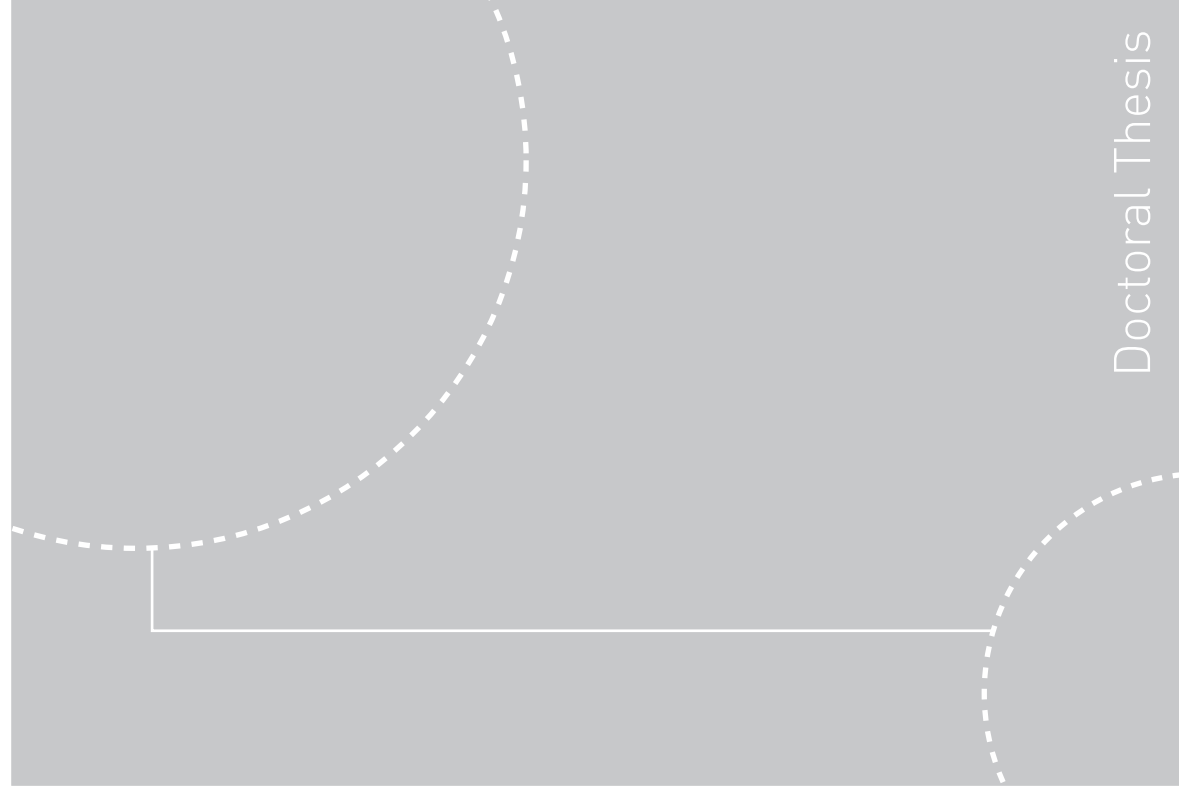


Doctoral Theses at NTNU, 2005:212

Alf Emil Løvgren  
**Reduced basis modeling of  
hierarchical flow systems**



Doctoral Thesis

ISBN 82-471-7322-0 (printed ver.)  
ISBN 82-471-7321-2 (electronic ver.)  
ISSN 1503-8181

Doctoral Theses at NTNU, 2005:212

**NTNU**  
Norwegian University of  
Science and Technology  
Doctoral thesis  
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Printed by Tapir Uttrykk

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# Acknowledgments

This is my thesis for the Doctor Scientiarum degree at the Department of Mathematical Sciences, Norwegian University of Science and Technology. The work in the thesis was conducted between August 2002 and September 2005, at the Department of Mathematical Sciences, and at Simula Research Laboratories. The project has been funded by the Norwegian Research Council (NRC) through the BeMatA programme under contract 147044/431.

First of all I would like to thank the Department of Mathematical Sciences and NRC for making the project possible, and Simula Research Laboratories for providing an excellent working environment.

I would like to thank my supervisor Professor Einar Rønquist, whose encouragement and guidance has been invaluable to me throughout these years. I am thankful to Professor Yvon Maday for offering me his professional and moral support. I would also like to thank Einar and his wife Liz for opening their home in Trondheim to me, and for making me feel as a part of their family during my stays there.

I have benefited from the influence of Professor Ragnar Winther, and Professor Aslak Tveito made my stay at Simula possible and enjoyable. While in Trondheim I had the pleasure of acquainting Xavier Raynaud, Tormod Bjøntegaard, Vegard Kippe, and Håvard Berland. In the positive environment at Simula I enjoyed the company of Gunnar Staff, Kent-Andre Mardal, Trygve Kastberg Nilssen, Ola Skavhaug, Halvard Moe, Marius Lysaker, Bjørn Fredrik Nielsen, Åsmund Ødegård Tom Thorvaldsen, Trond Vidar Stensby, Christian Tarrou, and Joakim Sundnes.

Finally, I am very grateful to my family for supporting me in every choice I make. I especially thank my wife Mette and our two children Alfred and Matilda, for bringing so much joy and love into my life.

Alf Emil Løvgren  
Drammen, October 2005



## Contents

This thesis is a collection of four papers and an introduction

- **Paper I:** *A reduced basis element method for the steady Stokes problem*, by Alf Emil Løvgren, Yvon Maday and Einar Rønquist. Submitted to Mathematical Modelling and Numerical Analysis (M2AN).
- **Paper II:** *A reduced basis element method for the steady Stokes problem: Application to flow in bifurcations*, by Alf Emil Løvgren, Yvon Maday and Einar Rønquist. In preparation for submission as a journal paper.
- **Paper III:** *A reduced basis element method for the steady Stokes problem: Application to hierarchical flow systems*, by Alf Emil Løvgren, Yvon Maday and Einar Rønquist. Published in: Proceedings of SIMS 2005.
- **Paper IV:** *A reduced basis method for the steady Navier-Stokes problem*, by Alf Emil Løvgren, Yvon Maday and Einar Rønquist. In preparation for submission as a journal paper.





# Introduction

Alf Emil Løvgren

In this thesis we consider the reduced basis element method for approximating the solution of parameter dependent problems described by partial differential equations. In particular we focus on fluid flow in pipes, bifurcations and hierarchical systems, where the flow is described by the steady Stokes equations, or the steady Navier-Stokes equations. The thesis consists of four papers and this introduction.

The reduced basis element method is different from traditional reduced basis methods in that it combines these methods with domain decomposition. A given geometry is decomposed into building blocks with the same topology as a few reference domains, e.g. a rectangle and a reference bifurcation. Relative to each reference domain we precompute and store basis functions found on a preselected set of deformations of the respective reference domains. A reduced basis solution is then found by mapping the basis functions from their respective reference domains to corresponding domains in the domain decomposition. A local approximation of the “true” solution on one domain is found using the basis functions belonging to that specific domain, and the global approximation is found by “gluing” the local approximations together with constraints across domain interfaces. Geometries where building blocks of the same topology are reused many times are attractive candidates for the reduced basis element method. When there is only one domain in the geometry, the reduced basis element method is seen as a traditional reduced basis method where the geometry of the domain is one of the independent parameters.

In the first part of the introduction we present the reduced basis method and the reduced basis element method. In the second part of the introduction we motivate the work done in the thesis, and give a summary of the papers.

## The reduced basis method

The reduced basis method is not a new method, and it has been used with success in the nonlinear analysis of structures since 1980 [1, 14]. Other applications include the modeling of turbulent flows [4], control problems [18, 22], optimization [9], and aerodynamics [7, 24]. Typical for all applications is problems depending on one or more parameters. Often the same

problem need to be solved repeatedly for different instantiations of a parameter, each solve being time-consuming with standard solution methods. In control problems the solution is needed in real-time, and the input parameters change constantly.

A general parameter dependent problem on a domain  $\Omega \in \mathbb{R}^n$  may be written as: Find  $u \in X(\Omega)$  such that

$$F(u; \mu) = 0, \quad (1)$$

where  $X(\Omega)$  is the solution space,  $\mu$  is the independent parameter vector, and  $F$  is the operator acting on  $u$  and  $\mu$ . If (1) is well-defined, and if we have a suitable conventional solver for (1), we may for a given choice of parameter vector  $\mu_i$  find the corresponding solution  $u_i$ . A key point in the reduced basis method is that if a different parameter vector  $\mu_j$  is chosen such that it is close to  $\mu_i$  in the parameter space  $\mathcal{D}$ , then the corresponding solution  $u_j$  must be close to  $u_i$  in  $X$ . We also note that, although the solution space  $X$  is infinite, all realistic solutions of (1) reside within some lower-dimensional manifold contained in  $X$ . This is true for problems with smooth solutions.

To solve a parameter dependent problem like (1) numerically, we discretize the domain  $\Omega$ , by constructing a grid. On this grid we then define a discrete solution space  $X_{\mathcal{N}}$ , where  $\mathcal{N}$  resembles the number of degrees-of-freedom in this discrete space.

Basis functions defined on  $X_{\mathcal{N}}$  are used in the reduced basis method to produce a good approximation space

$$X_N = \text{span}\{u_i\}_{i=1}^N. \quad (2)$$

If the resolution of a solution of (1) with a conventional solver is  $\mathcal{N}$ , then  $N \ll \mathcal{N}$ . For example  $\mathcal{N}$  is  $\mathcal{O}(1000)$ , while  $N$  is  $\mathcal{O}(10)$ . For a generic parameter vector  $\mu$ , a reduced basis approximation to the high resolution solution  $u_{\mathcal{N}}(\mu)$  of (1), is then found as

$$u_N(\mu) = \sum_{i=1}^N \alpha_i(\mu) u_i, \quad (3)$$

where the coefficients  $\alpha_i(\mu)$  are determined through a Galerkin method.

The construction of the reduced basis space  $X_N$  is done in a preprocessing stage, before the actual computations start. The basis functions,  $\{u_i\}_{i=1}^N$ , resulting from this offline stage are then stored, and used in the online stage. For the reduced basis method to be efficient and reliable,  $X_N$  must be generated in such a way that  $N$  may be as small as possible and still enable  $u_N(\mu)$  to be a good approximation of the high resolution solution  $u_{\mathcal{N}}(\mu)$ . First of all,  $X_N$  must span a large portion of the part of the solution space  $X$  in which all realistic solutions of (1) lie. Second, the elements of  $X_N$  should contain as little redundant information as possible to keep  $N$

small, i.e., each  $u_i$  should span an unique part of  $X_N$ , only contributing new information. Third, for flexibility of the reduced basis method, the addition of more basis functions to  $X_N$  should be possible in a fairly simple manner.

The most common approach to the construction of  $X_N$  is to start with an ensemble of basis functions which surely span the relevant portion of  $X$ , and then process this ensemble such that the number of basis functions are reduced, while the quality of each is increased. The ensemble of basis functions is usually generated either as a Lagrange sample, or as a Taylor expansion.

To construct a Lagrange sample we choose  $M$  different instantiations of the parameter vector  $\mu \in \mathcal{D} \subset \mathbb{R}^P$ ,  $\{\mu_i\}_{i=1}^M$ , where  $P$  is the number of independent parameters. We then obtain a solution of the given problem (1) for each  $\mu_i$ , either by using a conventional solver, or by sampling a physical model of the problem. When  $\mathcal{D}$  is large, the corresponding relevant portion of  $X$  is large, and  $M$  quickly grows if we want the basis functions to span the space properly.

In a Taylor expansion approach, we choose only one instantiation of the parameter vector  $\mu_0 = (\mu^1, \dots, \mu^P)^T \in \mathcal{D} \subset \mathbb{R}^P$ , and solve the problem (1) to find the first basis function  $u_0$ . Next we denote  $u_k^i = \frac{\partial^k u}{\partial (\mu^i)^k}$  and obtain equations for  $u_k^i$  by differentiating (1) with respect to the parameter and then evaluating at  $\mu_0$ . The basis functions  $u_k^i$  are then found in a recursive manner by solving

$$\frac{\partial F}{\partial u}(u_0; \mu_0) u_k^i = G_k(u_0, u_1^i, \dots, u_{k-1}^i; \mu_0), \quad i = 1, \dots, P, \quad k = 1, \dots, K. \quad (4)$$

The basis functions generated in this way will constitute a good reduced basis approximation space for parameter vectors close to  $\mu_0$ . Note that the resulting coefficient-matrix is the same for all  $i$  and  $k$ , such that once the matrix has been factorized, only back-solves are needed to find more basis functions. On the other hand, the generation of the different right hand sides of (4) may be both complicated and costly in the general case. The Taylor approach is applied by Noor and Peters in [14] to solve finite-element discretizations of nonlinear analysis of structures, and by Peterson in [15] to generate finite-element solutions of the stationary Navier-Stokes equations.

The Taylor expansion space only spans a small part of the solution space, but more basis functions are easily added for increased local approximation abilities. The Lagrange sample space needs one solve of the original problem for each additional basis function, and is thus more tedious than the Taylor approach, but it easily spans a greater portion of the solution space, and the right-hand side of (1) only has to be constructed once.

As a third approach to generate an ensemble of basis functions Porsching describes a discrete least squares method in [16]. A combination of the Taylor expansion and the Lagrange approach is also possible. To produce a

Hermite basis several parameters  $\mu_i$  are used to find the basis functions  $u_i$  and their first derivatives with respect to the parameters.

## Offline/online decoupling

The key ingredient in the reduced basis method is an offline/online decoupling. In the offline stage we precompute  $N$  basis functions, and process them in order to obtain a high quality approximation space  $X_N$ . Depending on the problem at hand and the solution method used, each basis function requires at least  $\mathcal{O}(\mathcal{N})$  operations and storage.

In the process of determining  $u_N(\mu)$  in (3) by a Galerkin procedure, we need to compute inner products of the basis functions. Since each basis function is stored in high resolution, this yields  $\mathcal{O}(N^2\mathcal{N})$  operations. In [17] Prud'homme et al. show that for problems with affine parameter dependence, these inner products may be computed in the offline stage (i.e. the precomputational stage), and the operations in the online stage will only depend on  $N$  and a small integer  $Q$  from the affine decoupling of the problem. We assume for the moment that the weak form of the problem (1) may be written as: Find  $u \in X$  such that

$$a(u, v; \mu) = l(v), \quad \forall v \in X, \quad (5)$$

where  $a(u, v; \mu)$  is a bilinear form, and  $l(v)$  is a linear form.

If the problem has affine parameter dependence, we may decouple the bilinear form in (5) such that

$$a(u, v; \mu) = \sum_{q=1}^Q \sigma^q(\mu) a^q(u, v), \quad (6)$$

where each  $a^q(u, v)$  is a bilinear form independent of  $\mu$ . For many problems this is possible for relatively small  $Q$ , see [17] for examples of a thermal fin and a truss structure.

When we use the reduced basis method to find  $u_N = \sum_{i=0}^N \alpha_i u_i = (\underline{\alpha})^T \underline{u}$ , the decoupling (6) gives the problem: Find  $u_N$  such that

$$\sum_{q=1}^Q \sigma^q(\mu) a^q(u_N, v) = l(v), \quad \forall v \in X_N. \quad (7)$$

The resulting algebraic equations are

$$\left( \sum_{q=1}^Q \sigma^q(\mu) A^q \right) \underline{\alpha} = \underline{l}, \quad (8)$$

where each  $A^q \in \mathbb{R}^{N \times N}$  has entries  $A_{ij}^q = a^q(u_j, u_i)$ ,  $1 \leq i, j \leq N$ , and  $\underline{l} = [l(u_1), \dots, l(u_N)]^T$ . In the offline stage we thus compute the  $u_i$ ,  $1 \leq i \leq N$

and form  $A^q$ ,  $1 \leq q \leq Q$  and  $\underline{l}$ . This requires  $N$  solves of the high resolution system, and  $\mathcal{O}(QN^2)$  inner products in the high resolution basis. In the online stage, for a generic  $\mu$ , we then add the contributions to  $A$  together in  $\mathcal{O}(QN^2)$  operations, and solve for  $\underline{\alpha}$  using  $\mathcal{O}(\frac{2}{3}N^3)$  operations.

To assemble the reduced basis solution  $u_N$ , we have to add basis functions of length  $\mathcal{N}$ , see (3), and the number of online operations will thus be  $\mathcal{O}(\mathcal{N})$ . In many cases, however, we are not interested in the solution itself, but rather some quantity derived from it. These outputs may for example describe the drag forces when air flows past an object, or the volume flow rate of a fluid through a pipe. If we define  $s(u) = f(u)$  to be the output of interest for some bounded linear functional  $f : X \rightarrow \mathbb{R}$ , the reduced basis output of interest may be computed as

$$s(u_N) = f(u_N) = f\left(\sum_{i=1}^N \alpha_i u_i\right) = \sum_{i=1}^N \alpha_i f_i, \quad (9)$$

where  $f_i = f(u_i)$  is computed in the offline stage in  $\mathcal{O}(\mathcal{N})$  operations.

## Processing the basis functions

In multi-parameter problems, the number of basis functions generated by either the Lagrange approach or the Taylor-expansion approach quickly exceeds reasonable values. For the reduced basis method to remain efficient, the number of basis functions must be kept at a minimum, while still spanning the desired portion of  $X$ . Intuitively one might think that a simple orthogonalization, where the basis functions which are nearly spanned by the previous basis functions are thrown from the space, will suffice. It is desirable, however, to compress the information content of several precomputed basis functions into one basis function, and often we want to adapt the choice of basis functions according to the generic parameter.

**Proper orthogonal decomposition** The classic method for processing the precomputed basis functions is known as Proper Orthogonal Decomposition (POD), and it has been much used in, for example, reduced basis modeling of turbulence and non-linear analysis of structures; see [4, 14]. POD is based on the Karhunen-Loéve expansion theorem for second order stochastic processes, and the spectral theory of compact and self-adjoint operators; see e.g. [13].

For a given parameter dependent problem, we assume that an ensemble  $\{u_i(x, t)\}_{i=1}^n$  of random state variables is given. The POD basis vectors  $\phi$  are chosen such that they satisfy

$$\max_{\psi} \frac{\langle |u, \psi|^2 \rangle}{(\psi, \psi)} = \frac{\langle |u, \phi|^2 \rangle}{(\phi, \phi)}, \quad (10)$$

where  $(\cdot, \cdot)$  is an inner product, and  $\langle \cdot \rangle$  is an averaging operator. To do this they must satisfy

$$\int_{\Omega} K(x, x') \phi_k(x') dx' = \lambda_k \phi_k(x), \quad (11)$$

where  $K$  is the two-point correlation matrix defined by

$$K_{ij}(x, x') = \langle u_i(x, t) \bar{u}_j(x', t) \rangle, \quad i, j = 1, \dots, n. \quad (12)$$

The  $\{\phi_k(x)\}$  are called the empirical eigenmodes (or eigenvectors), and the eigenvalues are ordered such that  $\lambda_i \geq \lambda_{i+1}$ . The eigenmodes are orthonormal, and they are what is called the proper orthogonal decomposition. The sum of the eigenvalues  $\sum_k \lambda_k$  represents the average energy of the ensemble, and if only the first  $N$  eigenmodes are used as a basis for the solution of the given problem, POD is optimal in the sense that it maximizes the captured energy, on an average.

The dimension of the correlation matrix is  $D \times \mathcal{N} \times \mathcal{N}$ , where  $D$  is the spatial dimension, and  $\mathcal{N}$  is the number of spatial degrees of freedom. The computation of the eigenmodes of  $K$  thus becomes very expensive when decent accuracy is needed in the discretization. Sirovich proposed the method of snapshots in [21] in order to cope with this problem.

The idea behind the method of snapshots is to choose a parameter  $\tau$  such that the state-solutions

$$u_j(x) = u(x, j\tau), \quad (13)$$

are uncorrelated for different  $j$ . The eigenmodes are expressed as a linear combination of the snapshots, i.e.

$$\phi_k(x) = \sum_{j=1}^N a_j u_j(x), \quad (14)$$

where the constants  $a_j$  remain to be found. This problem reduces to the eigenvalue problem

$$C \underline{a} = \lambda \underline{a}, \quad (15)$$

where  $C$  is a  $N \times N$  matrix with entries  $\frac{1}{N} (u_i(x), u_j(x))$ , and  $\underline{a} = (a_1, \dots, a_N)^T$ . In [3] it is shown that the method of snapshots is equivalent to the original formulation of the eigenvalue problem (11). We remark that, if  $\{u_j\}_{j=1}^N$ , represent snapshots from a high resolution Navier-Stokes simulation, the POD basis will represent a low order basis already satisfying the incompressibility condition.

**Centroidal Voronoi tessellations** An alternative to POD, which has been much used in data compression, is Centroidal Voronoi Tessellations (CVT). A tessellation of a region  $\Omega \in \mathbb{R}^2$  is created by covering the region

with smaller shapes, like a mosaic. Similarly a tessellation of a set of basis functions  $W_N = \{u_i\}_{i=1}^N$  is defined as a set  $\{V_i\}_{i=1}^k$  which subdivides  $W_N$  into disjoint covering subsets. Formally we write

$$\begin{aligned} V_i &\subset W_N & i &= 1, \dots, k, \\ V_i \cap V_j &= \emptyset & i &\neq j, \\ \bigcup_{i=1}^k \overline{V_i} &= \overline{W_N}. \end{aligned} \quad (16)$$

For a given set of functions in  $X_N$ ,  $\{v_i\}_{i=1}^k$ , the Voronoi region corresponding to the function  $v_i$  is defined by

$$\hat{V}_i = \{u \in W_N, \|u - v_i\|_{X_N} \leq \|u - v_j\|_{X_N}, j = 1, \dots, k, j \neq i\}. \quad (17)$$

The set  $\{\hat{V}_i\}_{i=1}^k$  is called a Voronoi tessellation of  $W_N$  corresponding to the functions  $\{v_i\}_{i=1}^k$ , and the functions in the set are called the generators of the Voronoi tessellation. CVT is a Voronoi tessellation whose generating functions are also the centers of mass of the corresponding Voronoi regions.

For an ensemble of basis functions  $W_N = \{u_j\}_{j=1}^N$ , CVT are used to extract a smaller set of new basis functions  $Z_k = \{z_i\}_{i=1}^k$ , also belonging to  $X_N$ , but in general not belonging to  $W_N$ . A reduced-order modeling methodology based on CVT is introduced in [5, 6].

**An output based adaptive method** For problems where the output of some functional is of interest, there exists an adaptive method. In problems where we need to solve (1) many times, for example in control problems and optimization problems, the set of basis functions generated from the original ensemble of functions, either by POD or CVT, may not remain adequate. If it turns out that all new instantiations of the parameter  $\mu$  are located within a very small portion of  $\mathcal{D}$ , a new set of basis functions closer to  $u(\mu)$  would give better results.

In [23] Veroy et al. propose a method for adaptive construction of the set of basis functions. The main ingredient is an *a posteriori* error bound  $\Delta_N(\mu)$  for the output of interest (9), and the process is split in an offline/online decoupling. We let  $S_N$  denote the set of parameters used to generate  $N$  basis functions in  $X_N$  as a Lagrange sample. In the offline stage, we choose a random parameter  $\mu^1 \in S_N$  and define the new parameter set  $S_{N'=1} = \{\mu^1\}$ , and the new reduced basis  $X_{N'=1} = \text{span}\{u(\mu^1)\}$ . Amongst all  $\mu \in S_N \setminus S_{N'=1}$  we then find the parameter which maximizes  $\Delta_{N'=1}(\mu)$ . We denote this parameter  $\mu^2$ , and set  $S_{N'=2} = \{\mu^1, \mu^2\}$ , and  $X_{N'=2} = \text{span}\{u(\mu^1), u(\mu^2)\}$ . The procedure is then repeated until the maximum of  $\Delta_{N'=N^{prior}}(\mu)$  over  $S_N \setminus S_{N'=N^{prior}}$  is less than some  $\epsilon^{prior}$ .

In the online stage, for a new instantiation of the parameter,  $\mu$ , and a given tolerance  $\epsilon^{post}$ , we repeat the process described for the offline stage, but now the maximizing parameters are sought only in  $S_{N'=N^{prior}}$ , and we stop when the maximum of  $\Delta_{N'}(\mu)$  is less than  $\epsilon^{post}$ .



## Error estimation

When the reduced basis method is applied to general nonlinear problems depending on a single parameter, we may use error estimates presented by Fink and Rheinboldt in [8], and by Porsching in [16]. The estimates are local in parameter space, and states that for a generic parameter  $\mu$  in some interval  $[-\mu_*, \mu_*]$ , the error of the reduced basis solution  $u_N(\mu)$  scales with  $\mu_*^{N+1}$ , i.e.

$$\|u_N(\mu) - u(\mu)\|_\infty = \mathcal{O}(\mu_*^{N+1}). \quad (18)$$

In [19] the estimates are extended to the multi-parameter case.

For problems with affine parameter dependence, Maday, Patera, and Turinici demonstrate in [10] a way to achieve exponential convergence in the error of the reduced basis solution, uniformly over the parameter domain. The parameter  $\mu$  is taken to be in some interval  $\mathcal{D} = [0, \mu_{max}]$ , and the reduced basis space is constructed as a Lagrangian sample, where the parameters used to construct the space are logarithmically distributed in  $\mathcal{D}$ . For  $N > N_{crit}$  it is then shown that

$$\|u_N(\mu) - u(\mu)\|_X \leq C \|u(0)\|_X e^{\frac{-(N-1)}{N_{crit}-1}}, \quad \forall \mu \in \mathcal{D}, \quad (19)$$

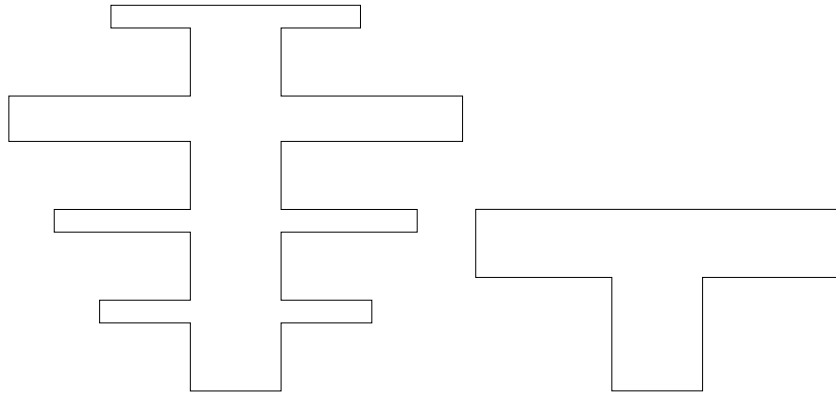
where  $C$  depends on the given problem and  $\mu_{max}$ . The theory is developed for a single parameter, but numerical results indicate exponential convergence also in the multi-parameter case.

Either way, we are ensured good approximation properties, but we can only say something about the error convergence rate, not the actual size of the error. To be certain that the error is small enough, too many basis functions are used in the reduced basis approximation, and unnecessary work is done.

For the reduced basis method to be useful, we need *a posteriori* error estimates as well. Based on the output of interest described earlier, Prud'homme et al. present in [17] rigorous error bounds for the output. In addition, they apply offline/online decoupling in the computation of the error bounds. We are thus able to certify the error in the reduced basis method, and may start with a modest number of basis functions in the reduced basis approximation. More basis functions are then included until the error bounds satisfy a given tolerance.

## The reduced basis element method

In the reduced basis element method decomposition of the domain into generic building blocks is applied in combination with the construction of a reduced basis approximation space for each generic block. In Figure 1(a) we see a thermal fin used to cool electronic equipment. The fin consists of



(a) The geometry of the thermal fin problem. The fin consists of 4 stages, and with the root of the fin at the bottom.

(b) The one stage fin.

Figure 1: Different geometries.

several stages, and each stage may have distinct conductivity, thickness and width. To apply the reduced basis element method, we recognize all stages of the fin as variations of the same generic building block: the one stage fin depicted in Figure 1(b). For the Laplace equation

$$\nabla^2 u = 0, \quad (20)$$

Maday and Rønquist show in [12] how a reduced basis approximation space constructed on the one stage fin may be used on each stage of the fin in Figure 1(a). To ensure continuity across the interfaces between the stages in the fin, Lagrange multipliers are used to glue the pieces together.

In the example of the thermal fin the geometric variation is described by pure stretching, or compression, and as described in [17], the equation (20) will have affine parameter dependence. The offline/online decomposition described above for the evaluation of all inner products between basis functions in the offline stage thus applies for these geometric parameters.

For more general geometries we refer to [11], where deformed pipes are used to demonstrate the reduced basis element method. The deformation of the pipes varies over the domain, and the weak form of the Laplace operator attends the following form when mapped to the reference domain,  $\hat{\Omega} = (-1, 1)^2$ ,

$$\int_{\hat{\Omega}} \mathcal{J}(\Phi^{-1}) \hat{\nabla} \hat{u} \cdot \mathcal{J}(\Phi^{-1}) \hat{\nabla} \hat{v} J(\Phi) d\hat{\Omega}, \quad (21)$$

where  $\Phi$  is the geometric mapping from  $\hat{\Omega}$  to a deformed pipe,  $\mathcal{J}$  is the

Jacobian matrix of the mapping, and  $J$  is the determinant of  $\mathcal{J}$ ,  $\hat{v} = v \circ \Phi$ , and  $\hat{\nabla}$  is the gradient operator with respect to the reference variables  $(\xi, \eta)$ . Since the mapping  $\Phi$  varies over  $\hat{\Omega}$ , the affine detachment presented in (6) cannot be used, and the attractive offline/online work distribution does not apply either. Still, the results presented in [11] confirm the approximation abilities of the reduced basis element method also in this case, and with the introduction of the empirical interpolation technique in [2], an offline/online decoupling is realizable.

In [20] Rozza presents a preliminary study of the reduced basis element method applied to a bypass configuration, by limiting the geometric variations to pure stretching and rotation. The parameter dependence is thus affine, as five parameters are used to model the diameters, lengths and relative angle of the bifurcation shaped bypass configuration.

## Summary of the papers

The work on the reduced basis element method for the Laplace problem in [11, 12] is used as a vantage point for the work in this thesis. We explore the possibility of using the geometry as a parameter to find reduced basis solutions of hierarchical flow systems, and our goal is to approximate the solution of systems too large for conventional methods. A hierarchical system, such as veins of blood, is decomposed into a number of pipes and bifurcations, and the idea is to use a reference pipe and a reference bifurcation as generic building blocks for the hierarchical system.

To model laminar flow we use the steady Stokes equations, and to model flow at low Reynolds numbers we use the steady Navier-Stokes equations. To use the solution of one of these equations as a basis function on a different geometry, special care has to be taken. Compared to the Laplace problem, we introduce many new aspects in the reduced basis modeling. One new aspect is the presence of both a vector velocity solution and a scalar pressure solution.

We use “truth” approximations of the solutions on the test geometries to investigate the quality of the reduced basis solution, but the focus is on problems where the output of some bounded linear functional is of interest. In this respect we adapt the *a posteriori* error bounds developed for problems with affine parameter dependence in [17] to suit our needs. This presents two challenges, first to find expressions for such error bounds when the geometry is used as a parameter, and second to develop offline/online decompositions of the expressions found.

The work in the thesis is decomposed into three blocks. In the first paper we focus on the steady Stokes problem on deformed pipes. The conceptual tools needed for using the reduced basis element method on vector fields on different geometries are presented. We also develop *a posteriori* error bounds

on a single domain. We use the conceptual tools in the second paper to find the reduced basis solution of the steady Stokes problem on systems of pipes and bifurcations. The generation of the bifurcation geometries is non-trivial, and a method for producing weak  $C^1$  mappings between two bifurcations is presented. The third paper is a combination of the two first papers, prepared for the proceedings of the SIMS 2005 conference. In the fourth paper we proceed to the steady Navier-Stokes problem on single domain deformed pipes. The *a posteriori* error bounds are adapted to this problem, and we present an offline/online decoupling of the computational effort.

### **Paper I:**

We use a set of deformed pipes to precompute solutions of the steady Stokes problem. We use the Piola transformation to map the precomputed solutions to a reference domain, where they are stored. On a generic deformed pipe we propose two methods to find the reduced basis solution of the steady Stokes problem, using the precomputed solutions as basis functions. Method 1 involves enrichment of the velocity basis, while Method 2 exploits the use of divergence free velocity basis functions to find the reduced basis solution in a decoupled fashion. We investigate the use of different Lagrange multipliers to “glue” the solution together across subdomain interfaces. In the single domain case we present *a posteriori* error bounds for the output of interest.

### **Paper II:**

A general bifurcation is discretized by decomposing it into six subdomains, where each subdomain is a one-to-one mapping of the reference square  $\hat{\Omega} = (-1, 1)^2$ . On different deformed bifurcations we precompute solutions of the steady Stokes problem. To store the precomputed solutions on a reference bifurcation  $\hat{B}$ , we propose a smoothing algorithm for the deformed bifurcations such that each bifurcation is seen as a weak  $C^1$  one-to-one mapping of  $\hat{B}$ . We notice that some accuracy is lost in the Piola transformation of divergence free velocity fields, and propose a regularization technique for Method 2. Furthermore, we use an adaptive selection method based on the output of interest to limit the number of basis functions needed in the approximation spaces. Finally, we use Lagrange multipliers and the basis functions used for deformed pipes together with the ones developed for bifurcations to find the reduced basis solution of a hierarchical flow system.

### **Paper III:**

This paper combines the results of Paper I and Paper II. The focus is on the reduced basis element solution of the steady Stokes problem on hierarchical flow systems, and a coherent approach is presented. We also present results for a bypass configuration.

## Paper IV:

We introduce non-linear and non-symmetric terms in the reduced basis method when the reduced basis solution of the steady Navier-Stokes problem is found on deformed pipes. We extend the *a posteriori* error analysis for non-affine operators to get error bounds for the output of interest also when we have non-symmetric operators. In addition, we address the offline/online decomposition of both the output of interest, and the error bounds. To this end we use the empirical interpolation technique developed in [2] to approximate the operators as affine sums of expensive parameter independent operators and cheap parameter dependent operators.

## References

- [1] B. O. Almroth, P. Stern, and F. A. Brogan. Automatic choice of global shape functions in structural analysis. *AIAA J.*, 16:525–258, 1978.
- [2] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera. An ‘empirical interpolation’ method: Application to efficient reduced-basis discretization of partial differential equations. *C. R. Acad. Sci. Paris, Serie I*, 339:667–672, 2004.
- [3] G. Berkooz. *Turbulence, coherent structures and low dimensional models*. PhD thesis, Cornell University, 1991.
- [4] G. Berkooz, P. Holmes, and J. L. Lumley. The proper orthogonal decomposition in the analysis of turbulent flows. *Annu.Rev.Fluid Mech.*, 25:539–575, 1993.
- [5] J. Burkhardt, Q. Du, M. D. Gunzburger, and H. Lee. Reduced order modeling of complex systems. In D. F. Griffiths and G. A. Watson, editors, *Proceedings of the 20th Biennial Conference on Numerical Analysis*. University of Dundee, 2003.
- [6] J. Burkhardt, M. Gunzburger, and H. Lee. Centroidal voronoi tessellations-based reduced-order modeling of complex systems. *Submitted to SIAM J. Sci. Comput.*, 2005.
- [7] E.H. Dowell, K.C. Hall, J.P. Thomas, R. Florea, B.I. Epureanu, and J. Heeg. Reduced order models in unsteady aerodynamics. AIAA Paper 99-1261, 1999.
- [8] J.P. Fink and W.C. Rheinboldt. On the error behavior of the reduced basis technique in nonlinear finite element approximations. *Z. Angew. Math. Mech.*, 63:21–28, 1983.

- [9] L. Machiels, Y. Maday, A. T. Patera, and D. V. Rovas. A blackbox reduced-basis output bound method for shape optimization. In *Proceedings 12th International Conference on Domain Decomposition Methods*. DDM org., 2001.
- [10] Y. Maday, A. T. Patera, and G. Turinici. *A Priori* convergence theory for reduced-basis approximations of single-parameter elliptic partial differential equations. *J. Sci. Comput.*, 17:437–446, 2002.
- [11] Y. Maday and E. M. Rønquist. A reduced-basis element method. *J. Sci. Comput.*, 17:447–459, 2002.
- [12] Y. Maday and E. M. Rønquist. The reduced-basis element method: application to a thermal fin problem. *SIAM J. Sci. Comput.*, 26(1):240–258, 2004.
- [13] A. Newman. Model reduction via the Karhunen-Loeve expansion; Part 1: An exposition. Technical Report 96-32, Institute for System Research, University of Maryland, 1996.
- [14] A. K. Noor and J. M. Peters. Reduced basis technique for nonlinear analysis of structures. *AIAA J.*, 18(4):455–462, 1980.
- [15] J. S. Peterson. The reduced basis method for incompressible viscous flow calculations. *SIAM J. Sci. Stat. Comput.*, 10(4):777–786, 1989.
- [16] T. A. Porsching. Estimation of the error in the reduced basis method solution of nonlinear equations. *Math. Comp.*, 45:487–496, 1985.
- [17] C. Prud’homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera, and G. Turinici. Reliable real-time solution of parametrized partial differential equations: Reduced basis output bound methods. *J. Fluids Engineering*, 124:70–80, 2002.
- [18] S.S. Ravindran. Reduced-order adaptive controllers for fluid flows using POD. *J. Sci. Comput.*, 15:457–478, 2000.
- [19] W. C. Rheinboldt. On the theory and error estimation of the reduced basis method for multi-parameter problems. *Nonlinear Analysis, Theory, Method & Applications*, pages 849–858, 1993.
- [20] G. Rozza. Reduced-basis methods for elliptic equations in sub-domains with a posteriori error bounds and adaptivity. *To appear in Applied Numerical Mathematics*, 2005.
- [21] L. Sirovich. Turbulence and the dynamics of coherent structures. Pt. I: Coherent structures. *Quarter. Appl. Math.*, 45(3):561–571, 1987.

- [22] K.Y. Tang, W.R. Graham, and J. Peraire. Active flow control using a reduced order model and optimum control. AIAA Paper 96-1946, 1996.
- [23] K. Veroy, C. Prud'homme, D. V. Rovas, and A. T. Patera. *A Posteriori* error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations (AIAA Paper 2003-3847). In *Proceedings of the 16th AIAA Computational Fluid Dynamics Conference*, June 2003.
- [24] K.E. Willcox, J.D. Paduano, J. Peraire, and K.C. Hall. Low order aerodynamic models for aeroelastic control of turbomachines. AIAA Paper 99-1467, 1999.

# I

## A reduced basis element method for the steady Stokes problem

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Submitted to Mathematical Modelling and Numerical  
Analysis (M2AN)





# A reduced basis element method for the steady Stokes problem

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## Abstract

The reduced basis element method is a new approach for approximating the solution of problems described by partial differential equations. The method takes its roots in domain decomposition methods and reduced basis discretizations. The basic idea is to first decompose the computational domain into a series of subdomains that are deformations of a few reference domains (or generic computational parts). Associated with each reference domain are precomputed solutions corresponding to the same governing partial differential equation, but solved for different choices of deformations of the reference subdomains and mapped onto the reference shape. The approximation corresponding to a new shape is then taken to be a linear combination of the precomputed solutions, mapped from the reference domain for the part to the actual computational part. We extend earlier work in this direction to solve incompressible fluid flow problems governed by the steady Stokes equations. Particular focus is given to satisfying the inf-sup condition, to *a posteriori* error estimation, and to “gluing” the local solutions together in the multidomain case.

## 1 Introduction

The reduced basis element method is a new approach for approximating the solution of problems described by partial differential equations. The method takes its roots in domain decomposition methods and in reduced basis discretizations.

Reduced basis methods have been introduced in [8] and [15] as a computational approach that makes it possible to obtain a good approximation by solving very small systems. The problem is described by partial differential equations, and the solution depends on one or more parameters. For many problems, the dependency of the solution on the parameters is often regular. It is thus possible to approximate the solution for general values of the parameters as a linear combination of precomputed solutions for certain instances (or “snapshots”) of the parameters. The best linear combination is often found by a Galerkin procedure.

The reduced basis framework may be applied to a particular region in physical space. The parameters may be material properties or geometric parameters, say. An extension of this framework to solve more complex problems is based on first decomposing the domain into generic building blocks, and then applying the reduced basis approach locally in the context of each building block. A global approximation is then constructed by gluing together the local approximations using a mortar type method. These are the essential ingredients of the reduced basis element method. The term element here refers to the notion of a generic building block or part.

The reduced basis element method was introduced in [13] in the context of solving the Laplace equation in a deformed geometry. In that work, the computational domain was broken into generic building blocks or parts (e.g., “pipes”), and the geometry of each part played the role of the parameters. The reduced basis approximations associated with all the parts were glued together through the use of Lagrange multipliers.

Further progress was made in [14] in the context of solving a thermal fin problem. In particular, *a posteriori* error bounds were developed for the output of interest. We remark that this aspect represents an essential ingredient in developing practical reduced basis methods. Since it is *a priori* difficult to guess the number of reduced elements required in practical computations, *a posteriori* error bounds represent practical tools which allow us to certify the computational results.

In this paper, we extend the results in [13] to solve incompressible fluid flow problems modeled by the steady Stokes equations. What is similar to earlier work is the construction of a reduced basis for a particular generic part based on “geometric snapshots,” and the representation of the individual basis functions on a reference domain associated with the generic part. However, there are also several new extensions and new aspects considered in this paper.

In Section 2 we state the steady Stokes problem we are considering together with the particular geometries we are dealing with. The spectral discretization used to construct the reduced basis is also briefly reviewed. In Section 3 we introduce the reduced basis method in the single domain case, and present numerical results. Two different methods are presented, one where the velocity-pressure pair is computed (Method 1), and one where only the velocity is computed (Method 2). We also extend the *a posteriori* analysis in [16] and [18] to the compliant case.

Finally, in Section 4, we turn our attention to the multiple domain case. Of particular interest is the proper gluing of the local approximations. As in the earlier work, we use a mortar type method based on Lagrange multipliers; see [4]. The pressure is only in  $L^2$ , so no particular problems occur here. However, special care has to be given to the definition of the Lagrange multiplier spaces associated with the velocity components.

## 2 The steady Stokes problem

### 2.1 Governing equations

We consider here the two-dimensional steady Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega, \end{aligned} \quad (2.1)$$

where  $\mathbf{u} = (u_1, u_2)$  is the velocity field,  $p$  is the pressure,  $\mathbf{f} = (f_1, f_2)$  is a prescribed volumetric body force, and  $\nu$  is the fluid viscosity; see [1]. The steady Stokes equations model laminar flow at very low Reynolds number; the nonlinear advection term of the Navier-Stokes equations is neglected, and we are left with a linear problem. The velocity formulation (2.1) is an appropriate model when  $\nu$  is constant and for certain types of boundary conditions. For all the problems studied in this paper, this model will suffice. For more general problems, the full stress formulation of the steady Stokes equations should be used.

We consider here a domain  $\Omega$  with an inflow boundary,  $\Gamma_{in}$ , an outflow boundary,  $\Gamma_{out}$ , and wall boundaries,  $\Gamma_w$ . On this domain we introduce the velocity space

$$X = \{\mathbf{v} \in (H^1(\Omega))^2, \quad \mathbf{v}|_{\Gamma_w} = 0 \text{ and } v_t|_{\Gamma_{in}} = v_t|_{\Gamma_{out}} = 0\}, \quad (2.2)$$

where  $v_t$  is the tangential velocity component. In addition, we have the Neumann type boundary conditions given by specifying  $\sigma_n = \nu \frac{\partial u_n}{\partial n} - p$  to be  $\sigma_n^{in} = -1$  along  $\Gamma_{in}$  and  $\sigma_n^{out} = 0$  along  $\Gamma_{out}$ ; here,  $u_n$  is the normal velocity component and  $\partial/\partial n$  denotes the derivative in the outward normal direction. For all the problems solved in this study, the exact solution of (2.1) satisfies  $\frac{\partial u_n}{\partial n} = 0$  along  $\Gamma_{in}$  and  $\Gamma_{out}$ , which implies that the Neumann conditions correspond to specifying the pressure along the inflow and outflow boundaries (in a weak sense).

With the given boundary conditions, we define the pressure space to be

$$M = L^2(\Omega). \quad (2.3)$$

In order to solve the steady Stokes equations we define the bilinear forms

$$a(\mathbf{v}, \mathbf{w}) = \nu \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{w} \, d\Omega = \nu \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\Omega, \quad (2.4)$$

$$b(\mathbf{v}, q) = - \int_{\Omega} q \nabla \cdot \mathbf{v} \, d\Omega = \sum_{i=1}^2 - \int_{\Omega} q \frac{\partial v_i}{\partial x_i} \, d\Omega, \quad (2.5)$$

and consider the weak form: Find  $\mathbf{u} \in X$  and  $p \in M$  such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= l(\mathbf{v}) \quad \forall \mathbf{v} \in X \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in M, \end{aligned} \quad (2.6)$$

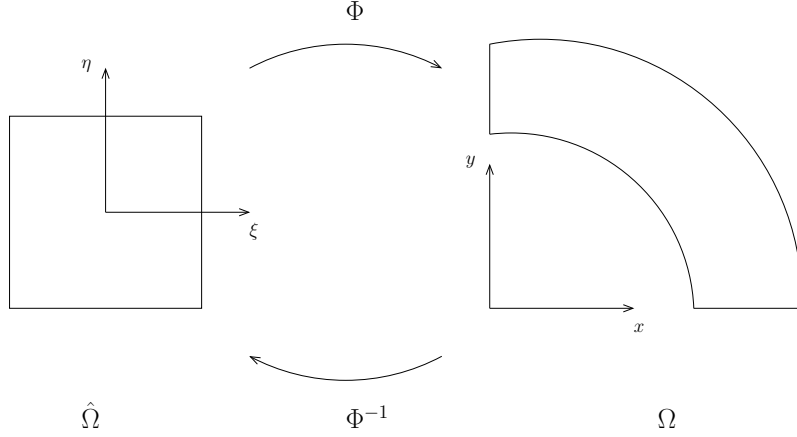


Figure 1: Mapping of the reference domain.

where

$$l(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \int_{\Gamma_{in}} \sigma_n^{in} \mathbf{v} \cdot \mathbf{n} ds + \int_{\Gamma_{out}} \sigma_n^{out} \mathbf{v} \cdot \mathbf{n} ds. \quad (2.7)$$

To ensure a unique solution of the steady Stokes problem (2.6), the coercivity condition

$$a(\mathbf{w}, \mathbf{w}) \geq \alpha \|\mathbf{w}\|_{H^1(\Omega)}^2, \quad \forall \mathbf{w} \in X, \quad \alpha > 0, \quad (2.8)$$

and the inf-sup condition

$$\inf_{q \in M} \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}, q)}{\|q\|_{L^2(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)}} = \beta > 0, \quad (2.9)$$

must be satisfied; see [2] and [6]. These conditions are fulfilled for our particular Stokes problem.

## 2.2 Mapping to a reference domain

In the following, we focus on two-dimensional problems where the domain  $\Omega$  is at least Lipschitz continuous, and can be considered as a mapping of a reference domain  $\hat{\Omega} = (-1, 1)^2$ , see Figure 1. We write  $\Omega = \Phi(\hat{\Omega})$ , where  $\Phi$  is an one-to-one mapping, and the left and right vertical edges on  $\hat{\Omega}$  correspond to the inflow and outflow boundaries on  $\Omega$ , respectively. In terms of the reference variables, we can then alternatively express (2.4) and (2.5) as

$$a(\mathbf{v}, \mathbf{w}) = \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \mathcal{J}^{-T} \hat{\nabla}(\mathbf{w} \circ \Phi) |J| d\hat{\Omega}, \quad (2.10)$$

$$b(\mathbf{v}, q) = - \int_{\hat{\Omega}} (q \circ \Phi) \hat{\nabla} \cdot [\mathcal{J}^{-1}(\mathbf{v} \circ \Phi)] |J| d\hat{\Omega}, \quad (2.11)$$

where  $\mathcal{J}$  is the Jacobian of  $\Phi$ ,

$$\mathcal{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}, \quad (2.12)$$

and  $J$  denotes the determinant of  $\mathcal{J}$ . A similar transformation can be done to the linear form (2.7); in the following, we set  $\mathbf{f} = \mathbf{0}$  and  $\nu = 1$ .

As is well known, this transformation is useful for the discretization of the Stokes problem. However, it will also prove very useful when we consider the reduced basis element method. As discussed in Section 3, we will exploit the fact that the solution  $\mathbf{u}$  and  $p$  can be considered dependent on the geometry (or shape) through the mapping  $\Phi$ , i.e.,  $\mathbf{u} = \mathbf{u}(\Phi)$  and  $p = p(\Phi)$ ; see [13].

### 2.3 Spectral discretization

We now consider a discretization of the steady Stokes problem (2.6) using a pure spectral method based on high order polynomials; see [11] and [12]. Let  $\mathbb{P}_n(\hat{\Omega})$  be the space of all functions which are polynomials of degree less than or equal to  $n$  in each spatial direction of the reference domain  $\hat{\Omega}$ . The discrete space for the velocity is then taken to be

$$X_{\mathcal{N}} = \{\mathbf{v} \in X, \quad \mathbf{v} \circ \Phi \in (\mathbb{P}_{\mathcal{N}}(\hat{\Omega}))^2\}, \quad (2.13)$$

while the discrete space for the pressure is

$$M_{\mathcal{N}} = \{q \in M, \quad q \circ \Phi \in \mathbb{P}_{\mathcal{N}-2}(\hat{\Omega})\}. \quad (2.14)$$

The bases for  $X_{\mathcal{N}}$  and  $M_{\mathcal{N}}$  are conveniently expressed in terms of the reference variables  $\xi$  and  $\eta$ . As a basis for  $X_{\mathcal{N}}$  we use a nodal basis through the tensor-product Gauss-Lobatto Legendre (GLL) points, while the basis for  $M_{\mathcal{N}}$  is a nodal basis through the tensor-product Gauss-Legendre (GL) points; see [11] and [12]. Specifically, we write  $\mathbf{u}_{\mathcal{N}} \in X_{\mathcal{N}}$  as

$$(\mathbf{u}_{\mathcal{N}} \circ \Phi)(\xi, \eta) = \sum_{i,j=0}^{\mathcal{N}} \mathbf{u}_{ij} \ell_i(\xi) \ell_j(\eta), \quad (2.15)$$

where  $\ell_i(\xi)$  refers to a one-dimensional  $\mathcal{N}$ -th order Lagrangian interpolant through the GLL points  $\xi_m$ ,  $m = 0, \dots, \mathcal{N}$ ; hence,  $\ell_i(\xi_m) \ell_j(\xi_n) = \delta_{im} \delta_{jn}$  for a given point  $(\xi_m, \xi_n)$  in the underlying tensor-product GLL grid.

In a similar fashion, we write  $p_{\mathcal{N}} \in M_{\mathcal{N}}$  as

$$(p_{\mathcal{N}} \circ \Phi)(\xi, \eta) = \sum_{i,j=0}^{\mathcal{N}-2} p_{ij} \tilde{\ell}_i(\xi) \tilde{\ell}_j(\eta), \quad (2.16)$$

where  $\tilde{\ell}_i(\xi)$  refers to a one-dimensional  $(\mathcal{N}-2)$ -th order Lagrangian interpolant through the (interior) GL points  $\zeta_m$ ,  $m = 0, \dots, \mathcal{N}-2$ ; hence,  $\tilde{\ell}_i(\zeta_m) \tilde{\ell}_j(\zeta_n) = \delta_{im} \delta_{jn}$  for a given point in the tensor-product GL grid.

The discrete velocity  $\mathbf{u}_N \in X_N$  is then uniquely determined by  $(N+1)^2$  coefficients for each spatial component, where some of the coefficients are fixed due to the prescribed Dirichlet boundary conditions. The discrete pressure is determined by  $(N-1)^2$  basis coefficients.

The mapping  $\Phi$  is realized computationally by using an isoparametric representation of the geometry. Each edge of  $\Omega$  is given as a one-to-one mapping of a corresponding edge  $[-1, 1]$  on  $\hat{\Omega}$ . Each edge of  $\Omega$  is approximated as an  $N$ -th order polynomial, and the location of the (interior) points  $(x_m, y_n) = \Phi(\xi_m, \xi_n)$  are found by a Gordon-Hall algorithm; see [9].

The bilinear forms and the linear form in (2.6) are expressed in terms of the reference variables, and the integrals are evaluated using GLL and GL quadrature. This gives us the following discrete system: Find  $\mathbf{u}_N \in X_N$  and  $p_N \in M_N$  such that

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{v}) + b_N(\mathbf{v}, p_N) &= l_N(\mathbf{v}) \quad \forall \mathbf{v} \in X_N \\ b_N(\mathbf{u}_N, q) &= 0 \quad \forall q \in M_N, \end{aligned} \quad (2.17)$$

where  $a_N$ ,  $b_N$  and  $l_N$  refer to integration of the bilinear and linear forms using Gauss-type quadrature. Using the chosen bases (2.15) and (2.16), we arrive at a system of algebraic equations for the unknown basis coefficients; this system is solved using the conjugate gradient method in the context of the Uzawa algorithm; see [12].

### 3 A reduced basis method

#### 3.1 Geometry as a parameter

In Section 2 we considered the numerical solution of a steady Stokes problem using a pure spectral method based on high order polynomials. We had given a domain  $\Omega$  as well as prescribed boundary conditions along  $\partial\Omega$ .

Imagine now that we would like to solve similar problems, i.e., the same governing equations, imposing the same boundary conditions, but now in a different domain. The only restriction is that the new domain should also be obtained as the deformation of the reference domain  $\hat{\Omega}$  through some regular one-to-one mapping; see Figure 1.

The mapping from  $\Omega$  onto the reference domain as explained in Section 2.2, suggests that the solution of the steady Stokes problem will depend on the geometry (or shape) of the domain in a fairly regular manner. The idea behind the reduced basis method is to exploit this fact. We will first solve the steady Stokes problem for different ‘‘snapshots’’ of possible shapes, and then express the solution corresponding to a new and untried shape as a linear combination of the precomputed solutions. To find the best linear combination, a Galerkin procedure will be used.

Note that the standard velocity and pressure bases introduced in Section 2.3 are expressed in terms of the reference variables. In the reduced basis

method, we will also express the bases with reference to  $\hat{\Omega}$ , but now the basis functions will be the precomputed steady Stokes solutions mapped to the reference domain  $\hat{\Omega}$ .

### 3.2 Precomputing the basis solutions

We propose to use the same technique as for the Poisson problem in [13], in order to generate basis functions for the steady Stokes problem. For a sample set of mappings of a reference domain,  $\{\Phi_i : \hat{\Omega} \rightarrow \Omega_i\}_{i=1}^N$ , we solve the steady Stokes problem (2.17). Again the left and right edges of  $\hat{\Omega}$  correspond to the inflow and outflow boundaries on  $\Omega_i$ , respectively. These solutions are then used as basis functions to find the solution for a generic deformation. In order to do this we must map the solutions found on the deformed geometries,

$$\{(\mathbf{u}_i, p_i) \in X_{\mathcal{N}}(\Omega_i) \times M_{\mathcal{N}}(\Omega_i)\}_{i=1}^N, \quad (3.1)$$

to the reference domain, where they are stored. The pressure is a scalar field and will be mapped as such,  $\hat{p}_i(\xi, \eta) = p_i \circ \Phi_i(\xi, \eta)$ . Since the regular mapping  $\Phi_i(\xi, \eta) = (x, y)$  is one-to-one from  $\hat{\Omega}$  to  $\Omega_i$ , this is well defined. The velocity solution,  $\mathbf{u}_i$ , is a divergence free vector field on the deformed domain, and we want the solution to keep this property when mapped to the reference domain. In particular, this means that if the flow is perpendicular to a surface of the deformed domain, the mapped flow should be perpendicular to the mapped surface. If we let  $\hat{\nabla}$  be the gradient operator on the reference domain, we have the following relationship with the gradient operator on the deformed domain

$$\hat{\nabla} = \mathcal{J}_i^T \nabla. \quad (3.2)$$

To find the proper mapping for the velocity, we start with the fact that

$$\int_{\Omega_i} q \nabla \cdot \mathbf{u}_i \, d\Omega_i = 0 \quad \forall q \in M_{\mathcal{N}}(\Omega_i), \quad (3.3)$$

and demand that  $\hat{\mathbf{u}}_i = \Psi_i(\mathbf{u}_i)$  satisfies

$$\int_{\hat{\Omega}} \hat{q} \hat{\nabla} \cdot \hat{\mathbf{u}}_i \, d\hat{\Omega} = 0 \quad \forall \hat{q} \in M_{\mathcal{N}}(\hat{\Omega}). \quad (3.4)$$

By mapping the integral in (3.4) to  $\Omega_i$ , we get

$$\int_{\Omega_i} q \nabla \cdot \mathcal{J}_i(\hat{\mathbf{u}}_i \circ \Phi_i^{-1}) |J_i^{-1}| \, d\Omega_i = 0 \quad \forall q \in M_{\mathcal{N}}(\Omega_i). \quad (3.5)$$

Equation (3.5) holds if  $\mathcal{J}_i(\hat{\mathbf{u}}_i \circ \Phi_i^{-1}) |J_i^{-1}| = \mathbf{u}_i$ , which is known as Piola's transformation; see [7] for general properties. The proper mapping of the velocity from  $\Omega_i$  to  $\hat{\Omega}$  will therefore be

$$\hat{\mathbf{u}}_i = \Psi_i(\mathbf{u}_i) = \mathcal{J}_i^{-1}(\mathbf{u}_i \circ \Phi_i) |J_i|, \quad (3.6)$$



where

$$\mathcal{J}^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix}. \quad (3.7)$$

Mapping the velocity-vectors to the reference domain in this way, will certainly keep the velocity perpendicular to the same lines on the reference domain as the lines' mapped counterparts on the deformed domain, and the velocity will be divergence free on the reference domain. We define the reduced basis reference spaces as

$$\begin{aligned} \hat{X}_N^0 = X_N^0(\hat{\Omega}) &= \text{span}\{\hat{\mathbf{u}}_i, \quad i = 1, \dots, N\} \\ \hat{M}_N = M_N(\hat{\Omega}) &= \text{span}\{\hat{p}_i, \quad i = 1, \dots, N\}. \end{aligned} \quad (3.8)$$

Some of the precomputed basis functions are not symmetric with respect to the horizontal center-line  $\eta = 0$  in the reference domain. These basis functions are reflected across the line  $\eta = 0$  in order to enrich the basis and to eliminate directional effects; see [13]. In the following, we let  $N$  denote the total number of basis functions such that the definition of the reduced basis reference spaces in (3.8) still holds.

For a generic domain  $\Omega = \Phi(\hat{\Omega})$  we will now define the reduced basis solution spaces  $X_N \subset X_N$  and  $M_N \subset M_N$  that we will use. Our objective is to find a unique reduced basis solution  $\mathbf{u}_N \in X_N$  and  $p_N \in M_N$  satisfying

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{v}) + b_N(\mathbf{v}, p_N) &= l_N(\mathbf{v}) \quad \forall \mathbf{v} \in X_N \\ b_N(\mathbf{u}_N, q) &= 0 \quad \forall q \in M_N. \end{aligned} \quad (3.9)$$

As before the coercivity of  $a(\cdot, \cdot)$  holds for all  $\mathbf{v} \in X_N$ . The inf-sup condition (2.9), however, depends strongly on  $X_N$  and  $M_N$ .

Since we have assumed that  $\Phi : \hat{\Omega} \rightarrow \Omega$  is one-to-one, and the pressure  $p_N$  is a scalar field over  $\Omega$ , we define the reduced basis pressure space  $M_N$  as

$$M_N = \text{span}\{\hat{p}_i \circ \Phi^{-1}, \quad i = 1, \dots, N\}. \quad (3.10)$$

We map the precomputed velocity fields  $\hat{\mathbf{u}}_i$  to the new geometry  $\Omega$  as

$$\tilde{\mathbf{u}}_i = \Psi^{-1}(\hat{\mathbf{u}}_i) \equiv \mathcal{J}(\hat{\mathbf{u}}_i \circ \Phi^{-1})|J^{-1}|, \quad i = 1, \dots, N, \quad (3.11)$$

and then we define the discrete velocity space  $X_N^0$  as

$$X_N^0 = \text{span}\{\tilde{\mathbf{u}}_i\} = \text{span}\{\Psi^{-1}(\hat{\mathbf{u}}_i), \quad i = 1, \dots, N\}. \quad (3.12)$$

With this mapping, all  $\mathbf{v} \in X_N^0$  will be weakly divergence free since

$$\int_{\Omega} q \nabla \cdot \tilde{\mathbf{u}}_i \, d\Omega = \int_{\Omega} q \nabla \cdot (\Psi^{-1}(\hat{\mathbf{u}}_i)) \, d\Omega = 0 \quad \forall q \in M_N. \quad (3.13)$$

In particular, this means that the inf-sup condition can not be fulfilled for the pair  $X_N^0 \times M_N$ . We now present two different methods to solve the steady Stokes problem (3.9).

### 3.3 Method 1

In our first method for choosing the reduced basis velocity space we will define  $X_N$  such that (3.9) can be solved directly for both the velocity and the pressure. In order to satisfy the inf-sup condition we have to enrich the velocity basis. One way of doing this is for each pressure solution,  $\hat{p}_i \in \hat{M}_N$ , to find  $\hat{\mathbf{v}}(\hat{p}_i) = \hat{\mathbf{v}}_i \in \hat{X}_N = X_N(\hat{\Omega})$  such that

$$\hat{\mathbf{v}}_i = \arg \max_{\hat{\mathbf{u}} \in \hat{X}_N} \frac{\int_{\hat{\Omega}} \hat{p}_i \hat{\nabla} \cdot \hat{\mathbf{u}} d\hat{\Omega}}{\|\hat{\mathbf{u}}\|_{H^1}}. \quad (3.14)$$

It is well known that the solution of (3.14) also satisfies

$$\int_{\hat{\Omega}} \hat{\nabla} \hat{\mathbf{v}}_i \cdot \hat{\nabla} \hat{\mathbf{w}} d\hat{\Omega} = \int_{\hat{\Omega}} \hat{p}_i \hat{\nabla} \cdot \hat{\mathbf{w}} d\hat{\Omega}, \quad \forall \hat{\mathbf{w}} \in \hat{X}_N. \quad (3.15)$$

We thus find the  $\hat{\mathbf{v}}_i$ 's by solving (3.15) for the precomputed  $\hat{p}_i, i = 1, \dots, N$ . In practice, we evaluate the integrals in (3.15) using GLL quadrature. We define the enriched velocity reference space as

$$\hat{X}_N^e = \text{span}\{\hat{\mathbf{v}}_i, \quad i = 1, \dots, N\}, \quad (3.16)$$

with corresponding enriched velocity space defined on  $\Omega$  as

$$X_N^e = \text{span}\{\Psi^{-1}(\hat{\mathbf{v}}_i), \quad i = 1, \dots, N\}. \quad (3.17)$$

If we then define

$$X_N = X_N^0 \oplus X_N^e, \quad (3.18)$$

the inf-sup condition will be satisfied since

$$\inf_{\hat{q} \in \hat{M}_N} \sup_{\hat{\mathbf{u}} \in \hat{X}_N^0 \oplus \hat{X}_N^e} \frac{\int_{\hat{\Omega}} \hat{q} \hat{\nabla} \cdot \hat{\mathbf{u}} d\hat{\Omega}}{\|\hat{q}\|_{L^2} \|\hat{\mathbf{u}}\|_{H^1}} = \inf_{\hat{q} \in \hat{M}_N} C \frac{\|\hat{\mathbf{v}}(\hat{q})\|_{H^1}}{\|\hat{q}\|_{L^2}} = \beta > 0, \quad (3.19)$$

where  $C$  is a constant, and  $\hat{\mathbf{v}}(\hat{q})$  is the solution of (3.15) with  $\hat{p} = \hat{q}$ . We remark that  $\beta$  may depend on  $N$ .

### 3.4 Method 2

In our second method we will explicitly use the fact that the basis functions in  $X_N^0$  are divergence free, see (3.12), and that any element in  $\mathbf{v} \in X_N^0$  can be written  $\mathbf{v} = \sum_{i=1}^N \alpha_i \Psi^{-1}(\hat{\mathbf{u}}_i)$ . It is well known that the solution of the steady Stokes problem (3.9) corresponds to solving an underlying min-max problem; see [7]. However, with the availability of a divergence free basis, this min-max problem reduces to the pure minimization problem

$$\mathbf{u}_N = \arg \min_{\mathbf{v} \in X_N^0} I(\mathbf{v}) = \frac{1}{2} a_N(\mathbf{v}, \mathbf{v}) - l_N(\mathbf{v}), \quad (3.20)$$

or, equivalently, to the (vector) Poisson problem: Find  $\mathbf{u}_N \in X_N^0$  such that

$$a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}) = l_{\mathcal{N}}(\mathbf{v}) \quad \forall \mathbf{v} \in X_N^0. \quad (3.21)$$

To solve this Poisson-type problem the inf-sup condition is irrelevant, and (3.21) has a unique solution  $\mathbf{u}_N \in X_N^0$  due to the coercivity of  $a_{\mathcal{N}}(\cdot, \cdot)$ .

We refer to this choice of reduced basis velocity space as Method 2. The advantage of this approach lies in the fact that the resulting algebraic system is very small. In fact, the dimension of this system is just  $N$ . An extension of this approach to three dimensions would also yield a system of dimension  $N$  for  $N$  precomputed basis functions.

The solution of (3.21) only gives the velocity. In order to compute the pressure, we proceed as follows: We know that for all  $\mathbf{v} \in X_N^0$  we have  $b_{\mathcal{N}}(\mathbf{v}, q) = 0, \forall q \in M_N$ . We also know that the inf-sup condition is fulfilled for the previously defined spaces  $X_N$  and  $M_N$ . Hence, the inf-sup condition must hold also for the spaces  $X_N^e$  and  $M_N$ . We can then compute the pressure  $p_N \in M_N$  uniquely by solving

$$b_{\mathcal{N}}(\mathbf{v}, p_N) = -a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}) + l_{\mathcal{N}}(\mathbf{v}) \quad \forall \mathbf{v} \in X_N^e. \quad (3.22)$$

The dimension of the algebraic system corresponding to (3.22) is just  $N$ , similar to the velocity system. Note that the space  $X_N^e$  is spanned by the precomputed, enriched velocity basis functions described in Method 1.

### 3.5 Equivalence between Method 1 and Method 2

We will now prove that Method 1 and Method 2 give the same reduced basis solution. Although  $\mathbf{u}_N$  according to Method 1 is an element in  $X_N^0 \oplus X_N^e$ , we will argue that  $\mathbf{u}_N \in X_N^0$  only. First we let

$$\mathbf{u}_N = \mathbf{u}_N^0 + \sum_{i=1}^N \alpha_i \mathbf{v}_i, \quad (3.23)$$

where  $\mathbf{u}_N^0 \in X_N^0$ , and  $\mathbf{v}_i \in X_N^e$  for  $i = 1, \dots, N$ . This gives the relation

$$b_{\mathcal{N}}(\mathbf{u}_N, q) = b_{\mathcal{N}}\left(\sum_{i=1}^N \alpha_i \mathbf{v}_i, q\right) \quad \forall q \in M_N. \quad (3.24)$$

Next we take  $q = \sum_{i=1}^N \alpha_i p_i$ , where  $\{p_i\}_{i=1}^N$  is the basis for  $M_N$ , and get

$$b_{\mathcal{N}}(\mathbf{u}_N, q) = b_{\mathcal{N}}\left(\sum_{i=1}^N \alpha_i \mathbf{v}_i, \sum_{i=1}^N \alpha_i p_i\right), \quad (3.25)$$

which according to (3.15) leads to

$$b_{\mathcal{N}}(\mathbf{u}_N, q) = a_{\mathcal{N}}\left(\sum_{i=1}^N \alpha_i \mathbf{v}_i, \sum_{i=1}^N \alpha_i \mathbf{v}_i\right) \geq 0. \quad (3.26)$$

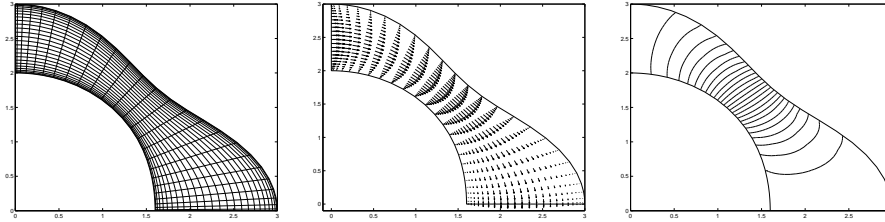


Figure 2: The domain  $\Omega$  with velocity and pressure solution.

We already know that

$$b_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, q) = 0 \quad \forall q \in M_{\mathcal{N}}, \quad (3.27)$$

so we must have  $\sum_{i=1}^{\mathcal{N}} \alpha_i \mathbf{v}_i = 0$ , which again means that  $\alpha_i = 0$  for  $i = 1, \dots, \mathcal{N}$  since the  $\mathbf{v}_i$ s are linearly independent. Hence,  $\mathbf{u}_{\mathcal{N}} \in X_{\mathcal{N}}^0 \subset X_{\mathcal{N}}$ .

From a mathematical point of view, Method 1 and Method 2 are thus equivalent (at least for the single domain case considered here), however, the two methods lead to different computational schemes with different computational complexities. Note that, since  $\mathbf{u}_{\mathcal{N}} \in X_{\mathcal{N}}^0$  for both methods,

$$b_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, q) = 0 \quad \forall q \in M_{\mathcal{N}}. \quad (3.28)$$

We also note that when the precomputed velocity basis functions are mapped to the reference domain through the Piola transformation, and then to the generic geometry through the inverse Piola transformation, some accuracy may be lost. This issue will be addressed in a later paper, however, the result is that (3.28) is generally only an approximation. Hence, the results from Method 2 may not always be as good as the results from Method 1, in particular, when the reduced basis error is very small.

Finally, we remark that for both methods, the exact (spectral) solution is recovered on the geometries that have been used to build the reduced basis (consistency of the approach), and also on the geometries obtained through symmetry acting on these geometries.

### 3.6 Numerical results

We now define the domain  $\Omega = \Phi(\hat{\Omega})$  to be the deformed annulus in Figure 2. For a polynomial degree  $\mathcal{N} = 25$  we compute the solution  $(\mathbf{u}_{\mathcal{N}}, p_{\mathcal{N}})$  of (2.17), and use it to assess the quality of our reduced basis approximation  $(\mathbf{u}_{\mathcal{N}}, p_{\mathcal{N}})$ . The velocity and pressure functions  $\{(\mathbf{u}_i, p_i) \in X_{\mathcal{N}}(\Omega_i) \times M_{\mathcal{N}}(\Omega_i)\}_{i=1}^8$  are found by solving (2.17) on the different geometries appearing in Figure 3. After mapping these functions to the reference domain as  $\hat{p}_i = p_i \circ \Phi_i$  and  $\hat{\mathbf{u}}_i = \Psi_i(\mathbf{u}_i)$ ,  $i = 1, \dots, 8$ , we have obtained the first eight basis functions. The velocity and pressure basis functions 2 through 8 are then reflected across the horizontal center line on the reference domain to produce basis

functions 9 through 15, i.e.  $N = 15$ . Finally, for each pressure basis function, one additional velocity basis function is constructed according to (3.15) to enrich the velocity basis. A precise definition of  $\Omega$  and  $\{\Omega_i\}_{i=1}^8$  is given in [14], where a scalar Poisson problem is solved with the reduced basis method using the same deformed geometries.

We use Method 1 for compatible spaces  $X_N$  and  $M_N$  to solve the steady Stokes equations (3.9) in the reduced basis context for different values of  $N$ . These results are presented in Table 1. We clearly see in Table 1 that the error between  $\mathbf{u}_N \in X_N$  and  $\mathbf{u}_N \in X_N$  decreases as  $N$  grows. Similar to the results for the scalar Poisson problem studied in [14], the reduced basis method gives good results also for incompressible vector fields.

The pressure solution is a scalar field, and we see that it converges somewhat faster than the velocity. We would like exponential convergence both for velocity and pressure, but if we study the results in Table 1 for  $N = 1, \dots, 4$ , the convergence is very slow. For  $N = 5, \dots, 8$ , however, the convergence is very good. A natural explanation for this is that the first four basis functions in  $X_N$  and  $M_N$  are constructed from the solutions of (3.9) on squares with deformed lower edges, see Figure 3. Since  $\Omega$  is a deformed annulus, the next four basis functions approximate the solution better. The effect of using only the four basis functions constructed from the deformed annuli in Figure 3 is shown in Figures 4(a) and 4(b). From this it is easy to see that a good choice of domains  $\Omega_i$  is essential in order to make our method effective. How to make this choice for a generic  $\Omega$  will be addressed in a later paper. The approach consists in combining the *a posteriori* tool presented in the next section in a recursive manner. This “greedy approach” follows the paper [19].

The results in Table 1 also confirm that Method 2 gives the same approximation for the velocity and the pressure as Method 1. The main difference between the two approaches lies in the computational complexity associated

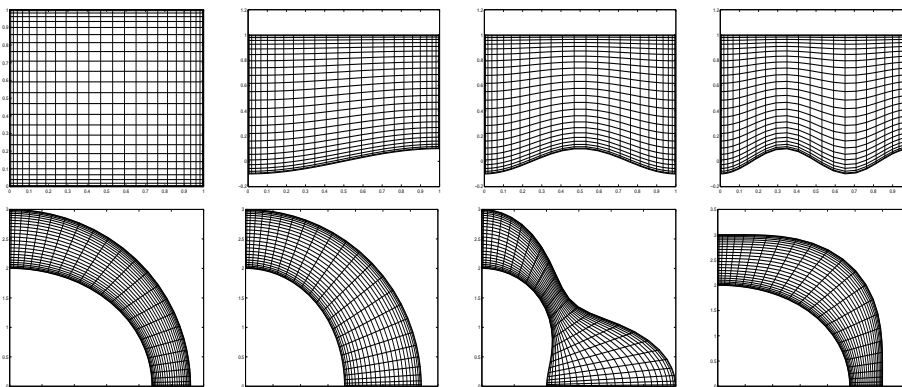
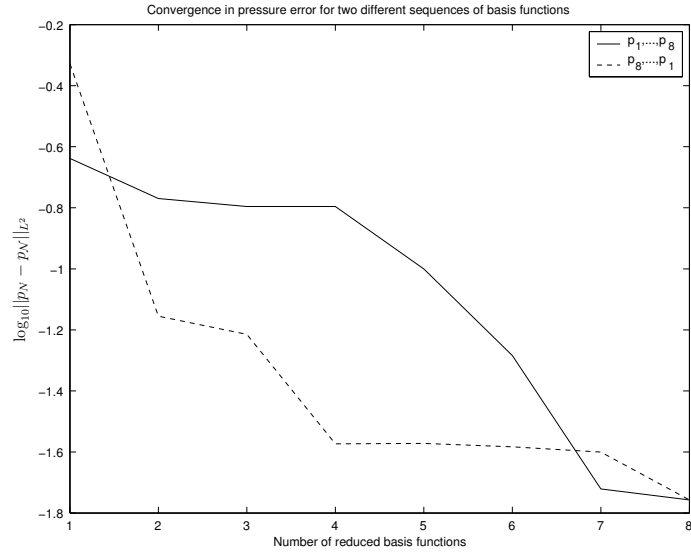
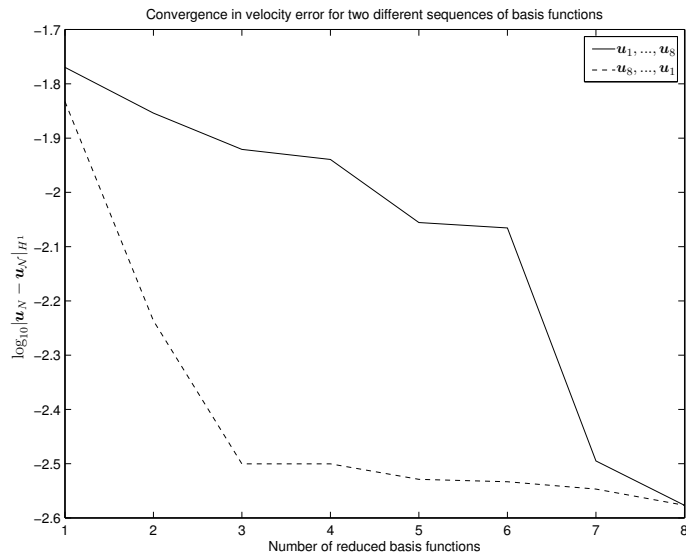


Figure 3: The deformed geometries used to construct the basis functions.



(a) Pressure



(b) Velocity

Figure 4: The effect of changing the order of the basis functions (Method 1).

$N$	Method 1		Method 2	
	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$
1	$1.7 \cdot 10^{-2}$	$2.3 \cdot 10^{-1}$	$1.7 \cdot 10^{-2}$	$2.3 \cdot 10^{-1}$
2	$1.4 \cdot 10^{-2}$	$1.7 \cdot 10^{-1}$	$1.4 \cdot 10^{-2}$	$1.7 \cdot 10^{-1}$
3	$1.2 \cdot 10^{-2}$	$1.6 \cdot 10^{-1}$	$1.2 \cdot 10^{-2}$	$1.6 \cdot 10^{-1}$
4	$1.2 \cdot 10^{-2}$	$1.6 \cdot 10^{-1}$	$1.2 \cdot 10^{-2}$	$1.6 \cdot 10^{-1}$
5	$8.8 \cdot 10^{-3}$	$1.0 \cdot 10^{-1}$	$8.8 \cdot 10^{-3}$	$1.0 \cdot 10^{-1}$
6	$8.6 \cdot 10^{-3}$	$5.2 \cdot 10^{-2}$	$8.6 \cdot 10^{-3}$	$5.2 \cdot 10^{-2}$
7	$3.2 \cdot 10^{-3}$	$1.9 \cdot 10^{-2}$	$3.2 \cdot 10^{-3}$	$1.8 \cdot 10^{-2}$
8	$2.7 \cdot 10^{-3}$	$1.8 \cdot 10^{-2}$	$2.7 \cdot 10^{-3}$	$1.7 \cdot 10^{-2}$
9	$2.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$	$2.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$
10	$2.0 \cdot 10^{-3}$	$9.7 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	$9.7 \cdot 10^{-3}$
11	$2.0 \cdot 10^{-3}$	$9.7 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	$9.7 \cdot 10^{-3}$
12	$1.8 \cdot 10^{-3}$	$9.5 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$	$9.5 \cdot 10^{-3}$
13	$1.7 \cdot 10^{-3}$	$9.3 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$	$9.2 \cdot 10^{-3}$
14	$1.4 \cdot 10^{-3}$	$9.2 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$	$9.2 \cdot 10^{-3}$
15	$1.4 \cdot 10^{-3}$	$8.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$	$8.6 \cdot 10^{-3}$

Table 1: Convergence results for Method 1 and Method 2. The dimension of  $X_N$  is  $2N$ , while the dimension of  $X_N^0$  is  $N$ .

with solving the resulting algebraic systems. In Method 1 we solve a coupled Stokes system involving  $2N$  degrees of freedom for the velocity and  $N$  degrees of freedom for the pressure. In Method 2 we first solve a system of dimension  $N$  for the velocity and then a system of dimension  $N$  for the pressure. In fact, if we don't need the pressure solution, Method 2 permits us to only solve the velocity system of dimension  $N$ .

Linear independence of the basis functions can generally be expected in most practical cases. However, some of the basis functions may be nearly linearly dependent, and this issue can be controlled by using an orthogonalization procedure. The orthogonalization also greatly reduces conditioning issues associated with the resulting systems of algebraic equations. For the velocity, orthogonalization is done on  $\hat{\Omega}$  with respect to  $a_N(\cdot, \cdot)$ , while the pressure is orthogonalized with respect to the discrete  $L^2$  inner product. Since this is done on the reference domain it can be part of the preprocessing stage. Note that in order to preserve the divergence free property of the original velocity basis functions, the orthogonalization must be done after ordering the basis functions in a proper way. First we orthogonalize the elements in  $X_N^0$ , followed by the elements in  $X_N^e$ .

For the fully coupled system of Method 1, the corresponding Uzawa pressure operator can be constructed explicitly, and the smallest eigenvalue of this operator is directly connected to the inf-sup parameter  $\beta$ ; see [10]. By

computing numerically the minimum eigenvalue and the condition number of the Uzawa pressure operator, the numerical results indicate that these are constant for our problem and equal to  $7.3 \cdot 10^{-2}$  and 14, respectively, independent of  $N$  (at least for the range of values we are dealing with).

### 3.7 *A posteriori* error estimation

In order to assess the quality of our reduced basis method we need *a posteriori* error estimation. For some specified output of interest,  $s(\mathbf{u}_N)$ , it will consist in providing lower and upper output bounds  $s^-(\mathbf{u}_N)$  and  $s^+(\mathbf{u}_N)$ , such that

$$s^-(\mathbf{u}_N) \leq s(\mathbf{u}_N) \leq s^+(\mathbf{u}_N). \quad (3.29)$$

In this work, we focus on compliant output, i.e.

$$s(\mathbf{u}) = l(\mathbf{u}). \quad (3.30)$$

We will follow the theory developed in [16] for operators which are continuous, coercive, symmetric and affine in terms of the parameter, in the similar way as has been done in [18] for the steady Stokes problem for more standard parameter dependencies. The steady Stokes operator is symmetric and continuous, but not coercive, and due to the geometric parameter it is not affine either.

We introduce the diffusion operator

$$\hat{a}(\mathbf{v}, \mathbf{w}) = \int_{\hat{\Omega}} g(\Phi) \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \hat{\nabla}(\mathbf{w} \circ \Phi) d\hat{\Omega}, \quad (3.31)$$

on the reference domain, where  $g(\Phi)$  is a geometry dependent positive function. The reconstructed error  $\hat{\mathbf{e}} \in \tilde{X}_N$  is then defined as the field that for some  $g(\Phi)$  satisfies

$$\hat{a}(\mathbf{e}, \mathbf{v}) = l(\mathbf{v}) - a(\mathbf{u}_N, \mathbf{v}) - b(\mathbf{v}, p_N) \quad \forall \mathbf{v} \in \tilde{X}_N, \quad (3.32)$$

where  $\tilde{X}_N = \{\mathbf{v} \circ \Phi \in (\mathbb{P}_N(\hat{\Omega}))^2, \mathbf{v}|_{\Gamma_w} = 0\}$ . The operator  $g(\Phi)$  is chosen such that

$$\alpha_0 \|\mathbf{v}\|_{X_N}^2 \leq \hat{a}(\mathbf{v}, \mathbf{v}) \leq a(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in X_N, \quad (3.33)$$

for some positive real constant  $\alpha_0$ . For this reconstructed error we claim that

$$s^-(\mathbf{u}_N) = l(\mathbf{u}_N), \text{ and} \quad (3.34)$$

$$s^+(\mathbf{u}_N) = l(\mathbf{u}_N) + \hat{a}(\mathbf{e}, \mathbf{e}) \quad (3.35)$$

are lower and upper bounds for  $s(\mathbf{u}_N)$ . We remark that we throughout this section will drop the subscript  $N$  indicating that we actually evaluate all the bilinear and linear forms using GLL quadrature.



Before we prove (3.34) and (3.35), we put  $\mathbf{v} = \mathbf{u}_N$  in (2.17) and (3.9) to derive that

$$a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) + b(\mathbf{u}_N, p_N) = 0. \quad (3.36)$$

According to (3.28) the last term is zero and thus (3.36) reduces to

$$a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) = 0. \quad (3.37)$$

For the lower bound we now get

$$\begin{aligned} s^-(\mathbf{u}_N) &= s(\mathbf{u}_N) + l(\mathbf{u}_N - \mathbf{u}_N) \\ &= s(\mathbf{u}_N) + a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) + b(\mathbf{u}_N - \mathbf{u}_N, p_N) \\ &= s(\mathbf{u}_N) + a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) \\ &= s(\mathbf{u}_N) + a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) + a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) \\ &= s(\mathbf{u}_N) + a(\mathbf{u}_N - \mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N). \end{aligned} \quad (3.38)$$

And we have the desired relationship

$$s^-(\mathbf{u}_N) \leq s(\mathbf{u}_N), \quad (3.39)$$

independent of  $g(\Phi)$ .

For the upper bound we denote the error on the deformed domain by  $\mathbf{e}^u = \mathbf{u}_N - \mathbf{u}_N$ , and find that

$$\begin{aligned} 2\hat{a}(\mathbf{e}, \mathbf{e}^u) &= l(\mathbf{u}_N - \mathbf{u}_N) + l(\mathbf{u}_N - \mathbf{u}_N) \\ &\quad - 2a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) \\ &= l(\mathbf{u}_N - \mathbf{u}_N) + a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) \\ &\quad + b(\mathbf{u}_N - \mathbf{u}_N, p_N) - 2a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) \\ &= l(\mathbf{u}_N - \mathbf{u}_N) + a(\mathbf{u}_N - \mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) \\ &\quad - a(\mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N) \\ &= l(\mathbf{u}_N - \mathbf{u}_N) + a(\mathbf{e}^u, \mathbf{e}^u). \end{aligned} \quad (3.40)$$

To prove that (3.35) is an upper bound we now use (3.40) to get

$$\begin{aligned} s^+(\mathbf{u}_N) &= l(\mathbf{u}_N) + \hat{a}(\mathbf{e}, \mathbf{e}) \\ &= l(\mathbf{u}_N) + \hat{a}(\mathbf{e}, \mathbf{e}) \\ &\quad - 2\hat{a}(\mathbf{e}, \mathbf{e}^u) + l(\mathbf{u}_N - \mathbf{u}_N) + a(\mathbf{e}^u, \mathbf{e}^u) \\ &\quad + \hat{a}(\mathbf{e}^u, \mathbf{e}^u) - \hat{a}(\mathbf{e}^u, \mathbf{e}^u) \\ &= l(\mathbf{u}_N) + \hat{a}(\mathbf{e} - \mathbf{e}^u, \mathbf{e} - \mathbf{e}^u) \\ &\quad + a(\mathbf{e}^u, \mathbf{e}^u) - \hat{a}(\mathbf{e}^u, \mathbf{e}^u) \\ &\geq s(\mathbf{u}_N), \end{aligned} \quad (3.41)$$

where the inequality is due to (3.33) and the coercivity of  $\hat{a}(\cdot, \cdot)$ .

It now remains to find a positive function  $g(\Phi)$ , such that

$$a(\mathbf{v}, \mathbf{v}) \geq \hat{a}(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in X_N. \quad (3.42)$$

For a constant  $g(\Phi) = \lambda$ , we can use the theory of [16] to see that  $\lambda$  should be chosen as large as possible without violating (3.42). This largest constant may be found, as in [14], by computing the smallest eigenvalue of the generalized symmetric eigenvalue problem

$$a(\mathbf{v}, \mathbf{v}) = \lambda \int_{\hat{\Omega}} \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \hat{\nabla}(\mathbf{v} \circ \Phi) d\hat{\Omega}. \quad (3.43)$$

We tried this approach also for the current problem, and used an inverse Rayleigh quotient iteration to estimate  $\lambda$ , but the resulting upper bound gap proved much too conservative.

To get a better estimate we consider the Jacobian,  $\mathcal{J}(\Phi)$ , of the mapping from  $\hat{\Omega}$  to  $\Omega$ . We start with the left hand side of (3.42), and use the fact that

$$\nabla = \mathcal{J}^{-T} \hat{\nabla} \quad (3.44)$$

to rewrite (2.4), like we did in (2.10), to get

$$\begin{aligned} a(\mathbf{v}, \mathbf{v}) &= \int_{\hat{\Omega}} (\hat{\nabla}(\mathbf{v} \circ \Phi))^T \mathcal{J}^{-1} \mathcal{J}^{-T} \hat{\nabla}(\mathbf{v} \circ \Phi) |J| d\hat{\Omega} \\ &= \int_{\hat{\Omega}} \mathbf{w}^T G \mathbf{w} d\hat{\Omega}, \end{aligned} \quad (3.45)$$

where  $\mathbf{w} = \hat{\nabla}[\mathbf{v} \circ \Phi]$  and  $G = G(\Phi) = (\mathcal{J}^T \mathcal{J})^{-1} |J|$ . At each point  $\hat{\mathbf{x}} \in \hat{\Omega}$  we diagonalize the  $2 \times 2$  symmetric positive-definite matrix  $G$ , that is, we write  $G(\Phi(\hat{\mathbf{x}})) = Q^T \Lambda Q$ , where  $Q$  consists of the orthonormal eigenvectors of  $G$ . If we (at each point  $\hat{\mathbf{x}} \in \hat{\Omega}$ ) replace the two diagonal elements of  $\Lambda$  with the smallest one,  $\Lambda_{min}$ , we get

$$\int_{\hat{\Omega}} \mathbf{w}^T G \mathbf{w} d\hat{\Omega} \geq \int_{\hat{\Omega}} \Lambda_{min} (Q\mathbf{w})^T Q\mathbf{w} d\hat{\Omega}. \quad (3.46)$$

Since  $Q$  consists of the orthonormal eigenvectors, the last expression is equivalent to  $\int_{\hat{\Omega}} \Lambda_{min} \mathbf{w}^T \mathbf{w} d\hat{\Omega}$ , and we end up with

$$a(\mathbf{v}, \mathbf{v}) \geq \int_{\hat{\Omega}} \Lambda_{min} \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \hat{\nabla}(\mathbf{v} \circ \Phi) d\hat{\Omega}. \quad (3.47)$$

This is just (3.31) with  $g(\Phi) = \Lambda_{min}(\Phi)$ , and thus (3.42) is satisfied.

If we replace  $\Lambda_{min}(\Phi)$  by  $\bar{\Lambda}_{min}(\Phi) = \min_{\hat{\mathbf{x}} \in \hat{\Omega}} \Lambda_{min}(\Phi)$ , we may put  $g(\Phi)$  outside the integral and apply the theory of [16]. This will produce a more conservative upper bound, but the calculation of (3.35) can then be split in an off-line/on-line procedure.

### 3.8 Numerical results for the output bounds

To compute the output bounds numerically, we will use the domain  $\Omega = \Phi(\hat{\Omega})$  defined in Figure 2. The basis functions for  $X_N$  and  $M_N$  are computed according to Section 3.6, which gives us 15 basis functions for  $X_N^0$ ,  $X_N^e$  and  $M_N$ .

$N$	$s(\mathbf{u}_N) - s^-(\mathbf{u}_N)$	$s^+(\mathbf{u}_N) - s(\mathbf{u}_N)$	$s_2^+(\mathbf{u}_N) - s(\mathbf{u}_N)$
1	$2.82 \cdot 10^{-4}$	$5.28 \cdot 10^{-2}$	$1.01 \cdot 10^{-1}$
2	$1.87 \cdot 10^{-4}$	$9.86 \cdot 10^{-2}$	$1.85 \cdot 10^{-1}$
3	$1.35 \cdot 10^{-4}$	$9.74 \cdot 10^{-2}$	$1.90 \cdot 10^{-1}$
4	$1.32 \cdot 10^{-4}$	$9.70 \cdot 10^{-2}$	$1.86 \cdot 10^{-1}$
5	$7.67 \cdot 10^{-5}$	$3.02 \cdot 10^{-2}$	$4.52 \cdot 10^{-2}$
6	$7.44 \cdot 10^{-5}$	$6.86 \cdot 10^{-3}$	$1.20 \cdot 10^{-2}$
7	$1.04 \cdot 10^{-5}$	$1.56 \cdot 10^{-3}$	$2.60 \cdot 10^{-3}$
8	$7.03 \cdot 10^{-6}$	$2.62 \cdot 10^{-3}$	$4.32 \cdot 10^{-3}$
9	$7.02 \cdot 10^{-6}$	$1.61 \cdot 10^{-3}$	$2.39 \cdot 10^{-3}$
10	$4.24 \cdot 10^{-6}$	$6.10 \cdot 10^{-4}$	$1.02 \cdot 10^{-3}$
11	$4.16 \cdot 10^{-6}$	$6.21 \cdot 10^{-4}$	$1.05 \cdot 10^{-3}$
12	$3.15 \cdot 10^{-6}$	$6.16 \cdot 10^{-4}$	$9.68 \cdot 10^{-4}$
13	$2.82 \cdot 10^{-6}$	$4.63 \cdot 10^{-4}$	$7.50 \cdot 10^{-4}$
14	$1.94 \cdot 10^{-6}$	$4.32 \cdot 10^{-4}$	$6.97 \cdot 10^{-4}$
15	$1.94 \cdot 10^{-6}$	$3.82 \cdot 10^{-4}$	$6.71 \cdot 10^{-4}$

Table 2: Convergence of the lower and the upper bound gaps. Here,  $s^+$  corresponds to the variable  $g(\Phi) = \Lambda_{min}(\Phi)$ , while  $s_2^+$  corresponds to the constant  $g(\Phi) = \bar{\Lambda}_{min}(\Phi)$ .

The upper bound for  $g(\Phi) = \Lambda_{min}(\Phi)$  is denoted  $s^+$ , while the upper bound for the constant  $g(\Phi) = \bar{\Lambda}_{min}(\Phi)$  is denoted  $s_2^+$ . The results for the previously defined deformed geometry are presented in Table 2. In Figure 5 we present the same results graphically, and in addition we compare the results with  $a(\mathbf{u}_N - \mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N)$ . As we should expect from (3.38), Figure 5 confirms that  $s(\mathbf{u}_N) - s^-(\mathbf{u}_N) = a(\mathbf{u}_N - \mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N)$ . The slight deviation that can be seen for some  $N$  is on the order of  $10^{-10}$ , which is smaller than the stopping criterion in the iterative method used to find the basis functions; see Section 2.3. From Figure 5 we also see that  $s_2^+$  is very close to  $s^+$ . Thus the approximation of  $\Lambda_{min}(\Phi)$  by a constant does not deteriorate the upper bound considerably in the case we have tested so far.

From both Table 2 and Figure 5 we see that the upper bound gap is relatively large compared to the lower bound gap. In the future, a different method to find an improved estimate of  $g(\Phi)$  is desirable to reduce the upper bound gap.

## 4 A reduced basis element method

We will in this section extend the tools developed for a single domain to multiple domains. This is done to provide enough geometrical flexibility in the solvers, and we get either a spectral element method or a reduced basis

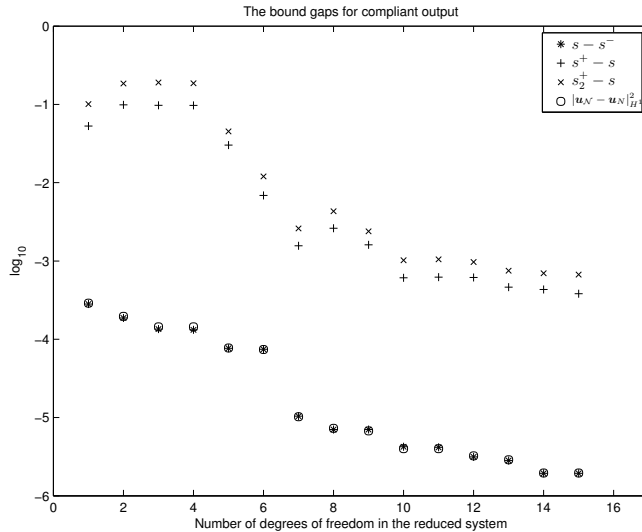


Figure 5: The bound gaps.

element method. By dividing the domain into separate subdomains, we may also exploit parallel processing techniques. For  $K$  subdomains the decomposition is performed in a non-overlapping way such that  $\Omega = \cup_{k=1}^K \Omega_k$ . For simplicity, the domain is only decomposed in the flow direction. In this way the inlet boundary will be entirely in one subdomain, the outflow boundary in another, and all other subdomains will have only two neighbouring subdomains. On the interface  $\bar{\Gamma}_{kl} = \bar{\Omega}_k \cap \bar{\Omega}_l$  between two neighbouring subdomains, the pressure may be discontinuous since it is only required to be in  $L^2$ . We use a conforming spectral element method to generate the reference solution  $(\mathbf{u}_N, p_N)$  on multiple domains. The velocity  $\mathbf{u}_N$  is then in  $H^1$  and the reduced basis element solution  $\mathbf{u}_N$  should thus be continuous in all of  $\Omega$ . This continuity is generally not possible to achieve exactly across the interfaces, and we get a nonconforming method with consistency error

$$\sum_{k,l} \int_{\Gamma_{kl}} \left( \nu \frac{\partial \mathbf{u}_N}{\partial \mathbf{n}} - p_N \mathbf{n} \right) \cdot (\mathbf{u}_N|_{\Omega_k} - \mathbf{u}_N|_{\Omega_l}) ds. \quad (4.1)$$

In the spirit of the mortar element method, we will minimize the jump across the interfaces through Lagrange multipliers; see [3] and [4].

#### 4.1 Precomputing the basis functions

In the single domain case we constructed precomputed velocity basis functions with zero tangential velocity on the inlet and outflow boundaries, and

precomputed pressure basis functions which all were close to one on the inlet boundary and zero on the outflow boundary. In the case of multiple subdomains, we need reduced basis functions with good approximation properties along the internal interfaces. To achieve this, we use two alternative approaches.

In the first approach we solve the steady Stokes problem on the eight predefined geometries using two subdomains. The restriction of these solutions to the subdomain associated with the outflow boundary comprise our initial basis functions. On the reference domain these functions are flipped across the horizontal axis ( $\eta = 0$ ) to produce their symmetric variants. The solution on the undeformed square is not flipped, since it is symmetric already, and we thus get  $N = 15$  basis functions for both the pressure and the velocity. To represent the solution on the inlet subdomain and on central subdomains, the functions are reflected across the vertical axis ( $\xi = 0$ ) on the reference domain. Thus we get a total of  $2N = 30$  available basis functions from the first approach. All the  $2N$  basis functions will be used on interior subdomains, while only half of these will be used on the subdomains corresponding to the inlet and outflow subdomains. In addition, one pressure basis function with a constant value equal to one will be used on the inlet subdomain in order to raise the level of the pressure solution on this subdomain. Since we use the restriction of the precomputed solutions to the outflow subdomain, all the pressure basis functions have values close to zero on the outflow edge. Without the additional constant function, this approximation is insufficient on the inlet domain. Alternatively, the reflected basis functions may be added for the inlet subdomain, but then more than one additional basis function is needed.

In the second approach we solve the steady Stokes problem on the eight predefined geometries using three subdomains instead of two. Now we use the restriction of the solutions to each of the subdomains as basis functions in similar subdomains in the generic problem. The idea is that the solution restricted to the inlet subdomain better represents what happens on the inlet subdomain in the generic problem, and similarly for the other subdomains. After flipping on the reference domain this approach produces a total of  $3N = 45$  basis functions for both the pressure and the velocity. In the second approach there is no need to reflect the basis functions across the vertical axis. There is also no need to use an additional constant pressure function on the inlet subdomain, since the restrictions of the pressure solutions to the inlet subdomain are clearly non-zero on all edges. Following this approach we will thus use  $N$  basis functions to approximate the pressure on the inlet subdomain, and  $N$  basis functions on any interior subdomain, and  $N$  basis functions on the outflow subdomain. A similar procedure is used to approximate the velocity.

To ensure stability of the reduced basis element solution, all basis functions are orthogonalized, but we keep the basis functions representing dif-

ferent subdomains separated. For the pressure basis functions the orthogonalization is done with respect to the discrete  $L^2$  inner product on the reference domain. The velocity basis functions are orthogonalized with respect to  $a_N(\cdot, \cdot)$  on the reference domain, while imposing the boundary conditions corresponding to the subdomain the given velocity basis function represents.

In both Method 1 and Method 2 we have to enrich the velocity basis in order to find both the pressure and the velocity solution in the reduced basis element context. In Method 1 it is crucial that the inf-sup condition is fulfilled, and we use the same technique as for the single-domain case to produce one additional velocity basis function for each pressure basis function. Again, the correct boundary conditions have to be used, according to which subdomain the velocity basis function represents. In Method 2 these additional velocity basis functions are used in the computation of the pressure solution only.

Using the first approach to construct the precomputed basis functions we define

$$\begin{aligned} \hat{X}_N = X_N(\hat{\Omega}) &= \begin{cases} \text{span}\{\hat{\mathbf{u}}_i, & i = 2N+1, \dots, 4N+1\}, & k = 1 \\ \text{span}\{\hat{\mathbf{u}}_i, & i = 1, \dots, 4N\}, & k = 2, \dots, K-1 \\ \text{span}\{\hat{\mathbf{u}}_i, & i = 1, \dots, 2N\}, & k = K \end{cases} \\ \hat{M}_N = M_N(\hat{\Omega}) &= \begin{cases} \text{span}\{\hat{p}_i, & i = N, \dots, 2N+1\}, & k = 1 \\ \text{span}\{\hat{p}_i, & i = 1, \dots, 2N\}, & k = 2, \dots, K-1 \\ \text{span}\{\hat{p}_i, & i = 1, \dots, N\}, & k = K. \end{cases} \end{aligned} \quad (4.2)$$

For the second approach we define

$$\begin{aligned} \hat{X}_N = X_N(\hat{\Omega}) &= \begin{cases} \text{span}\{\hat{\mathbf{u}}_i^{in}, & i = 1, \dots, 2N\}, & k = 1 \\ \text{span}\{\hat{\mathbf{u}}_i^c, & i = 1, \dots, 2N\}, & k = 2, \dots, K-1 \\ \text{span}\{\hat{\mathbf{u}}_i^o, & i = 1, \dots, 2N\}, & k = K \end{cases} \\ \hat{M}_N = M_N(\hat{\Omega}) &= \begin{cases} \text{span}\{\hat{p}_i^{in}, & i = 1, \dots, N\}, & k = 1 \\ \text{span}\{\hat{p}_i^c, & i = 1, \dots, N\}, & k = 2, \dots, K-1 \\ \text{span}\{\hat{p}_i^o, & i = 1, \dots, N\}, & k = K, \end{cases} \end{aligned} \quad (4.3)$$

where superscripts *in*, *c* and *o* indicates whether a basis function represents the inlet subdomain, a central subdomain, or the outflow subdomain. We also recall that the index  $k$  refers to the subdomain  $\Omega_k$ . The velocity basis functions are ordered such that the original  $N = 15$  have the indices  $i = 1, \dots, 15$ , and the enriched basis functions corresponding to the original  $N = 15$  pressure basis functions have the indices  $i = N + 1, \dots, 2N$ . The reflected original and the reflected enriched basis functions then follow. We remark that the constant pressure basis function added on the inlet domain ( $k = 1$ ) corresponds to the index  $i = 2N + 1$ , and the associated enriched velocity basis function corresponds to the index  $i = 4N + 1$ .

On a generic deformed geometry we can then define the reduced basis

pressure space as

$$M_N = M_N(\Omega) = \{q \in M, \quad q|_{\Omega_k} \circ \Phi^k \in \hat{M}_N, \quad k = 1, \dots, K\}, \quad (4.4)$$

where  $\Phi^k$  is the mapping associated with the deformation of the  $k$ 'th subdomain. We also define a preliminary reduced basis velocity space on the deformed geometry as

$$Y_N = Y_N(\Omega) = \{\mathbf{v} \in X, \quad |J^k|(\mathcal{J}^k)^{-1}(\mathbf{v}|_{\Omega_k} \circ \Phi^k) \in \hat{X}_N, \quad k = 1, \dots, K\}. \quad (4.5)$$

This space is preliminary since we have not yet imposed any continuity conditions across the subdomain interfaces.

## 4.2 Matching conditions

As mentioned earlier, we use a mortar element method to glue the velocities together. In practice we try to minimize the jump across internal interfaces by introducing the constraints

$$\int_{\Gamma_{kl}} (\mathbf{v}|_{\Omega_k} - \mathbf{v}|_{\Omega_l}) \cdot \mathbf{n} \psi ds = 0, \quad \forall \psi \in W_{k,l}^n, \quad \forall k, l \quad (4.6)$$

and

$$\int_{\Gamma_{kl}} (\mathbf{v}|_{\Omega_k} - \mathbf{v}|_{\Omega_l}) \cdot \mathbf{t} \psi ds = 0, \quad \forall \psi \in W_{k,l}^t, \quad \forall k, l, \quad (4.7)$$

where  $\bar{\Gamma}_{kl} = \bar{\Omega}_k \cap \bar{\Omega}_l$ ,  $\mathbf{n}$  is the unit normal vector of  $\Gamma_{kl}$ ,  $\mathbf{t}$  is the unit tangential vector, and  $W_{k,l}^n$  and  $W_{k,l}^t$  are spaces of low order polynomials defined on  $\Gamma_{kl}$ . Depending on the order of the polynomials in  $W_{k,l}^n$  and  $W_{k,l}^t$ , the jump across the interfaces can be controlled; see [14] for results of different polynomial degrees used on a thermal fin problem. Our reduced basis velocity space for Method 1 is then defined as

$$X_N = \{\mathbf{v} \in Y_N, \quad (4.6) \text{ and } (4.7) \text{ hold}\}. \quad (4.8)$$

For Method 2 we define the reduced basis velocity space as

$$X_N^0 = \{\mathbf{v} \in X_N, \quad \sum_{k=1}^K \int_{\Omega_k} q \nabla \cdot \mathbf{v} d\Omega = 0, \quad \forall q \in M_N\}, \quad (4.9)$$

which is just  $X_N$  without the enriched velocity basis functions.

To evaluate the integrals in (4.6) and (4.7), the  $x$ - and  $y$ -components of  $\mathbf{v}$  are rotated to find the corresponding normal and tangential components.

### 4.3 Numerical results

We define  $\Omega$  to be the deformed annulus in Figure 2 divided into three subdomains in the flow direction, i.e.  $K = 3$ . An initial choice of Lagrange multiplier spaces will be

$$W_{k,l}^n = W_{k,l}^t = \mathbb{P}_3(\Gamma_{kl}). \quad (4.10)$$

For Method 1 we solve the problem: Find  $\mathbf{u}_N \in X_N$  and  $p_N \in M_N$  such that

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{w}) + b_N(\mathbf{w}, p_N) &= l_N(\mathbf{w}) \quad \forall \mathbf{w} \in X_N \\ b_N(\mathbf{u}_N, q) &= 0 \quad \forall q \in M_N. \end{aligned} \quad (4.11)$$

In Table 3 we compare the results for the different choices of spaces  $\hat{X}_N$  and  $\hat{M}_N$  in (4.2) and (4.3).

If we use (4.2) to define  $\hat{X}_N$  and  $\hat{M}_N$ , the dimension of  $X_N \times M_N$  is  $12N + 2 - 2L$ , where  $L = \dim(W_{k,l}^n) + \dim(W_{k,l}^t)$  is the total dimension of the Lagrange multiplier spaces. If we instead use (4.3) to define  $\hat{X}_N$  and  $\hat{M}_N$  the dimension is only  $9N - 2L$ , but we still see from Table 3 that the approximation abilities, measured as the norm of the error, are better than the first choice when the dimension of the spaces is comparable. In Figures 6(a) and 6(b), we see that for both choices of reduced basis spaces the error of the pressure is more noticeable close to the internal interfaces relative to the rest of the domain. The velocity error for the first choice of basis spaces, which is shown in Figures 7(a) and 7(b), indicates that the error along the internal interfaces is no larger than in the rest of the domain. The velocity error for the second choice of basis spaces exhibit similar behaviour, which confirms the effect of the weak continuity imposed by the Lagrange multipliers.

In Method 2 we find the reduced basis velocity solution by solving: Find  $\mathbf{u}_N \in X_N^0$  such that

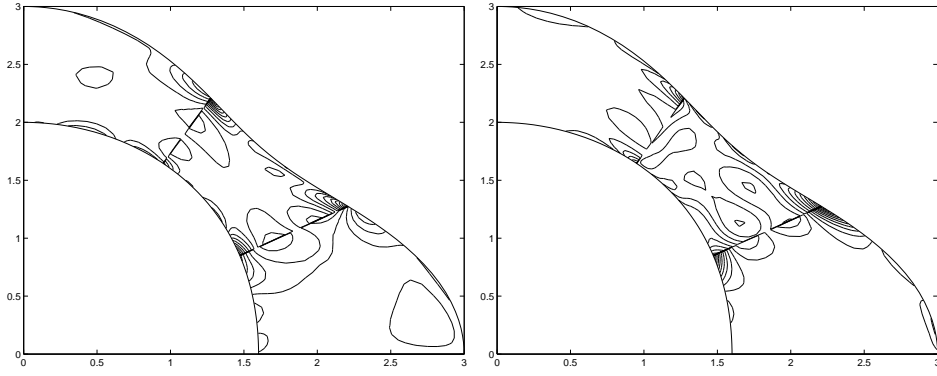
$$a_N(\mathbf{u}_N, \mathbf{v}) = l_N(\mathbf{v}) \quad \forall \mathbf{v} \in X_N^0. \quad (4.12)$$

If we use (4.3) to define  $\hat{X}_N$ , the dimension of  $X_N^0$  is  $3N - 2L$ , about one third of the corresponding dimension for Method 1. Having obtained the

$\hat{X}_N, \hat{M}_N$	$N$	$\dim(X_N \times M_N)$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1(\Omega)}$	$\ p_N - p_N\ _{L^2(\Omega)}$
(4.2)	11	118	$1.8 \cdot 10^{-3}$	$9.8 \cdot 10^{-3}$
(4.3)	15	119	$8.4 \cdot 10^{-4}$	$3.6 \cdot 10^{-3}$

Table 3: Comparison of the error for different choices of reduced basis spaces in Method 1.

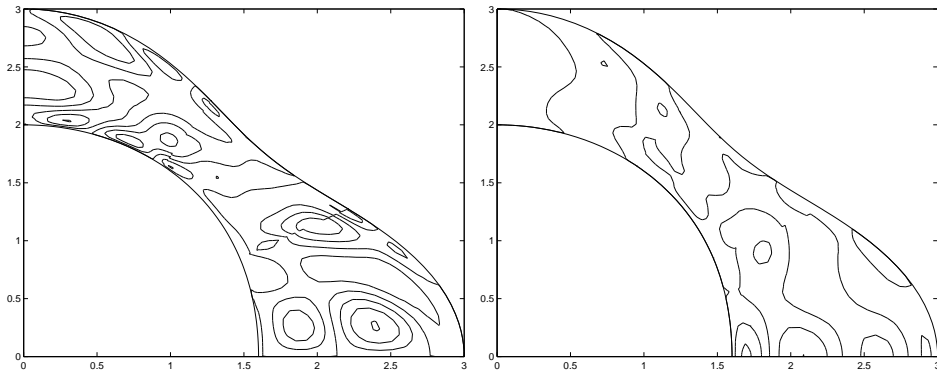




(a)  $\hat{X}_N$  and  $\hat{M}_N$  defined by (4.2)

(b)  $\hat{X}_N$  and  $\hat{M}_N$  defined by (4.3)

Figure 6: Contour of the pressure error for  $\hat{X}_N$  and  $\hat{M}_N$  defined as in (4.2) and (4.3).



(a)  $x$ -velocity

(b)  $y$ -velocity

Figure 7: Contour of the velocity error for  $\hat{X}_N$  and  $\hat{M}_N$  defined as in (4.2).

$N$	Method 1				Method 2	
	$d(X_N)$	$d(M_N)$	$ \mathbf{u}_N - \mathbf{u}_N _{H^1}$	$\ p_N - p_N\ _{L^2}$	$d(X_N^0)$	$ \mathbf{u}_N - \mathbf{u}_N _{H^1}$
6	20	18	$2.1 \cdot 10^{-2}$	$6.9 \cdot 10^5$	2	$2.5 \cdot 10^{-2}$
7	26	21	$3.7 \cdot 10^{-3}$	$2.2 \cdot 10^4$	5	$5.8 \cdot 10^{-3}$
8	32	24	$2.7 \cdot 10^{-3}$	$5.5 \cdot 10^{-1}$	8	$2.7 \cdot 10^{-3}$
9	38	27	$2.3 \cdot 10^{-3}$	$3.6 \cdot 10^{-1}$	11	$2.3 \cdot 10^{-3}$
10	44	30	$1.3 \cdot 10^{-3}$	$1.3 \cdot 10^{-1}$	14	$1.3 \cdot 10^{-3}$
11	50	33	$1.2 \cdot 10^{-3}$	$5.8 \cdot 10^{-2}$	17	$1.2 \cdot 10^{-3}$
12	56	36	$9.8 \cdot 10^{-4}$	$6.2 \cdot 10^{-3}$	20	$9.8 \cdot 10^{-4}$
13	62	39	$9.7 \cdot 10^{-4}$	$4.4 \cdot 10^{-3}$	23	$9.7 \cdot 10^{-4}$
14	68	42	$8.7 \cdot 10^{-4}$	$4.6 \cdot 10^{-3}$	26	$8.7 \cdot 10^{-4}$
15	74	45	$8.4 \cdot 10^{-4}$	$3.6 \cdot 10^{-3}$	29	$8.4 \cdot 10^{-4}$

Table 4: The error convergence of Method 1 and Method 2 in the multi domain case,  $d(\cdot)$  is the dimension of the space in question.

velocity from (4.12), the pressure is then found by solving the problem: Find  $p_N \in M_N$  such that

$$b_N(\mathbf{v}, p_N) = -a_N(\mathbf{u}_N, \mathbf{v}) + l_N(\mathbf{v}) \quad \forall \mathbf{v} \in X_N^e, \quad (4.13)$$

where  $X_N^e = \{\mathbf{v} \in X_N, \mathbf{v} \notin X_N^0\}$  contains the enriched velocity basis functions found earlier. On three subdomains we can find the pressure by solving three decoupled systems, each system being of dimension  $N$ .

To further study the convergence properties of Method 1 and 2, we use (4.3) for different choices of  $N$  to solve (4.11) and (4.12) respectively. In Table 4 we see that the error of the velocity converges well for both methods. As in the single domain case, Method 2 is just as accurate as Method 1. The pressure found in Method 1 also converges well, and if we use (4.13) to find the pressure in Method 2, it will exhibit similar properties.

#### 4.4 The effect of different Lagrange multipliers

We have already seen that the approximation properties are affected by the choice of basis functions in the reduced basis velocity and pressure spaces. In this section we will try different choices of Lagrange multiplier spaces used in the mortar element method. Previously we used  $W_{k,l}^n = W_{k,l}^t = \mathbb{P}_3(\Gamma_{kl})$  to get the results in Tables 3 and 4. However, it is not necessary to use the same space in the normal and tangential direction. Considering the incompressibility condition, the normal derivative is more important than the tangential condition since a good matching of the normal derivative along an interface is directly related to a good mass conservation (or continuity of the volume flow rate across an interface). It would also be advantageous

$W_{k,l} = W_{k,l}^n \times W_{k,l}^t$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$
$\mathbb{P}_3 \times \mathbb{P}_3$	$8.4 \cdot 10^{-4}$	$3.6 \cdot 10^{-3}$
$\mathbb{P}_3 \times \emptyset$	$6.6 \cdot 10^{-2}$	$1.4 \cdot 10^{-1}$
$\mathbb{P}_3 \times \mathbb{P}_1$	$7.2 \cdot 10^{-4}$	$2.1 \cdot 10^{-3}$
$\{\tilde{\psi}_{kl}^i\}$ from (4.18), $i = 1, 2, 3$	$1.5 \cdot 10^{-3}$	$6.7 \cdot 10^{-3}$
$\{\tilde{\psi}_{kl}^i\}$ from (4.18), $i = 2, 4, 6$	$6.8 \cdot 10^{-4}$	$1.3 \cdot 10^{-3}$
$\{\psi_{kl}\}$ from (4.17)	$6.0 \cdot 10^{-4}$	$1.1 \cdot 10^{-3}$

Table 5: The norm of the error for different choices of  $W_{k,l}^n$  and  $W_{k,l}^t$ .

to lower the dimension on the space in the tangential direction compared to the space in the normal direction in order to decrease the number of constraints imposed on the reduced basis system. In the three dimensional case there are two tangential directions and only one normal direction on each interface, and lowering the dimension of the tangential spaces becomes more important. The velocity components are connected through the divergence free property, but a minimization of the jump in the normal velocity component alone across an interface does not guarantee that the tangential velocity component points in the same direction along an interface. To illustrate this we first define

$$\begin{aligned} W_{k,l}^n &= \mathbb{P}_3(\Gamma_{kl}) \\ W_{k,l}^t &= \emptyset, \end{aligned} \tag{4.14}$$

where we have no constraint in the tangential direction, and then

$$\begin{aligned} W_{k,l}^n &= \mathbb{P}_3(\Gamma_{kl}) \\ W_{k,l}^t &= \mathbb{P}_1(\Gamma_{kl}), \end{aligned} \tag{4.15}$$

where we use linear functions in the tangential direction. The results are presented in Table 5 together with the results from the previous choice of  $W_{k,l}^n$  and  $W_{k,l}^t$ .

If we increase the polynomial degree in the Lagrange multiplier spaces, the discontinuities across the interfaces will decrease, but the number of degrees of freedom in the global reduced basis element problem will also decrease and hence the global error may increase. We keep the linear functions in the tangential direction, and calculate the error for different polynomial degrees in the normal direction, i.e.

$$\begin{aligned} W_{k,l}^n &= \mathbb{P}_m(\Gamma_{kl}) , \\ W_{k,l}^t &= \mathbb{P}_1(\Gamma_{kl}) . \end{aligned} \tag{4.16}$$

From Table 6 we see that the error has a minimum when the polynomial degree  $m$  in the normal direction is between two and seven. This is consistent

with the results presented in [14] for a scalar heat transfer problem with only one Lagrange multiplier space.

From the consistency error defined in (4.1), we see that the best Lagrange multipliers should be ones that are close to the exact stress vector

$$\boldsymbol{\psi}_{kl} = \left( \nu \frac{\partial \mathbf{u}_N}{\partial \mathbf{n}} - p_N \mathbf{n} \right)_{|\Gamma_{kl}}, \quad (4.17)$$

on each interface  $\Gamma_{kl}$ . We therefore calculate

$$\boldsymbol{\psi}_{kl}^i = \left( \nu \frac{\partial \mathbf{u}_i}{\partial \mathbf{n}_i} - p_i \mathbf{n}_i \right)_{|\Gamma_{kl}^i}, \quad (4.18)$$

for all  $\Omega_i$ . Similar to the velocity basis functions,  $\mathbf{u}_i$ , the stress vectors are mapped to a reference interface  $\hat{\Gamma}$  through the Piola transformation,  $\hat{\boldsymbol{\psi}}_{kl}^i = \Psi_i(\boldsymbol{\psi}_{kl}^i)$ . On the generic domain  $\Omega$  we first map each  $\hat{\boldsymbol{\psi}}_{kl}^i$  to corresponding interfaces  $\Gamma_{kl}$ , again using the Piola transformation  $\tilde{\boldsymbol{\psi}}_{kl}^i = \Psi^{-1}(\hat{\boldsymbol{\psi}}_{kl}^i)$ , and then rotate the coordinates to find the normal and tangential components. The space  $W_{k,l}^n$  will thus consist of the normal components of the  $\tilde{\boldsymbol{\psi}}_{kl}^i$ , and  $W_{k,l}^t$  of the tangential components. We impose (4.6) and (4.7), which in turn ensures that the vectorial jump

$$\sum_{k,l} \int_{\Gamma_{kl}} (\mathbf{u}_N|_{\Omega_k} - \mathbf{u}_N|_{\Omega_l}) \cdot \tilde{\boldsymbol{\psi}}_{kl}^i ds = 0, \quad \forall i. \quad (4.19)$$

In order to balance the number of constraints against the number of degrees of freedom, we choose the same number of constraints as for the  $\mathbb{P}_3 \times \mathbb{P}_1$  case, but now with  $\dim(W_{k,l}^n) = \dim(W_{k,l}^t) = 3$ ; the total number of constraints is thus six. Depending on which three multipliers are used, we get different results. The worst result is achieved for  $i = 1, 2$  and  $3$ , while the best result is achieved for  $i = 2, 4$  and  $6$ ; see Table 5. Compared to the

$m$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$
1	$4.94 \cdot 10^{-3}$	$4.66 \cdot 10^{-3}$
2	$8.22 \cdot 10^{-4}$	$2.24 \cdot 10^{-3}$
3	$7.17 \cdot 10^{-4}$	$2.06 \cdot 10^{-3}$
4	$7.17 \cdot 10^{-4}$	$2.07 \cdot 10^{-3}$
5	$6.91 \cdot 10^{-4}$	$2.00 \cdot 10^{-3}$
6	$6.93 \cdot 10^{-4}$	$2.00 \cdot 10^{-3}$
7	$7.00 \cdot 10^{-4}$	$2.17 \cdot 10^{-3}$
8	$7.15 \cdot 10^{-4}$	$2.24 \cdot 10^{-3}$
9	$8.25 \cdot 10^{-4}$	$2.74 \cdot 10^{-3}$

Table 6: The norm of the error when  $W_{k,l}^t$  and  $W_{k,l}^n$  are defined as in (4.16).

Lagrange multiplier spaces defined as low order polynomials, we see that the stress vectors (4.18) give better results, however, care has to be taken as to which three stress vectors are used. As a final test, we use the exact stress vector (4.17) to produce a single constraint in each of the spatial directions. This should give zero consistency error and result in a pure approximation error in the reduced basis element solution. As expected, the numerical result is also better than for all other Lagrange multipliers; see Table 5.

## 5 Conclusions

In this paper, we have proposed a reduced basis element method for the steady Stokes problem. The computational method is based on a domain decomposition into geometrically similar parts (denoted as “elements”), and local approximations based on precomputed steady Stokes solutions corresponding to different geometric shapes (denoted as geometric “snapshots”). The proposed method represents an extension of earlier work associated with solving the Laplace equation.

The precomputed velocity fields are (weakly) incompressible on the preselected geometries. We have used Piola’s transformation to ensure that this property is also satisfied on the reference domain associated with the computational parts, and also when mapped to a new and unknown geometric shape. The issue of satisfying the inf-sup condition has been addressed in two different ways.

One alternative, called Method 1, is based on enriching the reduced velocity basis in order to obtain a stable, coupled system for the velocity and the pressure. A second alternative, called Method 2, directly exploits the fact that the precomputed velocity fields are incompressible. Following this latter method, the Stokes system is reduced to solving a Poisson-like equation for the velocity. The pressure can then be computed in a postprocessing step. The advantages of this alternative are the facts that the velocity and the pressure can be computed separately, and very small systems need to be solved.

We have demonstrated the approximation and convergence properties of Method 1 and Method 2 on selected two-dimensional problems. The numerical results are almost identical, suggesting that Method 2 is very attractive due to the lower computational cost.

The methods have been tested both in the single-domain context and using multiple subdomains. In the latter case, the local velocity approximations are glued together using the mortar element method. Proper treatment of the continuity conditions for the normal and tangential velocity components is discussed. It has been found that it is sufficient to use a very low dimension of the Lagrange multiplier space associated with the tangential component.

In the single-domain context we have also developed *a posteriori* error estimators for the steady Stokes problem and the theoretical results have been confirmed by numerical experiments in the compliant case. For the particular test problem used, the lower bound gap appears to be very good, while the upper bound gap is somewhat conservative.

Future extensions will include the simulation of geometrically more complex systems involving several types of computational parts (e.g., pipes and bifurcations). In addition, it will be interesting to exploit the properties of Method 2 to solve unsteady problems, including problems modeled by the incompressible Navier-Stokes equations. Finally, the *a posteriori* error estimators developed here need to be extended to the general non-compliant multiple element case, as was done in [14].

**Acknowledgement.** *This work has been supported by the Research Council of Norway through the BeMatA programme under contract 147044/431, by the RTN project HaeMOdel HPRN-CT-2002-00270, and the ACI project "le-poumon-vous-dis-je" granted by the Fond National pour la Science. The support is gratefully acknowledged.*

## References

- [1] R. Aris. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Publications, 1989.
- [2] I. Babuska. Error-bounds for finite element method. *Numer. Math.*, 16:322–333, 1971.
- [3] B. F. Belgacem. The mortar finite element method with Lagrange multipliers. *Numer. Math.*, pages 173–197, 1999.
- [4] B. F. Belgacem, C. Bernardi, N. Chorfi, and Y. Maday. Inf-sup conditions for the mortar spectral element discretization of the Stokes problem. *Numer. Math.*, 85:257–281, 2000.
- [5] C. Bernardi and Y. Maday. Polynomial approximation of some singular functions. *Appl. Anal.*, 42:1–32, 1992.
- [6] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers. *Rairo Anal. Numer.*, 8 R2:129–151, 1974.
- [7] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer Verlag, 1991.
- [8] J.P. Fink and W.C. Rheinboldt. On the error behavior of the reduced basis technique in nonlinear finite element approximations. *Z. Angew. Math. Mech.*, 63:21–28, 1983.

- [9] W. J. Gordon and C. A. Hall. Construction of curvilinear co-ordinate systems and applications to mesh generation. *Int J. Numer. Methods Eng.*, 7:461–477, 1973.
- [10] Y. Maday, D. Meiron, A. T. Patera, and E. M. Rønquist. Analysis of iterative methods for the steady and unsteady Stokes problem: Application to spectral element discretizations. *SIAM J. Sci. Stat. Comp.*, pages 310–337, 1993.
- [11] Y. Maday and A. T. Patera. Spectral element methods for the Navier-Stokes equations. In Noor A. (ed) *State of the Art Surveys in Computational Mechanics*, pages 71–143, 1989.
- [12] Y. Maday, A. T. Patera, and E. M. Rønquist. The  $P_N \times P_{N-2}$  method for the approximation of the Stokes problem. *Technical Report No. 92009, Department of Mechanical Engineering, Massachusetts Institute of Technology*, 1992.
- [13] Y. Maday and E. M. Rønquist. A reduced-basis element method. *J. Sci. Comput.*, 17:447–459, 2002.
- [14] Y. Maday and E. M. Rønquist. The reduced-basis element method: application to a thermal fin problem. *SIAM J. Sci. Comput.*, 26(1):240–258, 2004.
- [15] A. K. Noor and J. M. Peters. Reduced basis technique for nonlinear analysis of structures. *AIAA J.*, 18(4):455–462, 1980.
- [16] C. Prud’homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera, and G. Turinici. Reliable real-time solution of parametrized partial differential equations: Reduced basis output bound methods. *J. Fluids Engineering*, 124:70–80, 2002.
- [17] P.A. Raviart and J.M. Thomas. A mixed finite element method for 2-nd order elliptic problems. In I. Galligani and E. Magenes, editors, *Mathematical Aspects of Finite Element Methods, Lecture Notes in Mathematics, Vol. 606*. Springer-Verlag, 1977.
- [18] D. V. Rovas. *Reduced-Basis Output Bound Methods for Parametrized Partial Differential Equations*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, October 2002.
- [19] K. Veroy, C. Prud’homme, D. V. Rovas, and A. T. Patera. *A Posteriori* error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations (AIAA Paper 2003-3847). In *Proceedings of the 16th AIAA Computational Fluid Dynamics Conference*, June 2003.

# II

## A reduced basis element method for the steady Stokes problem: Application to flow in bifurcations

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In preparation for submission as a journal paper.





# A reduced basis element method for the steady Stokes problem: Application to flow in bifurcations

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## Abstract

Within the reduced basis framework, we consider the geometry as a parameter for the steady Stokes problem. For different instantiations of the geometry, we precompute basis functions which are subsequently used to find the reduced basis solution on a generic geometry. The geometries in this article are chosen to be multidomain bifurcations, and we show how divergence free vector fields may be mapped from one bifurcation to another through a global Piola transformation in order to maintain the divergence free property. We also present an alternative reduced basis formulation, together with a particular regularization technique, which exploits this divergence free property of the velocity basis functions to lower the dimension of the reduced basis system even further. We use *a posteriori* error estimation for two purposes: (i) in a “greedy” algorithm to select the optimal basis functions for a given error reduction; and (ii) to certify the reduced basis solution. Finally, we show how bifurcations together with pipes can be used as generic building blocks to construct a hierarchical flow system in which the precomputed basis functions are reused within each building block to find a global reduced basis element solution.

## 1 Introduction

Consider a system governed by the parameter dependent problem: Find  $u \in X$  such that

$$F(u; \mu) = 0, \tag{1.1}$$

where  $u$  is the unknown solution and  $\mu$  is a known parameter vector. In a discrete setting, the resolution (in space and time) required to obtain a good approximation  $u_{\mathcal{N}}$  to the exact solution  $u$  is often very high, and the associated computational cost may be significant following a traditional approach based on finite elements or finite difference methods.

The main idea behind a reduced basis approach is to realize that the exact solution  $u$  often can be approximated very well in a low dimensional manifold in  $X$ . By constructing global functions  $\{u_i\}_{i=1}^N$  which capture the essential features of the problem, a solution may be sought in the low dimensional reduced basis space  $X_N = \text{span}\{u_i\}_{i=1}^N$ .

The reduced basis element method was introduced in [16] in the context of solving the Laplace equation in a deformed geometry. In that work, the computational domain was broken into generic building blocks or parts (e.g., “pipes”), and the geometry of each part was considered to be the parameters. The reduced basis approximations associated with all the parts were glued together through the use of Lagrange multipliers.

Further progress was made in [17] in the context of solving a thermal fin problem. In particular, *a posteriori* error bounds were developed for the output of interest. We remark that this aspect represents an essential ingredient in developing practical reduced basis methods. Since it is *a priori* difficult to guess the number of reduced elements required in practical computations, *a posteriori* error bounds represent practical tools which allow us to certify the computational results.

In [13], the reduced basis element method was extended to solve incompressible fluid flow problems modeled by the steady Stokes equations. In particular, a Piola transformation was introduced in order to maintain incompressibility when mapping the velocity field from one geometry to another one. Two methods were introduced, one where the velocity-pressure pair is computed (Method 1), and one where only the velocity is computed (Method 2). The *a posteriori* analysis presented in [17, 18] was also extended to deformed geometries in the compliant case. Finally, a special mortar type method was proposed to handle the multiple domain case.

The work in [13] focused on incompressible fluid flow in domains characterized as deformed “pipes.” Each basic building block was discretized using a pure spectral method, or more importantly, a deformed, structured discretization. In this work, we extend the previous work by considering incompressible fluid flow in bifurcations. This will necessitate that each pre-computed solution must be represented using a multi-domain discretization. The implication of this is not so dramatic when solving the scalar Poisson problem. However, maintaining an incompressible velocity field through a global Piola transformation becomes a challenge since this (ideally) necessitates global  $C^1$ -mappings. The work presented in this paper discusses this issue, in the context of both Method 1 and Method 2 introduced in [13].

In Section 2 we present the model problem on a single bifurcation. We discuss the challenges of having a multi-element building block for the reduced basis method compared to previous cases. The goals of this work are introduced together with the challenges to realize these goals.

The key challenge of realizing a global  $C^1$  mapping from one bifurcation to another is addressed in Section 3, where different approaches are discussed. Here, we restrict the possible geometric variation of a bifurcation in order to construct a finite dimensional parameter space.

In Section 4 we choose a reference bifurcation, as well as basis bifurcations within the chosen parameter space. We briefly present two different methods for finding the reduced basis solution, and discuss how errors intro-

duced by the Piola transformation may be taken care of by a regularization technique. We extend the *a posteriori* error bounds analysis from [13] to bifurcations, and present how an algorithm for picking the best basis functions works.

To illustrate the power of the reduced basis element method, we couple the basis functions for bifurcations together with basis functions for pipes. This is done in Section 5 where the different building blocks are glued together by proper Lagrange multipliers in order to simulate a hierarchical flow system.

## 2 The model problem

### 2.1 Governing equations

We consider here the two-dimensional steady Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega, \end{aligned} \quad (2.1)$$

where  $\mathbf{u} = (u_1, u_2)$  is the velocity field,  $p$  is the pressure,  $\mathbf{f} = (f_1, f_2)$  is a prescribed volumetric body force, and  $\nu$  is the fluid viscosity; see [1]. For all the problems studied in this paper, this model will suffice.

The domain  $\Omega$  has an inflow boundary,  $\Gamma_{in}$ , an outflow boundary,  $\Gamma_{out}$ , and wall boundaries,  $\Gamma_w$ . On this domain we introduce the velocity space

$$X(\Omega) = \{\mathbf{v} \in (H^1(\Omega))^2, \quad \mathbf{v}|_{\Gamma_w} = 0 \text{ and } v_t|_{\Gamma_{in}} = v_t|_{\Gamma_{out}} = 0\}, \quad (2.2)$$

where  $v_t$  is the tangential velocity component. In addition, we have the Neumann type boundary conditions given by specifying  $\sigma_n = \nu \frac{\partial u_n}{\partial n} - p$  to be  $\sigma_n^{in} = -1$  along  $\Gamma_{in}$  and  $\sigma_n^{out} = 0$  along  $\Gamma_{out}$ ; here,  $u_n$  is the normal velocity component and  $\partial/\partial n$  denotes the derivative in the outward normal direction. For all the problems solved in this study, the exact solution of (2.1) satisfies  $\frac{\partial u_n}{\partial n} = 0$  along  $\Gamma_{in}$  and  $\Gamma_{out}$ , which implies that the Neumann conditions correspond to specifying the pressure along the inflow and outflow boundaries (in a weak sense).

With the given boundary conditions, we define the pressure space to be

$$M(\Omega) = L^2(\Omega). \quad (2.3)$$

In order to solve the steady Stokes equations we define the bilinear forms

$$a(\mathbf{v}, \mathbf{w}) = \nu \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{w} \, d\Omega = \nu \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\Omega, \quad (2.4)$$

$$b(\mathbf{v}, q) = - \int_{\Omega} q \nabla \cdot \mathbf{v} \, d\Omega = \sum_{i=1}^2 \int_{\Omega} q \frac{\partial v_i}{\partial x_i} \, d\Omega, \quad (2.5)$$

and consider the weak form: Find  $\mathbf{u} \in X(\Omega)$  and  $p \in M(\Omega)$  such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= l(\mathbf{v}) \quad \forall \mathbf{v} \in X(\Omega) \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in M(\Omega), \end{aligned} \quad (2.6)$$

where

$$l(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \int_{\Gamma_{in}} \sigma_n^{in} \mathbf{v} \cdot \mathbf{n} ds + \int_{\Gamma_{out}} \sigma_n^{out} \mathbf{v} \cdot \mathbf{n} ds. \quad (2.7)$$

For all the problems considered in this work, the body force  $\mathbf{f}$  will be zero. To ensure a unique solution of the steady Stokes problem (2.6), the coercivity condition

$$a(\mathbf{w}, \mathbf{w}) \geq \alpha \|\mathbf{w}\|_{H^1(\Omega)}^2, \quad \forall \mathbf{w} \in X(\Omega), \quad \alpha > 0, \quad (2.8)$$

and the inf-sup condition

$$\inf_{q \in M(\Omega)} \sup_{\mathbf{v} \in X(\Omega)} \frac{b(\mathbf{v}, q)}{\|q\|_{L^2(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)}} = \beta > 0, \quad (2.9)$$

must be satisfied; see [2] and [6]. These conditions are fulfilled for our particular Stokes problem.

## 2.2 The bifurcation

The computational domain,  $\Omega$ , will in the following be a bifurcation, and we denote it  $\mathcal{B}$ . We assume that the domain  $\mathcal{B}$  is obtained as the deformation of a reference bifurcation  $\hat{\mathcal{B}}$ , see Figure 1, by a regular one-to-one transformation  $\mathcal{F}$ . On  $\hat{\mathcal{B}}$ , the inflow boundary,  $\Gamma_{in}$  is along the vertical edge of the body of the bifurcation, while the outflow boundary,  $\Gamma_{out}$  is along the vertical edges of each of the bifurcation's two legs. The rest of the outer edge is then wall boundary,  $\Gamma_w$ . We use spectral elements to discretize bifurcations. In particular, we decompose the domains  $\mathcal{B}$  and  $\hat{\mathcal{B}}$  in six subdomains in a non-overlapping way such that  $\overline{\hat{\mathcal{B}}} = \bigcup_{e=1}^6 \overline{\hat{\mathcal{B}}}_e$  and  $\overline{\mathcal{B}} = \bigcup_{e=1}^6 \overline{\mathcal{B}}_e$ . This is not the only way to decompose the domains; if low order finite element methods are used, this domain decomposition may not be necessary.

Each subdomain,  $\mathcal{B}_e$ , is also considered to be a one-to-one mapping,  $\Phi_e$ , of a reference square  $\hat{\Omega} = (-1, 1)^2$ , and we denote  $\overline{\mathcal{B}} = \overline{\Phi(\hat{\Omega})} = \bigcup_{e=1}^6 \overline{\Phi_e(\hat{\Omega})}$ . For an element  $\mathbf{v} \in X(\mathcal{B})$  and an element  $p \in M(\mathcal{B})$ , we denote the restrictions of these elements to  $\mathcal{B}_e$  as  $\mathbf{v}|_{\mathcal{B}_e} = \mathbf{v}_e$ , and  $p|_{\mathcal{B}_e} = p_e$ . The bilinear forms are then expressed in terms of the reference variables and the mapping  $\Phi$  as

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \sum_{e=1}^6 \int_{\hat{\Omega}} \mathcal{J}_e^{-T} \hat{\nabla}(\mathbf{v}_e \circ \Phi_e) \cdot \mathcal{J}_e^{-T} \hat{\nabla}(\mathbf{w}_e \circ \Phi_e) |J_e| d\hat{\Omega}, \quad (2.10)$$

$$b(\mathbf{v}, p; \Phi) = \sum_{e=1}^6 - \int_{\hat{\Omega}} (p_e \circ \Phi_e) \hat{\nabla} \cdot [\mathcal{J}_e^{-1}(\mathbf{v}_e \circ \Phi_e)] |J_e| d\hat{\Omega}, \quad (2.11)$$

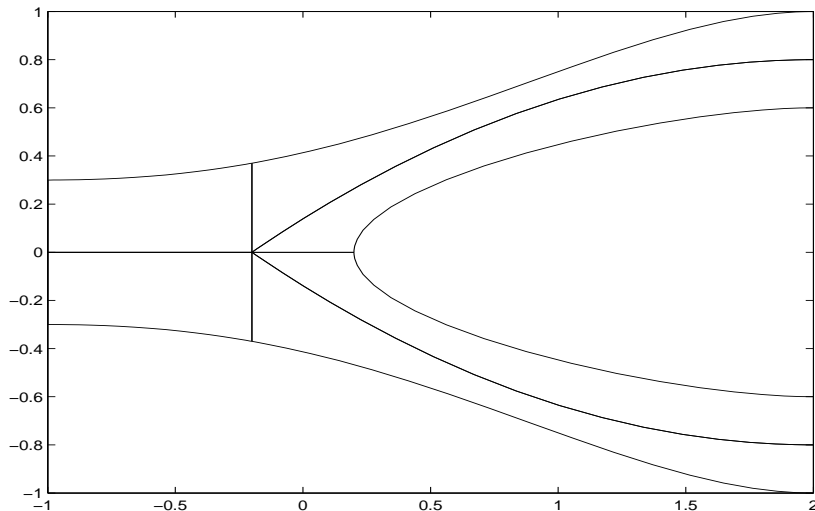


Figure 1: The reference bifurcation  $\hat{\mathcal{B}}$  constructed from six elements.

where  $\mathcal{J}_e$  is the Jacobian matrix of  $\Phi_e$ , and  $J_e$  its determinant. Furthermore we use the Piola transformation to maintain the divergence free property in a weak sense when mapping the velocity field from one geometry to another. From  $\mathcal{B}_e$  to  $\hat{\Omega}$  this transformation is defined by

$$\hat{\mathbf{u}}_e = \Psi_e(\mathbf{u}_e) = \mathcal{J}_e^{-1}(\mathbf{u}_e \circ \Phi_e)|J_e|, \quad (2.12)$$

where  $\mathbf{u}$  is the velocity solution of (2.6); see [7] for general properties of the Piola transformation. The restrictions of  $\mathbf{u}$  to the different subdomains,  $\mathcal{B}_e$ , are thus all transformed to the same domain  $\hat{\Omega}$ . This is fundamentally different from the case when  $\Omega$  consists of one element only. In the present setting, the weak solution of the steady Stokes equations only guarantees a continuous velocity field over  $\mathcal{B}$ , while the derivatives may be discontinuous. According to spectral theory however, if the underlying solution is regular enough, the derivatives of the solution converge exponentially to the derivatives of the exact solution; see [14].

Note that if we use the Piola transformation, (2.12), to map the velocity solution to the reference domain  $\hat{\Omega}$ , and then use the inverse Piola transformation to map the velocity solution from the reference domain to a new bifurcation, the resulting velocity field need not be globally continuous. In fact, if it were globally continuous, it would be a strike of luck. The reason for this discontinuity lies in the Jacobian matrix of  $\Phi$ . Since each  $\mathcal{B}_e$  is considered as a mapping of  $\hat{\Omega}$ , the global Jacobian matrix of  $\Phi$  will not, in general, be continuous on  $\mathcal{B}$  across internal element boundaries. By defining all the bifurcations as one-to-one regular deformations of the reference bifurcation  $\hat{\mathcal{B}}$  we circumvent this problem. In Section 3 we introduce a method

to achieve  $\mathcal{B} = \mathcal{F}(\hat{\mathcal{B}})$  in such a way that the velocity fields on  $\mathcal{B}$  are globally continuous in a weak sense also after the Piola transformation.

### 2.3 Spectral discretization

We now consider a discretization of the steady Stokes problem (2.6) using a spectral element method based on high order polynomials; see [14] and [15]. Let  $\mathbb{P}_n(\hat{\Omega})$  be the space of all functions which are polynomials of degree less than or equal to  $n$  in each spatial direction on the reference domain  $\hat{\Omega}$ . The discrete space for the velocity is then taken to be

$$X_{\mathcal{N}}(\mathcal{B}) = \{\mathbf{v} \in X(\mathcal{B}), \quad \mathbf{v}_e \circ \Phi_e \in (\mathbb{P}_{\mathcal{N}}(\hat{\Omega}))^2, \quad e = 1, \dots, 6\} \quad , \quad (2.13)$$

while the discrete space for the pressure is

$$M_{\mathcal{N}}(\mathcal{B}) = \{q \in M(\mathcal{B}), \quad q_e \circ \Phi_e \in \mathbb{P}_{\mathcal{N}-2}(\hat{\Omega}), \quad e = 1, \dots, 6\} \quad . \quad (2.14)$$

The bases for  $X_{\mathcal{N}}(\mathcal{B})$  and  $M_{\mathcal{N}}(\mathcal{B})$  are conveniently expressed in terms of the reference variables  $\xi$  and  $\eta$ . As a basis for  $X_{\mathcal{N}}(\mathcal{B})$  we use a nodal basis through the tensor-product Gauss-Lobatto Legendre (GLL) points, while the basis for  $M_{\mathcal{N}}(\mathcal{B})$  is a nodal basis through the tensor-product Gauss-Legendre (GL) points; see [14] and [15]. Specifically, we write

$$(\mathbf{u}_e \circ \Phi_e)(\xi, \eta) = \sum_{i,j=0}^{\mathcal{N}} \mathbf{u}_{ij}^e \ell_i(\xi) \ell_j(\eta), \quad (2.15)$$

where  $\ell_i(\xi)$  refers to a one-dimensional  $\mathcal{N}$ -th order Lagrangian interpolant through the GLL points  $\xi_m$ ,  $m = 0, \dots, \mathcal{N}$ ; here,  $\ell_i(\xi_m) \ell_j(\xi_n) = \delta_{im} \delta_{jn}$  for a given point  $(\xi_m, \xi_n)$  in the underlying tensor-product GLL grid.

In a similar fashion, we write

$$(p_e \circ \Phi_e)(\xi, \eta) = \sum_{i,j=0}^{\mathcal{N}-2} p_{ij}^e \tilde{\ell}_i(\xi) \tilde{\ell}_j(\eta), \quad (2.16)$$

where  $\tilde{\ell}_i(\xi)$  refers to a one-dimensional  $(\mathcal{N}-2)$ -th order Lagrangian interpolant through the (interior) GL points  $\zeta_m$ ,  $m = 0, \dots, \mathcal{N}-2$ ; here,  $\tilde{\ell}_i(\zeta_m) \tilde{\ell}_j(\zeta_n) = \delta_{im} \delta_{jn}$  for a given point in the tensor-product GL grid.

The discrete velocity  $\mathbf{u}_{\mathcal{N}} \in X_{\mathcal{N}}(\mathcal{B})$  is then uniquely determined by  $6(\mathcal{N}+1)^2$  coefficients, minus overlapping nodes, for each spatial component, where some of the coefficients are fixed due to the prescribed Dirichlet boundary conditions. The discrete pressure is determined by  $6(\mathcal{N}-1)^2$  basis coefficients.

The mapping  $\Phi_e$  is realized computationally by using an isoparametric representation of the geometry. Each edge of  $\mathcal{B}_e$  is given as a one-to-one mapping of a corresponding edge  $[-1, 1]$  on  $\hat{\Omega}$ , and is approximated as an

$\mathcal{N}$ -th order polynomial. The location of the (interior) points  $(x_m, y_n) = \Phi(\xi_m, \xi_n)$  are found by a Gordon-Hall algorithm; see [11].

The bilinear forms and the linear form in (2.6) are expressed in terms of the reference variables, and the integrals are evaluated using GLL and GL quadrature. This gives us the following discrete system: Find  $\mathbf{u}_{\mathcal{N}} \in X_{\mathcal{N}}(\mathcal{B})$  and  $p_{\mathcal{N}} \in M_{\mathcal{N}}(\mathcal{B})$  such that

$$\begin{aligned} a_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, \mathbf{v}; \Phi) + b_{\mathcal{N}}(\mathbf{v}, p_{\mathcal{N}}; \Phi) &= l_{\mathcal{N}}(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_{\mathcal{N}}(\mathcal{B}) \\ b_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, q; \Phi) &= 0 \quad \forall q \in M_{\mathcal{N}}(\mathcal{B}), \end{aligned} \quad (2.17)$$

where  $a_{\mathcal{N}}$ ,  $b_{\mathcal{N}}$  and  $l_{\mathcal{N}}$  refer to integration of the bilinear and linear forms using Gauss-type quadrature. Using the chosen bases (2.15) and (2.16), we arrive at a system of algebraic equations for the unknown basis coefficients; this system is solved using the conjugate gradient method in the context of the Uzawa algorithm; see [15].

### 3 Mapping strategy for bifurcations

As mentioned in Section 2 a solution of the steady Stokes equations on a bifurcation resides on a union of six elements, where each element is a one-to-one mapping of  $\hat{\Omega}$ , see Figure 1. Locally each mapping is reversible, with a continuous and invertible Jacobian. Globally however, the Jacobian is not continuous, and its elements and determinant will have discontinuities along the internal boundaries.

If we use the Piola transformation (2.12) on the velocity on each element, each part of the velocity will be divergence free on  $\hat{\Omega}$ . To find the representation of the velocity solution on a different bifurcation, the inverse Piola transformation, with the appropriate Jacobian, should be used to get a divergence free field. Since the global Jacobian of two different bifurcations will have different jump discontinuities along internal boundaries, the resulting velocity representation will be discontinuous. This means that the velocity solutions that are meant to be basis functions for the velocity on a generic bifurcation in the reduced basis framework will, in general, be discontinuous. One way around this problem is to define the restriction of the basis functions to each element in the bifurcation as separate basis functions. This will give us six times as many basis functions, and to find a reduced basis solution we would have to glue the solutions on each element together with Lagrange multipliers as was done in [13]. Note that in this case the reduced basis solution will only be weakly continuous across the interfaces. Using a direct solver based on LU-decomposition for the reduced basis algebraic system, the online computation would take  $\mathcal{O}(6^3)$  times longer for a six-element domain decomposition than if we could use the global basis functions directly.



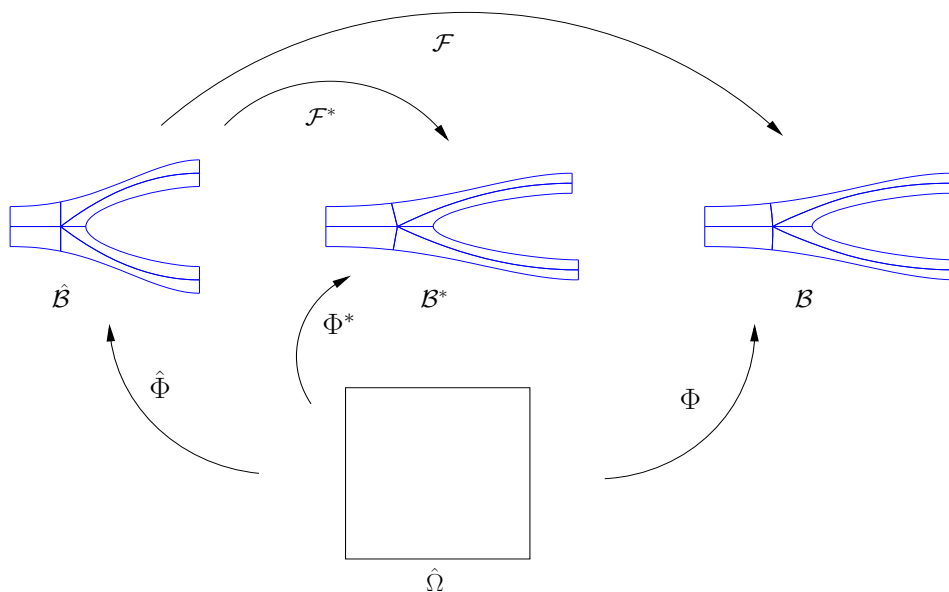


Figure 2: The relationships between the reference domain,  $\hat{\Omega}$ , the reference bifurcation,  $\hat{\mathcal{B}}$ , the intermediate bifurcation,  $\mathcal{B}^*$ , and the globally  $C^1$  bifurcation,  $\mathcal{B}$ .

To get velocity fields which are continuous over a generic bifurcation block, we must take care of the discontinuity in the Jacobian of the mapping,  $\mathcal{F}$ , from one bifurcation to another. If all bifurcations,  $\mathcal{B}$ , have the same ratio between the different elements, and the same angles towards the internal interfaces, chances are that the jump in the Jacobian of the mapping,  $\Phi$ , from  $\hat{\Omega}$  to  $\mathcal{B}$ , will be equal for two different bifurcations along the element interfaces. Thus taking the inverse Piola transformation, from  $\hat{\Omega}$  to a generic bifurcation,  $\mathcal{B}^*$ , will not produce discontinuities in the solution, but instead counter the effect of the original Piola transformation from  $\mathcal{B}$  to  $\hat{\Omega}$ . Assuming this kind of regularity between the different elements in all possible bifurcations would put severe restrictions on the flexibility of the geometries. One of the goals of the reduced basis method with geometric parameters is to achieve good approximations for fairly general geometries. The solution proposed above to avoid discontinuities in the Jacobian will clearly compromise this goal.

In order to keep the velocity basis functions globally continuous and divergence free, we make use of the reference bifurcation,  $\hat{\mathcal{B}}$ . The subdomains,  $\hat{\mathcal{B}}_e$ , of  $\hat{\mathcal{B}}$  are, as the subdomains of  $\mathcal{B}$ , all one-to-one mappings,  $\hat{\Phi}_e$ , of  $\hat{\Omega}$ , i.e.  $\hat{\mathcal{B}} = \hat{\Phi}(\hat{\Omega}) = \bigcup_{e=1}^6 \hat{\Phi}_e(\hat{\Omega})$ . The mapping from  $\hat{\mathcal{B}}$  to a generic bifurcation  $\mathcal{B}^* = \Phi^*(\hat{\Omega})$ , is then given by  $\mathcal{F}^* = \Phi^*(\hat{\Phi}^{-1}(\hat{\mathcal{B}}))$ , see Figure 2. The mapping  $\mathcal{F}^* : \hat{\mathcal{B}} \rightarrow \mathcal{B}^*$  is continuous and one-to-one, but the Jacobian of  $\mathcal{F}^*$  will have

discontinuous elements, and thus the resulting Piola transformation from  $\mathcal{B}^*$  to  $\hat{\mathcal{B}}$  will give discontinuous representation of the velocity field on  $\hat{\mathcal{B}}$ . The key to achieve continuous velocity fields after the Piola transformation lies in the elements of the Jacobian of  $\mathcal{F}^*$ . The Jacobian itself is given by

$$\mathcal{J}_{\mathcal{F}^*} = \begin{bmatrix} \frac{\partial x^*}{\partial \hat{x}} & \frac{\partial x^*}{\partial \hat{y}} \\ \frac{\partial y^*}{\partial \hat{x}} & \frac{\partial y^*}{\partial \hat{y}} \end{bmatrix}, \quad (3.1)$$

where  $(x^*, y^*)$  are the coordinates of a point in  $\mathcal{B}^*$ , and  $(\hat{x}, \hat{y})$  denote the corresponding point in  $\hat{\mathcal{B}}$ . To avoid jump discontinuities in the derivatives across internal boundaries we redistribute the internal points in  $\mathcal{B}^*$  by a smoothing process described below. The resulting bifurcation  $\mathcal{B} = \mathcal{F}(\hat{\mathcal{B}})$  has the same external boundary as  $\mathcal{B}^*$ , while the corresponding Jacobian  $\mathcal{J}_{\mathcal{F}}$  has weakly continuous elements.

### 3.1 Weakly continuous mappings

We consider  $x^*(\hat{x}, \hat{y})$  and  $y^*(\hat{x}, \hat{y})$  to be continuous functions on  $\hat{\mathcal{B}}$ , and define  $dx(\hat{x}, \hat{y}) = x^*(\hat{x}, \hat{y}) - \hat{x}$  and  $dy(\hat{x}, \hat{y}) = y^*(\hat{x}, \hat{y}) - \hat{y}$ . Both  $dx(\hat{x}, \hat{y})$  and  $dy(\hat{x}, \hat{y})$  are  $C^0$ , but not  $C^1$  functions on  $\hat{\mathcal{B}}$ . We need to redistribute the internal nodes in  $\mathcal{B}^*$  such that the transition is smooth, i.e., we need to find  $x_\delta(\hat{x}, \hat{y})$ , and  $y_\delta(\hat{x}, \hat{y}) \in C^1(\hat{\mathcal{B}})$  such that

$$\begin{aligned} x(\hat{x}, \hat{y}) &= \hat{x} + x_\delta(\hat{x}, \hat{y}) \\ y(\hat{x}, \hat{y}) &= \hat{y} + y_\delta(\hat{x}, \hat{y}), \end{aligned} \quad (3.2)$$

where

$$\begin{aligned} x|_{\partial\hat{\mathcal{B}}} &= x^*|_{\partial\hat{\mathcal{B}}} \\ y|_{\partial\hat{\mathcal{B}}} &= y^*|_{\partial\hat{\mathcal{B}}}. \end{aligned} \quad (3.3)$$

The coordinates  $(x, y)$  then define the bifurcation  $\mathcal{B}$ , such that the external boundaries of  $\mathcal{B}$  and  $\mathcal{B}^*$  are the same. The resulting Jacobian  $\mathcal{J}_{\mathcal{F}}$  will then have continuous elements since, for example,  $\frac{\partial x}{\partial \hat{x}} = 1 + \frac{\partial x_\delta}{\partial \hat{x}}$ . To find the proper  $x_\delta(\hat{x}, \hat{y})$  and  $y_\delta(\hat{x}, \hat{y})$ , we solve the problems: Find  $x_\delta \in H^1(\hat{\mathcal{B}})$  and  $y_\delta \in H^1(\hat{\mathcal{B}})$  such that

$$\begin{aligned} a(x_\delta, v) &= 0, & \forall v \in H_0^1(\hat{\mathcal{B}}), \\ x_\delta|_{\partial\hat{\mathcal{B}}} &= (x^* - \hat{x})|_{\partial\hat{\mathcal{B}}}, \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} a(y_\delta, v) &= 0, & \forall v \in H_0^1(\hat{\mathcal{B}}), \\ y_\delta|_{\partial\hat{\mathcal{B}}} &= (y^* - \hat{y})|_{\partial\hat{\mathcal{B}}}, \end{aligned} \quad (3.5)$$

where  $a(\cdot, \cdot)$  is the bilinear form associated with the scalar Laplace operator. Although these functions are not in  $C^1$ , they are very good approximations.

A velocity solution of the steady Stokes equation found on the generic bifurcation  $\mathcal{B} = \mathcal{F}(\hat{\mathcal{B}})$ , defined through the coordinates  $(x, y) = (\hat{x} + x_\delta, \hat{y} +$

$y_\delta$ ), can now be mapped in a continuous fashion by a Piola transformation to  $\hat{\mathcal{B}}$ . To realize the mapping we may also use  $\Phi : \hat{\Omega} \rightarrow \mathcal{B}$ , and define  $\mathcal{F}(\hat{\mathcal{B}}) = \Phi(\hat{\Phi}^{-1}(\hat{\mathcal{B}}))$ , see Figure 2. In fact, if we have two different bifurcations, and both are found as a  $C^1$  mapping of  $\hat{\mathcal{B}}$ , i.e.  $\mathcal{B}_1 = \mathcal{F}_1(\hat{\mathcal{B}})$  and  $\mathcal{B}_2 = \mathcal{F}_2(\hat{\mathcal{B}})$ , the map from  $\mathcal{B}_1$  to  $\mathcal{B}_2$  may be defined as

$$\mathcal{B}_2 = \Phi_2(\Phi_1^{-1}(\mathcal{B}_1)). \quad (3.6)$$

This implies that the velocity basis functions may be stored in a discontinuous fashion on  $\hat{\Omega}$ , and still be continuous when Piola transformed to a generic bifurcation.

### 3.2 Parameterizing the geometries

In (2.10) and (2.11) we see how the geometric mapping  $\Phi$  enters the steady Stokes equations in a natural way. To control different instantiations of this mapping we define  $\Phi(\hat{\Omega}; \mu)$ , where  $\mu \in \mathcal{D} \subset \mathbb{R}^P$ . The elements in  $\mu$  are parameters which describe, for example, the length, thickness and opening angle of a bifurcation. Given  $\mu$ , a bifurcation is constructed by defining its outer edges according to  $\mu$ , and then the internal nodes are found using a Gordon-Hall algorithm. Finally, all the internal nodes are adjusted according to the smoothing process described above. After all the points are found, the bifurcation may be rotated to any desired orientation. We note that all corners in the bifurcation are right angles.

Once the final values of all the nodal points have been computed, the Jacobian,  $\mathcal{J}$ , of the mapping  $\Phi(\hat{\Omega}; \mu)$  and its determinant,  $J$ , are calculated and stored for each node. If we again study (2.10) and (2.11), we see that these quantities appear nonlinearly in the equations. Thus we have non-linear parameter dependence, which is fundamentally different from most reduced basis applications. The equations are not affine in their parameter dependence either; see [18, 20] for affine, linear parameter dependence.

Since we are only interested in giving a proof of concept, we choose  $P = 2$ , and let  $\mu = (\mu^1, \mu^2)$ . We let the first parameter,  $\mu^1$ , define the difference in length between the upper leg of the bifurcation and the lower leg, i.e.  $\mu^1 = L_l - L_u$ ; see Figure 3. The second parameter,  $\mu^2$ , is taken to be the difference in the opening angle of two legs of the bifurcation, i.e.  $\mu^2 = \theta_1 - \theta_0$ .

The outline of the bifurcation is defined through its corner points and the length of the body relative to the length of the legs. The opening angle is adjusted by a rigid body rotation of the two cornerpoints of the upper leg around the centerpoint of the inflow boundary. Before any rotation, the difference in length between the two legs is defined by setting the  $x$ -coordinates of the corner points of the upper leg (both are the same before the rotation). The non-linear edges of the bifurcation are constructed such that they pass through the corner points and the common edge point of

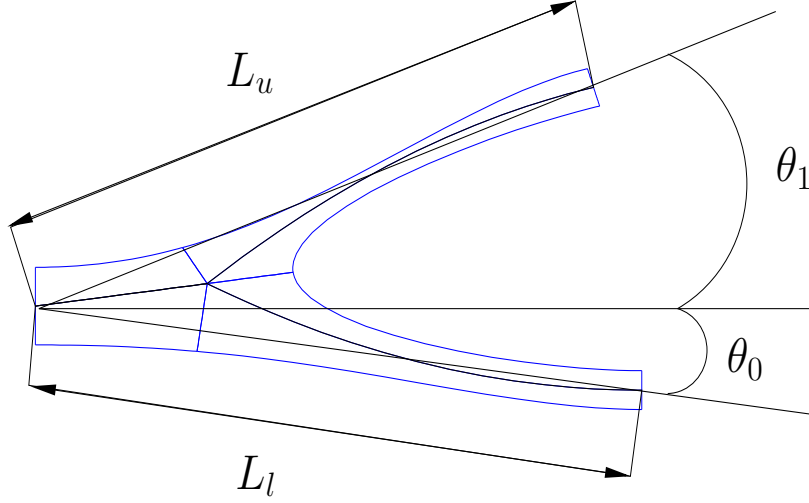


Figure 3: The parameters used to define the bifurcations.

the two elements sharing an edge, and such that they are perpendicular to the inflow and outflow boundaries. The upper and lower edges are fourth order polynomials, while the edge connecting the two legs is, for optimal flexibility, constructed by the use of cubic splines.

## 4 A reduced basis method

For a generic domain  $\mathcal{B} = \Phi(\hat{\Omega})$  we now define the reduced basis solution spaces  $X_N(\mathcal{B}) \subset X_N(\mathcal{B})$  and  $M_N(\mathcal{B}) \subset M_N(\mathcal{B})$ . Our objective is to find a unique reduced basis solution  $\mathbf{u}_N \in X_N(\mathcal{B})$  and  $p_N \in M_N(\mathcal{B})$  satisfying

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{v}; \Phi) + b_N(\mathbf{v}, p_N; \Phi) &= l_N(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_N(\mathcal{B}) \\ b_N(\mathbf{u}_N, q; \Phi) &= 0 \quad \forall q \in M_N(\mathcal{B}). \end{aligned} \quad (4.1)$$

As before the coercivity of  $a(\cdot, \cdot; \Phi)$  holds for all  $\mathbf{v} \in X_N(\mathcal{B})$  since it is a subset of  $X_N(\mathcal{B})$ . The inf-sup condition (2.9), however, depends strongly on  $X_N(\mathcal{B})$  and  $M_N(\mathcal{B})$ .

For a given set of  $N$  parameter vectors  $S_N = \{\mu_1 \in \mathcal{D}, \dots, \mu_N \in \mathcal{D}\}$ , we find the resulting geometries  $\{\mathcal{B}_i\}_{i=1}^N$  and solve the steady Stokes equations on each geometry. The resulting velocity fields,  $\{\mathbf{u}_i\}_{i=1}^N$ , are then mapped element by element to the reference domain  $\hat{\Omega}$  by the Piola transformation

$$\hat{\mathbf{u}}_{ie} = \Psi_i(\mathbf{u}_{ie}) = \mathcal{J}_{ie}^{-1}(\mathbf{u}_{ie} \circ \Phi_{ie})|J_{ie}|, \quad (4.2)$$

for  $e = 1, \dots, 6$  and  $i = 1, \dots, N$ . From Section 3 we know that each  $\{\hat{\mathbf{u}}_{ie}\}_{e=1}^6$  constitutes a weakly continuous function  $\tilde{\mathbf{u}}_i = \{\tilde{\mathbf{u}}_{i1}, \dots, \tilde{\mathbf{u}}_{i6}\}$  when mapped

through the inverse Piola transformation

$$\tilde{\mathbf{u}}_{ie} = \Psi^{-1}(\hat{\mathbf{u}}_{ie}) = \mathcal{J}_e(\hat{\mathbf{u}}_{ie} \circ \Phi_e^{-1})|J_e^{-1}|, \quad (4.3)$$

to a generic geometry  $\mathcal{B} = \Phi(\hat{\Omega})$ . The scalar pressure fields,  $p_i$ , are found in a nodal basis, and can be reused on a different geometry through the notion  $\tilde{p}_i = p_i \circ \Phi_i \circ \Phi^{-1}$ , for  $i = 1, \dots, N$ .

#### 4.1 Constructing compatible spaces

We define the spaces

$$\begin{aligned} X_N^0(\mathcal{B}) &= \text{span}\{\tilde{\mathbf{u}}_i, \quad i = 1, \dots, N\} \\ M_N(\mathcal{B}) &= \text{span}\{\tilde{p}_i, \quad i = 1, \dots, N\}, \end{aligned} \quad (4.4)$$

for which we know that the inf-sup condition (2.9) is not fulfilled since the velocity fields in  $X_N^0(\mathcal{B})$  are all divergence free, (or as we will discuss later, at least approximately divergence free). If we are not interested in the pressure on the generic domain, we may solve the problem: Find  $\mathbf{u}_N$  in  $X_N^0(\mathcal{B})$  such that

$$a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}; \Phi) = l_{\mathcal{N}}(\mathbf{v}; \Phi), \quad \forall \mathbf{v} \in X_N^0(\mathcal{B}). \quad (4.5)$$

In this case the inf-sup condition is insignificant. If we also want to find the pressure however, the reduced basis velocity space  $X_N(\mathcal{B})$  must be defined as  $X_N(\mathcal{B}) = X_N^0(\mathcal{B}) \oplus X_N^e(\mathcal{B})$ , where  $X_N^e$  consists of velocity fields constructed in order to guarantee the inf-sup condition. One way of constructing  $X_N^e(\mathcal{B})$ , is for each pressure solution  $\tilde{p}_i = (\hat{p}_i \circ \hat{\Phi}^{-1}) \in M_N(\hat{\mathcal{B}})$ , to find  $\mathbf{v}_i^e \in X_{\mathcal{N}}(\hat{\mathcal{B}})$  such that

$$\mathbf{v}_i^e = \arg \max_{\mathbf{u} \in X_{\mathcal{N}}(\hat{\mathcal{B}})} \frac{\int_{\hat{\mathcal{B}}} \tilde{p}_i \nabla \cdot \mathbf{u} d\hat{\mathcal{B}}}{|\mathbf{u}|_{H^1(\hat{\mathcal{B}})}}, \quad (4.6)$$

where  $\hat{\mathcal{B}} = \hat{\Phi}(\hat{\Omega})$  is the reference bifurcation. Like the velocity solutions of the steady Stokes equations, the  $\mathbf{v}_i^e$ 's are mapped element by element to the reference domain  $\hat{\Omega}$  using the Piola transformation, and then to the generic bifurcation  $\mathcal{B}$  using the inverse Piola transformation. The space  $X_N^e(\mathcal{B})$  is then defined as

$$X_N^e(\mathcal{B}) = \text{span}\{\tilde{\mathbf{v}}_i^e = \Psi^{-1}(\hat{\Psi}(\hat{\mathbf{v}}_i^e)), \quad i = 1, \dots, N\}. \quad (4.7)$$

The inf-sup condition (2.9) is then fulfilled for the spaces  $X_N(\mathcal{B}) = X_N^0(\mathcal{B}) \oplus X_N^e(\mathcal{B})$  and  $M_N(\mathcal{B})$ , and we may solve (4.1) to find both the velocity and the pressure simultaneously involving a system of size  $3N$ . Alternatively, we could solve the two separate  $N$ -sized problems (4.5) and

$$b_{\mathcal{N}}(\mathbf{v}, p_N; \Phi) = -a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}; \Phi) + l_{\mathcal{N}}(\mathbf{v}; \Phi), \quad \forall \mathbf{v} \in X_N^e(\mathcal{B}), \quad (4.8)$$

for the velocity and pressure, respectively. See [13] for more details in the context of deformed pipes (single-domain case).

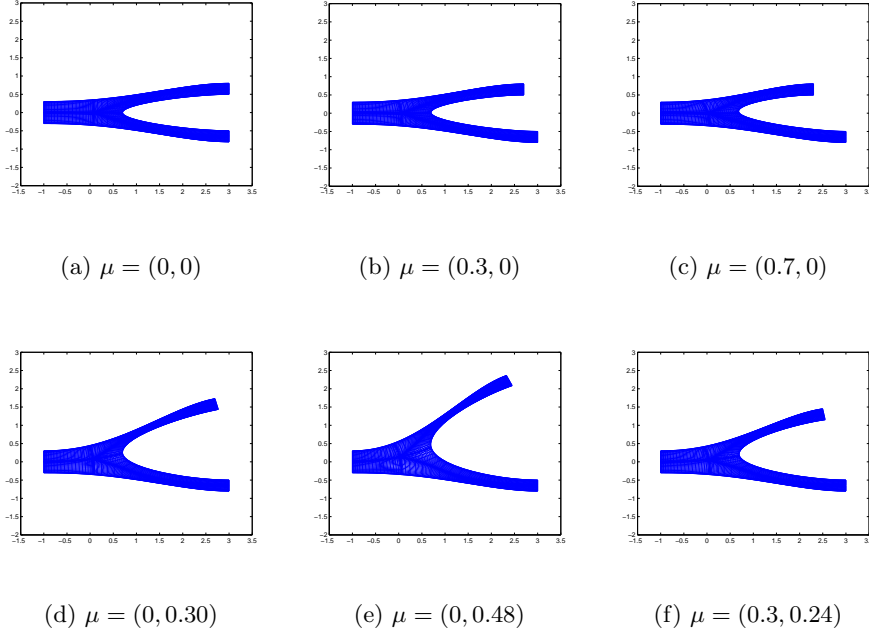


Figure 4: Examples of bifurcations corresponding to different parameter choices. Here, the two parameters represent the relative length of the legs of each bifurcation and the angle between the legs.

## 4.2 Choice of basis geometries and numerical results

The basis functions are found as the spectral element solution of the steady Stokes equations on bifurcations defined by different instantiations of the parameter vector  $\mu = (\mu^1, \mu^2)$ . For  $\mu = (0, 0)$  we get the bifurcation in Figure 4(a), where the length of the bifurcation is 4 spatial units, and the opening angle is  $\theta_0$  radians. For our experiments, we restrict the parameter space to  $\mathcal{D} = [0, 0.7] \times [0, 0.48]$ , and distribute  $N = 64$  points equidistantly in  $\mathcal{D}$  in a tensor product fashion. That is,  $\{\mu_i^1 = 0.1(i - 1)\}_{i=1}^8$  and  $\{\mu_i^2 = 0.06(i - 1)\}_{i=1}^8$ , and  $S_N$  is found as all possible combinations of  $\mu^1$  and  $\mu^2$ . Some bifurcations corresponding to a selection of parameter vectors in  $S_N$  are displayed in Figures 4(a)-4(f).

For the resulting  $N = 64$  basis bifurcations,  $\{\mathcal{B}_i\}_{i=1}^N$ , we thus compute the basis functions  $\{(\mathbf{u}_i, p_i)\}_{i=1}^N$ . The velocity basis functions are mapped to the reference domain  $\hat{\Omega}$ , and for each orthogonalized pressure basis function we generate one velocity function designed to fulfill the inf-sup condition. The inf-sup generated velocities are also mapped to the reference domain.

The generic bifurcation used to verify the reduced basis solution is defined by  $\mu = (0.45, 0.27)$ . This  $\mu$  is “in the middle of”  $\mathcal{D}$ , so it should

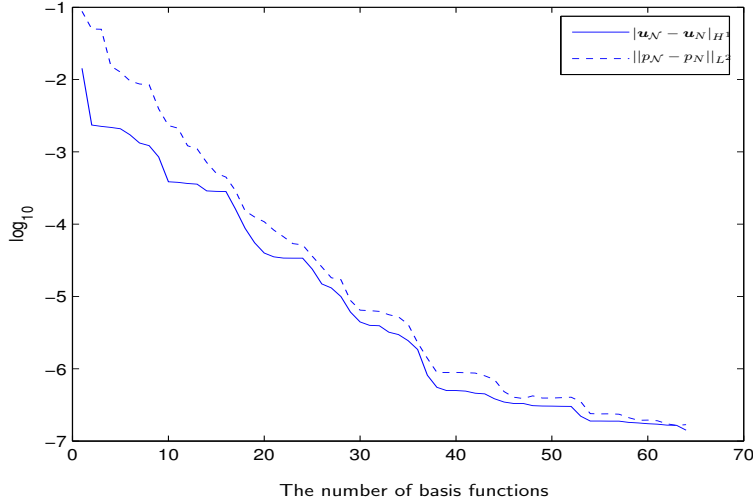


Figure 5: The reduced basis error in Method 1 when varying two geometric parameters.

be possible to achieve a good approximation. On the other hand, the distance from  $\mu$  to the closest parameter in  $S$  is the largest possible, so this should be a worst case. The reduced basis solution,  $(\mathbf{u}_N, p_N)$ , on the generic bifurcation  $\mathcal{B}$  is then found from (4.1) as

$$\mathbf{u}_N = \sum_{i=1}^N \alpha_i \tilde{\mathbf{u}}_i + \beta_i \tilde{\mathbf{u}}_i^e, \quad p_N = \sum_{i=1}^N \gamma_i \tilde{p}_i. \quad (4.9)$$

We remark that we have one degree-of-freedom per basis function, independent of the number of spatial dimensions

For an increasing number of basis functions, the reduced basis error is shown in Figure 5, and we see that the error converges exponentially down to  $10^{-7}$ , where it flattens a little. Since all basis functions are close in  $X_{\mathcal{N}}(\mathcal{B})$ , orthogonalization is necessary to avoid numerical instability, and we use the modified Gram-Schmidt algorithm for this. The orthogonalization is done as part of the preprocessing by mapping the basis functions to the reference bifurcation, orthogonalize them, and then map them back to  $\hat{\Omega}$ . The basis functions will not be orthogonal on the generic bifurcation, but they will be sufficiently orthogonal so as to avoid any numerical instability.

The results we get are very good, but in order to avoid using all 64 basis functions together with their enriched velocity basis functions, we will in a later section see how *a posteriori* error estimation can be applied to select the most significant basis functions; see also [21]. As more parameters are introduced, this optimization will be even more important. In the mean time we choose a new generic bifurcation defined by the parameter  $\mu = (0.45, 0)$ ,

$N$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$
1	$3.46 \cdot 10^{-3}$	$2.68 \cdot 10^{-2}$
2	$6.58 \cdot 10^{-4}$	$1.94 \cdot 10^{-3}$
3	$1.48 \cdot 10^{-4}$	$2.12 \cdot 10^{-4}$
4	$5.04 \cdot 10^{-5}$	$7.18 \cdot 10^{-5}$
5	$9.63 \cdot 10^{-6}$	$1.56 \cdot 10^{-5}$
6	$1.15 \cdot 10^{-6}$	$1.32 \cdot 10^{-6}$
7	$2.29 \cdot 10^{-7}$	$1.36 \cdot 10^{-7}$
8	$1.78 \cdot 10^{-7}$	$7.12 \cdot 10^{-8}$

Table 1: The convergence of the reduced basis error when  $\mu = (0.45, 0)$ , and only basis functions with  $\mu^2 = 0$  are used.

and approximate the corresponding solution of the steady Stokes problem only with basis functions with  $\mu^2 = 0$ . The resulting error between the reduced basis solution and the spectral element solution is shown in Table 1. We clearly observe that, although only a few basis functions are used, the tailored choice makes the approximation just as good as when we used all the basis functions in the previous, more general, case.

We have also tried to use a sub-parametric representation of the geometries in the smoothing process described in Section 3.2. The results are just as good as the results in Table 1. The advantage of this approach is the reduction in the necessary time for the *online* smoothing of the generic grid.

### 4.3 Method 2

In the above tests we used what in [13] is denoted Method 1 to find the reduced basis solution on a generic geometry  $\mathcal{B} = \Phi(\hat{\Omega})$ . If all velocities in the velocity basis are divergence free, i.e.

$$\mathbf{v} \in X_{\mathcal{N}}^0(\mathcal{B}) = \{\mathbf{v} \in X_{\mathcal{N}}(\mathcal{B}), \int_{\mathcal{B}} q \nabla \cdot \mathbf{v} d\mathcal{B} = 0, \quad \forall q \in M_{\mathcal{N}}(\mathcal{B})\}, \quad (4.10)$$

the steady Stokes problem may be decoupled. In particular, we replace  $X_{\mathcal{N}}(\mathcal{B})$  in (4.1) with  $X_{\mathcal{N}}^0(\mathcal{B})$ , the space spanned by the Piola transformation of the precomputed divergence free velocity basis functions. In Method 2 we then assume  $X_{\mathcal{N}}^0(\mathcal{B}) \subset X_{\mathcal{N}}^0(\mathcal{B})$  to get the (vector) Poisson problem: Find  $\mathbf{u}_N \in X_{\mathcal{N}}^0(\mathcal{B})$  such that

$$a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}; \Phi) = l_{\mathcal{N}}(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_{\mathcal{N}}^0(\mathcal{B}). \quad (4.11)$$

From  $X_{\mathcal{N}}^e(\mathcal{B})$ , the space spanned by the previously computed enriched velocity basis functions, we determine the reduced basis pressure uniquely by solving

$$b_{\mathcal{N}}(\mathbf{v}, p_N; \Phi) = -a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}; \Phi) + l_{\mathcal{N}}(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_{\mathcal{N}}^e(\mathcal{B}). \quad (4.12)$$



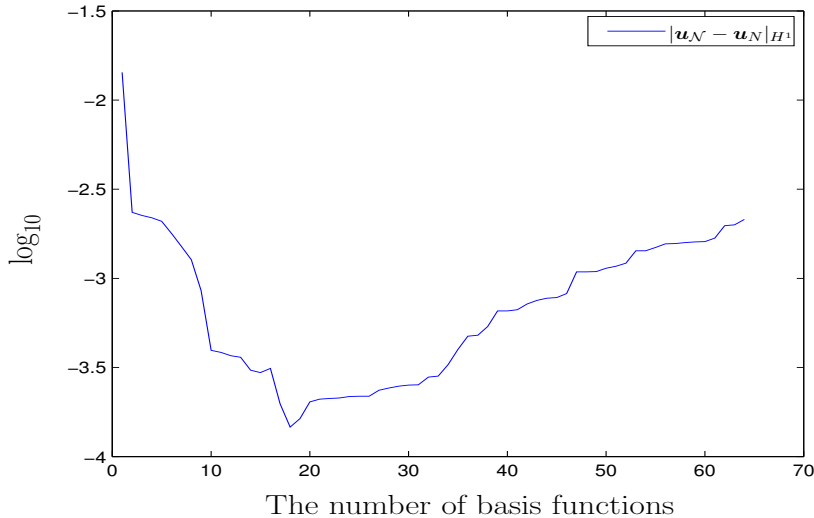


Figure 6: The reduced basis error in Method 2 when varying two geometric parameters.

This worked well on deformed pipes in [13], but on the bifurcations we are experiencing some difficulties. As can be seen in Figure 6, the reduced basis velocity error converges to a certain point before it starts diverging. Compared with Method 1, the stiffness matrix and right hand side of the reduced basis linear system have the exact same values for comparable entries. This could indicate that what is causing problems in Method 2 is also present in Method 1, just not obvious in the results. We now discuss this issue further.

When we precompute the basis functions as solutions of the steady Stokes problem on different geometries  $\mathcal{B}_i$ , we use the Uzawa nested conjugate gradients method. The stopping criterion in the outer loop of this method reflects the value of  $\int_{\mathcal{B}_i} q(\nabla \cdot \mathbf{v}_i) d\mathcal{B}_i$ , and in order to get

$$\mathbf{v}_i \in X_{\mathcal{N}}^0(\mathcal{B}_i) = \{\mathbf{v} \in X_{\mathcal{N}}(\mathcal{B}_i), \int_{\mathcal{B}_i} q \nabla \cdot \mathbf{v} d\mathcal{B}_i = 0, \quad \forall q \in M_{\mathcal{N}}(\mathcal{B}_i)\}, \quad (4.13)$$

this stopping criterion has to be very strict. Typically we use a relative stopping criterion of  $10^{-14}$  in the outer loop, and  $10^{-16}$  in the inner loop. The cause of the problem in Figure 6 is that, after the Piola transformation of the velocity basis functions from the basis geometries,  $\mathcal{B}_i, i = 1, \dots, N$ , to the reference domain,  $\hat{\Omega}$ , and then to the generic geometry,  $\mathcal{B}$ , the divergence of the velocities is of the order  $10^{-10}$ . Since the velocity basis functions are not strictly in  $X_{\mathcal{N}}^0(\mathcal{B})$ , we get a non-conforming method and the attempt to solve (4.11) does not give the correct answer. In Method 1, however, the velocity basis functions are not required to be in  $X_{\mathcal{N}}^0(\mathcal{B})$ , and thus the

correct solution is found.

To demonstrate where the difficulty lies, we refer to Figure 2 in Section 3. Analytically, the Piola transformation is constructed in order to preserve the identities

$$\int_{\mathcal{B}_i} q(\nabla \cdot \mathbf{v}_i) d\mathcal{B}_i = \sum_{e=1}^6 \int_{\hat{\Omega}} \hat{q}_e(\hat{\nabla} \cdot \hat{\mathbf{v}}_{ie}) d\hat{\Omega} = \int_{\mathcal{B}} \tilde{q}(\nabla \cdot \tilde{\mathbf{v}}_i) d\mathcal{B}, \quad (4.14)$$

where

$$\begin{aligned} \mathcal{B}_i &= \mathcal{F}_i(\hat{\mathcal{B}}) = \Phi_i(\hat{\Omega}), & i = 1, \dots, N \\ \mathcal{B} &= \mathcal{F}(\hat{\mathcal{B}}) = \Phi(\hat{\Omega}) \\ \hat{\mathcal{B}} &= \hat{\Phi}(\hat{\Omega}) = \sum_{e=1}^6 \hat{\Phi}_e(\hat{\Omega}) \end{aligned} \quad (4.15)$$

through globally one-to-one regular mappings  $\mathcal{F}_i$  and  $\mathcal{F}$ , and piecewise continuous mappings  $\Phi_i$ ,  $\Phi$  and  $\hat{\Phi}$ . We use (2.12) on each subdomain separately to obtain these identities. In the precomputation of the basis functions, the term  $\int_{\mathcal{B}_i} q(\nabla \cdot \mathbf{v}_i) d\mathcal{B}_i$  is calculated using Gauss Legendre quadrature on each subdomain, and we define  $b_{\mathcal{N}}(\mathbf{v}_i, q; \mathcal{B}_i)$  as

$$\sum_{e=1}^6 \sum_{\alpha\beta=1}^{\mathcal{N}-1} \omega_\alpha \omega_\beta (q_e \circ \Phi_{ie}(\zeta_\alpha, \zeta_\beta)) [\nabla \cdot (\mathbf{v}_{ie} \circ \Phi_{ie}(\zeta_\alpha, \zeta_\beta))] |J_{ie}(\zeta_\alpha, \zeta_\beta)|, \quad (4.16)$$

where  $(\zeta_\alpha, \zeta_\beta)$  is a Gauss Legendre point and  $\omega_\alpha \omega_\beta$  the corresponding Gauss Legendre weight. We use a nodal basis for the pressure, choosing

$$\hat{q}_{mn}^e(\zeta_\alpha, \zeta_\beta) = q_{mn}^e \circ \Phi_{ie}(\zeta_\alpha, \zeta_\beta) = \delta_{m\alpha} \delta_{n\beta}. \quad (4.17)$$

The precomputed velocity basis functions are thus divergence free on  $\mathcal{B}_i$  in the points corresponding to the mapping  $\Phi_i(\zeta_\alpha, \zeta_\beta)$ .

When the Piola transformation (2.12) is applied to  $\mathbf{v}_{ie}$ , the result is a polynomial of degree  $2\mathcal{N} - 1$  in each spatial direction. This is because  $\mathbf{v}_{ie}$  is a polynomial of degree  $\mathcal{N}$ , and each element in the matrix  $\mathcal{J}_{ie}^{-1} |J_{ie}|$  is a polynomial of degree  $\mathcal{N} - 1$  since we use an isoparametric representation of the geometry; see Section 2.3. For reasons that will be explained shortly, we perform the transformation on a higher order grid,

$$X_{3\mathcal{N}}(\mathcal{B}_i) = \{\mathbf{v} \in X(\mathcal{B}_i), \quad \mathbf{v}_e \circ \Phi_{ie} \in (\mathbb{P}_{3\mathcal{N}}(\hat{\Omega}))^2, \quad e = 1, \dots, 6\}. \quad (4.18)$$

That is, the geometry is interpolated to  $\mathbb{P}_{3\mathcal{N}}(\mathcal{B}_i)$ ,  $\mathbf{v}_i$  is interpolated to  $X_{3\mathcal{N}}(\mathcal{B}_i)$ , and the elements of  $\mathcal{J}_i^{-1} |J_i|$  are found using differentiation of degree  $3\mathcal{N}$ .

For the Piola transformation of  $\hat{\mathbf{v}}_i$  from  $\hat{\Omega}$  to the generic domain  $\mathcal{B} = \Phi(\hat{\Omega})$ , we have for each subdomain

$$\tilde{\mathbf{v}}_{ie} = \Psi_e^{-1}(\hat{\mathbf{v}}_{ie}) = \frac{1}{|J_e|} \mathcal{J}_e \hat{\mathbf{v}}_{ie}. \quad (4.19)$$

The division of  $|J_e|$  results in a non-polynomial function  $\tilde{\mathbf{v}}_{ie}$ . We use a higher order grid to apply the Piola transformation, and  $3\mathcal{N}$  should give a good approximation of the rational term  $\frac{1}{|J_e|}$ . We note that a grid of order  $2\mathcal{N} - 1$  is sufficient in the Piola transformation from  $\mathcal{B}_{ie}$  to  $\hat{\Omega}$ , but in order to keep things simple, i.e. to avoid too many grids, we use the same order for both Piola transformations. The incompressibility after the Piola transformation is still measured in the  $(\mathcal{N} - 1)^2$  low order Gauss Legendre points, since this is where we can expect the divergence to be zero.

The higher order grids are used for all geometries, and schematically we write the Piola transformation from  $X_{\mathcal{N}}^0(\mathcal{B}_i)$  via  $X_{3\mathcal{N}}^0(\hat{\Omega})$  to  $X_{3\mathcal{N}}^0(\mathcal{B})$  as

$$\mathbf{v} \xrightarrow{I_{\mathcal{N},3\mathcal{N}}} \mathbf{v}_{3\mathcal{N}} \xrightarrow{\Psi_{i,3\mathcal{N}}} \hat{\mathbf{v}}_{3\mathcal{N}} \xrightarrow{\Psi_{3\mathcal{N}}^{-1}} \tilde{\mathbf{v}}_{3\mathcal{N}}, \quad (4.20)$$

where

$$X_{3\mathcal{N}}^0(\mathcal{B}) = \{\mathbf{v} \in X_{3\mathcal{N}}(\mathcal{B}), \quad b_{\mathcal{N}}(\mathbf{v}, q; \mathcal{B}) = 0, \quad \forall q \in M_{\mathcal{N}}(\mathcal{B})\}. \quad (4.21)$$

This corresponds to solving the original steady Stokes problem in a  $\mathbb{P}_{3\mathcal{N}}/\mathbb{P}_{\mathcal{N}-2}$  setting. Numerical tests verify that  $\tilde{\mathbf{v}}_{3\mathcal{N}} \in X_{3\mathcal{N}}^0(\mathcal{B})$ , but when we try to project the Piola transformed velocity down to a grid of order  $\mathcal{N}$  again, the measured divergence is still of the order  $10^{-10}$ .

What we would like is to complete the line in (4.20) with

$$\tilde{\mathbf{v}}_{3\mathcal{N}} \xrightarrow{P_{3\mathcal{N},\mathcal{N}}} \tilde{\mathbf{v}} \in X_{\mathcal{N}}^0(\mathcal{B}), \quad (4.22)$$

where  $P_{3\mathcal{N},\mathcal{N}}$  is a projection operator from the higher order grid to the original grid. Alternatively we would like to find a way to map

$$\mathbf{v}_i \in X_{\mathcal{N}}^0(\mathcal{B}_i) \xrightarrow{\Psi} \tilde{\mathbf{v}}_i \in X_{\mathcal{N}}^0(\mathcal{B}) \quad (4.23)$$

directly. The work on this issue is on-going, but for the moment Method 2 has to be solved in the intuitive, but non-conforming  $X_{\mathcal{N}}^0(\mathcal{B}) \not\subseteq X_{\mathcal{N}}^0(\mathcal{B})$  setting, or in the conforming, but odd  $\mathbb{P}_{3\mathcal{N}}/\mathbb{P}_{\mathcal{N}-2}$  setting.

#### 4.4 Noise regularization approach

When solving inverse problems one typically seeks the solution inside a domain, based on measurements on the domain boundary. In practice all the measurements are obtained within some error tolerance. The inverse problems are often ill-conditioned, and the error introduced in the measurements typically affects the computed solution severely. To remedy this, it is common to regularize the solution; see [12].

The algebraic system in Method 2 corresponding to (4.11) is

$$A\alpha = b, \quad (4.24)$$

where  $A \in \mathbb{R}^{N \times N}$ , and  $\alpha$  and  $b$  are vectors in  $\mathbb{R}^N$ . The solution  $\alpha = A^{-1}b$  gives the reduced basis solution

$$\mathbf{u}_N = \sum_{i=1}^N \alpha_i \mathbf{u}_i. \quad (4.25)$$

If we let  $b_0$  be a vector with the exact values, and  $e$  a vector with the errors introduced, we may decompose the right hand side of (4.24) as  $b = b_0 + e$ . The naive solution  $\alpha_{naive} = A^{-1}b$  is not necessarily an approximation to the exact solution  $\alpha_0 = A^{-1}b_0$ , if  $A$  is ill-conditioned. Since all the velocity basis functions used to construct the elements of  $A$  are relatively close in  $X_N$ ,  $A$  is very ill-conditioned. Regularization is thus appropriate, and we apply the simplest regularization technique called the truncated singular value decomposition (TSVD).

We follow the outline given in [12] for singular value decomposition of rectangular matrices, but since we get a symmetric positive definite stiffness matrix from Method 2, we proceed as follows. First we do an eigenvalue decomposition of the matrix  $A$  such that

$$A = W \Lambda W^T = \sum_{i=1}^N w_i \lambda_i w_i^T, \quad (4.26)$$

where  $W$  consists of orthonormal eigenvectors. Next we expand  $b$  and  $\alpha$  in terms of the eigenvectors to get

$$b = \sum_{i=1}^N (w_i^T b) w_i, \quad \alpha = \sum_{i=1}^N (w_i^T \alpha) w_i. \quad (4.27)$$

Together with  $Aw_i = \lambda_i w_i$  we then get

$$A\alpha = \sum_{i=1}^N \lambda_i (w_i^T \alpha) w_i, \quad (4.28)$$

and when we equate the expressions for  $b$  and  $A\alpha$  we get the relations  $(w_i^T b) = \lambda_i (w_i^T \alpha)$  for  $i = 1, \dots, N$ . Hence, we may write  $\alpha = A^{-1}b$  as

$$\alpha = \sum_{i=1}^N \frac{w_i^T b}{\lambda_i} w_i. \quad (4.29)$$

When the eigenvalues are ordered in a decreasing fashion, the quantities  $|w_i^T b|$  should decay faster than the eigenvalues; see [12] for theory and figures. When there is noise in the right hand side, the quantities  $|w_i^T b|$  decay faster than the eigenvalues for a while before they level off. As a consequence the absolute value of the  $\alpha$ -coefficients  $|\frac{w_i^T b}{\lambda_i}|$  grow rapidly. To get a better

approximation of the true solution than we get from (4.29), we may now truncate the  $\alpha$ -expansion where the quantities  $|w_i^T b|$  level off. In this way we find a solution which keeps the best parts of the basis functions, while the noisy parts are thrown away.

To see how the “noise” is introduced in Method 2 we take a closer look at the velocity basis functions  $\tilde{\mathbf{v}}_i \in X_N^0(\mathcal{B})$ . In (4.11) we assumed  $\mathbf{v} \in X_N^0(\mathcal{B}) \subset X_{\mathcal{N}}^0(\mathcal{B})$  to get  $l_{\mathcal{N}}(\mathbf{v}; \Phi)$  on the right hand side of the equation. The Piola transformed velocity basis functions  $\tilde{\mathbf{v}}_i$  are, as we have seen, not in  $X_{\mathcal{N}}^0(\mathcal{B})$ , and instead we get

$$l_{\mathcal{N}}(\tilde{\mathbf{v}}_i; \Phi) - b_{\mathcal{N}}(\tilde{\mathbf{v}}_i, p_N; \Phi) = \int_{\partial\mathcal{B}} \sigma_n \tilde{\mathbf{v}}_i \cdot \mathbf{n} ds + \int_{\mathcal{B}} p_N (\nabla \cdot \tilde{\mathbf{v}}_i) d\mathcal{B} \quad (4.30)$$

on the right hand side. Each  $\tilde{\mathbf{v}}_i$  may be viewed as  $\tilde{\mathbf{v}}_i = \tilde{\mathbf{u}}_i + \mathbf{e}_i$ , where  $\tilde{\mathbf{u}}_i \in X_{\mathcal{N}}^0(\mathcal{B})$ , and  $\mathbf{e}_i$  is “noise”. The right hand side of the algebraic system (4.24) comes from (4.30), and since  $\mathbf{e}_i \notin X_{\mathcal{N}}^0(\mathcal{B})$ , we end up with

$$b_i = - \int_{\partial\mathcal{B}} \tilde{\mathbf{u}}_i \cdot \mathbf{n} ds - \int_{\partial\mathcal{B}} \mathbf{e}_i \cdot \mathbf{n} ds + \int_{\mathcal{B}} p_N (\nabla \cdot \mathbf{e}_i) d\mathcal{B} = b_{i0} + e_i, \quad (4.31)$$

where  $b_{i0} = - \int_{\partial\mathcal{B}} \tilde{\mathbf{u}}_i \cdot \mathbf{n} ds$ . Since the basis functions are used to construct the stiffness matrix, we get noise in the system as well,

$$A_{ij} = a_{\mathcal{N}}(\tilde{\mathbf{u}}_i + \mathbf{e}_i, \tilde{\mathbf{u}}_j + \mathbf{e}_j). \quad (4.32)$$

If we for the moment ignore the noise in the system, the truncated singular value decomposition applies to Method 2. The truncation is performed by omitting  $\alpha$ -coefficients in (4.29) corresponding to eigenvalues smaller than  $10^{-10}$ ,

$$\alpha_k = \sum_{i=1}^k \frac{w_i^T b}{\lambda_i} w_i, \quad 10^{-10} > \lambda_{k+1} \geq \dots \geq \lambda_N. \quad (4.33)$$

This value is chosen since this is the size of the error in the divergence of the basis functions. The TSVD is not a Galerkin method, but the solution found should be better than the naive solution found by solving  $A\alpha = b$  directly. More error is introduced in  $b$  for each basis function added to the system, alas  $\alpha_{naive}$  diverges from the true  $\alpha$ . As a result  $\mathbf{u}_{N,naive} = \sum_{i=1}^N \alpha_{i,naive} \mathbf{u}_i$  diverges from  $\mathbf{u}_{\mathcal{N}}$ .

In Figure 7 we present the error of  $\mathbf{u}_{N,naive}$  together with the error of  $\mathbf{u}_N$  from Method 1 and the error of  $\mathbf{u}_k = \sum_{i=1}^k \alpha_{i,k} \mathbf{u}_i$  from the TSVD method. We see that when few basis functions are used the naive solution from Method 2 is better than the TSVD solution, but Method 2 accumulates more and more error as the number of basis functions are increased. The solution from Method 1 is superior to both the naive solution and the TSVD solution. Although the TSVD is not a Galerkin method it stabilizes at an error level somewhat higher than Method 1.

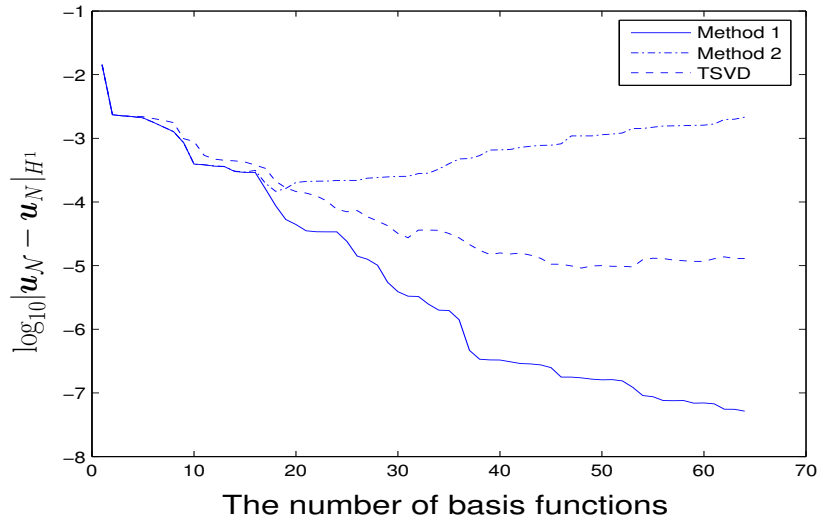


Figure 7: The reduced basis error in Method 1, Method 2 and TSVD when varying two geometric parameters.

We recall that the TSVD method applied here ignores the error in the stiffness matrix of the reduced basis system. To deal with the error in the stiffness matrix as well, we apply the truncated total least squares method presented in [10]. The results from this method are almost identical to the results from the TSVD method, indicating that the error in the stiffness matrix is negligible compared to the error in the right hand side. More sophisticated methods for regularization also exist, see [12] for an introduction to Tikhonov regularization. The motivation for applying more sophisticated methods would be to get results closer to the results from Method 1. This does not seem unreasonable, since the same basis functions are used to generate the reduced basis solutions in both Method 1 and Method 2.

#### 4.5 *A posteriori* error estimation

Since the approximation abilities of the reduced basis method strongly depends on the quality of the precomputed basis functions, we have no *a priori* knowledge of how well the reduced basis solution for a generic parameter will approximate the actual solution. To get an estimate of how good our solution is we need *a posteriori* error estimation. Based on the theory developed in [18], and following the strategies of [19] and [13], we have constructed the lower and upper output bounds  $s^-(\mathbf{u}_N)$  and  $s^+(\mathbf{u}_N)$  for the compliant output

$$s(\mathbf{u}) = l(\mathbf{u}). \quad (4.34)$$

On the reference bifurcation we introduce the diffusion operator

$$\hat{a}(\mathbf{v}, \mathbf{w}) = \int_{\hat{\mathcal{B}}} g(\mathcal{F}) \hat{\nabla}(\mathbf{v} \circ \mathcal{F}) \cdot \hat{\nabla}(\mathbf{w} \circ \mathcal{F}) d\hat{\mathcal{B}}, \quad (4.35)$$

where  $\mathbf{v}$  and  $\mathbf{w}$  are functions on  $\mathcal{B}$ , and  $g(\mathcal{F})$  is a positive function depending on the mapping  $\mathcal{F} : \hat{\mathcal{B}} \rightarrow \mathcal{B}$ . The reconstructed error is then defined as the field that for some  $g(\mathcal{F})$  satisfies

$$\hat{a}(\mathbf{e}, \mathbf{v}) = l(\mathbf{v}) - a(\mathbf{u}_N, \mathbf{v}) - b(\mathbf{v}, p_N) \quad \forall \mathbf{v} \in \tilde{X}_N, \quad (4.36)$$

where  $\tilde{X}_N = \{\mathbf{v}_e \circ \hat{\Phi}_e \in (\mathbb{P}_N(\hat{\Omega}))^2, \mathbf{v}|_{\Gamma_w} = 0\}$ . For this reconstructed error it is proven in [13] that if the operator  $g(\mathcal{F})$  is chosen such that

$$\alpha_0 \|\mathbf{v}\|_{\tilde{X}_N}^2 \leq \hat{a}(\mathbf{v}, \mathbf{v}) \leq a(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in \tilde{X}_N, \quad (4.37)$$

for some positive real constant  $\alpha_0$ , then the bounds defined by

$$s^-(\mathbf{u}_N) = l(\mathbf{u}_N), \quad \text{and} \quad (4.38)$$

$$s^+(\mathbf{u}_N) = l(\mathbf{u}_N) + \hat{a}(\mathbf{e}, \mathbf{e}), \quad (4.39)$$

satisfy

$$s^-(\mathbf{u}_N) \leq s(\mathbf{u}_N) \leq s^+(\mathbf{u}_N). \quad (4.40)$$

The positive function  $g(\mathcal{F})$  is defined through the Jacobian of  $\mathcal{F}$ ,  $\mathcal{J}_{\mathcal{F}}$ . The following relationship holds for  $\nabla = \mathcal{J}_{\mathcal{F}}^{-T} \hat{\nabla}$ ,

$$\begin{aligned} a(\mathbf{v}, \mathbf{v}) &= \int_{\mathcal{B}} \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathcal{B} \\ &= \int_{\hat{\mathcal{B}}} (\hat{\nabla}(\mathbf{v} \circ \mathcal{F}))^T \mathcal{J}_{\mathcal{F}}^{-1} \mathcal{J}_{\mathcal{F}}^{-T} \hat{\nabla}(\mathbf{v} \circ \mathcal{F}) |J_{\mathcal{F}}| d\hat{\mathcal{B}} \\ &= \int_{\hat{\mathcal{B}}} \mathbf{w}^T G \mathbf{w} d\hat{\mathcal{B}}, \end{aligned} \quad (4.41)$$

where  $\mathbf{w} = \hat{\nabla}(\mathbf{v} \circ \mathcal{F})$ , and  $G = G(\mathcal{F}) = (\mathcal{J}_{\mathcal{F}}^T \mathcal{J}_{\mathcal{F}})^{-1} |J_{\mathcal{F}}|$ . The smallest eigenvalue of  $G$  at each point  $\mathbf{x} \in \hat{\mathcal{B}}$  is then used to get

$$\int_{\hat{\mathcal{B}}} \mathbf{w}^T G \mathbf{w} d\hat{\mathcal{B}} \geq \int_{\hat{\mathcal{B}}} \Lambda_{\min}(Q\mathbf{w})^T Q\mathbf{w} d\hat{\mathcal{B}}. \quad (4.42)$$

We have that  $Q$  consists of the orthonormal eigenvectors of  $G$ , and we end up with

$$a(\mathbf{v}, \mathbf{v}) \geq \int_{\hat{\mathcal{B}}} \Lambda_{\min}(\mathcal{F}) \hat{\nabla}(\mathbf{v} \circ \mathcal{F}) \cdot \hat{\nabla}(\mathbf{v} \circ \mathcal{F}) d\hat{\mathcal{B}}. \quad (4.43)$$

By choosing  $g(\mathcal{F}) = \Lambda_{\min}(\mathcal{F})$  we thus satisfy (4.37) and (4.40). We may also choose  $g(\mathcal{F}) = \min_{\mathbf{x} \in \hat{\mathcal{B}}} \Lambda_{\min}(\mathcal{F})$ , and thereby put  $g(\mathcal{F})$  outside the integral. This is consistent with the theory of [18].

Using the opening angle as a parameter in the reduced basis method, we get the numerical results presented in Table 2. As we can see these results are astonishing, and much better than the results we got in [13]. The reason

$N$	$s(\mathbf{u}_N) - s^-(\mathbf{u}_N)$	$a(\mathbf{u}_N - \mathbf{u}_N, \mathbf{u}_N - \mathbf{u}_N)$	$s^+(\mathbf{u}_N) - s(\mathbf{u}_N)$
1	$4.22 \cdot 10^{-04}$	$4.22 \cdot 10^{-04}$	$1.72 \cdot 10^{-03}$
2	$6.10 \cdot 10^{-07}$	$6.10 \cdot 10^{-07}$	$2.10 \cdot 10^{-04}$
3	$4.48 \cdot 10^{-08}$	$4.49 \cdot 10^{-08}$	$2.02 \cdot 10^{-05}$
4	$3.32 \cdot 10^{-09}$	$3.34 \cdot 10^{-09}$	$3.13 \cdot 10^{-07}$
5	$7.95 \cdot 10^{-11}$	$7.93 \cdot 10^{-11}$	$3.31 \cdot 10^{-09}$
6	$1.84 \cdot 10^{-11}$	$1.82 \cdot 10^{-11}$	$4.87 \cdot 10^{-10}$
7	$1.81 \cdot 10^{-11}$	$1.79 \cdot 10^{-11}$	$4.21 \cdot 10^{-10}$
8	$1.77 \cdot 10^{-11}$	$1.75 \cdot 10^{-11}$	$4.20 \cdot 10^{-10}$

Table 2: The convergence of the lower output bound together with  $a(\mathbf{e}, \mathbf{e})$ , and the convergence of the upper output bound.

the results are so much better is the choice of basis functions used to generate the reduced basis solution. In [13] the underlying geometries of the basis functions were not optimized with respect to the generic geometry, while we in the present case vary only one parameter, and choose the generic geometry in the middle of the range of this single parameter. The best results from [13] was  $s(\mathbf{u}_N) - s^-(\mathbf{u}_N) = 1.94 \cdot 10^{-6}$  and  $s^+(\mathbf{u}_N) - s(\mathbf{u}_N) = 3.82 \cdot 10^{-4}$ . If we compare these results with the results on the bifurcations for  $N = 2$ , we see that the ratio between  $s(\mathbf{u}_N) - s^-(\mathbf{u}_N)$  and  $s^+(\mathbf{u}_N) - s(\mathbf{u}_N)$  is the same. This strengthens the reliability of the results, but more tests should be done in order to certify them. One such test could be to perform tests on a deformed pipe with similar simple geometry dependence, and see if the resulting bounds match the ones we get on the bifurcation. This has yet to be done.

For the reduced basis solution when varying two geometric parameters, the bound gaps converge as shown in Figure 8. Again the results are extremely good.

## 4.6 Output driven reduction

When we generate the reduced basis, the number of basis functions quickly increases when more parameters are introduced. Potentially we could end up with more than thousand basis functions, which would make the method rather costly and impractical. Fortunately, the basis functions typically contain much redundant information. Different post-processing techniques [4, 8] may be applied to reduce the necessary number of basis functions, while preserving the approximation capabilities of the generated basis.

We choose to find the basis functions best suited for the approximation of the output of interest  $s(\mathbf{u}) = l(\mathbf{u})$ . We follow the method presented in [21], where the output bound gap developed in Section 4.5 is used to reorder the



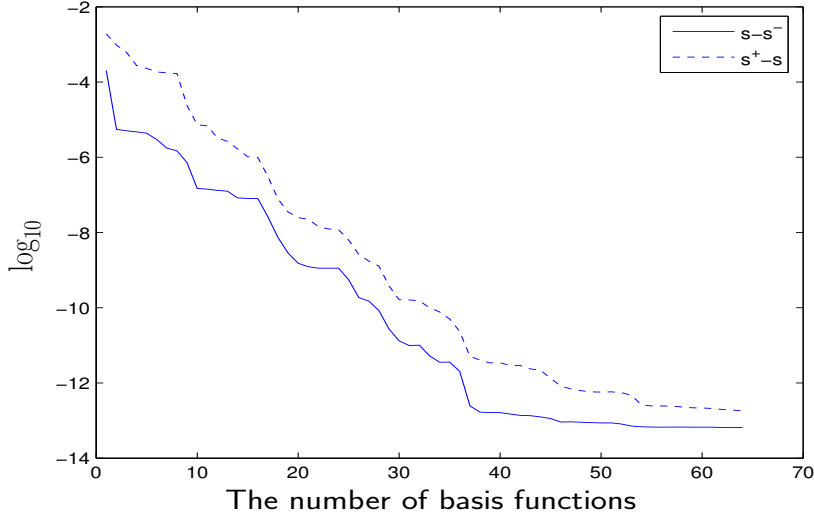


Figure 8: The bound gaps when varying two geometric parameters.

basis functions. Adapted to geometric parameters, we proceed as follows.

*Offline* we choose an arbitrary parameter  $\mu'_1 = \mu_i \in S_N$ , with corresponding geometry  $\mathcal{B}_i$  and basis functions  $\mathbf{u}_i, \mathbf{u}_i^e$ , and  $p_i$ . These basis functions are saved as  $\mathbf{v}_1, \mathbf{v}_1^e$ , and  $q_1$ , and they span the spaces  $X_{N'_1} = \{\mathbf{v}_1, \mathbf{v}_1^e\}$  and  $M_{N'_1} = \{q_1\}$ . For all  $\mu_j \in S_N \setminus \mu'_1$  we now solve

$$\begin{aligned} a_{\mathcal{N}}(\mathbf{u}_{N'_1}, \mathbf{v}; \Phi_j) + b_{\mathcal{N}}(\mathbf{v}, p_{N'_1}; \Phi_j) &= l_{\mathcal{N}}(\mathbf{v}; \Phi_j) & \mathbf{v} \in X_{N'_1}(\mathcal{B}_j) \\ b_{\mathcal{N}}(\mathbf{u}_{N'_1}, q; \Phi) &= 0 & q \in M_{N'_1}(\mathcal{B}_j). \end{aligned} \quad (4.44)$$

and calculate

$$\mu'_2 = \max_{\mu_j \in S_N \setminus \mu'_1} |s^+(\mathbf{u}_{N'_1}) - s^-(\mathbf{u}_{N'_1})|, \quad (4.45)$$

where  $\mathbf{u}_{N'_1}$  is the resulting reduced basis velocity. The basis functions corresponding to  $\mu'_2$  are saved as  $\mathbf{v}_2, \mathbf{v}_2^e$ , and  $q_2$ , and together with  $\mathbf{v}_1, \mathbf{v}_1^e$ , and  $q_1$  they span the spaces  $X_{N'_2}$  and  $M_{N'_2}$ . We denote  $S_{N'_2} = \{\mu'_1, \mu'_2\}$  and repeat the process above for all  $\mu_j \in S_N \setminus S_{N'_2}$ . In a recursive manner we thus choose  $\mu'_i$  with corresponding velocity and pressure basis functions until the maximum bound gap reaches a predefined level.

In the *online* computation of generic solutions, we then start with  $\mu'_1$  and its corresponding basis functions. We solve for the reduced basis solution, and calculate the bound gap. If the bound gap is larger than a specified limit, we include the basis functions corresponding to the next parameter in  $S_{N'}$ . The bound gap limit in the *online* case has to be larger than the bound gap limit used to sort the basis functions.

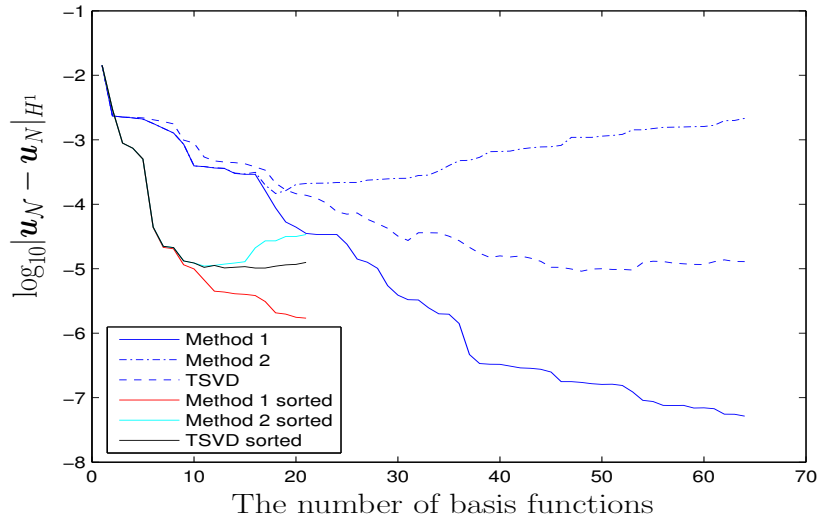


Figure 9: The reduced basis error when varying two geometric parameters.

In Figure 9 we see how the sorting of the basis affects the error convergence of the reduced basis solution. In all three methods, the error is of the order  $10^{-5}$  before 10 sorted basis functions are used. This plot shows the error compared to the truth solution  $\mathbf{u}_N$ , whereas we in real cases have to rely on the bound gap produced in the *a posteriori* error analysis.

## 5 A reduced basis element method

In this section we will combine the work from [13] on steady Stokes flow in pipes with the work on flow in bifurcations to realize a reduced basis element method with two fundamentally different building blocks: a pipe and a bifurcation; see Figure 10. To find the truth approximation in this case, we will use the mortar element method [5] to glue the two building blocks together. For the reduced basis element solution we also have to use Lagrange multipliers to glue the pieces together, and we extend the results from [13] to this case.

Since we have constructed the bifurcations with two elements on the inlet and outflow boundaries, the coupling with a single element pipe needs special care. Based on the mortar element method, we construct a conforming method for this non-conforming geometry. We refer to Figure 11, and let  $\mathbf{u}^e = \mathbf{u}|_{\Omega_e}$ , for  $e = 0, 1, 2$ . The pressure is in  $L^2$ , and does not need special care across this interface. We use polynomial degree  $\mathcal{N}$  on all elements. In order to get a conforming method, we thus choose  $\mathbf{u}^0$  to represent the true degrees of freedom, making  $\mathbf{u}^1$  and  $\mathbf{u}^2$  slave nodes in the jargon of the mortar element method. The values of  $\mathbf{u}^1$  and  $\mathbf{u}^2$  are determined by the

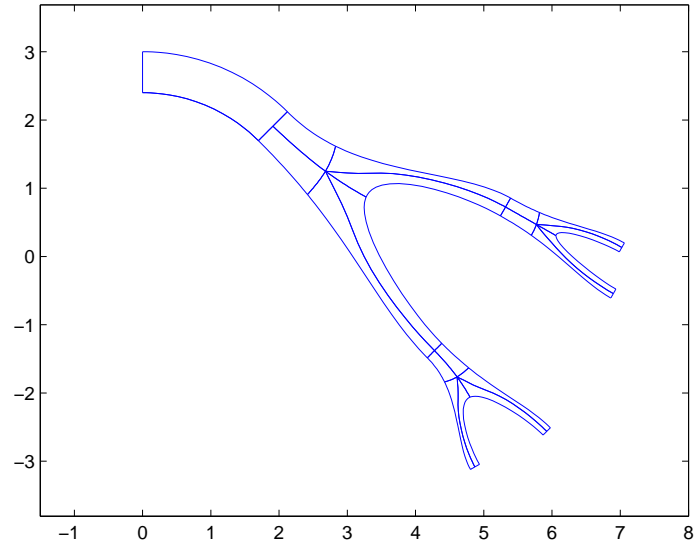


Figure 10: A system of one pipe and three bifurcations.

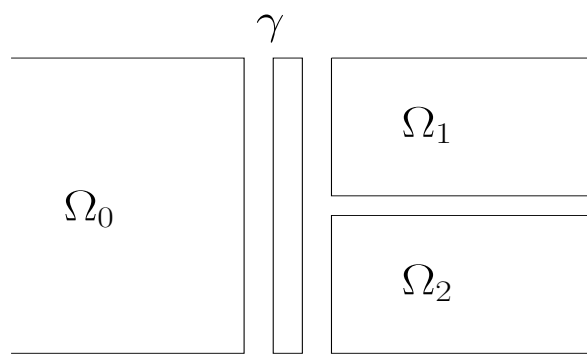


Figure 11: Interface in a non-conforming geometry.

mortar condition

$$\int_{\gamma} (\mathbf{u}^e - \mathbf{u}^0)|_{\gamma} \psi^e ds = 0 \quad \forall \psi^e \in \mathbb{P}_{\mathcal{N}-2}(\gamma_e), \quad e = 1, 2, \quad (5.1)$$

where  $\gamma_e = \gamma|_{\Omega_e}$ . See [9] for implementational details related to a similar domain decomposition.

For the hierarchical system shown in Figure 10, this procedure is used to compute a conforming reference solution to which the reduced basis solution will be compared. As usual, the reference solution is denoted  $(\mathbf{u}_{\mathcal{N}}, p_{\mathcal{N}})$ .

In order to use the reduced basis element method to find the solution of the steady Stokes equations on the hierarchical system in Figure 10, we need basis functions without prescribed boundary conditions. This is to allow unconstrained behaviour across the internal block boundaries, where a block refers to a pipe or an entire bifurcation. For pipes this was done in [13] and [16] by computing snapshot solutions on deformed pipes comprising two or three elements, and then restricting the solution to one of the elements. In a similar manner we now use snapshot solutions found on pipes consisting of three elements, restricted to the element containing the inlet boundary. This gives us  $N_P = 15$  pressure basis functions, 15 velocity basis functions and 15 enriched velocity basis functions for the pipe part of the hierarchical system.

To construct the basis functions for the bifurcation blocks without prescribed boundary conditions, we solve the steady Stokes equations on bifurcations defined as above, but now with a pipe added to both the inflow and the two outflow boundaries. The boundary conditions are then imposed on the outer perimeter of these geometries, and when the pipes are cut off, the solutions on the remaining bifurcations do not have any prescribed boundary conditions on the inflow and outflow boundaries. These solutions may now be used on the first bifurcation in Figure 10. For the two last bifurcations we construct solutions in the same way, but now with prescribed boundary conditions on the outflow boundaries. In addition to the solutions found, we reflect the solutions across a horizontal line through the bifurcations to produce their symmetric counterparts. Furthermore, we use the output driven selection method to find the basis functions that best approximates the steady Stokes solution on any given bifurcation within the span of the basis geometries. Initially we choose the same number of basis functions for the bifurcation blocks as we have for the pipe block, i.e.  $N_B = 15$ . To have the same number of basis functions for the bifurcation blocks as for the pipe block is convenient for implementational causes, but we may also choose a different number of basis functions for the bifurcation blocks; indeed, the same number of basis functions need not be used on the different bifurcation blocks.

In the reduced basis element method we have to glue the solution across interfaces together using Lagrange multipliers. This is similar to the mortar

element method used to get the conforming solution in the spectral element case described above. However, instead of using a high order Lagrange space we use linear Lagrange multipliers and end up with a non-conforming method. In [13] we argued that low order polynomials are easy to use as multipliers, and at the same time they give good results. The best results were achieved when using cubic polynomials in the normal direction, and linear polynomials in the tangential direction. We now choose to use linear polynomials in both directions, defined on one half of the interface, cf. Figure 11.

When Method 1 is used to solve the reduced basis element problem on the hierarchical system with one pipe and three bifurcations, we get  $N_P + 3N_B$  degrees of freedom from the pressure basis functions,  $N_P + 3N_B$  degrees of freedom from the original velocity basis functions, and  $N_P + 3N_B$  degrees of freedom from the velocity basis functions included to fulfill the inf-sup condition. Using four linear Lagrange multipliers in both the normal and tangential direction on each interface we get a total of  $N_C = 24$  constraints, all acting on the velocity basis functions. From spectral theory we know that the order of the velocity basis should be larger than the order of the pressure basis to avoid spurious pressure modes. This means that we have to choose  $N_P$  and  $N_B$  such that

$$N_P + 3N_B > N_C, \quad (5.2)$$

and for  $N_P = N_B$  we get  $N_P > 6$ . For  $N_P = N_B = 15$  the complete algebraic system thus has 204 unknowns, and we solve it directly. For comparison the corresponding spectral element system has more than 23000 degrees of freedom. In Figure 12 we see the contour of the error in the pressure between the spectral element solution and the reduced basis element solution when  $N_P = N_B = 15$ . In Figure 13 we see the contour of the corresponding velocity error. In both figures it seems like the error is no larger close to the block interfaces than elsewhere in the system, and it also seems like the error in the bifurcation blocks is somewhat larger than the error in the pipe block. If we double the number of basis functions used for the bifurcation blocks, we get the error contours shown in Figures 14 and 15 for the pressure and the velocity, respectively. We see now that at least the pressure error is much more evenly distributed throughout the hierarchical system. Also the velocity error behaves more uniformly, but we notice an increase in the error on the smallest bifurcation blocks.

In order to demonstrate the convergence of the reduced basis element method, we solve the steady Stokes equations in the reduced basis element setting for increasing  $N_P = N_B$ . The resulting error convergence is presented in Table 3, and we observe a nice and steady convergence rate.

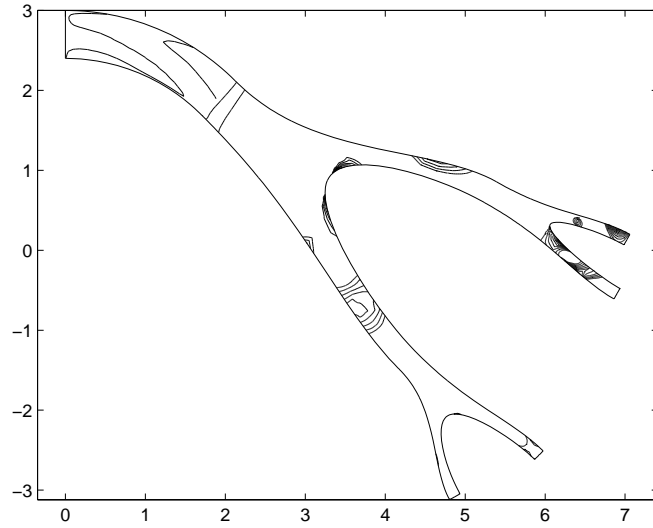


Figure 12: The contour of the pressure error in the reduced basis element solution for  $N_P = N_B = 15$ .

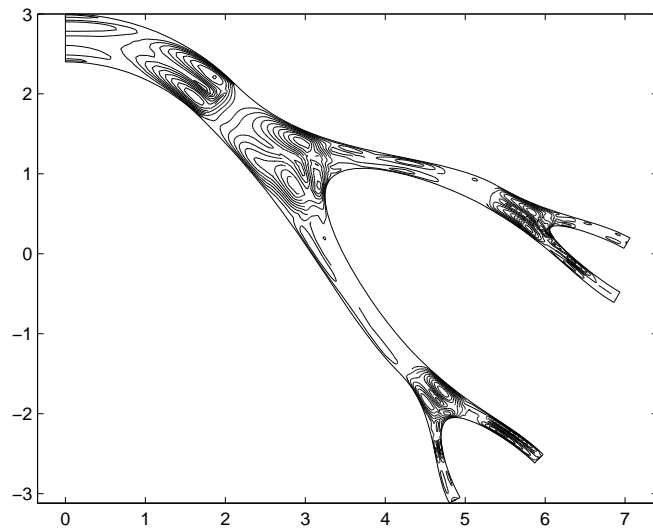


Figure 13: The contour of the velocity error in the reduced basis element solution for  $N_P = N_B = 15$ . The error in the velocity measured in the  $H^1$  seminorm is  $1.08 \cdot 10^{-3}$ , while the corresponding  $L^2$ -error in the pressure is  $3.69 \cdot 10^{-2}$ . The size of the algebraic system is 204.

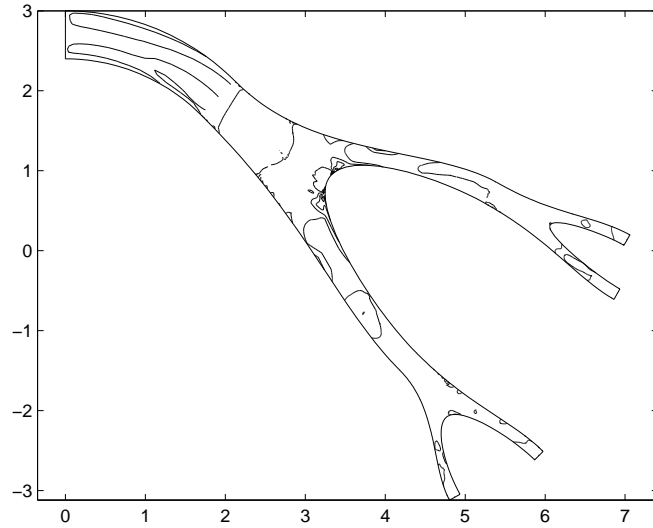


Figure 14: The contour of the pressure error in the reduced basis element solution for  $N_P = 15$  and  $N_B = 30$ .

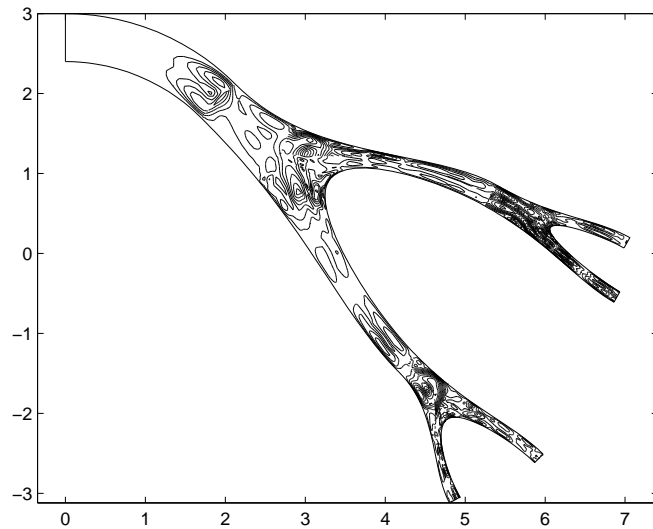


Figure 15: The contour of the velocity error in the reduced basis element solution for  $N_P = 15$  and  $N_B = 30$ . The error in the velocity measured in the  $H^1$  seminorm is  $4.18 \cdot 10^{-4}$ , while the corresponding  $L^2$ -error in the pressure is  $6.34 \cdot 10^{-3}$ . The size of the algebraic system is 339.

$N_P = N_B$	$ \mathbf{u}_N - \mathbf{u}_N _{H^1}$	$\ p_N - p_N\ _{L^2}$
7	$1.09 \cdot 10^{-2}$	$7.64 \cdot 10^0$
8	$4.00 \cdot 10^{-3}$	$8.20 \cdot 10^{-1}$
9	$2.60 \cdot 10^{-3}$	$3.99 \cdot 10^{-1}$
10	$1.81 \cdot 10^{-3}$	$1.24 \cdot 10^{-1}$
11	$1.72 \cdot 10^{-3}$	$6.56 \cdot 10^{-2}$
12	$1.25 \cdot 10^{-3}$	$5.16 \cdot 10^{-2}$
13	$1.22 \cdot 10^{-3}$	$4.87 \cdot 10^{-2}$
14	$1.19 \cdot 10^{-3}$	$3.80 \cdot 10^{-2}$
15	$1.08 \cdot 10^{-3}$	$3.69 \cdot 10^{-2}$

Table 3: The convergence of the reduced basis element error for the hierarchical system shown in Figure 10.

## 6 Conclusions

In this work we have demonstrated that the reduced basis method can be used to solve the steady Stokes problem on bifurcations, and where the geometry is considered to be the parameter. The steady Stokes problem is non-affine and non-linear with respect to the geometric parameters, thus some more computations have to be done *online* compared to the affine case; see [18].

Special attention has been given to the mapping of divergence free vector fields from one geometry to another. In particular, we have proposed a way to obtain global  $C^1$ -mappings (in a weak sense), thus allowing the Piola transformation to be used for complex geometries like bifurcations. We have also identified the limitations in maintaining the divergence free property in a discrete sense via such global mappings, and we have proposed an alternative way to handle this issue which is based on regularizing the resulting algebraic system of equations.

We have also exploited the “greedy” algorithm proposed in [21] for picking the basis functions best suited for a global approximation in a predefined subspace of the parameter space. This proved very useful for the reduced basis element method, where several building blocks were glued together in order to obtain the global solution of a hierarchical system of pipes and bifurcations.

Future work will focus on the divergence free mapping of vector fields, the extension of the proposed method to solve the Navier-Stokes equations, and the extension to the three-dimensional case. Further extensions include: solving unsteady problems in a reduced basis element framework; *a posteriori* error estimation in the multi-block case; and application of empirical interpolation [3] to improve the *online/offline* relationship.

**Acknowledgement.** *This work has been supported by the Research Council*



of Norway through the BeMatA programme under contract 147044/431, by the RTN project HaeMOdel HPRN-CT-2002-00270, and the ACI project "le-poumon-vous-dis-je" granted by the Fond National pour la Science. The support is gratefully acknowledged.

## References

- [1] R. Aris. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Publications, 1989.
- [2] I. Babuska. Error-bounds for finite element method. *Numer. Math.*, 16:322–333, 1971.
- [3] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera. An 'empirical interpolation' method: Application to efficient reduced-basis discretization of partial differential equations. *C. R. Acad. Sci. Paris, Serie I*, 339:667–672, 2004.
- [4] G. Berkooz, P. Holmes, and J. L. Lumley. The proper orthogonal decomposition in the analysis of turbulent flows. *Annu.Rev.Fluid Mech.*, 25:539–575, 1993.
- [5] C. Bernardi, Y. Maday, and A.T. Patera. A new nonconforming approach to domain decomposition: the mortar element method. *Nonlinear Partial Differential Equations and Their Applications, College de France seminar*, pages 13–51, 1990.
- [6] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers. *Rairo Anal. Numer.*, 8 R2:129–151, 1974.
- [7] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer Verlag, 1991.
- [8] J. Burkhardt, M. Gunzburger, and H. Lee. Centroidal voronoi tessellations-based reduced-order modeling of complex systems. *Submitted to SIAM J. Sci. Comput.*, 2005.
- [9] M.O. Deville, P.F. Fischer, and E.H. Mund. *High-Order Methods for Incompressible Fluid Flow*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge, 2002.
- [10] R.D. Fierro, G.H. Golub, P.C Hansen, and D.P. O'Leary. Regularization by truncated total least squares. *SIAM J. Sci. Comput.*, 18:1223–1241, 1997.

- [11] W. J. Gordon and C. A. Hall. Construction of curvilinear co-ordinate systems and applications to mesh generation. *Int J. Numer. Methods Eng.*, 7:461–477, 1973.
- [12] P. C. Hansen. Deconvolution and regularization with Toeplitz matrices. *Numer. Alg.*, 29:323–378, 2002.
- [13] A. E. Løvgrén, Y. Maday, and E. M. Rønquist. A reduced basis element method for the steady Stokes problem. *Submitted to M2AN*, 2004.
- [14] Y. Maday and A. T. Patera. Spectral element methods for the Navier-Stokes equations. In Noor A. (ed) *State of the Art Surveys in Computational Mechanics*, pages 71–143, 1989.
- [15] Y. Maday, A. T. Patera, and E. M. Rønquist. The  $P_N \times P_{N-2}$  method for the approximation of the Stokes problem. *Technical Report No. 92009, Department of Mechanical Engineering, Massachusetts Institute of Technology*, 1992.
- [16] Y. Maday and E. M. Rønquist. A reduced-basis element method. *J. Sci. Comput.*, 17:447–459, 2002.
- [17] Y. Maday and E. M. Rønquist. The reduced-basis element method: application to a thermal fin problem. *SIAM J. Sci. Comput.*, 26(1):240–258, 2004.
- [18] C. Prud’homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera, and G. Turinici. Reliable real-time solution of parametrized partial differential equations: Reduced basis output bound methods. *J. Fluids Engineering*, 124:70–80, 2002.
- [19] D. V. Rovas. *Reduced-Basis Output Bound Methods for Parametrized Partial Differential Equations*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, October 2002.
- [20] G. Rozza. Reduced-basis methods for elliptic equations in sub-domains with a posteriori error bounds and adaptivity. *To appear in Applied Numerical Mathematics*, 2005.
- [21] K. Veroy, C. Prud’homme, D. V. Rovas, and A. T. Patera. A *Posteriori* error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations (AIAA Paper 2003-3847). In *Proceedings of the 16th AIAA Computational Fluid Dynamics Conference*, June 2003.



# III

A reduced basis element method for  
the steady Stokes problem:  
Application to hierarchical flow  
systems

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Published in: Proceedings of SIMS 2005



# A reduced basis element method for the steady Stokes problem: Application to hierarchical flow systems

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## Abstract

The reduced basis element method is a new approach for approximating the solution of problems described by partial differential equations. The method takes its roots in domain decomposition methods and reduced basis discretizations [8, 16, 17], and its applications extend to, for example, control and optimization problems. The basic idea is to first decompose the computational domain into a series of subdomains that are similar to a few reference domains (or generic computational parts). Associated with each reference domain are pre-computed solutions corresponding to the same governing partial differential equation, but solved for different choices of some underlying parameter. In this work, the parameters are representing the geometric shape associated with a computational part. The approximation corresponding to a new shape is then taken to be a linear combination of the precomputed solutions, mapped from the reference domain for the part to the actual domain. We extend earlier work [14, 15] in this direction to solve incompressible fluid flow problems governed by the steady Stokes equations. Particular focus is given to constructing the basis functions, to the mapping of the velocity fields, to satisfying the inf-sup condition, and to “gluing” the local solutions together in the multidomain case [4]. We also demonstrate an algorithm for choosing the most efficient precomputed solutions. Two-dimensional examples are presented for pipes, bifurcations, and couplings of pipes and bifurcations in order to simulate hierarchical flow systems.

## 1 Introduction

The reduced basis element method is a new approach for approximating the solution of problems described by partial differential equations. The method takes its roots in domain decomposition methods and in reduced basis discretizations.

For a given parameter dependent problem: Find  $u \in X$  such that

$$F(u; \mu) = 0, \tag{1.1}$$

the computational effort needed to find an approximate discrete solution often makes the problem unsuitable for repetitive solves or real time control. Similarly, if (1.1) represents a very complex system, the resolution requirements may be so severe that even a single approximative solution may be hard to obtain.

The idea behind reduced basis methods is to precompute several solutions of (1.1),  $\{u_i\}_{i=1}^N$ , corresponding to a preselected set of parameter values,  $S_N = \{\mu_i\}_{i=1}^N$ . If the resolution of each  $u_i$  is represented by  $\mathcal{N}$ , then  $N \ll \mathcal{N}$ . These precomputed solutions (or “snapshots”) are then used as a basis for the solution space of (1.1) to find the reduced basis solution for a generic  $\mu$

$$u_N(\mu) = \sum_{i=1}^N \alpha_i(\mu) u_i, \quad (1.2)$$

where the coefficients  $\alpha_i(\mu)$  are determined through a Galerkin method. The error between the reduced basis solution  $u_N$  and the high resolution solution  $u_{\mathcal{N}}$  depends on the quality of the reduced basis space

$$V = \text{span}\{u_i, \quad i = 1, \dots, N\}, \quad (1.3)$$

and on the underlying regularity of  $u_{\mathcal{N}}$  with respect to  $\mu$ . As long as  $N$  is small, the work needed to find  $u_N$  is negligible. Examples of reduced basis methods following this computational approach include the Proper Orthogonal Decomposition [5], Centroidal Voronoi Tessellations [7], and Output Bound methods [11].

In the reduced basis element method we consider the geometry of the computational domain as the generic parameter. The domain is decomposed into smaller blocks, all of which can be considered to be deformations of a few reference shapes. Associated with each reference shape are precomputed solutions for different deformations of the shapes. The precomputed solutions are mapped from the reference shapes to the different blocks of the decomposed domain, and the solution on each block is found as a linear combination of the mapped precomputed solutions. The solutions on the different blocks are glued together using Lagrange multipliers.

We will in this work focus on hierarchical flow systems, which can be decomposed into pipes and bifurcations. We limit ourselves to the steady Stokes equations in the modeling of the flow through such systems, and in the next section we describe how the geometry enters the equations as a parameter. We use spectral elements in the modeling, but the method applies to other discretization techniques as well.

## 2 The steady Stokes problem

We consider here the two-dimensional steady Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega, \end{aligned} \quad (2.1)$$

where  $\mathbf{u} = (u_1, u_2)$  is the velocity field,  $p$  is the pressure,  $\mathbf{f} = (f_1, f_2)$  is a prescribed volumetric body force, and  $\nu$  is the fluid viscosity; see [1]. For all the problems studied in this paper, this model will suffice.

The domain  $\Omega$  has an inflow boundary  $\Gamma_{in}$ , an outflow boundary  $\Gamma_{out}$ , and wall boundaries  $\Gamma_w$ . On this domain we introduce the velocity space

$$X(\Omega) = \{\mathbf{v} \in (H^1(\Omega))^2, \mathbf{v}|_{\Gamma_w} = 0, v_{t|\Gamma_{in}} = v_{t|\Gamma_{out}} = 0\}, \quad (2.2)$$

where  $v_t$  is the tangential velocity component. In addition, we have the Neumann type boundary conditions given by specifying  $\sigma_n = \nu \frac{\partial u_n}{\partial n} - p$  to be  $\sigma_n^{in} = -1$  along  $\Gamma_{in}$  and  $\sigma_n^{out} = 0$  along  $\Gamma_{out}$ ; here,  $u_n$  is the normal velocity component and  $\partial/\partial n$  denotes the derivative in the outward normal direction. For all the problems solved in this study, the exact solution of (2.1) satisfies  $\frac{\partial u_n}{\partial n} = 0$  along  $\Gamma_{in}$  and  $\Gamma_{out}$ , which implies that the Neumann conditions correspond to specifying the pressure along the inflow and outflow boundaries (in a weak sense).

With the given boundary conditions, we define the pressure space to be

$$M(\Omega) = L^2(\Omega). \quad (2.3)$$

In order to solve the steady Stokes equations we define the bilinear forms

$$a(\mathbf{v}, \mathbf{w}) = \nu \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{w} \, d\Omega, \quad (2.4)$$

$$b(\mathbf{v}, q) = - \int_{\Omega} q \nabla \cdot \mathbf{v} \, d\Omega, \quad (2.5)$$

and consider the weak form: Find  $\mathbf{u} \in X(\Omega)$  and  $p \in M(\Omega)$  such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= l(\mathbf{v}) & \forall \mathbf{v} \in X(\Omega) \\ b(\mathbf{u}, q) &= 0 & \forall q \in M(\Omega), \end{aligned} \quad (2.6)$$

where

$$l(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \int_{\Gamma_{in}} \sigma_n^{in} \mathbf{v} \cdot \mathbf{n} \, ds + \int_{\Gamma_{out}} \sigma_n^{out} \mathbf{v} \cdot \mathbf{n} \, ds. \quad (2.7)$$

For all the problems considered in this work, the body force  $\mathbf{f}$  will be zero.

To ensure a unique solution of the steady Stokes problem (2.6), the coercivity condition

$$a(\mathbf{w}, \mathbf{w}) \geq \alpha \|\mathbf{w}\|_{H^1(\Omega)}^2, \quad \forall \mathbf{w} \in X(\Omega), \quad \alpha > 0, \quad (2.8)$$



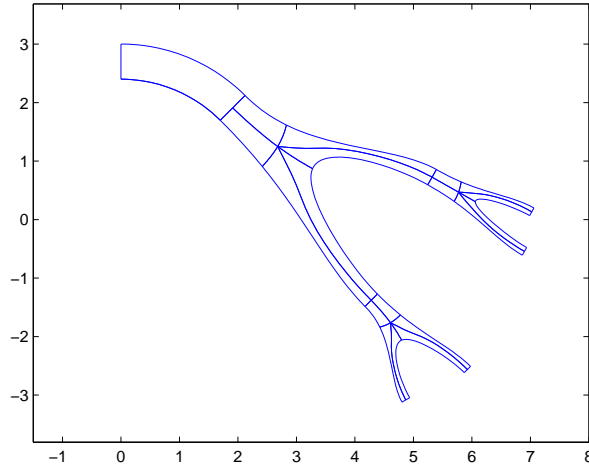


Figure 1: A hierarchical flow system with one pipe-block and three bifurcation-blocks.

and the inf-sup condition

$$\inf_{q \in M(\Omega)} \sup_{\mathbf{v} \in X(\Omega)} \frac{b(\mathbf{v}, q)}{\|q\|_{L^2(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)}} = \beta > 0, \quad (2.9)$$

must be satisfied; see [2] and [6]. These conditions are fulfilled for our particular Stokes problem.

## 2.1 Discretization

We consider here a two dimensional computational domain  $\Omega$ , which represents a flow system as depicted in Figure 1. We assume that the domain can be decomposed as a union of  $E$  non-overlapping subdomains  $\Omega_e$ ,  $e = 1, \dots, E$ , with each subdomain representing a deformed square. Each deformed square is again a regular one-to-one deformation,  $\phi_e$ , of the reference square  $\hat{\Omega} = (-1, 1)^2$ , i.e.,

$$\overline{\Omega} = \bigcup_{e=1}^E \overline{\Omega_e} = \bigcup_{e=1}^E \overline{\phi_e(\hat{\Omega})} \equiv \overline{\Phi(\hat{\Omega})}. \quad (2.10)$$

In our case, each subdomain will be considered to be a single spectral element; see [12]. Let  $\mathbb{P}_n(\hat{\Omega})$  be the space of all functions which are polynomials of degree less than or equal to  $n$  in each spatial direction on  $\hat{\Omega}$ . For  $\mathbf{v}_e = \mathbf{v}|_{\Omega_e}$ , the discrete space for the velocity is then taken to be

$$X_{\mathcal{N}}(\Omega) = \{\mathbf{v} \in X(\Omega), \quad \mathbf{v}_e \circ \phi_e \in (\mathbb{P}_{\mathcal{N}}(\hat{\Omega}))^2, \quad e = 1, \dots, E\}, \quad (2.11)$$

while the discrete space for the pressure is

$$M_{\mathcal{N}}(\Omega) = \{q \in M(\Omega), \quad q_e \circ \phi_e \in \mathbb{P}_{\mathcal{N}-2}(\hat{\Omega}), \quad e = 1, \dots, E\}. \quad (2.12)$$

The bases for  $X_{\mathcal{N}}(\Omega)$  and  $M_{\mathcal{N}}(\Omega)$  are conveniently expressed in terms of the reference variables  $\xi$  and  $\eta$ . As a basis for  $X_{\mathcal{N}}(\Omega)$  we use a nodal basis through the tensor-product Gauss-Lobatto Legendre (GLL) points, while the basis for  $M_{\mathcal{N}}(\Omega)$  is a nodal basis through the tensor-product Gauss-Legendre (GL) points; see [12] and [13]. Specifically, we write

$$(\mathbf{u}_e \circ \phi_e)(\xi, \eta) = \sum_{i,j=0}^{\mathcal{N}} \mathbf{u}_{ij}^e \ell_i(\xi) \ell_j(\eta) \quad (2.13)$$

where  $\ell_i(\xi)$  refers to a one-dimensional  $\mathcal{N}$ -th order Lagrangian interpolant through the GLL points  $\xi_m$ ,  $m = 0, \dots, \mathcal{N}$ ; here,  $\ell_i(\xi_m) \ell_j(\xi_n) = \delta_{im} \delta_{jn}$  for a given point  $(\xi_m, \xi_n)$  in the underlying tensor-product GLL grid.

In a similar fashion, we write

$$(p_e \circ \phi_e)(\xi, \eta) = \sum_{i,j=0}^{\mathcal{N}-2} p_{ij}^e \tilde{\ell}_i(\xi) \tilde{\ell}_j(\eta), \quad (2.14)$$

where  $\tilde{\ell}_i(\xi)$  refers to a one-dimensional  $(\mathcal{N} - 2)$ -th order Lagrangian interpolant through the (interior) GL points  $\zeta_m$ ,  $m = 0, \dots, \mathcal{N} - 2$ ; here,  $\tilde{\ell}_i(\zeta_m) \tilde{\ell}_j(\zeta_n) = \delta_{im} \delta_{jn}$  for a given point in the tensor-product GL grid.

Based on (2.10), where we have defined the global mapping  $\Omega = \Phi(\hat{\Omega})$ , we express the bilinear forms (2.4) and (2.5) in terms of the reference variables  $\xi$  and  $\eta$  as

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \sum_{e=1}^E a(\mathbf{v}, \mathbf{w}; \phi_e), \quad (2.15)$$

$$b(\mathbf{v}, p, \Phi) = \sum_{e=1}^E b(\mathbf{v}, p, \phi_e). \quad (2.16)$$

The elemental contributions to these sums are

$$a(\mathbf{v}, \mathbf{w}; \phi_e) = \nu \int_{\hat{\Omega}} \mathcal{J}_e^{-T} \hat{\nabla}(\mathbf{v}_e \circ \phi_e) \cdot \mathcal{J}_e^{-T} \hat{\nabla}(\mathbf{w}_e \circ \phi_e) |J_e| d\hat{\Omega} \quad (2.17)$$

$$b(\mathbf{v}, p; \phi_e) = - \int_{\hat{\Omega}} (p_e \circ \phi_e) \hat{\nabla} \cdot [\mathcal{J}_e^{-1}(\mathbf{v}_e \circ \phi_e)] |J_e| d\hat{\Omega}, \quad (2.18)$$

where  $\mathcal{J}_e$  is the Jacobian of  $\phi_e$ , and  $J_e$  its determinant. The operator  $\hat{\nabla} = [\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}]^T$ . This gives us the following discrete system: Find  $\mathbf{u}_{\mathcal{N}} \in X_{\mathcal{N}}(\Omega)$  and  $p_{\mathcal{N}} \in M_{\mathcal{N}}(\Omega)$  such that

$$\begin{aligned} a_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, \mathbf{v}; \Phi) + b_{\mathcal{N}}(\mathbf{v}, p_{\mathcal{N}}; \Phi) &= l_{\mathcal{N}}(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_{\mathcal{N}}(\Omega) \\ b_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, q; \Phi) &= 0 \quad \forall q \in M_{\mathcal{N}}(\Omega), \end{aligned} \quad (2.19)$$

where  $a_{\mathcal{N}}$ ,  $b_{\mathcal{N}}$  and  $l_{\mathcal{N}}$  refer to integration of the bilinear and linear forms using Gauss-type quadrature. We thus see that the geometry enters the equations as a parameter via the mapping  $\Phi$ , or more specifically, through the elemental mappings  $\phi_e$ ,  $e = 1, \dots, E$ .

### 3 The reduced basis

We now define the reduced basis solution spaces  $X_N(\Omega) \subset X_{\mathcal{N}}(\Omega)$  and  $M_N(\Omega) \subset M_{\mathcal{N}}(\Omega)$ . Our objective is to find a unique reduced basis solution  $\mathbf{u}_N \in X_N(\Omega)$  and  $p_N \in M_N(\Omega)$  satisfying

$$\begin{aligned} a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}; \Phi) + b_{\mathcal{N}}(\mathbf{v}, p_N; \Phi) &= l_{\mathcal{N}}(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_N(\Omega) \\ b_{\mathcal{N}}(\mathbf{u}_N, q; \Phi) &= 0 \quad \forall q \in M_N(\Omega). \end{aligned} \quad (3.1)$$

As before the coercivity of  $a(\cdot, \cdot; \Phi)$  holds for all  $\mathbf{v} \in X_N(\Omega)$ , since it is a subset of  $X_{\mathcal{N}}(\Omega)$ . The inf-sup condition (2.9), however, depends strongly on  $X_N(\Omega)$  and  $M_N(\Omega)$ .

In a hierarchical flow system we differ between building blocks with pipe structure, and building blocks with bifurcation structure. The precomputation of basis functions is done separately for the two types of building blocks. By grouping the spectral elements of  $\Omega$  into building blocks,  $\mathcal{B}^k = \Phi^k(\hat{\Omega})$ , each comprising  $E^k$  spectral elements, we may write

$$\overline{\Omega} = \left( \bigcup_{k=1}^{K_1} \overline{\mathcal{B}^k} \right) \bigcup \left( \bigcup_{k=K_1+1}^{K_1+K_2} \overline{\mathcal{B}^k} \right), \quad (3.2)$$

where  $k = 1, \dots, K_1$  indicates the pipe blocks, and  $k = K_1 + 1, \dots, K_1 + K_2$  indicates the bifurcation blocks. Each building block can again be expressed as

$$\overline{\mathcal{B}^k} = \bigcup_{e=1}^{E^k} \overline{\phi_e^k(\hat{\Omega})} \equiv \overline{\Phi^k(\hat{\Omega})} \quad (3.3)$$

for  $k = 1, \dots, K_1 + K_2$ , and with  $\phi_e^k(\hat{\Omega})$  denoting the mapping of the reference square to each individual spectral element in the building block. Since each block  $\mathcal{B}^k$  may consist of several spectral elements, we have, in general,  $K_1 + K_2 < E$ . We will also denote the restriction of a field  $\mathbf{v}$  to a block  $\mathcal{B}^k$  as  $\mathbf{v}^k$ .

In this work, a pipe building block consists of a single spectral element, i.e.,  $E^k = 1$ ,  $k = 1, \dots, K_1$ . In order to generate the basis functions to be used on a pipe, we solve the steady Stokes problem (2.19) with the same possible boundary conditions as on  $\{\mathcal{B}^k\}_{k=1}^{K_1}$  for a preselected set of deformations of the reference domain,  $\{\Phi_i : \hat{\Omega} \rightarrow \Omega_i\}_{i=1}^{N_1}$ . This is achieved by solving the steady Stokes problem on a deformed pipe comprising three spectral elements. The restriction of the solution to each of these three elements will thus take care of the three possible types of pipe segments

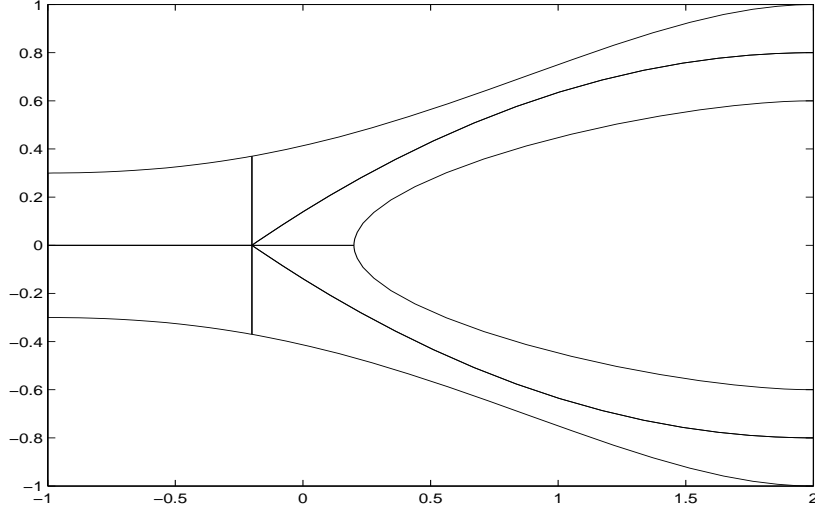


Figure 2: The reference bifurcation  $\hat{\mathcal{B}}$  constructed from six elements.

(inflow, interior, outflow) in the hierarchical flow system. The resulting velocity fields,  $\{\mathbf{u}_i\}_{i=1}^{3N_1}$ , are then mapped to the reference domain  $\hat{\Omega}$  by the Piola transformation

$$\hat{\mathbf{u}}_i = \Psi_i(\mathbf{u}_i) = \mathcal{J}_i^{-1}(\mathbf{u}_i \circ \Phi_i) |J_i|. \quad (3.4)$$

In this way, the direction of the velocity relative to the geometry is preserved, and by construction  $b(\mathbf{u}_i, q; \Phi_i) = b(\hat{\mathbf{u}}_i, q \circ \Phi_i; I)$ , where  $I$  is the identity mapping. Hence, each precomputed velocity field  $\hat{\mathbf{u}}_i$  is also incompressible on  $\hat{\Omega}$ . A similar result holds when the inverse Piola transformation is applied in order to map the velocity from the reference domain to the generic pipe block  $\mathcal{B}^k$ ,

$$\tilde{\mathbf{u}}_i^k = (\Psi^k)^{-1}(\hat{\mathbf{u}}_i) = \mathcal{J}^k(\hat{\mathbf{u}}_i \circ (\Phi^k)^{-1}) |(J^k)^{-1}|. \quad (3.5)$$

The pressure is a scalar field, and is mapped from  $\Omega_i$  to  $\mathcal{B}^k$  through  $\tilde{p}_i^k = p_i \circ \Phi_i \circ (\Phi^k)^{-1}$ . Only the  $N_1$  fields with the proper boundary conditions are used as basis functions on each  $\mathcal{B}^k$ .

A building block which represents a bifurcation must necessarily comprise several spectral elements. In this work, we use six elements to build each bifurcation; see Figure 2. Hence,  $E^k = 6$  for  $k = K_1 + 1, \dots, K_1 + K_2$  in (3.3). The basis functions associated with bifurcations are constructed by solving the steady Stokes equations on a preselected set of deformed bifurcations  $\{\mathcal{B}_i = \mathcal{F}_i(\hat{\mathcal{B}})\}_{i=1}^{N_2}$ , where  $\hat{\mathcal{B}}$  is a reference bifurcation and  $\mathcal{F}_i$  is a regular mapping. To take care of the different boundary conditions needed in the hierarchical flow system, basis functions are constructed with pipes added to either the inflow boundary, or the outflow boundaries of the bifurcations,

or both. Only the restriction of the solutions to the bifurcation blocks are used as basis functions. The resulting velocity solutions,  $\{\mathbf{u}_i\}_{i=1}^{3N_2}$ , are again mapped to the reference domain  $\hat{\Omega}$  by the Piola transformation. In contrast to the solutions found on pipes, all velocity (and pressure) solutions comprise six segments,  $\mathbf{u}_i = \{\mathbf{u}_{ie}\}_{e=1}^6$ , and they are mapped to the reference domain one element at a time. On each  $\{\mathcal{B}^k\}_{K_1+1}^{K_1+K_2}$  we use the  $N_2$  basis functions found on deformed bifurcations with the right boundary conditions, mapped to the generic domain by the inverse Piola transformation,

$$\tilde{\mathbf{u}}_i^k = \{\tilde{\mathbf{u}}_{ie}^k\}_{e=1}^6 = \{(\Psi_e^k)^{-1}(\hat{\mathbf{u}}_{ie})\}_{e=1}^6. \quad (3.6)$$

Now, on each of the six elements of a bifurcation the Jacobian is smooth and continuous, but across internal interfaces in a bifurcation it is not. To ensure that  $\tilde{\mathbf{u}}_i^k$  is a continuous function we must define both the preselected bifurcations  $\{\mathcal{B}_i\}_{i=1}^{N_2}$ , and the actual bifurcations in the hierarchical system  $\{\mathcal{B}^k\}_{k=1}^{K_2}$ , as  $C^1$  deformations of the same reference bifurcation  $\hat{B}$ . A method for achieving this is presented in [10].

The pressure solutions on the deformed bifurcations also comprise six elemental contributions, and on the generic bifurcation  $\mathcal{B}^k$ , these are evaluated as  $\tilde{p}_i^k = \{\tilde{p}_{ie}^k\}_{e=1}^6 = \{p_{ie} \circ \phi_{ie} \circ (\phi_e^k)^{-1}\}_{e=1}^6$ .

We define the spaces

$$\begin{aligned} Y_N^0(\Omega) &= \begin{cases} \text{span}\{\tilde{\mathbf{u}}_i^k\}_{i=1}^{N_1}, & k = 1, \dots, K_1 \\ \text{span}\{\tilde{\mathbf{u}}_i^k\}_{i=1}^{N_2}, & k = K_1 + 1, \dots, K_1 + K_2 \end{cases} \\ M_N(\Omega) &= \begin{cases} \text{span}\{\tilde{p}_i^k\}_{i=1}^{N_1}, & k = 1, \dots, K_1 \\ \text{span}\{\tilde{p}_i^k\}_{i=1}^{N_2}, & k = K_1 + 1, \dots, K_1 + K_2, \end{cases} \end{aligned} \quad (3.7)$$

for which we know that the inf-sup condition is not fulfilled since the velocity fields in  $Y_N^0(\Omega)$  are all divergence free. The index  $N$  denotes the dimension of  $Y_N^0(\Omega)$  and  $M_N(\Omega)$ , and may be expressed as

$$N = N_1 K_1 + N_2 K_2. \quad (3.8)$$

Recall that  $N_1$  is the number of precomputed basis functions for the  $K_1$  pipe blocks, and  $N_2$  is the number of precomputed basis functions for the  $K_2$  bifurcation blocks. We have that  $N_1 \ll \mathcal{N}$  and  $N_2 \ll \mathcal{N}$ , and both are independent of the number of spatial dimensions.

We also need to enforce a continuity condition across the block interfaces in  $\Omega$ ,  $\bar{\Gamma}_{kl} = \bar{\mathcal{B}}^k \cap \bar{\mathcal{B}}^l$ . We try to minimize the jump across these block interfaces by introducing the constraints

$$\int_{\Gamma_{kl}} (\mathbf{v}^k - \mathbf{v}^l) \cdot \mathbf{n} \psi \, ds = 0, \quad \forall \psi \in W_{k,l}^n, \quad \forall k, l, \quad (3.9)$$

and

$$\int_{\Gamma_{kl}} (\mathbf{v}^k - \mathbf{v}^l) \cdot \mathbf{t} \psi \, ds = 0, \quad \forall \psi \in W_{k,l}^t, \quad \forall k, l, \quad (3.10)$$

where  $\mathbf{n}$  is the unit normal vector of  $\Gamma_{kl}$ ,  $\mathbf{t}$  is the unit tangential vector, and  $W_{k,l}^n$  and  $W_{k,l}^t$  are spaces of low order polynomials defined on  $\Gamma_{kl}$ . In [9] it is shown that the order of these polynomial spaces can be used to control the jump across the interfaces for multi-element pipes. We thus define the reduced basis velocity space

$$X_N^0(\Omega) = \{ \mathbf{v} \in Y_N^0(\Omega), \text{ (3.9) and (3.10) hold } \}, \quad (3.11)$$

and remark that  $X_N^0(\Omega) \not\subset X_{\mathcal{N}}(\Omega)$  due to the jump across the block interfaces.

If we are not interested in the pressure on the generic domain, we may solve the problem: Find  $\mathbf{u}_N$  in  $X_N^0(\Omega)$  such that

$$a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}; \Phi) = l_{\mathcal{N}}(\mathbf{v}; \Phi), \quad \forall \mathbf{v} \in X_N^0(\Omega). \quad (3.12)$$

In this case the inf-sup condition is insignificant. If we also want to find the pressure however, the reduced basis velocity space  $X_N(\Omega)$  must be enriched. This is due to the fact that the space  $X_N^0$  is spanned by divergence free basis functions. We define the enriched space as  $X_N(\Omega) = X_N^0(\Omega) \oplus X_N^e(\Omega)$ , where  $X_N^e$  consists of velocity fields constructed in order to guarantee the inf-sup condition, together with the constraints in (3.9) and (3.10); see [9] and [10] for details on how to construct these velocity fields.

The inf-sup condition (2.9) is then fulfilled for the spaces  $M_N(\Omega)$  and  $X_N(\Omega)$ , and we may solve (3.1) to find both the velocity and the pressure simultaneously involving a system of size  $3N$ . Alternatively, we could solve the two separate  $N$ -sized problems (3.12) and

$$b_{\mathcal{N}}(\mathbf{v}, p_N; \Phi) = -a_{\mathcal{N}}(\mathbf{u}_N, \mathbf{v}; \Phi) + l_{\mathcal{N}}(\mathbf{v}; \Phi), \quad \forall \mathbf{v} \in X_N^e(\Omega), \quad (3.13)$$

for the velocity and pressure, respectively. Note that neither  $X_N^0(\Omega)$  nor  $X_N^e(\Omega)$  is a subset of  $X_{\mathcal{N}}(\Omega)$ , and that both methods are non-conforming.

## 4 *A posteriori* error estimation

Since the approximation abilities of the reduced basis method strongly depends on the quality of the precomputed basis functions, we have no *a priori* knowledge of how well the reduced basis solution for a generic parameter will approximate the actual solution. To get an estimate of how good our solution is we need *a posteriori* error estimation. Based on the theory developed in [17], and following the strategy of [18], the lower and upper output bounds,  $s^-(\mathbf{u}_N)$  and  $s^+(\mathbf{u}_N)$ , for the compliant output

$$s(\mathbf{u}) = l(\mathbf{u}), \quad (4.1)$$

was constructed in [9] in the single block case, where  $X_N(\Omega) \subset X_{\mathcal{N}}(\Omega)$ .

For a diffusion operator on the reference domain  $\hat{\Omega}$ ,

$$\hat{a}(\mathbf{v}, \mathbf{w}; \Phi) = \int_{\hat{\Omega}} g(\Phi) \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \hat{\nabla}(\mathbf{w} \circ \Phi) d\hat{\Omega}, \quad (4.2)$$

where  $\mathbf{v}$  and  $\mathbf{w}$  are functions on  $\mathcal{B}$ , and  $g(\Phi)$  is a positive function depending on the mapping  $\Phi : \hat{\Omega} \rightarrow \Omega$ ,  $g(\Phi)$  is chosen such that

$$\alpha_0 \|\mathbf{v}\|_{X_{\mathcal{N}}}^2 \leq \hat{a}(\mathbf{v}, \mathbf{v}) \leq a(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in X_{\mathcal{N}}(\Omega), \quad (4.3)$$

for some positive real constant  $\alpha_0$ . The reconstructed error  $\mathbf{e}$  is then defined as the field that satisfies

$$\hat{a}(\mathbf{e}, \mathbf{v}; \Phi) = l(\mathbf{v}; \Phi) - a(\mathbf{u}_N, \mathbf{v}; \Phi) - b(\mathbf{v}, p_N; \Phi) \quad \forall \mathbf{v} \in \tilde{X}_{\mathcal{N}}(\Omega), \quad (4.4)$$

where  $\tilde{X}_{\mathcal{N}}(\Omega) = \{\mathbf{v} \circ \Phi \in (\mathbb{P}_{\mathcal{N}}(\hat{\Omega}))^2, \mathbf{v}|_{\Gamma_w} = 0\}$ . For the bounds defined by

$$s^-(\mathbf{u}_N) = l(\mathbf{u}_N), \quad \text{and} \quad (4.5)$$

$$s^+(\mathbf{u}_N) = l(\mathbf{u}_N) + \hat{a}(\mathbf{e}, \mathbf{e}), \quad (4.6)$$

we then get

$$s^-(\mathbf{u}_N) \leq s(\mathbf{u}_N) \leq s^+(\mathbf{u}_N). \quad (4.7)$$

In the non-conforming setting when  $\Omega$  consists of several blocks, the work is on-going.

## 4.1 Output driven reduction

When we generate the reduced basis, the number of basis functions quickly increases when more parameters are introduced. Potentially we could end up with more than thousand basis functions, which would make the method rather costly and impractical. Fortunately, the basis functions typically contain much redundant information. Different post-processing techniques [5, 7, 11] may be applied to reduce the number of basis functions needed, while preserving the approximation capabilities of the generated basis.

We follow the method presented in [19], where the output bound gap developed in Section 4 is used to reorder the basis functions, such that the error in the output of interest,  $s(\mathbf{u})$ , is minimized. We also recall that  $S_N$  is the set of preselected parameter values. Adapted to geometric parameters, we proceed as follows, separately for the different block structures, i.e., pipe and bifurcation.

*Offline* we choose an arbitrary parameter  $\mu'_1 = \mu_i \in S_N$ , with corresponding geometry  $\mathcal{B}_i$  and basis functions  $\mathbf{u}_i, \mathbf{u}_i^e$ , and  $p_i$ . These basis functions are saved as  $\mathbf{v}_1, \mathbf{v}_1^e$ , and  $q_1$ , and they span the spaces  $X_{N'_1} = \{\mathbf{v}_1, \mathbf{v}_1^e\}$  and  $M_{N'_1} = \{q_1\}$ . For all  $\mu_j \in S_N \setminus \mu'_1$  we now solve

$$\begin{aligned} a_{\mathcal{N}}(\mathbf{u}_{N'_1}, \mathbf{v}; \Phi_j) + b_{\mathcal{N}}(\mathbf{v}, p_{N'_1}; \Phi_j) &= l_{\mathcal{N}}(\mathbf{v}; \Phi_j) \quad \mathbf{v} \in X_{N'_1}(\mathcal{B}_j) \\ b_{\mathcal{N}}(\mathbf{u}_{N'_1}, q; \Phi) &= 0 \quad q \in M_{N'_1}(\mathcal{B}_j). \end{aligned} \quad (4.8)$$

and calculate

$$\mu'_2 = \max_{\mu_j \in S_N \setminus \mu'_1} |s^+(\mathbf{u}_{N'_1}) - s^-(\mathbf{u}_{N'_1})|, \quad (4.9)$$

where  $\mathbf{u}_{N'_1}$  is the resulting reduced basis velocity. The basis functions corresponding to  $\mu'_2$  are saved as  $\mathbf{v}_2, \mathbf{v}_2^e$ , and  $q_2$ , and together with  $\mathbf{v}_1, \mathbf{v}_1^e$ , and  $q_1$  they span the spaces  $X_{N'_2}$  and  $M_{N'_2}$ . We denote  $S_{N'_2} = \{\mu'_1, \mu'_2\}$  and repeat the process above for all  $\mu_j \in S_N \setminus S_{N'_2}$ . In a recursive manner we thus choose  $\mu'_i$  with corresponding velocity and pressure basis functions until the maximum bound gap reaches a predefined level.

In the *online* computation of generic solutions, we then start with  $\mu'_1$  and its corresponding basis functions. We solve for the reduced basis solution, and calculate the bound gap. If the bound gap is larger than a specified limit, we include the basis functions corresponding to the next parameter in  $S_{N'}$ . The bound gap limit in the *online* case has to be larger than the bound gap limit used to sort the basis functions. If the number of basis functions in  $S_{N'}$  is not too large we may also include all of them to find the reduced basis solution in a non-adaptive fashion. Since the *a posteriori* analysis for the multi-block case is missing, this is what is done when solving the hierarchical flow system in Figure 1.

## 5 Parameterizing the geometries

In (2.17) and (2.18) we see how the geometric mapping  $\Phi$  enters the steady Stokes equations in a natural way. To control different instantiations of this mapping we define  $\Phi(\hat{\Omega}; \mu)$ , where  $\mu \in \mathcal{D} \subset \mathbb{R}^P$ . The  $P$  elements in  $\mu$  are parameters which describe, for example, the length, thickness and opening angle of a bifurcation block. Given  $\mu$ , a pipe or bifurcation block is constructed by defining its outer edges according to  $\mu$ , and then the internal nodes are found using a Gordon-Hall algorithm. Finally, for multi-element blocks, all the internal nodes are adjusted according to the smoothing process described above. After all the points are found, the blocks may be rotated to any desired orientation. We note that all corners in the blocks are right angles.

Once the final values of all the nodal points have been computed, the Jacobian,  $\mathcal{J}$ , of the mapping  $\Phi(\hat{\Omega}; \mu)$  and its determinant,  $J$ , are calculated and stored for each node. If we again study (2.17) and (2.18), we see that these quantities appear nonlinearly in the equations. Thus we have nonlinear parameter dependence, which is fundamentally different from most reduced basis applications. The equations are not affine in their parameter dependence either; see [17] for affine, linear parameter dependence, and [3] for non-affine parameter dependence.

Since we are only interested in giving a proof of concept, we choose  $P = 2$  for the bifurcations, and let  $\mu = (\mu^1, \mu^2)$ . We let the first parameter,  $\mu^1$ ,



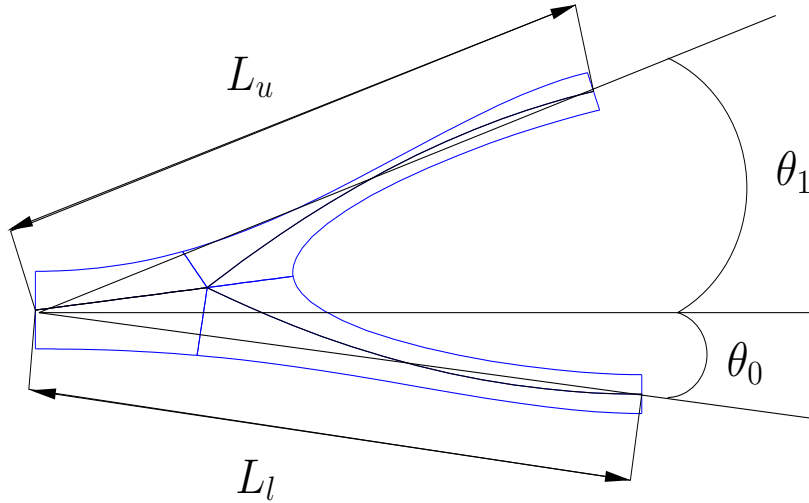


Figure 3: The parameters used to define the bifurcations.

define the difference in length between the upper leg of the bifurcation and the lower leg, i.e.,  $\mu^1 = L_l - L_u$ ; see Figure 3. The second parameter,  $\mu^2$ , is taken to be the difference in the opening angle of two legs of the bifurcation, i.e.,  $\mu^2 = \theta_1 - \theta_0$ .

The outline of the bifurcation is defined through its corner points and the length of the body relative to the length of the legs. The opening angle is adjusted by a rigid body rotation of the two cornerpoints of the upper leg around the centerpoint of the inflow boundary. Before any rotation, the difference in length between the two legs is defined by setting the  $x$ -coordinates of the corner points of the upper leg, (both are the same before the rotation). The non-linear edges of the bifurcation are constructed such that they pass through the corner points and the common edge point of the two elements sharing an edge, and such that they are perpendicular to the inflow and outflow boundaries. The upper and lower edges are fourth order polynomials, while the edge connecting the two legs is, for optimal flexibility, constructed by the use of cubic splines.

For the pipes we choose  $P = 4$ , and let  $\mu = (\mu^1, \dots, \mu^4)$ . We let the first parameter denote the rotation of the outflow boundary relative to the inflow boundary, see Figure 4 for  $\mu^1 = 0$  and  $\mu^1 = -\frac{\pi}{2}$ . The second and third parameters are used to define the length of the inflow and outflow boundaries, and the fourth parameter defines the fluctuation of the wall boundaries.

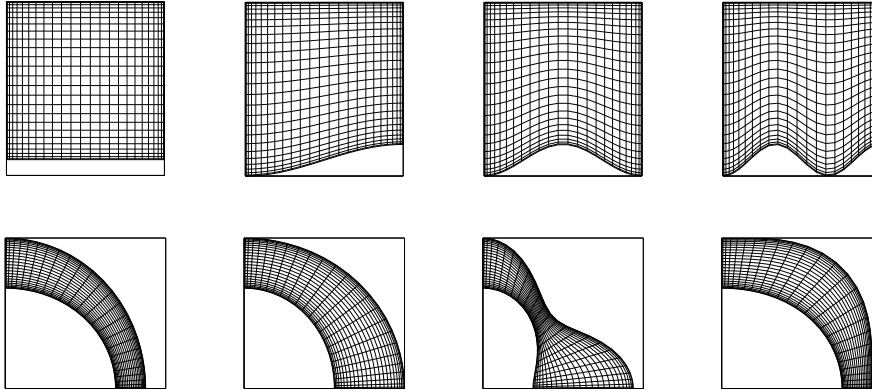


Figure 4: The deformed pipes used to construct the basis functions for the pipe blocks.

## 6 Numerical examples

We now present some examples of the method applied to different geometric structures. The first is a pipe consisting of three elements, where the solution is found as a linear combination on each element, glued together with Lagrange multipliers. The second is a six-element bifurcation, where the solution is found as a linear combination of global basis functions. The third structure is a hierarchical system consisting of one pipe and three bifurcations. The solution is now found as a linear combination on each block structure, i.e. pipe or six-element bifurcation, and glued together with Lagrange multipliers across the block interfaces. The final structure is a “bypass” system with three pipe blocks and two bifurcation blocks.

### 6.1 Pipes

We consider the eight geometries in Figure 4 to be pipes with inflow boundary along the left vertical edge, and outflow boundary along the opposite edge.

Each pipe is decomposed into three sub-domains, all of which are regular one-to-one deformations of the reference square  $\hat{\Omega} = (-1, 1)^2$ , and the restriction to each sub-domain of the steady Stokes solutions found on these geometries are stored on  $\hat{\Omega}$ . In addition we store the reflection of the solutions across the  $\xi$ -axis in  $\hat{\Omega}$ . This accounts to solving the steady Stokes equations on the reflection of the geometries across the  $x$ -axis. Since the first geometry is symmetric, we thus end up with 15 precomputed solutions for each sub-domain. We compute the associated enriched velocity solutions,

$N$	$N_1$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$
27	9	$2.3 \cdot 10^{-3}$	$3.6 \cdot 10^{-1}$
33	11	$1.2 \cdot 10^{-3}$	$5.8 \cdot 10^{-2}$
39	13	$9.7 \cdot 10^{-4}$	$4.4 \cdot 10^{-3}$
45	15	$8.4 \cdot 10^{-4}$	$3.6 \cdot 10^{-3}$

Table 1: The reduced basis error on a generic multi-block pipe with three blocks.  $N = 3N_1$  is the total number of degrees-of-freedom in the reduced basis spaces  $X_N^0$ ,  $X_N^e$ , and  $M_N$ .  $N_1$  is the number of basis geometries used to generate the basis functions.

$N$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$
1	$1.4 \cdot 10^{-2}$	$8.8 \cdot 10^{-2}$
5	$5.0 \cdot 10^{-4}$	$4.8 \cdot 10^{-3}$
10	$9.9 \cdot 10^{-6}$	$7.2 \cdot 10^{-5}$
15	$4.0 \cdot 10^{-6}$	$7.3 \cdot 10^{-6}$

Table 2: The reduced basis error on a single bifurcation.  $N$  is the total number of degrees-of-freedom in the reduced basis spaces  $X_N^0$ ,  $X_N^e$ , and  $M_N$ .

and solve (3.1) when  $\Omega$  is taken to be a generic deformed pipe, decomposed into three sub-domains. Since we only have  $N_1 = 15$  basis functions in this case, we do not apply the selection algorithm described in Section 4.1.

When we use cubic Lagrange multipliers in both the normal and tangential direction to glue the solution together across the block interfaces, the error of the reduced basis solution for an increasing number of basis functions is as presented in Table 1.

## 6.2 Bifurcations

We consider bifurcations characterized by the length and angle of the upper leg relative to the length and angle of the lower leg. In the tensor product parameter space generated by eight relative lengths and eight relative angles, we generate 64 bifurcations. We precompute the steady Stokes solutions on these bifurcations, and store them on  $\hat{\Omega}$ . Again we compute the associated enriched velocity solutions, but before we find the reduced basis solution we apply the selection algorithm described earlier.

The resulting errors in velocity and pressure are presented in Table 2, and we see that the convergence is very good. It is better than the convergence seen for a multi-element pipe in Table 1, both because we don't have any consistency error from the element interfaces, and because the basis bifurcations span the generic bifurcation better than the deformed pipes represent the generic pipe in the previous example.

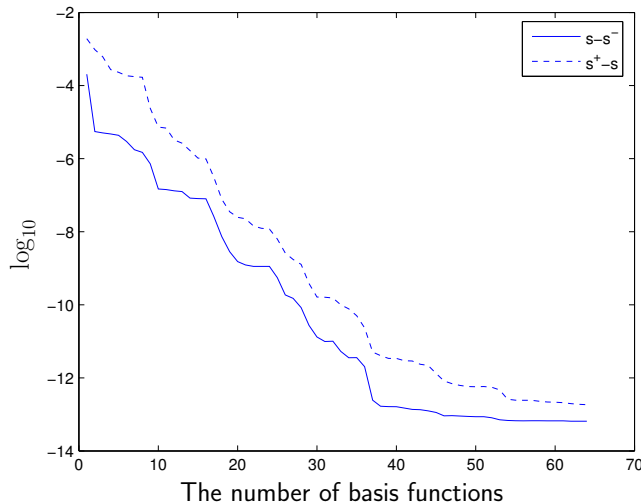


Figure 5: The bound gaps when varying two geometric parameters on a single bifurcation.

In this single-block case we may apply the *a posteriori* error analysis described in Section 4, and we compute both the upper and the lower bound gaps. We do this without using the selection algorithm, and the bound gaps converge as shown in Figure 5. Even without the selection algorithm the convergence is exponential.

### 6.3 Hierarchical flow system

An example of a multi-block domain comprising both pipe blocks and bifurcation blocks, is the complex flow system shown in Figure 1. To precompute

$N$	$N_1$	$N_2$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$
36	9	9	$2.6 \cdot 10^{-3}$	$4.0 \cdot 10^{-1}$
44	11	11	$1.7 \cdot 10^{-3}$	$6.6 \cdot 10^{-2}$
52	13	13	$1.2 \cdot 10^{-3}$	$4.9 \cdot 10^{-2}$
65	15	15	$1.1 \cdot 10^{-3}$	$3.7 \cdot 10^{-2}$
105	15	30	$4.2 \cdot 10^{-4}$	$6.3 \cdot 10^{-3}$

Table 3: The error in the reduced basis steady Stokes solution on a multi-block system corresponding to Figure 1.  $N = N_1 + 3N_2$  is the total number of degrees-of-freedom in the reduced basis spaces  $X_N^0, X_N^e$ , and  $M_N$ .  $N_1$  is the number of basis geometries used to generate the basis functions on the pipe block,  $N_2$  is the number of basis functions used on the bifurcation blocks.

$N$	$N_1$	$N_2$	$\ \mathbf{u}_N - \mathbf{u}_N\ _{H^1}$	$\ p_N - p_N\ _{L^2}$
45	9	9	$9.3 \cdot 10^{-3}$	$3.3 \cdot 10^{-1}$
55	11	11	$3.1 \cdot 10^{-3}$	$5.3 \cdot 10^{-1}$
65	13	13	$2.3 \cdot 10^{-3}$	$9.0 \cdot 10^{-2}$
75	15	15	$1.4 \cdot 10^{-3}$	$5.3 \cdot 10^{-2}$
105	15	30	$5.4 \cdot 10^{-4}$	$3.0 \cdot 10^{-2}$

Table 4: The error in the reduced basis steady Stokes solution on a multi-block bypass with three pipe blocks and two bifurcation blocks.  $N = 3N_1 + 2N_2$  is the total number of degrees-of-freedom in each of the reduced basis spaces  $X_N^0, X_N^e$ , and  $M_N$ .  $N_1$  is the number of basis geometries used to generate the basis functions on the pipe block,  $N_2$  is the number of basis functions used on the bifurcation blocks.

the basis solutions, we use the same geometries for both pipes and bifurcations as described above. For the pipes we only use the restrictions to the inflow element, while we for the bifurcations precompute the solutions by adding pipe elements to the inflow and outflow boundaries in order to get the right boundary conditions. Only the restrictions of the solutions to the bifurcation block are stored and used as basis solutions. For the pipe block we use all 15 precomputed solutions, while we for the bifurcation blocks again use the selection process to limit the number of precomputed solutions to 30. To glue the blocks together across block interfaces, we again use Lagrange multipliers. Since each bifurcation block consists of two elements on the interface to an adjacent block, we now use linear Lagrange multipliers defined on one half of the interface. In Table 3 we see how the errors in velocity and pressure behave as the number of basis functions increases.

#### 6.4 A “bypass”

As the final example we combine both block structures in the bypass system shown in Figure 6. Here the upper branch illustrates the effect of a clogged vein, while the lower branch is the bypass-vein. To model this domain with the reduced basis element method, we use snapshot solutions computed on three-domain pipes to generate the basis functions for the pipe blocks. The restriction of the snapshot solutions to each of the three sub-domains are now used as basis functions on their respective pipe block in the bypass system. As basis functions for the bifurcation blocks we use the same basis functions that were used on the hierarchical flow system in the previous example.

In this case we have two more block-interfaces compared to the hierarchical flow system, each contributing eight constraints on the reduced basis velocity solution  $\mathbf{u}_N$  (2 constraints in each spatial direction for each half of

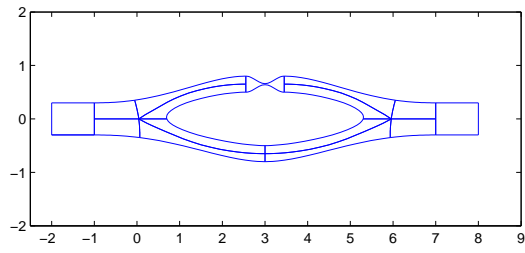


Figure 6: The bypass with three pipe blocks and two bifurcation blocks.

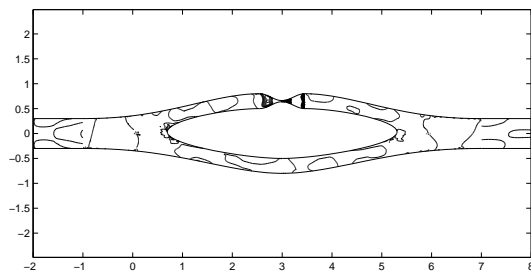


Figure 7: The contour of the error in the reduced basis pressure solution  $p_N$  when  $N_1 = 15$  and  $N_2 = 30$ .

one interface). We see in Table 4 that the error convergence is good, but if too few basis functions are used we get spurious pressure modes due to the severe constraints on the reduced basis velocity space  $X_N(\Omega)$ .

In Figure 6 we present a contour plot of the error in the reduced basis pressure solution  $p_N$  when  $N_1 = 15$  and  $N_2 = 30$ . Most of the error is located around the pipe block modeling the clogged vein. Compared to the deformed pipes in Figure 4, used to generate the basis functions for the pipe blocks, this pipe block differs significantly.

## 7 Future work

We have seen how the reduced basis element method works on the steady Stokes problem when the geometry is considered to be a parameter. In a forthcoming paper we will consider the steady Navier-Stokes equations, and theory for the *a posteriori* error estimation in the multi-block case. In addition we will incorporate the non-affine theory of [3] in order to do more of the necessary computations in the precomputation stage. Other issues to investigate include the extension to time-dependent problems, possibly with moving boundaries, and extension to three dimensional domains.

## References

- [1] R. Aris. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Publications, 1989.
- [2] I. Babuska. Error-bounds for finite element method. *Numer. Math.*, 16:322–333, 1971.
- [3] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera. An ‘empirical interpolation’ method: Application to efficient reduced-basis discretization of partial differential equations. *C. R. Acad. Sci. Paris, Serie I*, 339:667–672, 2004.
- [4] B. F. Belgacem, C. Bernardi, N. Chorfi, and Y. Maday. Inf-sup conditions for the mortar spectral element discretization of the Stokes problem. *Numer. Math.*, 85:257–281, 2000.
- [5] G. Berkooz, P. Holmes, and J. L. Lumley. The proper orthogonal decomposition in the analysis of turbulent flows. *Annu.Rev.Fluid Mech.*, 25:539–575, 1993.
- [6] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers. *Rairo Anal. Numer.*, 8 R2:129–151, 1974.

- [7] J. Burkhardt, M. Gunzburger, and H. Lee. Centroidal voronoi tessellations-based reduced-order modeling of complex systems. *Submitted to SIAM J. Sci. Comput.*, 2005.
- [8] J.P. Fink and W.C. Rheinboldt. On the error behavior of the reduced basis technique in nonlinear finite element approximations. *Z. Angew. Math. Mech.*, 63:21–28, 1983.
- [9] A. E. Løvgrén, Y. Maday, and E. M. Rønquist. A reduced basis element method for the steady Stokes problem. *Submitted to M2AN*, 2004.
- [10] A. E. Løvgrén, Y. Maday, and E. M. Rønquist. A reduced basis element method for the steady Stokes problem: Application to flow in bifurcations. *In progress, to be submitted in June*, 2005.
- [11] L. Machiels, Y. Maday, I. B. Olivera, A. T. Patera, and D. V. Rovas. Output bounds for reduced-basis approximations of symmetric positive definite eigenvalue problems. *C. R. Acad. Sci. Paris, Serie I*, 331:153–158, 2000.
- [12] Y. Maday and A. T. Patera. Spectral element methods for the Navier-Stokes equations. In *Noor A. (ed) State of the Art Surveys in Computational Mechanics*, pages 71–143, 1989.
- [13] Y. Maday, A. T. Patera, and E. M. Rønquist. The  $P_N \times P_{N-2}$  method for the approximation of the Stokes problem. *Technical Report No. 92009, Department of Mechanical Engineering, Massachusetts Institute of Technology*, 1992.
- [14] Y. Maday and E. M. Rønquist. A reduced-basis element method. *J. Sci. Comput.*, 17:447–459, 2002.
- [15] Y. Maday and E. M. Rønquist. The reduced-basis element method: application to a thermal fin problem. *SIAM J. Sci. Comput.*, 26(1):240–258, 2004.
- [16] A. K. Noor and J. M. Peters. Reduced basis technique for nonlinear analysis of structures. *AIAA J.*, 18(4):455–462, 1980.
- [17] C. Prud’homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera, and G. Turinici. Reliable real-time solution of parametrized partial differential equations: Reduced basis output bound methods. *J. Fluids Engineering*, 124:70–80, 2002.
- [18] D. V. Rovas. *Reduced-Basis Output Bound Methods for Parametrized Partial Differential Equations*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, October 2002.



- [19] K. Veroy, C. Prud'homme, D. V. Rovas, and A. T. Patera. *A Posteriori* error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations (AIAA Paper 2003-3847). In *Proceedings of the 16th AIAA Computational Fluid Dynamics Conference*, June 2003.

# IV

## A reduced basis method for the steady Navier-Stokes problem

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In preparation for submission as a journal paper.



# A reduced basis method for the steady Navier-Stokes problem

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October 31, 2005

## Abstract

A reduced basis method where the geometry of the computational domain is used as a parameter is presented. We extend earlier work on steady Stokes flow in deformed pipes [7], to the stationary Navier-Stokes problem for low Reynolds numbers. We thus introduce non-linear terms in the reduced basis element method, and we get a non-symmetric problem. This gives new challenges in the *a posteriori* error analysis, and we adapt and combine results from [7, 9, 14] to suit our needs. In addition we show that the offline/online decoupling developed in [14] for the output and reduced basis solution of linear problems with coercive operators and affine parameter dependence, can be achieved for the Navier-Stokes problem with non-affine parameter dependence. To this end we apply the empirical interpolation technique presented in [3].

## 1 Introduction

The reduced basis method is known for the rapid solution of parameter dependent problems of the form: Find  $u \in X$  when

$$F(u; \mu) = 0, \tag{1.1}$$

where  $\mu$  represents one or more independent parameters, and  $X$  is an appropriate solution space. When the same problem is to be solved repeatedly for different choices of independent parameters, i.e. in optimization and control problems, reduced basis methods together with error estimation provides a reliable alternative to classical methods. Typically, if  $\mathcal{N}$  is the number of degrees-of-freedom in the solution of (1.1), a basis of  $N \ll \mathcal{N}$  basis functions is precomputed, and the reduced basis solution is found by determining the coefficients needed in a linear combination of these  $N$  basis functions.

The reduced basis element method differs from traditional reduced basis methods in that it combines the reduced basis method with domain decomposition. The domain is decomposed into building blocks with the same

topology as a few reference domains. Basis functions are precomputed and stored separately for each distinct reference domain, and to find the reduced basis solution, the basis functions are mapped from their respective reference domain to each corresponding domain in the decomposition. A local approximation of the high resolution solution on one domain is found using the basis functions belonging to that specific domain, and the global approximation is found by “gluing” the local approximations together with constraints across domain interfaces. When there is only one domain, the reduced basis element method is seen as a traditional reduced basis method where the geometry of the domain is one of the independent parameters in (1.1).

The quality of the method is certified by *a posteriori* error estimation of certain outputs of interest relevant to the given problem. These outputs may for example describe the drag forces when air flows past an object, or the volume flow rate of a fluid through a pipe. It is shown in [14, 15] that, when the geometry is changed by pure stretching and rotation, and the problem at hand is affine in its parameter dependence, the online work needed to compute the output of interest in the reduced basis method may be independent of the resolution,  $\mathcal{N}$ , of the precomputed basis functions. This is achieved via an offline/online splitting of the computational effort.

We will in this work extend earlier work on steady Stokes flow in deformed pipes [7], to the stationary Navier-Stokes problem for low Reynolds numbers. We thus introduce non-linear terms in the reduced basis element method, and the necessary approximation space is more challenging to construct than for the steady Stokes problem. We also get a non-symmetric problem, and in the *a posteriori* error analysis we have to introduce a dual problem to get sharp bounds on the output error estimators.

Finally, we address the non-affine parameter dependence experienced for general geometric variations, and approximate the geometric operators using linear combinations of precomputed sets representative functions. By exploiting a technique known as empirical interpolation [3], we avoid using projection in the approximation of the geometric operators, and are able to decouple the problem of estimating the output of interest and its error estimators in offline/online computations. The online work needed to compute the output of interest is thus made independent of  $\mathcal{N}$ , as in the affine case.

## 2 The steady Navier-Stokes problem

On a two-dimensional domain  $\Omega$ , obtained as a regular one-to-one deformation,  $\Phi$ , of the reference domain  $\hat{\Omega} = (-1, 1)^2$ , we consider the steady incompressible Navier-Stokes equations

$$\begin{aligned} \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where  $\mathbf{u} = (u_1, u_2)$  is the velocity field,  $p$  is the pressure,  $\mathbf{f} = (f_1, f_2)$  is a prescribed volumetric body force, and  $\nu$  is the fluid viscosity; see [1]. The steady incompressible Navier-Stokes equations model the flow of fluids at low Reynolds numbers.

The domain  $\Omega = \Phi(\hat{\Omega})$  consists of an inflow boundary  $\Gamma_{in}$ , an outflow boundary  $\Gamma_{out}$ , and a walled boundary  $\Gamma_w$ . On this domain we introduce the velocity space

$$X(\Omega) = \{\mathbf{v} \in (H^1(\Omega))^2, \quad \mathbf{v}|_{\Gamma_w} = 0 \text{ and } v_t|_{\Gamma_{in}} = v_t|_{\Gamma_{out}} = 0\}, \quad (2.2)$$

where  $v_t$  is the tangential velocity component. In addition, we define  $\sigma_n = \nu \frac{\partial u_n}{\partial n} - p$ , where  $u_n$  is the normal component of the velocity relative to the boundary, and  $\frac{\partial}{\partial n}$  denotes the derivative in the outward normal direction. We then have the Neumann type boundary conditions given by specifying  $\sigma_n$  to be  $\sigma_n^{in} = -1$  along  $\Gamma_{in}$  and  $\sigma_n^{out} = 0$  along  $\Gamma_{out}$ . The second equation in (2.1) ensures that  $\frac{\partial u_n}{\partial n} + \frac{\partial u_t}{\partial t} = 0$ , and for all velocities  $\mathbf{u} \in X$ , we thus have  $\frac{\partial u_n}{\partial n} = 0$  along  $\Gamma_{in}$  and  $\Gamma_{out}$ . The Neumann conditions thus correspond to specifying the pressure along the inflow and outflow boundaries.

As our output of interest  $s$ , we choose the flow rate of the velocity solution of (2.1) across the inflow boundary of  $\Omega$ , which is found by the functional

$$s(\mathbf{u}; \Phi) = - \int_{\Gamma_{in}} \mathbf{u} \cdot \mathbf{n} ds, \quad (2.3)$$

where  $\mathbf{n}$  is the outward pointing normal of the boundary.

With the given boundary conditions, we define the pressure space to be  $M(\Omega) = L^2(\Omega)$ , and introduce the variational formulation of (2.1): Find  $\mathbf{u} \in X(\Omega)$  and  $p \in M(\Omega)$  such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}; \Phi) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}; \Phi) + b(\mathbf{v}, p; \Phi) &= l(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X(\Omega) \\ b(\mathbf{u}, q; \Phi) &= 0 \quad \forall q \in M(\Omega), \end{aligned} \quad (2.4)$$

where

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{w} d\Omega = \nu \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} d\Omega, \quad (2.5)$$

$$b(\mathbf{v}, q; \Phi) = - \int_{\Omega} q \nabla \cdot \mathbf{v} d\Omega = - \sum_{i=1}^2 \int_{\Omega} q \frac{\partial v_i}{\partial x_i} d\Omega, \quad (2.6)$$

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}; \Phi) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} d\Omega = \sum_{i,j=1}^2 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j d\Omega, \quad (2.7)$$

are bilinear and trilinear forms, and

$$l(\mathbf{v}; \Phi) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega + \int_{\Gamma_{in}} \sigma_n^{in} \mathbf{v} \cdot \mathbf{n} ds + \int_{\Gamma_{out}} \sigma_n^{out} \mathbf{v} \cdot \mathbf{n} ds, \quad (2.8)$$

is a linear form. Alternatively we may define (2.5), (2.6), and (2.7) in terms of the reference variables  $(\xi, \eta)$ , as

$$\begin{aligned} a(\mathbf{v}, \mathbf{w}; \Phi) &= \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \mathcal{J}^{-T} \hat{\nabla}(\mathbf{w} \circ \Phi) |J| d\hat{\Omega}, \\ b(\mathbf{v}, q; \Phi) &= - \int_{\hat{\Omega}} (q \circ \Phi) \hat{\nabla} \cdot [\mathcal{J}^{-1}(\mathbf{v} \circ \Phi)] |J| d\hat{\Omega}, \\ c(\mathbf{u}, \mathbf{v}, \mathbf{w}; \Phi) &= \int_{\hat{\Omega}} ((\mathbf{u} \circ \Phi) \cdot \mathcal{J}^{-T} \hat{\nabla})(\mathbf{v} \circ \Phi) \cdot (\mathbf{w} \circ \Phi) |J| d\hat{\Omega}, \end{aligned} \quad (2.9)$$

where  $\mathcal{J}$  is the Jacobian of  $\Phi : \hat{\Omega} \rightarrow \Omega$ ,

$$\mathcal{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}, \quad (2.10)$$

and  $J$  is the determinant of  $\mathcal{J}$ . A similar transformation can be done to the linear form (2.8); in the following, we set  $\mathbf{f} = \mathbf{0}$ . For this choice of  $\mathbf{f}$  we get  $s(\mathbf{u}; \Phi) = l(\mathbf{u}; \Phi)$ , which in [14] is denoted a compliant output.

The variational form (2.4) of the steady incompressible Navier-Stokes equations is non-linear and non-symmetric. For the spaces defined above, the inf-sup condition

$$\inf_{q \in M} \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}, q)}{\|q\|_{L^2(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)}} = \beta > 0, \quad (2.11)$$

is satisfied; see [2] and [4].

To estimate the error in the output of interest we also define the linear dual problem: Find  $\boldsymbol{\psi} \in X(\Omega)$  and  $\lambda \in M(\Omega)$  such that

$$\begin{aligned} a(\mathbf{v}, \boldsymbol{\psi}; \Phi) + c_1(\mathbf{u}, \mathbf{v}, \boldsymbol{\psi}; \Phi) + b(\mathbf{v}, \lambda; \Phi) &= -l(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X(\Omega) \\ b(\boldsymbol{\psi}, q; \Phi) &= 0 \quad \forall q \in M(\Omega), \end{aligned} \quad (2.12)$$

where

$$c_1(\mathbf{u}, \mathbf{v}, \boldsymbol{\psi}; \Phi) = c(\mathbf{u}, \mathbf{v}, \boldsymbol{\psi}; \Phi) + c(\mathbf{v}, \mathbf{u}, \boldsymbol{\psi}; \Phi), \quad (2.13)$$

and  $\mathbf{u}$  is the velocity solution of (2.4).

We now consider a discretization of the primal problem (2.4) and the linear dual problem (2.12) using a pure spectral method based on high order polynomials; see [10] and [11]. Let  $\mathbb{P}_n(\hat{\Omega})$  be the space of all functions which are polynomials of degree less than or equal to  $n$  in each spatial direction on the reference domain  $\hat{\Omega}$ . The discrete space for the velocity is then taken to be

$$X_{\mathcal{N}}(\Omega) = \{\mathbf{v} \in X(\Omega), \quad \mathbf{v} \circ \Phi \in (\mathbb{P}_{\mathcal{N}}(\hat{\Omega}))^2\}, \quad (2.14)$$

while the discrete space for the pressure is

$$M_{\mathcal{N}}(\Omega) = \{q \in M(\Omega), \quad q \circ \Phi \in \mathbb{P}_{\mathcal{N}-2}(\hat{\Omega})\}. \quad (2.15)$$

For this choice of discrete spaces, the inf-sup condition defined in (2.11) is satisfied. The bases for  $X_{\mathcal{N}}$  and  $M_{\mathcal{N}}$  are conveniently expressed in terms of the reference variables  $\xi$  and  $\eta$ . As a basis for  $X_{\mathcal{N}}$  we use a nodal basis through the  $(\mathcal{N} + 1)^2$  tensor-product Gauss-Lobatto Legendre (GLL) points  $(\xi_m, \xi_n), m, n = 0, \dots, \mathcal{N}$ , while the basis for  $M_{\mathcal{N}}$  is a nodal basis through the  $(\mathcal{N} - 1)^2$  tensor-product Gauss-Legendre (GL) points,  $(\zeta_m, \zeta_n), m, n = 0, \dots, \mathcal{N} - 2$ ; see [10] and [11].

The mapping  $\Phi$  is realized computationally by using an isoparametric representation of the geometry. Each edge of  $\Omega$  is given as a one-to-one mapping of a corresponding edge  $[-1, 1]$  on  $\hat{\Omega}$ . Each edge of  $\Omega$  is approximated as an  $\mathcal{N}$ -th order polynomial, and the location of the (interior) points  $(x_m, y_n) = \Phi(\xi_m, \xi_n)$  are found by a Gordon-Hall algorithm; see [6].

The linear, bilinear, and trilinear forms in (2.4) and (2.12) are expressed in terms of the reference variables, and the integrals are evaluated using GLL and GL quadrature. This gives us the following discrete system for the primal problem: Find  $\mathbf{u}_{\mathcal{N}} \in X_{\mathcal{N}}(\Omega)$  and  $p_{\mathcal{N}} \in M_{\mathcal{N}}(\Omega)$  such that

$$\begin{aligned} a_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, \mathbf{v}; \Phi) + c_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, \mathbf{u}_{\mathcal{N}}, \mathbf{v}; \Phi) + b_{\mathcal{N}}(\mathbf{v}, p_{\mathcal{N}}; \Phi) &= l_{\mathcal{N}}(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_{\mathcal{N}}(\Omega) \\ b_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, q; \Phi) &= 0 \quad \forall q \in M_{\mathcal{N}}(\Omega), \end{aligned} \quad (2.16)$$

where  $a_{\mathcal{N}}$ ,  $c_{\mathcal{N}}$ ,  $b_{\mathcal{N}}$  and  $l_{\mathcal{N}}$  refer to integration of the linear, bilinear, and trilinear forms using Gauss-type quadrature. Similarly, the discrete system for the linear dual problem is: Find  $\boldsymbol{\psi}_{\mathcal{N}} \in X_{\mathcal{N}}(\Omega)$  and  $\lambda_{\mathcal{N}} \in M_{\mathcal{N}}(\Omega)$  such that

$$\begin{aligned} a_{\mathcal{N}}(\mathbf{v}, \boldsymbol{\psi}_{\mathcal{N}}; \Phi) + c_{1\mathcal{N}}(\mathbf{u}_{\mathcal{N}}, \mathbf{v}, \boldsymbol{\psi}_{\mathcal{N}}; \Phi) + b_{\mathcal{N}}(\mathbf{v}, \lambda_{\mathcal{N}}; \Phi) &= -l_{\mathcal{N}}(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_{\mathcal{N}}(\Omega) \\ b_{\mathcal{N}}(\boldsymbol{\psi}_{\mathcal{N}}, q; \Phi) &= 0 \quad \forall q \in M_{\mathcal{N}}(\Omega). \end{aligned} \quad (2.17)$$

Finally, the output of interest is found by evaluating

$$s(\mathbf{u}_{\mathcal{N}}; \Phi) = l_{\mathcal{N}}(\mathbf{u}_{\mathcal{N}}; \Phi), \quad (2.18)$$

where  $\mathbf{u}_{\mathcal{N}}$  is the velocity solution found in (2.16).

The transformation of the operators to the reference domain defined in (2.9) shows how the mapping  $\Phi$  and the geometric variables  $\mathcal{J}$  and  $J$  enter the variational form of the Navier-Stokes equations. We see that since the geometric variables enter the variational form inside the integral operators, and since (except for pure stretching and rotation,) the geometric variables are non-constant over  $\hat{\Omega}$ , we are not able to express the operators as affine combinations of geometry dependent operators and velocity/pressure dependent operators. For parameter dependent problems where affine separation is possible, it is shown in [14] that the number of online operations needed to compute the reduced basis output of interest,  $s_{\mathcal{N}}$ , is independent of the resolution  $\mathcal{N}$ . We show in Section 5 however, that the operators in (2.9) may be approximated by affine combinations of certain geometric functions and velocity/pressure dependent operators. The  $\Phi$ -dependence is then taken care



of by evaluating these geometric functions only at isolated points. Using this approximation, we show in Section 3 that the number of online operations needed to compute  $s_N$  for geometric parameters is independent of  $\mathcal{N}$  to a great extent. For complete  $\mathcal{N}$ -independence, the geometric functions used in the affine approximation should be computed only in the isolated points in which they are evaluated, and not over the whole domain. Since we use spectral elements however, the underlying basis consists of polynomials of degree  $\mathcal{N}$ . To compute the derivative of a function at one point involves a linear combination of  $\mathcal{N} + 1$  functional values, as opposed to computing the derivative on the whole domain in  $\mathcal{O}(\mathcal{N}^3)$  operations.

### 3 The reduced basis method

The idea behind the reduced basis method is to construct an approximation space of  $N$  precomputed solutions, or their derivatives, for a sample of parameters. These precomputed solutions are then used as a basis with very high information content to find the solution of the same problem for a generic choice of parameters. The sample parameters are chosen such that they cover the domain of relevant parameters to some extent, either by distributing the parameters evenly throughout the domain, or by a log-random distribution; see [14]. For an approximation space consisting of only one precomputed solution and several derivatives (a Taylor expansion), only one parameter is chosen; see [13]. Typically  $N$  is much smaller than the resolution  $\mathcal{N}$  used in the underlying problem, and the reduced basis solution is found as a linear combination of the precomputed solutions.

When the geometry is used as the parameter, we use typical shape parameters to define the geometries. These shape parameters may be the diameter of a pipe, the deflection angle of a beam, or the height and width of rectangular parts; see [8] for shape parameters for a bifurcation. We note that different shape parameters, may constitute the same solution when “viewed” on the same geometry. For example, if the Laplace equation,  $-\Delta u = 0$ , is solved on two different rectangles, with Dirichlet boundary conditions on the vertical boundaries and homogeneous Neumann conditions on the horizontal boundaries, the solutions are identical when the results are presented on a reference rectangle, e.g.  $\hat{\Omega}$ . The solution is just a scalar field, varying linearly from the left vertical boundary to the right vertical boundary. In fact, the same solution is also obtained when the Laplace problem is solved on an undeformed quarter of an annulus; see [12] and Figure 1. Although the precomputed solutions of (2.16) present more variation than the solution of the Laplace problem, orthogonalization of the precomputed solutions is necessary.

For a generic domain  $\Omega = \Phi(\hat{\Omega})$ , we now define the reduced basis approximation spaces  $X_N(\Omega) \subset X_{\mathcal{N}}(\Omega)$  and  $M_N(\Omega) \subset M_{\mathcal{N}}(\Omega)$ . Here  $N$  denotes

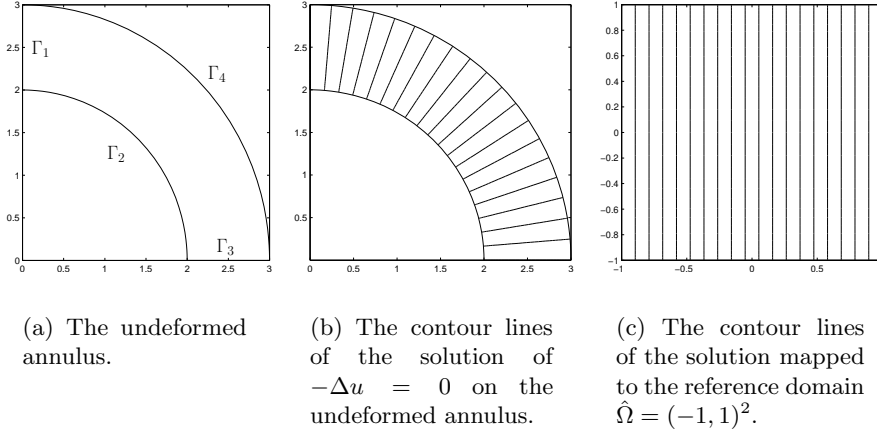


Figure 1: The solution of  $-\Delta u = 0$  with Dirichlet boundary conditions on the inflow and outflow, and isolated walls is a linear scalar field.

the number of precomputed solutions, corresponding to a set of preselected geometries. Our objective is to find the reduced basis output of interest

$$s(\mathbf{u}_N; \Phi) = l_N(\mathbf{u}_N; \Phi), \quad (3.1)$$

where  $\mathbf{u}_N$  is the velocity solution of the problem: Find  $\mathbf{u}_N \in X_N(\Omega)$  and  $p_N \in M_N(\Omega)$ , satisfying

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{v}; \Phi) + c_N(\mathbf{u}_N, \mathbf{u}_N, \mathbf{v}; \Phi) + b_N(\mathbf{v}, p_N; \Phi) &= l_N(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_N(\Omega) \\ b_N(\mathbf{u}_N, q; \Phi) &= 0 \quad \forall q \in M_N(\Omega). \end{aligned} \quad (3.2)$$

In addition, we want to find error bounds for the output of interest, for which we need the solution of the linear dual reduced basis problem: Find  $\psi_N \in X_N(\Omega)$  and  $\lambda_N \in M_N(\Omega)$ , satisfying

$$\begin{aligned} a_N(\mathbf{v}, \psi_N; \Phi) + c_{1N}(\mathbf{u}_N, \mathbf{v}, \psi_N; \Phi) + b_N(\mathbf{v}, \lambda_N; \Phi) &= -l_N(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_N(\Omega) \\ b_N(\psi_N, q; \Phi) &= 0 \quad \forall q \in M_N(\Omega). \end{aligned} \quad (3.3)$$

Since we intend to find the solution of both (3.2) and (3.3) in  $X_N(\Omega)$  and  $M_N(\Omega)$ , the spaces must contain precomputed solutions of both (2.16) and (2.17).

We proceed as follows. First we assume that we have a space  $S$  of regular one-to-one mappings of  $\hat{\Omega}$ , such that  $\Phi \in S$  for all relevant generic  $\Phi$ . We then take a sample of mappings, such that  $S_{N/2} = \{\Phi_i : \hat{\Omega} \rightarrow \Omega_i\}_{i=1}^{N/2} \subset S$ . On each domain  $\Omega_i$ , we find both the solution  $(\mathbf{u}_i, p_i)$  of the primal problem (2.16), and the dual solution  $(\psi_i, \lambda_i)$  of the linear dual problem (2.17). Thus the sample  $S_{N/2}$  provides  $N$  velocity basis functions and  $N$  pressure

basis functions. The precomputed solutions are mapped to the reference domain  $\hat{\Omega}$ , where they are stored. To use these precomputed solutions as basis functions on the generic geometry  $\Omega$ , they are mapped from the reference geometry to the generic geometry. The pressure functions are mapped through simple composition, such that on the reference domain we get  $\hat{p}_i = p_i \circ \Phi_i$ , and on the generic domain we get  $\tilde{p}_i = p_i \circ \Phi_i \circ \Phi^{-1}$ . The reduced basis approximation space for the pressure on a generic domain is thus

$$M_N(\Omega) = \text{span}\{\tilde{p}_i, \tilde{\lambda}_i, i = 1, \dots, N/2\}. \quad (3.4)$$

The velocity functions are mapped to the reference domain by the Piola transformation [5]

$$\hat{\mathbf{u}}_i = \Psi_i(\mathbf{u}_i) = \mathcal{J}_i^{-1}(\mathbf{u}_i \circ \Phi_i) |J_i|. \quad (3.5)$$

The Piola transformation ensures that the velocity basis functions remain divergence free, and on the generic domain we get the reduced basis approximation space

$$X_N^0(\Omega) = \text{span}\{(\Psi^{-1}(\hat{\mathbf{u}}_i), \Psi^{-1}(\hat{\boldsymbol{\psi}}_i)), i = 1, \dots, N/2\}, \quad (3.6)$$

where each basis function is divergence free. To solve (3.2) and (3.3), the inf-sup condition should conveniently be fulfilled for the reduced basis approximation spaces. We thus have to enrich the velocity space, and, as described in [7], for each pressure solution,  $\hat{p}_i \in M_N(\hat{\Omega})$ , we find  $\hat{\mathbf{v}}_i(\hat{p}_i) = \hat{\mathbf{v}}_i \in X_N(\hat{\Omega})$  such that

$$\hat{\mathbf{v}}_i = \arg \max_{\hat{\mathbf{u}} \in X_N(\hat{\Omega})} \frac{\int_{\hat{\Omega}} \hat{p}_i \hat{\nabla} \cdot \hat{\mathbf{u}} \, d\hat{\Omega}}{|\hat{\mathbf{u}}|_{H^1(\hat{\Omega})}}. \quad (3.7)$$

Similarly, we use (3.7) to find  $\hat{\boldsymbol{\phi}}_i(\hat{\lambda}_i) = \hat{\boldsymbol{\phi}}_i \in X_N(\hat{\Omega})$  for each  $\hat{\lambda}_i \in M_N(\hat{\Omega})$ . The enriched reduced basis approximation space for the velocity is then defined as

$$X_N(\Omega) = X_N^0(\Omega) \oplus X_N^e(\Omega), \quad (3.8)$$

where

$$X_N^e(\Omega) = \text{span}\{(\Psi^{-1}(\hat{\mathbf{v}}_i), \Psi^{-1}(\hat{\boldsymbol{\phi}}_i)), i = 1, \dots, N/2\}. \quad (3.9)$$

For the given reduced basis approximation spaces  $X_N(\Omega)$  and  $M_N(\Omega)$  we are then guaranteed unique solutions of (3.2) and (3.3).

Alternatively, we may exploit the fact that the velocity basis functions in  $X_N^0(\Omega)$  are divergence free, and find only the velocity solution of (3.2) by: Find  $\mathbf{u}_N \in X_N^0(\Omega)$  such that

$$a_N(\mathbf{u}_N, \mathbf{v}; \Phi) + c_N(\mathbf{u}_N, \mathbf{u}_N, \mathbf{v}; \Phi) = l_N(\mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_N^0(\Omega). \quad (3.10)$$

A thorough investigation of this alternate method is done in [8] for the steady Stokes problem.

In Section 5 we use the transformation of the operators defined in (2.9) to show that, despite the fact that the geometric variables are inside the integrals, we may approximate the operators by affine combinations of geometry dependent operators and velocity/pressure dependent operators. For the diffusion operator, say, we get

$$a(\mathbf{v}, \mathbf{w}; \Phi) \approx \nu \sum_{q=1}^Q \sum_{m=1}^{M_q} \beta_m^q(\Phi) \int_{\hat{\Omega}} \tilde{g}_m^q a^q(\hat{\mathbf{v}}, \hat{\mathbf{w}}) d\hat{\Omega}, \quad \forall \mathbf{v}, \mathbf{w} \in X(\Omega), \forall \Phi \in S, \quad (3.11)$$

where  $\tilde{g}_m^q$  and  $a^q(\hat{\mathbf{v}}, \hat{\mathbf{w}})$  are independent of  $\Phi$ , and  $Q$  and  $M_q$  are small integers. The  $\Phi$ -dependent coefficients  $\beta_m^q$  are constant over  $\hat{\Omega}$ . We also show in Section 5 that the number of operations needed to find the coefficients are independent of  $\mathcal{N}$ . This affine approximation of the operators allows for an offline/online decomposition of the computational work, and very efficient computation of the reduced basis output of interest.

To compute the reduced basis output of interest  $s(\mathbf{u}_N; \Phi)$  we only need the coefficients of the reduced basis velocity solution. To clarify, we write

$$\mathbf{u}_N = \sum_{j=1}^{2N} \alpha_j \tilde{\mathbf{u}}_j, \quad \text{and} \quad p_N = \sum_{j=1}^N \gamma_j \tilde{q}_j, \quad (3.12)$$

where each  $\tilde{\mathbf{u}}_j$  represents either  $\Psi^{-1}(\hat{\mathbf{u}}_i)$ ,  $\Psi^{-1}(\hat{\boldsymbol{\psi}}_i)$ ,  $\Psi^{-1}(\hat{\mathbf{v}}_i)$ , or  $\Psi^{-1}(\hat{\boldsymbol{\phi}}_i)$  for some  $i \in [1, N/2]$ , and  $\tilde{q}_j$  represents either  $\tilde{p}_i$ , or  $\tilde{\lambda}_i$  for some  $i \in [1, N/2]$ . The algebraic equations corresponding to (3.2) are

$$\begin{bmatrix} A + C & -D^T \\ -D & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad (3.13)$$

where  $A_{ij} = a_{\mathcal{N}}(\tilde{\mathbf{u}}_j, \tilde{\mathbf{u}}_j; \Phi)$  and  $C_{ij} = c_{\mathcal{N}}(\tilde{\mathbf{u}}_j, \tilde{\mathbf{u}}_j, \tilde{\mathbf{u}}_i; \Phi)$  for  $1 \leq i, j \leq 2N$ ,  $D_{ij} = -b_{\mathcal{N}}(\tilde{\mathbf{u}}_j, \tilde{q}_i)$  for  $1 \leq i \leq N$  and  $1 \leq j \leq 2N$ , and  $F_i = l_{\mathcal{N}}(\tilde{\mathbf{u}}_i; \Phi)$  for  $1 \leq i \leq 2N$ . In (3.13)  $\alpha$  and  $\gamma$  are vectors  $\alpha = [\alpha_1, \dots, \alpha_{2N}]^T$  and  $\gamma = [\gamma_1, \dots, \gamma_N]^T$ . The output of interest is found from

$$s_N = s(\mathbf{u}_N; \Phi) = l_{\mathcal{N}}(\mathbf{u}_N; \Phi) = \sum_{i=1}^{2N} \alpha_i l_{\mathcal{N}}(\tilde{\mathbf{u}}_i; \Phi) = \sum_{i=1}^{2N} \alpha_i F_i, \quad (3.14)$$

such that we do not need to assemble the reduced basis solution itself. In this way the output of interest is found independently of  $\mathcal{N}$ . In Section 4 we develop output error bounds which, by affine approximations like (3.11), are also found independently of  $\mathcal{N}$ .

We now use the affine approximation (3.11) to see that most of the computational effort needed to find the entries in  $A, C, D$ , and  $F$  may be spent in the precomputational stage. We get

$$A_{ij} \approx \nu \sum_{q=1}^Q \sum_{m=1}^M \beta_m^q(\Phi) A_m^q{}_{ij}, \quad 1 \leq i, j \leq 2N, \quad (3.15)$$

where all elements

$$A_{m\ ij}^q = \int_{\hat{\Omega}} \tilde{g}_m^q a^q(\hat{\mathbf{u}}_j, \hat{\mathbf{u}}_i) d\hat{\Omega}, \quad 1 \leq q \leq Q, \quad 1 \leq m \leq M_q, \quad (3.16)$$

only depend on the precomputed velocity basis functions and the preselected basis geometries, and may be computed in the offline stage. We thus have to compute and store  $M_q Q$  matrices  $A_m^q \in \mathcal{R}^{2N \times 2N}$ . Compared to the affine case described in [14], we need  $M_q$  times the storage, and in the online stage the assembly of  $A$  in (3.15) we also need  $M_q$  times the operations needed in the affine case. Similar approximations are found for the other operators.

## 4 *A posteriori* error estimation

To evaluate the error in the reduced basis output of interest  $s_N$ , we compute bounds for the error based on the reduced basis solution  $\mathbf{u}_N$  and the reduced basis linear dual solution  $\boldsymbol{\psi}_N$ . Since the output of interest is found independent of  $\mathcal{N}$ , so should the error bounds. In [14] *a posteriori* error estimation in the reduced basis context for linear problems is done for non-symmetric, coercive operators with affine parameter dependence, and the dual solution is used to achieve strict error bounds for the output of interest. The Navier-Stokes equations are both non-linear and non-symmetric, and in addition we have non-affine parameter dependence. We thus have to modify the error bounds found in [14] to suit our needs. Some work was done in [7] for *a posteriori* error estimation of the reduced basis solution of the Stokes equations with non-affine parameter dependence, and in [9] *a posteriori* error estimation for the Navier-Stokes equations is presented in the finite element context. The analysis below is a combination of these three works.

A given output of interest is used to evaluate the error, and as in [7], we consider the compliant output

$$s(\mathbf{u}; \Phi) = l(\mathbf{u}; \Phi). \quad (4.1)$$

For a given instantiation of the mapping  $\Phi$ , we wish to compute bounds for the output such that

$$s_N^- \leq s(\mathbf{u}; \Phi) \leq s_N^+. \quad (4.2)$$

Before we define the bounds we need some notation. We define the primary residual

$$\begin{aligned} \mathcal{R}^{pr}((\mathbf{u}_N, p_N); \mathbf{v}; \Phi) &= l(\mathbf{v}; \Phi) - a(\mathbf{u}_N, \mathbf{v}; \Phi) \\ &\quad - c(\mathbf{u}_N, \mathbf{u}_N, \mathbf{v}; \Phi) - b(\mathbf{v}, p_N; \Phi), \end{aligned} \quad (4.3)$$

and the generalized dual residual

$$\begin{aligned} \mathcal{R}^{du}(\mathbf{u}_N; (\boldsymbol{\psi}_N, \lambda_N); \mathbf{v}; \Phi) &= -l(\mathbf{v}; \Phi) - a(\mathbf{v}, \boldsymbol{\psi}_N; \Phi) \\ &\quad - b(\mathbf{v}, \lambda_N; \Phi) - c(\mathbf{u}_N, \mathbf{v}, \boldsymbol{\psi}_N; \Phi) \\ &\quad - c(\mathbf{v}, \mathbf{u}_N, \boldsymbol{\psi}_N; \Phi). \end{aligned} \quad (4.4)$$

It is shown in [7] that the diffusion operator defined on the reference domain by

$$\hat{a}(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\hat{\Omega}} g(\Phi) \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \hat{\nabla}(\mathbf{w} \circ \Phi) d\hat{\Omega}, \quad (4.5)$$

satisfies

$$\alpha_0 \|\mathbf{v}\|_{X_{\mathcal{N}}(\Omega)}^2 \leq \hat{a}(\mathbf{v}, \mathbf{v}; \Phi) \leq a(\mathbf{v}, \mathbf{v}; \Phi) \quad \forall \mathbf{v} \in X_{\mathcal{N}}(\Omega), \quad (4.6)$$

for some positive constant  $\alpha_0$ , and  $g(\Phi)$  defined as either the local or the global minimum eigenvalue of the  $2 \times 2$  symmetric positive-definite matrix  $G(\Phi) = (\mathcal{J}^T \mathcal{J})^{-1} |J|$ . Based on empirical interpolation, we may approximate  $G(\Phi)$  as the sum of precomputed matrices, and the evaluation of  $\hat{a}(\cdot, \cdot; \Phi)$  may thus be decoupled in an offline stage and an online stage as described earlier.

We define the primary and dual ‘‘reconstructed errors’’,  $\hat{e}^{pr}$  and  $\hat{e}^{du}$  from

$$2\hat{a}(\hat{e}^{pr}, \mathbf{v}; \Phi) = \mathcal{R}^{pr}((\mathbf{u}_N, p_N); \mathbf{v}; \Phi) \quad \forall \mathbf{v} \in \tilde{X}_{\mathcal{N}}(\Omega) \quad (4.7)$$

$$2\hat{a}(\hat{e}^{du}, \mathbf{v}; \Phi) = \mathcal{R}^{du}(\mathbf{u}_N; (\boldsymbol{\psi}_N, \lambda_N); \mathbf{v}; \Phi) \quad \forall \mathbf{v} \in \tilde{X}_{\mathcal{N}}(\Omega), \quad (4.8)$$

where  $\tilde{X}_{\mathcal{N}}(\Omega) = \{\mathbf{v} \circ \Phi \in (\mathbb{P}_{\mathcal{N}}(\hat{\Omega}))^2, \mathbf{v}|_{\Gamma_w} = 0\}$ .

Finally we define the output bounds as

$$s_N^\pm = l(\mathbf{u}_N; \Phi) - \mathcal{R}^{pr}((\mathbf{u}_N, p_N); \boldsymbol{\psi}_N; \Phi) \pm \kappa \hat{a}(\hat{e}^\pm, \hat{e}^\pm; \Phi), \quad (4.9)$$

where  $\kappa$  is a strictly positive number, and

$$\hat{e}^\pm = \hat{e}^{pr} \mp \frac{1}{\kappa} \hat{e}^{du}. \quad (4.10)$$

To see that the bounds defined in (4.9) really are bounds, we define  $\mathbf{e} = \mathbf{u} - \mathbf{u}_N$  and  $\epsilon = p - p_N$ , and use (4.10), (4.7), and (4.8) to see that

$$\begin{aligned} 2\kappa \hat{a}(\hat{e}^\pm, \mathbf{e}; \Phi) &= 2\kappa \hat{a}(\hat{e}^{pr} \mp \frac{1}{\kappa} \hat{e}^{du}, \mathbf{e}; \Phi) \\ &= 2\kappa \hat{a}(\hat{e}^{pr}, \mathbf{e}; \Phi) \mp 2\hat{a}(\hat{e}^{du}, \mathbf{e}; \Phi) \\ &= \kappa \mathcal{R}^{pr}((\mathbf{u}_N, p_N); \mathbf{e}; \Phi) \\ &\quad \mp \mathcal{R}^{du}(\mathbf{u}_N; (\boldsymbol{\psi}_N, \lambda_N); \mathbf{e}; \Phi). \end{aligned} \quad (4.11)$$

We then use  $\mathcal{R}^{pr}((\mathbf{u}_N + \mathbf{e}, p_N + \epsilon); \mathbf{v}; \Phi) = 0, \forall \mathbf{v} \in X_{\mathcal{N}}$  to expand the right hand side of (4.11) to

$$\begin{aligned} 2\kappa \hat{a}(\hat{e}^\pm, \mathbf{e}; \Phi) &= \kappa (l(\mathbf{e}; \Phi) - a(\mathbf{u}_N, \mathbf{e}; \Phi) - b(\mathbf{e}, p_N; \Phi) \\ &\quad - c(\mathbf{u}_N, \mathbf{u}_N, \mathbf{e}; \Phi)) \\ &\quad - \kappa (l(\mathbf{e}; \Phi) - a(\mathbf{u}_N + \mathbf{e}, \mathbf{e}; \Phi) - b(\mathbf{e}, p_N + \epsilon; \Phi) \\ &\quad - c(\mathbf{u}_N + \mathbf{e}, \mathbf{u}_N + \mathbf{e}, \mathbf{e}; \Phi)) \\ &\quad \mp (-l(\mathbf{e}; \Phi) - a(\mathbf{e}, \boldsymbol{\psi}_N; \Phi) - b(\mathbf{e}, \lambda_N; \Phi) \\ &\quad - c(\mathbf{u}_N, \mathbf{e}, \boldsymbol{\psi}_N; \Phi) - c(\mathbf{e}, \mathbf{u}_N, \boldsymbol{\psi}_N; \Phi)) \\ &\quad \mp (-l(\boldsymbol{\psi}_N; \Phi) + a(\mathbf{u}_N + \mathbf{e}, \boldsymbol{\psi}_N; \Phi) + b(\boldsymbol{\psi}_N, p_N + \epsilon; \Phi) \\ &\quad + c(\mathbf{u}_N + \mathbf{e}, \mathbf{u}_N + \mathbf{e}, \boldsymbol{\psi}_N; \Phi)). \end{aligned} \quad (4.12)$$

We then exploit that

$$\begin{aligned}
b(\mathbf{u}, q; \Phi) &= 0 \quad \forall q \in M_{\mathcal{N}}(\Omega) \\
b(\mathbf{u}_N, q; \Phi) &= 0 \quad \forall q \in M_{\mathcal{N}}(\Omega) \\
b(\boldsymbol{\psi}_N, q; \Phi) &= 0 \quad \forall q \in M_{\mathcal{N}}(\Omega),
\end{aligned} \tag{4.13}$$

to get

$$\begin{aligned}
2\kappa\hat{a}(\hat{\mathbf{e}}^\pm, \mathbf{e}; \Phi) &= \kappa(a(\mathbf{e}, \mathbf{e}; \Phi) + c(\mathbf{u}_N, \mathbf{e}, \mathbf{e}; \Phi) + c(\mathbf{e}, \mathbf{u}_N, \mathbf{e}; \Phi) \\
&\quad + c(\mathbf{e}, \mathbf{e}, \mathbf{e}; \Phi)) \\
&\quad \mp (-l(\mathbf{e}; \Phi) - \mathcal{R}^{pr}((\mathbf{u}_N, p_N); \boldsymbol{\psi}_N; \Phi) + c(\mathbf{e}, \mathbf{e}, \boldsymbol{\psi}_N; \Phi)).
\end{aligned} \tag{4.14}$$

With this expression for the right hand side, equation (4.11) is then added to the bound equation (4.9), and we end up with

$$\begin{aligned}
s_N^\pm &= l(\mathbf{u}; \Phi) \pm \kappa(\hat{a}(\hat{\mathbf{e}}^\pm, \hat{\mathbf{e}}^\pm; \Phi) - 2\hat{a}(\hat{\mathbf{e}}^\pm, \mathbf{e}; \Phi) + a(\mathbf{e}, \mathbf{e}; \Phi)) \\
&\quad \pm \kappa(c(\mathbf{u}_N, \mathbf{e}, \mathbf{e}; \Phi) + c(\mathbf{e}, \mathbf{u}_N, \mathbf{e}; \Phi) + c(\mathbf{e}, \mathbf{e}, \mathbf{e}; \Phi)) \\
&\quad - c(\mathbf{e}, \mathbf{e}, \boldsymbol{\psi}_N; \Phi).
\end{aligned} \tag{4.15}$$

Due to (4.6) we have

$$\begin{aligned}
&\hat{a}(\hat{\mathbf{e}}^\pm, \hat{\mathbf{e}}^\pm; \Phi) - 2\hat{a}(\hat{\mathbf{e}}^\pm, \mathbf{e}; \Phi) + a(\mathbf{e}, \mathbf{e}; \Phi) \\
&\quad \geq \hat{a}(\hat{\mathbf{e}}^\pm, \hat{\mathbf{e}}^\pm; \Phi) - 2\hat{a}(\hat{\mathbf{e}}^\pm, \mathbf{e}; \Phi) + \hat{a}(\mathbf{e}, \mathbf{e}; \Phi) \\
&\quad = \hat{a}(\mathbf{e} - \hat{\mathbf{e}}^\pm, \mathbf{e} - \hat{\mathbf{e}}^\pm; \Phi) \geq 0.
\end{aligned} \tag{4.16}$$

To finalize the argument we need to show that the indefinite terms are smaller, or converge faster, than the positive definite term. To do this we need the *a priori* error estimate for  $|\mathbf{e}|_{H^1}$ .

In Section 3 we saw that the output of interest may be computed independently of  $\mathcal{N}$  by approximating the operators in (2.4) by affine combinations of parameter dependent functions and velocity/pressure dependent functions, like (3.11). To achieve the same  $\mathcal{N}$  independence for the error bounds, we have to approximate  $\hat{a}(\mathbf{v}, \mathbf{w}; \Phi)$  by a similar affine combination. We write

$$\hat{a}(\mathbf{v}, \mathbf{w}; \Phi) \approx \nu \sum_{q'=1}^{Q'} \sum_{m'=1}^{M_{q'}} \hat{\beta}_{m'}^{q'}(\Phi) \int_{\hat{\Omega}} \hat{g}_{m'}^{q'} \hat{a}^{q'}(\hat{\mathbf{v}}, \hat{\mathbf{w}}) d\hat{\Omega}, \tag{4.17}$$

where all integrals may be precomputed, and the parameter dependent coefficients are found using the empirical interpolation technique [3] presented in the next section.

In the offline stage we also find the solutions  $\hat{\mathbf{z}}_{jm'm}^{q'q}$  of

$$\int_{\hat{\Omega}} \hat{g}_{m'}^{q'} \hat{a}^{q'}(\hat{\mathbf{z}}_{jm'm}^{q'q}, \hat{\mathbf{v}}) d\hat{\Omega} = \int_{\hat{\Omega}} \tilde{g}_m^q a^q(\hat{\mathbf{u}}_j, \hat{\mathbf{v}}) d\hat{\Omega}, \quad \forall \hat{\mathbf{v}} \in \tilde{X}_{\mathcal{N}}(\hat{\Omega}), \tag{4.18}$$

for  $j = 1, \dots, 2N$ ,  $q' = 1, \dots, Q'$ ,  $q = 1, \dots, Q$ ,  $m' = 1, \dots, M_{q'}$ , and  $m = 1, \dots, M_q$ . Fortunately  $Q', Q, M_{q'}$ , and  $M_q$  are small integers, and the storage needed is moderate. Similar solutions are found with respect to the other operators in (2.4) and (2.12). These solutions represent the solutions of (4.7) and (4.8) for each velocity and pressure basis function in the reduced basis spaces, and in the online stage the error bounds (4.9) are found by adding the contributions together, multiplied with the correct reduced basis coefficients. This is explained in detail in [14], and the application here is just an extension of the work done there.

## 5 Offline/online decoupling

We have seen in Section 3 that if we are able to write the operators in the problem at hand as sums of products between parameter independent operators and parameter dependent operators, the coefficients of the reduced basis solution, and thus the output of interest, may be found independent of  $\mathcal{N}$ . Furthermore, we have seen in Section 4, that also the error bounds for the output may be found independent of  $\mathcal{N}$ , if such sums are available.

When using the geometry as a parameter, however, we see from (2.9) that, since the Jacobian of the geometric mapping is present under the integrals, we are unable to express the operators in the variational form of the steady Navier-Stokes equations as sums of products between parameter dependent operators and operators acting only on the basis functions.

Instead we express the operators as the integral of a sum of products between parameter dependent operators and operators acting only on the basis functions. We use the diffusion operator as an example, and below we show that for  $Q = 17$  we may write

$$a(\mathbf{u}, \mathbf{v}; \Phi) = \nu \int_{\hat{\Omega}} \sum_{q=1}^Q g^q(\Phi) a^q(\hat{\mathbf{u}}, \hat{\mathbf{v}}) d\hat{\Omega}, \quad (5.1)$$

where all  $a^q(\cdot, \cdot)$  are independent of  $\Phi$ .

The next step is to develop a basis for each  $g^q(\Phi)$  based on a few sample mappings  $\Phi_i^q$ . These sample mappings should be chosen such that they span the space of mappings in which we find the generic mapping  $\Phi$ , but they need not be equal to the sample mappings used to generate the velocity and pressure basis functions, and different  $g^q(\Phi)$  might use different sets of sample mappings.

For each  $g^q(\Phi)$  we use a basis comprising  $M_q$  basis functions to find an approximation

$$g^q(\Phi) \approx \sum_{m=1}^{M_q} \beta_m^q(\Phi) \tilde{g}_m^q, \quad (5.2)$$



where  $\beta_m^q(\Phi)$  are constant coefficients specific to the generic geometry, and  $\tilde{g}_m^q = \tilde{g}^q(\Phi_m^q)$  are basis functions for this specific  $g^q(\Phi)$ . The tilde indicates that the basis functions are not just  $g^q(\Phi)$  sampled at  $\Phi_m^q$ , which would yield a projection of  $g^q(\Phi)$  onto space spanned by the basis functions, but a basis carefully constructed through empirical interpolation; see [3]. The method is described below, and the coefficients in (5.2) are found by evaluating  $g^q(\Phi)$  at  $M_q$  isolated points  $t_i^q \in \hat{\Omega}$ , and solving a precomputed lower triangular matrix  $B^q$ .

Although projection gives a more accurate result in the approximation of  $g^q(\Phi)$ , it requires inner products of functions with resolution  $\mathcal{N}$  in the online stage. In empirical interpolation one only needs to sample each  $g^q(\Phi)$  at given isolated points in  $\hat{\Omega}$ , and for smooth functions, not much accuracy is lost. Since the coefficients  $\beta_m^q(\Phi)$  are constant over  $\hat{\Omega}$ , we are able to approximate the diffusion operator on the generic geometry as

$$a(\mathbf{u}, \mathbf{v}; \Phi) \approx \nu \sum_{q=1}^Q \sum_{m=1}^{M_q} \beta_m^q(\Phi) \int_{\hat{\Omega}} \tilde{g}^q(\Phi_m) a^q(\hat{\mathbf{u}}, \hat{\mathbf{v}}) d\hat{\Omega}. \quad (5.3)$$

We follow the procedure developed for one coefficient function in [3], and assume that  $g^q(\Phi) \in L^\infty(\hat{\Omega})$  for  $q = 1, \dots, Q$ . We also assume that the sample  $S_{N/2} = \{\Phi_i\}_{i=1}^{N/2}$  is sufficiently rich, such that  $g^q(\Phi_i)$  may be used to approximate  $g^q(\Phi)$ .

For each  $q$  we choose an arbitrary  $\Phi_1^q \in S_{N/2}$ . We define the sample  $S_1^q = \{\Phi_1^q\}$ , the function  $\xi_1^q(x) = g^q(\Phi_1^q(x))$ , and the space  $W_1^q = \text{span}\{\xi_1^q\}$ . By induction, we find

$$\Phi_M^q = \arg \max_{\Phi \in S_{N/2}} \inf_{z \in W_{M-1}^q} \|g^q(\Phi) - z\|_{L^\infty(\hat{\Omega})}, \quad (5.4)$$

and define the new sample  $S_M^q = S_{M-1}^q \cup \Phi_M^q$ , the function  $\xi_M^q(x) = g^q(\Phi_M^q(x))$ , and the space  $W_M^q = \text{span}\{\xi_m^q, 1 \leq m \leq M\}$ . We proceed until

$$\max_{\Phi \in S_{N/2}} \inf_{z \in W_{M-1}^q} \|g^q(\Phi) - z\|_{L^\infty(\hat{\Omega})} \quad (5.5)$$

reaches a desired limit, and let  $M_q = M$ . So far this is standard procedure.

The empirical interpolation technique then defines the basis functions  $\{\tilde{g}_i^q\}_{i=1}^{M_q}$  as scaled linear combinations of the functions in  $W_{M_q}^q$ , and the interpolation points  $\{t_i^q\}_{i=1}^{M_q}$ , such that

$$\tilde{g}_i^q(t_i^q) = 1, \quad i = 1, \dots, M_q \quad (5.6)$$

$$\tilde{g}_i^q(t_j^q) = 0, \quad i < j. \quad (5.7)$$

We first set  $t_1^q = \arg \text{ess sup}_{x \in \hat{\Omega}} |\xi_1^q(x)|$ , and define the scaled function  $\tilde{g}_1^q = \xi_1^q(x)/\xi_1^q(t_1^q)$ , such that  $\tilde{g}_1^q(t_1^q) = 1$ . Again we proceed by induction. For

$2 \leq M \leq M_q$  we solve the linear system

$$\sum_{j=1}^{M-1} \sigma_j^{M-1} \tilde{g}_j(t_i^q) = \xi_M(t_i^q), \quad 1 \leq i \leq M-1, \quad (5.8)$$

for the unknown coefficients  $\{\sigma_j^{M-1}\}_{j=1}^{M-1}$ . We define the linear combination  $r_M^q(x) = \xi_M^q(x) - \sum_{j=1}^{M-1} \sigma_j^{M-1} \tilde{g}_j^q(x)$ , and find its maximizing point  $t_M^q = \arg \operatorname{ess\,sup}_{x \in \hat{\Omega}} |r_M^q(x)|$ . The next basis function is then found by scaling  $\tilde{g}_M^q(x) = r_M^q(x)/r_M^q(t_M^q)$ .

The basis functions  $\{\tilde{g}_i^q\}_{i=1}^{M_q}$  are used together with the reduced basis velocity basis functions to compute the integrals in (3.11) in the offline stage. The interpolation points  $\{t_i^q\}_{i=1}^{M_q}$  are stored together with the matrices  $B^q$  defined by  $B_{ij}^q = \tilde{g}_i^q(t_j^q)$ ,  $1 \leq i, j \leq M_q$ .

For  $Q = 1$ , it is shown in [3] that the construction of the interpolation points is well-defined, and the functions  $\tilde{g}_1^q, \dots, \tilde{g}_M^q$  form a basis for  $W_M^q$ . From the construction,  $B^q$  is lower triangular, and thus invertible for all  $q$ . In the online stage of the reduced basis element method we find the  $\beta_m^q(\Phi)$ -coefficients needed in (3.11) by solving

$$\sum_{j=1}^{M_q} B_{ij}^q \beta_j^q(\Phi) = g(\Phi(t_i^q)), \quad 1 \leq i \leq M_q. \quad (5.9)$$

This only involves sampling  $g(\Phi)$  in  $M_q$  points, and due to the lower triangularity of  $B^q$  the system is already solved.

Expressions similar to (5.1) are developed for  $c(\mathbf{u}, \mathbf{v}, \mathbf{w}; \Phi)$  and  $l(\mathbf{u}; \Phi)$ , and bases for the geometric variables are constructed using the empirical interpolation technique. Due to the Piola transformation (3.5), we get

$$\int_{\Omega} p \nabla \cdot \mathbf{v} \, d\Omega = \int_{\hat{\Omega}} \hat{p} \hat{\nabla} \cdot \hat{\mathbf{v}} \, d\hat{\Omega}, \quad \forall \mathbf{v} \in X(\Omega), \quad (5.10)$$

where  $\hat{\mathbf{v}} = \Psi(\mathbf{v})$ , and the contributions from  $b_{\mathcal{N}}(\tilde{\mathbf{u}}_j, \tilde{q}_i)$  may thus be computed directly on the reference domain. In the offline stage of the reduced basis element method we then compute and store

$$\int_{\hat{\Omega}} \tilde{g}_m^q a^q(\hat{\mathbf{u}}_j, \hat{\mathbf{u}}_i) d\hat{\Omega}, \quad i, j = 1, \dots, 2N, \quad q = 1, \dots, Q, \quad m = 1, \dots, M_q, \quad (5.11)$$

$$\int_{\hat{\Omega}} \tilde{h}_m^r c^r(\hat{\mathbf{u}}_i, \hat{\mathbf{u}}_j, \hat{\mathbf{u}}_k) d\hat{\Omega}, \quad i, j, k = 1, \dots, 2N, \quad r = 1, \dots, R, \quad m = 1, \dots, M_r, \quad (5.12)$$

$$\int_{\hat{\Omega}} \hat{q}_i \hat{\nabla} \cdot \hat{\mathbf{u}}_j d\hat{\Omega}, \quad i = 1, \dots, N, \quad j = 1, \dots, 2N, \quad (5.13)$$

$$\int_{\hat{\Omega}} \tilde{f}_m^s l^s(\hat{\mathbf{u}}_i) d\hat{\Omega}, \quad i = 1, \dots, N, \quad s = 1, \quad m = 1, \dots, M_s. \quad (5.14)$$

We have  $Q = 17$  and  $R = 12$ , and if we assume that  $M_q, M_r$ , and  $M_s$  are of the same order, the online assembly of the stiffness matrix is done in  $\mathcal{O}(RM_r N^3)$  operations, i.e. independent of  $\mathcal{N}$ . To construct the reduced basis solution itself we sum over basis functions of size  $\mathcal{N}$ , such that we here involve  $\mathcal{N}$ -dependence, but in Section 3 we show that only the coefficients of the reduced basis solution are needed to estimate the error of the output of interest. We thus find the output and the error estimate independent of  $\mathcal{N}$ .

To find the geometric operators  $g^q(\Phi)$  in (5.1) we start with the diffusion operator as defined in terms of the reference variables in (2.9). We express the velocities as the Piola transformation of velocities stored on the reference domain such that we get

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \hat{\nabla} \left( \frac{1}{|\mathcal{J}|} \mathcal{J} \hat{\mathbf{v}} \right) \cdot \mathcal{J}^{-T} \hat{\nabla} \left( \frac{1}{|\mathcal{J}|} \mathcal{J} \hat{\mathbf{w}} \right) |J| d\hat{\Omega}. \quad (5.15)$$

After multiplying the Jacobian with the reference velocities, we get the component form of (5.15)

$$\begin{aligned} & \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \hat{\nabla} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{v}_\xi + \mathcal{J}_{12} \hat{v}_\eta) \right) \cdot \mathcal{J}^{-T} \hat{\nabla} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{w}_\xi + \mathcal{J}_{12} \hat{w}_\eta) \right) |J| d\hat{\Omega} \\ & + \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \hat{\nabla} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{21} \hat{v}_\xi + \mathcal{J}_{22} \hat{v}_\eta) \right) \cdot \mathcal{J}^{-T} \hat{\nabla} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{21} \hat{w}_\xi + \mathcal{J}_{22} \hat{w}_\eta) \right) |J| d\hat{\Omega}. \end{aligned} \quad (5.16)$$

Next we note that  $\hat{\nabla} u = [\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}]^T$ , and use this to get the equivalent form

$$\begin{aligned} & \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \left[ \frac{\partial}{\partial \xi} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{v}_\xi + \mathcal{J}_{12} \hat{v}_\eta) \right) \right] \cdot \mathcal{J}^{-T} \left[ \frac{\partial}{\partial \xi} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{w}_\xi + \mathcal{J}_{12} \hat{w}_\eta) \right) \right] |J| d\hat{\Omega} \\ & + \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \left[ \frac{\partial}{\partial \eta} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{v}_\xi + \mathcal{J}_{12} \hat{v}_\eta) \right) \right] \cdot \mathcal{J}^{-T} \left[ \frac{\partial}{\partial \eta} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{w}_\xi + \mathcal{J}_{12} \hat{w}_\eta) \right) \right] |J| d\hat{\Omega} \\ & + \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \left[ \frac{\partial}{\partial \xi} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{21} \hat{v}_\xi + \mathcal{J}_{22} \hat{v}_\eta) \right) \right] \cdot \mathcal{J}^{-T} \left[ \frac{\partial}{\partial \xi} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{21} \hat{w}_\xi + \mathcal{J}_{22} \hat{w}_\eta) \right) \right] |J| d\hat{\Omega} \\ & + \nu \int_{\hat{\Omega}} \mathcal{J}^{-T} \left[ \frac{\partial}{\partial \eta} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{21} \hat{v}_\xi + \mathcal{J}_{22} \hat{v}_\eta) \right) \right] \cdot \mathcal{J}^{-T} \left[ \frac{\partial}{\partial \eta} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{21} \hat{w}_\xi + \mathcal{J}_{22} \hat{w}_\eta) \right) \right] |J| d\hat{\Omega}. \end{aligned} \quad (5.17)$$

After multiplying each vector in (5.17) with  $\mathcal{J}^{-T}$  and writing out the inner products, we get the sum of four products under the integral. The first of these four products is as follows,

$$\begin{aligned} & \frac{1}{|\mathcal{J}|} \left( J_{22} \left( \frac{\partial}{\partial \xi} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{v}_\xi + \mathcal{J}_{12} \hat{v}_\eta) \right) \right) - J_{21} \left( \frac{\partial}{\partial \eta} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{v}_\xi + \mathcal{J}_{12} \hat{v}_\eta) \right) \right) \right) \\ & * \left( J_{22} \left( \frac{\partial}{\partial \xi} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{w}_\xi + \mathcal{J}_{12} \hat{w}_\eta) \right) \right) - J_{21} \left( \frac{\partial}{\partial \eta} \left( \frac{1}{|\mathcal{J}|} (\mathcal{J}_{11} \hat{w}_\xi + \mathcal{J}_{12} \hat{w}_\eta) \right) \right) \right). \end{aligned} \quad (5.18)$$

The other three products are similar, only the indices are different. To separate the elements of the Jacobian from the components of the reference

velocity, we first differentiate with respect to  $\xi$  and  $\eta$  inside the four products. From the first product (5.18) we then get

$$\begin{aligned} & \frac{1}{|J|} \left( \left( \mathcal{J}_{22} \frac{\partial}{\partial \xi} \left( \frac{\mathcal{J}_{11}}{|J|} \right) - \mathcal{J}_{21} \frac{\partial}{\partial \eta} \left( \frac{\mathcal{J}_{11}}{|J|} \right) \right) \hat{v}_\xi + \left( \mathcal{J}_{22} \frac{\partial}{\partial \xi} \left( \frac{\mathcal{J}_{12}}{|J|} \right) - \mathcal{J}_{21} \frac{\partial}{\partial \eta} \left( \frac{\mathcal{J}_{12}}{|J|} \right) \right) \hat{v}_\eta \right. \\ & \quad \left. + \frac{1}{|J|} \left( \mathcal{J}_{22} \left( \mathcal{J}_{11} \frac{\partial \hat{v}_\xi}{\partial \xi} + \mathcal{J}_{12} \frac{\partial \hat{v}_\eta}{\partial \xi} \right) - \mathcal{J}_{21} \left( \mathcal{J}_{11} \frac{\partial \hat{v}_\xi}{\partial \eta} + \mathcal{J}_{12} \frac{\partial \hat{v}_\eta}{\partial \eta} \right) \right) \right) \\ & * \left( \left( \mathcal{J}_{22} \frac{\partial}{\partial \xi} \left( \frac{\mathcal{J}_{11}}{|J|} \right) - \mathcal{J}_{21} \frac{\partial}{\partial \eta} \left( \frac{\mathcal{J}_{11}}{|J|} \right) \right) \hat{w}_\xi + \left( \mathcal{J}_{22} \frac{\partial}{\partial \xi} \left( \frac{\mathcal{J}_{12}}{|J|} \right) - \mathcal{J}_{21} \frac{\partial}{\partial \eta} \left( \frac{\mathcal{J}_{12}}{|J|} \right) \right) \hat{w}_\eta \right. \\ & \quad \left. + \frac{1}{|J|} \left( \mathcal{J}_{22} \left( \mathcal{J}_{11} \frac{\partial \hat{w}_\xi}{\partial \xi} + \mathcal{J}_{12} \frac{\partial \hat{w}_\eta}{\partial \xi} \right) - \mathcal{J}_{21} \left( \mathcal{J}_{11} \frac{\partial \hat{w}_\xi}{\partial \eta} + \mathcal{J}_{12} \frac{\partial \hat{w}_\eta}{\partial \eta} \right) \right) \right). \end{aligned} \quad (5.19)$$

After carrying out the multiplications for all four products, and collecting terms corresponding to the same velocity components, we find that we may decouple the diffusion operator as in (5.1) for  $Q = 17$ . As an example of the operators in the decoupling, we let

$$a^1(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = \frac{\partial \hat{u}_\xi}{\partial \xi} \frac{\partial \hat{v}_\xi}{\partial \xi} + \frac{\partial \hat{u}_\eta}{\partial \eta} \frac{\partial \hat{v}_\eta}{\partial \eta}, \quad (5.20)$$

with corresponding

$$g^1(\Phi) = \frac{1}{|J|^3} (\mathcal{J}_{11}^2 + \mathcal{J}_{21}^2) (\mathcal{J}_{12}^2 + \mathcal{J}_{22}^2). \quad (5.21)$$

## 6 Numerical examples

To generate basis functions for the reduced basis space, we need to define the basis geometries. Based on experience from deformed pipes and bifurcations in [7, 8], we choose deformed pipes defined by two parameters with limited variation. As a point of departure, we choose the quarter of an annulus depicted in Figure 1(a), with inner radius 2 and outer radius 3. All corners will always be perpendicular, the inflow boundary,  $\Gamma_{in} = \Gamma_1$ , will be vertical, and the outflow boundary,  $\Gamma_{out} = \Gamma_3$ , will be horizontal. The walled part of the boundary,  $\Gamma_w = \Gamma_2 \cup \Gamma_4$ , is curved. As was done in [12], we deform the inner curved boundary,  $\Gamma_2$ , and the outer curved boundary,  $\Gamma_4$ , by the following functional forms.

On  $\Gamma_2$ , the deformation from a quarter annulus is described as

$$\Delta x = a(1 - \cos(2\theta)) \sin(\theta) \quad (6.1)$$

$$\Delta y = a(1 - \cos(2\theta)) \cos(\theta), \quad (6.2)$$

where  $0 \leq \theta \leq \pi/2$ . This deformation of  $\Gamma_2$  also affects  $\Gamma_3$  by moving its left corner from  $(2, 0)$  to  $(2 + 2a, 0)$ , i.e., we must choose  $a < 0.5$  since the right corner is located at  $(3, 0)$ .

On  $\Gamma_4$ , the deformation from a quarter annulus is described as

$$\Delta x = b(1 - \cos(4\theta)) \sin(\theta) \quad (6.3)$$

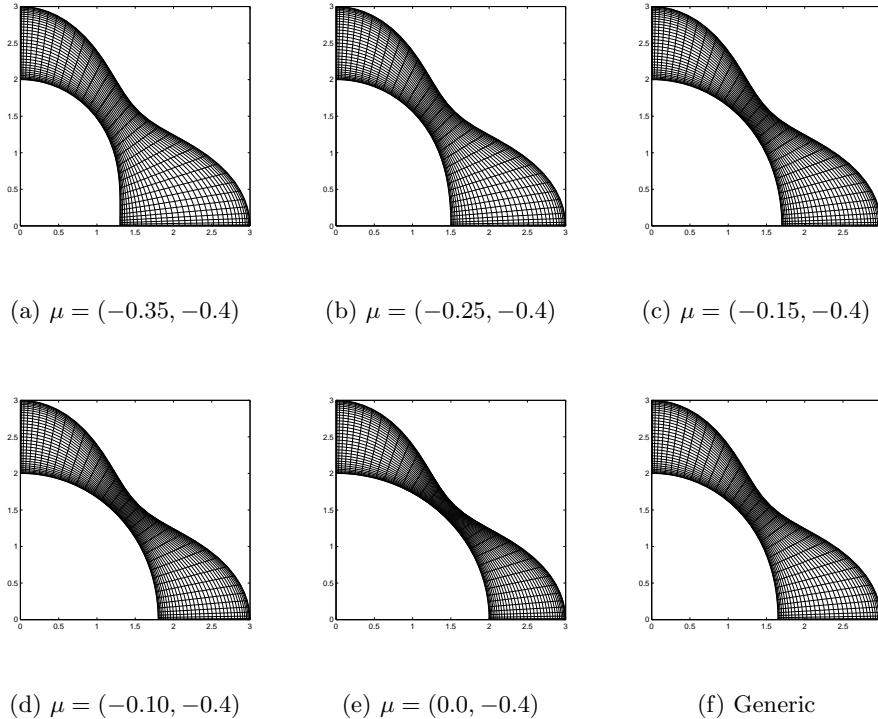


Figure 2: Some of the geometries used to construct basis functions. The geometries are defined by the parameter vector  $\mu = (a, b)$ .

$$\Delta y = b(1 - \cos(4\theta)) \cos(\theta), \quad (6.4)$$

where  $0 \leq \theta \leq \pi/2$ . This deformation of  $\Gamma_4$  does not affect neither  $\Gamma_1$  nor  $\Gamma_3$ , but the maximum amplitude of the deformation is at  $\theta = \pi/4$  and we must choose  $b$  such that  $\Gamma_4$  does not intersect  $\Gamma_2$ . We note that for this choice of parameterization of the deformations, we are able to write (2.4) as an affine combination of parameter independent operators and products of the constants  $a$  and  $b$ . We will however, use the non-affine decoupling developed in Section 5 together with empirical interpolation as a proof of concept for geometries where affine decoupling is not an option.

We define eight basis geometries,  $\{\Omega_i\}_{i=1}^8$ , by setting  $b = -0.4$  and  $a_i = -0.4 + 0.05i$ , and choose the generic geometry by setting  $b = -0.4$  and  $a = -0.175$ . In all numerical experiments we use  $\nu = 0.04$ , corresponding to a Reynolds number between one and 10, based on the maximum velocity across the inflow boundary, and the diameter of this boundary. Each basis function for the primal problem is found by solving the unsteady Navier-Stokes equations until the residual of the corresponding stationary Navier-Stokes problem (2.16) reaches  $10^{-6}$ . This stopping criterion does, in theory,

not need to be very strict, since the overall behaviour of the solutions only have to span a large portion of the possible solution space. If this behaviour comes from the steady state, or not, really does not matter. We also note that the incompressibility condition is satisfied at each time-step in the unsteady Navier-Stokes problem. For the linear dual problem we could use an iterative method such as minres, to find a basis function for each basis geometry, but since the framework for solving the unsteady Navier-Stokes equations was already there, we used a similar approach also for the dual basis functions. The resulting  $N = 16$  pressure basis functions are orthogonalized on the reference domain with respect to the  $L^2$  norm, and then we find the corresponding enriched velocity basis functions. All  $N = 32$  velocity basis functions are then orthogonalized with respect to the  $H^1$  semi-norm, and after mapping all basis functions to the generic geometry, we get

$$X_N(\Omega) = \text{span}\{ \{ (\Psi^{-1}(\hat{\mathbf{u}}_i), \Psi^{-1}(\hat{\boldsymbol{\psi}}_i), \Psi^{-1}(\hat{\mathbf{v}}_i), \Psi^{-1}(\hat{\boldsymbol{\phi}}_i)) \}_{i=1}^8 \} \quad (6.5)$$

$$M_N(\Omega) = \text{span}\{ \{ (\tilde{p}_i, \tilde{\boldsymbol{\lambda}}_i) \}_{i=1}^8 \}. \quad (6.6)$$

We first find the reduced basis solution without using the offline/online decoupling outlined in Section 5, but rather solve (3.2) and (3.3) using basis functions mapped to the generic domain. For the reduced basis solution  $(\mathbf{u}_N, p_N)$ , we use a Newton iteration to find the solution of the nonlinear problem (3.2). The solution of the reduced linear dual problem (3.3) is found by LU-factorization. The reconstructed errors used in the *a posteriori* error estimation, are found by a conjugate gradients solver on the given Poisson problems (4.7) and (4.8).

We also compute the reference solution  $(\mathbf{u}_{\mathcal{N}}, p_{\mathcal{N}})$  on the generic geometry by iterating the unsteady Navier-Stokes equations to steady state. The solution on the generic geometry, however, has to approximate the steady state very well, and the stopping criterion is now set to  $10^{-10}$ . The reduced basis solution  $(\mathbf{u}_N, p_N)$  is then compared to the reference solution by taking the  $H^1$  semi-norm of the error in the velocity, and the  $L^2$  norm of the error in the pressure. For  $\mathcal{N} = 20$ , the norms of the errors decrease rapidly as the number of basis functions used in the reduced basis approximation increases. This can be seen in Table 1, where the number  $N/2$  denotes the number of geometries used to generate the basis functions.

A closer look at the convergence of the norms of the errors indicate that the convergence is very fast for the four first basis geometries, whereas it levels off for the next four. This behaviour is similar to what we experienced for bifurcations in [8], and is due to our choice of the parameter  $a$  for the test geometry, in the middle of the  $a$ -values used to construct the fourth and fifth geometries.

The quality of the output bounds is measured by computing  $(s_N^+ - s)$  and  $(s - s_N^-)$ , and the results are presented in the two last columns of Table 1. Also these quantities converge fastest for the four first basis geometries,

$N/2$	$ u_N - u_{\mathcal{N}} _{H^1}$	$\ p_N - p_{\mathcal{N}}\ _{L^2}$	$s_N^+ - s$	$s - s_N^-$
1	$1.5 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	1.1	$3.9 \cdot 10^{-3}$
2	$1.7 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$1.4 \cdot 10^{-2}$	$1.0 \cdot 10^{-3}$
3	$2.1 \cdot 10^{-3}$	$8.7 \cdot 10^{-4}$	$3.7 \cdot 10^{-4}$	$2.7 \cdot 10^{-4}$
4	$6.7 \cdot 10^{-5}$	$2.4 \cdot 10^{-5}$	$1.0 \cdot 10^{-5}$	$5.4 \cdot 10^{-6}$
5	$2.2 \cdot 10^{-5}$	$1.3 \cdot 10^{-6}$	$8.6 \cdot 10^{-7}$	$5.9 \cdot 10^{-7}$
6	$2.1 \cdot 10^{-5}$	$1.3 \cdot 10^{-6}$	$6.4 \cdot 10^{-7}$	$4.7 \cdot 10^{-7}$
7	$2.1 \cdot 10^{-5}$	$1.2 \cdot 10^{-6}$	$2.4 \cdot 10^{-7}$	$1.3 \cdot 10^{-7}$
8	$1.6 \cdot 10^{-5}$	$8.3 \cdot 10^{-7}$	$-1.9 \cdot 10^{-9}$	$2.2 \cdot 10^{-9}$

Table 1: Results from the standard reduced basis element method when the polynomial degree  $\mathcal{N} = 20$ . The error in the reduced basis solution of the Navier-Stokes problem when the stopping criterion for the truth solution is  $10^{-10}$ . The test geometry is defined by  $a = -0.175$ .

before they level off for the next four. For the last basis geometry included we experience some trouble, as the upper bound suddenly appears to be lower than the actual output of interest. This contradicts the theory developed for the bound gaps, but can be explained by errors in the reference solution. Recall that the stopping criterion for the reference solution was set to  $10^{-10}$ , such that the local error along the  $\Gamma_{in}$  could reach levels large enough to distort the results in the output bounds.

When we next apply the offline/online decoupling for the reduced basis solution and the output of interest, the results for  $\mathcal{N} = 20$  are somewhat deteriorated. We see in Table 2 that the error in the velocity increases after the four first basis geometries. This was noticed already when the operators were written on the form (5.1) without offline/online decoupling and empirical interpolation. The reason seems to be the definition of the basis geometries through sine and cosine functions. The products of four sine or cosine functions with one full period each, are not resolved well enough. To see how this affects the offline/online decoupling and the empirical interpolation, we first produce results similar to Table 1, but now for  $\mathcal{N} = 34$ . This should resolve the geometric variables well, and the results without decoupling are presented in Table 3. As we see the results are quite similar to the results when  $\mathcal{N} = 20$ , and the convergence is again very good. In Table 4 we present the results from the offline/online decoupled case for  $\mathcal{N} = 34$ , and we see that they are just as good as the results in Table 3

## 7 Conclusions

In this paper we have used the reduced basis method with the geometry of the computational domain as a parameter to solve the steady Navier-Stokes problem on a deformed pipe. We have used the concepts of earlier

$N/2$	$ u_N - u_{\mathcal{N}} _{H^1}$	$\ p_N - p_{\mathcal{N}}\ _{L^2}$	$s_N^+ - s$	$s - s_N^-$
1	$1.5 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	1.1	$3.9 \cdot 10^{-3}$
2	$1.7 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$1.0 \cdot 10^{-3}$
3	$2.3 \cdot 10^{-3}$	$8.8 \cdot 10^{-4}$	$3.7 \cdot 10^{-4}$	$2.8 \cdot 10^{-4}$
4	$6.0 \cdot 10^{-4}$	$1.1 \cdot 10^{-4}$	$4.3 \cdot 10^{-5}$	$5.4 \cdot 10^{-5}$
5	$1.8 \cdot 10^{-3}$	$2.9 \cdot 10^{-4}$	$9.3 \cdot 10^{-5}$	$1.7 \cdot 10^{-4}$
6	$4.3 \cdot 10^{-3}$	$4.2 \cdot 10^{-4}$	$1.4 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$
7	$3.6 \cdot 10^{-3}$	$3.9 \cdot 10^{-4}$	$1.9 \cdot 10^{-5}$	$3.9 \cdot 10^{-5}$
8	$4.4 \cdot 10^{-3}$	$4.6 \cdot 10^{-4}$	$7.1 \cdot 10^{-7}$	$7.4 \cdot 10^{-7}$

Table 2: Results when we use offline/online decoupling, and the polynomial degree  $\mathcal{N} = 20$ . The error in the reduced basis solution of the Navier-Stokes problem when the stopping criterion for the truth solution is  $10^{-10}$ . The test geometry is defined by  $a = -0.175$ .

$N/2$	$ u_N - u_{\mathcal{N}} _{H^1}$	$\ p_N - p_{\mathcal{N}}\ _{L^2}$	$s_N^+ - s$	$s - s_N^-$
1	$1.5 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	1.1	$3.9 \cdot 10^{-3}$
2	$1.7 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$1.0 \cdot 10^{-3}$
3	$2.0 \cdot 10^{-3}$	$8.8 \cdot 10^{-4}$	$3.8 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$
4	$1.1 \cdot 10^{-4}$	$2.5 \cdot 10^{-5}$	$1.1 \cdot 10^{-5}$	$5.7 \cdot 10^{-6}$
5	$5.3 \cdot 10^{-5}$	$3.6 \cdot 10^{-6}$	$2.5 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$
6	$5.1 \cdot 10^{-5}$	$3.4 \cdot 10^{-6}$	$1.3 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$
7	$3.7 \cdot 10^{-5}$	$2.0 \cdot 10^{-6}$	$2.3 \cdot 10^{-7}$	$4.2 \cdot 10^{-7}$
8	$2.5 \cdot 10^{-5}$	$1.5 \cdot 10^{-6}$	$4.6 \cdot 10^{-9}$	$-4.0 \cdot 10^{-9}$

Table 3: Results from the standard reduced basis element method when the polynomial degree  $\mathcal{N} = 34$ . The error in the reduced basis solution of the Navier-Stokes problem when the stopping criterion for the truth solution is  $10^{-10}$ . The test geometry is defined by  $a = -0.175$ .

$N/2$	$ u_N - u_{\mathcal{N}} _{H^1}$	$\ p_N - p_{\mathcal{N}}\ _{L^2}$	$s_N^+ - s$	$s - s_N^-$
1	$1.5 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	1.1	$3.9 \cdot 10^{-3}$
2	$1.7 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$1.0 \cdot 10^{-3}$
3	$2.0 \cdot 10^{-3}$	$8.8 \cdot 10^{-4}$	$3.8 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$
4	$1.1 \cdot 10^{-4}$	$2.5 \cdot 10^{-5}$	$1.0 \cdot 10^{-5}$	$5.8 \cdot 10^{-6}$
5	$5.3 \cdot 10^{-5}$	$3.6 \cdot 10^{-6}$	$2.5 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$
6	$5.1 \cdot 10^{-5}$	$3.4 \cdot 10^{-6}$	$1.3 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$
7	$3.9 \cdot 10^{-5}$	$2.1 \cdot 10^{-6}$	$2.2 \cdot 10^{-7}$	$4.2 \cdot 10^{-7}$
8	$2.9 \cdot 10^{-5}$	$1.6 \cdot 10^{-6}$	$1.4 \cdot 10^{-9}$	$-1.4 \cdot 10^{-8}$

Table 4: Results when we use offline/online decoupling, and the polynomial degree  $\mathcal{N} = 34$ . The error in the reduced basis solution of the Navier-Stokes problem when the stopping criterion for the truth solution is  $10^{-10}$ . The test geometry is defined by  $a = -0.175$ .



work on the steady Stokes problem to show that even in the non-linear case, the reduced basis solution may provide a good approximation for very few precomputed basis functions.

We have developed *a posteriori* error bounds for the reduced basis output of interest, and presented numerical results to confirm the analysis. In addition, we have used empirical interpolation to decouple the operators with non-affine parameter dependence in an offline/online decomposition. The online effort to compute the output of interest and the error bounds is then independent of the high resolution  $\mathcal{N}$ . As long as the resolution of the geometry is good enough, very little accuracy is lost in this decoupling of the operators.

Future work will include the extension of the *a posteriori* analysis to multi-domain geometries. We will also consider time-dependent problems, and reduced basis modeling of fluid-structure interaction.

**Acknowledgement.** *This work has been supported by the Research Council of Norway through the BeMatA programme under contract 147044/431, by the RTN project HaeMOdel HPRN-CT-2002-00270, and the ACI project "le-poumon-vous-dis-je" granted by the Fond National pour la Science. The support is gratefully acknowledged.*

## References

- [1] R. Aris. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Publications, 1989.
- [2] I. Babuska. Error-bounds for finite element method. *Numer. Math.*, 16:322–333, 1971.
- [3] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera. An 'empirical interpolation' method: Application to efficient reduced-basis discretization of partial differential equations. *C. R. Acad. Sci. Paris, Serie I*, 339:667–672, 2004.
- [4] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers. *Rairo Anal. Numer.*, 8 R2:129–151, 1974.
- [5] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer Verlag, 1991.
- [6] W. J. Gordon and C. A. Hall. Construction of curvilinear co-ordinate systems and applications to mesh generation. *Int J. Numer. Methods Eng.*, 7:461–477, 1973.

- [7] A. E. Løvgrén, Y. Maday, and E. M. Rønquist. A reduced basis element method for the steady Stokes problem. *Submitted to M2AN*, 2004.
- [8] A. E. Løvgrén, Y. Maday, and E. M. Rønquist. A reduced basis element method for the steady Stokes problem: Application to flow in bifurcations. *In progress, to be submitted in June*, 2005.
- [9] L. Machiels, J. Peraire, and A. T. Patera. *A Posteriori* finite-element output bounds for the incompressible Navier-Stokes equations: Application to a natural convection problem. *J. Comput. Phys.*, 172:401–425, 2001.
- [10] Y. Maday and A. T. Patera. Spectral element methods for the Navier-Stokes equations. In *Noor A. (ed) State of the Art Surveys in Computational Mechanics*, pages 71–143, 1989.
- [11] Y. Maday, A. T. Patera, and E. M. Rønquist. The  $P_N \times P_{N-2}$  method for the approximation of the Stokes problem. *Technical Report No. 92009, Department of Mechanical Engineering, Massachusetts Institute of Technology*, 1992.
- [12] Y. Maday and E. M. Rønquist. A reduced-basis element method. *J. Sci. Comput.*, 17:447–459, 2002.
- [13] J. S. Peterson. The reduced basis method for incompressible viscous flow calculations. *SIAM J. Sci. Stat. Comput.*, 10(4):777–786, 1989.
- [14] C. Prud’homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera, and G. Turinici. Reliable real-time solution of parametrized partial differential equations: Reduced basis output bound methods. *J. Fluids Engineering*, 124:70–80, 2002.
- [15] G. Rozza. Reduced-basis methods for elliptic equations in sub-domains with a posteriori error bounds and adaptivity. *To appear in Applied Numerical Mathematics*, 2005.