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# Narratives constructed in the discourse on early fractions

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*Fractions are one of the most demanding topics for teachers to teach and for students to learn. In this paper, we examine narratives about general properties of fractions constructed in a class when they were introduced through an equal-sharing context. The students' work and discussions constitute the starting point in planning further teaching, moving from lesson to lesson. Three episodes are presented in order to illustrate and discuss our findings. We argue that the analysis of the narratives provides insights into opportunities for students to learn as well as regarding the complexity of the topic.*

## Introduction and theoretical framework

The aim of this paper is to contribute to the understanding of opportunities and constraints that may occur when teaching and learning fractions. A class of Norwegian 4th grade students worked on fractions over a five-week period, which was their introduction to the concept in school. In this study, we are particularly interested in statements about general properties of fractions and the relations between fractions that were discussed in the class when the topic was introduced through a context of equal sharing. We analyze video recordings of lessons and illustrate and discuss our findings through three selected episodes.

From a pedagogical point of view, fractions and rational numbers take on many "personalities". Kieren (1976) recommends that work on fractions should be conceptualized as a set of interrelated meanings, which he calls subconstructs: part-whole, ratio, operator, quotient and measure. Behr, Harel, Post and Lesh (1993) have further developed Kieren's model, connecting it to operations on fractions, equivalence and problem solving. However, Olive and Laboto (2008) argue that the model is a semantic top-down analysis, which represents the adult view on fractions, and that it is not certain that it describes children's construction of fractional knowledge. Thompson and Saldanha (2003) have also been critical of the model, but their critique arose from a mathematical point of view: the mathematical motivation for rational numbers did not emerge from meanings, but from arithmetic and calculus. They suggest that fractional reasoning is tightly connected to multiplicative reasoning, arguing that fractional reasoning develops concurrently with reasoning on measurement, multiplication and division.

Nevertheless, Kieren's model has had a great influence on developments in the area of research. Lamon (2007) suggests that it can be important to choose one of the subconstructs as a starting point in the instruction and gradually include the others. In a realistic mathematics educational tradition (see Streefland, 1993), the notion of fractions is usually introduced through the context of equal sharing, i.e. as a quotient. Streefland (1993) argues that equal sharing gives rich learning opportunities regarding different aspects of fractions and that part-whole and operators appear naturally in this context. In their approach to fractions through cognitively guided instruction, Empson and Levi (2011) also started by working on equal-sharing contexts. In the class that is the focus of this study, the instruction started with an equal-sharing context, but the direction of further teaching was not decided a priori. Rather, each lesson was designed based on the students' work and the classroom discussions that occurred in the previous lesson.

How do students learn mathematical ideas? Sfard (2008) takes the position that learning mathematics is learning to participate in a particular discourse, where discourse is a special type of communication within a particular community. A discourse is made mathematical by a community's *use of words, visual mediators, narratives* and *routines*. The *use of words* in mathematics includes the use of ordinary words that are given special meaning in mathematics, such as function and ring, and mathematical words such as fractions and trapezium. In mathematical communication, participants use *visual mediators* to identify the object of their talk. These visual mediators are often symbolic, such as mathematical symbols, graphs, illustrations (e.g. number lines) and physical artefacts (e.g. centicubes). Within discourses, any spoken or written text that discusses properties of objects or relationships between objects is called a *narrative*. Narratives can be numerical, e.g. ½ is equivalent to 2/4, or more general, e.g. addition is commutative (see Sfard, 2015). Narratives are subject to endorsement or rejection, which is labelled true or false based on specific rules defined by the community. *Routines* are well-defined practices that are regularly employed in a discourse by a given community. These include how one talks about geometrical objects, how one performs calculations, how one substantiates a calculation, how to generalize and justify as well as when to use a particular action. A routine is called an *exploration* if it produces an endorsable narrative. Examples of explorations are numerical calculations, such as $21 \cdot 19$, the generalizing of patterns and the justification of these generalizations.

Several studies have reported on initial fraction learning through equal sharing (e.g. Empson, 1999), but none have used Sfard's framework for learning. We argue that Sfard's thinking—regarding the learning of mathematics as learning to participate in a particular discourse—is suitable to describe and analyze students' learning processes as well as opportunities for learning. Students gradually start to use fraction words and develop routines and narratives about the properties of and relations between fractions as they engage in work on fractions. Constructions of narratives and their rejections or endorsements are central to mathematical discourse. We argue, therefore, that an analysis of constructed narratives can provide insights into opportunities for learning in a given discourse. Narratives about general properties and relations between objects are of particular interest here, as they can be lifted above the numerical situations and used in new situations. Aiming to gain additional insight into learning opportunities, our research question is: "What narratives about general properties of fractions and relations between fractions can be constructed in an early discourse about fractions when they are introduced through an equal-sharing context?"

## Method

The study stems from a collaboration between an elementary teacher and two researchers, the authors of the paper. The teacher has been teaching the class in all subjects from their first grade. She was concerned about the students' participation and understanding in mathematics. There were 20 9-10-year-old students in the class, who attended a conventional Norwegian school.

The class worked on fractions over a period of five weeks: two 70-minutes lessons per week. The teacher´s motivation for the collaboration was further development of her teaching practice. She suggested that the researchers sketched ideas for lessons. The ideas were then discussed with the teacher. The teacher´s comments and suggestions on the researchers' ideas were built on the students´ prior knowledge and the way of working they were used to. The instruction on fractions began with a problem about a school trip, whereby different groups of students shared sandwiches: one group of four students shared three sandwiches, and another group of five students also shared three sandwiches, etc. (inspired by Fosnot & Dolk, 2002). The first activity set the basis for the series of lessons, as all other lessons, and are connected to the students' work on this first problem. The researchers were present as participants observers during the lessons, videotaping, observing and sometimes talking to individual students or even leading the instruction for short periods. After the lessons, the researchers and the teacher discussed students' work. Based on these discussions, they sought to identify areas that should subsequently be emphasized and how.

### Data and data analysis

The data that is the focus of this paper is the video recordings of the class discussion. Starting the analysis together, we watched through the recordings and marked out all utterances, spoken and written, that could be considered true or false. These utterances made up the set of all narratives discussed in the class. Most of the narratives were numerical, such as "one-fourth is half of one-half" or "three children get more than four children when they share a chocolate". As our research question is about narratives concerning general properties of fractions or relations between fractions, the next step in the data analysis was to identify these instances. We discussed the generality of each of the narratives marked out in the data and ended up with five narratives:

A. When we share equally, we can express shares as fractions
B. When we talk about a fractional part of something, it is crucial to be aware of what it is a part of
C. The more parts we divide something into, the smaller the parts that we get
D. When we want to express part of something as a fraction, the parts have to be equal
E. Fractions can be the same even though they do not look the same

Finally, in order to present our findings, we agreed on three episodes (not sequential) in which all five narratives were evident to illustrate the way they were constructed and discussed in the class. The first episode is from week two, the second from week three and the third from week four.

*Episode 1* is an excerpt from a classroom discussion on sharing two chocolates among three children. Two solutions that the students worked out before the discussion are presented in Figure 1.

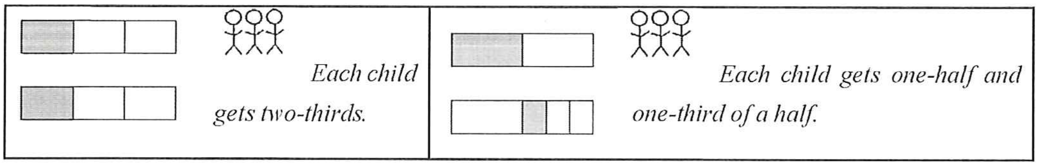


**Figure 1: Students' solutions to the problem of sharing 2 chocolates among 3 children**

The teacher wants to discuss with the students whether the answers in the two solutions are equal and how this is so. She also wants to discuss which fractional part of a chocolate "one-third of a half" is. The teacher begins the discussion by asking the students to talk about their solutions. Thomas suggests dividing the first chocolate into three equal parts, giving one part to each child; the same with the second chocolate. He concludes that each child gets one-third of the first chocolate and one-third of the second chocolate. The other students agree. The teacher asks how much of one chocolate one child gets in total.

| | |
|---|---|
| John: | I think it is two-thirds. |
| Teacher: | Two-thirds of one chocolate? Is that what you think? |
| John: | Yes… Two chocolates are divided in three parts [each], and each child gets two of them. |
| Lena: | Yeah, but… If we take them together, then each child gets two-sixths too? |

Some students agree with John, while some agree with Lena. If one considers the two chocolates as one unit (as Lena suggests when saying "take them together"), then the first picture in Figure 1 shows a unit divided in six parts, two of them are shaded – so 2/6 of the unit. In order to emphasize this change in the unit, the teacher writes both suggestions, "2/3 of 1 chocolate" and "2/6 of 1 chocolate" on the blackboard, one below the other, and asks whether both can be right. Lena suggests that it is the same. John and several other students agree.

| | |
|---|---|
| Teacher: | So, two-thirds are the same as two-sixths? Can it be? Here, we divide each chocolate in three parts (the first example in figure 1). If we divide it in six parts, then the parts are smaller, right? Remember that the question is how much of *one* chocolate each child gets. In what parts is *one* chocolate divided? |
| James: | Thirds. |
| Teacher: | And how many of such thirds does each child get? |
| James: | Two. |
| Teacher: | So, each child gets two-thirds of one chocolate. We can say that each child gets two-sixths, but then it is not of one chocolate. Two-sixths of what is it? |
| Lena: | If you take two chocolates [as a unit], then it is two-sixths. If you take one chocolate [as a unit], it is two-thirds. |
| Teacher: | Right. When we talk about fractional parts, then we have to say parts *of what*. It makes a difference. If you are about to get one-third of one chocolate, or one-third of a big bag full of chocolates, it is different, right [students nod and smile]? Shall we try to find out how much chocolate we actually get if we get one-third of a big bag of chocolates, 100 chocolates in the bag? |

The narrative "When we share equally, we can express shares as fractions" is central in the given context. Two chocolates are to be shared equally among three children, and the students share the chocolates in one of the two ways presented in Figure 1. They have worked with similar tasks several times before in the series of the lessons in this research, and the use of "fraction words" (two-thirds, one-half, third of a half) to describe shares seems to have become part of their routine in such tasks. They emphasize that each chocolate is shared equally among the three children and that each part is one-third of the chocolate.

Thomas suggests that "each child gets one-third of the first chocolate and one-third of the second chocolate". In the written work, many students suggest that "each child gets one-half of the first chocolate and one-third of a half of the second chocolate". There is nothing in the context that makes it necessary to consider these parts together as a fractional part of "one chocolate", which is emphasized by the teacher in order to compare the two different solutions. The teacher presses on, expressing the share as a fractional part of one chocolate, and another narrative is being constructed in the process: "When we talk about a fractional part of something, it is crucial to be aware of what it is a part of". In other words, the role of the unit is emphasized by the teacher.

In order to challenge the students' claim that both "2/3 of 1 chocolate" and "2/6 of 1 chocolate" can be right answers, the teacher points out that one-sixth of a chocolate is smaller than one-third. Here, she brings in another narrative, which was discussed earlier in the context of sharing one sandwich among five and six kids: "The more parts we divide something into, the smaller the parts we get".

*Episode 2* is an excerpt from a classroom discussion on a task in Figure 2. Tasks of this type were designed to discuss problems regarding type "one-third of a half", which appear in the context of equal sharing (of chocolates and such), as in Episode 1.

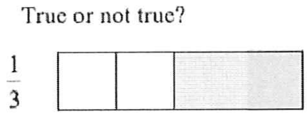


**Figure 2: Task discussed in Episode 2**

Nelly: Not true. Because... it is ... there are three parts, but if we are to share equally, it will not be equal ... hmm ... it will be unfair because one part is big.

Several students agree and suggest dividing the shaded part into two equal parts. Martin says that he does not understand.

Teacher: Ann, can you explain to Martin why we need to divide the big part into two parts?

Ann: Mmm. Because that part is too big if it is to be shared equally.

Teacher: That part is too big if it is to be shared equally. Do you understand it, Martin? [Martin nods.] Can you explain it in your own words?

Martin: Mmm. If it was one chocolate and three kids, then ... well ... two kids would get equal parts, but the last one would get bigger than the others.

Teacher: Yes. If it was a chocolate, as Martin says, it would be rather unfair because one would get more. Exactly as you say, it would be unfair.

The problem in Episode 2 is without context, but the students independently connected it to an equal-sharing context. Nelly starts to talk about equal sharing and uses the word "unfair" – the shaded part cannot be 1/3 because it would be unfair. This is supported by Ann and expounded by Martin. This indicates that the meaning of fractions for students is strongly connected to equal-sharing situations and "fairness". The teacher supports the argumentation, and the narrative emphasized is that if we want to express part of something as a fraction, the parts have to be equal.

*Episode 3* is part of a whole-class discussion on "tell me what you see". The aim was to discuss different partitioning of a figure. The tasks were designed so as to discuss solutions of the form: "each child gets one-half and one-third of a half", as in Episode 1. The students discuss Figure 3a) and find that it is "divided into two equal parts, one of which is shaded" and "we have one of two". The teacher then adds Figure 3b) so that both figures are on the blackboard.

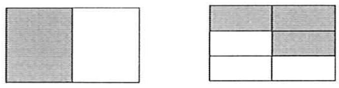

**Figure 3: Figures a) and b) discussed in Episode 3**

| | |
|---|---|
| Teacher: | What does this one look like? Mary [meaning 3b]? |
| Mary: | It is six parts, and three are shaded. |
| Teacher: | Yes, what about you, Ben? |
| Ben: | One of two [several students: What?] |
| Teacher: | How do you see one of two? |
| Ben: | I move the one shaded from the left down to the right and then one of two. |
| Teacher: | Ok. Someone disagrees? Nelly? |
| Nelly: | I see three of six. |
| Teacher: | You see three of six. But can it be both one of two and three of six? Can it be? Is it the same figure? Lena? |
| Lena: | Yes, it is true if the parts are equally big. |
| Teacher: | It can be true if the parts are equally big [writes this on the blackboard]. What does that mean, Lena? If all the parts are equally big? |
| Lena: | If the parts are not equally big, it is not a real fraction. |
| Teacher: | Then I challenge you, one of two and three of six, they do not look the same. How can they be the same? |
| Lena: | If we move some parts, they become the same. |
| Teacher: | So, one-half can look different; is that what you mean [Lena nods]? |

The discussion is about Figure 3b), which can be seen as *three of six*, but also *one of two* if "we move some parts" and then "we erase some lines", i.e. the idea of equivalent fractions is discussed. The constructed narrative is "the fractions can be the same even though they do not look the same".

## Results and discussion

In Episode 1, two earlier-discussed narratives came up again: *When we share equally, we can express shares as fractions* and *the more parts we divide something into, the smaller the parts we get.* Both narratives emerged from the equal-sharing context that was initially used in the teaching. In addition, the narrative *When we talk about a fractional part of something, it is crucial to be aware of what it is a part of* was constructed as a consequence of an operator subconstruct appearing in the quotient context (one-third of a half chocolate). The constructed narrative emphasizes the role of the unit, which is one of the critical aspects of learning fractions (see, e.g. Lamon, 2007).

In Episode 2, the narrative *When we want to express a part of something as a fraction, the parts have to be equal* was constructed and endorsed by the students by referring to "fair sharing". The task used in the episode was designed as it was shown to be important in emphasizing partitioning so as to illuminate a challenge that came up in a quotient context. The task can be seen as a part-whole subconstruct, but the constructed narrative was endorsed by connecting the situation to the equal-sharing, i.e. quotient, subconstruct.

In Episode 3, the narrative *Fractions can be the same even though they do not look the same* was constructed. The idea of equivalent fractions, another important aspect of fractions, were discussed in the episode. The need to discuss tasks as the one in the episode is imbedded naturally in the equal sharing context, as different ways to share two chocolates among three children. The task was given in a part-whole context, and the students endorsed the narrative by partitioning.

The teaching period started with a quotient subconstruct of fractions (Kieren, 1976). However, both the operator (as one-third of a half) and the part-whole construct (as one of three parts) appeared almost immediately in the students' work. Their work and discussions were the starting point in teaching planning from lesson to lesson. Moreover, looking back, we see an interplay between the quotient, part-whole and operator subconstructs throughout, as illustrated in the three episodes presented in this paper. This contradicts Lemons' (2007) recommendation that the initial instruction should concentrate on one subconstruct, indicating that focusing only on one subconstruct can be restrictive and unnatural in teaching. In our study, the context of equal sharing was shown to be a rich starting point that brought out many important aspects of fractions, as suggested by many researchers (e.g. Empson & Levi, 2011; Streefland, 1993). However, it also seemed to be highly complex for teaching and learning for the same reason, and one can say that the class worked on basically the same problem for the whole teaching period, as the teacher tried to help the students delve deeper into the emerging ideas.

It is well known that a teacher plays an important role in creating leaning opportunities for students. The three episodes illustrate, in particular, the teacher's crucial role in the process of constructing narratives. The equal-sharing (like sharing chocolates among some kids) situations were imaginable for the students, as they constituted part of their everyday experiences, and they had no difficulty suggesting a solution. However, as everyday experiences, there is no need for students to dwell on moments as "what part of a chocolate is one-third of a half" or "is one-half the same as three-sixths". The equal-sharing situation was moved into a new, mathematical, discourse in the teaching. It was the teacher who pressed with new questions in the situation and tried to emphasize narratives on properties and relations between fractions, making the equal-sharing a context for learning fractions.

In the process of discursive learning, the use of words, routines and narratives developed in a community are in continual flux and refinement (Sfard, 2008). We started our teaching on fractions by equal sharing, and after a while, it became a routine for the students to use fraction words to denote shares. Fractions became related to equal sharing and fairness, which constitute everyday experiences for students. This made way for several explorations and narrative constructions. Fractions are complex, both in terms of teaching and learning, and the question is how to make the concept more accessible without oversimplifying it. We hope that our paper and analyses of the general narratives constructed in the discourse can contribute to research on this question. However, our study was conducted over a short time period and further longitudinal studies on the construction of narratives are needed to gain more insights. We suggest that the episodes presented in the paper can be used in teacher education to discuss the complexity of teaching fractions with pre-service teachers.

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