COMPARISON OF INVARIANT METRICS AND DISTANCES ON STRONGLY PSEUDOCONVEX DOMAINS AND WORM DOMAINS

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Abstract. We prove that for a strongly pseudoconvex domain $D \subset \mathbb{C}^n$, the infinitesimal Carathéodory metric $g_C(z, v)$ and the infinitesimal Kobayashi metric $g_K(z, v)$ coincide if z is sufficiently close to bD and if v is sufficiently close to being tangential to bD. Also, we show that every two close points of D sufficiently close to the boundary and whose difference is almost tangential to bD can be joined by a (unique up to reparameterization) complex geodesic of D which is also a holomorphic retract of D.

The same continues to hold if D is a worm domain, as long as the points are sufficiently close to a strongly pseudoconvex boundary point. We also show that a strongly pseudoconvex boundary point of a worm domain can be globally exposed; this has consequences for the behavior of the squeezing function.

1. Introduction

For a domain $D \subset \mathbb{C}^n$, a point $z \in D$, and a vector $v \in T_z D = \mathbb{C}^n$, $v \neq 0$, we say that a holomorphic map $f_{z,v} : \triangle \to D$ is *extremal* with respect to (z, v), if $f_{z,v}(0) = z, f'(0) = \lambda v$, for some $\lambda > 0$ and for any holomorphic $g : \triangle \to D, g(0) = z$, and $g'(0) = \tilde{\lambda} v$, we have that $|\tilde{\lambda}| \leq \lambda$. A subset $S \subset D$ is a holomorphic retract, if there exists a holomorphic map $r : D \to D$ such that $r(D) = S$ and $r(z) = z$ for all $z \in S$. For a bounded strongly convex domain D of class $C^k, k \geq 3$, the following is due to Lempert [17, 18]:

(1) the extremal map $f_{z,v}$ is unique,
(2) $f_{z,v}$ extends to a C^{k-2} map on $\overline{\triangle}$, and $f_{z,v}$ embeds $\overline{\triangle}$ into $\overline{D}$ with $f(b\triangle) \subset bD$,
(3) the corresponding extremal disc $S = f_{z,v}(\triangle)$ is a holomorphic retract of D,

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This article was written as part of the international research program "Several Complex Variables and Complex Dynamics" at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the academic year 2016/2017.

† supported by the NRC grant number 240569.

* Partially supported by GNSAGA of INDAM.

(4) the extremal map $f_{z,v}$ is a complex geodesic, *i.e.*, it is an isometry between the Poincaré distance of $\triangle$ and the Kobayashi distance of D,
(5) any two points $z, w \in D, z \neq w$, can be joined by a complex geodesic f, which is unique up to pre-composition with automorphisms of $\triangle$. Such a complex geodesic f is an extremal map with respect to $(f(\zeta), f'(\zeta))$ for all $\zeta \in \triangle$ and its image is a holomorphic retract of D.

A straightforward consequence is that for a strongly convex domain D, the Carathéodory infinitesimal metric g_C and the Kobayashi infinitesimal metric g_K coincide, *i.e.*, the quotient $Q(z,v) = g_C(z,v)/g_K(z,v)$ is identically equal to one. This is no longer the case for non-convex domains, and our main question here is to what extent it does continue to hold near strictly pseudoconvex boundary points of pseudoconvex domains in general.

For a domain $D \subset \mathbb{C}^n$ of class C^2 we let $\delta(z)$ denote the distance from z to bD. If $\delta(z)$ is small enough, there is a unique point $\pi(z) \in bD$ closest to z, and for any vector $v \in T_z D = \mathbb{C}^n$ there is an orthogonal decomposition $v = v_N + v_T$, with $v_T \in T^{\mathbb{C}}_{\pi(z)} bD$. Our first two results are the following:

Theorem 1.1. *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain of class C^k for $k \geq 3$. Then there exists $\epsilon > 0$ such that the following hold:*

a) *for any $z \in D$ with $\delta(z) < \epsilon$ and $v \in T_z D \setminus \{0\}$ with $\|v_N\| < \epsilon \|v_T\|$, the extremal map $f_{z,v}$ satisfies (1), (2), (3) and (4) above.*
b) *for any $z, w \in D$ such that $\max\{\delta(z), \delta(w), \mathrm{dist}(z,w)\} < \epsilon$ and $\|(w-z)_N\| < \epsilon \|(w-z)_T\|$, there exists a complex geodesic f joining z and w which satisfies (5) above (in particular, z, w are contained in a one-dimensional holomorphic retract of D). As a consequence, the Kobayashi distance between z and w equals the Carathéodory distance between z and w.*

Corollary 1.2. *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain of class C^k for $k \geq 3$. Then there exists $\epsilon > 0$ such that $Q(z,v) = 1$ if $\delta(z) < \epsilon$ and if $\|v_N\| < \epsilon \|v_T\|$.*

The theorem generalises a theorem of Kosinski [16], in which he shows that the invariant metrics coincide on *some* open set. Also, as pointed out by the referee, a version of the theorem for strongly pseudoconvex domains with boundaries of class C^6, follows from the arguments in Section 4 in [4].

Considering domains more general than strictly pseudoconvex domains, a natural class of domains to consider is the class of Worm-domains (see Section 3 for the definition); these domains are pseudoconvex but without Stein neighbourhood bases, and provide counterexamples to many complex analytic problems of a global character.

Theorem 1.3. *Let Ω_r be a Worm domain, and let $p \in b\Omega_r$ be a strongly pseudoconvex boundary point. Let D be small open neighborhood of p. Then there exists $\epsilon > 0$ such that the following hold:*

a) *for any $z \in D \cap \Omega_r$ with $\delta(z) < \epsilon$ and $v \in T_z D \setminus \{0\}$ with $\|v_N\| < \epsilon \|v_T\|$, the extremal map $f_{z,v}$ satisfies (1), (2), (3) and (4) above.*
b) *for any $z, w \in D \cap \Omega_r$ such that $\max\{\delta(z), \delta(w), \mathrm{dist}(z,w)\} < \epsilon$ and $\|(w-z)_N\| < \epsilon \|(w-z)_T\|$, there exists a complex geodesic f joining z and w which satisfies (5) above (in particular, z, w are contained in a one-dimensional holomorphic retract of Ω_r). As a consequence, the Kobayashi distance between z and w equals the Carathéodory between z and w.*

The corresponding corollary continues to hold. The main new theorem needed in order to prove these results for Worm-domains is the following:

Theorem 1.4. *Let $p \in b\Omega_r$ be a strongly pseudoconvex boundary point of a Worm-domain Ω_r. Then for any $k \in \mathbb{N} \cup \{\infty\}$ there exists a C^k-smooth embedding $\phi : \overline{\Omega}_r \to \overline{\mathbb{B}^2}$ such that*

(i) *$\phi : \Omega_r \to \mathbb{B}^2$ is holomorphic, and*
(ii) *$\phi(p) \in b\mathbb{B}^2$.*

The point $\phi(p)$ is said to be *exposed*. Theorem 1.3 was proved in [8] in the case of strongly pseudoconvex domains. The relevant difference between that and the present case, is that $\overline{\Omega}_r$ does not have a strongly pseudoconvex neighbourhood basis, and so we do not have access to a key ingredient in [8], which is the existence of a certain compositional splitting of injective holomorphic maps, the proof of which relied on certain solution operators for the $\overline{\partial}$-equation (see e.g. [12], 8.7). Instead, we will prove the existence of such a splitting using Hörmander's L^2-theory, see Theorem 4.1 below.

A consequence of Theorem 1.3 and the work [6] is the following.

Theorem 1.5. *Let $p \in b\Omega_r$ be a strongly pseudoconvex boundary point. Then*

$$\lim_{z \to p} S_{\Omega_r}(z) = 1, \tag{1.1}$$

where $S_{\Omega_r}(z)$ denotes the squeezing function on Ω_r.

The question whether any strongly pseudoconvex boundary point on a Worm domain can be exposed was raised in [10].

2. The Proof of Theorem 1.1 and Theorem 1.3

In this section we provide the proof of Theorem 1.1 and Theorem 1.3. In order to do this, we need some preliminaries about real and complex geodesics.

For a domain $\Omega \subset \mathbb{C}^n$ and a piecewise C^1-smooth curve $\gamma : [0,1] \to \Omega$, we let $l_K(\gamma)$ denote the Kobayashi length of the image γ, *i.e.*,

$$l_K(\gamma) := \int_0^1 g_K(\gamma(t); \gamma'(t)) dt. \tag{2.1}$$

For points $z, w \in \Omega$ we let $d_K(z,w)$ denote the induced distance between z and w; the Kobayashi distance.

A *real geodesic* (for the Kobayashi distance) is a piecewise C^1-smooth map $\gamma : [a,b] \to \Omega$ such that $d_K(\gamma(t), \gamma(s)) = |t-s|$ for all $t, s \in [a,b]$. Here, $-\infty < a < b < +\infty$.

A *complex geodesic* is a holomorphic map $\varphi : \triangle \to \Omega$ such that $d_P(\zeta, \eta) = d_K(\varphi(\zeta), \varphi(\eta))$ for all $\zeta, \eta \in \triangle$, where d_P denotes the Poincaré distance.

That (1)-(4) are satisfied in Theorem 1.1 is a consequence of the fact that boundary points of strongly pseudoconvex domains can be globally exposed (see Theorem 2.6 below) and the following result due to X. Huang (see [15, Corollary 1]).

Proposition 2.1 (Huang)**.** *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with C^3-smooth boundary. Let $p \in bD$ and let U be an open neighborhood of p. Then there exist an open neighborhood V of p and $\epsilon_0 > 0$ such that for every $z \in V$ and for all $v \in \mathbb{C}^n \setminus \{0\}$ with $\|v_N\| < \epsilon_0 \|v_T\|$, the complex geodesic $\varphi : \triangle \to D$ such that $\varphi(0) = p$ and $\varphi'(0) = \lambda v$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ satisfies $\varphi(\triangle) \subset U$.*

(As a matter of notation, if D is a bounded domain with C^2-smooth boundary and $p \in bD$, we have an orthogonal splitting $\mathbb{C}^n = T_p^{\mathbb{C}} bD \oplus N$, with N a one dimensional complex space. If $v \in \mathbb{C}^n$ is a vector, we let v_N be the projection of v on N and v_T the projection of v on $T_p^{\mathbb{C}} bD$.)

To get (5) we need to extend Huang's result to complex geodesics connecting the two points - see Proposition 2.5 below. Finally, to get (1)-(5) in Theorem 1.3, we need to extend results on exposing points to Worm domains — this will be done in Section 3.

We proceed to give some intermediate results before we prove Proposition 2.5, and then we will prove Theorem 1.1 and Theorem 1.3.

Proposition 2.2. *Let Ω be a bounded strongly pseudoconvex domain with C^2-smooth boundary. Let $p \in b\Omega$ and let U be an open neighborhood of p. Then there exist an open neighborhood V of p and a compact set $K \subset \Omega$ such that for every real geodesic $\gamma : [a,b] \to \Omega$ with $\gamma(a) \in V$ and $\gamma(b) \notin U$, we have that $\gamma([a,b]) \cap K \neq \emptyset$.*

Proof. For $j \in \mathbb{N}$ we set $K_j := \{z \in \Omega : \operatorname{dist}(z, b\Omega) \geq 1/j\}$. Assume to get a contradiction that there exist sequences of points $z_j \to p, w_j \in \Omega \setminus U$, and geodesics γ_j connecting z_j and w_j with $\gamma_j \in \Omega \setminus K_j$. Without loss of generality we may assume that $w_j \to q \in b\Omega \setminus U$. Fix a point $z_0 \in \Omega$. By [1, Thm. 2.3.51]) there exists a constant $C_1 > 0$ such that $d_K(z_0, z) \leq$

$C_1 - \frac{1}{2}\log\delta(z)$ for $z \in \Omega$. Set $\delta := \mathrm{dist}(p,q)/6$. By [1, Thm. 2.3.54], there exist a constant $C_2 \in \mathbb{R}$ and $0 < \epsilon_1 < \epsilon_2 < \delta$ such that if $\mathrm{dist}(z,p) < \epsilon_1$ (resp. q) and if $\mathrm{dist}(w,p) \geq 2\epsilon_2$ (resp. q) then $d_K(z,w) \geq C_2 - \frac{1}{2}\log\delta(z)$.

For each j set $\tilde{\gamma}_j := \gamma_j \setminus (B_{2\delta}(p) \cup B_{2\delta}(q))$. For large enough j we get that

$$l_K(\gamma_j) \geq 2C_2 - \frac{1}{2}\log(\delta(z_j)) - \frac{1}{2}\log\delta(w_j) + l_K(\tilde{\gamma}_j), \tag{2.2}$$

and

$$\mathrm{dist}(z_j, z_0) + \mathrm{dist}(w_j, z_0) \leq 2C_1 - \frac{1}{2}\log(\delta(z_j)) - \frac{1}{2}\log\delta(w_j). \tag{2.3}$$

Since the Euclidean length of $\tilde{\gamma}_j$ is longer than $\mathrm{dist}(p,q)/6$ for large j, and since $\tilde{\gamma}_j \subset \Omega \setminus K_j$ we have that $l_K(\tilde{\gamma}_j) \to \infty$ as $j \to \infty$. So for large enough j we have that $2C_2 + l_K(\tilde{\gamma}_j) \geq 2C_1$, contradicting the assumption that γ_j is a geodesic. □

Proposition 2.3. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with C^3-smooth boundary. Let $\{\varphi_n\}$ be a sequence of complex geodesics parameterized so that $\delta(\varphi_n(0)) = \sup_{\zeta\in\triangle}\delta(\varphi_n(\zeta))$. If there exists $p \in \partial D$ such that $\lim_{n\to\infty}\varphi_n(0) = p$, then $\{\varphi_n\}$ converges uniformly on $\overline{\triangle}$ to the constant map $\overline{\triangle} \ni \zeta \mapsto p$.*

Proof. It is enough to show that for every neighborhood U of p there exists $n_0 \in \mathbb{N}$ such that $\varphi_n(\overline{\triangle}) \subset U$ for all $n \geq n_0$. Assume to get a contradiction that this is not the case. Then without loss of generality, we may assume that there exists an open neighborhood U of p such that $\varphi_n(\triangle) \not\subset U$ for all n.

Let V and K be given by Proposition 2.2. For n large enough, $\varphi_n(0) \in V$ and there exists $\zeta_n \in \triangle$ such that $\varphi_n(\zeta_n) \notin U$. By pre-composing φ_n with a rotation, we can assume that $\zeta_n \in (0,1)$. The curve $\gamma_n : [0, \frac{1}{2}\log\frac{1+\zeta_n}{1-\zeta_n}] \ni t \mapsto \varphi_n(\tanh(t))$ is a real geodesic such that $\gamma_n(0) \in V$ and $\gamma_n(\frac{1}{2}\log\frac{1+\zeta_n}{1-\zeta_n}) = \varphi_n(\zeta_n) \notin U$. Hence, by Proposition 2.2, we have that $\gamma_n([0, \frac{1}{2}\log\frac{1+\zeta_n}{1-\zeta_n}]) \cap K \neq \emptyset$ for all k. This implies that there exists a constant $C > 0$ such that for all n large enough we have that

$$\delta(\varphi_n(0)) = \sup_{\zeta\in\triangle}\delta(\varphi_n(\zeta)) \geq C,$$

against the hypothesis $\lim_{n\to\infty}\varphi_n(0) = p$. □

The next proposition follows from [15, Proposition 1] (see also [3, Section 2]):

Proposition 2.4 (Huang)**.** *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with C^3-smooth boundary. Let $\{\varphi_n\}$ be a sequence of complex geodesics in D converging uniformly on compacta to a complex geodesic $\varphi : \triangle \to D$. Then, $\{\varphi_n\}$ converges to φ in the C^1-topology of $\overline{\triangle}$.*

Proposition 2.5. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with C^3-smooth boundary. Then for every $p \in bD$ and for every open neighborhood U of p there exist an open neighborhood $V \subset U$ of p and $\epsilon > 0$ such that for all $z, w \in V$ with $\|(z-w)_N\| < \epsilon \|(z-w)_T\|$ the complex geodesic $\varphi : \triangle \to D$ containing z and w is contained in U.*

Proof. Assume to get a contradiction the result is not true. Then, there exist an open neighborhood U of p and two sequences $\{z_n\}, \{w_n\} \subset D$ converging to p with $\lim_{n\to\infty} \frac{z_n - w_n}{\|z_n - w_n\|} = v$ for some $v \in T_p^{\mathbb{C}} bD$, such that for every $n \in \mathbb{N}$, the complex geodesic $\varphi_n : \triangle \to D$ which contains z_n and w_n satisfies $\varphi_n(\triangle) \not\subset U$. We can assume that φ_n is parameterized in such a way that $\delta(\varphi_n(0)) = \max_{\zeta\in\triangle} \delta(\varphi_n(\zeta))$ for all n. Up to subsequences, we can also assume that $\{\varphi_n\}$ converges uniformly on compacta to some holomorphic map $\varphi : \triangle \to \overline{D}$. By Proposition 2.3, $\varphi(\triangle) \subset D$ and hence, since $\lim_{n\to\infty} d_K(\varphi_n(\zeta), \varphi_n(\eta)) = d_K(\varphi(\zeta), \varphi(\eta))$ for all $\zeta, \eta \in \mathbb{D}$, it follows that φ is a complex geodesic. By Proposition 2.4, $\{\varphi_n\}$ converges uniformly to φ in C^1-norm on $\overline{\triangle}$ and $\{\varphi_n'\}$ converges uniformly on $\overline{\triangle}$ to φ'.

Since $\{z_n\}$ and $\{w_n\}$ are converging to p, it follows that there exists $\zeta \in \partial\triangle$ such that $\varphi(\zeta) = p$. Moreover, since $\lim_{n\to\infty} \frac{z_n - w_n}{\|z_n - w_n\|} = v$, then $\varphi'(\zeta) = \lambda v$ for some $\lambda \in \mathbb{C}$. However, since $v \in T_p^{\mathbb{C}} bD$, this contradicts the Hopf Lemma. □

In order to prove Theorem 1.1, we need one more fact:

Theorem 2.6. *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain of class C^k, $k \geq 2$ and let $p \in bD$, or let D be a Worm-domain, and $p \in bD$ a strictly pseudoconvex boundary point. Then there exists a bounded strongly convex domain W with C^k-smooth boundary and a biholomorphism $\phi : D \to \mathbb{C}^n$ such that*

1. *ϕ extends as a diffeomorphism of class C^k on $\overline{D}$,*
2. *$\phi(D) \subset W$,*
3. *there exists an open neighborhood U of $\phi(p)$ such that $U \cap \phi(D) = U \cap W$.*

After applying [8, Thm. 1.1] or Theorem 1.4 below, this result follows from techniques in [9]. For the convenience of the reader, we include here a complete proof.

Proof of Theorem 2.6. By [8, Thm. 1.1] in case $D \subset \mathbb{C}^n$ is a bounded strongly pseudoconvex domain, or by Theorem 1.4 in case D is a Worm-domain, there exists a C^k-smooth embedding embedding $\phi : \overline{D} \to \mathbb{C}^n$, holomorphic on D, $\phi(D) \subset \mathbb{B}^n := \{z \in \mathbb{C}^n : \|z\| = 1\}$, $\phi(p) \in b\mathbb{B}^n$ and $\phi(\overline{D} \setminus \{p\}) \subset \mathbb{B}^n$.

Let $\widehat{D}$ be the convex hull of $\phi(D)$. Then there exists an open neighbourhood U of $\phi(p)$ such that $\widehat{D} \cap U = \phi(D) \cap U$. Let ψ be the signed distance function to $b\widehat{D}$. Then ψ is convex, and near the point p it is strictly convex. Let χ a non-negative smooth function, $\chi(x) = 0$ near the origin, and $\chi = 1$ near $\mathbb{R}^n \setminus \mathbb{B}^n$. Then for small enough $\epsilon > 0$ and small enough $\delta = \delta(\epsilon) > 0$ the function $\tilde{\psi} = \psi - \delta\chi(x/\epsilon)$ is convex with $\tilde{\psi} = \psi$ near the origin, $\tilde{\psi} < \psi$ for $\|x\| \geq \epsilon$, and $\tilde{\psi}$ is strictly convex for $\epsilon \leq \|x\| \leq 2\epsilon$. Let $\tilde{\chi}$ be a non-negative smooth function which is 0 near $\epsilon\overline{\mathbb{B}^n}$ and 1 on $\mathbb{R}^n \setminus 2\epsilon\mathbb{B}^n$. Smoothing, we may obtain a sequence of strictly convex functions $\tilde{\psi}_j$ converging to $\tilde{\psi}$ on a neighbourhood of $\overline{\{\tilde{\psi} < 0\}}$, and the convergence is in C^2-norm on $\mathbb{B}^n_{2\epsilon}(0)$. By Morse theory we may assume that $\nabla\tilde{\psi}_j$ is non-vanishing on $b\{\tilde{\psi}_j < 0\}$. So for sufficiently large j we have that $\tilde{\psi} + \tilde{\chi}(\tilde{\psi}_j - \tilde{\psi})$ defines a smoothly bounded strictly convex domain which agrees with $\phi(D)$ near $\phi(p)$. $\square$

Proof of Theorem 1.1 and Theorem 1.3: Let ϕ and W be given by Theorem 2.6. The orthogonal splitting $\mathbb{C}^n = T^{\mathbb{C}}_p bD + N$ might not be preserved under ϕ. Indeed, $d\phi_p(T^{\mathbb{C}}_p bD) = T^{\mathbb{C}}_{\phi(p)} b\phi(D)$ but $d\phi_p(N)$ might not be orthogonal to $T^{\mathbb{C}}_{\phi(p)} b\phi(D)$. Let N' be the orthogonal complement of $T^{\mathbb{C}}_{\phi(p)} b\phi(D)$ in $\mathbb{C}^n$. Then there exists a constant $C > 0$ such that, if $v \in T_zD$ and $\|v_N\| < \epsilon\|v_T\|$ (in the splitting $\mathbb{C}^n = T^{\mathbb{C}}_p bD + N$), then $\|d\phi_z(v)_N\| < C\epsilon\|d\phi_z(v)_T\|$ (in the splitting $\mathbb{C}^n = T^{\mathbb{C}}_{\phi(p)} b\phi(D) + N'$).

Therefore, without loss of generality, we can assume that there exists a bounded strongly convex domain W with C^3 boundary such that $D \subset W$ and an open neighborhood U of p such that $D \cap U = W \cap U$.

a) Let $z \in D$ and $v \in T_zD \setminus \{0\}$. Let $f_{z,v}$ be the extremal map for W with respect to (z, v). By Proposition 2.1 there exist an open neighborhood V of p and $\epsilon_0 > 0$ such that $f_{z,v}(\triangle) \subset U$ provided $z \in V$ and $\|v_N\| < \epsilon_0\|v_T\|$. Since $g_W(z, v) \leq g_D(z, v)$, it follows that $f_{z,v}$ is an extremal map for D as well. It is also unique: otherwise, since $D \subset W$, W would have two different extremal maps with respect to (z, v).

Finally, according to Lempert's theory, every extremal disc of W is a holomorphic retract of W, and, since $\phi(D) \subset W$ and $W \cap U = \phi(D) \cap U$, it follows that $f_{z,v}(\triangle)$ is a holomorphic retract of $\phi(D)$ as well. From this it is easy to see that $f_{z,v}$ is a complex geodesic of D.

Hence, $f_{z,v}$ satisfies (1), (2), (3) and (4). By the compactness of $\overline{D}$, one can choose a uniform $\epsilon > 0$ for every $p \in bD$ and the result is proved.

b) The argument is similar to a), using Proposition 2.5 instead of Proposition 2.1. $\square$

3. Worm Domains

3.1. **The Worm.** We recall the definition of the Worm domains Ω_r from [7]. Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-smooth function such that

a) $\lambda(x) = 0$ if $x \leq 0$,
b) $\lambda(x) > 1$ if $x > 1$,

c) $\lambda''(x) \geq 100\lambda'(x)$ for all x,
d) $\lambda'(x) > 0$ if $x > 0$, and
e) $\lambda'(x) > 100$ if $\lambda(x) > 1/2$.

Let $r > 1$. We set

$$\rho_r(z,w) := |w + e^{i\log|z|^2}|^2 - 1 + \lambda(\frac{1}{|z|^2} - 1) + \lambda(|z|^2 - r^2), \tag{3.1}$$

and then $\Omega_r := \{(z,w) \in \mathbb{C}^* \times \mathbb{C} : \rho_r(z,w) < 0\}$. We have that Ω_r is pseudoconvex, and that $b\Omega_r$ is strongly pseudoconvex away from the variety $Z_r := b\Omega_r \cap \{w = 0\}$.

3.2. **A Stein semi-neighbourhood basis of Ω_r.** For $a, b, c > 1$ we set

$$\phi_{a,c}(z,w) = (z/a, ce^{-i\log|a|^2}w) \tag{3.2}$$

and

$$\Omega_{r,a,b,c} := \phi_{a,c}(\Omega_{br}) \tag{3.3}$$

Lemma 3.1. *For $1 < b << 2$ there exist $1 < a, c < b$ such that $\Omega_r \subset \Omega_{r,a,b,c}$, and the boundaries of the two domains agree only along $\{w = 0\}$. Moreover, there exists $A > 0$ such that for any $q \in b\Omega_r$ we have that*

$$\mathrm{dist}(q, b\Omega_{r,a,b,c}) \geq A \cdot \mathrm{dist}(q, \{w = 0\})^2. \tag{3.4}$$

Proof. Let us first set $c = 1$. Then the defining function for $\Omega_{r,a,b,c}$ becomes

$$\begin{aligned}\tilde{\rho}(z,w) &= |e^{i\log|a|^2}w + e^{i\log a^2|z|^2}|^2 - 1\\ &+ \lambda(\frac{1}{a^2|z|^2} - 1) + \lambda(a^2|z|^2 - b^2r^2)\\ &= |w + e^{i\log|z|^2}|^2 - 1\\ &+ \lambda(\frac{1}{a^2|z|^2} - 1) + \lambda(a^2|z|^2 - b^2r^2).\end{aligned}$$

Then the centers of the disc-fibers given by the projection $(z,w) \mapsto z$, remain the same, but the radii of some of them change. The radii of the disc fibers of Ω_r start decreasing with decreasing $|z|$ at $|z| = 1$, whereas the radii of the disc fibers of $\Omega_{r,a,b,c}$ start decreasing at $|z| = 1/a$. And for $|z| < 1/a$ their radii will always be strictly larger than those of Ω_r. Next the radii of the disc fibers of Ω_r are 1 for $1 < |z| \leq r$ and start decreasing at $|z| = r$. The radii of the disc fibers of $\Omega_{r,a,b,c}$ are 1 for $1/a < |z| \leq (rb/a)$, and start decreasing at $|z| = (rb/a)$. Furthermore, if a is chosen close enough to 1, then the radii of the disc fibers of $\Omega_{r,a,b,c}$ as strictly larger than those of Ω_r for $|z| \geq (rb/a)$. For this it is enough that

$$\begin{aligned}&0 < |z|^2 - r^2 - (a^2|z|^2 - b^2r^2)\\ &\Leftrightarrow a^2 - 1 < \frac{r^2(b^2-1)}{|z|^2},\end{aligned}$$

which clearly holds for all relevant z if a is close to 1. After having fixed a, we get the conclusions of the lemma except for (3.4) by choosing c close enough to 1.

To see that we have (3.4) we apply the global change of coordinates $\psi(z,w) := (z, e^{-i\log|z|^2}w)$ defined on $\mathbb{C}^* \times \mathbb{C}$. An application of ψ only changes distances by a factor, so it suffices to consider instead the domains $\psi(\Omega_r)$ and $\psi(\Omega_{r,a,b,c})$. For these domains all disc fibers have the same centers. Moreover, the worst case to consider is when $1 \leq |z| \leq r$, so it is enough to consider the two products $\{1 \leq |z| \leq r\} \times D_1$ and $\{1 \leq |z| \leq r\} \times D_c$, where D_1 is the disc of radius 1 centered at the point 1, and D_c is the disc of radius c centered at the point c. And now we need only to show that for $w \in bD_1$ we have that $\mathrm{dist}(w, bD_c) \geq \tilde{A} \cdot |w|^2$, which is easy to check. □

4. A splitting lemma for biholomorphic maps on worm domains

The following is the key technical ingredient in the proof of Theorem 1.5. It was originally proved by Forstnerič for strongly pseudoconvex Cartan pairs in complex Stein manifolds (See e.g. [12]), and more recently on Stein spaces [13].

Theorem 4.1. *Suppose $\overline{\Omega}_r = A \cup B$, where A and B are closed sets such that*

(i) $\overline{(A \setminus B)} \cap \overline{(B \setminus A)} = \emptyset$ *(separation condition),*
(ii) $C := A \cap B$ *is a Stein compact, and*
(iii) $A \cap Z_r = \emptyset$.

Then for any open set $\tilde{C} \supset C$ and $k \in \mathbb{N} \cup \{\infty\}$, there exist open sets $A' \supset A, \tilde{C} \supset C' \supset C$ such that for any injective holomorphic map $\gamma : C' \to \mathbb{C}^2$ sufficiently close to the identity, there exist

(a) *an injective holomorphic map $\alpha : A' \to \mathbb{C}^2$, and*
(b) *an injective map $\beta : B \to \mathbb{C}^2$, $\beta \in C^k(B) \cap \mathcal{O}(B \setminus Z_r)$,*

such that $\beta|_C = \gamma \circ \alpha|_C$. Moreover, α and β may be chosen to depend continuously on γ, such that

$$\|\alpha - \mathrm{id}\|_{A'} \to 0 \text{ and } \|\beta - \mathrm{id}\|_{C^k(B)} \to 0 \text{ as } \gamma \to \mathrm{id}. \tag{4.1}$$

Finally, α and β may be chosen to vanish to any given order along a given subvariety W of $\mathbb{C}^2$ not intersecting C.

Proof. We start by doing some preparation for the proof of the theorem. First we will construct a sequence of domains Ω_j that decreases in a controlled way to a neighbourhood of Ω_r as in Lemma 3.1. We will then be in a position to give a rough sketch of the proof. Then we will prepare for the vanishing on W. Finally we need good estimates for solutions of linear Cousin problems on the Ω_j's.

4.1. **A decreasing sequence of domains.** For any closed set K and $\nu > 0$ we define

$$K(\nu) := \{q \in \mathbb{C}^2 : \mathrm{dist}(z, K) \leq \nu\}. \tag{4.2}$$

There exists $\nu_0 > 0$ small enough (See *e.g.* the proof of Lemma 5.7.4 in [12]) such that for any $0 < \nu \leq \nu_0$ the pair $(A(\nu), B(\nu))$ still satisfies the separation condition corresponding to (i), and

$$C(\nu) = A(\nu) \cap B(\nu) \subset U, \tag{4.3}$$

where U is a fixed Stein neighborhood of C.

We set $C' := \mathrm{int}(C(\nu_0))$.

Next choose $\tilde{\Omega} := \Omega_{r,a,b,c} \subset A(\nu_0) \cup B(\nu_0)$ with $\Omega_{r,a,b,c}$ as in Lemma 3.1.

If ν_0 is small enough, we may let $\epsilon > 0$ be small enough such that $\mathrm{dist}(A(\nu_0), \{w = 0\}) > \epsilon$, and let ψ be a nonnegative smooth function on $\mathbb{C}^2$, identically equal to 0 on a neighbourhood of $|w| \leq \epsilon/4$ and identically equal to 1 on a neighbourhood of $|w| \geq \epsilon/2$. Let $\tilde{\rho}$ be a smooth defining function for $\tilde{\Omega}$ which is strictly plurisubharmonic near $\mathrm{Supp}(\psi) \cap b\tilde{\Omega}$. Then for sufficiently small s_0 we have that

$$\tilde{\Omega}_s := \{\rho_s < 0\} \text{ with } \rho_s := \tilde{\rho} - s\psi, \tag{4.4}$$

is a pseudoconvex semi-neighbourhood of $\tilde{\Omega}$, for all $s < s_0$. And if a, b, c and s_0 are small enough, we have that $\tilde{\Omega}_{s_0} \subset A(\nu_0) \cup B(\nu_0)$.

For $\widehat{t} << \nu_0$ we set $A' = \tilde{\Omega} \cap A(\nu_0 - \widehat{t})$. For an increasing sequence $t_j > 0$ (to be constructed later) with $t_j < \min\{\widehat{t}, s_0\}$ for all j, we define

(i) $\Omega_j := \tilde{\Omega}_{s_0 - t_j}$,
(ii) $A_j := \Omega_j \cap A(\nu_0 - t_j)$,
(iii) $B_j := \Omega_j \cap B(\nu_0 - t_j)$, and
(iv) $C_j := A_j \cap B_j$.

We also set $t_0 = 0$ and $\Omega_0 = \tilde{\Omega}_{s_0}$.

Lemma 4.2. *There exists a constant $k > 0$ such that*

$$\mathrm{dist}(C_{j+1}, \mathbb{C}^2 \setminus C_j) \geq k \cdot (t_{j+1} - t_j). \tag{4.5}$$

Proof. Note that $C_j = C(\nu_0 - t_j) \cap \Omega_j$. Let $p \in bC_{j+1}$ and let $q \in bC_j$. Then we have four possibilities

1) $p \in bC(\nu_0 - t_{j+1})$ and $q \in bC(\nu_0 - t_j)$,
2) $p \in bC(\nu_0 - t_{j+1})$ and $q \in b\Omega_j$,
3) $p \in b\Omega_{j+1}$ and $q \in bC(\nu_0 - t_j)$, or
4) $p \in b\Omega_{j+1}$ and $q \in b\Omega_j$,

and it is easy to check from these cases and the very definition of the sets that the statement holds. ☐

4.2. **Outline of the proof.** We will now give an outline of the proof. Start by setting $c_0 = c|_{C_0}$ where $\gamma = \mathrm{id} + c$. We will assume $\|c_0\|_{C_0} < \epsilon_0$ for some $\epsilon_0 > 0$ to be determined. Seeking a compositional splitting $c_0 = \beta \circ \alpha^{-1}$ we first find a linear splitting $c_0 = b_0 - a_0$, where b_0 and a_0 has good L^2-estimates on B_0 and A_0 respectively, depending on ϵ_0. Provided that ϵ_0, t_1, t_2 are chosen carefully (to be explained in detail below), we will then get good sup-norm estimates for b_0, a_0 on B_1 and A_1 respectively, and then an estimate

$$\|\beta_1^{-1} \circ \gamma \circ \alpha_1 - \mathrm{id}\|_{C_2} \leq \epsilon_2, \tag{4.6}$$

where $\alpha_1 = \mathrm{id} + a_0, \beta_1 = \mathrm{id} + b_0$, and ϵ_2 is considerably smaller than ϵ_0. We then set $\gamma_2 := \beta_1^{-1} \circ \gamma \circ \alpha_1|_{C_2} = \mathrm{id} + c_2$, and repeat the process. Repeating the process indefinitely, provided we choose the sequences $\{\epsilon_{2j}\}_{\mathrm{J}\in\mathbb{N}}, \{t_k\}_{k\in\mathbb{N}}$ carfully and interdependently, we will obtain sequences $\alpha_{2j+1}, \beta_{2j+1}$ of injective holomorphic maps defined on A_{2j+2} and B_{2j+2} respectively, such that

$$\beta_{2k+1}^{-1} \circ \beta_{2k-1} \circ \cdots \circ \beta_1^{-1} \circ \gamma \circ \alpha_1 \circ \cdots \circ \alpha_{2k-1} \circ \alpha_{2k+1} \to \mathrm{id} \tag{4.7}$$

on C' as $k \to \infty$, and such that the compositions

$$\alpha_1 \circ \cdots \circ \alpha_{2k-1} \circ \alpha_{2k+1} \text{ and } \beta_1 \circ \cdots \circ \beta_{2k-1} \circ \beta_{2k+1} \tag{4.8}$$

converge to our desired maps α and β on A' and B respectively.

4.3. **Preparation for vanishing on W.** Let $f_1, ..., f_N$ be entire holomorphic functions vanishing to a given order $k \in \mathbb{N}$ along W, with the property that they have no common zeroes on an open Stein neighbourhood $U \subset \widetilde{C}$ of C. Then by Cartan's division theorem there exist $g_j \in \mathcal{O}(U), j = 1, ..., N$, such that

$$1 = \sum_{j=1}^{N} g_j \cdot f_j. \tag{4.9}$$

We will use these functions later in the proof.

4.4. **Estimates of splittings.** Note that for $k \in \mathbb{N}$ large enough, a suitable kth root $f_k(w)$ of $1/w$ will satisfy $\mathrm{Re}(f_k) < 0$ on Ω_r, and $\mathrm{Re}(f_k(w)) \leq -C|w|^{-1/k}$. So $\exp(-\mathrm{Re}(f_k(w))) \geq C_m \frac{1}{|w|^m}$ for any given m. We set

$$\psi(z, w) := -\mathrm{Re}(f_k(w)) + |z|^2 + |w|^2. \tag{4.10}$$

By Hörmander [14] there exists a constant $C > 0$, independent of j and m, such that for any $\overline{\partial}$-closed $\omega \in L^2_{0,1}(\Omega_j, \psi)$ there exists $h \in L^2(\Omega_j, \psi)$ with $\overline{\partial} h = \omega$, and

$$\int_{\Omega_j} |h|^2 e^{-\psi} dV \leq C \cdot \int_{\Omega_j} |\omega|^2 e^{-\psi} dV. \tag{4.11}$$

Lemma 4.3. *Let $s \in \mathbb{N}$. There exists a constant $M_1 > 0$ such that the following holds. Assume that we are given a holomorphic map $c_j : C_j \to \mathbb{C}^2$ with $\|c_j\|_{C_j} \leq \epsilon_j$, and set $t_{j+1} = t_j + \delta_{j+1}$. Then there exist holomorphic maps $a_j : A_j \to \mathbb{C}^2$ and $b_j : B_j \to \mathbb{C}^2$, such that $c_j = b_j - a_j$, and such that*

$$\int_{A_j} |a_j|^2 e^{-\psi} dV \leq M_1 \epsilon_j^2, \text{ and } \int_{B_j} |b_j|^2 e^{-\psi} dV \leq M_1 \epsilon_j^2, \tag{4.12}$$

and

$$\|a_j\|_{C_{j+1}} \leq \frac{M_1 \epsilon_j}{\delta_{j+1}^2} \text{ and } \|b_j\|_{C_{j+1}} \leq \frac{M_1 \epsilon_j}{\delta_{j+1}^2}. \tag{4.13}$$

Moreover, the functions a_j, b_j vanish to order s along W.

Proof. Choose a nonegative smooth χ which is identically equal to zero in a neighbourhood of $\overline{A_{r_0} \setminus B_{r_0}}$ and identically equal to one in a neighbourhood of $\overline{B_{r_0} \setminus A_{r_0}}$. If $\widehat{\tau}$ is chosen small enough, the corresponding separation conditions will hold with r_0 replaced by j for all j.

Write

$$c_j = c_j \cdot \sum_{i=1}^{N} g_i \cdot f_i = \sum_{i=1}^{N} f_i \cdot (g_i c_j). \tag{4.14}$$

We obtain first smooth splittings of the maps $g_i c_j$ by setting $\tilde{a}_{ij} := -\chi \cdot g_i c_j$ on A_j and $\tilde{b}_{ij} := (1-\chi) \cdot g_i c_j$ on B_j. Now $\overline{\partial} \tilde{a}_{ij} = \overline{\partial} \tilde{b}_{ij}$ on C_j, and so we have a well defined closed $(0,1)$-form ω_{ij} on Ω_j, whose sup-norm is proportional to that of c_j independently of j.

Now, the support of ω_{ij} is unformly bounded away from the singularity of the weight ψ, and so we obtain solutions $\overline{\partial} h_{ij} = \omega_{ij}$ with

$$\int_{\Omega_j} |h_{ij}|^2 e^{-\psi} dV \leq C' \epsilon_j^2, \tag{4.15}$$

where C' is independent of j. Setting $b_j := \sum_{i=1}^{N} f_i \cdot (\tilde{b}_{ij} - h_{ij})$ and $a_j := \sum_{i=1}^{N} f_i \cdot (\tilde{a}_{ij} - h_{ij})$, we obtain a holomorphic splitting $c_j = b_j - a_j$ with

$$\int_{A_j} |a_j|^2 e^{-\psi} dV \leq C'' \epsilon_j^2, \text{ and } \int_{B_j} |b_j|^2 e^{-\psi} dV \leq C'' \epsilon_j^2 \tag{4.16}$$

And passing from the L^2-estimate to a sup-norm estimate on C_{j+1} using Lemma 4.2 we get

$$\|a_j\|_{C_{j+1}} \leq \frac{C''' \epsilon_j}{\delta_{j+1}^2} \text{ and } \|b_j\|_{C_{j+1}} \leq \frac{C''' \epsilon_j}{\delta_{j+1}^2} \tag{4.17}$$

where again C''' is independent of j. □

The following lemma follows immediately from Lemma 8.7.4 in [12].

Lemma 4.4. *There exists a constant $M_2 > 0$ such that the following holds. Starting with the map c_j on C_j from the previous lemma, assume that also $t_{j+2} = t_{j+1} + \delta_{j+1}$, and assume that $\frac{4M_1}{\delta_{j+1}^2}\epsilon_j < \delta_{j+1}$. Set $\alpha_{j+1} = \mathrm{id} + a_j, \beta_{j+1} = \mathrm{id} + b_j$ and $\gamma_j = \mathrm{id} + c_j$. Set $\gamma_{j+2} = \beta_{j+1}^{-1}\gamma_j\alpha_{j+1} =: \mathrm{id} + c_{j+2}$. Then*

$$\|c_{j+2}\|_{C_{j+2}} \leq \frac{M_2}{\delta_{j+1}^5}\epsilon_j^2. \tag{4.18}$$

4.5. **The proof of Theorem 4.1.** Set $\gamma_0 = \gamma|_{C_0}$, and write $\gamma_0 = \mathrm{id} + c_0$. To start an inductive construction, choose first $0 < \delta_1 << 1$ such that $4M_1\delta_1^5 < 1$, and $M_2\delta_1 < 1$ (δ_1 will be further decreased several times throughout the proof). Set $\epsilon_0 = \delta_1^8$. Set $\delta_2 = \delta_1, t_1 = \delta_1, t_2 = \delta_1 + \delta_2$. By Lemma 4.3 there exist $a_0 : A_0 \to \mathbb{C}^2, b_0 : B_0 \to \mathbb{C}^2$, such that $c_0 = b_0 - a_0$, and such that

$$\|a_0\|_{C_1} \leq \frac{M_1\epsilon_0}{\delta_1^2} \text{ and } \|b_0\|_{C_1} \leq \frac{M_1\epsilon_0}{\delta_1^2}. \tag{4.19}$$

Set $\alpha_1 := \mathrm{id} + a_0, \beta_1 := \mathrm{id} + b_0$. Now

$$\frac{4M_1}{\delta_1^2}\epsilon_0 = 4M_1\delta_1^6 < \delta_1, \tag{4.20}$$

so we are in the setting of Lemma 4.4. So writing $\gamma_2 := \beta_1^{-1}\gamma_0\alpha_1 = \mathrm{id} + c_2$, we get that

$$\|c_2\|_{C_2} \leq \frac{M_2}{\delta_1^5}\epsilon_0^2 = M_2\delta_1^{11} = M_2\delta_1(\delta_1^{5/4})^8 < (\delta_1^{5/4})^8. \tag{4.21}$$

This suggests how to define the sequences δ_j, ϵ_j further to enable an inductive construction. Assume that we have defined δ_i for $i \leq 2k$ (which we have now done for $k = 1$). Set $\delta_{2k+1} = \delta_{2k}^{5/4}$, and $\delta_{2k+2} = \delta_{2k+1}$. Set $\epsilon_{2k} := \delta_{2k+1}^8$, and for all i set $t_i = \sum_{j=1}^i \delta_j$. Then (4.21) reads

$$\|c_2\|_{C_2} \leq \delta_3^8 = \epsilon_2. \tag{4.22}$$

Now assume as our inductive hyposesis that we have constructed $\gamma_{2k} : C_{2k} \to \mathbb{C}^2, \gamma_{2k} = \mathrm{id} + c_{2k}$, with $\|c_{2k}\|_{C_{2k}} \leq \epsilon_{2k}$ for $k = 1, ..., j$. We complete the inductive step by repeating the above arguments essentially verbatim, only changing indices. By Lemma 4.3 there exist $a_{2j} : A_{2j} \to \mathbb{C}^2, b_{2j} : B_{2j} \to \mathbb{C}^2$, such that $c_{2j} = b_{2j} - a_{2j}$, and such that

$$\|a_{2j}\|_{C_{2j+1}} \leq \frac{M_1\epsilon_{2j}}{\delta_{2j+1}^2} \text{ and } \|b_{2j}\|_{C_{2j+1}} \leq \frac{M_1\epsilon_{2j}}{\delta_{2j+1}^2}. \tag{4.23}$$

Set $\alpha_{2j+1} := \mathrm{id} + a_{2j}, \beta_{2j+1} := \mathrm{id} + b_{2j}$. Now

$$\frac{4M_1}{\delta_{2j+1}^2}\epsilon_{2j} = 4M_1\delta_{2j+1}^6 < \delta_{2j+1}, \tag{4.24}$$

so we are in the setting of Lemma 4.4. So writing $\gamma_{2j+2} := \beta_{2j+1}^{-1}\gamma_{2j}\alpha_{2j+1} = \mathrm{id} + c_{2j+2}$, we get that

$$\begin{aligned}\|c_{2j+2}\|_{C_{2j+2}} &\leq \frac{M_2}{\delta_{2j+1}^5}\epsilon_{2j}^2 = M_2\delta_{2j+1}^{11}\\ &= M_2\delta_{2j+1}(\delta_{2j+1}^{5/4})^8 < \delta_{2j+3}^8 = \epsilon_{2j+2}.\end{aligned}$$

This shows that the induction can go on indefinitely.

By the construction we see that, near C we have that

$$\lim_{j\to\infty}\beta(j)^{-1}\gamma_0\alpha(j) := \lim_{j\to\infty}\beta_{2j+1}^{-1}\circ\cdots\circ\beta_1^{-1}\circ\gamma_0\circ\alpha_1\circ\cdots\circ\alpha_{2j+1} = \mathrm{id}. \tag{4.25}$$

So it remains to show that $\beta(j)$ converges to a smooth injective map on B, and that $\alpha(j)$ converges to an injective map on A'.

In the construction we defined $\alpha_{2j+1} = \mathrm{id} + a_{2j}$ and $\beta_{2j+1} = \mathrm{id} + b_{2j}$ on A_{2j} and B_{2j} respectively, and the estimates from Lemma 4.3 leading to the estimate (4.23) were

$$\int_{A_{2j}}|a_{2j}|^2e^{-\psi}dV \leq M_1\epsilon_{2j}^2, \text{ and } \int_{B_{2j}}|b_{2j}|^2e^{-\psi}dV \leq M_1\epsilon_{2j}^2. \tag{4.26}$$

We start by considering the simplest case, the convergence of the sequence $\alpha(j)$. Note first that there exists an $\eta > 0$ such that if δ_1 is chosen small enough, then $\mathrm{dist}(A', \mathbb{C}^2\setminus A_{2j}) > \eta$ for all j. Together with (4.26) this shows that each α_{2j+1} is injective holomorphic on $A'(\eta/2)$ provided δ_1 is chosen small enough, and we get that the family $\alpha(i,j) := \alpha_{2i+1}\circ\cdots\circ\alpha_{2j+1}, i < j$, is uniformly Lipschitz on $A'(\eta/4)$, with Lipschitz constant decreasing to zero with decreasing δ_1. Then

$$\begin{aligned}\|\alpha(j+1)(z) - \alpha(j)(z)\|_{A'} &= \|\alpha(j)(\alpha_{2j+1}(z)) - \alpha(j)(z)\|_{A'}\\ &\leq C(\delta_1)\|a_{2j}(z)\|_{A'} \leq C(\delta_1)\delta_{2j+1},\end{aligned}$$

where the last inequality follows from (4.26) provided δ_1 is small enough. This shows that $\alpha(j)$ converges to an injective holomorphic map on A' provided δ_1 is chosen small enough, and $\|c_0\|_{C'} \leq \epsilon_0 = \delta_1^8$. It is clear from the construction that the limit maps depend continuously on the input γ.

Now the same type of argument shows that for any $\epsilon > 0$, the sequence $\beta(j)$ will converge uniformly to an injective holomorphic map on $B\cap\{|w| \geq \epsilon/2\}$, provided δ_1 is chosen small enough. So it remains to show that the sequence $\beta(j)$ converges to an injective holomorphic map β on $B\cap\{|w|\leq\epsilon\}$.

Notice first that all maps β_{2j+1} are defined on $\Omega_{r,a,b,c}\cap\{|w|\leq\epsilon\}$. So for all $q \in B\cap\{|w|\leq\epsilon\}$, the ball $B_{A|w|^2}(q)$ is contained in B_{2j}. We get that

$$\begin{aligned}\int_{B_{A|w|^2}(q)}|b_{2j}|^2dV &\sim |w|^m\int_{B_{A|w|^2}(q)}|b_{2j}|^2e^{-\psi}dV\\ &\leq |w|^mM_1\epsilon_{2j}^2.\end{aligned}$$

This shows that the sup-norm of b_{2j} is comparable to $|w|^{m/2-4}\epsilon_{2j}$ and that the C^1-norm is comparable to $|w|^{m/2-5}\epsilon_{2j}$. In particular we see that β_{2j} is injective if δ_1 was chosen small enough.

Next we need to show that the sequence $\beta(j)$ of compositions is well defined near Z_r. We have that

$$\beta(j)(w) = \beta(j-1)(\beta_{2j+1}(q)) = \beta(j-1)(q + b_{2j}(q)),$$

with $\|b_{2j}(q)\| \leq B \cdot |w|^k \delta_1^{8(5/4)^j}$ with $k = m/2-4$. Because of (4.2) the well definedness then follows from the following lemma (in which we drop restricting to odd indices), as long as we set $m \geq 12$.

Lemma 4.5. *Let $B_k > 0$ for $k \in \mathbb{N}, k \geq 2$ and let $\alpha > 1$. Then there exist $C_k > 0, a(k) > 0$ and $\delta_0 > 0$ such that the following holds. For any k define $f_1^k(x) := x + B_k x^k \delta^\alpha$, and define f_j^k inductively by*

$$f_{j+1}^k(x) := f_j^k(x + B_k x^k \delta^{\alpha^{j+1}}). \tag{4.27}$$

Then $f_j^k(x) - x \leq C_k x^k \delta$ for all $x \in [0, a(k)]$ and $\delta \leq \delta_0$.

Remark 4.6. For simplicity we dropped the factor 8 in the power - it would only make the estimates better.

Proof. We have that

$$(f_{j+1}^k)'(x) := (f_j^k)'(x + B_k x^k \delta^{\alpha^{j+1}}) \cdot (1 + B_k k x^{k-1} \delta^{\alpha^{j+1}}). \tag{4.28}$$

For $\tilde{a}(k) > 0$ small, we prove by induction the statement I_j that

$$(f_j^k)'(x) \leq \Pi_{i=1}^j (1 + B_k k \delta^{\alpha^i}) \text{ for all } x \in [0, \tilde{a}(k) - B_k \tilde{a}(k)^k \sum_{i=1}^j \delta^{\alpha^i}]. \tag{4.29}$$

First of all $f_1'(x) \leq 1 + B_k k \delta^\alpha$ for all $x \in [0,1]$, so I_1 holds. The map $x + B_k x^k \delta^{\alpha^{j+1}}$ maps the intervall $[0, \tilde{a}(k) - B_k \tilde{a}(k)^k \sum_{i=0}^j \delta^{\alpha^i}]$ into the interval $[0, \tilde{a}(k) - B_k \tilde{a}(k)^k \sum_{i=0}^{j-1} \delta^{\alpha^i}]$ and so by (4.28) we get I_{j+1}. This shows that the family $\{f_j^k\}$ is uniformly Lipschitz on $[0, a(k)]$, where $a(k) = \tilde{a}(k) - B_k \tilde{a}(k)^k \sum_{j=1}^\infty \delta^{\alpha^j}$. So we get

$$\begin{aligned} f_{j+1}(x) - x &= f_j(x + B_k x^k \delta^{\alpha^{j+1}}) - f_j(x) + f_j(x) - x \\ &= (\sum_{i=2}^j f_i(x + B_k x^k \delta^{\alpha^{i+1}}) - f_i(x)) + f_1(x) - x \\ &\leq C_k x^k \sum_{i=1}^j \delta^{\alpha^i} \leq C_k' x^k \delta. \end{aligned}$$

□

So the limit $\beta = \mathrm{id} + b$ exists (near Z_r), and β extends to the identity map on Z_r. Finally, by the same scheme as above, we may control any finite C^k-norm up to Z_r by increasing the integer m in the application of the weight ψ. □

5. Exposing points on Worm domains

We will here briefly eplain how to prove Theorem 1.4 following [8], after having established Theorem 4.1 above. The first steps in [8] provide an element $\Phi \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^2$ such that

(1) $\Phi(p) = 0$,
(2) $T_0(b\Phi(\Omega_r)) = \{\mathrm{Re}(z) = 0\}$,
(3) $\Phi(\overline{\Omega}_r) \cap \Gamma = \{0\}$, where $\Gamma := \{w = \mathrm{Im}(z) = 0, \mathrm{Re}(z) \geq 0\}$, and
(4) $b\Phi(\Omega_r)$ is strongly convex at 0.

The existence of such a Φ relies only on the strict pseudoconvexity of $b\Omega_r$ at p, and no other global assumption about Ω_r.

Next set $U_\delta := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0, |z| < \delta\}$ for some $0 < \delta << 1$. For a large $R > 0$ we let, for each $j \in \mathbb{N}$, g_j be a smooth map such that $g_j(z) = z$ for all z near $\overline{U}_\delta$, and such that g_j embeds the interval $I_j = [0, 1/j]$ onto the interval $[0, R]$. By Mergelyan's theorem we may approximate g_j by a holomorphic map $\tilde{g}_j$ in C^1-norm on $\overline{U}_\delta \cup I_j$, and we set $f_j(z) := \tilde{g}_j(z) + \overline{\tilde{g}_j(\overline{z})}$. Then by adding thinner and thinner strips around the I_j's, we obtain a sequence of domains $U_{\delta,j}$, symmetric with respect to the x-axis, such that f_j embeds $U_{\delta,j}$ onto a domain V_j, such that $f_j(I_j) = [0, R]$, and such that R is a strongly convex globally exposed point for V_j, and we may achieve that $f_j \to \mathrm{id}$ uniformly on $\overline{U}_\delta$. Moreover, we may achieve that $\overline{U}_{\delta,j}$ converges to the domain U_δ in the sense of Goluzin, and so letting $\psi_j : U_\delta \to U_{\delta,j}$ be the Riemann map symmetric with respect to the x-axis, we have that $\tilde{f}_j := f_j \circ \psi_j$ converges uniformly to the identity map on $\overline{U}_\delta \setminus D_\mu(0)$ for any $\mu > 0$.

Now for $0 < \sigma << 1$ we set

$$A_\sigma := \{(z, w) \in \Phi(\Omega_r) : \mathrm{Re}(z) \geq -\sigma, (z, w) \text{ close to the origin}\}, \tag{5.1}$$

and further $B_\sigma := \overline{\Phi(\Omega_r)} \setminus A_{\sigma/2}$. Then (A_σ, B_σ) satisfies the hypotheses of Theorem 4.1. And setting $F_j(z, w) := (\tilde{f}_j(z), w)$, we have that $F_j \to \mathrm{id}$ uniformly on a fixed neighbourhood $\tilde{C}$ of $A_\sigma \cap B_\delta$. We set $\gamma_j = F_j|_{\tilde{C}}$, and let α_j, β_j provide splittings as in Theorem 4.1, such that α_j vanishes to order two at the origin. Then the maps Ψ_j defined as β_j on B_σ and $F_j \circ \alpha_j$ near A_σ will for a sufficiently large j have the property that $\phi_j = \Psi_j \circ \Phi$ embeds Ω_r into a sufficiently large ball with $(R, 0)$ on its boundary (not necessarily centred at the origin), and with $\phi_j(p) = (R, 0)$ a globally exposed point. A scaling and a translation then gives the conclusion of the theorem.

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