mathbf{x} - \mathbf{y}) + \mathbf{y}. \tag{4.12}$$

And the inverse of a set E with respect to $S(\mathbf{y}, R)$ is the set $E^* = \{\mathbf{x}^* : \mathbf{x} \in E \setminus \{\mathbf{y}\}\}$. If f is a function defined on at least E, then we define f^* on E^* by

$$f^*(\mathbf{x}) = \frac{R}{|\mathbf{x} - \mathbf{y}|} f(\mathbf{x}^*). \tag{4.13}$$

The mapping $f \to f^*$ is called the Kelvin transform *(with respect to $S(\mathbf{y}, R)$).*

Theorem 21. *If $f \in C^2(E)$ and f^* is the image of f under the Kelvin transform with respect to S, then*

$$\Delta f^*(\mathbf{x}) = |\mathbf{x}|^{-5} \Delta f(\mathbf{x}^*) \quad \mathbf{x} \in E^*. \tag{4.14}$$

Proof. The result follows by direct calculation (see [AG01, Theorem 1.6.3]). □

Corollary 22. *If $h \in \mathcal{H}(E)$ and h^* is the image of h under the Kelvin transform with respect to $S(\mathbf{y}, a)$, then $h^* \in \mathcal{H}(E^*)$*

Proof. This follows from Theorem 21 by using a dilation and an isometry. □

Theorem 23. *If h is harmonic in $B(\mathbf{y}, R) \setminus \{\mathbf{y}\}$ and $h(\mathbf{x})|\mathbf{x} - \mathbf{y}| \to 0$ as $\mathbf{x} \to \mathbf{y}$, then h has a harmonic continuation to $B(\mathbf{y}, R)$.*

Proof. See [AG01, Theorem 1.3.7]). □

Estimate

We will estimate ∇h point-wise on S_{R_0} by using the L^2-estimate on $S_{R_{-1/2}}$. To do this we are going to apply the Poisson integral formula, so we project our field ∇h onto a vector $\mathbf{p}$. We write $g = \nabla h \cdot \mathbf{p}$. Then g is obviously harmonic since

$$\Delta g = \Delta(\nabla h \cdot \mathbf{p}) = \nabla(\Delta h) \cdot \mathbf{p}.$$

Let g^* be the Kelvin transform of g with respect to $S_{R_{-1/2}}$. Then by Corollary 22 we have $g^* \in \mathcal{H}(B_{R_{-1/2}} \setminus \{0\})$. Also $g \in C(\mathbb{R}^3 \setminus B_{R_{-1/2}})$ so we have $g \in C(\overline{B}_{R_{-1/2}} \setminus \{0\})$ since $g = g^*$ on $S_{R-1/2}$. Now ∇h vanishes at infinity, this means that $g(\mathbf{x}) = o(1)$ when $|\mathbf{x}| \to \infty$, thus $g^*(\mathbf{x}^*) = o(1)$ when $|\mathbf{x}^*| \to 0$. Therefore it follows from Theorem 23 that g^* has a harmonic continuation to $B_{R_{-1/2}}$, we denote this by g_c^*. Now we apply Poisson's integral formula,

$$g_c^*(\mathbf{x}) = \int_{S_{R_{-1/2}}} g_c^*(\mathbf{y}) P(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

Now $g_c^*(\mathbf{x}) = g^*(\mathbf{x}) = \frac{R_{-1/2}}{|\mathbf{x}|} g(\mathbf{x}^*)$ for $\mathbf{x} \in B_{R_{-1/2}} \setminus \{0\}$. Using $g_c^* = g^* = g$ on $S_{R_{-1/2}}$ and $(\mathbf{x}^*)^* = \mathbf{x}$ we get

$$g(\mathbf{x}) = \frac{|\mathbf{x}^*|}{R_{-1/2}} \int_{S_{R_{-1/2}}} g(\mathbf{y}) P(\mathbf{x}^*, \mathbf{y}) d\mathbf{y}, \qquad \text{for } \mathbf{x} \in \mathbb{R}^3 \setminus \overline{B}_{R_{-1/2}}.$$

By Cauchy-Schwartz's inequality we have

$$|g(\mathbf{x})| = \frac{|\mathbf{x}^*|}{R_{-1/2}} \left| \int_{S_{R_{-1/2}}} g(\mathbf{y}) P(\mathbf{x}^*, \mathbf{y}) d\mathbf{y} \right| \leq \frac{|\mathbf{x}^*|}{R_{-1/2}} \|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})} \|g\|_{L^2(S_{R_{-1/2}})}.$$

This holds specially for all $\mathbf{x} \in S_{R_0}$, thus

$$\|g\|_{L^\infty(S_{R_0})} \leq \frac{|\mathbf{x}^*|}{R_{-1/2}} \|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})} \|g\|_{L^2(S_{R_{-1/2}})}.$$

Since $g = \nabla h \cdot \mathbf{p}$ where $\mathbf{p}$ was an arbitrary vector, we have the following

$$\begin{aligned} \|\nabla h\|_{L^\infty(S_{R_0})} &= \max_{\mathbf{x} \in S_{R_0}} |\nabla h(\mathbf{x})| \\ &= \max_{\mathbf{x} \in S_{R_0}} (|\nabla h \cdot \hat{\mathbf{x}}|^2 + |\nabla h \cdot \hat{\mathbf{y}}|^2 + |\nabla h \cdot \hat{\mathbf{z}}|^2)^{1/2} \\ &\leq (\max_{\mathbf{x} \in S_{R_0}} |\nabla h \cdot \hat{\mathbf{x}}|^2 + \max_{\mathbf{x} \in S_{R_0}} |\nabla h \cdot \hat{\mathbf{y}}|^2 + \max_{\mathbf{x} \in S_{R_0}} |\nabla h \cdot \hat{\mathbf{z}}|^2)^{1/2} \\ &\leq \frac{|\mathbf{x}^*|}{R_{-1/2}} \|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})} \left(\int_{S(R_{-1/2})} |\partial_x h|^2 + |\partial_y h|^2 + |\partial_z h|^2 d\mathbf{s} \right)^{1/2} \\ &\leq \frac{|\mathbf{x}^*|}{R_{-1/2}} \|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})} \|\nabla h\|_{L^2(S_{R_{-1/2}})}. \end{aligned}$$

Now we calculate $\|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})}$,

$$\|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})} = \left(\int_{S_{R_{-1/2}}} |P(\mathbf{x}^*, \mathbf{y})|^2 d\mathbf{y} \right)^{1/2}.$$

We have $P(\mathbf{x}^*, \mathbf{y}) = \frac{1}{4\pi R_{-1/2}} \frac{R_{-1/2}^2 - |\mathbf{x}^*|^2}{|\mathbf{x}^* - \mathbf{y}|^3}$, $\mathbf{x}^* = (\frac{R_{-1/2}}{R_0})^2 \mathbf{x}$ and $|\mathbf{x}| = R_0$, so setting $a = \frac{R_{-1/2}}{R_0}$ we get

$$\|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})} = \frac{R_{-1/2}}{4\pi} (1 - a^2) \left(\int_{S_{R_{-1/2}}} \frac{d\mathbf{y}}{|a^2 \mathbf{x} - \mathbf{y}|^6} \right)^{1/2}.$$

We normalize the variables by setting $\mathbf{u} = \mathbf{x}/R_0$ and $\mathbf{v} = \mathbf{y}/R_{-1/2}$, then

$$\|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})} = \frac{1}{4\pi R_{-1/2}} (1 - a^2) \left(\int_S \frac{d\mathbf{v}}{|a\mathbf{u} - \mathbf{v}|^6} \right)^{1/2}.$$

To calculate the integral we parameterize the unit sphere by

$$\mathbf{r}(\xi, \theta) = (\sqrt{1 - \xi^2} \cos\theta, \sqrt{1 - \xi^2} \sin\theta, \xi).$$

If we let the z-axis go along u we can think of this as slicing the sphere up along this axis. Then first we integrate the slices, and then we sum them up along the z-axis. This is illustrated in Figure 4.4 where ξ is the integration variable along the z-axis.

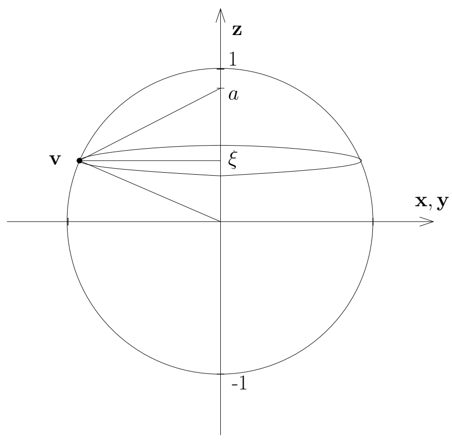

Figure 4.4: Sketch of how we integrate.

Now our surface element is $d\mathbf{v} = |\partial_\theta \mathbf{r} \times \partial_\xi \mathbf{r}| d\theta d\xi$, see [Kre99, p. 501] for details. Elementary calculations shows that $|\partial_\theta \mathbf{r} \times \partial_\xi \mathbf{r}| = 1$. Also we see from the figure that if we use the Pythagoras' theorem twice we have

$$|a\mathbf{u} - \mathbf{v}|^2 = (\xi - a)^2 + 1 - \xi^2 = 1 + a^2 - 2a\xi.$$

Then

$$\int_S \frac{d\mathbf{v}}{|a\mathbf{u} - \mathbf{v}|^6} = \int_{-1}^{1} \int_0^{2\pi} \frac{|\partial_\theta \mathbf{r} \times \partial_\xi \mathbf{r}| d\theta d\xi}{(1 + a^2 - 2a\xi)^3} = 2\pi \int_{-1}^{1} \frac{d\xi}{(1 + a^2 - 2a\xi)^3}.$$

Substituting $w = 1 + a^2 - 2a\xi$ we get $dw = -2ad\xi$, so

$$2\pi \int_{-1}^{1} \frac{d\xi}{(1 + a^2 - 2a\xi)^3} = \frac{-\pi}{a} \int_{(1+a)^2}^{(1-a)^2} \frac{dw}{w^3} = \frac{\pi}{2a} \left(\frac{1}{(1-a)^4} - \frac{1}{(1+a)^4} \right) = 4\pi \frac{1 + a^2}{(1 - a^2)^4}.$$

Now

$$\begin{aligned} \|P(\mathbf{x}^*, \mathbf{y})\|_{L^2(S_{R_{-1/2}})} &= \frac{1}{4\pi R_{-1/2}} (1 - a^2) \sqrt{4\pi \frac{1 + a^2}{(1 - a^2)^4}} \\ &= \frac{\sqrt{1 + a^2}}{\sqrt{4\pi} R_{-1/2} (1 - a^2)}. \end{aligned}$$

Then we have the following estimate

$$\begin{aligned}\|\nabla h\|_{L^\infty(S_{R_0})} &\leq \frac{\sqrt{1+a^2}}{4\pi R_0(1-a^2)}\|\nabla h\|_{L^2(S_{R_{-1/2}})} \\ &= \frac{\sqrt{R_0^2+R_{-1/2}^2}}{4\pi(R_0^2-R_{-1/2}^2)}\|\nabla h\|_{L^2(S_{R_{-1/2}})}.\end{aligned}$$

4.2.5 Result

Collecting all the estimates we have

$$\begin{array}{lll}\text{R1:} & \|\nabla h\|_{L^\infty(S_{R_0})} & \leq \frac{\sqrt{R_0^2+R_{-1/2}^2}}{4\pi(R_0^2-R_{-1/2}^2)}\|\nabla h\|_{L^2(S_{R_{-1/2}})} \\ \text{R2:} & \|\nabla h\|_{L^2(S_{R_{-1/2}})} & \leq (2\sqrt{\pi}B_{\max}R_{-1})^\alpha\|\nabla h\|^{1-\alpha}_{L^2(S_{R_1})} \\ \text{R3:} & \|\nabla h\|_{L^2(S_{R_1})} & \leq \frac{\sqrt{2}}{R_1-R_0}\|h\|_{L^2(S_{R_0})} \\ \text{R4:} & \|h\|_{L^2(S_{R_0})} & \leq 2\sqrt{\pi}R_0\|h\|_{L^\infty(S_{R_0})} \\ \text{R5:} & \|h\|_{L^\infty(S_{R_0})} & \leq 6(K|\nabla h(\mathbf{x}_0)|)^{2/3} \\ \text{R6:} & |\nabla h(\mathbf{x}_0)| & < \max\{\sqrt{\epsilon^2+2\delta B_{\max}}-\epsilon, 4\epsilon\} = R(\epsilon,\delta)\end{array}$$

We want a uniform estimate for ∇h on S_{R_0} and using the results above we have

$$\begin{aligned}\|\nabla h\|_{L^\infty(S_{R_0})} &\leq \frac{\sqrt{R_0^2+R_{-1/2}^2}}{4\pi(R_0^2-R_{-1/2}^2)}(2\sqrt{\pi}B_{\max}R_{-1})^\alpha\left(\frac{\sqrt{2}}{R_1-R_0}2\sqrt{\pi}R_0 6(KR(\epsilon,\delta))^{2/3}\right)^{1-\alpha} \\ &= 6\sqrt{2}K^{2/3}\left(\frac{B_{\max}}{12K^{2/3}}\right)^\alpha\sqrt{1+\left(\frac{R_{-1/2}}{R_0}\right)^2}\frac{R_{-1}^\alpha}{R_0}\left(1-\left(\frac{R_{-1/2}}{R_0}\right)^2\right)^{-1}\left(\frac{R(\epsilon,\delta)}{\frac{R_1}{R_0}-1}\right)^{1-\alpha}\end{aligned}$$

We have the freedom to choose $R_{-1/2}, R_1$ in order to minimize the expression above. We assume that $B_{\max}, 12K^{2/3} > 1$ and we have $R_{-1}, R_{-1/2} < R_0$, thus

$$\|\nabla h\|_{L^\infty(S_{R_0})} \leq 12K^{2/3}B_{\max}\left(1-\frac{R_{-1/2}}{R_0}\right)^{-1}\left(\frac{R(\epsilon,\delta)^{2/3}}{\frac{R_1}{R_0}-1}\right)^{1-\alpha}.$$

Now we want to choose $R_{-1/2}, R_1$ such that

$$\left(1-\frac{R_{-1/2}}{R_0}\right)^{-1}\left(\frac{R(\epsilon,\delta)^{2/3}}{\frac{R_1}{R_0}-1}\right)^{1-\alpha} \tag{4.15}$$

goes to zero in a relatively fast way when $R(\epsilon,\delta) \to 0$. We will set $R(\epsilon,\delta) = \varepsilon$, then we will have

$$\frac{R(\epsilon,\delta)^{2/3}}{\frac{R_1}{R_0}-1} = \varepsilon^\gamma \tag{4.16}$$

for some γ depending on the choice of R_1. Note that if we choose R_1 such that $\gamma \leq 0$, thus $\frac{R(\epsilon,\delta)^{2/3}}{\frac{R_1}{R_0}-1} \geq 1$, then R_{-1} will be the optimal choice for $R_{-1/2}$, but then $\alpha = 0$, so the expression in (4.15) will not go to zero when $R(\epsilon, \delta) \to 0$. Therefore we will choose R_1 such that $\gamma > 0$. Now (4.16) gives

$$\frac{R_1}{R_0} = 1 + \varepsilon^{2/3-\gamma}.$$

We set $y = \frac{2}{3} - \gamma$. Then using the connection between $R_{-1/2}$ and R_1 we get

$$R_{-1/2} = R_{-1}^{\alpha} R_1^{1-\alpha} = R_{-1}^{\alpha}(1+\varepsilon^y)^{1-\alpha} R_0^{1-\alpha}.$$

Thus

$$\frac{R_{-1/2}}{R_0} = \left(\frac{R_{-1}}{R_0}\right)^{\alpha} (1+\varepsilon^y)^{1-\alpha}.$$

Set $\frac{R_{-1}}{R_0} = 1 - x$, then

$$(1-x)^{\alpha}(1+\varepsilon^y)^{1-\alpha} = (1+\varepsilon^y)\left(\frac{1-x}{1+\varepsilon^y}\right)^{\alpha} = (1+\varepsilon^y)\left(1 - \frac{x+\varepsilon^y}{1+\varepsilon^y}\right)^{\alpha}.$$

For $0 \leq \alpha, z \leq 1$ we have that

$$(1-z)^{\alpha} \leq 1 - \alpha z. \tag{4.17}$$

This clearly holds for $\alpha = 0, 1$, and for $0 < \alpha < 1$ we have equality when $z = 0$. So by differentiation we see that the left hand side decays faster then the right. Clearly $x < 1$ and when $y > 0$ we may assume $x + \varepsilon^y < 1$, so using (4.17) we get

$$\begin{aligned} \frac{R_{-1/2}}{R_0} &\leq (1+\varepsilon^y)\left(1 - \alpha\frac{x+\varepsilon^y}{1+\varepsilon^y}\right) \\ &= 1 + \varepsilon^y - \alpha(x+\varepsilon^y). \end{aligned}$$

Then we have

$$1 - \frac{R_{-1/2}}{R_0} \geq \alpha(x+\varepsilon^y) - \varepsilon^y \geq 0.$$

So we need $\alpha > \frac{\varepsilon^y}{x+\varepsilon^y}$. We choose $\alpha = 2\frac{\varepsilon^y}{x+\varepsilon^y}$. Then

$$(1 - \frac{R_{-1/2}}{R_0})^{-1}\varepsilon^{\gamma(1-\alpha)} \leq [\alpha(x+\varepsilon^y) - \varepsilon^y]^{-1}\varepsilon^{\gamma(1-\alpha)} = \varepsilon^{\gamma(1-\alpha)-y}.$$

Now we want to choose $\gamma \in (0, \frac{2}{3})$ such that

$$f(\gamma) = \gamma(1-\alpha) - y = 2\gamma\left(1 - \frac{\varepsilon^y}{x+\varepsilon^y}\right) - \frac{2}{3} > 0.$$

Thus we need

$$1 - \frac{\varepsilon^y}{x+\varepsilon^y} = \frac{x}{x+\varepsilon^y} > \frac{1}{3\gamma}.$$

$$\frac{1}{3}\left(\frac{x+\varepsilon^y}{x}\right) = \frac{1}{3}(1+\varepsilon^y/x) < \gamma.$$

So at least we need $\gamma > 1/3$. For $\gamma = \frac{1}{2}$ we get

$$f(1/2) = \frac{1}{3} - \frac{\varepsilon^{1/6}}{x+\varepsilon^{1/6}} = \frac{1}{3} - \frac{1}{1+x/\varepsilon^{1/6}}.$$

We see that in order for $f(1/2)$ to be positive we need $\varepsilon < (x/2)^6$, this also makes $\alpha < 1$. If we for example assume $\varepsilon < (x/5)^6$ we get a convergence rate of $\varepsilon^{\frac{1}{6}}$. We sum this up in a result.

The Stability Result. *Let $u, v \in M$ have the same signed ϵ-dip equator and*

$$|\nabla u(x)| - |\nabla v(x)|| < \delta \quad \textit{for every } x \in S_{R_0}.$$

If $\varepsilon = \max\{\sqrt{\epsilon^2 + 2\delta B_{\max}} - \epsilon, 4\epsilon\} < \left(\frac{1-R_{-1}/R_0}{5}\right)^6$, *then*

$$\|\nabla u - \nabla v\|_{L^\infty(S_{R_0})} \leq 12K^{2/3}B_{\max}\varepsilon^{1/6}.$$

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