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Correction Classes and Elliptic Cohomology

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Abstract. After introducing the general machinery needed, we consider Landweber exact cohomology theories and prove a result by H. Miller [**Mil89**]. This enables us to manufacture multiplicative natural transformations of said theories. In particular, we use this to make the elliptic character, which relates elliptic cohomology and real K-theory. We compute several correction classes for various Riemann–Roch formulas, focusing on the ones obtained from the elliptic character.

Foreword

This thesis represents the work of my last year as a student for the degree of Master of Science in Mathematics at NTNU. It was written under supervision of Idar Hansen in the field of algebraic topology.

There are many people that deserve my thanks. I am very grateful towards Idar Hansen, for his encouragement, helpfulness and discussions during the past two years. Thanks to Haynes Miller, for sharing his knowledge and insight when I visited MIT for five weeks last semester.

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Introduction

In the eighties topologists constructed a cohomology theory called elliptic cohomology [**Fra92, LRS95**]. This theory was constructed from complex cobordism by means of a criterion called the Landweber exact functor theorem [**Lan76**], and as such, was not geometrically defined. Coming up with a satisfactory geometric interpretation of this cohomology theory was, and still is, an unresolved problem, and this is a major disadvantage in understanding the theory. However, elliptic cohomology is still very useful. It does a better job than both K-theory and singular cohomology in telling spaces apart, and can therefore be used to attack problems out of reach of the more classical cohomology theories.

Work by several people (see for instance [**Zag88**]) suggested that the coefficient ring of elliptic cohomology should be interpreted as a ring of modular functions. This enabled Haynes Miller [**Mil89**], via a result of his (see Theorem 2.6.1 below), to produce a multiplicative natural transformation of cohomology theories relating elliptic cohomology and real K-theory. Applying the topological Riemann–Roch theorem he recovered formulas that Ed Witten obtained by physical considerations in [**Wit88**].

Purpose

The goal of this thesis was to study the construction of elliptic cohomology and H. Miller's paper [**Mil89**], and in particular account for Miller's construction of the elliptic character. Another point was to apply this construction in different settings and thereafter compute the Riemann–Roch correction classes that arise.

Overview of thesis

First I will say a few words about the general style of the thesis. Throughout the text I have made considerable effort trying to make things come in a natural order. Chapter 1 and the generalities about formal group laws are meant to give a solid foundation for the later parts. For the parts that does not make up the core of the thesis, I have tried to say just enough to give the reader an idea of what is going on. Most of the material I have had to study and that has been relevant for the results has been included in some form.

Chapter 1 will be devoted to introducing terminology and notation. We will briefly discuss orientations of vector bundles and manifolds, and state a topological version of the Riemann–Roch theorem which will play an important role later. We introduce real and complex K-theory, as well as the theories obtained by localizing away from the prime 2. We then devote some time introducing operations in K-theory, and we will use the Adams operations to give a criterion for telling when an element lies in the image of the complexification $c\colon KO[\frac{1}{2}]^*(-) \to K[\frac{1}{2}]^*(-)$. Thereafter spectra will be introduced, and we take a rough and ready approach, avoiding most technicalities. We mention how spectra give rise to (co)homology theories, and vice versa. We end this chapter by giving an example of a very important spectrum which gives rise to complex bordism.

Chapter 2 consists of material related to formal group laws. After having made the preliminary definitions, we turn our attention to the complex oriented cohomology theories, because they come equipped with formal group laws. Next follows a discussion about universal formal group laws. Manufacturing a ring with a universal formal group law is not hard, but determining the structure of the ring is complicated. This is Lazard's theorem (Theorem 2.4.2 below). Quillen's theorem now states that the formal group law attached to complex cobordism is in fact the universal formal group law. We state Landweber's exact functor theorem (Theorem 2.5.2) which gives a criterion enabling us to produce new homology theories from complex bordism, and we point out that Landweber exactness is a property independent of the choice of orientation. We end this chapter by proving a theorem of H. Miller (see Theorem 2.6.1) which was stated without proof in [**Mil89**]. Coming up with a proof of this theorem (together with the necessary preliminaries) constitutes a considerable portion of the work with this thesis.

In Chapter 3 we introduce elliptic genera from which we obtain various elliptic cohomology theories via the exact functor theorem; the verification of the Landweber conditions relies on properties of elliptic curves. Next we turn to introducing modular functions, because the coefficient rings of the elliptic cohomology theories can be interpreted as rings of such functions.

Chapter 4 is devoted to computations. Specifically, we compute the correction classes that are part of the Riemann–Roch theorem for several multiplicative natural transformations. Notably, we use the theorem of Miller to obtain transformations from the newly acquired elliptic cohomology theories to singular cohomology, and compute their correction classes. We conclude by looking at how the Weierstrass $\wp$-function gives rise to the elliptic genera we have considered, and thereby relates the calculations we have performed.

Prerequisites, notation and conventions

Throughout the text we will assume familiarity with (generalized) homology and cohomology theories for (pointed) CW-complexes. The suspension isomorphism in reduced cohomology theories will always be denoted by

$$\sigma\colon \tilde{h}^*(X) \to \tilde{h}^{*+1}(\Sigma X).$$

When we speak of a natural transformation of cohomology theories we mean a degree 0 stable natural transformation $h^*(-) \to k^*(-)$; stable meaning that it commutes with the coboundary homomorphism (or the suspension isomorphism if the theory is reduced). A natural transformation of cohomology theories with products is called multiplicative if it respects the external products. The coefficient ring of a multiplicative cohomology theory is a graded ring, and we say that its grading is cohomological. The ring with the grading reversed is said to have homological grading.

Further, we assume basic knowledge of vector bundles. Once and for all we remark that when we speak of the canonical line bundle η over $\mathbb{C}P^\infty$, we mean the complex line bundle with total space $\{(l, x) \in \mathbb{C}P^\infty \times \mathbb{C}^\infty \mid x \in l\}$, and *not* its dual. A trivial vector bundle of rank n over a space X is denoted by $\mathbf{n}$. The context (or a subscript) will make it clear if we are speaking of real or complex rank. When ξ is a complex vector bundle, we write $\bar{\xi}$ for its complex conjugate. We also assume that the reader is familiar with the classifying spaces $BO(n)$, $BSO(n)$ and $BU(n)$, resp., for real, oriented and complex vector bundles, resp.

Recall that a power series $f(x) \in R[\![x]\!]$ over a commutative ring has a multiplicative inverse if and only if it is of the form $f(x) = u + \cdots$ and a functional inverse if and only if $f(x) = ux + \cdots$; here $u \in R$ is some unit. Whenever elementary functions appear in the text, it is to ease notation; we really mean their power series expansions about

0. Furthermore, we expect the reader to interpret expressions in such a way that they make sense. For instance, even though $\sinh(x) = x + \cdots$ has no multiplicative inverse, the expression $x/\sinh(x)$ is meaningful. In particular, differentiation and integration of some function always denote formal differentiation and integration of the corresponding power series expansions.

Finally, the one-point space will be denoted by pt and we let $X_+ := X \sqcup pt$.

CHAPTER 1

Review

1. Orientations of manifolds

By means of the tangent bundle of a smooth manifold one can speak about orientations with respect to a cohomology theory $h^*(-)$, as well as orientations of maps between manifolds. We will see that oriented maps give rise to covariant homomorphisms in cohomology, which do not necessarily commute with natural transformations. The Riemann–Roch theorem will tell us how to modify the maps to obtain commutativity.

1.1. Orientations of vector bundles. Let $\xi \downarrow X$ be a vector bundle over a CW-complex. The **Thom space** is defined by $X^\xi := D\xi/S\xi$ (also written $\mathrm{Th}(\xi)$), where $D\xi$ and $S\xi$ are the associated disk and sphere bundles. Let $\xi' \downarrow Y$ be another vector bundle and $f\colon \xi \to \xi'$ a bundle map. The Thom spaces are canonically pointed, and f induces a pointed map on Thom spaces,

$$\mathrm{Th}(f)\colon X^\xi \to Y^{\xi'}.$$

We recall that there is a homeomorphism

$$\psi\colon X^\xi \wedge Y^{\xi'} \xrightarrow{\approx} (X \times Y)^{\xi \times \xi'}. \tag{1.1.1}$$

We recall what it means for a vector bundle to be oriented with respect to a cohomology theory.

Definition 1.1.2. Let $h^*(-)$ be a multiplicative cohomology theory. A real vector bundle of rank n, $\xi \downarrow X$, is said to be **h-orientable** if there is an element $u_\xi \in \tilde{h}^n(X^\xi)$ such that for all x in X, the inclusion of the fiber

$$j_x\colon (D^n, S^{n-1}) \to (D\xi, S\xi),$$

pulls u_ξ back to an $\tilde{h}^*(S^0)$-module generator $j_x^* u_\xi \in \tilde{h}^n(S^n) \cong \tilde{h}^0(S^0)$.

The element u_ξ is called a **Thom class** for ξ, and when a specific choice of Thom class for ξ has been made, we say that ξ is **h-oriented**.

Let $g\colon Y \to X$ be a continuous map. For $\xi \downarrow X$, we get an induced bundle map $\bar{g}\colon g^*\xi \to \xi$. This in turn gives a map of Thom spaces $\mathrm{Th}(\bar{g})\colon Y^{g^*\xi} \to X^\xi$ called the **Thomification** of g. We will often denote the Thomification of g simply by g to ease notation.

Lemma 1.1.3. *Let $g\colon Y \to X$ be continuous and $\xi \downarrow X$ an h-oriented vector bundle of rank n with Thom class $u_\xi \in \tilde{h}^n(X^\xi)$. Then $g^*\xi \downarrow Y$ is h-oriented with Thom class $g^*u_\xi \in \tilde{h}^n(Y^{g^*\xi})$.*

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd}
\tilde{h}^n(X^\xi) \arrow[d, "g^*"'] \arrow[dr, "j^*_{g(y)}"] & \\
& \tilde{h}^n(S^n). \\
\tilde{h}^n(Y^{g^*\xi}) \arrow[ur, "j^*_y"'] &
\end{tikzcd}$$

∎

The projection map $\pi\colon \xi \to X$ for a vector bundle restricts to a map $D\xi \to X$ which we also denote by π. This map and the cup product give $\tilde{h}^*(X^\xi)$ an $h^*(X)$-module structure by the composition

$$h^m(X) \otimes \tilde{h}^n(X^\xi) \xrightarrow{\pi^*\otimes 1} h^m(D\xi, \varnothing) \otimes h^n(D\xi, S\xi) \xrightarrow{\smile} h^{m+n}(D\xi, S\xi) \xrightarrow{\cong} \tilde{h}^{m+n}(X^\xi).$$

We write $x \cdot y = \pi^* x \smile y$ for this product. For h-oriented vector bundles, this module structure gives a very important isomorphism, described in the following theorem. See [**Swi02**] for a proof.

Theorem 1.1.4 (Thom Isomorphism). *Let $\xi \downarrow X$ be an h-oriented vector bundle with Thom class u_ξ. The $h^*(X)$-module homomorphism*

$$\varphi_\xi\colon h^*(X) \to \tilde{h}^{*+n}(X^\xi)$$

defined by

$$\varphi_\xi(x) = x \cdot u_\xi$$

is an isomorphism. □

Remark. Let $\xi \downarrow X$ be an h-oriented vector bundle with Thom class u_ξ, and assume that for each $g\colon Y \to X$ the vector bundle $g^*\xi$ has Thom class g^*u_ξ, i.e. $g^*u_\xi = u_{g^*\xi}$, then the diagram

$$\begin{tikzcd} h^*(X) \arrow[r, "\varphi_\xi"] \arrow[d, "g^*"] & \tilde{h}^{*+n}(X^\xi) \arrow[d, "g^*"] \\ h^*(Y) \arrow[r, "\varphi_{g^*\xi}"] & \tilde{h}^{*+n}(Y^{g^*\xi}) \end{tikzcd}$$

commutes. Thus assuming we have such a system of Thom classes, the Thom isomorphism is natural in $\xi \downarrow X$. As we will see, this is always the case in the examples we will investigate.

Let $h^*(-)$ be a cohomology theory, and let $\xi \downarrow X$ be a rank n h-oriented vector bundle with Thom class u_ξ in $\tilde{h}^n(X^\xi)$. We take z to be the zero section into the Thom space given by the composition $z\colon X \hookrightarrow D\xi \to X^\xi$. The first map sends each point to the zero vector in the corresponding fiber while the second collapses $S\xi$ to a point.

In cohomology this induces a ring homomorphism

$$\tilde{h}^n(X^\xi) \subseteq h^n(X^\xi) \xrightarrow{z^*} h^n(X),$$

and we define the **Euler class** of ξ to be

$$e(\xi) = z^* u_\xi \in h^n(X). \tag{1.1.5}$$

This class clearly depends on the h-orientation chosen for ξ.

1.2. The Pontrjagin–Thom construction and umkehrs. Let $f\colon M^m \to N^n$ be a smooth map of closed manifolds of dimension m and n, and put $d = m - n$. Fix a smooth embedding $e\colon M \hookrightarrow \mathbb{R}^p$, for p sufficiently large, and define the **diagonal embedding**, $e_f\colon M \to N \times \mathbb{R}^p$, as the composition

$$M \xrightarrow{\Delta} M \times M \xrightarrow{f\times e} N \times \mathbb{R}^p.$$

The normal bundle of this embedding, $\nu(e_f)$, is of rank $p - m + n = p - d$.

Still letting $h^*(-)$ be a multiplicative cohomology theory, we now can speak of orientations of manifolds.

Definition 1.1.6. M^m is called **h-orientable** if there is some smooth embedding $e\colon M \hookrightarrow \mathbb{R}^p$ such that the normal bundle of this embedding, $\nu(e)$, is h-orientable. An **h-orientation** of M is a specific choice of embedding and a choice of Thom class $u_{\nu(e)}$ in $\tilde{h}^{p-m}(M^{\nu(e)})$.

In the same vein, a smooth map of manifolds $f\colon M \to N$ is **h-orientable** if for some smooth embedding $e\colon M \hookrightarrow \mathbb{R}^p$, the normal bundle of the diagonal embedding e_f is h-orientable. An **h-orientation** of f is a choice of embedding e and a Thom class for $\nu(e_f)$.

Remark. In the following sense, Section I.D of [**Dye69**] tells us that the choice of orientation of M does not depend on the embedding chosen: the normal bundles of any two embeddings are **stably equivalent**, i.e. if $e\colon M \to \mathbb{R}^p$ and $e'\colon M \to \mathbb{R}^{p'}$ are smooth embeddings, then there are integers q and q' such that $\nu(e) \oplus \mathbf{q} \cong \nu(e') \oplus \mathbf{q}'$. This bundle isomorphism gives a correspondence of Thom classes by passing to Thom spaces. Because of this, we suppress the choice of embedding from the notation, and write ν_M to denote the normal bundle of some embedding.

Similar comments hold for orientations of $f\colon M \to N$, and thus we often will speak of an h-orientation without specific mention of a diagonal embedding.

It is shown in [**Dye69**] that the h-orientations of

(1) ν_M and the tangent bundle τ_M
(2) f and $\nu_M \oplus f^*\tau_N$

correspond bijectively.

Consider the disk bundle $D\nu(e_f)$ as a tubular neighborhood of $e_f(M)$ contained in $N \times D^p \subseteq N \times \mathbb{R}^p$. By collapsing $N \times S^{p-1}$, one can view the tubular neighborhood as a subset

$$D\nu(e_f) \subseteq \Sigma^p N_+.$$

The **Pontrjagin–Thom construction** is the "collapse map"

$$c\colon \Sigma^p N_+ \to M^{\nu(e_f)},$$

defined by

$$c(x) = \begin{cases} x, & x \in \operatorname{int} D\nu(e_f) \\ *, & x \in \Sigma^p N_+ - \operatorname{int} D\nu(e_f), \end{cases}$$

where $*$ is the base point of $M^{\nu(e_f)}$.

We now construct the homomorphism mentioned in the introduction at the beginning of this section.

Definition 1.1.7. Let $f\colon M^m \to N^n$ be smooth and h-oriented, and let $e_f\colon M \to N \times \mathbb{R}^p$ be a choice of diagonal embedding with normal bundle $\nu = \nu(e_f)$. The **umkehr homomorphism** associated to f is denoted by $f^!$, and is defined as the composition in the diagram below.

$$\begin{tikzcd} h^*(M) \arrow[r, dashed, "f^!"] \arrow[d, "\varphi_\nu", "\cong"'] & h^{*-d}(N) \\ \tilde{h}^{*+p-d}(M^\nu) \arrow[r, "c^*"] & \tilde{h}^{*+p-d}(\Sigma^p N_+) \arrow[u, "\sigma^{-p}"', "\cong"] \end{tikzcd}$$

Here c^* is the map induced by the Pontrjagin–Thom construction, and $d = m - n$.

It is clear that $f^!$ depends on the choice of h-orientation for f, but the choice of diagonal embedding is irrelevant. For properties concerning the umkehr homomorphisms, see [**Dye69**].

1.3. Riemann–Roch. Let $h^*(-)$ and $k^*(-)$ be multiplicative cohomology theories and let $\lambda\colon h^*(-) \to k^*(-)$ be a multiplicative natural transformation. We first show how this transformation makes h-oriented vector bundles k-oriented.

LEMMA 1.1.8. *Let $\xi \downarrow X$ be a rank d vector bundle with Thom class u_ξ in $\tilde{h}^d(X^\xi)$. Then λu_ξ is a Thom class orienting ξ in k-cohomology.*

PROOF. By Definition 1.1.2, we need to check that for all x in X, $j_x^*(\lambda u_\xi)$ is a generator of $\tilde{k}^d(S^d)$, where j_x is the fiber inclusion

$$(D^n, S^{n-1}) \hookrightarrow (D\xi, S\xi).$$

Since λ is multiplicative, it becomes a ring homomorphism on coefficient rings and thus takes units (which are precisely the generators) of $\tilde{h}^0(S^0)$ to units of $\tilde{k}^0(S^0)$. The diagram

$$\begin{tikzcd}
\tilde{h}^d(X^\xi) \arrow[r, "j_x^*"] \arrow[d, "\lambda"] & \tilde{h}^d(S^d) \arrow[d, "\lambda"] & \tilde{h}^0(S^0) \arrow[l, "\cong"'] \arrow[d, "\lambda"] \\
\tilde{k}^d(X^\xi) \arrow[r, "j_x^*"] & \tilde{k}^d(S^d) & \tilde{k}^0(S^0) \arrow[l, "\cong"']
\end{tikzcd}$$

commutes because λ commutes with both induced maps and the suspension isomorphism. The result follows. $\square$

Suppose now that $\xi \downarrow X$ has both an h- and a k-orientation given by the Thom classes u_ξ^h and u_ξ^k respectively. From the Thom isomorphism theorem 1.1.4,

$$\varphi_\xi^k = - \cdot u_\xi^k \colon k^*(X) \to \tilde{k}^{*+d}(X^\xi)$$

is an isomorphism of $k^*(X)$-modules, so in particular, there is a unique element ρ_ξ in $k^0(X)$ such that

$$\rho_\xi \cdot u_\xi^k = \pi^*\rho_\xi \smile u_\xi^k = \lambda u_\xi^h. \tag{1.1.9}$$

We see that

$$\rho_\xi = (\varphi_\xi^k)^{-1}(\lambda u_\xi^h) = (\varphi_\xi^k)^{-1}\lambda\varphi_\xi^h(1),$$

and this indicates that the diagram

$$\begin{tikzcd}
h^*(X) \arrow[r, "\varphi_\xi^h", "\cong"'] \arrow[d, "\lambda"] & \tilde{h}^{*+d}(X^\xi) \arrow[d, "\lambda"] \\
k^*(X) \arrow[r, "\varphi_\xi^k", "\cong"'] & \tilde{k}^{*+d}(X^\xi)
\end{tikzcd} \tag{1.1.10}$$

does in general not commute. In fact, ρ_ξ is the class repairing this defect. That is, the diagram

$$\begin{tikzcd} h^*(X) \arrow[rr, "\varphi_\xi^h", "\cong"'] \arrow[d, "\lambda"] & & \tilde{h}^{*+d}(X^\xi) \arrow[dd, "\lambda"] \\ k^*(X) \arrow[d, "- \smile \rho_\xi"] & & \\ k^*(X) \arrow[rr, "\varphi_\xi^k", "\cong"'] & & \tilde{k}^{*+d}(X^\xi) \end{tikzcd}$$

commutes. This is shown by the following calculation:

$$\begin{aligned} \lambda\varphi_\xi^h(x) &= \lambda(\pi^* x \smile u_\xi^h) \\ &= \pi^*(\lambda x) \smile \rho_\xi \cdot u_\xi^k \\ &= \pi^*(\lambda x \smile \rho_\xi) \smile u_\xi^k \\ &= \varphi_\xi^k(\lambda x \smile \rho_\xi) \end{aligned}$$

One therefore refers to ρ_ξ as the **correction class** of ξ with respect to λ.

Just as (1.1.10) is non-commutative, the following diagram relating multiplicative natural transformations and umkehr homomorphisms does not commute.

$$\begin{tikzcd} h^*(M) \arrow[r, "f_h^!"] \arrow[d, "\lambda"] & h^{*-d}(N) \arrow[d, "\lambda"] \\ k^*(M) \arrow[r, "f_k^!"] & k^{*-d}(N) \end{tikzcd}$$

It turns out that ρ again does the job as a correction class. We make this precise in the theorem below. A proof can be found in [**Dye69**].

Theorem 1.1.11 (Riemann–Roch). *Let $\lambda\colon h^*(-) \to k^*(-)$ be a multiplicative transformation of cohomology theories, and let $f\colon M \to N$ be an h-oriented map of smooth and closed manifolds. Then for all α in $h^*(M)$*

$$\lambda f_h^!(\alpha) = f_k^!(\lambda\alpha \smile \rho_\nu),$$

where $\nu = \nu(e_f)$ is the normal bundle orienting f, and ρ_ν is the correction class for this vector bundle. □

The correspondence between h-orientations of f and $\nu_M \oplus f^*\tau_N$ in the remark following Definition 1.1.6 is actually given by a stable equivalence between this vector bundle and $\nu(e_f)$. This allows us to restate the theorem in the following form.

Corollary 1.1.12. *Let $\lambda\colon h^*(-) \to k^*(-)$ be a multiplicative transformation of cohomology theories, and let $f\colon M \to N$ be map of smooth manifolds such that $\nu_M \oplus f^*\tau$ is h-oriented. Then f has a corresponding orientation, and*

$$\lambda f_h^!(\alpha) = f_k^!(\lambda\alpha \smile \rho_{\nu_M \oplus f^*\tau_N}),$$

for all $\alpha \in h^*(M)$. □

It is clearly important to get some hold on what this correction class is. This will be investigated in Chapter 4, where we compute specific examples of these classes.

2. Characteristic classes

Let $BU(n)$ be the classifying space for complex vector bundles of rank n and let $\xi_n \downarrow BU(n)$ be the universal bundle. There are canonical inclusions $j_n \colon BU(n) \hookrightarrow BU(n+1)$ classifying $\xi_n \oplus \mathbf{1} \downarrow BU(n)$, for all n. From these maps, one obtains inclusions $i_n \colon BU(n) \hookrightarrow BU = \operatorname{colim}_{n\to\infty} BU(n)$.

Definition 1.2.1. Let $f \colon X \to BU(n)$ classify a rank n complex vector bundle ξ over X.

Any element $\gamma \in h^*\big(BU(n)\big)$ gives a class

$$\gamma(\xi) := f^*\gamma \in h^*(X)$$

which is called a **characteristic class** for ξ.

A **universal characteristic class** for complex vector bundles is an element $\gamma \in h^*(BU)$. Such an element gives rise to a characteristic class

$$\gamma(\xi) := f^* i_n^* \gamma \in h^*(X),$$

which is an example of a **stable characteristic class**, i.e. for all $\xi \downarrow X$, $\gamma(\xi) = \gamma(\xi \oplus \mathbf{1})$.

Remark. If $f \colon X \to BU(n)$ classifies ξ, then $j_n \circ f \colon X \to BU(n+1)$ classifies $\xi \oplus \mathbf{1}$. Hence it follows that if $\gamma \in h^*(BU)$ is a universal characteristic class, then

$$\gamma(\xi \oplus \mathbf{1}) = f^* j_n^* i_{n+1}^* \gamma = f^* i_n^* \gamma = \gamma(\xi),$$

so that γ is stable.

Two isomorphic complex vector bundles $\xi \cong \xi'$ are classified by homotopic maps. Therefore the induced maps coincide, and it follows that the characteristic classes of ξ and ξ' coincide.

The characteristic classes are natural. More precisely, let $\xi \downarrow X$ be a rank n complex vector bundle classified by $f \colon X \to BU(n)$ and let $g \colon Y \to X$ be continuous. Then $f \circ g$ classifies the pullback bundle $g^*\xi \downarrow Y$, and hence

$$\gamma(g^*\xi) = (f \circ g)^* i_n^* \gamma = g^* f^* i_n^* \gamma = g^* \gamma(\xi).$$

In other words, characteristic classes are certain cohomology classes which we associate to vector bundles over a space X. They help us tell two vector bundles apart, since two vector bundles with different characteristic classes necessarily must be non-isomorphic.

Recall that in singular cohomology $H^*(-)$,

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[\omega], \tag{1.2.2}$$

where $\omega \in \widetilde{H}^2(\mathbb{C}P^\infty)$ is the generator chosen such that $j^* \colon \widetilde{H}^2(\mathbb{C}P^\infty) \to \widetilde{H}^2(S^2)$, where $j \colon S^2 \approx \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$, sends ω to the suspension $\sigma^2(1)$.

This choice of ω determines very important examples of characteristic classes for complex vector bundles, namely the **Chern classes**. They are uniquely determined by the following theorem. A proof can be found in for instance [**Swi02**, Theorem 16.2] and [**Sto68**, p. 63].

Theorem 1.2.3. *Let ξ be a complex vector bundle over the CW-complex X. There are unique cohomology classes $c_i(\xi)$ in $H^{2i}(X)$, determined by the isomorphism class of ξ such that for all complex vector bundles*

(1) $c_0(\xi) = 1$.
(2) $c_1(\eta) = \omega$, *where η is the canonical line bundle over $\mathbb{C}P^\infty$.*
(3) $c_i(\xi) = 0$, *for $i < 0$ and $i > \operatorname{rank} \xi$.*
(4) *The c_i are natural, i.e. given $f \colon Y \to X$, $c_i(f^*\xi) = f^* c_i(\xi)$.*

(5) *Whenever* ξ *and* ξ' *are complex vector bundles over the same base space,* $c_k(\xi \oplus \xi') = \sum_{i+j=k} c_i(\xi) \smile c_j(\xi')$. □

c_i is called the i**th Chern class** of ξ. The sum

$$c(\xi) = \sum_{i \geq 0} c_i(\xi)$$

is the **total Chern class**. Note that this is a well-defined sum, since $c_i(\xi) = 0$ when $i > \operatorname{rank} \xi$. Considering the total Chern class, the last characterizing property in the theorem above takes the form

$$c(\xi \oplus \xi') = c(\xi) \smile c(\xi'). \tag{1.2.4}$$

We make the following convention: We speak of sums of characteristic classes as a "characteristic class" as well. In particular, the total Chern class will be called a characteristic class.

A characteristic class κ (possibly a sum) is called **exponential** if it satisfies $\kappa(\xi \oplus \xi') = \kappa(\xi) \smile \kappa(\xi')$ for all complex vector bundles ξ and ξ'.

Example 1.2.5. The total Chern class is such an exponential characteristic class.

Let $\mathbf{n} \downarrow X$ be any trivial complex vector bundle. Then $c_i(\mathbf{n}) = 0$ for $i > 0$, because $\mathbf{n}$ is the pullback of a rank n bundle over pt, and $H^*(pt)$ is 0 in positive degrees. This implies that $c(\mathbf{n}) = 1$, so in particular $c(\xi \oplus \mathbf{1}) = c(\xi) \smile c(\mathbf{1}) = c(\xi)$. Therefore the total Chern class is stable. For the same reasons, each c_i is stable as well.

For a complex vector bundle ξ of rank n, denote by $c_{\text{top}}(\xi) = c_n(\xi)$, the top Chern class. By the properties of the Chern classes, we see that

$$c_{\text{top}}(\xi \oplus \xi') = c_{n+m}(\xi \oplus \xi') = c_n(\xi) \smile c_m(\xi') = c_{\text{top}}(\xi) \smile c_{\text{top}}(\xi'),$$

i.e. c_{top} is exponential. However, c_{top} is not stable, since

$$c_{\text{top}}(\xi \oplus \mathbf{1}) = c_n(\xi) \smile c_1(\mathbf{1}) = 0.$$

When dealing with complex vector bundles, and in particular when considering characteristic classes of vector bundles, the following theorem is of great importance. Note that we are a bit ahead of ourselves; we shall state the theorem for so-called "complex orientable cohomology theories" which we define in Chapter 2. In particular, the theorem holds for $H^*(-)$. A proof can be found in for instance [**LM89, Sto68**].

Theorem 1.2.6 (Splitting Principle). *Let* $E^*(-)$ *be a complex orientable cohomology theory. For any complex vector bundle* ξ *over a CW-complex* X*, there is a continuous map* $f\colon Y \to X$ *such that*

(1) $f^*\xi$ *is isomorphic to a sum of complex line bundles over* Y*,*

$$f^*\xi \cong \ell_1 \oplus \cdots \ell_n,$$

where $n = \operatorname{rank} \xi$*,*

(2) $f^*\colon E^*(X) \to E^*(Y)$ *is a monomorphism.* □

To illustrate why this principle is helpful, consider a characteristic class κ of a complex vector bundle ξ, and by the theorem, write

$$f^*\kappa(\xi) = \kappa(f^*\xi) = \kappa(\ell_1 \oplus \cdots \oplus \ell_n).$$

Since f^* is a monomorphism, $\kappa(\ell_1 \oplus \cdots \oplus \ell_n)$ uniquely determines $\kappa(\xi)$. This allows us to view any complex vector bundle as a sum of complex line bundles when working with characteristic classes. We will frequently do so from now on.

If κ is an exponential characteristic class, then $\kappa(\xi) = \kappa(\ell_1) \smile \cdots \smile \kappa(\ell_n)$, so $\kappa(\xi)$ is determined by the value on the line bundles. Appealing to naturality and the universality of the canonical line bundle $\eta \downarrow \mathbb{C}P^\infty$, one sees that $\kappa(\xi)$ is in fact determined by $\kappa(\eta)$ for all complex vector bundles ξ. Most characteristic classes we encounter will be exponential, and so are determined by their value on η.

EXAMPLE 1.2.7. As an illustration, let $\xi = \ell_1 \oplus \cdots \oplus \ell_n$, and put $x_i = c_1(\ell_i)$. By exponentiality of the total Chern class,

$$c(\xi) = \prod c(\ell_i) = \prod (1 + x_i),$$

and hence, for reasons of dimension, one sees that $c_j(\xi) = \sigma_j(x_1, \ldots, x_n)$, the jth symmetric polynomial in the x_i.

Since $c_1(\bar{\ell}) = -c_1(\ell)$ (see [**MS74**, Lemma 14.9]), we immediately obtain that for any complex vector bundle ξ

$$c_i(\bar{\xi}) = \sigma_i(-x_1, \ldots, -x_n) = (-1)^i \sigma_i(x_1, \ldots, x_n) = (-1)^i c_i(\xi). \tag{1.2.8}$$

There is a classical approach to making new exponential characteristic classes from Chern classes, namely via "multiplicative sequences"; see [**Hir66, LM89**]. For brevity we will present a reverse engineered account that is sufficient for our purposes.

To this end, let $\xi = \ell_1 \oplus \cdots \oplus \ell_n$, and $x_i = c_1(\ell_i)$. Let $Q(x) \in \mathbb{Q}[\![x]\!]$ be a power series with leading term 1. We shall refer to this as the **characteristic power series**, but in the literature it is often also called the "Hirzebruch Q-series". We define a characteristic class

$$F_Q(\xi) \in \prod_{n \geq 0} H^{2n}(X; \mathbb{Q})$$

by letting it take the value $F_Q(\ell) = Q\big(c_1(\ell)\big)$ on a complex line bundle, and then extending to all vector bundles by exponentiality and naturality. Specifically, viewing ξ as a sum of line bundles, we define F_Q by

$$F_Q(\xi) = F_Q(\ell_1 \oplus \cdots \oplus \ell_n) = \prod F_Q(\ell_i) = \prod Q(x_i) \in \prod_{n \geq 0} H^{2n}(X; \mathbb{Q}).$$

DEFINITION 1.2.9. The exponential characteristic class F_Q is called the **total F_Q-class**.

These characteristic classes are stable.

EXAMPLE 1.2.10. Let $Q(x) = 1 + x$. Clearly, $F_Q(\xi) = c(\xi)$.

Define the power series $t(x) = \frac{x}{1-e^{-x}}$. The associated characteristic class,

$$\mathrm{td}(\xi) = \prod \frac{x_i}{1 - e^{-x_i}} \in \prod_{n \geq 0} H^{2n}(X; \mathbb{Q}),$$

is called the **total Todd class**.

Any complex vector bundle ξ has an underlying real bundle $\xi_{\mathbb{R}}$. This assignment is called **realification**, and if ξ is a complex vector bundle of rank n, then $\xi_{\mathbb{R}}$ has (real) rank $2n$. This is a canonically oriented vector bundle. Conversely, if ξ is real, we get a complex vector bundle $\xi \otimes \mathbb{C}$ by tensoring each fiber with $\mathbb{C}$ (over $\mathbb{R}$). This assignment is called **complexification**. Let ξ be a complex and ζ be a real vector bundle. Then there are natural isomorphisms ([**Kar78, Sto68**])

$$\begin{aligned} \zeta \otimes \mathbb{C} &\cong \overline{\zeta \otimes \mathbb{C}} \\ (\zeta \otimes \mathbb{C})_{\mathbb{R}} &\cong \zeta \oplus \zeta \\ \xi_{\mathbb{R}} \otimes \mathbb{C} &\cong \xi \oplus \bar{\xi}. \end{aligned} \tag{1.2.11}$$

We would like to define characteristic classes for vector bundles that are not necessarily complex, and to do so, we will need the following variant of the splitting principle.

THEOREM 1.2.12 ([**LM89**]). *Let $E^*(-)$ be a complex orientable cohomology theory and let ξ be a real oriented vector bundle of rank $2n$ over a CW-complex X. Then there is a map $f\colon Y \to X$ such that*

$$f^*(\xi \otimes \mathbb{C}) \cong \ell_1 \oplus \bar{\ell}_1 \oplus \cdots \oplus \ell_n \oplus \bar{\ell}_n,$$

where the ℓ_i are complex line bundles, and the induced map

$$f^*\colon E^*(X) \to E^*(Y)$$

is a monomorphism. □

This means that we can think of the complexification of a rank $2n$ oriented real vector bundle as a sum of conjugate pairs of complex vector bundles. Note that this is consistent with the first version of the splitting principle, for if ξ is a complex vector bundle of rank n we have from (1.2.11) that $\xi_{\mathbb{R}} \otimes \mathbb{C} \cong \xi \oplus \bar{\xi}$, and splitting ξ into complex line bundles thus makes $\xi_{\mathbb{R}} \otimes \mathbb{C}$ split into conjugate pairs of line bundles.

Fix an oriented, real vector bundle ζ of rank $2n$, and by the splitting principle write

$$\zeta \otimes \mathbb{C} = \ell_1 \oplus \bar{\ell}_1 \oplus \cdots \oplus \ell_n \oplus \bar{\ell}_n.$$

Let $Q(x) \in \mathbb{Q}[\![x]\!]$ be an even power series with leading term 1. On a rank 2 oriented, real vector bundle ζ_2 (with $\zeta_2 \otimes \mathbb{C} = \ell \oplus \bar{\ell}$) define $F_Q(\zeta_2) = Q\big(c_1(\ell)\big)$. By forcing exponentiality and naturality as in the complex case, we obtain the **total F_Q-class**

$$F_Q(\zeta) = \prod Q(x_i) \in \prod_{n \geq 0} H^{4n}(X;\mathbb{Q}),$$

where $x_i = c_1(\ell_i)$. Since $c_1(\bar{\ell}) = -c_1(\ell)$, we are dependent on the fact that $Q(x)$ is even for this to be well-defined.

EXAMPLE 1.2.13. For further reference, we give some important examples of characteristic classes. Let ζ be as above, that is, real and oriented of rank $2n$.

$$\begin{aligned} A(\zeta) &= \prod \frac{2x_j}{\sinh(2x_j)} \\ \hat{A}(\zeta) &= \prod \frac{x_j}{2\sinh(x_j/2)} \\ L(\zeta) &= \prod \frac{x_j}{\tanh(x_j)} \\ \hat{L}(\zeta) &= \prod \frac{x_j}{2\tanh(x_j/2)} \end{aligned} \tag{1.2.14}$$

The first three classes have a central role in the literature, but the total $\hat{L}$-class is somewhat non-standard. It is given its name because it is related to the L-class in a similar way $\hat{A}$ is related to the A-class.

3. K-theory

In this section we briefly review some basic facts about K-theory and establish the notation that will be used later. We will also describe a few constructions on vector bundles that give rise to operations in K-theory. Unless stated otherwise, the vector bundles of this section may both be real or complex.

3.1. Generalities. Any multiplicative (unreduced) cohomology theory $h^*(-)$ comes equipped with an external cross product

$$\times\colon h^m(X)\otimes h^n(Y)\to h^{m+n}(X\times Y).$$

The reduced cohomology theory $\tilde{h}^*(-)$ then has a corresponding external smash product

$$\wedge\colon \tilde{h}^m(X)\otimes \tilde{h}^n(Y)\to \tilde{h}^{m+n}(X\wedge Y).$$

This product is characterized by the commutative diagram

$$\begin{tikzcd}
\tilde{h}^m(X)\otimes \tilde{h}^n(Y) \arrow[r, "\wedge"] \arrow[dd] & \tilde{h}^{m+n}(X\wedge Y) \arrow[d, "q^*"] \\
& \tilde{h}^{m+n}(X\times Y) \arrow[d] \\
h^m(X)\otimes h^n(Y) \arrow[r, "\times"] & h^{m+n}(X\times Y)
\end{tikzcd} \tag{1.3.1}$$

where X and Y are pointed spaces, the unnamed vertical arrows are induced by the inclusions and q is the projection $q\colon X\times Y\to X\wedge Y$ collapsing $X\vee Y$. This characterizes the product, for if $x\in\tilde{h}^m(X)$ and $y\in\tilde{h}^m(Y)$ then there is a unique element $x\wedge y\in\tilde{h}^{m+n}(X\wedge Y)$ such that $q^*(x\wedge y)=x\times y$. (Cf. [**May99**].)

Now recall (from [**Bot69**] for instance) real and complex K-theory. These are contravariant functors from the category of finite CW-complexes into the category of rings, defined by

$$K(X):=K\big(\operatorname{Vect}(X)\big).$$

Here $\operatorname{Vect}(X)$ denotes the set of isomorphism classes of (either real or complex) vector bundles over X. It is a commutative semiring under the operations of Whitney sum ($\oplus$) and tensor product ($\otimes$). Applying Grothendieck's K-construction then gives a ring. A continuous map $f\colon X\to Y$ is taken to a ring homomorphism $f^*\colon K(Y)\to K(X)$ which is induced by the pullback of vector bundles. We will not make any notational difference between an honest vector bundle and the class it represents in $K(X)$. By definition, the elements of $K(X)$, called **virtual bundles**, are of the form $\xi-\xi'$. The addition and multiplication in $K(X)$ is usually denoted by $+$ and juxtaposition, but we will sometimes take the liberty to use the symbols $\oplus$ and $\otimes$. We write $KO(X)$ to denote real, and $K(X)$ to denote complex K-theory of a space X. When there is no need to differ, we will just use $K(X)$ to denote either.

For a pointed finite CW-complex X with basepoint x_0 the reduced K-theory group $\widetilde{K}(X)$ is (as usual) defined as the kernel of the homomorphism $K(X)\to K(\{x_0\})$ induced by the inclusion $\{x_0\}\hookrightarrow X$. The geometric interpretation of these groups is that they consist of virtual bundles $\xi-\xi'$ where ξ and ξ' are of the same rank. We keep in mind that we can always pass back to unreduced K-theory by $K(X)=\widetilde{K}(X_+)$.

The complexification and realification now gives additive homomorphisms

$$\begin{aligned} r&\colon \widetilde{K}(X)\to\widetilde{KO}(X)\\ c&\colon \widetilde{KO}(X)\to\widetilde{K}(X)\end{aligned} \tag{1.3.2}$$

and in fact, c is a ring homomorphism, since tensoring by $\mathbb{C}$ distributes over tensor products over $\mathbb{R}$. One also sees that c and r are natural in X.

For vector bundles $\xi \downarrow X$ and $\xi' \downarrow Y$ define the **external tensor product** to be the vector bundle over $X \times Y$ given by

$$\xi \mathbin{\widehat{\otimes}} \xi' := \mathrm{pr}_X^* \xi \otimes \mathrm{pr}_Y^* \xi',$$

where pr_X and pr_Y are the obvious projections. This is the external cross product in this cohomology theory;

$$\widehat{\otimes}\colon K(X) \otimes K(Y) \to K(X \times Y).$$

In light of the discussions above, we get a corresponding external smash product.

So far we have treated real and complex K-theory simultaneously, but when describing their periodic properties, we are forced to consider them separately. See [**Bot69**] for a proof of the following theorem.

THEOREM 1.3.3 (Periodicity theorem). *Let X be a pointed finite CW-complex. In complex K-theory, the homomorphism*

$$\widetilde{K}(X) \otimes \widetilde{K}(S^2) \to \widetilde{K}(X \wedge S^2)$$

defined by $x \otimes y \mapsto x \wedge y$ is an isomorphisms of rings. Analogously,

$$\widetilde{KO}(X) \otimes \widetilde{KO}(S^8) \to \widetilde{KO}(X \wedge S^8)$$

is an isomorphism of rings in real K-theory.

Both $\widetilde{K}(S^2)$ and $\widetilde{KO}(S^8)$ are infinite cyclic, and fixing generators u and v_8 gives group isomorphisms for all finite pointed CW-complexes,

$$\beta\colon \widetilde{K}(X) \to \widetilde{K}(X \wedge S^2)$$

by $x \mapsto x \wedge u$ and

$$\beta_{\mathbb{R}}\colon \widetilde{KO}(X) \to \widetilde{KO}(X \wedge S^8)$$

by $x \mapsto x \wedge v_8$. □

The isomorphisms β and $\beta_{\mathbb{R}}$ are called the **Bott isomorphisms**, and the elements u and v_8 are called **Bott elements**. For complex K-theory we make a specific choice for u: Let $\eta_1 \downarrow \mathbb{C}P^1$ be the restriction of $\eta \downarrow \mathbb{C}P^\infty$ and put $u = \mathbf{1} - \eta_1 \in \widetilde{K}(S^2)$. It is shown in [**Bot69**] that this really is a generator.

The K-theories are concentrated in degree 0. One extends to non-positive degrees by putting $\widetilde{K}^{-n}(X) := \widetilde{K}(\Sigma^n X)$ for $n \geq 0$, and

$$\widetilde{K}^\sharp(X) := \bigoplus_{n \geq 0} \widetilde{K}^{-n}(X).$$

The external smash product gives an external product on $\widetilde{K}^\sharp(X)$ by

$$\widetilde{K}^{-m}(X) \otimes \widetilde{K}^{-n}(Y) \xrightarrow{\wedge} \widetilde{K}(X \wedge S^m \wedge Y \wedge S^n) \cong \widetilde{K}^{-n-m}(X \wedge Y), \tag{1.3.4}$$

which in turn makes $\widetilde{K}^\sharp(X)$ into a graded ring.

Restricting to complex K-theory (the real case is similar), the Bott isomorphism allows the extension to positive degrees, by inductively putting $\widetilde{K}^n(X) = \widetilde{K}^{n-2}(X)$ as groups. Then

$$\widetilde{K}^*(X) := \bigoplus_{n \in \mathbb{Z}} \widetilde{K}^n(X)$$

can be made into a cohomology theory and we will refer to this cohomology theory as ($\mathbb{Z}$-graded) K-theory. The composition in (1.3.4) gives $\widetilde{K}^*(X)$ both a ring structure and a $\widetilde{K}^*(S^0)$-module structure. We denote both products by juxtaposition. The coefficient ring is

$$\widetilde{K}^*(S^0) = \mathbb{Z}[u, u^{-1}],$$

and by virtue of the Bott isomorphism, we can (and shall often) write any element of $\widetilde{K}^{-2n-\nu}(X)$, where $\nu = 0, 1$, on the form $u^n x$ where $x \in \widetilde{K}^{-\nu}(X)$. One should think of multiplication by u (or u^{-1}) as a shift of degree.

Similarly, we have the ($\mathbb{Z}$-graded) real K-theory. In this case the coefficient ring is ([**ABS64**])

$$\widetilde{KO}^*(S^0) = \mathbb{Z}[v_1, v_4, v_8, v_8^{-1}]/(2v_1, v_1^3, v_1v_4, v_4^2 - 4v_8),$$

where $v_i \in KO^{-i}(S^0)$ are some particular choices of elements. In particular, $v_4 \in KO(S^4)$ complexifies to $2u^2$, as shown in [**ABS64**].

Proposition 1.3.5. *The complexification* $c\colon \widetilde{KO}(X) \to \widetilde{K}(X)$ *extends to a multiplicative natural transformation*

$$c\colon \widetilde{KO}^*(-) \to \widetilde{K}^*(-)$$

of cohomology theories.

Proof. We have already seen that c is natural and multiplicative with respect to the tensor product. It follows that it also preserves the external smash product, since

$$c(x \wedge y) = c\big(q^*(\mathrm{pr}_1^* x \otimes \mathrm{pr}_2^* y)\big) = q^*\big(\mathrm{pr}_1^* c(x) \otimes \mathrm{pr}_2^* c(y)\big) = c(x) \wedge c(y).$$

It only remains to show that c commutes with the Bott isomorphisms, that is

$$\begin{tikzcd}
\widetilde{KO}(X) \arrow[r, "c"] \arrow[d, "- \wedge v_8"] & \widetilde{K}(X) \arrow[d, "- \wedge u^4"] \\
\widetilde{KO}(X \wedge S^8) \arrow[r, "c"] & \widetilde{K}(X \wedge S^8)
\end{tikzcd}$$

commutes. This follows from

$$c(v_8) = c\left(\frac{1}{4}v_4^2\right) = c\left(\frac{1}{2}v_4\right)^2 = u^4,$$

using that $c(v_4) = 2u^2$. $\square$

From these cohomology theories we produce two more by localizing away from 2. Define

$$\widetilde{K[\tfrac{1}{2}]}^*(-) := \widetilde{K}^*(-) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}]$$

and

$$\widetilde{KO[\tfrac{1}{2}]}^*(-) := \widetilde{KO}^*(-) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}].$$

These are indeed cohomology theories, since localizations are flat. Clearly,

$$\widetilde{K[\tfrac{1}{2}]}^*(S^0) = \mathbb{Z}[\tfrac{1}{2}][u, u^{-1}],$$

and inverting 2 in the real case kills v_1, and one has $v_8 = \frac{1}{4}v_4^2$. Put

$$u^2 := \frac{1}{2}v_4 \in \widetilde{KO[\tfrac{1}{2}]}^{-4}(S^0).$$

Then the coefficient ring is generated by u^2 and is periodic with period 4:

$$\widetilde{KO[\tfrac{1}{2}]}^*(S^0) = \mathbb{Z}[\tfrac{1}{2}][u^2, u^{-2}]. \tag{1.3.6}$$

The reason for choosing to call this generator u^2 is apparent when we note that the complexification

$$c\colon \widetilde{KO[\tfrac{1}{2}]}^*(-) \to \widetilde{K[\tfrac{1}{2}]}^*(-) \tag{1.3.7}$$

then sends u^2 to u^2.

3.2. Operations. In this section we describe some constructions on vector bundles. As a rule of thumb, it will be sufficient to do the construction on vector spaces and extend fiberwise. The following discussion holds in both the complex and real case unless otherwise noted.

Let V be a finite dimensional vector space over some field k. For $n \geq 0$, denote by $T^n V = V \otimes \cdots \otimes V$ the n-fold tensor product over k. Note that $T^0 V = k$. Let $I_n \subseteq T^n V$ be the subspace

$$\langle v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid v_i \in V, \sigma \in \Sigma_n \rangle,$$

where Σ_n is the symmetric group on n letters. Define the n**th symmetric power** of V to be

$$S^n V = T^n V / I_n.$$

Similarly, we define the n**th exterior power** of V to be

$$\Lambda^n V = T^n V / J_n,$$

where J_n is the subspace

$$\langle v_1 \otimes \cdots \otimes v_n - \operatorname{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid v_i \in V, \sigma \in \Sigma_n \rangle.$$

We note that $\dim S^n V = \binom{\dim V + n - 1}{n}$ and $\dim \Lambda^n V = \binom{\dim V}{n}$.

Both the symmetric and exterior power has a multiplicative property.

Proposition 1.3.8. *Let V and W be vector spaces. There are natural isomorphisms*

$$S^n(V \oplus W) \cong \bigoplus_{i+j=n} S^i V \otimes S^j W \tag{1.3.9}$$

and

$$\Lambda^n(V \oplus W) \cong \bigoplus_{i+j=n} \Lambda^i V \otimes \Lambda^j W. \tag{1.3.10}$$

Proof. See Propositions XVI.8.2 and XIX.1.2 of [**Lan02**]. □

These constructions, as well as the proposition, carry over to vector bundles. In particular, for any vector bundle ξ

$$T^0 \xi = S^0 \xi = \Lambda^0 \xi = \mathbf{1}, \tag{1.3.11}$$

and

$$T^1 \xi = S^1 \xi = \Lambda^1 \xi = \xi. \tag{1.3.12}$$

We now extend these operations to K-theory. Form power series in the indeterminate t,

$$s_t \xi = \sum_{n \geq 0} S^n(\xi) t^n, \quad \lambda_t \xi = \sum_{n \geq 0} \Lambda^n(\xi) t^n,$$

which, by (1.3.11), are both elements of $1 + tK(X)[\![t]\!]$, the abelian group of power series with leading term 1 under multiplication. By Proposition 1.3.8 the maps

$$s_t, \lambda_t \colon \operatorname{Vect}(X) \to 1 + tK(X)[\![t]\!]$$

are monoid homomorphisms into an abelian group, and thus, by the universal property of the K-construction, factors uniquely through $K(X)$ to give group homomorphisms

$$s_t, \lambda_t \colon K(X) \to 1 + tK(X)[\![t]\!].$$

For any element $x \in K(X)$ one then defines $s^n x$ to be the coefficient of t^n in $s_t x$, and similarly for $\lambda^n x$.

By tracing through the definitions, one can see that s_t and λ_t are both natural, i.e. for $x \in K(X)$ and $f\colon Y \to X$ continuous, $f^*s_tx = s_t(f^*x)$ and $f^*\lambda_tx = \lambda_t(f^*x)$.

LEMMA 1.3.13. *Let $x, y \in K(X)$. Then $s_t(x+y) = s_t(x)s_t(y)$ and $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$.*

PROOF. We consider only the "symmetric" case. By writing both x and y as differences of vector bundles one sees that it sufficient to show this for sums of vector bundles, since s_t is a group homomorphism (and therefore $s_t(-\xi) = s_t(\xi)^{-1}$). Now

$$s_t(\xi+\xi') = \sum_{n\geq 0} s^n(\xi+\xi')t^n$$

and

$$s_t(\xi)s_t(\xi') = \Big(\sum_{m\geq 0} s^m(\xi)t^m\Big)\Big(\sum_{n\geq 0} s^n(\xi')t^n\Big) = \sum_{k\geq 0}\Big(\sum_{m+n=k} s^m(\xi)s^n(\xi')\Big)t^k,$$

which are equal by Proposition 1.3.8. □

The operations λ_t and s_t are closely connected, as the following results indicate.

LEMMA 1.3.14. *Let ℓ be a line bundle. Then $s_t(\ell) = \lambda_{-t}(\ell)^{-1} = \lambda_{-t}(-\ell)$.*

PROOF. Since ℓ is 1-dimensional, $\lambda^n\ell = 0$ for all $n > 1$, and thus $\lambda_t\ell = 1 + \ell t$. We remark that $s^n\ell = \ell^n$, the n-fold tensor product. It follows that

$$s_t(\ell) = \sum_{n\geq 0} \ell^n t^n = \frac{1}{1-\ell t} = \lambda_{-t}(\ell)^{-1}. \qquad \square$$

COROLLARY 1.3.15. *Any element of complex K-theory, $x \in K(X)$, satisfies $s_t(x) = \lambda_{-t}(x)^{-1} = \lambda_{-t}(-x)$.*

PROOF. Write $x = \xi - \xi'$. Choose a map $f\colon Y \to X$ such that both $f^*\xi$ and $f^*\xi'$ splits as complex line bundles. The result follows from the two lemmas and naturality. □

The following theorem determines some important operations in complex K-theory. See [**Kar78**, Theorem IV.7.13] for a proof.

THEOREM 1.3.16 (Adams operations). *For all integers k, there are uniquely determined natural maps*

$$\psi^k\colon K(X) \to K(X)$$

which are ring homomorphisms and additionally satisfy

$$\psi^k(\ell) = \ell^k,$$

when ℓ is the class of a complex line bundle over X. □

ψ^k is called the k**th Adams operation**. Naturality of these operations give rise to natural ring homomorphisms

$$\psi^k\colon \widetilde{K}(X) \to \widetilde{K}(X).$$

The Adams operations respect the external products, as is readily shown by using naturality and multiplicativty with respect to the tensor product.

The following proposition is a standard fact, but we include the proof because of its importance in the following discussion.

PROPOSITION 1.3.17. *$\psi^k\colon \widetilde{K}(S^{2n}) \to \widetilde{K}(S^{2n})$ is multiplication by k^n.*

PROOF. We prove this using induction. Let $x \in \widetilde{K}(S^2) \cong \mathbb{Z}$. Then $x = mu$, for some integer m, and

$$\psi^k(x) = m\psi^k(1-\eta_1) = m(1-\eta_1^k) = m\big(1-(1-u)^k\big) = m(ku) = kx.$$

Here we used that the internal product in reduced K-theory on a suspension is trivial.

Assume that $\psi^k(x) = k^{n-1}x$ for $x \in \widetilde{K}(S^{2(n-1)})$. The external product

$$\wedge \colon \widetilde{K}(S^{2(n-1)}) \otimes \widetilde{K}(S^2) \to \widetilde{K}(S^{2n})$$

is an isomorphism, and since the Adams operations respect it, we obtain

$$\psi^k(x) = \psi^k(mu^{n-1} \wedge u) = \psi^k(mu^{n-1}) \wedge \psi^k(u) = k^{n-1}(mu^{n-1}) \wedge ku = k^n x. \qquad \square$$

COROLLARY 1.3.18. *For any $x \in \widetilde{K}(X)$, the kth Adams operation*

$$\psi^k \colon \widetilde{K}(\Sigma^{2n}X) \to \widetilde{K}(\Sigma^{2n}X)$$

satisfies $\psi^k(xu^n) = k^n\psi^k(x)u^n$. $\square$

Returning to the relation of real and complex K-theory, (1.2.11) shows that $r\colon K(X) \to KO(X)$ and $c\colon KO(X) \to K(X)$ satisfy $r \circ c(x) = 2x$ and $c \circ r(x) = x + \bar{x}$.

Let $K(X)^0 = \{x \in K(X) \mid \bar{x} = x\}$ be the subgroup of $K(X)$ consisting of the elements that are left invariant under the $\mathbb{Z}_2$-action of complex conjugation, and let $K(X)^1 = \{x \in K(X) \mid \bar{x} = -x\}$. The following result describes the image of the complexification away from 2.

PROPOSITION 1.3.19. *$c\colon KO(X) \to K(X)$ induces a group isomorphism*

$$KO(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \to K(X)^0 \otimes \mathbb{Z}[\tfrac{1}{2}].$$

PROOF. If ξ is a real vector bundle, then $\xi \otimes \mathbb{C} \cong \overline{\xi \otimes \mathbb{C}}$, so an element x of $KO(X)$ is mapped to $c(x) \in K(X)^0$. As before $r \circ c(x) = 2x$, and taking an element $y \in K(X)^0$ we see that $c \circ r(y) = y + \bar{y} = 2y$. Since we localize away from 2, c (and r) induces additive isomorphisms. $\square$

Using $\ell \otimes \bar{\ell} \cong \mathbf{1}$ for complex line bundles ℓ and Theorem 1.3.16 we see that

$$\psi^{-1}(\ell) = \ell^{-1} = \bar{\ell}$$

in $K(X)$. Thus, by Theorem 1.2.6, for any $x \in \widetilde{K}(X)$ we have $\psi^{-1}(x) = \bar{x}$. It follows from Proposition 1.3.19 that the image of the complexification

$$c\colon KO[\tfrac{1}{2}](X) \to K[\tfrac{1}{2}](X)$$

is precisely those elements invariant under the action of ψ^{-1}

From the above corollary, one sees that $\psi^{-1}(xu^n) = (-1)^n\bar{x}u^n$ on $\widetilde{K}(\Sigma^{2n}X)$, and we obtain the following more general result.

PROPOSITION 1.3.20. *The image of*

$$c\colon \widetilde{KO[\tfrac{1}{2}]}^*(X) \to \widetilde{K[\tfrac{1}{2}]}^*(X)$$

is the elements invariant under ψ^{-1} on $\widetilde{K}^*(X)$, or equivalently those elements fixed under the $\mathbb{Z}_2$-action of $x \mapsto \bar{x}$, $u \mapsto -u$.

PROOF. It is clear that invariance under the $\mathbb{Z}_2$-action and ψ^{-1} is equivalent.

We describe the image in one degree at a time, say $-2n$. From Proposition 1.3.19

$$\operatorname{im} c = \widetilde{K}(\Sigma^{2n}X)^0 \otimes \mathbb{Z}[\tfrac{1}{2}].$$

By [**Kar78**, Proposition III.2.10], the Bott isomorphism induces isomorphisms

$$\beta^i \colon \widetilde{K}(X)^i \to \widetilde{K}(\Sigma^2 X)^{i+1}$$

for $i \in \mathbb{Z}_2$, so multiplication with u^n induces the isomorphism

$$\widetilde{K}(X)^\nu \xrightarrow{\cong} \widetilde{K}(\Sigma^{2n}X)^0,$$

where ν is the mod 2 reduction of n. From this the following equivalences follow,

$$u^n x \in \operatorname{im} c \iff x \in \widetilde{K}(X)^\nu \otimes \mathbb{Z}[\tfrac{1}{2}] \iff \bar{x} = (-1)^n x,$$

and hence $u^n x$ lies in the image of c if and only if $u^n x = (-u)^n \bar{x}$. Said differently; if and only if the element is invariant under the $\mathbb{Z}_2$-action described. The result then follows for each even degree, and the odd degrees can be treated similarly. □

4. Spectra and their relation to cohomology theories

In this section we will give a brief introduction to spectra. Rather than getting into the fine-grained details, we shall agree on notation and list formal properties that will be relevant later. Most results are collected from [**Ada95, Rud98, Swi02**].

DEFINITION 1.4.1. A **spectrum** E is a sequence $\{E_n, s_n \colon \Sigma E_n \to E_{n+1}\}_{n\in\mathbb{Z}}$ of pointed CW-complexes along with pointed maps s_n embedding ΣE_n as a subcomplex of E_{n+1}. The maps s_n are called the **structure maps**. A spectrum is called an **Ω-spectrum** if the adjoint maps $\sigma_n \colon E_n \to \Omega E_{n+1}$ of s_n are homotopy equivalences for all n.

A spectrum F is a **subspectrum** of E if F_n is a pointed subcomplex of E_n and the structure maps are the restrictions of those in E for all n.

For a spectrum E, let the **kth suspension** of E to be spectrum denoted by $\Sigma^k E$ and defined by $(\Sigma^k E)_n = E_{n+k}$ where the structure map $\Sigma(\Sigma^k E)_n \to (\Sigma^k E)_{n+1}$ is s_{n+k}.

The **suspension spectrum** of a pointed CW-complex X is written $\Sigma^\infty X$ and defined by $(\Sigma^\infty X)_n = \Sigma^n X$ for $n \geq 0$. For $n < 0$ we take $(\Sigma^\infty X)_n = pt$. The structure maps are identities for $n \geq 0$ and just the inclusion of the base point otherwise. The suspension spectrum will often just be denoted by X, when there is low risk of confusion. In particular, let the **sphere spectrum** S^0 be defined as the suspension spectrum $\Sigma^\infty S^0$. Note also that we shall use the notation S^n to denote $\Sigma^n S^0$.

A spectrum E is called **finite** if it is of the form

$$E = \Sigma^k \Sigma^\infty X,$$

for some finite CW-complex X and an integer k.

The spectra constitute the objects of a category, so we should look for morphisms. For this we refer the reader to [**Ada95, Rud98, Swi02**], but we note that the naive definition, i.e. demanding that a morphism $E \to F$ should be a sequence of pointed, cellular maps $\{f_n \colon E_n \to F_n\}$ such that $f_{n+1} \circ s_n = t_n \circ \Sigma f_n$ for all n greater than some n_0, is not good enough. A morphism should be a particular equivalence class of such sequences.

We can take the smash product between a spectrum E and a pointed CW-complex X, by defining $(E \wedge X)_n = E_n \wedge X$ and letting the structure maps be the obvious ones. In particular this allows us to form $E \wedge I_+$, where I_+ is the unit interval with a disjoint base point, which in turn will lead to the notion of homotopy. In effect, one gets the notion of homotopy classes of morphisms, and the set of these from E to F will be denoted $[E, F]$. We say that E and F are equivalent, and write $E \simeq F$, if there are morphisms $E \to F$

and $F \to E$ such that both compositions are homotopic to the identities. It can be shown that $E \wedge S^1 \simeq \Sigma E$; the spectrum obtained from spacewise suspension is equivalent to the suspended spectrum.

EXAMPLE 1.4.2. Let E_n be the nth space of a spectrum E. We have a canonical morphism of spectra $\Sigma^\infty E_n \to \Sigma^n E$ represented by the sequence of maps $f_k \colon (\Sigma^\infty E_n)_k \to (\Sigma^n E)_k$, where

$$f_k = \begin{cases} \mathrm{id}_{E_n}, & k = 0 \\ s_{n+k-1} \circ \cdots \circ s_n, & k > 0 \\ *, & k < 0. \end{cases}$$

This morphism is usually denoted by just $E_n \to \Sigma^n E$.

Define the **wedge** of two spectra $E \vee F$ by putting $(E \vee F)_n = E_n \vee F_n$ and letting the structure maps be $s_n = s_n^E \vee s_n^F$. One could hope that a similar definition would work for smash products as well, but it turns out that defining a good smash product in this category is nontrivial to say the least. Several constructions exist, and we recollect the formal properties in the following theorem. (See [**Rud98**, II.2]).

THEOREM 1.4.3. *There is an assignment sending spectra E and F to a spectrum $E \wedge F$ such that it is an equivalence-preserving covariant functor in each argument. Moreover, for any spectra E, F, G and pointed CW-complexes X, there are natural equivalences*

$$\begin{aligned} a &\colon (E \wedge F) \wedge G \to E \wedge (F \wedge G) \\ \tau &\colon E \wedge F \to F \wedge E \\ \Sigma &\colon \Sigma E \wedge F \to \Sigma(E \wedge F) \\ s &\colon E \wedge X \to E \wedge \Sigma^\infty X \end{aligned}$$

and natural equivalences $S^0 \wedge E \simeq E \simeq E \wedge S^0$. □

We are interested in spectra with additional structure.

DEFINITION 1.4.4. A **ring spectrum** E is a spectrum equipped with a **unit** $\iota \colon S^0 \to E$ and a **multiplication** $\mu \colon E \wedge E \to E$, such that the following diagrams commute up to homotopy.

$$\begin{tikzcd} S^0 \wedge E \arrow[r, "\iota \wedge 1"] \arrow[dr, "\simeq"'] & E \wedge E \arrow[d, "\mu"] & E \wedge S^0 \arrow[l, "1 \wedge \iota"'] \arrow[dl, "\simeq"] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} (E \wedge E) \wedge E \arrow[rr, "\mu \wedge 1"] \arrow[d, "a"] & & E \wedge E \arrow[d, "\mu"] \\ E \wedge (E \wedge E) \arrow[r, "1 \wedge \mu"] & E \wedge E \arrow[r, "\mu"] & E \end{tikzcd}$$

A ring spectrum E is called **commutative** if the diagram

$$\begin{tikzcd} E \wedge E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

is homotopy commutative.

A **ring morphism** $\varphi\colon E \to F$ of ring spectra is a morphism such that

$$\begin{tikzcd} E \wedge E \arrow[r, "\varphi \wedge \varphi"] \arrow[d, "\mu^E"] & F \wedge F \arrow[d, "\mu^F"] \\ E \arrow[r, "\varphi"] & F \end{tikzcd} \qquad \begin{tikzcd} & S^0 \arrow[dl, "\iota^E"'] \arrow[dr, "\iota^F"] & \\ E \arrow[rr, "\varphi"'] & & F \end{tikzcd}$$

commute up to homotopy.

Let E be a ring spectrum. F is called an E**-module spectrum** if it comes equipped with a morphism $m\colon E \wedge F \to F$ such that the diagrams

$$\begin{tikzcd} S^0 \wedge F \arrow[r, "\iota \wedge 1"] \arrow[dr, "\simeq"'] & E \wedge F \arrow[d, "m"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} (E \wedge E) \wedge F \arrow[rr, "\mu \wedge 1"] \arrow[d, "a"] & & E \wedge F \arrow[d, "m"] \\ E \wedge (E \wedge F) \arrow[r, "1 \wedge m"] & E \wedge F \arrow[r, "m"] & F \end{tikzcd}$$

commute up to homotopy. $g\colon F \to F'$ is an E**-module morphism** if the following diagram commutes up to homotopy,

$$\begin{tikzcd} E \wedge F \arrow[r, "1 \wedge g"] \arrow[d, "m"] & E \wedge F' \arrow[d, "m'"] \\ F \arrow[r, "g"] & F' \end{tikzcd}$$

where m and m' give F and F' their respective E-module structures.

Let E be a commutative ring spectrum. F is an E**-algebra spectrum** if it is a ring spectrum and a module spectrum over E.

To simplify diagrams, we will from now on omit mention of $a\colon (E \wedge F) \wedge G \xrightarrow{\simeq} E \wedge (F \wedge G)$ and write just $E \wedge F \wedge G$ for both of these spectra.

Example 1.4.5. The sphere spectrum S^0 is a ring spectrum. The multiplication is given by the homotopy equivalence $S^0 \wedge S^0 \xrightarrow{\simeq} S^0$ and the unit $S^0 \to S^0$ is just the identity.

For any ring spectrum E, the unit $\iota\colon S^0 \to E$ is a ring morphism. Furthermore, if $\varphi\colon E \to F$ and $\varphi'\colon E' \to F'$ are ring morphisms, then $\varphi \wedge \varphi'\colon E \wedge E' \to F \wedge F'$ is as well. In particular, assuming that E and F are ring spectra, the compositions

$$E \simeq E \wedge S^0 \xrightarrow{1 \wedge \iota^F} E \wedge F$$

and

$$F \simeq S^0 \wedge F \xrightarrow{\iota^E \wedge 1} E \wedge F$$

are ring morphisms.

For any pair E, F of spectra, the composition

$$\begin{aligned}[E,F] \times [E,F] \cong [E \wedge S^2, \Sigma^2 F] \times [E \wedge S^2, \Sigma^2 F] &\cong [E \wedge (S^2 \vee S^2), \Sigma^2 F] \\ &\to [E \wedge S^2, \Sigma^2 F] \cong [E,F],\end{aligned}$$

induced by the pinch map $S^2 \to S^2 \vee S^2$, gives $[E,F]$ the structure of an abelian group. In particular, the abelian group $\pi_n E := [S^n, E]$ is called the n**th homotopy group** of E. Analogous to the case with spaces, there is the concept of (co)homology theories on the category of spectra. We do not repeat the definition here, and refer the reader to [**Rud98**, Definition III.3.10]. Every spectrum gives rise to both a homology and a cohomology theory:

THEOREM 1.4.6. *For any spectrum E, the covariant functor from the category of spectra to the category of abelian groups defined by*

$$E_n(X) := \pi_n(E \wedge X),$$

for all n, gives rise to a homology theory denoted by $E_*(-)$. Similarly, the contravariant functors

$$E^n(X) := [X, \Sigma^n E]$$

define a cohomology theory $E^*(-)$. □

EXAMPLE 1.4.7. If E is a ring spectrum and X is any spectrum, the morphism $X \simeq S^0 \wedge X \xrightarrow{\iota\wedge 1} E \wedge X$ induces a homomorphism $\pi_*(X) \to E_*(X)$ which is called the **Hurewicz homomorphism**.

If X is merely a pointed CW-complex, we define

$$\widetilde{E}^*(X) := E^*(\Sigma^\infty X),$$

and $\widetilde{E}^*(-)$ becomes a reduced cohomology theory for pointed CW-complexes. We similarly define $\widetilde{E}_*(-)$ for pointed CW-complexes, and this becomes a homology theory. (See [**Rud98**, Construction 3.13].)

Demanding that E is a ring spectrum, the associated homology and cohomology theories get external products. Specifically, let X and Y be spectra and let $f \in E_p(X)$, $g \in E_q(Y)$. The external product $f \wedge g \in E_{p+q}(X \wedge Y)$ is defined as the element induced by the composition

$$S^p \wedge S^q \xrightarrow{f\wedge g} E \wedge X \wedge E \wedge Y \simeq E \wedge E \wedge X \wedge Y \xrightarrow{\mu\wedge 1\wedge 1} E \wedge X \wedge Y.$$

The product in cohomology is given in a similar way: For $f \in E^p(X)$ and $g \in E^q(Y)$, the product $f \wedge g \in E^{p+q}(X \wedge Y)$ is induced by

$$\Sigma^{-p}X \wedge \Sigma^{-q}Y \xrightarrow{f\wedge g} E \wedge E \xrightarrow{\mu} E.$$

It is immediate that the coefficient groups are related by (interpreting S^0 as a CW-complex)

$$\widetilde{E}_*(S^0) = \pi_* E = \widetilde{E}^{-*}(S^0),$$

and that $\pi_* E$ becomes a ring under these products, as is seen by setting $X = Y = S^0$. It is unital, and $\iota\colon S^0 \to E$ represents 1 in the ring. If the ring spectrum is commutative, then $\pi_* E$ is as well. We will use the notation E_* as a synonym for $\pi_* E$, while E^{-*} is $\pi_* E$ with the grading reversed.

Any morphism of spectra $\varphi\colon E \to F$ induces natural transformations of both homology and cohomology theories; specifically by

$$E_p(X) = [S^p, E \wedge X] \xrightarrow{(\varphi\wedge 1)_*} [S^p, F \wedge X] = F_p(X)$$

and

$$E^p(X) = [\Sigma^{-p}X, E] \xrightarrow{\varphi_*} [\Sigma^{-p}X, F] = F^p(X).$$

Moreover, if φ is a ring morphism, then it preserves external products.

The **Kronecker pairing** in E-theory is the homomorphism

$$\langle -, - \rangle\colon E^p(X) \otimes E_q(X) \to \pi_{q-p} E \tag{1.4.8}$$

defined by letting $\langle g, f \rangle$ be the element induced by the composition

$$S^{-p} \wedge S^q \xrightarrow{1\wedge f} S^{-p} \wedge E \wedge X \simeq S^{-p} \wedge X \wedge E \xrightarrow{g\wedge 1} E \wedge E \xrightarrow{\mu} E.$$

LEMMA 1.4.9. *Let $\varphi\colon E \to F$ be a ring morphism. Then the Kronecker pairing commutes with φ, that is, $\varphi\langle g, f\rangle = \langle \varphi g, \varphi f\rangle$.*

PROOF. By definition $\varphi\langle g, f\rangle$ is the upper and $\langle \varphi g, \varphi f\rangle$ the lower row in the following diagram.

$$\begin{tikzcd}
S^{-p}\wedge S^q \arrow[r, "1\wedge f"] \arrow[d, "="] & S^{-p}\wedge E\wedge X \arrow[r, "\simeq"] \arrow[d, "1\wedge\varphi\wedge 1"] & S^{-p}\wedge X\wedge E \arrow[r, "g\wedge 1"] \arrow[d, "1\wedge 1\wedge\varphi"] & E\wedge E \arrow[r, "\mu^E"] \arrow[d, "\varphi\wedge\varphi"] & E \arrow[r, "\varphi"] & F \arrow[d, "="] \\
S^{-p}\wedge S^q \arrow[r, "(\varphi\wedge 1)_* f"] & S^{-p}\wedge F\wedge X \arrow[r, "\simeq"] & S^{-p}\wedge X\wedge F \arrow[r, "\varphi_* g\wedge 1"] & F\wedge F \arrow[rr, "\mu^F"] & & F
\end{tikzcd}$$

All squares commute up to homotopy; the rightmost because φ is a ring morphism. $\square$

We now introduce some examples that will be important later.

EXAMPLE 1.4.10. For all abelian groups A and all $n \geq 1$, there are CW-complexes $K(A, n)$ called **Eilenberg–Mac Lane spaces**, which satisfy

$$\pi_m\bigl(K(A,n)\bigr) = \begin{cases} A, & m = n \\ 0, & m \neq n. \end{cases}$$

(Here π_m denotes ordinary homotopy of a space.) There are homotopy equivalences $K(A,n) \to \Omega K(A, n+1)$, so these spaces make up an Ω-spectrum with nth space $K(A,n)$ which we denote by HA. This is the **Eilenberg–Mac Lane spectrum**. When R is a ring, HR can be given the structure of a ring spectrum. (See [**Ada95**, Part 3].)

We note that for pointed CW-complexes X, we have natural isomorphisms $\widetilde{H}^*(X; R) \cong HR^*(\Sigma^\infty X)$, so the spectrum HR induces singular cohomology with coefficients in R.

EXAMPLE 1.4.11. There is a homotopy equivalence $\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}$ ([**Swi02**]). This allows us to define an Ω-spectrum by letting $E_{2n} = BU \times \mathbb{Z}$ and $E_{2n+1} = \Omega BU$, where the structure maps are given by the adjoints of the homotopy equivalences. This is a ring spectrum, and for finite pointed CW-complexes it is known that $\widetilde{K}^*(X) \cong E^*(\Sigma^\infty X)$, so the cohomology theory obtained from E coincides with complex K-theory. Therefore we rename E and call it K. This spectrum extends K-theory to both the category of CW-complexes and spectra, and thus when speaking of K-theory from now on, we will speak of this extended theory obtained from the spectrum K. We are therefore no longer restricted to the category of finite CW-complexes.

Similarly, there is a spectrum denoted KO which has period 8 extending real K-theory. The zeroth space of this spectrum is $BO \times \mathbb{Z}$.

4.1. Representability. We just saw that we can construct a homology and a cohomology theory from a spectrum. In addition, the examples of the previous section show that some familiar theories can actually be defined from spectra. It turns out that this is true in general: To a (co)homology theory, there is a spectrum which gives rise to the same (co)homology theory. The theorems are recalled below, and are often called the Brown–Adams representability theorems.

We start by considering cohomology theories defined on different categories, namely those having as objects pointed CW-complexes ($\mathscr{C}$), finite pointed CW-complexes ($\mathscr{C}_f$), spectra ($\mathscr{S}$) and finite spectra ($\mathscr{S}_f$). For $\mathscr{C}$ and $\mathscr{C}_f$, we demand that the cohomology theory is reduced. We use $\mathscr{D}$ to denote any of these four categories.

A cohomology theory $h^*(-)$ on $\mathscr{D}$ is **additive** if for any collection $\{X_\alpha\}$ of objects from $\mathscr{D}$ the homomorphism induced by the inclusions

$$h^*(\bigvee X_\alpha) \to \prod h^*(X_\alpha)$$

is an isomorphism, provided that $\vee X_\alpha$ is again an object of $\mathscr{D}$. A cohomology theory $h^*(-)$ on $\mathscr{D}$ is **represented** by the spectrum E if there is a natural isomorphism $h^*(-) \cong E^*(-)$ of cohomology theories on $\mathscr{D}$.

THEOREM 1.4.12. *Let $h^*(-)$ be an additive cohomology theory on $\mathscr{D}$. There is a spectrum E representing the cohomology theory, i.e. $h^*(-) \cong E^*(-)$, and this spectrum is unique up to equivalence.*

PROOF. See e.g. [**Rud98**, III.3] □

THEOREM 1.4.13. *Let $E^*(-)$ and $F^*(-)$ be cohomology theories on $\mathscr{D}$ represented by the spectra E and F respectively. Then any natural transformation $E^*(-) \to F^*(-)$ can be induced by a morphism of spectra $E \to F$. For cohomology theories on $\mathscr{S}$ this morphism is unique up to homotopy. On $\mathscr{S}_f$ this morphism is unique up to weak homotopy, i.e. if g and g' are two morphisms inducing this natural transformation, then for all finite spectra X and all morphisms $f\colon X \to E$ the compositions $g \circ f$ and $g' \circ f$ are homotopic.*

PROOF. Again, see [**Rud98**, III.3]. □

So in the case of cohomology theories on finite spectra, the representability theorem only provides a morphism that is "nearly" unique up to homotopy. The obstruction to guarantee uniqueness up to homotopy is the following class of morphisms.

DEFINITION 1.4.14. A morphism of spectra $\varphi\colon E \to F$ is a **phantom** if for all finite spectra X, any composition $X \to E \xrightarrow{\varphi} F$ is nullhomotopic. Write $\mathrm{Ph}^n(X, Y)$ for the set of homotopy classes of phantoms $X \to \Sigma^n Y$.

In terms of homotopy classes in $[E, F]$, two weakly homotopic morphisms g, g' satisfy $h^*[g - g'] = 0$ for all $h\colon X \to E$ where X is finite. Under the assumption that there are no phantoms except the null-homotopic morphisms, we may conclude that $[g - g] = 0$ and hence $g \simeq g'$. Therefore the non-existence of (non-trivial) phantoms, that is $\mathrm{Ph}^0(X, Y) = 0$, will imply that weakly homotopic morphisms are homotopic.

A **duality** between the spectra X and Y is a morphism $e\colon S^0 \to X \wedge Y$ such that for any spectrum W, the homomorphisms $[X, W] \to [S^0, W \wedge Y]$ and $[Y, W] \to [S^0, X \wedge W]$ defined by $f \mapsto (f \wedge 1) \circ e$ and $g \mapsto (1 \wedge g) \circ e$ are isomorphisms. X and Y are called **dual** if such a duality exists.

THEOREM 1.4.15. *Let X be a finite spectrum. There is up to homotopy a unique finite spectrum DX which is dual to X. Furthermore we have*

(1) *$DDX \simeq X$*
(2) *If both X and Y are finite spectra, then there is an equivalence*

$$h\colon D(X \wedge Y) \xrightarrow{\simeq} DX \wedge DY,$$

and this equivalence makes the diagram, where the e's are the indicated dualities,

$$\begin{tikzcd} S^0 \simeq S^0 \wedge S^0 \arrow[r, "e_X \wedge e_Y"] \arrow[d, "e_{X\wedge Y}"] & X \wedge DX \wedge Y \wedge DY \arrow[d, "1 \wedge \tau \wedge 1"] \\ X \wedge Y \wedge D(X \wedge Y) \arrow[r, "1 \wedge 1 \wedge h"] & X \wedge Y \wedge DX \wedge DY \end{tikzcd}$$

homotopy commutative.

PROOF. See [**Rud98**, II.2] and [**Ada95**, (II.5.6)]. □

We call DX the **Spanier–Whitehead dual** of X.

Let E be any spectrum. The Spanier–Whitehead dual gives a way to pass from E-homology to E-cohomology and vice versa. Using only properties of suspensions and smash products along with properties of the Spanier–Whitehead dual, we see that for a finite spectrum X

$$\begin{aligned} E_n(DX) = [S^n, E \wedge DX] \cong [S^0, \Sigma^{-n}(E \wedge DX)] \cong [S^0, \Sigma^{-n}E \wedge DX] \\ \cong [X, \Sigma^{-n}E] = E^{-n}(X). \end{aligned}$$

We say that a homology theory on $\mathscr{D}$ is **additive** if for any collection $\{X_\alpha\}$ of objects from $\mathscr{D}$ such that $\vee X_\alpha$ is an object of $\mathscr{D}$, the inclusions induce an isomorphism

$$\bigoplus h_*(X_\alpha) \to h_*(\bigvee X_\alpha).$$

Moreover, if $h_*(-)$ is an additive homology theory on $\mathscr{S}_f$, then $h^*(X) := h_{-*}(DX)$ defines an additive cohomology theory on $\mathscr{S}_f$. There are some technical points to made here; see [**Rud98**, Proposition II.2.10]. The point is that this cohomology theory is represented by a spectrum, and that this spectrum extends the homology and cohomology theories to $\mathscr{S}$.

We remark that Spanier–Whitehead duality behaves nicely with respect to the external products. To see this, assume that $\alpha\colon E_*(-) \to F_*(-)$ is a multiplicative natural transformation of represented homology theories. By Spanier–Whitehead duality, one obtains a natural transformation of cohomology theories on the category of finite spectra as the composition

$$\beta\colon E^*(X) \cong E_{-*}(DX) \xrightarrow{\alpha} F_{-*}(DX) \cong F^*(X).$$

From the diagram in Theorem 1.4.15 one readily sees that β is multiplicative as well, since the diagram

$$\begin{tikzcd} E^*(X) \otimes E^*(Y) \arrow[rr, "\wedge"] \arrow[dd, "\cong"] & & E^*(X \wedge Y) \arrow[d, "\cong"] \\ & & E_{-*}\big(D(X \wedge Y)\big) \arrow[d, "h_*"', "\cong"] \\ E_{-*}(DX) \otimes E_{-*}(DY) \arrow[rr, "\wedge"] & & E_{-*}(DX \wedge DY) \end{tikzcd}$$

commutes. This implies that Spanier–Whitehead duality is compatible with the external products, and that $\beta\colon E^*(-) \to F^*(-)$ is a multiplicative natural transformation of cohomology theories on finite spectra.

We conclude this section with the following remark. In light of the preceding comments, we will from now on treat the homology theory $h_*(-)$, the cohomology theory $h^*(-)$ and the representing spectrum as one. The convention will be to name the spectra according to the names of the (co)homology theories. In particular, the spectra representing the cohomology theories $K[\frac{1}{2}]^*(-)$ and $KO[\frac{1}{2}]^*(-)$ will be named $K[\frac{1}{2}]$ and $KO[\frac{1}{2}]$ respectively.

5. Bordism theories

We start this section by describing the representing spectra of some very important (co)homology theories.

Example 1.5.1. Consider the classifying spaces $BU(n)$. For each n there is a universal vector bundle $\xi_n \downarrow BU(n)$, such that for any complex rank n vector bundle $\xi \downarrow X$, there is a map $g\colon X \to BU(n)$ which is unique up to homotopy and classifies ξ, i.e. $g^*\xi_n \cong \xi$. We write $MU(n) := BU(n)^{\xi_n}$ for the associated Thom space.

In particular, the universality gives maps $c_{m,n}\colon BU(m)\times BU(n) \to BU(m+n)$, unique up to homotopy, such that $c^*_{m,n}\xi_{m+n} \cong \xi_m \times \xi_n$. Thomifying and using the homeomorphism (1.1.1) gives maps

$$\lambda_{m,n}\colon MU(m) \wedge MU(n) \to MU(m+n). \tag{1.5.2}$$

The inclusion $pt \hookrightarrow BU(1)$ classifies the trivial complex line bundle $\mathbf{1} \downarrow pt$. We get a map

$$j\colon S^2 \approx pt^{\mathbf{1}} \to BU(1)^{\xi_1} = MU(1),$$

and we form the composition

$$\Sigma^2 MU(n) = MU(n) \wedge S^2 \xrightarrow{1\wedge j} MU(n) \wedge MU(1) \xrightarrow{\lambda_{n,1}} MU(n+1). \tag{1.5.3}$$

We now define the spectrum MU. Let $MU_{2n} = MU(n)$ and $MU_{2n+1} = \Sigma MU(n)$. The structure maps s_{2n} are the identities and the s_{2n+1} are as in (1.5.3). This becomes a commutative ring spectrum, and it is the maps (1.5.2) that glue together to form the multiplication $\mu\colon MU \wedge MU \to MU$. The resulting (co)homology theory is called **complex (co)bordism**.

The same constructions carries over to give ring spectra MO and MSO from the classifying spaces $BO(n)$ and $BSO(m)$. Here the universal bundles are real, so every structure map is of the form (1.5.3), but with a single instead of a double suspension. Respectively, these spectra represent **unoriented** and **oriented (co)bordism**. Spectra obtained this way are called **Thom spectra**. See [**Rud98, Swi02**] for several more examples.

We take some time to mention that there is a canonical morphism of spectra $MU \to MSO$. We will outline how to obtain it, as it will be used later. The underlying real vector bundle of the universal bundle $\xi^U_n \downarrow BU(n)$ is canonically oriented and of real rank $2n$. Thus it is classified by a map $f_n\colon BU(n) \to BSO(2n)$, and this map can be taken to be the map of Grassmanns induced by taking a complex n-plane in $\mathbb{C}^\infty$ to the underlying oriented $2n$-plane of $\mathbb{R}^\infty$. (I.e. this map forgets the complex structure.) One can check that the diagram

$$\begin{tikzcd}
BU(n) \times pt \arrow[r] \arrow[d, "f_n \times 1"] & BU(n) \times BU(1) \arrow[r] \arrow[d, "f_n \times f_1"] & BU(n+1) \arrow[d, "f_{n+1}"] \\
BSO(2n) \times pt \arrow[r] & BSO(2n) \times BSO(2) \arrow[r] & BSO(2n+2)
\end{tikzcd}$$

commutes up to homotopy, where the horizontal maps on the right classify the Cartesian product of the universal bundles, and the ones on the left classify $\xi^U_n \times \mathbf{1}_{\mathbb{C}}$ and $\xi^{SO}_{2n} \times \mathbf{2}_{\mathbb{R}}$. Applying $\mathrm{Th}(-)$ to the f_n, we get vertical maps $g_n\colon MU(n) \to MSO(2n)$ which by this diagram commute with the structure maps. This collection of maps gives the desired morphism of spectra $\psi\colon MU \to MSO$.

Things are even better. One can show that this morphism is really a ring morphism. The clue to realizing this is to consider the homotopy commutative diagram

$$\begin{tikzcd}
BU(m) \times BU(n) \arrow[r] \arrow[d, "f_m \times f_n"] & BU(m+n) \arrow[d, "f_{m+n}"] \\
BSO(2m) \times BSO(2n) \arrow[r] & BSO(2m+2n)
\end{tikzcd}$$

where the horizontal maps classifies the products of universal bundles. Again, after Thomification one sees that the maps $g_n\colon MU(n) \to MSO(2n)$ are compatible with the $\lambda_{m,n}$ for both MU and MSO. The resulting ring morphism $\psi\colon MU \to MSO$ induces a multiplicative natural transformation in homology and cohomology, which we will call the **forgetful natural transformation**.

Analogously, one can obtain "forgetful" ring morphisms $MU \to MO$ and $MSO \to MO$.

Historically, the theories in this example arose geometrically in connection with smooth manifolds. We will discuss this geometric construction briefly, in the case of complex bordism. This construction carry over to give a lot of interesting variants of bordism (see [**Sto68, Swi02**]). In particular one can obtain unoriented and oriented bordism this way, but we note that in these two cases, one can do the construction more directly. This is done in [**Con79, Swi02**].

Any real vector bundle $\zeta \downarrow X$ of rank k is classified by a map $\zeta\colon X \to BO(k)$, which is unique up to homotopy. Here we have adopted the convention of using the same letter to denote both the vector bundle and the classifying map. As we remarked earlier, [**Dye69**] shows that any two normal bundles of a smooth manifold are stably equivalent, meaning that $\nu'\colon M \to BO(k')$ and $\nu''\colon M \to BO(k'')$ followed by the inclusions into $BO(l)$, for some $l \geq \max\{k', k''\}$, become homotopic. This yields a well-defined homotopy class of maps $M \to BO$, i.e. an element $\nu_M \in [M, BO]$ called the **stable normal bundle** of M.

There are maps $f_n\colon BU(n) \to BO(2n)$ analogous to those described in the example above, and these give a map $f\colon BU \to BO$ which can be taken to be a fibration.

Definition 1.5.4. Let M be a compact, smooth manifold with stable normal bundle ν_M. Assume that ν_M admits a lifting $\tilde{\nu}_M\colon M \to BU$ such that $f \circ \tilde{\nu}_M = \nu_M$. The homotopy class of $\tilde{\nu}_M$, $[\tilde{\nu}_M]$, is called a **stable complex structure** on M. The pair $(M, [\nu_M])$ is called a **stably complex** manifold.

We will on occasions write just M to denote the pair $(M, [\tilde{\nu}_M])$.

The fact that $f\colon BU \to BO$ is a fibration is what guarantees that a stable complex structure is independent on the representative chosen for $\nu\colon M \to BO$. To see this, consider homotopic representatives ν_0 and ν_1 via the homotopy $H\colon \nu_0 \simeq \nu_1$ and a lifting $\tilde{\nu}_0$ of ν_0. Then the outer diagram commutes in

$$\begin{tikzcd} & & BU \arrow[d, "f"] \\ M \arrow[urr, "\tilde{\nu}_0"] \arrow[r, hook, "i_0"'] & M \times I \arrow[ur, dashed, "\bar{H}"'] \arrow[r, "H"'] & BO, \end{tikzcd} \tag{1.5.5}$$

and since f is a fibration, H lifts to $\bar{H}\colon M \times I \to BU$. One thus obtains a lifting of ν_1, and this lifting is homotopic to $\tilde{\nu}_0$.

Definition 1.5.6. A closed stably complex manifold $(M, [\tilde{\nu}_M])$ of dimension m is called **null-bordant**, written $(M, [\tilde{\nu}_M]) \sim \varnothing$, if there is a compact $m+1$-dimensional manifold W having M as boundary, such that the stable normal bundle ν_W of W lifts in the diagram

$$\begin{tikzcd} & & BU \arrow[d, "f"] \\ M \arrow[urr, "\tilde{\nu}_M"] \arrow[r, hook] & W \arrow[ur, dashed, "\tilde{\nu}_W"'] \arrow[r, "\nu_W"'] & BO. \end{tikzcd}$$

We have omitted some technicalities regarding transversality from this definition; see [**Swi02**].

The set of stably complex manifolds of a fixed dimension is closed under disjoint union:

$$(M, [\tilde{\nu}_M]) \sqcup (N, [\tilde{\nu}_N]) := (M \sqcup N, [\tilde{\nu}_M \sqcup \tilde{\nu}_N])$$

Given $(M, [\tilde{\nu}_M])$ one obtains a stable complex structure on $M \times I$ by choosing the constant homotopy $H\colon \nu_M \simeq \nu_M$ in (1.5.5). The stable complex structure $\bar{H}$ restricts to give a new lifting of ν_M by which we denote $-\tilde{\nu}_M$ and which is defined by the composition

$$-\tilde{\nu}_M\colon M \xrightarrow{i_1} M \times I \xrightarrow{\bar{H}} BU.$$

From this and $\partial(M \times I) = M \sqcup M$, it follows from the previous definition that $(M, [\tilde{\nu}_M]) \sqcup (M, [-\tilde{\nu}_M]) \sim \varnothing$. If M has stable complex structure given by $[\nu_M]$, we use the notation $-M$ to denote $(M, [-\tilde{\nu}_M])$. Now define a relation on the set of stably complex manifolds of dimension n by letting $M \sim N$ if and only if $M \sqcup -N \sim \varnothing$. If this is the case, we say that M and N are **bordant**. By convention, one views $\varnothing$ as a stably complex manifold of any dimension, equipped with its unique stable complex structure. By [**Swi02**, Lemma 12.22] $\sim$ is an equivalence relation on this set, and we write Ω_n^U for the set of equivalence classes with respect to $\sim$. Write $[M]$ for the equivalence class of M. The following result is [**Swi02**, Lemma 12.23].

Proposition 1.5.7. *For all n, disjoint union makes Ω_n^U into an abelian group. The addition is given by $[M]+[N] = [M \sqcup N]$, the neutral element is $[\varnothing]$ and $-[M] = [-M]$.* □

We remark that the graded group $\Omega_*^U := \bigoplus_{n\geq 0} \Omega_n^U$ actually becomes a ring, where the multiplication is induced by the Cartesian product of stably complex manifolds (see [**Swi02**, (13.89)]).

Definition 1.5.8. A **singular (stably complex) manifold** in a topological space X is a pair (M, f) where M is stably complex and $f\colon M \to X$ is a continuous map. A singular manifold (M, f) is **null-bordant** if there is a pair $(W, F\colon W \to X)$, where W is a smooth, compact manifold such that M is null-bordant with respect to W in the sense of Definition 1.5.6, and F is a continuous map such that $F|_M = f$.

As above, the set of singular stably complex manifolds of dimension n is closed under disjoint union $(M, f) \sqcup (N, g) := \big(M \sqcup N, (f, g)\big)$, where (f, g) is the composition $M \sqcup N \xrightarrow{f \sqcup g} X \sqcup X \to X$. The manifold $(M, f) \sqcup (-M, f)$ is null-bordant. We say that (M, f) and (N, g) are **bordant**, and write $(M, f) \sim (N, g)$, if and only if $(M, f) \sqcup (-N, g)$ is null-bordant. Again, this is an equivalence relation, and the set of equivalence classes $[M, f]$ is written $\Omega_n^U(X)$. As before, this becomes an abelian group, where the addition is $[M, f] + [N, g] = [M \sqcup N, (f, g)]$, $[\varnothing, \varnothing]$ acts as a neutral element, and the inverse is given by $-[M, f] = [-M, f]$. Write

$$\Omega_*^U(X) := \bigoplus_{n\geq 0} \Omega_n^U(X)$$

for the graded abelian group.

Theorem 1.5.9 (Thom). *There is a natural isomorphism*

$$\Omega_*^U(X) \cong MU_*(X).$$

In particular, $\Omega_*^U \cong \pi_* MU$. □

We shall only sketch the construction of the homomorphism, as this makes use of the Pontrjagin–Thom construction. For details, see [**Swi02**, Theorem 12.30].

Let $(M, f\colon M \to X)$ be a singular n-dimensional manifold in X. The stable normal bundle $\nu\colon M \to BO$ lifts to $\tilde{\nu}\colon M \to BU$, and so there is some p such that $\nu\colon M \to BO(2p)$

lifts to $\tilde{\nu}\colon M \to BU(p)$. Here we are abusing notation and use ν (resp., $\tilde{\nu}$) to denote both the stable normal bundle and the representative (resp., the lifting and a representative of the lifting). The composition

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\tilde{\nu}\times f} BU(p) \times X$$

induces bundle maps

$$\nu \xrightarrow{\bar{\Delta}} \nu \times \nu \xrightarrow{\bar{\tilde{\nu}}\times f'} \xi_p \times \mathbf{0}$$

where ξ_p is the universal bundle over $BU(p)$, $\mathbf{0}$ is really just X and f' is the projection map $\nu \to M$ followed by $f\colon M \to X$. Applying $\mathrm{Th}(-)$ yields

$$\mathrm{Th}\big((\bar{\tilde{\nu}} \times f') \circ \bar{\Delta}\big)\colon M^\nu \to \big(BU(p) \times X\big)^{\xi_p \times \mathbf{0}} \approx MU(p) \wedge X_+. \tag{1.5.10}$$

We now apply the Pontrjagin–Thom construction to the smooth map $M \to pt$ and the embedding $e\colon M \to \mathbb{R}^{n+2p}$ with normal bundle ν. The collapse map becomes

$$c\colon S^{n+2p} \to M^\nu.$$

Composing with the map $M^\nu \to MU(p) \wedge X_+$ we obtain

$$S^{n+2p} \to MU(p) \wedge X_+.$$

It is possible to show that the choice of p is not critical; had one started with $p+1$ one would obtain a map $S^{n+2p+2} \to MU(p+1) \wedge X_+$, which makes the diagram

$$\begin{tikzcd}
\Sigma^2 S^{n+2p} \arrow[r] \arrow[d, "\approx"] & \Sigma^2 MU(p) \wedge X_+ \arrow[d, "s_{2n+1} \wedge 1"] \\
S^{n+2p+2} \arrow[r] & MU(p+1) \wedge X_+
\end{tikzcd}$$

commute. Thus these maps represent a morphism $g_f\colon S^n \to MU \wedge X_+$ which is an element of $\pi_n(MU \wedge X_+) = MU_n(X)$. The set map $(M, f) \mapsto g_f$ induces the isomorphism above.

As we mentioned in the discussion following Example 1.5.1, the construction of $\Omega_*^U(X)$ can be done for other structure groups as well, and the obvious analogues for Theorem 1.5.9 hold. This means that elements of bordism theories can be viewed as singular manifolds, and this is useful from time to time. In particular the coefficient rings $\pi_* MU$ and $\pi_* MSO$ are known:

Theorem 1.5.11. *The coefficient ring for complex bordism is a polynomial ring*

$$\pi_* MU \cong \Omega_*^U \cong \mathbb{Z}[x_2, x_4, \ldots]$$

with one generator in each even, positive dimension. Tensoring with $\mathbb{Q}$, it is the polynomial ring

$$\pi_* MU \otimes \mathbb{Q} \cong \Omega_*^U \otimes \mathbb{Q} \cong \mathbb{Q}\big[[\mathbb{C}P^1], [\mathbb{C}P^2], \ldots\big].$$

The coefficient ring for oriented bordism has only 2-primary torsion, and

$$\pi_* MSO/\text{torsion} \cong \Omega_*^{SO}/\text{torsion} \cong \mathbb{Z}[x_4, x_8, \ldots]$$

with one generator in each positive dimension divisible by 4. Tensoring with $\mathbb{Q}$, it is the polynomial ring

$$\pi_* MSO \otimes \mathbb{Q} \cong \Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}\big[[\mathbb{C}P^2], [\mathbb{C}P^4], \ldots\big].$$

Proof. See [**Ada95**], [**Sto68**] and [**Swi02**]. ∎

CHAPTER 2

Formal group laws

1. Definitions and properties

In this section we will define formal group laws and look at some of their general properties. These are formal power series, so we will not worry about convergence, and all functions we mention are identified with their power series expansion about 0.

Definition 2.1.1. Let R be a commutative ring with 1. A **(commutative) formal group law** over R is a power series

$$F(x,y)=\sum_{i,j\geq 0} a_{ij}x^i y^j \in R[\![x,y]\!]$$

subject to the following conditions:

(1) $F\big(F(x,y),z\big)=F\big(x,F(y,z)\big)$ (associativity)
(2) $F(x,0)=x=F(0,x)$ (identity)
(3) $F(x,y)=F(y,x)$ (commutativity)

Some comments are in order. From the identity axiom of the above definition, it follows directly that the coefficients $a_{i0}=\delta_{i1}$ and $a_{0j}=\delta_{j1}$. In other words we may write $F(x,y)=x+y+\sum_{i,j\geq 1} a_{ij}x^iy^j$. From the commutativity axiom, we have that $a_{ij}=a_{ji}$, for all i and j.

Note that commutativity is usually not a part of the definition of a formal group law. We will only encounter commutative formal group laws, however, and therefore we include this as an axiom. In our applications, R will always be a graded ring, and whenever this is the case, we demand that x and y are of homological degree 2 and that the formal group law is homogeneous. This means that the coefficients a_{ij} lie in $R_{2(i+j-1)}$.

Given a formal group law F, there is a unique power series $i(x)=\sum_{i\geq 1} a'_i x^i$ in $R[\![x]\!]$, called the **inverse** of F such that $F\big(x,i(x)\big)=0$. The proof of this is straight forward: consider

$$F(x,y)=x+\sum_{k\geq 1} a'_k x^k+\sum_{i,j\geq 1} a_{ij}x^i\Big(\sum_{k\geq 1} a'_k x^k\Big)^j.$$

Setting this equal to 0, we see that the coefficients a'_k are uniquely determined by the a_{ij} and lower a'_k. In particular note that the leading term of $i(x)$ is $-x$. If need be, we will use the notation $i_F(x)$ to indicate which formal group law this power series is the inverse to.

Example 2.1.2. The **additive formal group law**, denoted by F_a, is the simplest possible formal group law and is given by $F_a(x,y)=x+y$. The inverse is clearly $i_a(x)=-x$.

The **multiplicative formal group law**, F_m, is given by $F_m(x,y)=x+y+axy$, for some $a\in R$. Its inverse is $i_m(x)=-x+ax^2-a^2x^3+\cdots$.

To a formal group law F over R and any integer $n\geq 0$ one associates the n**-series**, denoted by

$$[n]_F(x)\in R[\![x]\!],$$

and defined recursively as

$$[n]_F(x) := \begin{cases} F\big([n-1]_F(x), x\big), & n > 0 \\ 0, & n = 0 \end{cases} \tag{2.1.3}$$

When there is only one formal group law around, we take the liberty to denote the n-series by $[n](x)$.

The n-series extend to negative integers as well, by letting

$$[-n](x) := [n]\big(i(x)\big).$$

These power series satisfy the formulas

$$\begin{aligned} [-n](x) &= i\big([n](x)\big) \\ [m+n](x) &= F\big([m](x), [n](x)\big) \\ [mn](x) &= [m]\big([n](x)\big) \end{aligned}$$

for all integers m and n. We also note that $[n](x)$ has leading term nx.

EXAMPLE 2.1.4. The n-series associated to F_a is simply $[n]_a(x) = nx$, and a simple calculation shows that the n-series for $F_m(x,y) = x + y + axy$ is

$$[n]_m(x) = \frac{(1+ax)^n - 1}{a}.$$

Let two formal group laws F and F' over R be given. A power series $f(x)$ of $R[\![x]\!]$ is said to be a **homomorphism** of formal group laws from F to F', written $f\colon F \to F'$, if it satisfies

$$f\big(F(x,y)\big) = F'\big(f(x), f(y)\big).$$

If in addition $f(x)$ has an inverse under composition, we say that $f(x)$ is an **isomorphism** of formal group laws. If $f(x)$ has leading term x, we call it a **strict isomorphism**. A homomorphism $f\colon F \to F$ is called an **endomorphism** of F.

Whenever $f(x)$ is an invertible power series over R, we may "twist" the formal group laws over R by it. We define

$$F^f(x,y) := f\Big(F\big(f^{-1}(x), f^{-1}(y)\big)\Big),$$

and f becomes an isomorphism of formal group laws $F \to F^f$. Note that by definition, $F^{g\circ f} = (F^f)^g$.

EXAMPLE 2.1.5. The n-series of a formal group law F is an endomorphism of F. We prove this using induction. Clearly,

$$[0]\big(F(x,y)\big) = 0 = F\big([0](x), [0](y)\big),$$

so assume that $[n-1]$ is an endomorphism. Then, using the properties of a formal group law, the calculation

$$\begin{aligned} [n]\big(F(x,y)\big) &= F\Big([n-1]\big(F(x,y)\big), F(x,y)\Big) \\ &= F\Big(F\big([n-1](x), [n-1](y)\big), F(y,x)\Big) \\ &= F\Big(F\big([n-1](x), [n](y)\big), x\Big) \\ &= F\Big([n](y), F\big([n-1](x), x\big)\Big) \\ &= F\big([n](x), [n](y)\big) \end{aligned}$$

completes the proof.

The following proposition describes homomorphisms in characteristic p, where p is any prime.

PROPOSITION 2.1.6. *Let R be a commutative $\mathbb{F}_p$-algebra. Let $f(x)$ be a non-zero power series such that $f\colon F \to F'$ is a homomorphism of formal group laws. Then $f(x) = g(x^{p^n})$, with $g'(0) \neq 0$, for some $n \geq 0$. In particular, $f(x)$ has leading term of degree p^n.*

PROOF. (See [**Rav86**, A2].) We construct a power series $f_n(x)$ by induction such that $f_n'(0) \neq 0$. Assume that we may write $f(x) = f_i(x^{p^i})$ for some $i \geq 0$. For $i = 0$ this is trivial: take $f_0(x) = f(x)$. If $f'(0) \neq 0$, then we are done, so assume that this is not the case, i.e. $f_i'(0) = 0$.

Every term of $F(x,y)^{p^i}$ is a monomial in x^{p^i} and y^{p^i}, so we may define a power series $F^{(i)}(x,y)$ by

$$F^{(i)}(x^{p^i}, y^{p^i}) := F(x,y)^{p^i}.$$

This becomes a formal group law over R, and f_i is a homomorphism of formal group laws $F^{(i)} \to F'$ since

$$\begin{aligned} f_i\big(F^{(i)}(x^{p^i}, y^{p^i})\big) = f_i\big(F(x,y)^{p^i}\big) = f\big(F(x,y)\big) &= F'\big(f(x), f(y)\big) \\ &= F'\big(f_i(x^{p^i}), f_i(y^{p^i})\big). \end{aligned}$$

Setting $u = x^{p^i}$ and $v = y^{p^i}$ and differentiating both sides of the previous equation with respect to v, we obtain

$$\begin{aligned} f_i'\big(F^{(i)}(u,v)\big)F_2^{(i)}(u,v) = \frac{\partial}{\partial v} f_i\big(F^{(i)}(u,v)\big) &= \frac{\partial}{\partial v} F'\big(f_i(u), f_i(v)\big) \\ &= F_2'\big(f_i(u), f_i(v)\big) f_i'(v). \end{aligned}$$

Now we evaluate in $v = 0$, and see that the right hand side is 0 because $f_i'(0) = 0$. Further note that $F_2^{(i)}(u,0)$ has leading term 1 and is therefore non-zero in R. This shows that $f_i'(x) = 0$ over R, and thus $f_i(x)$ is a polynomial in x^p.

Put $f_{i+1}(x^p) = f_i(x)$. Then $f(x) = f_{i+1}(x^{p^{i+1}})$, and we repeat the process. Since $f(x) \neq 0$, this process must stop, and thus we produce a power series $f_n(x)$, for some n, such that $f(x) = f_n(x^{p^n})$ and with $f_n'(0) \neq 0$. This is the desired $g(x)$. □

Since $[p](x) = px + \cdots$, the following corollary is immediate.

COROLLARY 2.1.7. *With R a commutative $\mathbb{F}_p$-algebra, the p-series associated to a formal group law over R is a power series in x^{p^n}, for some $n \geq 1$.* □

The number n is called the **height** of the formal group law over $\mathbb{F}_p$. By convention, one says that the formal group law has height ∞ if $[p](x) = 0$.

Sometimes there is a strict isomorphism $l\colon F \to F_a$ over R from a formal group law to the additive, but in general it does not exist. To see this, assume that $l(x)$ has leading term x and satisfies $l\big(F(x,y)\big) = l(x) + l(y)$. Differentiating with respect to y, and then putting $y = 0$ yields

$$l'(x)F_2(x,0) = l'\big(F(x,y)\big)F_2(x,y)|_{y=0} = l'(y)|_{y=0} = 1.$$

One obtains the unique solution

$$l(x) = \int_0^x \frac{\mathrm{d}t}{F_2(t,0)}$$

which in general is a power series over $R \otimes \mathbb{Q}$ (but not necessarily over R).

By analogy to calculus, one calls this power series the **logarithm** of F, and writes $\log_F(x) = x + \cdots \in (R \otimes \mathbb{Q})[\![x]\!]$. The inverse under composition is written $\exp_F$ and called the **exponential function** associated to F. These functions provide strict isomorphisms $\log_F : F \to F_a$ and $\exp_F : F_a \to F$ over $R \otimes \mathbb{Q}$. In particular,

$$F(x, y) = \exp_F\big(\log_F(x) + \log_F(y)\big),$$

and starting with any power series $l(x) = x + \cdots$ over $R \otimes \mathbb{Q}$, one can make a formal group law by defining

$$G(x, y) = l^{-1}\big(l(x) + l(y)\big) \in (R \otimes \mathbb{Q})[\![x, y]\!].$$

This discussion leads to the following observation.

LEMMA 2.1.8. *Assume that R is a $\mathbb{Q}$-algebra. There is a one-to-one correspondence between formal group laws over R and power series $l(x)$ with leading term x in $R[\![x]\!]$.* □

EXAMPLE 2.1.9. We find the logarithms and exponential functions of our two familiar examples.

(1) The logarithm and exponential function of the additive formal group law is $\log_a(x) = x = \exp_a(x)$.
(2) Let $F_m(x, y) = x + y + xy$ be the multiplicative formal group law with $a = 1$. We have

$$\log(1 + x + y + xy) = \log(1 + x) + \log(1 + y)$$

where $\log(x)$ is (the power series expansion of) the natural logarithm. Hence

$$\log_m(x) = \log(1 + x) = \sum_{n \geq 0} \frac{(-1)^n}{n + 1} x^{n+1}.$$

A substitution shows that $\log_m(x) = a^{-1} \log(1 + ax)$ for arbitrary a in R. It follows that the exponential function is $\exp_m(x) = a^{-1}(e^{ax} - 1)$.

Having established some facts about about formal group laws over rings, the natural thing to do next is to examine how they behave under ring homomorphisms. Let $g : R \to S$ be a homomorphism of commutative rings. It has canonical extensions to ring homomorphisms $R[\![x_1, \ldots, x_n]\!] \to S[\![x_1, \ldots, x_n]\!]$ by pushing the coefficients of a power series to S. Explicitly, let

$$F(x, y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j$$

be a formal group law over R and define a power series over S by

$$gF(x, y) := \sum_{i,j \geq 0} g(a_{ij}) x^i y^j.$$

The following properties follows readily from the definitions and uniqueness of inverses and logarithms.

PROPOSITION 2.1.10. *Let $g : R \to S$ be a ring homomorphism, and let F be a formal group law over R. Then the following statements are true.*

(1) *$gF(x, y)$ is a formal group law over S.*
(2) *Let $i_F(x) = \sum_{i \geq 1} a'_i x^i$ be the inverse of $F(x, y)$. The inverse of $gF(x, y)$ is given by*

$$i_{gF}(x) = \sum_{i \geq 1} g(a'_i) x^i.$$

(3) *The logarithm of $gF(x, y)$ is given by*

$$\log_{gF}(x) = \sum_{i \geq 0} (g \otimes 1)(m_i) x^i \in (S \otimes \mathbb{Q})[\![x]\!],$$

where $\log_F(x) = \sum_{i \geq 1} m_i x^i \in (R \otimes \mathbb{Q})[\![x]\!]$.

(4) *If $f\colon F \to F'$ is a homomorphism of formal group laws over R, then by pushing the coefficients of $f(x)$ one obtains a homomorphism $gF \to gF'$ over S.* □

2. Complex orientations of cohomology theories

We now introduce the main object of study, namely the complex orientable cohomology theories. We will recall some very important results from [**Ada95**].

DEFINITION 2.2.1. A ring spectrum E (and the cohomology theory it represents) are called **complex orientable** if there is an element $\omega \in \widetilde{E}^2(\mathbb{C}P^\infty)$ such that the inclusion $j\colon S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ induces the restriction

$$j^*\omega = \sigma^2(1) \in \widetilde{E}^2(S^2).$$

ω is called an **orientation class** and E (and $E^*(-)$) along with a choice of such an element are called **complex oriented**.

The choice of ω has some remarkable implications which we will soon get to. First we list a few examples, and these choices of orientations will be fixed throughout.

EXAMPLE 2.2.2. In singular cohomology theory $H^*(-)$ we choose $\omega_H = c_1(\eta)$, where $\eta \downarrow \mathbb{C}P^\infty$ is the canonical line bundle. This is an orientation class by Theorem 1.2.3.

Complex K-theory is complex orientable, and we choose the orientation class to be $\omega_K = \frac{1-\eta}{u}$ where $u \in \widetilde{K}(S^2)$ is the Bott element. It becomes an orientation class because $u = 1 - \eta_1$ where η_1 is the pullback of η by j.

Let the zero section be given by the map $\mathbb{C}P^\infty \to D\eta \to D\eta/S\eta = MU(1)$. It induces a canonical morphism of spectra $\mathbb{C}P^\infty \xrightarrow{\simeq} MU(1) \to \Sigma^2 MU$. This morphism represents the orientation class ω_{MU} in $\widetilde{MU}^2(\mathbb{C}P^\infty)$ (where $\mathbb{C}P^\infty$ is regarded as a space). $j^*\omega_{MU}$ is the composition $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \to \Sigma^2 MU$, which by desuspension becomes the unit $S^0 \to MU$.

One of the advantages of complex orientable theories is that spectral sequences often provide an efficient means of computing the cohomology of different spaces. For instance, we have analogues of (1.2.2) in such theories.

THEOREM 2.2.3. *Let E be complex oriented by ω. Then as E^*-algebras*

$$E^*(\mathbb{C}P^\infty) \cong E^*[\![\omega]\!]$$

and

$$E^*\big((\mathbb{C}P^\infty)^n\big) \cong E^*[\![\omega_1, \dots, \omega_n]\!],$$

where $\omega_i = \mathrm{pr}_i^*\,\omega$ is the pullback along the projection $\mathrm{pr}_i\colon (\mathbb{C}P^\infty)^n \to \mathbb{C}P^\infty$ onto factor i.

PROOF. See [**Ada95**, (II.2.5)]. □

We will later need to know the E-homology of $\mathbb{C}P^\infty$ and MU. We state this now as the following theorem.

THEOREM 2.2.4. *If E is complex oriented with orientation class ω, then*

(1) *$E_*(\mathbb{C}P^\infty)$ is the free E_*-module generated by β_n, $n \geq 0$, where $\beta_n \in E_{2n}(\mathbb{C}P^\infty)$ is the unique dual element of ω^n with respect to the Kronecker pairing (1.4.8)*

$$\langle -, - \rangle\colon E^*(\mathbb{C}P^\infty) \otimes E_*(\mathbb{C}P^\infty) \to \pi_* E,$$

that is, $\langle \omega^m, \beta_n \rangle = \delta_{mn}$ for all m.

(2) *The morphism of spectra $\mathbb{C}P^\infty \to \Sigma^2 MU$ induces a homomorphism*

$$\widetilde{E}_k(\mathbb{C}P^\infty) \to E_{k-2}(MU).$$

Define $b_n \in E_{2n}(MU)$ to be the image of β_{n+1}. Then $b_0 = 1$ and

$$E_*(MU) \cong E_*[b_1, b_2, \dots]$$

as E_*-algebras.

PROOF. See (II.4.1) and (II.4.5) of [**Ada95**]. □

Given one orientation of $E^*(-)$, it is easy to describe all orientations.

LEMMA 2.2.5. *Let E be a ring spectrum oriented by ω. The orientations of E are in one-to-one correspondence with the homogeneous power series $f(x)$ over E^* with leading term x.*

PROOF. By Theorem 2.2.3 any element of $\widetilde{E}^2(\mathbb{C}P^\infty)$ is of the form $f(\omega) = a_0\omega + a_1\omega^2 + a_2\omega^3 + \cdots$, where $a_i \in E^{-2i}$. Now $j\colon S^2 \hookrightarrow \mathbb{C}P^\infty$ induces a ring homomorphism in cohomology, so $j^*f(\omega) = a_0 j^*\omega + a_1(j^*\omega)^2 + \cdots$. The product in $\widetilde{E}^2(S^2)$ is trivial, so $j^*f(\omega) = a_0 j^*\omega = a_0\sigma^2(1)$, and hence $f(\omega)$ is an orientation class if and only if $a_0 = 1$. □

Now we state a remarkable result which characterizes all ring morphisms of ring spectra $MU \to E$ up to homotopy.

THEOREM 2.2.6. *Given an orientation ω_E of E there is up to homotopy a unique ring morphism of ring spectra $\varphi\colon MU \to E$ such that $\varphi\colon \widetilde{MU}^2(\mathbb{C}P^\infty) \to \widetilde{E}^2(\mathbb{C}P^\infty)$ maps ω_{MU} to ω_E.*

PROOF. See [**Ada95**, (II.4.6)]. □

Therefore, agreeing that ω_{MU} is to be kept fixed, choosing an orientation of E is equivalent to choosing such a homotopy class of ring morphisms.

$MU^*(-)$ carries universal Thom classes in a natural way. Specifically, the canonical morphisms $MU(n) \to \Sigma^{2n}MU$ represent elements

$$u_n \in \widetilde{MU}^{2n}\big(MU(n)\big),$$

that are Thom classes for the universal bundles $\xi_n \downarrow BU(n)$. Extending by naturality, one obtains Thom classes for all complex vector bundles.

These universal Thom classes are multiplicative in a specific sense. Recall from Example 1.5.1 the maps

$$\lambda_{m,n}\colon MU(m) \wedge MU(n) \to MU(m+n).$$

Then the induced map acts on Thom classes by

$$\lambda^*_{m,n}(u_{m+n}) = u_m \wedge u_n.$$

(See [**KT06**].) Extending by naturality, we have the following:

LEMMA 2.2.7. *Let $\xi \downarrow X$ and $\xi' \downarrow Y$ be complex vector bundles. Then the Thom classes in MU-theory satisfy*

$$\psi^* u_{\xi\times\xi'} = u_\xi \wedge u_{\xi'}. \tag{2.2.8}$$

where $\psi\colon X^\xi \wedge Y^{\xi'} \to (X\times Y)^{\xi\times\xi'}$ is as in (1.1.1).

PROOF. Let $f\colon X \to BU(m)$ and $g\colon Y \to BU(n)$ classify the respective vector bundles. Then $c_{m,n}\circ(f\times g)$ classifies $\xi\times\xi'$, and thus

$$\psi^*u_{\xi\times\xi'} = \psi^*(f\times g)^*c^*_{m,n}u_{m+n} = (f\wedge g)^*\lambda^*u_{m+n} = (f\wedge g)^*(u_m\wedge u_n) = u_\xi \wedge u_\xi,$$

completing the proof. □

A ring morphism of spectra $\varphi\colon MU \to E$ therefore gives $E^*(-)$ Thom classes $u_\xi^E = \varphi(u_\xi)$ which satisfy (2.2.8).

We briefly explain how a morphism of spectra can be constructed from a choice of $\omega_E \in \widetilde{E}^2(\mathbb{C}P^\infty)$: By the homotopy equivalence $\mathbb{C}P^\infty \simeq MU(1)$ the element u_1^E in $\widetilde{E}^2\big(MU(1)\big)$ corresponding to ω_E serves as a Thom class for the canonical line bundle $\eta \downarrow \mathbb{C}P^\infty$. Using the splitting principle and universality, one can choose Thom classes u_n^E for the universal bundles $\xi_n \downarrow BU(n)$ such that

$$\lambda_{m,n}^*\colon \widetilde{E}^*\big(MU(m+n)\big) \to \widetilde{E}^*\big(MU(m) \wedge MU(n)\big)$$

takes u_{m+n}^E to $u_m^E \wedge u_n^E$ (as in the MU-case). Each of the u_n^E define homotopy classes $[MU(n), \Sigma^{2n}E]$ and the multiplicative property ensures that these maps patch together to make a ring morphism $MU \to E$. See [**KT06**] for details.

2.1. Euler classes revisited. Let E be complex oriented, such that all complex vector bundles have Thom classes. Since we have defined Thom classes from the universal case, they are automatically natural, and we are in the setting of the remark following Theorem 1.1.4.

The existence of Thom classes for all complex vector bundles implies the existence of Euler classes. Moreover, we have the following

Proposition 2.2.9. *Let E be complex oriented. The Euler classes are both natural and multiplicative with respect to the external product, i.e. $e(\xi \times \xi') = e(\xi) \times e(\xi')$.*

Proof. This follows from the properties of Thom classes in E-theory. Specifically, let $g\colon Y \to X$ be continuous and let ξ be a vector bundle over X. The diagram, where z is the zero section

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "z"] & X \arrow[d, "z"] \\ Y^{g^*\xi} \arrow[r, "g"] & X^\xi \end{tikzcd}$$

commutes and this implies that

$$e(g^*\xi) = z^*g^*u_\xi = g^*z^*u_\xi = g^*e(\xi),$$

so the Euler classes are natural.

Now let $\xi \downarrow X$ and $\xi' \downarrow Y$ be complex vector bundles, and denote by z the zero sections in all the Thom spaces X^ξ, $Y^{\xi'}$ and $(X \times Y)^{\xi\times\xi'}$. Further, let q denote the projection $A \times B \to A \wedge B$ of based spaces which relates the external products $\times$ and $\wedge$ by $q^*(a \wedge b) = a \times b$ as in (1.3.1). We have a commutative diagram

$$\begin{tikzcd} X \times Y \arrow[r, "z"] \arrow[d, "z \times z"] & (X \times Y)^{\xi\times\xi'} \\ X^\xi \times Y^{\xi'} \arrow[r, "q"] & X^\xi \wedge Y^{\xi'} \arrow[u, "\psi"'] \end{tikzcd},$$

and by it we see that

$$e(\xi \times \xi') = z^*(u_\xi \times u_{\xi'}) = (z\times z)^*q^*\psi^*(u_\xi \times u_{\xi'}) = (z\times z)^*(u_\xi \times u_{\xi'}) = e(\xi) \times e(\xi'),$$

which shows the multiplicative property. □

Letting $X = Y$ and using naturality with respect to the diagonal map $\Delta\colon X \to X \times X$, one has:

Corollary 2.2.10. *The Euler class is exponential, that is, for ξ and ξ' complex vector bundles over X, the Euler class satisfies $e(\xi \oplus \xi') = e(\xi) \smile e(\xi')$.* □

In particular this means that the Euler classes in E-theory are determined by naturality and the splitting principle from $e(\eta) = \omega_E$.

Example 2.2.11. In singular cohomology, the Euler class of a line bundle is given by $e_H(\ell) = c_1(\ell)$. Applying the splitting principle on a rank n complex vector bundle ξ, one readily sees that $e_H(\xi) = c_n(\xi) \in H^{2n}(X)$.

The Euler class in K-theory on a line bundle is $e_K(\ell) = \frac{1-\ell}{u} = u^{-1}\lambda_{-1}(\ell)$. Since λ_t is natural and exponential (Lemma 1.3.13) the splitting principle now yields

$$e_K(\xi) = u^{-n}\lambda_{-1}(\xi) = u^{-n}\big(1 - \Lambda^1(\xi) + \Lambda^2(\xi) - \cdots + (-1)^n\Lambda^n(\xi)\big) \in K^{2n}(X)$$

for any complex vector bundle of rank n.

3. Formal group laws associated to cohomology theories

We are about to see that we can associate formal group laws to complex orientable cohomology theories. This is really due to Theorem 2.2.3, which told us that

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[\![\omega_1, \omega_2]\!]$$

where the ω_i are pullbacks of the orientation class ω.

Since $\mathbb{C}P^\infty$ is the classifying space for complex line bundles, one may in particular find a map

$$m\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

(unique up to homotopy) classifying the line bundle $\eta_1 \otimes \eta_2$, where η_i as usual denotes the canonical line bundle over the ith factor $\mathbb{C}P^\infty$. Any choice of representative for this map induces the same E^*-algebra homomorphism

$$m^*\colon E^*[\![x]\!] \to E^*[\![x_1, x_2]\!]$$

where we have written x for $\omega = e(\eta)$. m^* is uniquely determined by its value on the generator x of $E^*(\mathbb{C}P^\infty)$, and

$$m^*(x) = \sum_{i,j\geq 0} a_{ij}x_1^i x_2^j =: F(x_1, x_2)$$

is a homogeneous power series of cohomological degree 2. We claim that $F(x_1, x_2)$ is actually a formal group law.

Lemma 2.3.1. *Assume that $\ell \downarrow X$ and $\ell' \downarrow Y$ are complex line bundles. The power series F obtained from the orientations of $E^*(-)$ satisfies*

$$e(\ell \mathbin{\widehat{\otimes}} \ell') = F\big(e(\ell), e(\ell')\big)$$

as elements of $E^*(X \times Y)$, where $e(-)$ is the Euler class associated to the orientation.

Proof. In the universal case, we have

$$e(\eta_1 \otimes \eta_2) = e(m^*\eta) = m^*\big(e(\eta)\big) = m^*(x) = F(x_1, x_2) = F\big(e(\eta_1), e(\eta_2)\big)$$

using naturality of Euler classes. Letting $f \times g\colon X \times Y \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ pull $\eta \mathbin{\widehat{\otimes}} \eta$ back to $\ell \mathbin{\widehat{\otimes}} \ell'$, we see that

$$\begin{aligned} e(\ell \mathbin{\widehat{\otimes}} \ell') = e\big((f\times g)^*(\eta \mathbin{\widehat{\otimes}} \eta)\big) &= (f\times g)^* F\big(e(\eta_1), e(\eta_2)\big) \\ &= F\big(e((f\times g)^*\eta_1), e((f\times g)^*\eta_2)\big) = F\big(e(\ell), e(\ell')\big) \end{aligned}$$

and this completes the proof. □

PROPOSITION 2.3.2. *Every complex oriented cohomology theory $E^*(-)$ determines a formal group law*

$$F_E(x,y) = x + y + \sum_{i,j\geq 1} a_{ij}x^i y^j,$$

which is homogeneous of cohomological degree 2 in $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ and determined by $m^*(x)$. The coefficients a_{ij} lie in $E^{2(1-i-j)}$.

PROOF. Everything except that F is a formal group law is clear. The calculation

$$\begin{aligned} F\big(x_1, F(x_2,x_3)\big) &= F\big(e(\eta_1), e(\eta_2\otimes\eta_3)\big) = e\big(\eta_1\otimes(\eta_2\otimes\eta_3)\big) \\ &= e\big((\eta_1\otimes\eta_2)\otimes\eta_3\big) = F\big(e(\eta_1\otimes\eta_2), e(\eta_3)\big) = F\big(F(x_1,x_2),x_3\big) \end{aligned}$$

shows that F is associative, using associativity of the tensor product. Similarly, commutativity follows from commutativity of tensor products. The identity axiom is verified by using the trivial line bundle over $\mathbb{C}P^\infty$. □

REMARK. It may be convenient to view an associated formal group law as a power series over E_* rather than E^*. In this case, let x and y have homological degree -2 and let the coefficients a_{ij} lie in $E_{2(i+j-1)}$.

We will often use the following corollary.

COROLLARY 2.3.3. *Let $E^*(-)$ be a complex orientable cohomology theory. If ω and $\omega' = \theta(\omega)$ are orientations giving rise to formal group laws F and F' respectively, then $\theta\colon F \to F'$ is a strict isomorphism of formal group laws.*

PROOF. Let e and e' denote the respective Euler classes. It follows that

$$\theta\big(F(\omega_1,\omega_2)\big) = \theta\big(e(\eta_1\otimes\eta_2)\big) = e'(\eta_1\otimes\eta_2) = F'(\omega_1',\omega_2') = F'\big(\theta(\omega_1),\theta(\omega_2)\big),$$

and since $\theta(x) = x + \cdots$ the result follows. □

We make the following useful observation, which is a consequence of the universality of $MU^*(-)$ as a complex oriented cohomology theory.

PROPOSITION 2.3.4. *Let $E^*(-)$ be complex oriented and let $\varphi\colon MU \to E$ be the corresponding ring morphism. The induced map on coefficients $\varphi\colon MU^* \to E^*$ classifies the formal group law determined by the orientation class of $E^*(-)$.*

PROOF. This is shown by the calculation

$$\begin{aligned} \varphi F_{MU}\big(e_E(\eta_1), e_E(\eta_2)\big) &= \varphi\big(F_{MU}\big(e_{MU}(\eta_1), e_{MU}(\eta_2)\big)\big) \\ &= \varphi\big(e_{MU}(\eta_1\otimes\eta_2)\big) \\ &= e_E(\eta_1\otimes\eta_2) \\ &= F_E\big(e_E(\eta_1), e_E(\eta_2)\big). \end{aligned}$$

□

We end this section with two examples.

EXAMPLE 2.3.5. As H^* only has non-zero coefficients in degree 0, the associated formal group law must be the additive; $F_H(x,y) = x + y$.

Recall that the Euler class for complex K-theory is $e_K(\ell) = u^{-1}(1-\ell)$ on a complex line bundle. As the ring structure in K-theory is given by tensor product of vector bundles, we obtain

$$\begin{aligned} e_K(\eta_1\otimes\eta_2) &= \frac{1-\eta_1\eta_2}{u} \\ &= \frac{(1-\eta_1)+(1-\eta_2)-(1-\eta_1)(1-\eta_2)}{u} \\ &= e_K(\eta_1) + e_K(\eta_2) - u e_K(\eta_1)e_K(\eta_2). \end{aligned}$$

In other words, the formal group law associated to K-theory is the multiplicative formal group law:

$$F_K(x,y) = x + y - uxy.$$

4. The universality of complex bordism

We have seen that $MU^*(-)$ is the universal complex oriented cohomology theory, and that a consequence of this is that the ring morphism $\varphi\colon MU \to E$ orienting E classifies the formal group law associated to $E^*(-)$ with this orientation. A surprising fact is that not only does F_{MU} determine all formal group laws associated to complex oriented cohomology theories, F_{MU} determines *all* formal group laws. This is the main result in [**Qui69**], and it is stated below in Theorem 2.4.8. Unfortunately, the proof of this important theorem is not in the scope of this thesis. However, we will consider the algebraic side of things, and construct such a universal formal group law, which must necessarily be the same (up to isomorphism). This will reveal to us valuable information about MU_* and its formal group law.

4.1. Lazard's ring. We construct the universal ring for formal group laws.

THEOREM 2.4.1. *There is a commutative ring L and a formal group law F^u over L, with the following universal property: For any unital commutative ring R and G any formal group law over R, there is a unique ring homomorphism $g\colon L \to R$ such that $gF^u = G$.*

REMARK. The standard argument shows that any ring equipped with a formal group law with this universal property must be isomorphic to L and that this isomorphism is unique. The isomorphism preserves the formal group law, and so it makes sense to talk about *the* universal formal group law. Moreover, if $g\colon L \to R$ is an isomorphism of rings, R and gF^u attains this universal property.

PROOF. We construct this ring directly. Let P be the polynomial algebra

$$P = \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots],$$

and let F be the power series over P defined by

$$F(x,y) = x + y + \sum_{i,j\geq 1} a_{ij} x^i y^j.$$

This is not a formal group law (it is neither associative nor commutative), but we can make it become one over a certain quotient ring.

Write

$$F\big(F(x,y),z\big) - F\big(x,F(y,z)\big) = \sum_{i,j,k\geq 1} c_{ijk} x^i y^j z^k,$$

where each c_{ijk} is a polynomial in a_{ij}. We let I be the ideal of P generated by the c_{ijk} and all polynomials of the form $a_{ij} - a_{ji}$. Now we form the quotient ring $L = P/I$, and by pushing the coefficients with the projection $P \to L$, F becomes the formal group law F^u over L.

To see that F^u is the universal formal group law, let R be any ring and G a formal group law over R. We write

$$G(x,y) = x + y + \sum_{i,j\geq 0} b_{ij} x^i y^j.$$

In order for g to be a ring homomorphism $L \to R$ such that $gF^u = G$, we have no choice but to send 1 to 1 and the a_{ij} to b_{ij}. Thus g exists and is unique. $\square$

L is known as **Lazard's ring**, and we call F^u the **universal formal group law**. The above construction is done without mention of grading. We will usually impose a grading on L by choosing the a_{ij} to be of homological degree $2(i+j-1)$, making the universal formal group law homogeneous of homological degree -2.

The construction of L in the proof above gives explicit generators and relations in the different degrees, so we may in principle compute the structure of L in each degree. In particular we immediately note that $L_0 \cong \mathbb{Z}$ (generated by 1). Assuming Lemma 2.4.3 below, we shall prove:

THEOREM 2.4.2 (Lazard). *L is a polynomial algebra over $\mathbb{Z}$,*

$$L \cong \mathbb{Z}[x_1, x_2, \ldots],$$

where the generators x_n have homological degree $2n$.

The proof will be done in several intermediate steps. First we recall a fact about binomial coefficients ([**Fin47**]). If $d_n := \gcd\left(\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}\right)$, then

$$d_n = \begin{cases} p, & n = p^s \\ 1, & \text{otherwise.} \end{cases}$$

For $n \geq 1$, we define homogeneous power series by

$$C_n(x,y) := \frac{1}{d_n}\left((x+y)^n - x^n - y^n\right),$$

which evidently are primitive (i.e. the greatest common divisor of the coefficients is 1).

LEMMA 2.4.3 (Lazard Comparison Lemma). *If F and G are formal group laws over R such that $F \equiv G \pmod{(x,y)^n}$ then there exists an element $a \in R$ such that*

$$F \equiv G + aC_n \pmod{(x,y)^{n+1}}.$$

PROOF. See [**Rav86**, A2.1.12]. $\square$

Let $R = \mathbb{Z}[b_1, b_2, \ldots]$ where b_n is of homological degree $2n$, and write $\exp(x) = \sum_{n \geq 0} b_n x^{n+1}$ where $b_0 = 1$. The reason for the name exp is clear in light of the following lemma: exp will become the universal exponential function for formal group laws strictly isomorphic to the additive.

LEMMA 2.4.4. *The formal group law $F_a^{\exp}$ over R is universal for formal group laws $G \downarrow S$ over any commutative ring which are strictly isomorphic to $F_a \downarrow S$.*

PROOF. Let $g\colon F_a \to G$ be the isomorphism in question, and write

$$g(x) = \sum_{n \geq 0} c_n x^{n+1},$$

where $c_0 = 1$. There is a unique ring homomorphism $\psi\colon R \to S$ mapping 1 to 1 and b_n to c_n. Let $\log = \exp^{-1}$. Then

$$\begin{aligned} \psi F_a^{\exp}(x,y) &= \psi\big(\exp\big(\log(x) + \log(y)\big)\big) \\ &= \psi \exp\big(\psi \log(x) + \psi \log(y)\big) \\ &= g\big(g^{-1}(x) + g^{-1}(y)\big) \\ &= G(x,y). \end{aligned}$$

$\square$

This lemma allows us to determine $L \otimes \mathbb{Q}$, which in turn will be an important ingredient to finding the structure of L:

PROPOSITION 2.4.5. *$L \otimes \mathbb{Q}$ is isomorphic to the polynomial ring $\mathbb{Q}[b_1, b_2, \ldots]$.*

PROOF. Let $L \to R \otimes \mathbb{Q} \cong \mathbb{Q}[b_1, b_2, \ldots]$ be the unique ring homomorphism classifying $F_a^{\exp}$. This homomorphism factors uniquely as $L \to L \otimes \mathbb{Q} \xrightarrow{f} R \otimes \mathbb{Q}$ to give $fF^u = F_a^{\exp}$. We now recall that over $\mathbb{Q}$-algebras, every formal group law is strictly isomorphic to the additive, and so there is a unique homomorphism $\psi\colon R\otimes\mathbb{Q} \to L\otimes\mathbb{Q}$ such that $\psi F_a^{\exp} = F^u$. By uniqueness of the classifying homomorphisms, $f\psi = 1$ and $\psi f = 1$. $\square$

For any abelian group A, and $n > 1$, let $A[2n-2]$ be the graded abelian group having A in degree $2n-2$ and 0 in all other degrees. One makes $\mathbb{Z} \oplus A[2n-2]$ into a graded commutative ring by defining the multiplication to be $(m, a)(n, b) = (mn, na + mb)$. By considering degrees, one sees that any formal group law over this ring is of the form

$$F(x, y) = x + y + \sum_{i+j=n} a_{ij} x^i y^j.$$

Using the comparison lemma, we will make a much sharper statement.

LEMMA 2.4.6. *Every formal group law F over $\mathbb{Z} \oplus A[2n-2]$ is of the form*

$$F(x, y) = x + y + aC_n(x, y),$$

where a is some element in A. In effect, there is a bijection between the elements of A and formal group laws over $\mathbb{Z} \oplus A[2n-2]$.

PROOF. Taking an arbitrary formal group law $F(x, y) = x + y + \sum_{i+j=n} a_{ij} x^i y^j$ over $\mathbb{Z} \oplus A[2n-2]$ one has $F \equiv F_a \pmod{(x, y)^n}$. By the comparison lemma, there is an element a in A such that $F \equiv F_a + aC_n \pmod{(x, y)^{n+1}}$.

We now show that the right hand side of this congruence is a formal group law. We view F as a formal group law over the ring localized away from d_n (the greatest common divisor of the coefficients of $C_n(x, y)$). For $b \in A$, define the power series

$$g_b(x) = x + bx^n \in (\mathbb{Z} \oplus A[2n-2])[d_n^{-1}][\![x]\!],$$

and note that $g_{-b} = g_b^{-1}$. We produce a formal group law over this localized ring by

$$F_a^{g_b}(x, y) = g_b\big(g_{-b}(x) + g_{-b}(y)\big) = x + y + b\big((x + y)^n - x^n - y^n\big).$$

In particular, for the element $a \in A$ from above,

$$F_a^{g_{a/d_n}}(x, y) = x + y + \frac{a}{d_n}\big((x + y)^n - x^n - y^n\big),$$

which actually has coefficients in $\mathbb{Z} \oplus A[2n-2]$ and hence is a formal group law over this ring. Furthermore, it coincides with $F_a + aC_n$. This implies that $F = F_a + aC_n$, as we wanted to show. $\square$

Let R be a graded ring which has $R_0 = \mathbb{Z}$ and $R_k = 0$ for $k < 0$. Such a ring is called a **connected graded ring**. The ideal of positively graded elements I is the kernel of the augmentation $R \to \mathbb{Z}$. Define the **indecomposable module** of R to be the graded module $Q_*(R) := I/I^2$. Examples of connected graded rings are L and $\mathbb{Z} \oplus A[2n-2]$. Denote by $\varepsilon\colon L \to \mathbb{Z}$ and $\varepsilon_{2n-2}\colon \mathbb{Z} \oplus A[2n-2] \to \mathbb{Z}$ the respective augmentation homomorphisms.

LEMMA 2.4.7. *The graded ring homomorphisms $L \to \mathbb{Z} \oplus A[2n-2]$ correspond bijectively to the group homomorphisms $Q_{2n-2}(L) \to A$.*

PROOF. Assume that a group homomorphism $Q_{2n-2}(L) \to A$ has been given. The composition

$$L \to \mathbb{Z} \oplus Q_{2n-2}(L) \to \mathbb{Z} \oplus A[2n-2]$$

gives a map of graded rings, where the first map is the projection of generators and the second is induced in the obvious way from the group homomorphism.

For any graded ring homomorphism $f\colon L \to \mathbb{Z} \oplus A[2n-2]$ the diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[dr, "\varepsilon"'] & \mathbb{Z} \oplus A[2n-2] \arrow[d, "\varepsilon_{2n-2}"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes and induces a homomorphism $I \to A[2n-2]$ of the kernels of ε and ε_{2n-2}. Since $A[2n-2]^2 = 0$, the map factors via $Q_*(L) = I/I^2$ to give a group homomorphism $Q_{2n-2}(L) \to A$.

These assignments are inverse to one another, as is seen by unraveling the definitions. □

The previous lemmas and the universal property of L imply that for any abelian group A we have one-to-one correspondences of sets:

(1) Elements of A.
(2) Formal group laws over $\mathbb{Z} \oplus A[2n-2]$.
(3) Ring homomorphisms $L \to \mathbb{Z} \oplus A[2n-2]$.
(4) Group homomorphisms $Q_{2n-2}(L) \to A$.

Proof of Theorem 2.4.2. For all abelian groups A and all $n > 0$, there is a bijection of sets $\mathrm{Hom}_{\mathbb{Z}}(Q_{2n}(L), A) \approx A$. $Q_{2n}(L)$ is finitely generated, so putting $A = \mathbb{Z}/(p)$ for all primes p shows that $Q_{2n}(L) \cong \mathbb{Z}$.

$L \to Q_*(L)$ is graded and an epimorphism, so we can pick an $x_n \in L_{2n}$ projecting to a generator of $Q_{2n}(L)$ for all n. This gives a ring homomorphism

$$f\colon \mathbb{Z}[x_1, x_2, \ldots] \to L.$$

$R := \mathbb{Z}[x_1, x_2, \ldots]$ is also a connected graded ring, so we can form $Q_*(R)$. By construction, f induces an epimorphism $Q_*(R) \to Q_*(L)$, and it is known that this implies that f is an epimorphism itself. (For a proof, see [**NS02**, Appendix A].) Tensoring with $\mathbb{Q}$ gives another epimorphism

$$\mathbb{Q}[x_1, x_2, \ldots] \to L \otimes \mathbb{Q} \cong \mathbb{Q}[b_1, b_2, \ldots].$$

In each degree, this is a linear surjection of vector spaces over $\mathbb{Q}$ having the same dimension, and hence this is an isomorphism. The diagram

$$\begin{tikzcd} \mathbb{Z}[x_1, x_2, \ldots] \arrow[r] \arrow[d, hook] & L \arrow[d] \\ \mathbb{Q}[x_1, x_2, \ldots] \arrow[r, "\cong"] & L \otimes \mathbb{Q} \end{tikzcd}$$

commutes, showing that $\mathbb{Z}[x_1, x_2, \ldots] \to L$ must be injective. From this follows that $L \cong \mathbb{Z}[x_1, x_2, \ldots]$. □

4.2. Universality of MU_* and MU_*MU. The next theorem tells us that Lazard's ring can be identified with MU_*. For a proof see [**Qui69, Ada95**].

Theorem 2.4.8 (Quillen). *The unique ring homomorphism $L \to MU_*$ characterizing F_{MU} is an isomorphism.* □

Remark. This theorem implies that the MU formal group law is universal. Moreover, by Theorem 2.4.2, the coefficient ring of MU has the structure of an evenly graded polynomial algebra over $\mathbb{Z}$. (This was also claimed in Theorem 1.5.11.) The coefficient ring is actually generated by the coefficients of F_{MU}, by construction of the Lazard ring. We adopt the notation F^u to denote the universal formal group law over MU_* (and MU^*).

In his proof, Quillen used the following theorem characterizing the logarithm of the formal group law for MU. We shall make use of this result later. A proof can be found in [**Ada95**].

THEOREM 2.4.9 (Mischenko). *The logarithm of the universal formal group law F^u is given by*

$$\log_{MU}(x) = \sum_{n\geq 0} \frac{[\mathbb{C}P^n]}{n+1} x^{n+1}$$

viewed as an element of $(MU_* \otimes \mathbb{Q})[\![x]\!]$. □

Recall from Theorem 2.2.4 that $E_*(MU) \cong E_*[b_1, b_2, \ldots]$. Define the **exponential function** associated to E to be the power series

$$\exp^E(x) = \sum_{n\geq 0} b_n x^{n+1} \in E_*(MU)[\![x]\!].$$

It should not be confused with the exponential function $\exp_E$, which is a power series defined on $\pi_* E \otimes \mathbb{Q}$ associated to the formal group law of $E^*(-)$. As usual, the inverse of $\exp^E$ is called the **logarithm** and denoted by $\log^E$.

The canonical morphisms from Example 1.4.5

$$E \longrightarrow E \wedge E' \longleftarrow E'$$

induces ring homomorphisms $\eta_L \colon E_* \to E_*(E')$ and $\eta_R \colon E'_* \to E_*(E')$ in homotopy. When E and E' are complex oriented, this procedure yields two formal group laws over $E_*(E')$, namely $\eta_L F_E$ and $\eta_R F_{E'}$.

If $E' = MU$ we are able to say a lot about how these formal group laws are connected.

THEOREM 2.4.10. *Let $E^*(-)$ be complex oriented. $\exp^E$ is a strict isomorphism $\eta_L F_E \to \eta_R F^u$ of formal group laws over $E_*(MU) \cong E_*[b_1, b_2, \ldots]$.*

PROOF. [**Ada95**, (6.5)]. The diagram

$$\begin{tikzcd}
{[\mathbb{C}P^\infty, \Sigma^k MU]} \arrow[rr] \arrow[dr, "\alpha"'] & & {[\mathbb{C}P^\infty, E \wedge \Sigma^k MU]} \arrow[dl, "p"] \\
& \mathrm{Hom}_{\pi_* E}\big(E_*(\mathbb{C}P^\infty), E_*(\Sigma^k MU)\big) &
\end{tikzcd}$$

where $\alpha(f)(h) = f_*(h)$ and $p(g)(h)$ is given by the composition

$$S^p \xrightarrow{h} E \wedge \mathbb{C}P^\infty \xrightarrow{1\wedge g} E \wedge E \wedge \Sigma^k MU \xrightarrow{\mu\wedge 1} E \wedge \Sigma^k MU$$

commutes. $E \wedge MU$ is a module spectrum over E the obvious way, and then we may apply [**Ada95**, (4.2)] to conclude that p is an isomorphism. In particular, we may choose $\omega_{MU} \colon \mathbb{C}P^\infty \to \Sigma^2 MU$ and go around the diagram. By definition of the b_i, we have $\alpha(\omega_{MU})(\beta_{i+1}) = b_i$, so we obtain $p(\bar{\omega}_{MU})(\beta_{i+1}) = b_i$ where we have written $\bar{\omega}_{MU}$ for the image of ω_{MU} in $[\mathbb{C}P^\infty, E \wedge \Sigma^2 MU]$.

On the other hand, (by definition of the β_i) we have $p(\bar{\omega}_E^j)(\beta_i) = \delta_{ij}$, so

$$p(\bar{\omega}_{MU}) = \sum_{j\geq 0} b_j p(\bar{\omega}_E^{j+1}).$$

Since p is an isomorphism, $\bar{\omega}_{MU} = \exp^E(\bar{\omega}_E)$, hence $\exp^E$ is the desired strict isomorphism. □

Now we put $E = MU$. η_R is the Hurewicz homomorphism, and it gives MU_*MU a right MU_*-module structure. Similarly, η_L gives MU_*MU a left MU_*-module structure. η_L sends elements a of MU_* to the constant polynomial a in $MU_*MU \cong MU_*[b_1, b_2, \ldots]$. (See [**Rav86**] for more on this.)

We are now ready to discuss the universality of MU_*MU in the context of formal group laws. Consider the commutative ring R and two ring homomorphisms $f, g\colon MU_* \to R$, classifying the formal group laws F and G respectively. We assume further that these formal group laws are related by a strict isomorphism $\theta\colon F \to G$.

PROPOSITION 2.4.11. *The strict isomorphism* $\exp^{MU}\colon \eta_L F^u \to \eta_R F^u$ *is universal amongst strict isomorphisms of formal group laws; specifically:*

Let two formal group laws F and G over R be classified by f and g and strictly isomorphic via $\theta\colon F \to G$. Then there is a unique ring homomorphism $\rho\colon MU_*MU \to R$ such that $\rho \exp^{MU}(x) = \theta(x)$ (by pushing coefficients) and such that

$$\begin{tikzcd} MU_* \arrow[r, "\eta_L"] \arrow[dr, "f"'] & MU_*MU \arrow[d, dashed, "\rho"] & MU_* \arrow[l, "\eta_R"'] \arrow[dl, "g"] \\ & R & \end{tikzcd}$$

commutes.

PROOF. Write
$$\exp^{MU}(x) = \sum_{n \geq 0} b_n x^{n+1} \in MU_*MU[\![x]\!]$$
and
$$\theta(x) = \sum_{n \geq 0} c_n x^{n+1} \in R[\![x]\!].$$
For this proof we put $\exp = \exp^{MU}$ and $\log = \log^{MU}$ to simplify notation.

We construct ρ directly. Since $MU_*MU \cong MU_*[b_1, b_2, \ldots]$ is a polynomial algebra, it is sufficient to specify what ρ does to the coefficients and the indeterminates. In order to take $\exp$ to θ, it must necessarily send b_n to c_n. Since η_L embeds MU_* as the constant polynomials in MU_*MU, commutativity of the left triangle implies that $\rho(a) = f(a)$ for all $a \in MU_*$. This uniquely determines a ring homomorphism $\rho\colon MU_*MU \to R$.

It remains to show that this ρ also makes the right triangle commute. The idea is to show that $\rho\eta_R F^u = G$. If this is the case, then both g and $\rho\eta_R$ are ring homomorphisms $MU_* \to R$ classifying G, and by uniqueness of the classifying homomorphism, we may conclude that $\rho\eta_R = g$. This is shown by the following calculation,
$$\begin{aligned} \rho\eta_R F^u(x, y) &= \rho\big(\exp\big(\eta_L F^u(\log x, \log y)\big)\big) \\ &= \rho \exp\big(\rho\eta_L F^u(\rho \log x, \rho \log y)\big) \\ &= \theta\big(F(\rho \log x, \rho \log y)\big) \\ &= G\big(\theta(\rho \log x), \theta(\rho \log y)\big) \\ &= G\big(\rho\big(\exp(\log x)\big), \rho\big(\exp(\log y)\big)\big) \\ &= G(x, y). \end{aligned}$$
□

5. The exact functor theorem

Take a homology theory $E_*(-)$ and an E_*-module G. Then we can define a candidate for a new homology theory by putting
$$G_*(-) = G \otimes_{E_*} E_*(-).$$

This will in general not be a homology theory, because tensor product does not preserve exactness. However, if $E = MU$, Theorem 2.5.2 below will provide us with a purely algebraic criterion for when $G_*(-)$ behaves correctly on cofiber sequences, and the candidate functor actually is a homology theory.

Let R be a commutative graded ring with 1 which is concentrated in even degrees and is a MU_*-module via the ring homomorphism $\varphi\colon MU_* \to R$. Then φ classifies the formal group law F over R obtained by pushing the coefficients of F^u over MU_*. Fix a prime p and let $u_n \in R$ be the coefficient of x^{p^n} in the p-series

$$[p]_F(x) = px + \cdots + u_1 x^p + \cdots + u_2 x^{p^2} + \cdots . \tag{2.5.1}$$

Recall that a sequence of elements $(a_1, a_2, \ldots)$ from R is said to be **regular** if multiplication by a_1 in R and a_{n+1} in $R/(a_1, \ldots, a_n)$ is injective for all $n \geq 1$.

We now state the exact functor theorem.

Theorem 2.5.2 (Landweber exact functor theorem, [**Lan76, LRS95**]). *As above, let $\varphi\colon MU_* \to R$ be a ring homomorphism, where R is graded and concentrated in even degrees. If the sequence $(p, u_1, u_2, \ldots)$ is regular in R for all primes p, then the functor*

$$R_*(-) = R \otimes_{MU_*} MU_*(-)$$

is a homology theory with coefficients R. □

Remark. The theorem is still valid without the assumption that R is concentrated in even degrees, but this restriction does not exclude any examples that are relevant to us. Additionally, this assumption allows us to make use of some nice results obtained in [**HS99**].

The homology theory produced is multiplicative; $R_*(-)$ gets the external product from $MU_*(-)$, namely by $(r \otimes x) \wedge (s \otimes y) = rs \otimes (x \wedge y)$. The canonical natural transformation $MU_*(-) \to R_*(-)$, $x \mapsto 1 \otimes x$, is multiplicative.

Spanier–Whitehead duality and Theorem 1.4.12 now gives a representing spectrum R. We note that the corresponding cohomology theory is explicitly given on finite spectra by

$$R^*(X) \cong R_{-*}(DX) = R \otimes_{MU_*} MU_{-*}(DX) \cong R^\bullet \otimes_{MU^*} MU^*(X),$$

where $R^\bullet$ is the ring R with the grading reversed. The Milnor short exact sequence [**Swi02**, (10.4)] applied to the system $\mathbb{C}P^1_+ \hookrightarrow \mathbb{C}P^2_+ \hookrightarrow \cdots \hookrightarrow \mathbb{C}P^\infty_+$ is

$$0 \to \lim{}^1 R^{*-1}(\mathbb{C}P^n) \to R^*(\mathbb{C}P^\infty) \to \lim R^*(\mathbb{C}P^n) \to 0,$$

and the $\lim^1$-term can be seen to vanish by the Mittag-Leffler condition, since for all n, the induced map

$$R^*(\mathbb{C}P^{n+1}) \cong R^\bullet \otimes_{MU^*} MU^*(\mathbb{C}P^{n+1}) \to R^\bullet \otimes_{MU^*} MU^*(\mathbb{C}P^n) \cong R^*(\mathbb{C}P^n)$$

is an epimorphism. It follows that

$$\begin{aligned} R^*(\mathbb{C}P^\infty) &\cong \lim R^*(\mathbb{C}P^n) \\ &\cong \lim R^\bullet \otimes_{MU^*} MU^*(\mathbb{C}P^n) \\ &\cong \lim R^\bullet \otimes_{MU^*} MU^*[\![\omega]\!]/(\omega^{n+1}) \\ &\cong \lim R^\bullet[\![\omega]\!]/(\omega^{n+1}) \\ &\cong R^\bullet[\![\omega]\!] \\ &\cong R^\bullet \otimes_{MU^*} MU^*(\mathbb{C}P^\infty). \end{aligned}$$

Proposition 2.20 of [**HS99**] now guarantees the existence of a unique commutative ring spectrum structure on the representing spectrum R, such that R is a complex oriented

MU-algebra spectrum and such that the multiplication $\mu\colon R\wedge R \to R$ induces the external product in homology:

$$R_*(X) \otimes R_*(Y) \to R_*(X \wedge Y).$$

An orientation class for $R^*(-)$ is $\omega_R := 1 \otimes \omega$, and we write $\varphi\colon MU \to R$ for the ring morphism corresponding to this orientation class. The associated multiplicative natural transformation on finite spectra (and $\mathbb{C}P^\infty$) is the composition

$$MU^*(-) \cong MU^* \otimes_{MU^*} MU^*(-) \xrightarrow{\varphi\otimes 1} R^\bullet \otimes_{MU^*} MU^*(-) = R^*(-). \tag{2.5.3}$$

Also note that by Corollary 2.3.4, this implies that $F_R(x, y) = \varphi F^u(x, y)$.

A ring homomorphism $\varphi\colon MU_* \to R$ satisfying the theorem will be called **Landweber exact**. We shall also occasionally call the resulting (co)homology theories **Landweber exact**. We remark that the coefficient ring of any Landweber exact (co)homology theory has no torsion. This is obvious from the requirement that multiplication by p should act injectively for all primes.

The exact functor theorem will be crucial when constructing elliptic cohomology theories in Chapter 3. We now give a couple of examples of how the theorem gives rise to homology theories that are already familiar to us.

EXAMPLE 2.5.4. Let $\varphi\colon MU_* \to \mathbb{Q}$ classify the additive formal group law $F_a(x, y) = x + y$. Multiplication by p on $\mathbb{Q}$ is an isomorphism, so $\mathbb{Q}/(p) = 0$. Thus the criteria of the exact functor theorem are met, so $\mathbb{Q} \otimes_{MU_*} MU_*(-)$ is a homology theory. By taking X to be a point, we see that the coefficient ring is $\mathbb{Q}$.

On the other hand, consider the Thom homomorphism $\mu\colon MU_*(X) \to H_*(X)$. By definition, it takes equivalence classes $[M^n, f\colon M \to X]$ of singular manifolds in $MU_n(X)$ to $f_*([M]) \in H_n(X)$. Here $[M] \in H_n(X)$ is the fundamental class of M. The Thom homomorphism is natural in X, and composing with $H_*(-) \to H\mathbb{Q}_*(-)$ we obtain a natural transformation $MU_*(-) \to H\mathbb{Q}_*(-)$. This natural transformation factors through the natural transformation $t\colon \mathbb{Q} \otimes_{MU_*} MU_*(-) \to H\mathbb{Q}_*(-)$ defined by $q \otimes x \mapsto q \cdot \mu(x)$. t is an isomorphism for X a point, and thus t is a natural isomorphism of homology theories

$$\mathbb{Q} \otimes_{\pi_* MU} MU_*(-) \cong H\mathbb{Q}_*(-).$$

EXAMPLE 2.5.5. As a second example, let $\varphi\colon MU_* \to \mathbb{Z}[u, u^{-1}]$ classify the multiplicative formal group law $F_m(x, y) = x + y - uxy$. We see that $[p]_m(x) = px + \cdots + (-u)^{p-1}x^p$. Further, multiplication by p on $\mathbb{Z}[u, u^{-1}]$ is injective, and so is multiplication by the unit $(-1)u^{p-1}$ on $\mathbb{Z}_p[u, u^{-1}]$. We apply Theorem 2.5.2 and conclude that $\mathbb{Z}[u, u^{-1}] \otimes_{MU_*} MU_*(-)$ is a homology theory.

Again, evaluating on a point reveals that the coefficient ring coincide with the coefficients $\pi_* K$ of K-theory. As in the case with rational cohomology above, there is a natural transformation

$$\mathbb{Z}[u, u^{-1}] \otimes_{MU_*} MU_*(-) \to K_*(-),$$

which is an isomorphism on a point. See [**Lan76**] for details.

EXAMPLE 2.5.6. Example 3.4 in [**Lan76**] shows that $KO[\frac{1}{2}]_*(-)$ is a Landweber exact homology theory. This fact will be crucial later, and we consider some of the details in Corollary 3.1.25. We also note that $KO_*(-)$ is not Landweber exact, since the coefficient ring has torsion. (In fact, it is not even complex orientable.)

If $E^*(-)$ is a complex orientable cohomology theory oriented by ω, we have seen that homogeneous power series of the form $\theta(x) = x + \cdots$ give every complex orientation of $E^*(-)$. It is natural to ask if a Landweber exact cohomology theory gotten from $\varphi\colon MU_* \to R$ stays Landweber exact upon change of orientation. This is indeed the case.

More precisely, denote by t the multiplicative natural transformation $t\colon MU^*(-) \to R^*(-)$ such that $t(\omega_{MU}) = \omega_R$. Further, let t_θ be the multiplicative natural transformation $t_\theta\colon MU^*(-) \to R^*(-)$ sending ω_{MU} to the orientation class $\theta(\omega_R)$. By assumption, $t\colon MU_* \to R$ is Landweber exact, and we will show that this implies that $t_\theta\colon MU_* \to R$ is Landweber exact as well.

t classifies the formal group law $F_R(x,y) = tF^u(x,y)$. Moreover, Corollary 2.3.3 shows that the formal group law classified by $t_\theta\colon MU_* \to R$ is precisely

$$F_R^\theta(x,y) = \theta\big(F_R\big(\theta^{-1}(x), \theta^{-1}(y)\big)\big).$$

LEMMA 2.5.7. *Let F and F' be strictly isomorphic formal group laws over a ring R via $f\colon F \to F'$. Then the n-series are related by*

$$f\big([n]_F(x)\big) = [n]_{F'}\big(f(x)\big).$$

PROOF. This is obviously true for $n = 0$, so we assume that it also holds for $n-1$ with $n \geq 1$. Then

$$f\big([n]_F(x)\big) = F'\big(f\big([n-1]_F(x)\big), f(x)\big) = F'\big([n-1]_{F'}\big(f(x)\big), f(x)\big) = [n]_{F'}\big(f(x)\big),$$

and so the claim holds by induction. $\square$

It follows from this lemma that

$$[p]_{F_R^\theta}(x) = \theta\big([p]_{F_R}\big(\theta^{-1}(x)\big)\big),$$

and so Landweber exactness of $t_\theta\colon MU_* \to R$ will follow from the fact that the specific sequence of coefficients from $[p]_{F_R}(x)$ stays regular in R under the conjugation with a strict isomorphism.

PROPOSITION 2.5.8. *If a complex oriented cohomology theory $R^*(-)$ is Landweber exact, then $R^*(-)$ is Landweber exact for all complex orientations.*

PROOF. Fix an arbitrary prime p and denote by ω the chosen orientation class for $R^*(-)$. With everything as above, we have, by assumption of Landweber exactness, that the sequence $(p, u_1, u_2, \ldots)$ of coefficients from $[p]_{F_R}(x)$ is regular in R. We want to show that for a homogeneous power series $\theta(x)$ with leading term x, the sequence $(p, u_1^\theta, u_2^\theta, \ldots)$ is regular in R, where u_i^θ is the coefficient of x^{p^i} in $[p]_{F_R^\theta}(x)$.

Recall from Example 2.1.5 that the p-series are endomorphisms of formal group laws. $R/(p)$ is a commutative $\mathbb{F}_p$-algebra and since the p-series remain endomorphisms after taking quotients, we may apply Corollary 2.1.7 to see that $[p]_{F_R}(x)$ (resp., $[p]_{F_R^\theta}(x)$) is a power series in x^p over $R/(p)$ with the coefficient of x^p being u_1 (resp., u_1^θ). As $\theta(x)$ has leading term x, we may conclude that $u_1 = u_1^\theta$ in $R/(p)$.

We proceed inductively. Assume that $u_i = u_i^\theta$ in $R/(p, u_1, \ldots, u_{i-1})$ for $1 \leq i \leq n-1$. Again using Corollary 2.1.7, $[p]_{F_R}(x)$ is a power series in x^{p^n} over $R/(p, u_1, \ldots, u_{n-1})$. It has no lower terms, since we have forced the coefficients of the lower x^{p^i} to be zero. Furthermore, the coefficient of x^{p^n} is u_n.

Similar considerations hold for $[p]_{F_R^\theta}(x)$: the coefficient of x^{p^n} is u_n^θ and lies in the ring $R/(p, u_1^\theta, \ldots, u_{n-1}^\theta)$, which by induction is equal to $R/(p, u_1, \ldots, u_{n-1})$. Thus in this ring, $u_n = u_n^\theta$, since $\theta(x)$ has leading term x.

It follows that the sequence $(p, u_1^\theta, u_2^\theta, \ldots)$ is regular. Since this holds for all primes p, $R^*(-)$ is Landweber exact with the orientation class $\theta(\omega)$. $\square$

6. A natural transformation of Landweber exact theories

In this section we shall review the construction of a natural transformation which is due to [**Mil89**]. Miller states, without giving a proof, that this natural transformation has certain properties. In this section we shall provide proofs, using results of [**HS99**]. Specifically, we shall prove:

THEOREM 2.6.1 (Miller [**Mil89**]). *Let R and S be commutative graded rings with* 1 *that are concentrated in even degrees. Assume that $\varphi_R\colon MU_* \to R$ and $\varphi_S\colon MU_* \to S$ are ring homomorphisms which are Landweber exact and let F_R and F_S denote the associated formal group laws. Assume further that $\lambda\colon R \to S$ is a ring homomorphism and that $\theta(x) = x + \cdots \in S[\![x]\!]$ is a strict isomorphism of formal group laws $\theta\colon F_S \to \lambda F_R$. Then there is a multiplicative natural transformation*

$$\hat{\lambda}\colon R^*(-) \to S^*(-),$$

which is induced by a unique ring morphism of representing spectra $R \to S$, subject to the following requirements:

(1) *On coefficients, $\hat{\lambda} = \lambda\colon R^* \to S^*$.*
(2) *On orientation classes, $\hat{\lambda}(\omega_R) = \theta(\omega_S)$.*

To this end, let the setting be as in the theorem. Note that the assumption that R and S are concentrated in even degrees is essential for our proof, but was not assumed in [**Mil89**].

Using the universality of MU_*MU, we construct a candidate for the natural transformation in homology. The idea is then to dualize with Spanier–Whitehead duality to get a natural transformation in cohomology with the desired properties.

By Proposition 2.4.11, we obtain a unique ring homomorphism $\rho\colon MU_*MU \to S$ pushing $\exp^{MU}(x)$ to $\theta^{-1}(x)$ such that the diagram commutes.

(2.6.2)
$$\begin{tikzcd} MU_* \arrow[r, "\eta_L"] \arrow[d, "\varphi_R"] & MU_*MU \arrow[d, dashed, "\rho"] & MU \arrow[l, "\eta_R"'] \arrow[dl, "\varphi_S"] \\ R \arrow[r, "\lambda"] & S & \end{tikzcd}$$

Recall from [**Ada69**, Lecture 3] the existence of a "coaction map"

$$\psi_X\colon MU_*(X) \to MU_*MU \otimes_{MU_*} MU_*(X),$$

defined for all spectra X. We give its definition here. Firstly, the composition

$$MU \wedge X \simeq MU \wedge S^0 \wedge X \xrightarrow{1\wedge\iota\wedge 1} MU \wedge MU \wedge X$$

induces a homomorphism $h\colon MU_*(X) \to MU_*(MU \wedge X)$. Secondly, the homomorphism

$$m\colon MU_*MU \otimes_{MU_*} MU_*(X) \to MU_*(MU \wedge X)$$

defined by sending $f \otimes g$ to

$$S^p \wedge S^q \xrightarrow{f\wedge g} MU \wedge MU \wedge MU \wedge X \xrightarrow{1\wedge\mu\wedge 1} MU \wedge MU \wedge X$$

is an isomorphism by [**Rav86**, Lemma 2.2.7]. One defines $\psi_X := m^{-1} \circ h$, which is natural in X. ψ_X is a left MU_*-module homomorphism.

We now use $\psi = \psi_X$ to produce a natural transformation between the Landweber exact homology theories. Consider the diagram

$$\begin{tikzcd}
MU_*(X) \arrow[r, "\psi"] \arrow[d, "\cong"] & MU_*MU \otimes_{\eta_R} MU_*(X) \arrow[dd, "\rho \otimes 1"] \\
MU_* \otimes_{MU_*} MU_*(X) \arrow[d, "\varphi_R \otimes 1"] & \\
R \otimes_{\varphi_R} MU_*(X) \arrow[r, dashed, "\hat{\lambda}"] & S \otimes_{\varphi_S} MU_*(X)
\end{tikzcd} \tag{2.6.3}$$

where all the tensor products are over MU_*, and the indices on the tensor products reflects which module structure is being used. Define $\hat{\lambda}$ to be the composition $(\rho\otimes 1)\circ\psi$ on $MU_*(X)$ extended linearly to all of $R \otimes_{\varphi_R} MU_*(X)$. Explicitly, put

$$\hat{\lambda}(r \otimes x) = \lambda(r) \cdot (\rho \otimes 1) \circ \psi(x). \tag{2.6.4}$$

Lemma 2.6.5. *$\hat{\lambda}\colon R_*(X) \to S_*(X)$ is well-defined.*

Proof. Let $x \in MU_*(X)$ and $m \in MU_*$. Write $\psi(x) = \sum p_i \otimes x_i$, and use that ψ is a left MU_*-module homomorphism to get

$$\begin{aligned}
\hat{\lambda}(r \cdot m \otimes x) &= \lambda\big(r\varphi_R(m)\big) \cdot (\rho \otimes 1)\Big(\sum p_i \otimes x_i\Big) \\
&= \lambda(r)\lambda\big(\varphi_R(m)\big) \sum \rho(p_i) \otimes x_i
\end{aligned}$$

and

$$\begin{aligned}
\hat{\lambda}(r \otimes m \cdot x) &= \lambda(r) \cdot (\rho \otimes 1)\psi(m \cdot x) \\
&= \lambda(r) \cdot (\rho \otimes 1)\Big(m \cdot \sum p_i \otimes x_i\Big) \\
&= \lambda(r) \sum \rho\big(\eta_L(m)p_i\big) \otimes x \\
&= \lambda(r)\rho\big(\eta_L(m)\big) \sum \rho(p_i) \otimes x.
\end{aligned}$$

By (2.6.2) we see that they are indeed equal. □

Lemma 2.6.6. *$\hat{\lambda}\colon R_*(-) \to S_*(-)$ is a multiplicative natural transformation.*

Proof. It is clear that $\hat{\lambda}$ is a natural transformation, since ψ is natural in X. We will therefore only have to show that it respects the external product in homology. Consider the diagram

$$\begin{tikzcd}
MU_*(X) \otimes MU_*(Y) \arrow[r, "\wedge"] \arrow[d, "\psi_X \otimes \psi_Y"'] & MU_*(X \wedge Y) \arrow[d, "\psi_{X\wedge Y}"] \\
\begin{array}{c}\big(MU_*MU \otimes_{MU_*} MU_*(X)\big) \\ \otimes\big(MU_*MU \otimes_{MU_*} MU_*(Y)\big)\end{array} \arrow[r, "\Phi"] \arrow[d, "(\rho\otimes 1)\otimes(\rho\otimes 1)"'] & MU_*MU \otimes_{MU_*} MU_*(X \wedge Y) \arrow[d, "\rho \otimes 1"] \\
\begin{array}{c}\big(S \otimes_{MU_*} MU_*(X)\big) \\ \otimes\big(S \otimes_{MU_*} MU_*(Y)\big)\end{array} \arrow[r, "\wedge"] & S \otimes_{MU_*} MU_*(X \wedge Y),
\end{tikzcd}$$

where $\Phi\colon (a \otimes b) \otimes (c \otimes d) \mapsto ac \otimes (b \wedge d)$. The commutativity of the upper square is shown in [**Ada69**, Lecture 3], and the lower commutes because

$$\big(\rho(a) \otimes b\big) \wedge \big(\rho(c) \otimes d\big) = \rho(a)\rho(c) \otimes (b \wedge d) = \rho(ac) \otimes (b \wedge d).$$

This shows that $(\rho \otimes 1)\psi\colon MU_*(-) \to S_*(-)$ is a multiplicative natural transformation. Extending to $\hat{\lambda}$ we see that

$$\begin{aligned}
\hat{\lambda}\big((r \otimes x) \wedge (s \otimes y)\big) &= \hat{\lambda}\big(rs \otimes (x \wedge y)\big) \\
&= \lambda(rs) \otimes (\rho \otimes 1)\psi_{X \wedge Y}(x \wedge y) \\
&= \big(\lambda(r) \otimes (\rho \otimes 1)\psi_X(x)\big) \wedge \big(\lambda(s) \otimes (\rho \otimes 1)\psi_Y(y)\big) \\
&= \hat{\lambda}(r \otimes x) \wedge \hat{\lambda}(s \otimes y)
\end{aligned}$$

and this completes the proof. □

Let R and S be spectra representing $R^*(-)$ and $S^*(-)$. Recall from earlier discussions that $R^*(-)$ and $S^*(-)$ are complex oriented with orientation classes $\omega_R = 1 \otimes \omega \in \widetilde{R}^2(\mathbb{C}P^\infty)$ and $\omega_S = 1 \otimes \omega \in \widetilde{S}^2(\mathbb{C}P^\infty)$. The associated formal group laws are F_R and F_S.

LEMMA 2.6.7. *The natural transformation $\hat{\lambda}\colon R^*(-) \to S^*(-)$ is induced by a morphism $\lambda\colon R \to S$ of spectra which is unique up to homotopy. This morphism is a ring morphism.*

PROOF. Propositions 2.12 and 2.18 and Corollary 2.15 of [**HS99**] show that

$$\mathrm{Ph}^n(R, S) = \begin{cases} 0, & n = 2k, \\ [R, \Sigma^{2k+1} S], & n = 2k+1, \end{cases}$$

and

$$\mathrm{Ph}^n(R \wedge R, S) = \begin{cases} 0, & n = 2k, \\ [R \wedge R, \Sigma^{2k+1} S], & n = 2k+1. \end{cases}$$

In particular, there are no phantoms $R \to S$ and $R \wedge R \to S$.

There is a morphism $\lambda\colon R \to S$ inducing $\hat{\lambda}\colon R^*(-) \to S^*(-)$ which is unique up to weak homotopy. Due to the non-existence of non-trivial phantoms, it follows it is actually unique up to homotopy.

By Lemma 2.6.6 the morphism $\lambda\colon R \to S$ is a "quasi-ring morphism" in the sense of [**Rud98**, Definition III.7.1]. Since $\mathrm{Ph}^0(R \wedge R, S) = 0$, [**Rud98**, Proposition III.7.5] shows that $\lambda\colon R \to S$ is a ring morphism. □

REMARK. We note that the idea to prove that λ is a ring morphism is itself simple. We know that the two possible compositions in

$$\begin{tikzcd}
X \wedge Y \arrow[r, "f \wedge g"] & R \wedge R \arrow[r, "\lambda \wedge \lambda"] \arrow[d, "\mu"] & S \wedge S \arrow[d, "\mu"] \\
& R \arrow[r, "\lambda"] & S
\end{tikzcd}$$

are homotopic whenever X and Y are finite spectra. The trick is show that this implies that the square commutes up to weak homotopy. Then the non-existence of phantoms guarantees that it actually commutes up to homotopy.

PROOF OF THEOREM 2.6.1. We are almost done. The only thing remaining is to check that $\hat{\lambda}$ acts as claimed on coefficients and on the orientation class.

Recall that the coaction

$$\psi_X \colon MU_*(X) \to MU_*MU \otimes_{MU_*} MU_*(X)$$

is induced from the composition

$$MU \wedge X \simeq MU \wedge S^0 \wedge X \xrightarrow{1\wedge\iota\wedge 1} MU \wedge MU \wedge X,$$

so in the case $X = S^0$, we see that $\psi = \eta_L \colon MU_* \to MU_*MU$. Thus diagram (2.6.3) reduces to the square in (2.6.2), and hence $\hat{\lambda} = \lambda$ on coefficient rings.

We now show that in cohomology

$$\hat{\lambda} \colon \widetilde{R}^2(\mathbb{C}P^\infty) \to \widetilde{S}^2(\mathbb{C}P^\infty)$$

takes ω_R to $\theta(\omega_S)$. Write

$$\hat{\lambda}(\omega_R) = \sum_{i\geq 0} c_i \omega_S^{i+1}$$

and

$$\theta^{-1}(t) = \sum_{i \geq 0} d_i t^{i+1},$$

where $c_0 = 1 = d_0$. By [**Ada95**, (11.4)], the coaction map for $X = \mathbb{C}P^\infty$ is determined by

$$\psi(\beta_m) = \sum_{j \leq m} b^j_{(m-j)} \otimes \beta_j,$$

where $b = b_0 + b_1 + \cdots$ is the formal sum of the b_i (cf. Theorem 2.2.4) and the subscript $(m-j)$ denotes the homogeneous part of weight $m-j$. (b_i has weight i.)

By Lemma 1.4.9 the Kronecker pairing commutes with multiplicative natural transformations, so

$$\langle \hat{\lambda}(\omega_R)^n, \hat{\lambda}(\beta^R_m)\rangle = \lambda\langle \omega^n_R, \beta^R_m\rangle = \delta_{nm},$$

where $\beta^R_m = 1 \otimes \beta_m$. We also find that

$$\hat{\lambda}(\beta^R_m) = (\rho \otimes 1)\psi(\beta_m) = \sum_{j\leq m} \rho(b^j_{(m-j)}) \otimes \beta_j = \sum_{j \leq m} d^j_{(m-j)} \beta^S_j,$$

where $d = d_0 + d_1 + \cdots$ is the formal sum of the coefficients of $\theta^{-1}(x)$.

In particular, we see that

$$\begin{aligned}
\delta_{1m} &= \langle \hat{\lambda}(\omega_R), \hat{\lambda}(\beta^R_m)\rangle \\
&= \Big\langle \sum_{i\geq 0} c_i \omega_S^{i+1}, \sum_{j \leq m} d^j_{(m-j)} \beta^S_j \Big\rangle \\
&= \sum_{i \geq 0} \sum_{j \leq m} c_i d^j_{(m-j)} \langle \omega_S^{i+1}, \beta^S_j\rangle \\
&= \sum_{1 \leq j \leq m} c_{j-1} d^j_{(m-j)}.
\end{aligned}$$

Introducing the indeterminate t,

$$\delta_{1m} t = \sum_{1 \leq j \leq m} c_{j-1} d^j_{(m-j)} t^m = \sum_{j \geq 0} c_j \big(\theta^{-1}(t)\big)^{j+1}_{(m)}.$$

Summing over all m, we see that $t = \sum_{j\geq 0} c_j \big(\theta^{-1}(t)\big)^{j+1}$, so by uniqueness of inverse power series one has that $\theta(t) = \sum_{j \geq 0} c_j t^{j+1}$, and therefore $\hat{\lambda}(\omega_R) = \theta(\omega_S)$. □

Note that Theorem 2.6.1 only guarantees the existence of a multiplicative natural transformation with the desired properties by means of an explicit construction. This raises a natural question: Do these properties uniquely determine such a multiplicative natural transformation? Unfortunately the answer to this question is at present unknown to the author.

We make some comments about this problem, and we first consider an alternative construction of multiplicative natural transformations. Let $R^*(-)$ be Landweber exact via the ring homomorphism $\varphi\colon MU_* \to R$ and let $E^*(-)$ be a complex oriented cohomology theory with orientation class ω_E. Assume as above that we have a ring homomorphism $\lambda\colon R \to E_*$ such that $\theta\colon F_E \to \lambda F_R$ is a strict isomorphism of formal group laws. This means that $\lambda F_R = F_E^\theta$, the formal group law associated to $E^*(-)$ with orientation class $\theta(\omega_E)$. Let $t_\theta\colon MU \to E$ be the corresponding ring morphism. We have a commutative diagram

$$\begin{tikzcd} MU^*(-) \arrow[r, "t_\theta"] \arrow[d, "\varphi\otimes 1"'] & E^*(-) \\ R^*(-) \arrow[ur, dashed, "\hat\lambda"'] & \end{tikzcd}$$

where $\hat\lambda$ is defined by $\hat\lambda(r \otimes x) := \lambda(r)t_\theta(x)$. It is not hard to verify that this becomes a multiplicative natural transformation. Returning to the question; we can now ask if one can choose $s \neq s'\colon MU^*(-) \to E^*(-)$ agreeing on the image of $\varphi \otimes 1$. Since $\varphi\colon MU_* \to R$ is not in general surjective, the author sees no reason why this should not be the case.

On the other hand, Landweber exact cohomology theories with coefficients in even degrees are very well-behaved. In fact, if one in addition assume that $E^*(-)$ is Landweber exact, as is indeed the case in our situation, there are indications in the litterature that under some further assumptions the multiplicative natural transformations are uniquely determined by their action on $\mathbb{C}P^\infty$. Our reference here is [**Kas94**], where T. Kashiwabara studies *unstable* natural transformations and shows that they, under certain conditions, are determined by how they behave on finite Cartesian products consisting of copies of $\mathbb{C}P^\infty$ and S^1. It seems to the author that it should be possible to make his arguments fit our situation, making use of multiplicativity and stability of the natural transformations we consider, but he has not worked out the details.

It thus remains an interesting question to decide whenever there is only one multiplicative natural transformation with the prescribed action on coefficients and on the Euler class. We remark that for purely topological reasons, this is the case if $R^*(-)$ is complex K-theory: Suppose given two multiplicative natural transformations $s, t\colon K^*(-) \to E^*(-)$ agreeing on coefficients and on $\omega_K = \frac{1-\eta}{u}$. Then $s(\eta) = t(\eta)$ and in effect $s(x) = t(x)$ for any $x \in K^*(X)$ by naturality and the splitting principle.

CHAPTER 3

Elliptic cohomology

1. The construction of the cohomology theory

The main goal of this section is to introduce the so-called elliptic genus and show that it satisfies the exact functor theorem of the previous chapter. By definition, this is a ring homomorphism $MSO_* \to R$ into a $\mathbb{Q}$-algebra. One cannot apply the exact functor theorem directly, since this is a criterion for homomorphisms $MU_* \to R$, but we will see that in this particular setting, we may precompose with the forgetful homomorphism $MU_* \to MSO_*$ without losing any information, and thus obtain a ring homomorphism $MU_* \to R$ to which the exact functor theorem applies.

Definition 3.1.1. A **complex genus** is a ring homomorphism $MU_* \to R$. An **oriented genus** is a ring homomorphism $MSO_* \to R$. In both cases we demand that the ring homomorphisms preserve 1.

Let $\varphi\colon MU_* \to R$ be a complex genus into a $\mathbb{Q}$-algebra R. Since φ factors in the diagram

$$\begin{tikzcd}
MU_* \arrow[r, "\varphi"] \arrow[d] & R \\
MU_* \otimes \mathbb{Q} \arrow[r, "\varphi \otimes 1"] & R \otimes \mathbb{Q} \arrow[u, "\cong"']
\end{tikzcd}$$

it is determined by the image of MU_* in $MU_* \otimes \mathbb{Q}$. By Theorem 1.5.11

$$MU_* \otimes \mathbb{Q} \cong \mathbb{Q}\big[[\mathbb{C}P^1], [\mathbb{C}P^2], \dots\big],$$

hence any $\varphi\colon MU_* \to R$ is uniquely determined by its value on the $\mathbb{C}P^n$ for $n \geq 0$. By Mischenko's theorem 2.4.9, $\log_{MU}(x)$ is pushed by φ to the power series

$$\log_\varphi(x) = \sum_{n\geq 0} \frac{\varphi(\mathbb{C}P^n)}{n+1} x^{n+1} \in R[\![x]\!].$$

(Here we are interpreting $\mathbb{C}P^n$ as the class it represents in MU_*, and we will continue to do so throughout this section whenever it simplifies the notation.) This motivates the following:

Definition 3.1.2. The power series $\log_\varphi(x)$ is called the **logarithm** of the complex genus $\varphi\colon MU_* \to R$. The inverse under composition is called the **exponential function** of φ and is denoted by $\exp_\varphi$.

Since we are working over a $\mathbb{Q}$-algebra R, there is a one-to-one correspondence of power series $l(x)$ with leading term x and genera $\varphi\colon MU_* \to R$, by demanding that

$$l(x) = \sum_{n\geq 0} \frac{\varphi(\mathbb{C}P^n)}{n+1} x^{n+1},$$

so that $l(x) = \log_\varphi(x)$. This power series gives rise to a formal group law over R the usual way;

$$F_\varphi(x, y) = \exp_\varphi\big(\log_\varphi(x) + \log_\varphi(y)\big).$$

Also note that $F_\varphi = \varphi F^u$, so that over R it is equivalent whether we choose a formal group law, a logarithm or a genus.

Recall from Example 1.5.1 the forgetful natural transformation

$$\psi\colon MU_*(-) \to MSO_*(-),$$

coming from a ring morphism of ring spectra. It provides a complex orientation of $MSO^*(-)$, and thus F_{MSO} is classified by $\psi\colon MU_* \to MSO_*$. In particular, the logarithm is

$$\log_{MSO}(x) = \sum_{n\geq 0} \frac{\psi(\mathbb{C}P^n)}{n+1} x^{n+1} = \sum_{n\geq 0} \frac{[\mathbb{C}P^{2n}]}{2n+1} x^{2n+1} \in (MSO_* \otimes \mathbb{Q})[\![x]\!], \tag{3.1.3}$$

because $[\mathbb{C}P^{2n+1}]$ is 0 in MSO_*. This can be proved using characteristic numbers; see [**Sto68**].

Based on this, we make an analogous definition for the logarithm of an oriented genus. Recall, again from Theorem 1.5.11, that $MSO_* \otimes \mathbb{Q} = \mathbb{Q}\big[[\mathbb{C}P^2], [\mathbb{C}P^4], \dots\big]$.

Definition 3.1.4. Let $\varphi\colon MSO_* \to R$ be an oriented genus into a $\mathbb{Q}$-algebra. The **logarithm** of φ is defined to be

$$\log_\varphi(x) = \sum_{n\geq 0} \frac{\varphi(\mathbb{C}P^{2n})}{2n+1} x^{2n+1} \in R[\![x]\!].$$

Just as in the case of a complex genus, φ is uniquely determined by where it sends the $\mathbb{C}P^{2n}$. Conversely, any odd power series with leading term x can be chosen to be the logarithm, and thus determine an oriented genus $MSO_* \to R$.

Now the total F_Q-classes of Chapter 1 provide concrete examples of genera. First consider a closed, oriented n-dimensional manifold with tangent bundle τ_M and fundamental class $[M]$. For any even power series $Q(x) = 1 + \cdots \in \mathbb{Q}[\![x]\!]$ we define

$$\varphi_Q(M) := \begin{cases} \langle F_Q(\tau_M), [M]\rangle, & 4 \mid n \\ 0, & \text{otherwise.} \end{cases}$$

This function can be shown to be bordism invariant, and it respects disjoint union and Cartesian product (see [**HBJ92**, 1.6]), and so $\varphi_Q\colon MSO_* \to \mathbb{Q}$ is actually an oriented genus.

Now let M be a compact, stably complex manifold of real dimension n. The stable normal bundle lifts to BU, so there is a normal bundle ν that lifts to $BU(p)$, for some p. We choose a complementary complex vector bundle ξ such that $\nu \oplus \xi \cong \mathbf{p}_\mathbb{C}$. M has tangent bundle τ, and the direct sum $\tau \oplus \nu$ is a trivial bundle of real rank $n + 2p$. Let ε denote the trivial bundle of real rank 0 if n is even and of real rank 1 if n is odd. Then, as real vector bundles,

$$\tau \oplus \mathbf{p}_\mathbb{C} \oplus \varepsilon \cong \tau \oplus \nu \oplus \xi \oplus \varepsilon \cong \xi \oplus \mathbf{n}_\mathbb{R} \oplus \varepsilon \oplus \mathbf{p}_\mathbb{C},$$

which shows that $\tau \oplus \mathbf{p}_\mathbb{C} \oplus \varepsilon$ admits a complex structure. In other words, the stable tangent bundle $\tau_M\colon M \xrightarrow{\tau} BO(n) \hookrightarrow BO$ admits a lifting to BU. One can show that any power series $Q(x) = 1 + \cdots \in \mathbb{Q}[\![x]\!]$ defines a complex genus $\varphi_Q\colon MU_* \to \mathbb{Q}$ by

$$\varphi_Q(M) := \langle F_Q(\tau_M), [M]\rangle,$$

where τ_M is the stable tangent bundle.

The following result is proved in [**Hir66**].

PROPOSITION 3.1.5. *Let φ_Q be the genus obtained from the power series $Q(x) = 1+\cdots$ and write $Q(x) = x/f(x)$. Then*

$$f^{-1}(x) = \sum_{n\geq 0} \frac{\varphi_Q(\mathbb{C}P^n)}{n+1} x^{n+1},$$

i.e. the exponential function of φ_Q is $\exp_{\varphi_Q}(x) = f(x)$. □

EXAMPLE 3.1.6. Let M be a closed, stably complex manifold. Let $\mathrm{Td}(M)$ be the complex genus obtained from the characteristic power series $t(x) = \frac{x}{1-e^{-x}}$ as in Example 1.2.10. This is called the **Todd genus**. To calculate the value of Td on complex projective spaces, we see that $\exp_{\mathrm{Td}}(x) = 1 - e^{-x}$, so

$$\log_{\mathrm{Td}}(x) = -\log(1-x) = \sum_{n\geq 0} \frac{1}{n+1} x^{n+1}$$

and hence $\mathrm{Td}(\mathbb{C}P^n) = 1$ for all n. In fact, $\mathrm{Td}\colon MU_* \to \mathbb{Q}$ takes only integer values. To see this, we first modify Td to be a grading preserving ring homomorphism. This can be done by starting with the power series $t(x) = \frac{ux}{1-e^{-ux}}$, where $|u| = 2$. The resulting logarithm is $\log_{\mathrm{Td}}(x) = -u^{-1}\log(1-ux)$. On the other hand one has a ring morphism $\varphi\colon MU \to K$ which on coefficients classify the formal group law $F_K(x,y) = x + y - uxy$ over $\mathbb{Z}[u, u^{-1}]$. Since $\log_{\mathrm{Td}} = \log_K$, $\mathrm{Td}(\mathbb{C}P^n) = \varphi(\mathbb{C}P^n)$ for all n, so Td and φ coincide on MU_*. Since φ maps into $\pi_*K = \mathbb{Z}[u, u^{-1}]$ it follows that Td is integral. Another proof is given in [**Hir66**, Theorem 24.5.4].

EXAMPLE 3.1.7. For a closed, oriented manifold M, we take the **L-genus** to be the genus gotten from the characteristic power series $l(x) = \frac{x}{\tanh(x)}$ of Example 1.2.13. As the case was with the Todd genus, knowledge about the power series expansion of $\tanh^{-1}(x)$ reveals that $L(\mathbb{C}P^{2n}) = 1$ and $L(\mathbb{C}P^{2n+1}) = 0$. Thus the geometrically defined signature (see for instance [**May99**]) coincides with the L-genus on the $\mathbb{C}P^n$. Since an oriented genus is uniquely determined by its value on the $\mathbb{C}P^n$, it follows that $L(M)$ equals the signature of M, for any closed, oriented manifold. In particular, this implies that $L\colon MSO_* \to \mathbb{Q}$ is integral, since this is true for the signature. (This is known as Hirzebruch's signature theorem.)

We remark that similar integrality results exist for the $\hat{A}$- and A-genera as well. (See [**Hir66, LM89**].)

We will now make a specific choice of genus, or rather, a class of genera. Let $r(t)$ be the polynomial

$$r(t) = 1 - 2\delta t^2 + \varepsilon t^4.$$

DEFINITION 3.1.8. An **elliptic genus** is an oriented genus $\varphi\colon MSO_* \to R$ into a $\mathbb{Q}$-algebra such that

$$\log_\varphi(x) = \int_0^x r(t)^{-1/2}\,\mathrm{d}t,$$

for some choices of δ and ε in R.

If δ and ε are algebraically independent over $\mathbb{Q}$ and $R = \mathbb{Q}[\delta, \varepsilon]$, then we call φ a **universal elliptic genus**.

The following observation gives an equivalent characterization of elliptic genera.

Lemma 3.1.9. *Let $h(x) = x + \cdots$ be an odd power series over R, a $\mathbb{Q}$-algebra containing the elements δ and ε. h satisfies the differential equation*

$$(h')^2 = 1 - 2\delta h^2 + \varepsilon h^4$$

if and only if h is the exponential function of the elliptic genus with parameters δ and ε.

Proof. Note that $h^{-1}(x)$ is an odd power series with leading term x and that $(h^{-1})'\big(h(x)\big) \cdot h'(x) = 1$. For $y = h(x)$, we see that

$$(h^{-1})'(y) = \frac{1}{h'(x)} = \big(h'(x)^2\big)^{-1/2} = \big(1 - 2\delta h(x)^2 + \varepsilon h(x)^4\big)^{-1/2}$$
$$= (1 - 2\delta y^2 + \varepsilon y^4)^{-1/2}.$$

Integrating shows that $h^{-1}(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-1/2}\,\mathrm{d}t$, since $h^{-1}(x)$ is odd. The other direction is proved by differentiation. $\square$

The formal group law obtained from an elliptic genus is called the **Euler formal group law** and denoted by F_{Ell}. By definition, F_{Ell} is determined by the fact that it should be an addition formula for the elliptic integral as made precise by

$$\int_0^{F_{Ell}(x,y)} r(t)^{-1/2}\,\mathrm{d}t = \int_0^x r(t)^{-1/2}\,\mathrm{d}t + \int_0^y r(t)^{-1/2}\,\mathrm{d}t.$$

F_{Ell} is non-trivial to determine, and we state its formula as the next theorem. A proof is given in the appendix of [**Lan88**].

Theorem 3.1.10. *The elliptic genus $\varphi\colon MSO_* \to R$ has associated formal group law*

$$F_{Ell}(x,y) = \exp_\varphi\big(\log_\varphi(x) + \log_\varphi(y)\big)$$

which is given explicitly by the formula

$$F_{Ell}(x,y) = \frac{x\sqrt{r(y)} + y\sqrt{r(x)}}{1 - \varepsilon x^2 y^2}, \tag{3.1.11}$$

where $r(t) = 1 - 2\delta t^2 + \varepsilon t^4$. $\square$

We may now reconstruct the genera obtained from the characteristic series in (1.2.14) as (non-universal) elliptic genera.

Example 3.1.12. For $\delta^2 = \varepsilon$, we see that the elliptic genus degenerates to a genus with $\exp_\varphi(x) = \frac{1}{\sqrt{\delta}}\tanh(\sqrt{\delta}x)$. The formal group law becomes

$$F_\varphi(x,y) = \frac{x(1-\delta y^2) + y(1-\delta x^2)}{1 - \delta^2 x^2 y^2} = \frac{x+y}{1+\delta xy},$$

which is the addition law for tanh when $\delta = 1$. In particular, $\delta = \varepsilon = 1$ yields the L-genus and $\delta = 1/4$, $\varepsilon = 1/16$ yield the $\hat{L}$-genus.

Similar considerations show that for $\delta = -\gamma/2$ and $\varepsilon = 0$, the genus in question has $\exp_\varphi(x) = \frac{1}{\sqrt{\gamma}}\sinh(\sqrt{\gamma}x)$, with associated formal group law

$$F_\varphi(x,y) = x\sqrt{1+\gamma y^2} + y\sqrt{1+\gamma x^2}.$$

$\delta = -2$, $\varepsilon = 0$ corresponds to the A-genus, while $\delta = -1/8$, $\varepsilon = 0$ yield the $\hat{A}$-genus.

From (3.1.11), it is clear that the coefficients of F_{Ell} lie in $\mathbb{Q}[\delta, \varepsilon]$, but later in this section, it will be important to know that they are contained in a particular subring.

Proposition 3.1.13. *The formal group law F_{Ell} has coefficients in $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$.*

Proof. The Taylor series expansion of $(1+x)^{-1/2}$ is easily computed and found to be

$$\frac{1}{\sqrt{1+x}} = \sum_{n\geq 0}(-1)^n \frac{(2n-1)!!}{n!2^n} x^n,$$

where '!!' denotes the multifactorial notation. Inserting $r(t)$ for $1+x$ we have

$$\begin{aligned} r(t)^{-1/2} &= \sum_{n\geq 0}(-1)^n \frac{(2n-1)!!}{n!2^n}\Bigg(\sum_{0\leq k\leq n}\binom{n}{k}(-1)^k(2\delta t^2)^k(\varepsilon t^4)^{n-k}\Bigg) \\ &= \sum_{0\leq k\leq n<\infty}(-1)^{n-k}\binom{n}{k}\frac{(2n-1)!!(2n)!!}{(n!)^2 2^{2n-k}}\delta^k\varepsilon^{n-k}t^{4n-2k} \\ &= \sum_{0\leq k\leq n<\infty}(-1)^{n-k}\binom{n}{k}\binom{2n}{n}2^{k-2n}\delta^k\varepsilon^{n-k}t^{4n-2k}. \end{aligned}$$

We see that the coefficient of the powers of t are polynomials from $\mathbb{Z}[\frac{1}{2}][\delta,\varepsilon]$. Moreover, $r(t)^{-1/2}$ has leading term 1, and thus its multiplicative inverse $\sqrt{r(t)}$ exists and lies in $\mathbb{Z}[\frac{1}{2}][\delta,\varepsilon][\![t]\!]$. It follows that $F_{Ell}(x,y) \in \mathbb{Z}[\frac{1}{2}][\delta,\varepsilon][\![x,y]\!]$. □

The reason we considered the power series expansion of $r(t)^{-1/2}$ and not $\sqrt{r(t)}$ directly is the following application: We can find the coefficient of t^{2m} (which by definition is $\varphi(\mathbb{C}P^{2m})$) by setting $4n-2k = 2m$ in the last expression above. After some minor rearrangements, we find that

$$\big(r(t)^{-1/2}\big)_{2m} = \sum_{m/2\leq n\leq m}(-1)^{m-n}\binom{n}{2n-m}\binom{2n}{n}\frac{1}{2^m}\delta^{2n-m}\varepsilon^{m-n}t^{2m}.$$

Setting $m=1$ and $m=2$ respectively, we obtain $\varphi(\mathbb{C}P^2) = \delta$ and $\varphi(\mathbb{C}P^4) = \frac{3}{2}\delta^2 - \frac{1}{2}\varepsilon$. Solving for ε, we get (in the oriented bordism ring MSO_*) that

$$\varepsilon = \varphi(3[\mathbb{C}P^2]^2 - 2[\mathbb{C}P^4]).$$

It is possible to show, using characteristic numbers (see [**Sto68**]), that the bordism classes $3[\mathbb{C}P^2]^2 - 2[\mathbb{C}P^4]$ and $[\mathbb{H}P^2]$ coincide in $MSO_* \otimes \mathbb{Q}$. This leads to the following

Proposition 3.1.14. *Let $\varphi\colon MSO_* \to R$ be an elliptic genus. Then $\varphi(\mathbb{C}P^2) = \delta$ and $\varphi(\mathbb{H}P^2) = \varepsilon$.* □

Corollary 3.1.15. *An elliptic genus is determined by its image on $\mathbb{C}P^2$ and $\mathbb{H}P^2$.*

Proof. Let $\varphi\colon MSO_* \to R$ be an elliptic genus. It has logarithm

$$\log_\varphi(x) = \int_0^x (1-2\delta t^2+\varepsilon t^4)^{-1/2}\,\mathrm{d}t = \sum_{n\geq 0}\frac{\varphi(\mathbb{C}P^{2n})}{2n+1}x^{2n+1},$$

so each $\varphi(\mathbb{C}P^{2n})$ is a polynomial in δ and ε. Since φ is determined by its image on the $\mathbb{C}P^{2n}$, the result follows. □

We need the following result, found in [**Sto68**, p. 180].

Theorem 3.1.16. *Let $\psi\colon MU_*(-) \to MSO_*(-)$ be the forgetful natural transformation. The composition*

$$MU_* \xrightarrow{\psi} MSO_* \xrightarrow{\pi} MSO_*/\text{torsion}$$

is onto. □

Proposition 3.1.17. *An elliptic genus $\varphi\colon MSO_* \to R$ maps into $\mathbb{Z}[\frac{1}{2}][\delta,\varepsilon]$.*

PROOF. Since R is a $\mathbb{Q}$-algebra, it is torsion-free and therefore all torsion elements of MSO_* go to 0 under φ, so we can factor through $MSO_*/\text{torsion}$.

$$\begin{tikzcd}
MU_* \arrow[r, "\psi"] \arrow[dr, "\pi \circ \psi"'] & MSO_* \arrow[r, "\varphi"] \arrow[d, "\pi"] & R \\
& MSO_*/\text{torsion} \arrow[ur, dashed, "\bar{\varphi}"'] &
\end{tikzcd}$$

By Theorem 3.1.16,

$$\operatorname{im} \varphi = \operatorname{im} \bar{\varphi} \circ \pi = \operatorname{im} \bar{\varphi} \circ \pi \circ \psi = \operatorname{im} \varphi \circ \psi,$$

and $\varphi \circ \psi$ classifies F_{Ell}. Since MU_* is generated by the coefficients of the universal formal group law, the image of the composition $\varphi \circ \psi$ is generated by the coefficients of F_{Ell}, which all lie in $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ by Proposition 3.1.13. □

Let $\varphi\colon MSO_* \to R$ be a universal elliptic genus. In light of this proposition, we from now on view the universal elliptic genus as mapping into $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$,

$$\varphi\colon MSO_* \to \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon],$$

where δ and ε are indeterminates. Moreover, we assign degrees $|\delta| = 4$ and $|\varepsilon| = 8$. Define the **discriminant** to be the element $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$ of degree 24. To ease notation, let

$$M_* := \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon],$$

whenever δ and ε are given homological degree 4 and 8. M^* will be taken to denote this ring with cohomological grading.

We set out to show that certain localizations of M_* give rise to homology theories via the universal elliptic genus. We want to prove

THEOREM 3.1.18. *With γ equal to either Δ, ε or $\delta^2 - \varepsilon$, the functor*

$$M_*[\gamma^{-1}] \otimes_{MSO_*} MSO_*(-) \tag{3.1.19}$$

is a multiplicative homology theory with coefficient ring $M_*[\gamma^{-1}]$.

REMARK. Franke proved a better version of this theorem in [**Fra92**]: one can take γ to be any homogeneous element of M_* of positive degree and obtain a homology theory. For our purposes, however, it will be convenient to restrict our attention to these three choices.

The idea is apply the Landweber exact functor theorem 2.5.2, and therefore we must relate the functors (3.1.19) to $MU_*(-)$.

It is shown in [**Lan76**] that the forgetful natural transformation

$$\psi\colon MU_*(-) \to MSO_*(-)$$

induces an isomorphism

$$MU_*(-) \otimes_{MU_*} (MSO_* \otimes \mathbb{Z}[\tfrac{1}{2}]) \xrightarrow{\cong} MSO_*(-) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Thus it follows that we have natural isomorphisms

$$\begin{aligned} M_*[\gamma^{-1}] \otimes_{MU_*} MU_*(-) &\cong M_*[\gamma^{-1}] \otimes_{MSO_*} MSO_* \otimes_{MU_*} MU_*(-) \\ &\cong M_*[\gamma^{-1}] \otimes_{MSO_*} MSO_*(-), \end{aligned} \tag{3.1.20}$$

since 2 is already inverted in $M_*[\gamma^{-1}]$. This implies that the proof of exactness of the functor in (3.1.19) is equivalent to showing that

$$M_*[\gamma^{-1}] \otimes_{MU_*} MU_*(-)$$

is a homology theory, and this can be done using the exact functor theorem.

For the proof of Theorem 3.1.18, we will need the following facts, collected from [**Lan88**].

THEOREM 3.1.21. *Let F_{Ell} be the Euler formal group law over M_*, and let u_n be the coefficients of the p-series of F_{Ell} as in* (2.5.1)*. Then*

$$u_2 \equiv (-1)^{\frac{p-1}{2}} \varepsilon^{\frac{p^2-1}{4}} \pmod{p, u_1}$$
$$\varepsilon^{\frac{p^2-1}{4}} \equiv (\delta^2-\varepsilon)^{\frac{p^2-1}{4}} \pmod{p, u_1}$$

for all odd primes p. □

PROOF OF THEOREM 3.1.18. We verify that the sequence $(p, u_1, u_2, \ldots)$ of elements from $R := M_*[\gamma^{-1}]$ is regular for all primes p. Since 2 is a unit, the sequence is regular for $p = 2$, so we need only consider odd primes.

Fix a prime $p > 2$. It is clear that p acts injectively on R by multiplication. Next we must show that u_1 is not a zero-divisor in $R/(p) = \mathbb{F}_p[\delta, \varepsilon, \gamma^{-1}]$. First we consider F_{Ell} modulo ε, which becomes the formal group law

$$F(x, y) = x\sqrt{1-2\delta y^2} + y\sqrt{1-2\delta x^2}$$

over $\mathbb{Z}[\frac{1}{2}][\delta]$. Its logarithm is

$$\log_F(x) = \frac{1}{\sqrt{-2\delta}} \sinh^{-1}(\sqrt{-2\delta}x) = \frac{1}{\sqrt{-2\delta}} \log(\sqrt{-2\delta}x + \sqrt{1-2\delta x^2}).$$

View this as a power series over the bigger ring $\mathbb{Z}[\frac{1}{2}][\sqrt{-2\delta}]$, and compose this logarithm with the power series

$$\exp_m(x) = \frac{1}{\sqrt{-2\delta}}(e^{\sqrt{-2\delta}x} - 1),$$

which is the exponential function associated to $F_m(x, y) = x + y + \sqrt{-2\delta}xy$. This composition is a strict isomorphism $f\colon F \to F_m$, explicitly given by

$$f(x) = \frac{1}{\sqrt{-2\delta}}(\sqrt{-2\delta}x + \sqrt{1-2\delta x^2} - 1).$$

Hence $f\big([p]_F(x)\big) = [p]_m\big(f(x)\big)$. Over $\mathbb{F}_p[\sqrt{-2\delta}]$, $[p]_F(x)$ and $[p]_m(x)$ thus shares the coefficient in degree p, as in the proof of Proposition 2.5.8. This coefficient is $(\sqrt{-2\delta})^{p-1}$, and we conclude that for the Euler formal group law

$$u_1 \equiv (-2\delta)^{(p-1)/2} \not\equiv 0 \pmod{(p, \varepsilon)}. \tag{3.1.22}$$

In particular, $u_1 \neq 0$ in $R/(p)$, so multiplication by this element is injective.

This conclusion also follows by considering the case $\delta^2 = \varepsilon$. Then the Euler formal group law degenerates to $F(x, y) = \frac{x+y}{1+\delta xy}$, and over $\mathbb{Z}[\frac{1}{2}][\sqrt{\delta}]$ this formal group law is strictly isomorphic to $F_m(x, y) = x + y + \sqrt{\delta}xy$. As above, it will follow that

$$u_1 \equiv \delta^{(p-1)/2} \not\equiv 0 \pmod{(p, \delta^2-\varepsilon)}. \tag{3.1.23}$$

If $\Delta = \varepsilon(\delta^2-\varepsilon)^2$ is a unit of $R/(p, u_1)$, then so are ε and $\delta^2-\varepsilon$. Conversely, by virtue of the last congruence in the previous theorem, ε is a unit of $R/(p, u_1)$ if and only if $\delta^2-\varepsilon$ is. Therefore, inverting any of ε, $\delta^2-\varepsilon$ or Δ in R automatically implies the existence of an inverse of the other two in $R/(p, u_1)$.

We combine the two congruences of the previous theorem to get

$$u_2 \equiv (-1)^{\frac{p-1}{2}} (\delta^2-\varepsilon)^{\frac{p^2-1}{4}} \pmod{p, u_1}.$$

Therefore u_2 is a unit in $R/(p, u_1)$, so $R/(p, u_1, u_2) = 0$ and consequently (p, u_1, u_2) is a regular sequence for all primes p. Invoking Theorem 2.5.2, we see that

$$M_*[\gamma^{-1}] \otimes_{MU_*} MU_*(-) \cong M_*[\gamma^{-1}] \otimes_{MSO_*} MSO_*(-)$$

is a homology theory, for all of our choices of γ. □

Remark. This proof relies heavily on the congruences found by Landweber. This is a non-trivial result, and Landweber uses the theory of elliptic curves to arrive at his conclusion. There is another, more conceptual proof of the fact that u_2 is a unit in $R/(p, u_1)$. This proof also uses elliptic curves, and we will sketch this proof; see [**Fra92, LRS95**].

Following Franke, we will assume that $\gamma = \gamma(\delta, \varepsilon)$ is any homogeneous element of positive degree in M_*. In anticipation of a contradiction, assume that u_2 is not a unit in $R/(p, u_1)$. Then it is not a unit in R, and thus there is a maximal ideal $\mathfrak{m} \subseteq R$ containing p, u_1 and u_2. $\mathfrak{m}$ cannot contain $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$, however, because then either ε or $\delta^2 - \varepsilon$ would lie in this ideal. Then the congruences (3.1.22) and (3.1.23) show that $\delta \in \mathfrak{m}$ (since $u_1 \in \mathfrak{m}$), and consequently $\gamma(\delta, \varepsilon) \in \mathfrak{m}$. But γ is a unit in R, so $\Delta \notin \mathfrak{m}$.

Therefore $\Delta \neq 0$ in the residue field $R/\mathfrak{m}$. It is known that the equation

$$y^2 = 1 - 2\delta x^2 + \varepsilon x^4$$

can be considered an elliptic curve over $R/\mathfrak{m}$, because the discriminant Δ is non-zero, and that the formal group law of this elliptic curve (see [**Sil09**]) is $F_{Ell}(x, y)$. But in $R/\mathfrak{m}$, $u_1 = u_2 = 0$, so the height of the formal group law is greater than 2. This contradicts the fact that for any elliptic curve over a field of characteristic $p > 0$, the height is either 1 or 2. It follows that u_2 must be a unit in $R/(p, u_1)$. See [**Fra92**] for references on the claims put forward in this discussion.

Definition 3.1.24. The homology theory

$$Ell_*(-) = M_*[\gamma^{-1}] \otimes_{MU_*} MU_*(-)$$

with γ equal to ε, $\delta^2 - \varepsilon$ or Δ is called **elliptic homology** with coefficient ring $M_*[\gamma^{-1}]$.

The associated cohomology theory, which on finite spectra is given by

$$Ell^*(-) = M^*[\gamma^{-1}] \otimes_{MU^*} MU^*(-),$$

is called **elliptic cohomology**.

We write *Ell* for the representing ring spectrum, and thus usually suppress the choice of γ from the notation. As the elliptic cohomology theories we have constructed are Landweber exact theories, they are complex orientable, and we choose the orientations

$$\omega_{Ell} := 1 \otimes \omega \in \widetilde{Ell}^2(\mathbb{C}P^\infty).$$

In the sections that follow, we will identify the coefficient ring of this cohomology theory with a graded ring of complex-valued functions.

The following is a corollary to the proof of Theorem 3.1.18.

Corollary 3.1.25. $KO[\frac{1}{2}]^*(-)$ *is Landweber exact via the* $\hat{A}$*-genus*

$$\varphi_{\hat{A}} \colon MSO_* \to \mathbb{Z}[\tfrac{1}{2}][u^2, u^{-2}]$$

which has been modified to keep track of grading, i.e. $M^{4n} \mapsto \hat{A}(M)u^{2n}$.

PROOF. Since 2 is inverted, (3.1.20) and the comments following justify that we only have to check the Landweber condition for this oriented genus. Recall that $\hat{A}(\mathbb{C}P^2) = -1/8$ and $\hat{A}(\mathbb{H}P^2) = 0$. It follows that the Euler formal group law in this case becomes

$$F(x, y) = x\sqrt{1 + \frac{1}{4}u^2y^2} + y\sqrt{1 + \frac{1}{4}u^2x^2}.$$

The aforementioned proof carries over to show that in $\mathbb{F}_p[u^2, u^{-2}]$, the coefficient of x^p in the p-series is $u_1 = (u^2/4)^{(p-1)/2}$ which is a unit, and therefore the Landweber conditions are satisfied.

It remains to exhibit a natural transformation

$$\mathbb{Z}[\tfrac{1}{2}][u^2, u^{-2}] \otimes_{MSO_*} MSO_*(-) \to KO[\tfrac{1}{2}]_*(-)$$

which is an isomorphism on coefficients. We leave this to [**ABS64**]; see also [**LRS95**, Proposition 4.9] and [**Lan76**, Example 3.4]. □

2. Modular forms and functions

Recall the Möbius functions $f\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, defined by

$$f(\tau) = \frac{a\tau + b}{c\tau + d},$$

and where a, b, c and d are complex coefficients such that $ad - bc \neq 0$. Instead of fixing the coefficients, we may fix τ and obtain a group action of $\mathrm{GL}_2(\mathbb{C})$ on $\bar{\mathbb{C}}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}. \tag{3.2.1}$$

In the current section, we will consider this action to describe a special class of complex-valued functions. The exposition given here is heavily influenced by [**Sil09**] and [**HBJ92**], and one should confer these sources for details.

Recall that $\mathrm{SL}_2(\mathbb{Z})$ is the group of 2-by-2 matrices with integer coefficients and determinant 1. Consider a subgroup $G \subseteq \mathrm{SL}_2(\mathbb{Z})$ of finite index. We will assume that it contains the element $-I_2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ throughout. Let

$$\mathcal{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\} \subseteq \mathbb{C}$$

denote the upper half-plane, which we view as embedded $\mathcal{H} \hookrightarrow \mathbb{C}P^1$ by $\tau \mapsto (\tau, 1)$. The canonical left action on $\mathbb{C}P^1$ is matrix multiplication, and upon restriction to elements from $\mathrm{SL}_2(\mathbb{Z})$, the upper half-plane is invariant. This action is precisely the one given in (3.2.1). In $\mathbb{C}P^1$ we put $\infty = (1, 0)$ and $0 = (0, 1)$.

The action restricts to $\mathbb{Q}P^1 \subseteq \mathbb{C}P^1$, and we call the orbits under the action of G the **cusps** of G. Note that we will also call a representative for each orbit a cusp. We further note that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{Q}P^1$, so every element $s \in \mathbb{Q}P^1$ may be written $s = S\infty$ for some $S \in \mathrm{SL}_2(\mathbb{Z})$, and thus $\mathrm{SL}_2(\mathbb{Z})$ has only the cusp ∞.

Let $f\colon \mathcal{H} \to \mathbb{C}$ be a meromorphic function. Define, for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and all integers k,

$$f|^k_A(\tau) := (c\tau + d)^{-k} f(A\tau).$$

Note that this is a right action on the set of meromorphic functions $\mathcal{H} \to \mathbb{C}$; we have $(f|^k_A)|^k_{A'} = f|^k_{AA'}$.

DEFINITION 3.2.2. Let $G \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and k an integer. A meromorphic function $f\colon \mathcal{H} \to \mathbb{C}$ is called a **modular function** of weight k for G if

(1) $f|^k_A = f$ for all $A \in G$; i.e. $f(A\tau) = (c\tau + d)^k f(\tau)$, for all $\tau \in \mathcal{H}$.

(2) for all $S \in \mathrm{SL}_2(\mathbb{Z})$ one can write

$$f|^k_S(\tau) = \sum_{n \geq n_S} a_n q^{n/N},$$

for some $a_n \in \mathbb{C}$, $N \in \mathbb{N}$ and $n_S \in \mathbb{Z}$, and where $q = q(\tau) := e^{2\pi i \tau}$. This power series is called the **q-expansion** at the cusp $S\infty$.

If a q-expansion exists at a cusp $S\infty$ and $n_S \in \mathbb{Z}$ (that is, we do not allow $a_n \neq 0$ for arbitrarily small n), one says that f is **meromorphic** at the cusp $S\infty$. If $n_S \geq 0$, f is said to be **holomorphic** at $S\infty$. If $n_S > 0$ (resp., $n_S < 0$) we say that f has a **zero** (resp., **pole**) of order $|n_S|$ at $S\infty$.

If f is a modular function of weight k for G which is in addition holomorphic on $\mathcal{H}$ and holomorphic at each cusp, we say that f is a **modular form** of weight k for G.

LEMMA 3.2.3. *Let G and H be subgroups of $\mathrm{SL}_2(\mathbb{Z})$. If f is a modular function (resp., form) of weight k for G, and $S \in \mathrm{SL}_2(\mathbb{R})$ such that $H \subseteq S^{-1}GS$, then $f|^k_S$ is a modular function (resp., form) of weight k for H.*

PROOF. Let $A \in H \subseteq S^{-1}GS$. Then with $B \in G$ such that $A = S^{-1}BS$,

$$(f|^k_S)|^k_A = f|^k_{SA} = f|^k_{BS} = (f|^k_B)|^k_S = f|^k_S. \qquad \square$$

LEMMA 3.2.4. *Let $S \in \mathrm{SL}_2(\mathbb{Z})$ and f a modular function of weight k for G. Then $f|^k_S$ is periodic with period N, for some $N \in \mathbb{N}$. In other words,*

$$f|^k_S(\tau + N) = f|^k_S(\tau).$$

PROOF. $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is one of the generators of $\mathrm{SL}_2(\mathbb{Z})$ (see [**Sil09**]). Hence for every subgroup of finite index, there must be some N such that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^N$ is contained in this subgroup. In particular, $S^{-1}GS$ has finite index, so we can choose an N such that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in S^{-1}GS$.

Put $A = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$. The lemma gives

$$f|^k_S(\tau + N) = (f|^k_S)|^k_A(\tau) = f|^k_S(\tau). \qquad \square$$

This shows that the existence of an N in the q-expansion ($n_S = -\infty$ allowed) follows from the assumption on the finiteness of the index of G in $\mathrm{SL}_2(\mathbb{Z})$. Thus (2) in the definition could be replaced with the requirement that f is meromorphic at each cusp. Moreover, if $S\infty$ and $S'\infty$ lie in the same orbit of $\mathbb{Q}P^1$ under the action of G, then there exists some constant $c > 0$ such that $n_S = cn_{S'}$, so in order to show that f is a modular function (resp., modular form) it is sufficient to check that f is meromorphic (resp., holomorphic) at only one representative for each cusp. (See [**HBJ92**, Appendix I] for a discussion of this fact.)

We will mainly be concerned with modular forms and functions for some specific subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

EXAMPLE 3.2.5. Let $\Gamma_0(2) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid c \equiv 0 \pmod 2\}$ be the subgroup of matrices where the lower left entry is an even integer. This is a non-normal subgroup of index 3 in $\mathrm{SL}_2(\mathbb{Z})$, and it has two cusps, namely $\infty = (1, 0)$ and $0 = (0, 1) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\infty$.

Note that since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(2)$, we have that for any modular function f of weight k for $\Gamma_0(2)$,

$$f(\tau + 1) = f\big(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\tau\big) = f(\tau),$$

so the q-expansion of f at ∞ is of the form

$$f(\tau) = \sum_{n \geq n_\infty} a_n q^n.$$

Since $0 = S\infty$ where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $S\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)S^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right) \in \Gamma_0(2)$, it follows from the lemma above that

$$f|_S^k(\tau + 2) = f|_S^k(\tau),$$

so that the q-expansion of f at $0 = S\infty$ can be written in the form

$$f|_S^k(\tau) = \sum_{n \geq n_0} b_n q^{n/2}.$$

We will frequently write

$$f^0 := f|_S^k \tag{3.2.6}$$

to denote the expansion at 0.

We remark that f^0 is not a modular function for $\Gamma_0(2)$; rather it is a modular function for the group

$$\Gamma^0(2) := S^{-1}\Gamma_0(2)S = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid b \equiv 0 \pmod 2\}$$

by Lemma 3.2.3. If one now puts $T = \left(\begin{smallmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{smallmatrix}\right)$, and let $A \in \Gamma^0(2)$, then a simple calculation shows that $T^{-1}AT \in \Gamma_0(2)$ so if g is any weight k modular function for $\Gamma^0(2)$ then $g|_T^k$ is a modular function for $\Gamma_0(2)$. In particular, this shows that

$$g(2\tau) = g\big(\left(\begin{smallmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{smallmatrix}\right)\tau\big) = 2^{-k/2} g|_T^k(\tau)$$

is a modular function for $\Gamma_0(2)$.

Observe that for a modular function f of weight k for G, we have

$$f(\tau) = f\Big(\frac{\tau + 0}{0\tau + 1}\Big) = f\Big(\frac{-\tau - 0}{-0\tau - 1}\Big) = (-1)^k f(\tau),$$

since $-I_2 \in G$. In effect, there are no non-zero modular functions of odd weight.

Write $M_k(G)$ for the vector space over $\mathbb{C}$ of weight k modular forms for G, where pointwise addition gives the group structure. Pointwise multiplication of modular forms gives a pairing $M_m(G) \otimes M_n(G) \to M_{m+n}(G)$ and this makes

$$M_*(G) = \bigoplus_{k \in \mathbb{Z}} M_k(G)$$

a commutative graded ring. The same way one can make rings of modular functions.

2.1. Relating elliptic cohomology and modular functions. Consider the lattice $L = \mathbb{Z}\omega_2 + \mathbb{Z}\omega_1$, where $\omega_2/\omega_1 = \tau \in \mathcal{H}$, and a function f which is elliptic with respect to L. The **order of** f, n_p, at a point p is defined to be

$$n_p = \begin{cases} n, & f \text{ has a zero of order } n \text{ at } p \\ -n, & f \text{ has a pole of order } n \text{ at } p \\ 0, & \text{otherwise.} \end{cases}$$

A **divisor** of $\mathbb{C}/L$ is an element of the free abelian group $\mathrm{Div}(\mathbb{C}/L)$ generated by the points of $\mathbb{C}/L$. We write an element as $d = \sum_{p \in \mathbb{C}/L} n_p \cdot (p)$, where the notation (p) is to emphasize that $\cdot$ is not scalar multiplication. The **divisor of** f is the sum $\operatorname{div} f := \sum_{p \in \mathbb{C}/L} n_p \cdot (p)$, where p runs over all points in $\mathbb{C}$ modulo L and n_p is the order of f at the point p.

The following theorem is proved in Section 5 of [**HBJ92**, Appendix I].

THEOREM 3.2.7. *Let $d = \sum_{p \in \mathbb{C}/L} n_p \cdot (p)$ be a divisor such that $\sum_{p \in \mathbb{C}/L} n_p = 0$ and $\sum_{p \in \mathbb{C}/L} n_p \cdot p \in L$ (this is scalar multiplication). Then there is an elliptic function f with respect to L, unique up to normalization, such that* $\operatorname{div} f = d$. □

For $\tau \in \mathcal{H}$, let as usual $q = q(\tau) = e^{2\pi i\tau}$, let L be the lattice $4\pi i(\mathbb{Z}\tau + \mathbb{Z})$, and choose the divisor $d = (0) + (2\pi i) - (2\pi i\tau) - \big(2\pi i(\tau+1)\big)$. By the previous theorem there is a unique elliptic function $f(z) = z + \cdots$ with respect to L having this divisor. [**Zag88**] now gives the following.

THEOREM 3.2.8. *This function $f(z)$ is odd, and is the exponential function of an elliptic genus $\varphi\colon MSO_* \to \mathbb{Z}[\frac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}]$. The characteristic power series $Q(z) := z/f(z)$ is an element of $\mathbb{Q}[\![q]\!][\![z]\!]$ given by both of the following formulas*

$$Q(z) = \frac{z/2}{\sinh(z/2)} \prod_{n\geq 1} \left(\frac{(1-q^n)^2}{(1-q^ne^z)(1-q^ne^{-z})} \right)^{(-1)^n},$$
$$Q(z) = z/\exp_\varphi(z),$$

where $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ are explicitly given by

$$\delta_{\hat{A}} = -\frac{1}{8} - 3\sum_{n\geq 1}\Big(\sum_{2\nmid d|n} d\Big) q^n,$$
$$\varepsilon_{\hat{A}} = \sum_{n\geq 1}\Big(\sum_{\substack{d|n \\ 2\nmid n/d}} d^3\Big) q^n.$$

$\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ are modular forms for $\Gamma_0(2)$ of weight 2 and 4, respectively. The coefficient of z^{2k} in $Q(z)$ is a homogeneous polynomial in $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ of weight $2k$, and so a modular form of weight $2k$ for $\Gamma_0(2)$. □

The subscript $\hat{A}$ decorating the δ and ε is meant to reflect the fact that the constant term of the q-expansions are precisely those values yielding the $\hat{A}$-genus. In particular, when $\tau \to \infty$ ($q \to 0$) this elliptic genus degenerates to the $\hat{A}$-genus. These formulas will be essential in the following discussion. For now we will just apply them, but at the end of Chapter 4 we will comment on how they are related to the Weierstrass $\wp$-function.

The modular forms $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ are algebraically independent and it is known that (see [**LRS95, HBJ92**])

$$M_*\big(\Gamma_0(2)\big) = \mathbb{C}[\delta_{\hat{A}}, \varepsilon_{\hat{A}}]. \tag{3.2.9}$$

Thus letting $M_*^R(G)$ denote the ring of modular forms with coefficients in R, one can show that $M_*^{\mathbb{Q}}\big(\Gamma_0(2)\big) = \mathbb{Q}[\delta_{\hat{A}}, \varepsilon_{\hat{A}}]$. It follows that the elliptic genus of the previous theorem takes values in this ring. We obtain a sharper result:

LEMMA 3.2.10. *Let R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$. Then*

$$M_*^R\big(\Gamma_0(2)\big) = R[8\delta_{\hat{A}}, \varepsilon_{\hat{A}}].$$

PROOF. First we note that from Theorem 3.2.8, $8\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ are modular forms with q-expansion coefficients in $\mathbb{Z}$, and so the inclusion $\supseteq$ follows.

Now take a modular form in $M_*^R\big(\Gamma_0(2)\big)$ of, say, weight $2k$. Then, by (3.2.9),

$$f = \sum_{0\leq j\leq k/2} a_j(-8\delta_{\hat{A}})^{k-2j}\varepsilon_{\hat{A}}^j$$

for some complex numbers a_j, and $-8\delta_{\hat{A}} = 1 + \sum_{n\geq 1} d_nq^n$, $\varepsilon_{\hat{A}} = \sum_{n\geq 1} e_nq^n$, where d_n and e_n are integers. Let $f = \sum b_nq^n$ be the q-expansion at ∞ with $b_n \in R$. Using this, we have that

$$\begin{aligned}\sum_{n\geq 0} b_n q^n &= \sum_{0\leq j\leq k/2} a_j(-8\delta_{\hat{A}})^{k-2j}\varepsilon_{\hat{A}}^j \\ &= \sum_{0\leq j\leq k/2} a_j\left(q^j + \sum_{m>j} c(j)_m q^m\right) \\ &= \sum_{n\geq 0}\left(a_n q^n + \sum_{j<n} a_j c(j)_n q^n\right),\end{aligned}$$

for integers $c(j)_m$. Picking out the coefficients in degree n, we have

$$b_n = a_n + \sum_{j<n} a_j c(j)_n.$$

Because $c(j)_n \in \mathbb{Z}$, $b_n \in R$ and $a_0 = b_0$, this shows that $a_n \in R$ by induction, and in effect we have shown $\subseteq$. □

PROPOSITION 3.2.11 ([**LRS95**, Proposition 5.7]). *The modular form $\varepsilon_{\hat{A}}$ is non-zero on $\mathcal{H}$, at 0 and has a simple zero at ∞. $\delta_{\hat{A}}^2 - \varepsilon_{\hat{A}}$ is non-zero on $\mathcal{H}$, at ∞ and has a simple zero at 0. $\Delta_{\hat{A}} := \varepsilon_{\hat{A}}(\delta_{\hat{A}}^2 - \varepsilon_{\hat{A}})^2$ thus is non-zero on $\mathcal{H}$ with zeros at the cusps.* □

The previous two results allows us to describe the coefficient ring of elliptic cohomology.

THEOREM 3.2.12. *$M_* = \mathbb{Z}[\frac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}]$ is the ring of modular forms $M_*^{\mathbb{Z}[\frac{1}{2}]}(\Gamma_0(2))$. $M_*[\varepsilon_{\hat{A}}^{-1}]$, $M_*[(\delta_{\hat{A}}^2 - \varepsilon_{\hat{A}})^{-1}]$ and $M_*[\Delta_{\hat{A}}^{-1}]$, respectively, are the rings of modular functions with q-expansion coefficients of $\mathbb{Z}[\frac{1}{2}]$ which are holomorphic on $\mathcal{H}$ and which may only have poles at ∞, 0 and both $\infty, 0$, respectively.*

PROOF. The first claim was proved in the lemma. We turn to the second part, but we only prove this for one of the localizations. The proofs for the other two are analogous.

For $M_*[\Delta_{\hat{A}}^{-1}]$, recall from Proposition 3.2.11 that $\Delta_{\hat{A}}$ is a modular form with q-expansion coefficients from $\mathbb{Z}[\frac{1}{2}]$ which is non-zero on $\mathcal{H}$ and zero at the cusps. Hence formally inverting $\Delta_{\hat{A}}$ introduces what we may consider a modular function of negative weight with poles at the cusps. Thus the elements of $M_*[\Delta_{\hat{A}}^{-1}]$ can be viewed as modular functions with q-expansion coefficients in $\mathbb{Z}[\frac{1}{2}]$ that are holomorphic on $\mathcal{H}$ and possibly have poles at the cusps (but nowhere else). Conversely, if f is such a modular function, the poles (if they exist) have finite order, and so we may kill them by multiplying with the modular form $\Delta_{\hat{A}}$ a sufficient number of times. So for some N,

$$f\Delta_{\hat{A}}^N \in M_*^{\mathbb{Z}[\frac{1}{2}]}\big(\Gamma_0(2)\big) = M_*$$

and it follows that $f \in M_*[\Delta_{\hat{A}}^{-1}]$. □

Therefore, letting $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ be the modular forms above, the elliptic homology and cohomology theories of Definition 3.1.24 have rings of modular functions as coefficient rings. This will be a useful point of view in the sections to come.

CHAPTER 4

Computing correction classes

1. Riemann–Roch for complex orientable theories

In anticipation of explicit computations, we restate the Riemann–Roch theorem for cohomology theories with complex orientations. In these theories we have a firm grasp on how the Thom classes behave, as we discussed in Chapter 2, and it is fair to believe that this behavior should be reflected in the correction classes. Before stating the special case of the Riemann–Roch theorem, we will show that the correction classes indeed act nicely.

LEMMA 4.1.1. *The correction classes for multiplicative natural transformations*

$$\lambda\colon E^*(-) \to F^*(-)$$

of complex orientable cohomology theories are natural. That is, for continuous maps $g\colon Y \to X$ *and complex vector bundles* $\xi \downarrow X$ *of rank* n,

$$g^*\rho_\xi = \rho_{g^*\xi}.$$

PROOF. Naturality of the Thom classes for $E^*(-)$ and $F^*(-)$ along with the naturality of λ implies that the following diagram commutes. (Recall that we write g for the Thomification of the induced bundle map $\mathrm{Th}(\bar{g})$.) The diagram

$$\begin{tikzcd}
E^*(X) \arrow[r, "\varphi_\xi^E", "\cong"'] \arrow[d, "g^*"] & \widetilde{E}^{*+2n}(X^\xi) \arrow[r, "\lambda"] \arrow[d, "g^*"] & \widetilde{F}^{*+2n}(X^\xi) \arrow[d, "g^*"] & F^*(X) \arrow[l, "\varphi_\xi^F"', "\cong"] \arrow[d, "g^*"] \\
E^*(Y) \arrow[r, "\varphi_{g^*\xi}^E", "\cong"'] & \widetilde{E}^{*+2n}(Y^{g^*\xi}) \arrow[r, "\lambda"] & \widetilde{F}^{*+2n}(Y^{g^*\xi}) & F^*(Y) \arrow[l, "\varphi_{g^*\xi}^F"', "\cong"]
\end{tikzcd}$$

commutes, and taking $1 \in E^0(X)$ and going to $F^0(Y)$ along the top and along the bottom proves the claim. $\square$

LEMMA 4.1.2. *The correction class is multiplicative: Let* $\xi \downarrow X$ *and* $\xi' \downarrow Y$ *be complex vector bundles and let* $\lambda\colon E^*(-) \to F^*(-)$ *be a multiplicative natural transformation. Then*

$$\rho_{\xi\times\xi'} = \rho_\xi \times \rho_{\xi'} \in E^0(X \times Y).$$

PROOF. Recall that in any complex oriented cohomology theory $\psi^*(u_{\xi\times\xi'}) = u_\xi \wedge u_{\xi'}$, where $\psi\colon X^\xi \wedge Y^{\xi'} \xrightarrow{\approx} (X \times Y)^{\xi\times\xi'}$. Using this, one sees that

$$\begin{aligned}
(\rho_\xi \times \rho_{\xi'}) \cdot u^F_{\xi\times\xi'} &= \pi^*_{\xi\times\xi'}(\rho_\xi \times \rho_{\xi'}) \smile (\psi^{-1})^*(u^F_\xi \wedge u^F_{\xi'}) \\
&= (\psi^{-1})^*\big((\pi^*_\xi\rho_\xi \smile u^F_\xi) \wedge (\pi^*_{\xi'}\rho_{\xi'} \smile u^F_{\xi'})\big) \\
&= (\psi^{-1})^*(\lambda u^E_\xi \wedge \lambda u^E_{\xi'}) \\
&= \lambda(\psi^{-1})^*(u^E_\xi \wedge u^E_{\xi'}) \\
&= \lambda u^E_{\xi\times\xi'}.
\end{aligned}$$

Here π denotes the projection from the indicated disc bundle. This shows that $\rho_\xi \times \rho_{\xi'}$ acts as a correction class for $\xi \times \xi'$, and is thus equal to $\rho_{\xi\times\xi'}$ by uniqueness. $\square$

LEMMA 4.1.3. *The correction class of a multiplicative natural transformation of complex oriented cohomology theories is a stable, exponential characteristic class.*

PROOF. Using the diagonal $\Delta\colon X \to X \times X$ and the previous two lemmas, one sees that $\rho_\xi \smile \rho_{\xi'} = \rho_{\xi\oplus\xi'}$, so ρ is exponential.

Denote the natural transformation by $\lambda\colon E^*(-) \to F^*(-)$. Since the cohomology theories are assumed to be complex oriented, the Thom class of the trivial line bundle $\mathbf{1} \downarrow pt$ is $\sigma^2(1)$. The trivial line bundle $\mathbf{1} \downarrow X$ is isomorphic to the pullback of $\mathbf{1} \downarrow pt$ along the projection $p\colon X \to pt$, so $u^E_{\mathbf{1}\downarrow X} = p^*\sigma^2(1) \in \widetilde{E}^2(X^{\mathbf{1}})$, and similarly for F. Since λ commutes with both induced maps and suspensions,

$$\lambda u^E_{\mathbf{1}} = \lambda p^*\sigma^2(1) = p^*\sigma^2\big(\lambda(1)\big) = u^F_{\mathbf{1}},$$

thus $\rho_{\mathbf{1}} = 1$. So $\rho_{\xi\oplus\mathbf{1}} = \rho_\xi$, i.e. ρ is stable. □

A consequence of this lemma is that ρ is determined by its value on the canonical line bundle over $\mathbb{C}P^\infty$.

From now on, we shall consider the following setting. Let D^d, M and N^n be smooth, closed manifolds and $p\colon M \to N$ a smooth map such that $p\colon M \to N$ becomes a fiber bundle with fiber D. Then p is a **smooth fiber bundle**, and consequently, M is $n+d$-dimensional. The normal bundle of D, τ_D, can be viewed as a subbundle of τ_M, and then one has that

$$\tau_D \oplus p^*\tau_N \cong \tau_M.$$

Let ν_M denote the tangent bundle of M and consider

$$\tau_D \oplus \nu_M \oplus p^*\tau_N \cong \tau_M \oplus \nu_M \cong \mathbf{p}.$$

The embedding can be chosen such that $\mathbf{p} \downarrow M$ is even dimensional and therefore can be endowed with a complex structure. Therefore it is E-orientable, and by Theorem 6, Section I.C.3 of [**Dye69**], $\nu_M \oplus p^*\tau_N$ is E-orientable if and only if τ_D is. Assuming this is the case, we obtain

$$\rho_{\tau_D} \smile \rho_{\nu_M\oplus p^*\tau_N} = \rho_{\mathbf{p}} = 1.$$

Hence we may formulate the Riemann–Roch theorem the following way.

COROLLARY 4.1.4. *Let $\lambda\colon E^*(-) \to F^*(-)$ be a multiplicative transformation of complex oriented cohomology theories, and let $p\colon M \to N$ be a smooth fiber bundle with fiber D as before. Assume that D is a stably complex manifold, with tangent bundle τ. Then for all $\alpha \in E^*(M)$*

$$\lambda p^!_E(\alpha) = p^!_F(\lambda\alpha \smile \rho_\tau^{-1}),$$

where ρ_τ^{-1} denotes the multiplicative inverse of the correction class ρ_τ. □

We remark that these Riemann–Roch formulas may be "composed" the obvious way: Let multiplicative natural transformations $\lambda\colon E^*(-) \to F^*(-)$ and $\mu\colon F^*(-) \to G^*(-)$ be given. In the setting of Corollary 4.1.4 we get

$$\begin{aligned} \mu\lambda p^!_E(\alpha) &= \mu p^!_F(\lambda\alpha \smile \rho_\tau^{-1}) \\ &= p^!_G(\mu\lambda\alpha \smile \mu\rho_\tau^{-1} \smile \sigma_\tau^{-1}), \end{aligned} \tag{4.1.5}$$

where ρ and σ are the correction classes for λ and μ respectively. Thus the correction class of the composition $\mu\lambda\colon E^*(-) \to G^*(-)$ is $\mu\rho_\tau^{-1} \smile \sigma_\tau^{-1}$.

We make a final observation. As one specifies an orientation for a complex orientable cohomology theory by choosing an Euler class for $\eta \downarrow \mathbb{C}P^\infty$, it would be convenient to express the correction class for vector bundles directly in terms of Euler classes.

LEMMA 4.1.6. *The correction class for the canonical line bundle with respect to the multiplicative natural transformation* $\lambda\colon E^*(-) \to F^*(-)$ *is* $\rho_\eta = \frac{\lambda\omega_E}{\omega_F}$.

PROOF. By definition, ρ_η is the unique element satisfying $\rho_\eta \cdot u_\eta^F = \lambda u_\eta^E$. With the notation of Chapter 1, we get that

$$\rho_\eta \smile e_F(\eta) = z^*\pi^*\rho_\eta \smile z^*u_\eta^F = z^*(\rho_\eta \cdot u_\eta^F) = z^*\lambda u_\eta^E = \lambda e_E(\eta).$$

Note that $\lambda\omega_E$ is an orientation class of $F^*(\mathbb{C}P^\infty)$, and so it is a power series with leading term ω_F, and thus the division makes sense. □

2. Examples of correction classes

From now on, we will assume the setting from Corollary 4.1.4. In other words, we consider complex oriented cohomology theories $E^*(-)$ and $F^*(-)$ and a multiplicative natural transformation $\lambda\colon E^*(-) \to F^*(-)$. By the last lemma of the previous section, it suffices to compute $\rho_\eta^{-1} = \frac{\omega_F}{\lambda\omega_E}$ and extend by the splitting principle to all (stably) complex vector bundles; such as the tangent bundle of the fiber D, as above.

In what follows, we compute several examples of correction classes.

EXAMPLE 4.2.1 (Chern character). The first example we consider is associated to the **Chern character**, which by definition is the ring morphism of ring spectra

$$ch\colon K \simeq S^0 \wedge K \xrightarrow{\iota\wedge 1} H \wedge K.$$

In cohomology it induces a multiplicative natural transformation (see [**Swi02**])

$$ch\colon K^*(-) \to H^*(-; H_*K)$$

where $H_*K \cong \mathbb{Q}[u, u^{-1}]$ and u is the Bott element of cohomological degree -2. Explicitly, the Chern character is determined by the splitting principle and naturality by

$$ch(\eta) = e^{-uc_1(\eta)}$$
$$ch(u) = u.$$

Recall that for rational cohomology we have chosen the orientation class $\omega_H = c_1(\eta)$ in $H^2(\mathbb{C}P^\infty; H_*K)$, while we for K-theory chose $\omega_K = \frac{1-\eta}{u}$. Thus,

$$ch(\omega_K) = \frac{1 - ch(\eta)}{u} = \frac{1 - e^{-uc_1(\eta)}}{u} = \frac{1 - e^{-u\omega_H}}{u},$$

so the correction class ρ_η is

$$\rho_\eta = \frac{ch\,\omega_K}{\omega_H} = \frac{u^{-1}(1 - e^{-ux})}{x},$$

where we have put $x = \omega_H$.

Thus ρ_η is the inverse of the Todd class $\mathrm{td}(\eta)$ which has been modified by u to be a homogeneous element of degree 0 in $H^*(\mathbb{C}P^\infty; H_*K)$. Appealing to Corollary 4.1.4 we conclude that the Chern character has associated Riemann–Roch formula

$$ch\, p_K^!(\alpha) = p_H^!\big(\, ch\,\alpha \smile \mathrm{td}(\tau)\big),$$

where $\mathrm{td}(\eta) = \frac{ux}{1-e^{-ux}} \in H^0(\mathbb{C}P^\infty; H_*K)$.

Because multiplicative natural transformations $K^*(-) \to H^*(-; H_*K)$ are determined by their action on coefficients and on the orientation class, we could have described the Chern character with an application of Theorem 2.6.1: Let $\lambda\colon \mathbb{Z}[u, u^{-1}] \hookrightarrow \mathbb{Q}[u, u^{-1}] \cong H_*K$ be the inclusion of the coefficient ring of $K^*(-)$ into the coefficient ring of $H^*(-; H_*K)$. We saw in Example 3.1.6 that the Todd genus classifies the formal group law F_K of K-theory. Let $\varphi\colon MU_* \to H_*K$ classify the additive formal group law F_H. These are the ring homomorphisms realizing K-theory and singular cohomology as

Landweber exact theories. Thus $\exp_K\colon F_H \to F_K$ is a strict isomorphism of formal group laws over H_*K, and by Theorem 2.6.1 there is a multiplicative natural transformation $\hat{\lambda}\colon K^*(-) \to H^*(-;H_*K)$ such that $\hat{\lambda}(\omega_K) = \exp_K(\omega_H)$. Since $ch(\omega_K) = \exp_K(\omega_H)$, it follows that $\hat{\lambda} = ch$.

Recall from Chapter 1 the multiplicative natural transformation

$$c\colon KO[\tfrac{1}{2}]^*(-) \to K[\tfrac{1}{2}]^*(-)$$

given by complexification of vector bundles. We showed in Proposition 1.3.20 that an element lies in the image of c if and only if it is invariant under ψ^{-1}; that is $\xi \mapsto \bar{\xi}$ and $u \mapsto -u$. We are interested in choosing an orientation class for $KO[\frac{1}{2}]^*(-)$, so in particular this class must map to an element of $K[\frac{1}{2}]^2(\mathbb{C}P^\infty)$ invariant under this action. We see that the orientation class for $K[\frac{1}{2}]$-theory,

$$\omega_K = \frac{1-\eta}{u} \in K[\tfrac{1}{2}]^2(\mathbb{C}P^\infty),$$

is not invariant under ψ^{-1} and thus does not lie in the image of c, so in particular it is not the image of an orientation class for $KO[\frac{1}{2}]^*(-)$.

In fact,

$$\psi^k(\omega_K) = \psi^k\left(\frac{1-\eta}{u}\right) = \frac{1-\eta^k}{ku} = \frac{1}{k}\frac{(1-u\omega_K)^k - 1}{-u} = \frac{1}{k}[k]_K(\omega_K)$$

(using Example 2.1.4), so in particular we have

$$\psi^{-1}(\omega_K) = -[-1]_K(\omega_K) = \frac{\omega_K}{1-u\omega_K}. \tag{4.2.2}$$

Example 4.2.3 (Characterizing an orientation of $KO[\frac{1}{2}]^*(-)$). In what follows we shall characterize an Euler class $e(\xi)$ of $KO[\frac{1}{2}]^*(-)$ by considering its complexification, $ce(\xi)$. Again using Proposition 1.3.20, we see that the class

$$e_K(\ell)e_K(\bar{\ell}) = \frac{(1-\ell)(1-\bar{\ell})}{u^2} \in K[\tfrac{1}{2}]^4(X),$$

where ℓ is a complex line bundle over X, is in the image of c since it is invariant under the $\mathbb{Z}_2$-action in question. Trying to get a hold of $ce(\xi)$, we demand both that

$$ce(\ell)ce(\bar{\ell}) = e_K(\ell)e_K(\bar{\ell})$$

and that ce is **odd**, i.e.

$$ce(\bar{\ell}) = -ce(\ell).$$

Combining these two equations with (4.2.2), we now obtain

$$ce(\eta)^2 = -e_K(\eta)e_K(\bar{\eta}) = \omega_K\psi^{-1}(\omega_K) = \frac{\omega_K^2}{1-u\omega_K}.$$

Taking square roots, we get the expression

$$ce(\eta) = \frac{\omega_K}{\sqrt{1-u\omega_K}} = \frac{1-\eta}{u\sqrt{\eta}}. \tag{4.2.4}$$

We have $ce(\eta) = \varphi(\omega_K)$, where $\varphi(x) = \frac{x}{\sqrt{1-ux}}$ is a homogeneous power series with leading term x in $\pi_*K[\frac{1}{2}][\![x]\!]$. Therefore, $ce(\eta)$ is another orientation class for $K[\frac{1}{2}]^*(-)$. To see that this really is the image of a class from $KO[\frac{1}{2}]^2(\mathbb{C}P^\infty)$ under complexification, we apply the usual criterion:

$$\psi^{-1}\big(ce(\eta)\big) = \psi^{-1}\left(\frac{1-\eta}{u\sqrt{\eta}}\right) = \frac{1-\bar{\eta}}{-u\sqrt{\bar{\eta}}} = \frac{\eta-\eta\bar{\eta}}{-u\eta\sqrt{\bar{\eta}}} = \frac{1-\eta}{u\sqrt{\eta}} = ce(\eta)$$

Since the complexification is injective (Proposition 1.3.19), $e(\eta)$ is uniquely determined by (4.2.4). This class is actually an orientation class for $KO[\frac{1}{2}]^*(-)$; to see why this is true, consider the computation

$$c\big(j^*e(\eta)\big) = j^*\big(ce(\eta)\big) = j^*\big(\varphi(\omega_K)\big) = \sigma^2(1) = \sigma^2\big(c(1)\big) = c\big(\sigma^2(1)\big),$$

where $j\colon S^2 \hookrightarrow \mathbb{C}P^\infty$, and then use injectivity of c.

Write $\omega_{KO} := e(\eta) \in KO[\frac{1}{2}]^2(\mathbb{C}P^\infty)$ for this orientation class and $e_{KO}(\xi)$ for the Euler class $e(\xi)$. Although this uniquely determines the orientation class, we seek to extract information more suitable for our purposes.

We write F for the formal group law over $\pi_*K[\frac{1}{2}]$ determined by the Euler class ce_{KO}. As φ transforms the Euler class e_K into ce_{KO}, we have a strict isomorphism of formal group laws $\varphi\colon F_K \to F$. In particular, the exponential function associated to this formal group law is given by

$$\begin{aligned}\exp_F(x) = \varphi \circ \exp_K(x) = \varphi\left(\frac{1-e^{-ux}}{u}\right) &= \frac{1-e^{-ux}}{ue^{-ux/2}} \\ &= 2u^{-1}\sinh(ux/2) \in (\pi_*K[\tfrac{1}{2}] \otimes \mathbb{Q})[\![x]\!].\end{aligned}$$

As $c\exp_{KO} = \exp_F$, this gives the exponential series obtained from the $KO[\frac{1}{2}]$-orientation:

$$\exp_{KO}(x) = 2u^{-1}\sinh(ux/2) \in (\pi_*KO[\tfrac{1}{2}] \otimes \mathbb{Q})[\![x]\!]. \tag{4.2.5}$$

We note that this makes sense; this power series is odd, and therefore there will only be occurrences of even powers of u.

Example 4.2.6 (Complexification). Consider the complexification

$$c\colon KO[\tfrac{1}{2}]^*(-) \to K[\tfrac{1}{2}]^*(-),$$

where the cohomology theories have Euler classes e_{KO} and e_K respectively. The Riemann–Roch theorem takes the form

$$cp^!_{KO}(\alpha) = p^!_K\big(c\alpha \smile \rho_\tau^{-1}\big),$$

where the characteristic class ρ_τ^{-1} is given by

$$\rho_\eta^{-1} = \frac{\omega_K}{c\omega_{KO}} = \sqrt{\eta},$$

on the canonical line bundle $\eta \downarrow \mathbb{C}P^\infty$.

Example 4.2.7 (Complexified Chern character). Composing the Chern character and the complexification, we now obtain a multiplicative natural transformation we call the **complexified Chern character**

$$chc\colon KO[\tfrac{1}{2}]^*(-) \to H^*(-;H_*K).$$

Using (4.1.5), we see that the correction class of this composition is given on the canonical line bundle by

$$\begin{aligned}\rho_\eta^{-1} &= ch(\sqrt{\eta}) \smile \mathrm{td}(\eta) \\ &= \sqrt{ch(\eta)}\frac{ux}{1-e^{-ux}} \\ &= e^{-ux/2}\frac{ux}{1-e^{-ux}} \\ &= \frac{x}{2u^{-1}\sinh(ux/2)}\end{aligned}$$

where $x = c_1(\eta)$.

Thus the correction class for $chc\colon KO[\frac{1}{2}]^*(-) \to H^*(-;H_*K)$ is the total $\hat{A}$-class from (1.2.14) modified to land in degree 0. This shows that the Riemann–Roch theorem in this case reads

$$chc\, p^!_{KO}(\alpha) = p^!_H\big(\, chc\,\alpha \smile \hat{A}(\tau)\big),$$

where the subscript H denotes singular cohomology with coefficients H_*K. Because of this occurrence of $\hat{A}$, we shall refer to this specific orientation of $KO[\frac{1}{2}]$ as the **$\hat{A}$-orientation**. We will write $\omega_{\hat{A}}$ for this orientation class and $e_{\hat{A}}$ for the corresponding Euler class from this point on.

3. The elliptic character

In this section we return to elliptic cohomology, and the goal of is to compute the correction class for a multiplicative natural transformation between elliptic and rational cohomology. This natural transformation is the "elliptic character", and it is produced using Theorem 2.6.1.

Recall that Theorem 3.2.12 enables us to view the ring $M_* = \mathbb{Z}[\frac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}]$ as the ring of modular forms for $\Gamma_0(2)$ with coefficients in $\mathbb{Z}[\frac{1}{2}]$. Theorem 3.2.8 gave explicit formulas for $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ as modular forms, and their constant terms motivates us to try relating elliptic cohomology and $KO[\frac{1}{2}]^*(-)$ with the $\hat{A}$-orientation.

Any modular form f in $\mathbb{Z}[\frac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}]$ has a q-expansion $\tilde{f}(q) = f(\tau)$ at ∞ which is an element of $\mathbb{Z}[\frac{1}{2}][\![q]\!]$. Recall from (1.3.6) that with 2 inverted, $\pi_*KO[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}][u^2, u^{-2}]$, where u has cohomological degree -2. Giving $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ homological degrees 4 and 8 respectively, we construct a grading preserving ring homomorphism

$$\lambda\colon \mathbb{Z}[\tfrac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}] \to \pi_*KO[\tfrac{1}{2}][\![q]\!]$$

by sending a weight $2k$ modular form f to $u^{2k}\tilde{f}(q) \in \pi_{4k}KO[\frac{1}{2}][\![q]\!]$, its q-expansion at ∞ which has been shifted by u to land in the correct degree.

We make the following easy observation.

Lemma 4.3.1. *λ factors through π_*Ell to give a ring homomorphism on coefficient rings*

$$\lambda\colon \pi_*Ell \to \pi_*KO[\tfrac{1}{2}][\![q]\!] \tag{4.3.2}$$

if and only if we choose the localization $\pi_*Ell = M_*[(\delta^2_{\hat{A}} - \varepsilon_{\hat{A}})^{-1}]$.

Proof. The q-expansion of $\varepsilon_{\hat{A}}$ has no multiplicative inverse in $\pi_*KO[\frac{1}{2}][\![q]\!]$ as it has no constant term. Thus for λ to factor through π_*Ell we must choose a localization where $\varepsilon_{\hat{A}}$ is not inverted. This excludes $\gamma = \varepsilon_{\hat{A}}$ and $\gamma = \Delta_{\hat{A}} = \varepsilon_{\hat{A}}(\delta^2_{\hat{A}} - \varepsilon_{\hat{A}})^2$. □

Let λ denote the ring homomorphism (4.3.2). Since $\mathbb{Z}[\frac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}]$ is generated by $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$, λ is determined by the q-expansions of $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$. From Theorem 3.2.8,

$$\begin{aligned} \lambda(\delta_{\hat{A}}) &= u^2\Big(-\frac{1}{8} - 3\sum_{n\geq 1}\Big(\sum_{2\nmid d\mid n} d\Big)q^n\Big), \\ \lambda(\varepsilon_{\hat{A}}) &= u^4\Big(\sum_{n\geq 1}\Big(\sum_{\substack{d\mid n \\ 2\nmid n/d}} d^3\Big)q^n\Big). \end{aligned} \tag{4.3.3}$$

We have found $Ell^*(-) = Ell^* \otimes_{MU^*} MU^*(-)$ to be a Landweber exact cohomology theory via the ring homomorphism $\varphi\colon MU_* \to Ell_*$ classifying the Euler formal group law. By Corollary 3.1.25 also $KO[\frac{1}{2}]^*(-)$ is Landweber exact, with formal group law $F_{\hat{A}}$ given by the exponential power series $\exp_{\hat{A}}(x)$ in (4.2.5). Setting the stage for Theorem 2.6.1, we have:

PROPOSITION 4.3.4. *The power series $\theta(x)$ over $\pi_* KO[\frac{1}{2}][\![q]\!]$ given by*

$$\theta(x) = x \prod_{n\geq 1} \left(1 - \frac{q^n u^2 x^2}{(1-q^n)^2}\right)^{(-1)^n}$$

is a strict isomorphism of formal group laws $\theta\colon F_{\hat{A}} \to \lambda F_{Ell}$.

PROOF. It is equivalent to show that the diagram

$$\begin{tikzcd} & F_a \arrow[dl, "\exp_{\hat{A}}"'] \arrow[dr, "\lambda\exp_{Ell}"] & \\ F_{\hat{A}} \arrow[rr, "\theta"] & & \lambda F_{Ell} \end{tikzcd}$$

commutes.

Recall that the logarithm of the elliptic genus $\varphi\colon MSO_* \to \mathbb{Z}[\frac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}]$ is $\log_\varphi(x) = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-2\delta_{\hat{A}}t^2+\varepsilon_{\hat{A}}t^4}}$. As λ merely introduces the appropriate power of u^2 and identifies the modular forms with their q-expansions at ∞ we see that (by slightly abusing notation and writing $\delta_{\hat{A}}, \varepsilon_{\hat{A}}$ for both the modular forms and their q-expansions)

$$\lambda \log_{Ell}(x) = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-2\delta_{\hat{A}}u^2t^2+\varepsilon_{\hat{A}}u^4t^4}} = \frac{1}{u}\int_0^{ux} \frac{\mathrm{d}t}{\sqrt{1-2\delta_{\hat{A}}t^2+\varepsilon_{\hat{A}}t^4}} = \frac{1}{u}\log_\varphi(ux).$$

Thus $\lambda\exp_{Ell}(x) = u^{-1}\exp_\varphi(ux)$, and by applying Theorem 3.2.8 we now obtain

$$\begin{aligned} \lambda\exp_{Ell}(x) &= 2u^{-1}\sinh(ux/2) \prod_{n\geq 1}\left(\frac{(1-q^n e^{ux})(1-q^n e^{-ux})}{(1-q^n)^2}\right)^{(-1)^n} \\ &= 2u^{-1}\sinh(ux/2) \prod_{n\geq 1}\left(1 - \frac{q^n u^2 \big(2u^{-1}\sinh(ux/2)\big)^2}{(1-q^n)^2}\right)^{(-1)^n} \\ &= \exp_{\hat{A}}(x) \prod_{n\geq 1}\left(1 - \frac{q^n u^2 \exp_{\hat{A}}(x)^2}{(1-q^n)^2}\right)^{(-1)^n}. \end{aligned}$$

This shows that $\theta \circ \exp_{\hat{A}} = \lambda\exp_{Ell}$, i.e. the diagram commutes. □

Now we are in a position to apply Miller's construction: By Theorem 2.6.1 there is a multiplicative natural transformation

$$eh := \hat{\lambda}\colon Ell^*(-) \to KO[\tfrac{1}{2}]^*(-)[\![q]\!]$$

such that $eh(\omega_{Ell}) = \theta(\omega_{\hat{A}})$ and identifying modular forms with their q-expansion on coefficients. This natural transformation is called the **elliptic character**. Clarke and Johnson justifies this name in [**CJ92**], where they show that this natural transformation is induced by the ring morphism of spectra

$$Ell \simeq S^0 \wedge Ell \to KO \wedge Ell$$

by analogy with the Chern character.

With the chosen orientations, the Riemann–Roch theorem states that

$$eh\, p^!_{Ell}(\alpha) = p^!_{KO}\big(\, eh\,\alpha \smile \rho_\tau^{-1}\big),$$

where (on the canonical line bundle)

$$\rho_\eta^{-1} = \frac{\omega_{\hat{A}}}{eh(\omega_{Ell})} = \frac{\omega_{\hat{A}}}{\theta(\omega_{\hat{A}})} = \prod_{n\geq 1}\left(1 - \frac{q^n u^2 \omega_{\hat{A}}^2}{(1-q^n)^2}\right)^{(-1)^{n+1}}. \tag{4.3.5}$$

Composing eh with the complexified Chern character, we obtain a Riemann–Roch formula for the multiplicative natural transformation

$$chc\,eh\colon Ell^*(-) \to H^*(-;H_*K)[\![q]\!],$$

where the correction class is

$$chc\,\rho_\xi^{-1} \smile \hat{A}(\xi).$$

This expression can be better appreciated in an alternative form. Specifically we use the constructions $\lambda_t\xi$ and $s_t\xi$ of Chapter 1 and their exponentiality to compute $c\rho_\xi^{-1}$ in complex K-theory. Since the complexification is both additive and multiplicative, we get that

$$\begin{aligned}
c\left(1 - \frac{q^n u^2 \omega_{\hat{A}}^2}{(1-q^n)^2}\right) &= 1 - \frac{q^n c(u^2) c(\omega_{\hat{A}})^2}{(1-q^n)^2} \\
&= 1 - \frac{q^n u^2 \left(\frac{1-\eta}{u\sqrt{\eta}}\right)^2}{(1-q^n)^2} \\
&= 1 - \frac{q^n \bar{\eta}(1-\eta)^2}{(1-q^n)^2} \\
&= \frac{(1-q^n\eta)(1-q^n\bar{\eta})}{(1-q^n)^2} \\
&= \frac{\lambda_{-q^n}\eta \otimes \lambda_{-q^n}\bar{\eta}}{(\lambda_{-q^n}\mathbf{1})^2} \\
&= \lambda_{-q^n}(\eta - \mathbf{1}) \otimes \lambda_{-q^n}(\overline{\eta - \mathbf{1}}) \\
&= \lambda_{-q^n}\big((\eta - \mathbf{1})_{\mathbb{R}} \otimes \mathbb{C}\big)
\end{aligned}$$

Using Lemma 1.3.14,

$$c\rho_\eta^{-1} = c\left(\bigotimes_{n\geq 1} s_{q^{2n}}(\eta - \mathbf{1})_{\mathbb{R}} \otimes \lambda_{-q^{2n-1}}(\eta - \mathbf{1})_{\mathbb{R}}\right),$$

so via the splitting principle, we get the following correction class for any complex vector bundle ξ:

$$R_q^{\hat{A}}(\xi) := c\rho_\xi^{-1} = c\left(\bigotimes_{n\geq 1} s_{q^{2n}}(\xi - \dim\xi)_{\mathbb{R}} \otimes \lambda_{-q^{2n-1}}(\xi - \dim\xi)_{\mathbb{R}}\right)$$

We conclude that with the orientations chosen as above, the multiplicative natural transformation

$$chc\,eh\colon Ell^*(-) \to H^*(-;H_*K)[\![q]\!]$$

gives rise to the Riemann–Roch formula

$$chc\,eh\,p^!_{Ell}(\alpha) = p^!_H\big(\,chc\,eh\,\alpha \smile ch\,R_q^{\hat{A}}(\tau) \smile \hat{A}(\tau)\big).$$

4. Choosing another orientation of $KO[\frac{1}{2}]$

It is interesting to see what happens to the formulas of the previous section if one tries to change the orientations. Specifically, we try to characterize another orientation class for $KO[\frac{1}{2}]$, as we did in the preceding discussion.

For a complex line bundle $\ell \downarrow X$ consider the class $f(\ell) = \frac{1}{u}\frac{1-\ell}{1+\ell}$ of $K[\frac{1}{2}]^2(X)$. It is easily seen to be invariant under the $\mathbb{Z}_2$-action ψ^{-1}, and so lies in the image of the complexification. One can verify that

$$f(\ell \otimes \ell') = \frac{f(\ell) + f(\ell')}{1 + u^2 f(\ell) f(\ell')},$$

which is (up to correction with u) the addition law for tanh.

It may seem that we have characterized the L-genus, but unfortunately this class does not give an orientation of $K[\frac{1}{2}]$. One sees this by expressing $f(\ell)$ in terms of $e_K(\ell)$; with $\varphi(x) = \frac{x}{2-ux}$ one has

$$\varphi\big(e_K(\ell)\big) = \frac{1}{u}\frac{1-\ell}{1+\ell} = f(\ell).$$

This is a power series of $\pi_* K[\frac{1}{2}][\![x]\!]$, but its leading term is $\frac{1}{2}x$, so $f(\eta)$ is not an orientation class for $K[\frac{1}{2}]^*(-)$. This problem is only a minor one, however, and starting with $2f(\eta)$ in place of $f(\eta)$ corrects this.

Thus we characterize an Euler class $e_{\hat{L}}$ in $KO[\frac{1}{2}]^*(-)$ by demanding that on line bundles

$$ce_{\hat{L}}(\ell) = \frac{2}{u}\frac{1-\ell}{1+\ell}.$$

Applying the Chern character to this, we see that

$$chc\, e_{\hat{L}}(\eta) = \frac{2}{u}\frac{1-e^{-ux}}{1+e^{-ux}} = 2u^{-1}\tanh(ux/2),$$

where $x = e_H(\eta)$. It follows that the correction class is given by

$$\rho_\eta^{-1} = \frac{e_H(\eta)}{chc\, e_{\hat{L}}(\eta)} = \hat{L}(\eta),$$

where $\hat{L}(\eta)$ is modified by u to land in degree 0. We put $\omega_{\hat{L}} = e_{\hat{L}}(\eta)$ and call this the **$\hat{L}$-orientation** for $KO[\frac{1}{2}]^*(-)$. Using this orientation, the Riemann–Roch formula is

$$chc\, p^!_{KO}(\alpha) = p^!_H\big(\, chc\,\alpha \smile \hat{L}(\tau)\big).$$

Recall from Corollary 3.1.25 that $KO[\frac{1}{2}]^*(-)$ is a Landweber exact cohomology theory with the $\hat{A}$-orientation. Now the formal group laws $F_{\hat{L}}$ and $F_{\hat{A}}$ are strictly isomorphic over $\pi_* KO[\frac{1}{2}]$, for

$$\theta(x) := \exp_{\hat{A}}\big(\log_{\hat{L}}(x)\big) = 2u^{-1}\sinh\big(\tanh^{-1}(ux/2)\big) = \frac{x}{1-(ux/2)^2}$$

is a power series of $\pi_* KO[\frac{1}{2}]$ with leading term x. By Proposition 2.5.8, $KO[\frac{1}{2}]^*(-)$ is Landweber exact with respect to the $\hat{L}$-orientation as well.

In light of Theorem 2.6.1 and by analogy to the case with the $\hat{A}$-orientation, we try to describe a ring homomorphism

$$\lambda\colon \pi_* Ell \to \pi_* KO[\tfrac{1}{2}]$$

for a suitable choice of localization.

Again, [**Zag88**] provides useful formulas.

THEOREM 4.4.1. *Let $\delta_{\hat{L}}$ and $\varepsilon_{\hat{L}}$ be modular forms of weight 2 and 4 for $\Gamma_0(2)$, defined by the q-expansions at ∞;*

$$\delta_{\hat{L}} = \frac{1}{4} + 6\sum_{n\geq 1}\Big(\sum_{2\nmid d|n} d\Big)q^n$$

$$\varepsilon_{\hat{L}} = \frac{1}{16} + \sum_{n\geq 1}\Big(\sum_{d|n}(-1)^d d^3\Big)q^n.$$

The exponential series of the elliptic genus $\varphi\colon MSO_* \to \mathbb{Z}[\frac{1}{2}][\delta_{\hat{L}}, \varepsilon_{\hat{L}}]$ *satisfies*

$$\frac{z}{\exp_\varphi(z)} = \frac{z/2}{\tanh(z/2)} \prod_{n\geq 1} \frac{(1+q^n e^z)(1+q^n e^{-z})/(1+q^n)^2}{(1-q^n e^z)(1-q^n e^{-z})/(1-q^n)^2}.$$

The coefficient of z^{2k} *is a weight* $2k$ *modular form for* $\Gamma_0(2)$. □

Again, the subscript $\hat{L}$ reflects the constant terms of these q-expansions.

The analogue of Lemma 4.3.1 shows that we have a ring homomorphism

$$\lambda\colon \pi_* Ell \to \pi_* KO[\tfrac{1}{2}][\![q]\!]$$

identifying modular forms with their q-expansions, if and only if we choose the localization where $\varepsilon_{\hat{L}}$ is inverted. This is because the q-expansion of $\delta_{\hat{L}}^2 - \varepsilon_{\hat{L}}$ has no constant term, and therefore no multiplicative inverse.

Letting $F_{\hat{L}}$ be the formal group law over $\pi_* KO[\frac{1}{2}][\![q]\!]$ with exponential series $\exp_{\hat{L}}(x) = 2u^{-1}\tanh(ux/2)$, we get a result analogous to Proposition 4.3.4.

Proposition 4.4.2. *The formal group laws* $F_{\hat{L}}$ *and* λF_{Ell} *over* $\pi_* KO[\frac{1}{2}][\![q]\!]$ *are strictly isomorphic, and an isomorphism* $\theta\colon F_{\hat{L}} \to \lambda F_{Ell}$ *is explicitly given by*

$$\theta(y) = y \prod_{n\geq 1}\left(1 - \frac{8q^n(1+q^{2n})u^2y^2}{4(1-q^{2n})^2 - (1-q^n)^4u^2y^2}\right).$$

Proof. Arguing as we did in Proposition 4.3.4,

$$\lambda \exp_{Ell}(x) = 2u^{-1}\tanh\left(\frac{ux}{2}\right) \prod_{n\geq 1} \frac{(1-q^n e^{ux})(1-q^n e^{-ux})/(1-q^n)^2}{(1+q^n e^{ux})(1+q^n e^{-ux})/(1+q^n)^2},$$

where we have used the previous theorem.

Now we put $y = \exp_{\hat{L}}(x) = 2u^{-1}\tanh(ux/2)$. It is easily verified that

$$e^{ux} = \frac{2+uy}{2-uy},$$

and so the expression becomes

$$\theta(y) := \lambda \exp_{Ell}(x) = y \prod_{n\geq 1} \frac{(1-q^n\frac{2+uy}{2-uy})(1-q^n\frac{2-uy}{2+uy})/(1-q^n)^2}{(1+q^n\frac{2+uy}{2-uy})(1+q^n\frac{2-uy}{2+uy})/(1+q^n)^2}. \tag{4.4.3}$$

The proof can be completed by a gruesome expansion. □

As in the previous case, Theorem 2.6.1 shows that there is a multiplicative natural transformation

$$eh\colon Ell^*(-) \to KO[\tfrac{1}{2}]^*(-)[\![q]\!]$$

with the specified orientations which coincides with λ on coefficients and satisfies $eh(\omega_{Ell}) = \theta(\omega_{\hat{L}})$. The correction class for this natural transformation thus satisfies

$$\rho_\eta^{-1} = \frac{\omega_{\hat{L}}}{eh(\omega_{Ell})} = \frac{\omega_{\hat{L}}}{\theta(\omega_{\hat{L}})}.$$

Again we complexify and use the operations of K-theory to express the correction class in a nicer way. The most suitable form of $\theta(y)$ for this purpose is the one in (4.4.3). Putting $y = \omega_{\hat{L}} = e_{\hat{L}}(\eta)$ and complexifying, the nth factor becomes

$$\frac{\left(1-q^n\frac{2+uc(\omega_{\hat{L}})}{2-uc(\omega_{\hat{L}})}\right)\left(1-q^n\frac{2-uc(\omega_{\hat{L}})}{2+uc(\omega_{\hat{L}})}\right)/(1-q^n)^2}{\left(1+q^n\frac{2+uc(\omega_{\hat{L}})}{2-uc(\omega_{\hat{L}})}\right)\left(1+q^n\frac{2-uc(\omega_{\hat{L}})}{2+uc(\omega_{\hat{L}})}\right)/(1+q^n)^2}.$$

Recalling that $c(\omega_{\hat{L}}) = \frac{2}{u}\frac{1-\eta}{1+\eta}$ and then using that

$$\frac{2 - uc(\omega_{\hat{L}})}{2 + uc(\omega_{\hat{L}})} = \frac{1 - \frac{1-\eta}{1+\eta}}{1 + \frac{1-\eta}{1+\eta}} = \eta,$$

the nth term is seen to be

$$\frac{\frac{1-q^n\bar{\eta}}{1-q^n}\frac{1-q^n\eta}{1-q^n}}{\frac{1+q^n\bar{\eta}}{1+q^n}\frac{1+q^n\eta}{1+q^n}} = \frac{\lambda_{-q^n}(\overline{\eta - \mathbf{1}}) \otimes \lambda_{-q^n}(\eta - \mathbf{1})}{\lambda_{q^n}(\overline{\eta - \mathbf{1}}) \otimes \lambda_{q^n}(\eta - \mathbf{1})} = c\left(\frac{\lambda_{-q^n}\big((\eta - \mathbf{1})_{\mathbb{R}}\big)}{\lambda_{q^n}\big((\eta - \mathbf{1})_{\mathbb{R}}\big)}\right).$$

It follows that

$$c\rho_\eta^{-1} = \frac{c(\omega_{\hat{L}})}{\theta\big(c(\omega_{\hat{L}})\big)} = c\left(\bigotimes_{n\geq 1} \lambda_{q^n}\big((\eta - \mathbf{1})_{\mathbb{R}}\big) \otimes s_{q^n}\big((\eta - \mathbf{1})_{\mathbb{R}}\big)\right).$$

Consequently, for any complex vector bundle ξ, the correction class is

$$R_q^{\hat{L}}(\xi) := c\rho_\xi^{-1} = c\left(\bigotimes_{n\geq 1} \lambda_{q^n}\big((\xi - \dim\xi)_{\mathbb{R}}\big) \otimes s_{q^n}\big((\xi - \dim\xi)_{\mathbb{R}}\big)\right).$$

The Riemann–Roch formula for $\mathit{chc}\,\mathit{eh}\colon \mathit{Ell}^*(-) \to H^*(-;H_*K)[\![q]\!]$ with these choices of orientations thus becomes

$$\mathit{chc}\,\mathit{eh}\, p^!_{\mathit{Ell}}(\alpha) = p^!_H\big(\,\mathit{chc}\,\mathit{eh}\,\alpha \smile \mathit{ch}\, R_q^{\hat{L}}(\tau) \smile \hat{L}(\tau)\big).$$

5. Comparing the calculations

In the preceding sections we blindly used formulas of [**Zag88**] without providing any insight. There are some things that should be said about how these formulas are related, however, and it will be the goal of this section to shed some light on this. To do so, we consider an explicit construction of the universal elliptic genus.

Let $L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ be a lattice depending on $\tau \in \mathcal{H}$. The **Weierstrass $\wp$-function** for the lattice L is given by

$$\wp(\tau, x) := \frac{1}{x^2} + \sum_{0\neq\omega\in L} \frac{1}{(x-\omega)^2} - \frac{1}{\omega^2}.$$

We recall some standard facts about this function. Proofs can be found in [**HBJ92**]. $\wp(\tau, x)$ is elliptic in x with respect to L and has poles of order 2 precisely at every lattice point. Now let

$$d_1(\tau) := \pi i, \quad d_2(\tau) := \pi i\tau, \quad d_3(\tau) := \pi i(\tau + 1)$$

be the half-division points of the lattice. Putting $e_i(\tau) := \wp\big(\tau, d_i(\tau)\big)$ for $i = 1, 2, 3$, it can be shown that $\wp(\tau, x)$ takes these values with order 2, i.e. $\wp(\tau, x) - e_i(\tau)$ has a double zero at $d_i(\tau)$, and this is thus the only zero modulo L.

See [**HBJ92**, Chapter 2] for a proof of the following theorem.

THEOREM 4.5.1. *The function $f(\tau, x) := 1/\sqrt{\wp(\tau, x) - e_1(\tau)} = x + \cdots$ is odd and elliptic for the lattice $2\pi i(\mathbb{Z}\cdot 2\tau + \mathbb{Z})$. It has divisor*

$$\operatorname{div} f = (0) + (2\pi i\tau) - (\pi i) - \big(\pi i(2\tau + 1)\big).$$

Moreover, f is the exponential function of an elliptic genus; it satisfies

$$(f')^2 = 1 - 2\delta_f f^2 + \varepsilon_f f^4$$

where $\delta_f = -\frac{3}{2}e_1$ and $\varepsilon_f = (e_1 - e_2)(e_2 - e_3)$ (as functions of τ). □

The analogous result with e_2 in place of e_1 is as follows.

THEOREM 4.5.2. *The function* $g(\tau, x) := 1/\sqrt{\wp(\tau, x) - e_2(\tau)} = x + \cdots$ *is odd and elliptic for* $2\pi i(\mathbb{Z}\tau + \mathbb{Z} \cdot 2)$, *and the divisor of this function is*

$$\operatorname{div} g = (0) + (2\pi i) - (\pi i \tau) - \big(\pi i(\tau + 2)\big).$$

g satisfies

$$(g')^2 = 1 - 2\delta_g g^2 + \varepsilon_g g^4,$$

with $\delta_g = -\frac{3}{2}e_2$ *and* $\varepsilon_g = (e_2 - e_1)(e_1 - e_3)$. □

The dependence on τ begs the question: What happens when τ varies? From [**HBJ92**, Theorem I.3.6] we collect several facts regarding the modular properties of the e_i.

THEOREM 4.5.3. *The functions* $e_1(\tau)$, $e_2(\tau)$ *and* $e_3(\tau)$, *resp., are modular forms of weight* 2 *for* $\Gamma_0(2)$, $\Gamma^0(2) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma_0(2)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma_0(2)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, *resp. Their respective q-expansions are*

$$e_1(\tau) := -\frac{1}{6}\bigg(1 + 24\sum_{n\geq 1}\Big(\sum_{2\nmid d|n} d\Big)q^n\bigg)$$

$$e_2(\tau) := \frac{1}{12}\bigg(1 + 24\sum_{n\geq 1}\Big(\sum_{2\nmid d|n} d\Big)q^{n/2}\bigg)$$

$$e_3(\tau) := \frac{1}{12}\bigg(1 + 24\sum_{n\geq 1}(-1)^n\Big(\sum_{2\nmid d|n} d\Big)q^{n/2}\bigg),$$

and for all $A \in \mathrm{SL}_2(\mathbb{Z})$ *the* e_i *are permuted by the* $|^2_A$*-action;* e_1 *is fixed under the action of* $\Gamma_0(2)$ *and* e_2 *is fixed under the action of* $\Gamma^0(2)$. *Recalling the notation from* (3.2.6), *we have* $e_1^0 = e_2$ *and* $e_3^0 = e_3$, *that is, the expansion at* 0 *interchanges* e_1 *and* e_2, *but keeps* e_3 *fixed.* □

By comparing q-expansions, it for instance immediately follows that $e_1 + e_2 + e_3 = 0$ and $e_1(\tau) = -2e_2(2\tau)$. Recalling that $\delta_f = -\frac{3}{2}e_1$, shows that δ_f is the same modular form as $\delta_{\hat{L}}$ given in Theorem 4.4.1. Furthermore, the expansion of ε_f can be shown to coincide with $\varepsilon_{\hat{L}}$. Thus it follows that the elliptic genera of Theorem 4.4.1 and Theorem 4.5.1 coincide.

By the theorem above, $\delta_f^0 = \delta_g$ and $\varepsilon_f^0 = \varepsilon_g$, which are modular forms for $\Gamma^0(2)$. We get back to $\Gamma_0(2)$ by $\tau \mapsto 2\tau$, and by comparison of q-expansions with Theorem 3.2.8 one finds that $\delta_g(2\tau) = \delta_{\hat{A}}(\tau)$ and $\varepsilon_g(2\tau) = \varepsilon_{\hat{A}}(\tau)$. This shows that

$$\delta_{\hat{A}}(\tau) = \delta_{\hat{L}}^0(2\tau) \quad \text{and} \quad \varepsilon_{\hat{A}}(\tau) = \varepsilon_{\hat{L}}^0(2\tau).$$

In other words, the elliptic genera with parameters $\delta_{\hat{L}}, \varepsilon_{\hat{L}}$ and $\delta_{\hat{A}}, \varepsilon_{\hat{A}}$ are the same, but expanded at the two different cusps of $\Gamma_0(2)$ interchanged by $\tau \mapsto -1/2\tau$.

6. Final remarks

Through the course of the last two chapters we have seen how elliptic integrals, curves, functions and modular forms are all important aspects of elliptic cohomology. We have focused heavily on the algebraic topological side of things, but it is interesting to remark that (notably) Witten [**Wit88**] shows that there also is a tight connection between elliptic cohomology and differential geometry. In fact, the correction classes of the type $R_q^{\hat{A}}$ and $R_q^{\hat{L}}$ are often accredited him, as he shows how they arise in connection with the equivariant index of the Dirac operator; see the survey article by Segal [**Seg88**] and also [**HBJ92**].

As a last remark, we try to compensate for the fact that so far, we have not motivated the use of the Riemann–Roch theorem. We end this section by expressing the elliptic genus in another way, more true to the setting of Witten, et. al. The basic observation is as follows.

LEMMA 4.6.1. *Let $\varphi\colon MU \to E$ be a ring morphism giving E a complex orientation. Then the genus on coefficients, $\varphi\colon MU_* \to E_*$ coincides with the assignment*

$$M \mapsto p^!_E(1),$$

where $p\colon M \to pt$ is the projection, and $p^!_E\colon E^*(M) \to E^{*-m}(pt) = \pi_{m-*}E$. □

There is a ring morphism $\varphi\colon MU \to \mathit{Ell}$ orienting elliptic cohomology and classifying the Euler formal group law, where Ell is the representing spectrum of elliptic cohomology with coefficient ring

$$\mathbb{Z}[\tfrac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}][(\delta^2_{\hat{A}} - \varepsilon_{\hat{A}})^{-1}].$$

Let $p\colon M^n \to pt$, where M is closed, stably complex; this is a smooth fiber bundle with fiber M, and this fiber in turn has tangent bundle τ. We previously have investigated the non-commutative diagram

$$\begin{array}{ccc} \mathit{Ell}^0(M) & \xrightarrow{\quad p^!_{\mathit{Ell}}\quad} & \mathit{Ell}^{-n}(pt) \\ \Big\downarrow{\scriptstyle chc\,eh} & & \Big\downarrow{\scriptstyle chc\,eh} \\ H^0(M, H_*K)[\![q]\!] & \xrightarrow{\quad p^!_H\quad} & H^{-n}(pt, H_*K)[\![q]\!] \end{array}$$

where the correction class is $R^{\hat{A}}_q(\tau) \smile \hat{A}(\tau)$. The right vertical arrow is the embedding of coefficients

$$\mathbb{Z}[\tfrac{1}{2}][\delta_{\hat{A}}, \varepsilon_{\hat{A}}][(\delta^2_{\hat{A}} - \varepsilon_{\hat{A}})^{-1}] \hookrightarrow \mathbb{Q}[u, u^{-1}][\![q]\!]$$

in degree n, where $\delta_{\hat{A}}$ and $\varepsilon_{\hat{A}}$ are sent to their q-expansions multiplied with u^2 and u^4 respectively. In particular, if $n = 4k$, then $chc\,eh\,p^!_{\mathit{Ell}}(1) = \varphi(M)u^{2k}$, where $\varphi(M)$ is identified with its q-expansion. Going the other way in the diagram, and using the fact that $p^!_H(-) = \langle -, [M]\rangle$, the Kronecker pairing with the fundamental class of M, we see that

$$\varphi(M) = p^!_H\big(ch\,R^{\hat{A}}_q(\tau) \smile \hat{A}(\tau)\big)u^{-2k} = \langle ch\,R^{\hat{A}}_q(\tau) \smile \hat{A}(\tau), [M]\rangle u^{-2k} \in \mathbb{Q}[u, u^{-1}][\![q]\!].$$

We define the **twisted $\hat{A}$-genus** of M by ξ as

$$\hat{A}(M, \xi) := \langle ch(\xi) \smile \hat{A}(\tau), [M]\rangle$$

where $\xi \downarrow M$ is a complex vector bundle and τ is the tangent bundle of M. Thus we can express the elliptic genus by means of an $\hat{A}$-genus twisted in $R^{\hat{A}}_q(\tau)$:

$$\varphi(M) = \hat{A}\big(M, R^{\hat{A}}_q(\tau)\big) \in \mathbb{Q}[\![q]\!].$$

Here we have omitted the Bott element.

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