Adaptive Output-Feedback Stabilization of 2×2 Linear Hyperbolic PDEs with Actuator and Sensor Delay

Henrik Anfinsen and Ole Morten Aamo

***Abstract*— We derive an adaptive output-feedback stabilizing controller for a system of 2×2 linear hyperbolic partial differential equations (PDEs) with delayed, anti-collocated sensing and control. This is done by using a series of transformations to show that the system is equivalent to delay-free systems for which such controllers have been derived. The only required knowledge of the system is the system's transport delays, the sensor and actuation delays and the sign of the product of the actuation and sensing scaling constants. The theory is verified in a simulation.**

I. Introduction

Linear first order hyperbolic partial differential equations (PDEs) describe transport phenomena with finite propagation speeds. Consequently, many processes arising in applications can be modeled by them, for instance sensor and actuation delays [1], oil wells [2] and predator-prey systems [3]. Linear first order hyperbolic PDEs have therefore been subject to extensive research regarding estimation and control.

In the last years, infinite-dimensional backstepping, originally derived for parabolic PDEs in [4], has been further developed for use on hyperbolic PDEs of increasing complexity. When using this method for controller design, an invertible Volterra transformation and a control law are used to map the original system of PDEs into a simpler target system whose stability is easier to establish. By invertibility of the Volterra transformation, stability of the original system then follows. The first use of backstepping on hyperbolic PDEs was for a single, first order PDE in [1]. The extension to 2×2 systems of the same type considered in the current paper was done in [5], while extensions to more complicated systems of linear hyperbolic PDEs have been done in [6] and [7].

Infinite-dimensional backstepping has also recently been used to derive adaptive controllers for hyperbolic PDEs, with the first result being presented in [8] where a 1–D system with a single, uncertain parameter was adaptively stabilized using boundary sensing only. This result was later extended in [9] to a slightly more general class of systems, offering a solution to a model reference adaptive control (MRAC) problem for which the stabilization problem is a subproblem. The only required knowledge of the system was its total transport delay and the sign of the product of the actuation and sensing scaling constants. The method in [8] has also been extended to 2×2 systems simultaneously in [10] and [11], where in both these papers a 2×2 system of linear hyperbolic PDEs with uncertain in-domain coefficients was adaptively stabilized using boundary sensing only. The methods from [9] and [10] were later combined to solve both an MRAC and a stabilization problem for 2×2 systems in [12]. However, the controllers of all these three papers had a higher dynamical order than the solution in [8]. Other variations of adaptive controllers for PDEs based on backstepping can be found in [13], [14], [15], [16].

The work presented here is an extension of the result from [10] and [11]: we derive an adaptive output-feedback stabilizing controller for a class of systems of 2×2 linear hyperbolic PDEs with actuator delay, using a single boundary sensing which is also allowed to be delayed. We will through a series of transformations show that the 2×2 system, with actuator and sensor delays, is for small sensor delays equivalent to a delay-free 2×2 system, while for large sensor delays, it is equivalent to a delay-free 1–D system. The only required knowledge of this system is the same that was assumed in [9], as well as the magnitude of the actuator and sensor delays.

Notation: We define the following domains $\mathcal{D} = \{x \mid x \in [0,1]\}$, $\mathcal{D}_1 = \{(x,t) \mid x \in \mathcal{D}, t \geq 0\}$ and $\mathcal{T} = \{(x,\xi) \mid 0 \leq \xi \leq x \leq 1\}$. For the variable $u : \mathcal{D} \to \mathbb{R}$ (or $u : \mathcal{D}_1 \to \mathbb{R}$), we define $||u|| = \sqrt{\int_0^1 u^2(x)dx}$, $||u||_\infty = \sup_{x \in \mathcal{D}} |u(x)|$, and $\mathcal{B}(\mathcal{D}) = \{u \mid ||u||_\infty < \infty\}$. For a function $f(t)$ we define $f \in \mathcal{L}_p \Leftrightarrow \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty$, for $p = 1, 2$, and $f \in \mathcal{L}_\infty \Leftrightarrow \sup_{t \geq 0} |f(t)| < \infty$.

II. Problem statement

Consider a system of 2×2 linear hyperbolic PDEs with time-delayed, scaled anti-collocated actuation and sensing

$$u_t(x,t) + \lambda(x) u_x(x,t) = c_1(x) v(x,t) \tag{1a}$$
$$v_t(x,t) - \mu(x) v_x(x,t) = c_2(x) u(x,t) \tag{1b}$$
$$u(0,t) = q v(0,t) \tag{1c}$$
$$v(1,t) = k_1 U(t - d_1) \tag{1d}$$
$$y(t) = k_2 v(0, t - d_2) \tag{1e}$$

for u, v defined over $\mathcal{D}_1$. The parameters μ, λ, c_1, c_2, q, k_1, k_2 are unknown, but assumed to satisfy

$$\mu, \lambda \in C^1(\mathcal{D}), \qquad \mu(x), \lambda(x) > 0 \ \forall x \in \mathcal{D} \tag{2a}$$
$$c_1, c_2 \in C^0(\mathcal{D}), \qquad q, k_1, k_2 \in \mathbb{R}\backslash\{0\} \tag{2b}$$
$$d_1, d_2 \in \mathbb{R}, \qquad d_1, d_2 \geq 0. \tag{2c}$$

The quantities d_1 and d_2 are the actuation and measurement delays, respectively, which we for natural reasons assume

The authors are with the Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim N-7491, Norway. (e-mail: henrik.anfinsen@ntnu.no; aamo@ntnu.no).

nonnegative. The system's initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ are assumed to satisfy $u_0, v_0 \in \mathcal{B}(\mathcal{D})$.

The goal is to design a control law $U(t)$ in (1d) so that system (1) is adaptively stabilized. Moreover, all additional variables in the closed loop system should be bounded pointwise in space. We seek to achieve this from minimal knowledge of the system parameters. Specifically, the only knowledge required of the system is stated in the following assumption.

Assumption 1: We assume that the following quantities are known

1) the actuator and sensor delays

$$d_1 = \epsilon_1^{-1} \qquad\qquad d_2 = \epsilon_2^{-1}, \tag{3}$$

2) the transport delays in each direction, that is

$$d_\alpha = \bar{\lambda}^{-1} = \int_0^1 \frac{d\gamma}{\lambda(\gamma)}, \quad d_\beta = \bar{\mu}^{-1} = \int_0^1 \frac{d\gamma}{\mu(\gamma)}, \tag{4}$$

3) the sign of the product $k_1 k_2$.

Remark 2: We have for simplicity restricted ourselves to systems with no reflection term at $x = 1$, and where $q \neq 0$, which was also assumed in [12]. However, the method extends to the case $q = 0$ and nonzero reflection coefficient at $x = 1$ by using the swapping method proposed for $n+1$ systems in [17].

III. Mapping to canonical form

We will in this section introduce a series of transformations that brings the system to a canonical form, which can be simplified if $d_2 \geq d_\alpha$. This canonical form is known from previous literature.

A. Decoupling by backstepping

The following result was proved in [5].

Lemma 3: The system (1) is through an invertible backstepping transformation equivalent to the following system

$$\check{\alpha}_t(x,t) + \lambda(x)\check{\alpha}_x(x,t) = 0,\ \check{\alpha}(x,0) = \check{\alpha}_0(x) \tag{5a}$$
$$\check{\beta}_t(x,t) - \mu(x)\check{\beta}_x(x,t) = 0,\ \check{\beta}(x,0) = \check{\beta}_0(x) \tag{5b}$$
$$\check{\alpha}(0,t) = q\check{\beta}(0,t) \tag{5c}$$
$$\check{\beta}(1,t) = k_1 U(t-d_1) - \int_0^1 L_1(\xi)\check{\alpha}(\xi,t)d\xi - \int_0^1 L_2(\xi)\check{\beta}(\xi,t)d\xi \tag{5d}$$
$$y(t) = k_2\check{\beta}(0,t-d_2) \tag{5e}$$

where L_1, L_2, L_3 are (continuous) functions of the unknown parameters λ, μ, c_1, c_2, q, and $\check{\alpha}_0, \check{\beta}_0 \in \mathcal{B}(\mathcal{D})$.

Proof: See [5]. ∎

B. Constant transport speeds and scaling

Lemma 4: The invertible mapping

$$\alpha(x,t) = \frac{k_2}{q}\check{\alpha}(h_\alpha^{-1}(x),t), \quad \beta(x,t) = k_2\check{\beta}(h_\beta^{-1}(x),t) \tag{6}$$

where

$$h_\alpha(x) = \frac{1}{d_\alpha}\int_0^x \frac{d\gamma}{\lambda(\gamma)}, \qquad h_\beta(x) = \frac{1}{d_\beta}\int_0^x \frac{d\gamma}{\mu(\gamma)} \tag{7}$$

transforms (5) into

$$\alpha_t(x,t) + \bar{\lambda}\alpha_x(x,t) = 0,\ \alpha(x,0) = \alpha_0(x) \tag{8a}$$
$$\beta_t(x,t) - \bar{\mu}\beta_x(x,t) = 0,\ \beta(x,0) = \beta_0(x) \tag{8b}$$
$$\alpha(0,t) = \beta(0,t) \tag{8c}$$
$$\beta(1,t) = \rho U(t-d_1) + \int_0^1 \sigma_1(\xi)\alpha(\xi,t)d\xi + \int_0^1 \sigma_2(\xi)\beta(\xi,t)d\xi, \tag{8d}$$
$$y(t) = \beta(0,t-d_2) \tag{8e}$$

where $\bar{\lambda}, \bar{\mu}$ are defined in (4), and σ_1, σ_2, ρ are functions of L_1, L_2, q, k_1, k_2, with $\alpha_0, \beta_0 \in \mathcal{B}(\mathcal{D})$.

Proof: We note from (7) that h_α and h_β are strictly increasing and thus invertible. The invertiblility of the transform (6) therefore follows. The rest of the proof follows immediately from insertion and noting that

$$h'_\alpha(x) = \frac{1}{d_\alpha}\frac{1}{\lambda(x)}, \qquad h'_\beta(x) = \frac{1}{d_\beta}\frac{1}{\mu(x)} \tag{9a}$$
$$h_\alpha(0) = h_\beta(0) = 0, \qquad h_\alpha(1) = h_\beta(1) = 1, \tag{9b}$$

and is therefore omitted. The new parameters are given as

$$\sigma_1(x) = -k_2 d_\alpha \lambda(h_\alpha^{-1}(x))L_1(h_\alpha^{-1}(x)) \tag{10a}$$
$$\sigma_2(x) = -d_\beta\mu(h_\beta^{-1}(x))L_2(h_\beta^{-1}(x)), \qquad \rho = k_1 k_2. \tag{10b}$$

∎

C. Actuator and sensor dynamics

We now augment the system with an additional filter ν that can be used to represent the actuator delay, and also split the variable α in two variables w and ζ.

Lemma 5: System (8) can be represented as

$$w_t(x,t) + \epsilon_3 w_x(x,t) = 0,\ w(x,0) = w_0(x) \tag{11a}$$
$$\zeta_t(x,t) + \epsilon_2\zeta_x(x,t) = 0,\ \zeta(x,0) = \zeta_0(x) \tag{11b}$$
$$\beta_t(x,t) - \bar{\mu}\beta_x(x,t) = 0,\ \beta(x,0) = \beta_0(x) \tag{11c}$$
$$\nu_t(x,t) - \epsilon_1\nu_x(x,t) = 0,\ \nu(x,0) = \nu_0(x) \tag{11d}$$
$$w(0,t) = \zeta(1,t) \tag{11e}$$
$$\zeta(0,t) = \beta(0,t) \tag{11f}$$
$$\beta(1,t) = \nu(0,t) + \int_0^1 \sigma_3(\xi)\zeta(\xi,t)d\xi + \int_0^1 \sigma_4(\xi)w(\xi,t)d\xi + \int_0^1 \sigma_2(\xi)\beta(\xi,t)d\xi \tag{11g}$$
$$\nu(1,t) = \rho U(t) \tag{11h}$$
$$y(t) = \zeta(1,t) + e(1,t) \tag{11i}$$

where e is governed by

$$e_t(x,t) + \epsilon_2 e_x(x,t) = 0, \qquad e(0,t) = 0 \tag{12a}$$

$$e(x,0) = e_0(x) \tag{12b}$$

with ϵ_1, ϵ_2 defined in (3), and

$$\epsilon_3 = \max\{(d_\alpha - d_2)^{-1}, 0\} \tag{13}$$

with σ_3 and σ_4 being functions of d_α, d_2 and σ_1, and $w_0, \zeta_0, \beta_0, \nu_0, e_0 \in \mathcal{B}(\mathcal{D})$. Moreover, if $d_2 \geq d_\alpha$, then

$$\sigma_4 \equiv 0 \tag{14}$$

and (11a) and (11e) can be discarded from the set of equations.

Proof: The variable α satisfying (8a) is a pure transport equation. If $d_2 < d_\alpha$, we simply split the transport equation into two parts, ζ, representing a delay of d_2, and w representing a delay of $d_3 = d_\alpha - d_2$, so that the total delay of ζ and w is d_α. Specifically, ζ and w are given from α through

$$\zeta(x,t) = \alpha\left(\frac{d_2}{d_\alpha}x, t\right) \tag{15a}$$

$$w(x,t) = \alpha\left(\frac{d_3}{d_\alpha}x + \frac{d_2}{d_\alpha}, t\right) \tag{15b}$$

or, reversely

$$\alpha(x,t) = \begin{cases} \zeta\left(\frac{d_\alpha}{d_2}x, t\right) & \text{for } x \in [0, \frac{d_2}{d_\alpha}] \\ w\left(\frac{d_\alpha}{d_3}x - \frac{d_2}{d_3}, t\right) & \text{for } x \in [\frac{d_2}{d_\alpha}, 1]. \end{cases} \tag{16}$$

Now, if on the other hand $d_\alpha \leq d_2$, the variable ζ will contain all the information in the variable α. Specifically, α can be reconstructed from ζ alone as

$$\alpha(x,t) = \zeta\left(\frac{d_\alpha}{d_2}x, t\right), \tag{17}$$

provided the initial conditions match. In either case, we can insert for α in the integral in (8d), and find a boundary condition on the form (11g). Specifically, σ_3 and σ_4 are given as

$$\sigma_3(x) = \begin{cases} \frac{d_\alpha}{d_2}\sigma_1\left(\frac{d_2}{d_\alpha}x\right) & \text{if } d_2 < d_\alpha \\ \sigma_{3,2}(x) & \text{otherwise} \end{cases} \tag{18a}$$

$$\sigma_4(x) = \begin{cases} \frac{d_3}{d_\alpha}\sigma_1\left(\frac{d_3}{d_\alpha}x + \frac{d_2}{d_\alpha}\right) & \text{if } d_2 < d_\alpha \\ 0 & \text{otherwise} \end{cases} \tag{18b}$$

where

$$\sigma_{3,2}(x) = \begin{cases} \frac{d_2}{d_\alpha}\sigma_1\left(\frac{d_2}{d_\alpha}x\right) & \text{for } 0 \leq x \leq \frac{d_\alpha}{d_2} \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

The delays in the actuation and sensing signals can be represented using linear hyperbolic PDEs. We have represented the actuation delay using the PDE ν in (11d), (11h), while the sensor delay can be represented using a PDE on the form

$$\pi_t(x,t) + \epsilon_2\pi_x(x,t) = 0, \qquad \pi(x,0) = \pi_0(x) \tag{20a}$$

$$\pi(0,t) = \beta(0,t), \qquad y(t) = \pi(1,t). \tag{20b}$$

with $\pi_0 \in \mathcal{B}(\mathcal{D})$. These representations are valid subject to the restriction that the initial conditions $\nu_0(x)$ and $\pi_0(x)$ are chosen to match past values of $U(t)$ and $\beta(0,t)$. Lastly, in (11), we have used the PDE ζ to generate a non-adaptive estimate of the PDE π as

$$\pi(x,t) = \zeta(x,t) + e(x,t), \tag{21}$$

from which we straight forwardly find that e satisfies the dynamics (12a), and the measurement (20b) becomes (11i).

It is obvious that if $d_2 \geq d_\alpha$, the variable w is surplus, and can hence be removed from the equations (by for instance choosing $||w_0||_\infty = 0$).

We note that in any case, $e(1,t) = 0$ for $t \geq d_2$, so that $y(t) = \zeta(1,t)$ for $t \geq d_2$. ■

D. Merging two states

System (11) is a cascade in the following order: ν, β, ζ and w, with the latter being identically zero if $d_2 \geq d_\alpha$. We will in the next lemma merge ζ and β into a single PDE.

Lemma 6: System (11) is equivalent to

$$w_t(x,t) + \epsilon_3 w_x(x,t) = 0, \; w(x,0) = w_0(x) \tag{22a}$$

$$\varphi_t(x,t) - \epsilon_4\varphi_x(x,t) = 0, \; \varphi(x,0) = \varphi_0(x) \tag{22b}$$

$$\nu_t(x,t) - \epsilon_1\nu_x(x,t) = 0, \; \nu(x,0) = \nu_0(x) \tag{22c}$$

$$w(0,t) = \varphi(0,t) \tag{22d}$$

$$\varphi(1,t) = \nu(0,t) + \int_0^1 \sigma_5(\xi)\varphi(\xi,t)d\xi + \int_0^1 \sigma_4(\xi)w(\xi,t)d\xi \tag{22e}$$

$$\nu(1,t) = \rho U(t) \tag{22f}$$

$$y(t) = \varphi(0,t) + e(1,t) \tag{22g}$$

where

$$\epsilon_4 = d_4^{-1} = (d_2 + d_\beta)^{-1} \tag{23}$$

with $w_0, \varphi_0, \nu_0 \in \mathcal{B}(\mathcal{D})$, and with $||w||_\infty = 0$ if $d_2 \geq d_\alpha$, and $e(1,t) = 0$ for $t \geq d_2$.

Proof: The variable φ is defined from ζ and β as

$$\varphi(x,t) = \begin{cases} \zeta(k_1(x_1 - x), t) & \text{for } x \in [0, x_1] \\ \beta(k_2(x - x_1), t) & \text{for } x \in [x_1, 1] \end{cases} \tag{24}$$

where

$$k_1 = \frac{\epsilon_2}{\epsilon_4}, \qquad k_2 = \frac{\bar{\mu}}{\epsilon_4}, \qquad x_1 = k_1^{-1}. \tag{25}$$

Straightforward calculations, using the dynamics (11b)–(11a) give the dynamics (22b) with boundary condition (22c) when

$$\sigma_5(x) = \begin{cases} k_1\sigma_3(1 - k_1 x) & \text{for } x \in [0, x_1] \\ k_2\sigma_2(k_2(x - x_1)) & \text{for } x \in (x_1, 1] \end{cases} \tag{26}$$

The measurement (22g) comes from (11i) and noting that

$$\zeta(1,t) = \varphi(0,t). \tag{27}$$

■

E. Mixed Volterra-Fredholm transformation

Lemma 7: System (22) is equivalent to the following system

$$w_t(x,t) + \epsilon_3 w_x(x,t) = 0,\ w(x,0) = w_0(x) \tag{28a}$$

$$\omega_t(x,t) - \epsilon_4\omega_x(x,t) = \theta_1(x)\varphi(0,t),\ \omega(x,0) = \omega_0(x) \tag{28b}$$

$$\eta_t(x,t) - \epsilon_1\eta_x(x,t) = \theta_2(x)\varphi(0,t),\ \eta(x,0) = \eta_0(x) \tag{28c}$$

$$w(0,t) = \omega(0,t) \tag{28d}$$

$$\omega(1,t) = \eta(0,t) \tag{28e}$$

$$\eta(1,t) = \rho U(t) + \int_0^1 \kappa(\xi)w(\xi,t)d\xi \tag{28f}$$

where θ_1, θ_2 and κ are functions of σ_4, σ_5, and where $w_0, \omega_0, \eta_0 \in \mathcal{B}(\mathcal{D})$.

Proof: Consider now the transformation $(w,\varphi,\nu) \to (w,\omega,\eta)$, given as

$$\omega(x,t) = \varphi(x,t) - \int_0^x A(x,\xi)\varphi(\xi,t)d\xi \tag{29a}$$

$$\eta(x,t) = \nu(x,t) - \int_0^1 B(x,\xi)w(\xi,t)d\xi \tag{29b}$$

where A, B satisfy the PDE

$$A_x(x,\xi) + A_\xi(x,\xi) = 0, \quad A(1,\xi) = \sigma_5(\xi) \tag{30a}$$

$$\epsilon_1 B_x(x,\xi) - \epsilon_3 B_\xi(x,\xi) = 0, \quad B(x,1) = 0 \tag{30b}$$

$$B(0,\xi) = -\sigma_4(\xi). \tag{30c}$$

Here, A is defined over the triangular domain $\mathcal{T}$ while B is defined over a square domain $\mathcal{S} = \mathcal{D}^2$. Transformation (29a) with a kernel satisfying (30a) is a standard, invertible Volterra backstepping transformation, and therefore φ is uniquely determined from ω. Since (29b) trivially provides ν from η and w, the existence of the inverse $(w,\omega,\eta) \to (w,\phi,\nu)$ is established. The well-posedness of (30) is trivial to prove by considering its characteristics, and we omit further details due to page limitations.

Next, we prove that (w,ω,η) has the dynamics (28) provided A and B are selected according to (30). Differentiating (29) with respect to time and space, inserting the dynamics (22a)–(22b), integration by parts and inserting the boundary condition (22d), we obtain

$$\begin{aligned}\varphi_t(x,t) &= \omega_t(x,t) + \epsilon_4 A(x,x)\varphi(x,t)\\ &- \epsilon_4 A(x,0)\varphi(0,t) - \epsilon_4\int_0^x A_\xi(x,\xi)\varphi(\xi,t)d\xi\end{aligned} \tag{31a}$$

$$\begin{aligned}\nu_t(x,t) &= \eta_t(x,t) - \epsilon_3 B(x,1)w(1,t)\\ &+ \epsilon_3 B(x,0)w(0,t) + \epsilon_3\int_0^1 B_\xi(x,\xi)w(\xi,t)d\xi\end{aligned} \tag{31b}$$

and

$$\begin{aligned}\varphi_x(x,t) &= \omega_x(x,t) + A(x,x)\varphi(x,t)\\ &+ \int_0^x A_x(x,\xi)\varphi(\xi,t)d\xi\end{aligned} \tag{32a}$$

$$\nu_x(x,t) = \eta_x(x,t) + \int_0^1 B_x(x,\xi)w(\xi,t)d\xi. \tag{32b}$$

Inserting (31) and (32) into the dynamics (22b)–(22c), we find

$$\begin{aligned}\varphi_t(x,t) - \epsilon_4\varphi_x(x,t) &= \omega_t(x,t) - \epsilon_4\omega_x(x,t)\\ &- \epsilon_4\int_0^x (A_x(x,\xi) + A_\xi(x,\xi))\varphi(\xi,t)d\xi\\ &- \epsilon_4 A(x,0)\varphi(0,t) = 0\end{aligned} \tag{33a}$$

$$\begin{aligned}\nu_t(x,t) - \epsilon_1\nu_x(x,t) &= \eta_t(x,t) - \epsilon_1\eta_x(x,t)\\ &- \int_0^1 (\epsilon_1 B_x(x,\xi) - \epsilon_3 B_\xi(x,\xi))w(\xi,t)d\xi\\ &- \epsilon_3 B(x,1)w(1,t) + \epsilon_3 B(x,0)\varphi(0,t) = 0.\end{aligned} \tag{33b}$$

Using (30a)–(30b) yields (28b)–(28c) with

$$\theta_1(x) = \epsilon_4 A(x,0) \qquad \theta_2(x) = -\epsilon_3 B(x,0). \tag{34}$$

Inserting (29) into (22d)–(22f), we obtain

$$w(0,t) = \omega(0,t) \tag{35a}$$

$$\begin{aligned}\omega(1,t) &= \eta(0,t) - \int_0^1 [A(1,\xi) - \sigma_5(\xi)]\,\varphi(\xi,t)d\xi\\ &+ \int_0^1 [B(0,\xi) + \sigma_4(\xi)]\,w(\xi,t)d\xi\end{aligned} \tag{35b}$$

$$\eta(1,t) = \rho U(t) - \int_0^1 B(1,\xi)w(\xi,t)d\xi. \tag{35c}$$

Using (30a) and (30c) results in the boundary conditions (28d)–(28f), with

$$\kappa(\xi) = -B(1,\xi). \tag{36}$$

■

F. Canonical form

System (28) is a cascade in the following order: η, ω, and w, with the latter being identically zero if $d_2 \geq d_\alpha$. This will be utilized in the following lemma, bringing the system to a canonical form

Lemma 8: System (28) is equivalent to

$$w_t(x,t) + \epsilon_3 w_x(x,t) = 0,\ w(x,0) = w_0(x) \tag{37a}$$

$$z_t(x,t) - \epsilon_5 z_x(x,t) = \theta(x)z(0,t),\ z(x,0) = z_0(x) \tag{37b}$$

$$w(0,t) = z(0,t) \tag{37c}$$

$$z(1,t) = \rho U(t) + \int_0^1 \kappa(\xi)w(\xi,t)d\xi \tag{37d}$$

$$y(t) = z(0,t) + e(1,t), \tag{37e}$$

where

$$\epsilon_5 = d_5^{-1} = (d_1 + d_4)^{-1}, \tag{38}$$

with $w_0, z_0 \in \mathcal{B}(\mathcal{D})$, and with $||w||_\infty = 0$ if $d_2 \geq d_\alpha$, and $e(1,t) = 0$ for $t \geq d_2$.

Proof: We define the new variable z as

$$z(x,t) = \begin{cases}\omega(k_2 x,t) & \text{for } x \in [0,x_2]\\ \eta(k_3(x - x_2),t) & \text{for } x \in (x_2,1]\end{cases} \tag{39}$$

where

$$k_2 = \frac{\epsilon_4}{\epsilon_5}, \qquad k_3 = \frac{\epsilon_1}{\epsilon_5}, \qquad x_2 = k_2^{-1}. \tag{40}$$

Straightforward calculations, using the dynamics (28b)–(28c) give the dynamics (37b) with θ given as

$$\theta(x) = \begin{cases} \theta_1(k_2 x) & \text{for } x \in [0, x_2] \\ \theta_2(k_3(x - x_2)) & \text{for } x \in (x_2, 1] \end{cases} \tag{41}$$

The boundary condition (37d) comes from noting that

$$z(1,t) = \eta(1,t) \tag{42}$$

and using the boundary condition (28f). ■

IV. Adaptive control

Since the term $e(1,t)$ in (37e) is zero for $t \geq d_2$, system (37) is for $d_2 < d_\alpha$ on the same form as the system which was adaptively stabilized in [12], while for $d_2 \geq d_\alpha$, $w \equiv 0$ and the system reduces to the form which was adaptively stabilized in [9]. Both controllers, however, require the following assumption.

Assumption 9: Bounds on ρ, θ and κ, are known. That is, we are in knowledge of some constants $\underline{\rho}, \bar{\rho}, \underline{\theta}, \bar{\theta}, \underline{\kappa}, \bar{\kappa}$, so that

$$\underline{\rho} \leq \rho \leq \bar{\rho} \qquad \underline{\theta} \leq \theta(x) \leq \bar{\theta} \qquad \underline{\kappa} \leq \kappa(x) \leq \bar{\kappa} \tag{43}$$

for all $x \in \mathcal{D}$, where

$$0 \notin [\underline{\rho}, \bar{\rho}]. \tag{44}$$

This assumption is not a limitation, since the bounds are arbitrary. The assumption (44) requires the sign on the product $k_1 k_2$ to be known, which is ensured by Assumption 1.

Consider the filters

$$\psi_t(x,t) - \epsilon_5 \psi_x(x,t) = 0, \qquad \psi(1,t) = U(t) \tag{45a}$$
$$\phi_t(x,t) - \epsilon_5 \phi_x(x,t) = 0, \qquad \phi(1,t) = y(t) \tag{45b}$$
$$P_t(x,\xi,t) + \epsilon_3 P_\xi(x,\xi,t) = 0, \quad P(x,0,t) = \phi(x,t) \tag{45c}$$

with initial conditions $\psi(x,0) = \psi_0(x)$, $\phi(x,0) = \phi_0(x), P(x,\xi,0) = P_0(x,\xi)$ satisfying

$$\psi_0, \phi_0 \in \mathcal{B}(\mathcal{D}), \qquad P_0 \in \mathcal{B}(\mathcal{D}^2). \tag{46}$$

Consider also the adaptive laws

$$\dot{\hat{\rho}}(t) = \text{proj}_{\underline{\rho},\bar{\rho}} \left\{ \gamma_1 \frac{\hat{\epsilon}(0,t)\psi(0,t)}{1 + f^2(t)}, \hat{\rho}(t) \right\} \tag{47a}$$
$$\hat{\theta}_t(x,t) = \text{proj}_{\underline{\theta},\bar{\theta}} \left\{ \gamma_2(x) \frac{\hat{\epsilon}(0,t)\phi(1-x,t)}{1 + f^2(t)}, \hat{\theta}(x,t) \right\} \tag{47b}$$
$$\hat{\kappa}_t(x,t) = \text{proj}_{\underline{\kappa},\bar{\kappa}} \left\{ \gamma_3(x) \frac{\hat{\epsilon}(0,t)p_0(x,t)}{1 + f^2(t)}, \hat{\kappa}(x,t) \right\} \tag{47c}$$

where

$$\hat{\epsilon}(x,t) = z(x,t) - \hat{z}(x,t) \tag{48a}$$
$$\hat{z}(x,t) = \hat{\rho}(t)\psi(x,t) + \int_x^1 \hat{\theta}(\xi,t)\phi(1-(\xi-x),t)d\xi + \int_x^1 \hat{\theta}(\xi,t)P(x,\xi,t)d\xi \tag{48b}$$

and

$$p_0(x,t) = P(0,x,t) \tag{49}$$

with

$$f^2(t) = \psi^2(0,t) + ||\phi(t)||^2 + ||p_0(t)||^2 \tag{50}$$

and $\gamma_1 > 0, \gamma_2(x), \gamma_3(x) > 0,\ \forall x \in \mathcal{D}$ as design gains. The initial guesses must be chosen inside the feasible domain, i.e. $\underline{\rho} \leq \hat{\rho}(0) \leq \bar{\rho}, \underline{\theta} \leq \hat{\theta}_0(x,0) \leq \bar{\theta}, \underline{\kappa} \leq \hat{\kappa}_0(x,0) \leq \bar{\kappa}$, $\forall x \in \mathcal{D}$. The projection operator is defined as

$$\text{proj}_{a,b}(\tau,\omega) = \begin{cases} 0 & \text{if } \omega \geq a \text{ and } \tau \leq 0 \\ 0 & \text{if } \omega \leq b \text{ and } \tau \geq 0 \\ \tau & \text{otherwise.} \end{cases} \tag{51}$$

Consider also the control law

$$U(t) = \frac{1}{\hat{\rho}(t)} \left(\int_0^1 \hat{k}(1-\xi,t)\hat{z}(\xi,t)d\xi - \int_0^1 \hat{\kappa}(\xi,t)p_1(\xi,t)d\xi \right) \tag{52}$$

where $\hat{z}$ is generated using (48b),

$$p_1(x,t) = P(1,x,t), \tag{53}$$

and $\hat{k}$ is the on-line solution to the Volterra equation

$$\hat{k}(x,t) = \int_0^x \hat{k}(x-\xi,t)\hat{\theta}(\xi,t)d\xi - \hat{\theta}(x,t) \tag{54}$$

with $\hat{\rho}$, $\hat{\theta}$ and $\hat{\kappa}$ generated from the adaptive laws (47).

Theorem 10: Consider the system (1), the filters (45) and the adaptive laws (47), where if $d_2 \geq d_\alpha$ we let $\hat{\kappa}_0 \equiv 0$ and $\gamma_3 \equiv 0$. The control law (52) guarantees

$$||u||, ||v||, ||u||_\infty, ||v||_\infty \in \mathcal{L}_2 \cap \mathcal{L}_\infty \tag{55a}$$
$$||u||, ||v||, ||u||_\infty, ||v||_\infty \to 0 \tag{55b}$$

Moreover, all additional signals in the closed loop system are bounded.

Proof: It was proved in [12] and [9] for the cases $d_2 < d_\alpha$ and $d_2 \geq d_\alpha$, respectively, that the controller achieves $||z||, ||z||_\infty \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. From the invertibility of the transforms of Lemmas 3–8, the result follows. ■

V. Simulation

The system (1) and controller of Theorem (10) are implemented MATLAB using the system parameters

$$\lambda(x) = 2 + x,\ \mu(x) = \frac{1}{2}e^{\frac{1}{2}x},\ d_1 = \frac{3}{4},\ d_2 = \frac{1}{2} \tag{56a}$$
$$c_1(x) = x,\ c_2(x) = \frac{1}{2}(1 + \sin(x)),\ q = 2 \tag{56b}$$
$$k_1 = \frac{3}{2},\ k_2 = \frac{1}{2}. \tag{56c}$$

We note that

$$d_\alpha = \bar{\lambda}^{-1} = \int_0^1 \frac{ds}{\lambda(s)} \approx 0.4055 \leq d_2 = 0.5 \tag{57}$$

so that the 1–D controller from [9] (i.e. $||w||_\infty = 0$) can be used. The adaptation parameters are set to

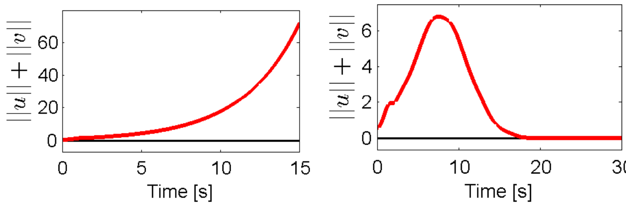

Fig. 1: State norm with the controller inactive (left) and active (right).

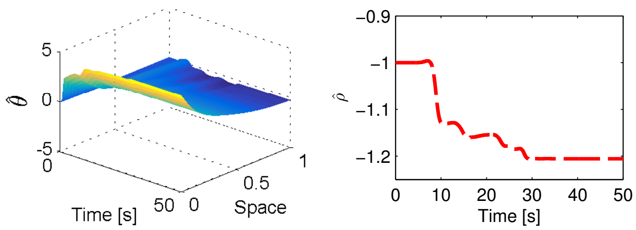

Fig. 2: Estimated parameters during stabilization.

$$\gamma_1 = \gamma_2(x) = 5, \ \forall x \in \mathcal{D} \tag{58a}$$

$$\bar{\theta} = -\underline{\theta} = 100, \quad \underline{\rho} = 0.1, \quad \bar{\rho} = 10 \tag{58b}$$

while adaptation of κ is not required. It is observed from Figure 1 that the system states diverge in the open loop case. In the closed loop case, the controller successfully manages to stabilize the system, and the system norms converge to zero, as seen from Figure 1. The estimated parameters are seen in Figure 2 to be bounded. The actuation signal is also bounded, as seen from Figure 3.

VI. Conclusions

Using a series of transformations, we showed that a system of linear 2×2 hyperbolic PDEs with delayed, anti-collocated actuation and sensing is for small sensor delays equivalent to a delay-free 2×2 system, while for large sensor delays, it is equivalent to a delay-free 1–D system. Already established controllers were then applied to adaptively stabilize the system from the single, delayed boundary sensing only. The only required knowledge of the system was the various delays involved, as well as the sign of the product of the actuation and sensing scaling constants. The theory was verified in a simulation.

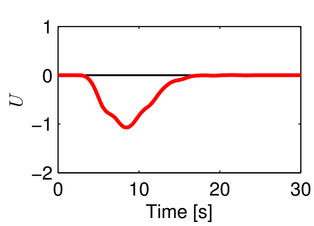

Fig. 3: Actuation signal during stabilization.

A natural direction for future work is to extend the method to more general systems consisting of several coupled PDEs; or derive under what conditions such an extension is possible. Another unsolved problem, is to adaptively stabilize system (1) from sensing restricted to the boundary collocated with the actuation. This latter problem has been solved for the case where the only uncertain boundary parameter was q, but no results exist for the case of uncertain in-domain parameters.

References

[1] M. Krstić and A. Smyshlyaev, "Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays," *Systems & Control Letters*, vol. 57, no. 9, pp. 750–758, 2008.

[2] I. S. Landet, A. Pavlov, and O. M. Aamo, "Modeling and control of heave-induced pressure fluctuations in managed pressure drilling," *IEEE Transactions on Control Systems and Technoloy*, vol. 21, no. 4, pp. 1340–1351, 2013.

[3] D. J. Wollkind, "Applications of linear hyperbolic partial differential equations: Predator-prey systems and gravitational instability of nebulae," *Mathematical Modelling*, vol. 7, pp. 413–428, 1986.

[4] W. Liu, "Boundary feedback stabilization of an unstable heat equation," *SIAM Journal on Control and Optimization*, vol. 42, pp. 1033–1043, 2003.

[5] R. Vazquez, M. Krstić, and J.-M. Coron, "Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, December 2011, pp. 4937 – 4942.

[6] F. Di Meglio, R. Vazquez, and M. Krstić, "Stabilization of a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input," *IEEE Transactions on Automatic Control*, vol. 58, no. 12, pp. 3097–3111, 2013.

[7] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstić, "Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3301–3314, 2016.

[8] P. Bernard and M. Krstić, "Adaptive output-feedback stabilization of non-local hyperbolic PDEs," *Automatica*, vol. 50, pp. 2692–2699, 2014.

[9] H. Anfinsen and O. M. Aamo, "Model reference adaptive control of an unstable 1–D hyperbolic PDE," in *56th Conference on Decision and Control, Melbourne, Victoria, Australia*, 2017.

[10] ——, "Adaptive output-feedback stabilization of linear 2×2 hyperbolic systems using anti-collocated sensing and control," *Systems & Control Letters*, vol. 104, pp. 86–94, 2017.

[11] H. Yu, R. Vazquez, and M. Krstić, "Adaptive output feedback for hyperbolic PDE pairs with non-local coupling," in *2017 American Control Conference, Seattle, WA, USA*, 2017.

[12] H. Anfinsen and O. M. Aamo, "Model reference adaptive control of 2×2 coupled linear hyperbolic PDEs," 2018, *IEEE Transactions on Automatic Control*, to appear.

[13] O. M. Aamo, "Disturbance rejection in 2×2 linear hyperbolic systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 5, pp. 1095–1106, 2013.

[14] S. Tang and M. Krstić, "Sliding mode control to the stabilization of a linear 2×2 hyperbolic system with boundary input disturbance," in *American Control Conference 2014, Portland, Oregon*, 2014.

[15] H. Anfinsen and O. M. Aamo, "Stabilization of linear 2×2 hyperbolic systems with uncertain coupling coefficients - Part I: Identifier-based design," in *Australian Control Conference 2016, Newcastle, New South Wales, Australia*, 2016.

[16] ——, "Stabilization of linear 2×2 hyperbolic systems with uncertain coupling coefficients - Part II: Swapping design," in *Australian Control Conference 2016, Newcastle, New South Wales, Australia*, 2016.

[17] ——, "Model reference adaptive control of $n + 1$ coupled linear hyperbolic PDEs," *Systems & Control Letters*, vol. 109, pp. 1–11, 2017.