Small-amplitude fully localised solitary waves for the full-dispersion Kadomtsev–Petviashvili equation

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Abstract. The KP-I equation

$$(u_t - 2uu_x + \tfrac{1}{2}(\beta - \tfrac{1}{3})u_{xxx})_x - u_{yy} = 0$$

arises as a weakly nonlinear model equation for gravity-capillary waves with strong surface tension (Bond number $\beta > 1/3$). This equation admits — as an explicit solution — a 'fully localised' or 'lump' solitary wave which decays to zero in all spatial directions. Recently there has been interest in the *full-dispersion KP-I equation*

$$u_t + m(\mathrm{D})u_x + 2uu_x = 0,$$

where $m(\mathrm{D})$ is the Fourier multiplier with symbol

$$m(k) = \left(1 + \beta|k|^2|\right)^{\frac{1}{2}} \left(\frac{\tanh|k|}{|k|}\right)^{\frac{1}{2}} \left(1 + \frac{2k_2^2}{k_1^2}\right)^{\frac{1}{2}},$$

which is obtained by retaining the exact dispersion relation from the water-wave problem. In this paper we show that the FDKP-I equation also has a fully localised solitary-wave solution. The existence theory is variational and perturbative in nature. A variational principle for fully localised solitary waves is reduced to a locally equivalent variational principle featuring a perturbation of the variational functional associated with fully localised solitary-wave solutions of the KP-I equation. A nontrivial critical point of the reduced functional is found by minimising it over its natural constraint set.

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1. Introduction

There has recently been considerable interest in 'full dispersion' versions of model equations obtained by modifying their dispersive terms so that their dispersion relation coincides with that of the original physical problem. The method has been used for some time in engineering and oceanography, but has become more attractive to mathematicians interested in nonlocal equations in view of improved use of harmonic analysis in partial differential equations. The prototypical example is the full-dispersion equation derived by Whitham [22] as an alternative to the celebrated Korteweg–de Vries equation for water waves by incorporating the same linear dispersion relation as the full two-dimensional water-wave problem. It was shown by Ehrnström, Groves & Wahlén [10] that the Whitham equation admits small-amplitude solitary-wave solutions which are approximated by scalings of the Korteweg–de Vries solitary wave; these waves are symmetric and of exponential decay rate (Bruell, Ehrnström & Pei [2]). Other examples of current interest in fully dispersive equations include analytical investigations of bidirectional models in the spirit of Whitham (Ehrnström, Johnson & Claassen [11], Hur & Tao [13]) and Green-Naghdi (Duchene, Nilsson & Wahlén [9]), as well as studies of the numerical, laboratory and modelling properties of these equations (see respectively Claassen & Johnson [6], Carter [4] and Klein *et al.* [14]). The monograph by Lannes [15] has a separate section on the subject of improved frequency dispersion. From a mathematical point of view, such equations often pose extra challenges arising from their more complicated symbols (which are typically inhomogeneous).

A higher-dimensional example is given by the full-dispersion Kadomtsev–Petviashvili (FDKP) equation

$$u_t + m(\mathrm{D})u_x + 2uu_x = 0, \tag{1}$$

where the Fourier multiplier m is given by

$$m(\mathrm{D}) = \left(1+\beta|\mathrm{D}|^2\right)^{\frac{1}{2}} \left(\frac{\tanh|\mathrm{D}|}{|\mathrm{D}|}\right)^{\frac{1}{2}} \left(1+\frac{2\mathrm{D}_2^2}{\mathrm{D}_1^2}\right)^{\frac{1}{2}}$$

with $\mathrm{D} = -\mathrm{i}(\partial_x, \partial_y)$, which was introduced by Lannes [15] (see also Lannes & Saut [16]) as an alternative to the classical KP equation

$$(u_t - 2uu_x + \tfrac{1}{2}(\beta - \tfrac{1}{3})u_{xxx})_x - u_{yy} = 0. \tag{2}$$

Equation (2) arises as a weakly nonlinear approximation for three-dimensional gravity-capillary water waves, the parameter $\beta > 0$ measuring the relative strength of surface tension; the cases $\beta > \frac{1}{3}$ ('strong surface tension') and $\beta < \frac{1}{3}$ ('weak surface tension') are termed respectively KP-I and KP-II.

A *(fully localised) FDKP solitary wave* is a nontrivial, evanescent solution of (1) of the form $u(x, y, t) = u(x - ct, y)$ with wave speed $c > 0$, that is, a homoclinic solution of the equation

$$-cu + m(\mathrm{D})u + u^2 = 0. \tag{3}$$

Similarly, a *(fully localised) KP solitary wave* is a nontrivial, evanescent solution of (2) of the form $u(x, y, t) = u(x - \tilde{c}t, y)$ with wave speed $\tilde{c} > 0$, that is, a homoclinic solution of the equation

$$(\tilde{c} - 1)u + \tilde{m}(\mathrm{D})u + u^2 = 0, \tag{4}$$

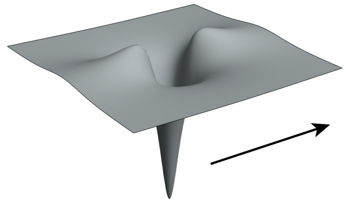

Figure 1. Sketch of the KP-I solitary wave given by (6)

where

$$\tilde{m}(\mathrm{D}) = 1 + \frac{D_2^2}{D_1^2} + \tfrac{1}{2}(\beta - \tfrac{1}{3})D_1^2.$$

Note that the KP wave speed $\tilde{c}$ can be normalised to unity by the transformation $u(x,y) \mapsto \tilde{c}u(\tilde{c}^{\frac{1}{2}}x, \tilde{c}y)$, which converts (4) into the equation

$$\tilde{m}(\mathrm{D})u + u^2 = 0. \tag{5}$$

It is known that the KP-II equation does not admit any solitary waves (de Bouard & Saut [7]), while the explicit solutions

$$u(x,y) = -12\frac{3 - X^2 + Y^2}{(3 + X^2 + Y^2)^2}, \qquad (X,Y) = \left(\tfrac{1}{2}(\beta - \tfrac{1}{3})\right)^{-\frac{1}{2}}(x,y) \tag{6}$$

of (5) define KP-I solitary waves (see Figure 1). In this paper we demonstrate the existence of solitary-wave solutions to the FDKP-I equation and show how they are approximated by scalings of KP-I solitary waves. (It is not known whether the latter are given by the explicit formula (6), but recent evidence points in this direction (see Chiron & Scheid [5] and Liu & Wei [18]).)

Theorem 1.1 *There exists a solitary-wave solution of the FDKP-I equation with speed $c = 1 - \varepsilon^2$ for each sufficiently small value of $\varepsilon > 0$. This solution belongs to $H^\infty(\mathbb{R}^2)$.*

An FDKP solitary wave is characterised as a critical point of the Hamiltonian

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |(m(\mathrm{D}))^{\frac{1}{2}}u|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{1}{3}\int_{\mathbb{R}^2} u^3 \,\mathrm{d}x\,\mathrm{d}y \tag{7}$$

subject to the constraint that the momentum

$$\mathcal{M}(u) = \frac{1}{2}\int_{\mathbb{R}^2} u^2 \,\mathrm{d}x\,\mathrm{d}y \tag{8}$$

is fixed; the Lagrange multiplier is the wave speed c. Using this observation we may reformulate the existence statement in Theorem 1.1 in terms of the calculus of variations. Let X denote the completion of $\partial_x\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$|u|_X^2 = \int_{\mathbb{R}^2}\left(1 + \frac{k_2^2}{k_1^2} + \frac{k_2^4}{k_1^2} + |k|^{2s}\right)|\hat{u}(k)|^2 \,\mathrm{d}k,$$

where $s > \frac{3}{2}$ and $\mathcal{S}(\mathbb{R}^2)$ is the two-dimensional Schwartz space.

Theorem 1.2 *Suppose that $\beta > \frac{1}{3}$. The formula $\mathcal{I}_\varepsilon = \mathcal{E} - c\mathcal{M}$ with $c = 1 - \varepsilon^2$ defines a smooth functional $\mathcal{I}_\varepsilon : X \to \mathbb{R}$ which has a nontrivial critical point for each sufficiently small value of $\varepsilon > 0$.*

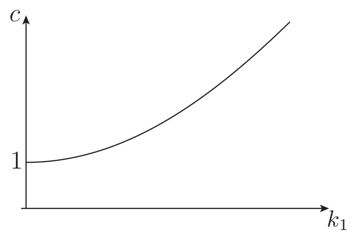

Figure 2. FKDP-I dispersion relation for two-dimensional wave trains

To motivate our main result it is instructive to review the formal derivation of the (normalised) steady KP equation (5) from the steady FDKP equation (3). We begin with the linear dispersion relation for a two-dimensional sinusoidal travelling wave train with wave number k_1 and speed c, namely

$$c = \left(1 + \beta |k_1|^2\right)^{\frac{1}{2}} \left(\frac{\tanh |k_1|}{|k_1|}\right)^{\frac{1}{2}}$$

The function $k_1 \mapsto c(k_1)$, $k_1 \geq 0$ has a unique global minimum at $k_1 = 0$ with $c(0) = 1$ (see Figure 2). Bifurcations of nonlinear solitary waves are expected whenever the linear group and phase speeds are equal, so that $c'(k_1) = 0$ (see Dias & Kharif [8, §3]); one therefore expects bifurcation of small-amplitude solitary waves from uniform flow with unit speed. Furthermore, observing that m is an analytic function of k_1 and $\frac{k_2}{k_1}$ (note that $|k|^2 = k_1^2 + \frac{k_2^2}{k_1^2} k_1^2$), one finds that

$$m(k) = \tilde{m}(k) + O(|(k_1, \tfrac{k_2}{k_1})|^4) \tag{9}$$

as $(k_1, \frac{k_2}{k_1}) \to 0$. We therefore make the steady-wave *Ansatz* $u(x, y, t) = \tilde{u}(x - ct, y)$ and substitute $c = 1 - \varepsilon^2$ and

$$\tilde{u}(x, y) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 y) \tag{10}$$

into equation (3). This calculation shows that to leading order ζ satisfies

$$\tilde{m}(D)\zeta + \zeta^2 = 0, \tag{11}$$

which is the Euler–Lagrange equation for the (smooth) functional $\mathcal{T}_0 : \tilde{Y} \to \mathbb{R}$ given by

$$\mathcal{T}_0(\zeta) = \frac{1}{2} \int_{\mathbb{R}^2} |(\tilde{m}(\mathrm{D}))^{\frac{1}{2}} \zeta|^2 \, \mathrm{d}x \, \mathrm{d}y + \frac{1}{3} \int_{\mathbb{R}^2} \zeta^3 \, \mathrm{d}x \, \mathrm{d}y,$$

where $\tilde{Y}$ is the completion of $\partial_x \mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$|\zeta|_{\tilde{Y}}^2 = \int_{\mathbb{R}^2} \left(1 + \tfrac{k_2^2}{k_1^2} + k_1^2\right) |\hat{\zeta}|^2 \, \mathrm{d}k.$$

We proceed by performing a rigorous local variational reduction which converts $\mathcal{I}_\varepsilon$ to a perturbation $\mathcal{T}_\varepsilon$ of $\mathcal{T}_0$ (Section 3).

The estimate (9) suggests that the spectrum of a solitary wave $u(x, y)$ is concentrated in the region $|k_1|, |\frac{k_2}{k_1}| \ll 1$. We therefore decompose u into the sum of functions u_1 and u_2 whose spectra are supported in the region

$$C = \left\{(k_1, k_2) \colon |k| \leq \delta, \left|\tfrac{k_2}{k_1}\right| \leq \delta\right\}$$

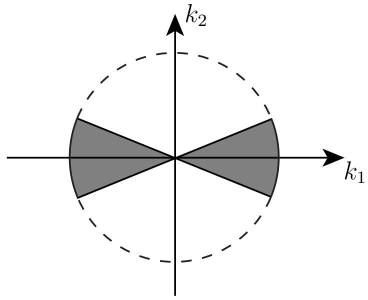

Figure 3. The cone $C = \{k \in \mathbb{R}^2 \colon |k| \leq \delta, \frac{|k_2|}{|k_1|} \leq \delta\}$ cut out of the closed ball $\{k \in \mathbb{R}^2 \colon |k| \leq \delta\}$ in $\mathbb{R}^2$.

and its complement, where δ is a small positive number (see Figure 3), so that

$$u_1 = \chi(\mathrm{D})u, \qquad u_2 = (1 - \chi(\mathrm{D}))u,$$

in which χ is the characteristic function of C. In Section 3 we employ a method akin to the variational Lyapunov-Schmidt reduction to determine u_2 as a function of u_1 and thus obtain the reduced functional $\mathcal{J}_\varepsilon : U \to \mathbb{R}$ given by

$$\mathcal{J}_\varepsilon(u_1) = \mathcal{I}_\varepsilon(u_1 + u_2(u_1));$$

here $U = \{u_1 \in X_1 : |u_1|_\varepsilon \leq 1\}$ is the unit ball in the space $(X_1, |\cdot|_\varepsilon)$, in which $X_1 = \chi(D)X$ and $|\cdot|_\varepsilon$ is the scaled norm

$$|u_1|_\varepsilon^2 = \int_{\mathbb{R}^2} \left(1 + \varepsilon^{-2}\frac{k_2^2}{k_1^2} + \varepsilon^{-2}k_1^2\right)|\hat{u}_1(k)|^2 \,\mathrm{d}k.$$

Applying the KP scaling (10) to u_1, one finds that $\mathcal{J}_\varepsilon(u_1) = \varepsilon^3 \mathcal{T}_\varepsilon(\zeta)$, where

$$\mathcal{T}_\varepsilon(\zeta) = \mathcal{T}_0(\zeta) + \varepsilon^{\frac{1}{2}}\mathcal{R}_\varepsilon(\zeta), \qquad \mathcal{R}_\varepsilon(\zeta) \lesssim |\zeta|_{\tilde{Y}}^2$$

(with corresponding estimates for the derivatives of the remainder term). Each critical point ζ_∞ of $\mathcal{T}_\varepsilon$ with $\varepsilon > 0$ corresponds to a critical point $u_{1,\infty}$ of $\mathcal{J}_\varepsilon$, which in turn defines a critical point $u_{1,\infty} + u_2(u_{1,\infty})$ of $\mathcal{I}_\varepsilon$.

We study $\mathcal{T}_\varepsilon$ in a fixed ball

$$B_M(0) = \{\zeta \colon |\zeta|_{\tilde{Y}} < M\},$$

in the space $(\tilde{Y}_\varepsilon, |\cdot|_{\tilde{Y}})$, where $\tilde{Y}_\varepsilon = \chi_\varepsilon(\mathrm{D})\tilde{Y}$ and $\chi_\varepsilon(k_1, k_2) = \chi(\varepsilon k_1, \varepsilon^2 k_2)$. The parameters M and ε are related in the following manner: for any $M > 1$ there exists $\varepsilon_M \lesssim M^{-2}$ such that all estimates hold uniformly over $\varepsilon \in [0, \varepsilon_M]$. We do not make these threshold values of ε explicit; it is simply assumed that ε_M is taken sufficiently small. In the limit $\varepsilon = 0$ we can set $M = \infty$ and recover the KP variational functional $\mathcal{T}_0 : \tilde{Y} \to \mathbb{R}$ (note that $\tilde{Y}_0 = \tilde{Y}$). In fact $\mathcal{T}_\varepsilon : B_M(0) \to \mathbb{R}$ may be considered as a perturbation of the 'limiting' functional $\mathcal{T}_0 : \tilde{Y} \to \mathbb{R}$. More precisely $\varepsilon^{\frac{1}{2}}\mathcal{R}_\varepsilon \circ \chi_\varepsilon(\mathrm{D})$ (which coincides with $\varepsilon^{\frac{1}{2}}\mathcal{R}_\varepsilon$ on $B_M(0) \subset \tilde{Y}_\varepsilon$) converges uniformly to zero over $B_M(0) \subset \tilde{Y}$ (with corresponding uniform convergence for its derivatives), and we study $\mathcal{T}_\varepsilon$ by perturbative arguments in this spirit.

In Section 4 we seek critical points of $\mathcal{T}_\varepsilon$ by minimising it on its *natural constraint set*

$$N_\varepsilon = \{\zeta \in B_M(0) : \zeta \neq 0, \mathrm{d}\mathcal{T}_\varepsilon[\zeta](\zeta) = 0\},$$

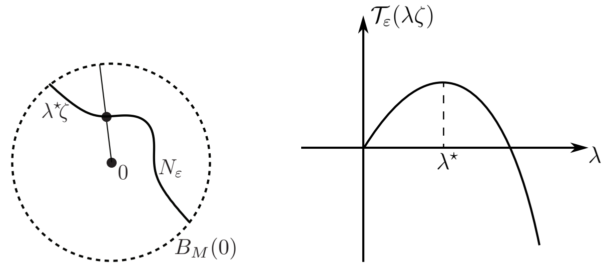

Figure 4. Any ray intersects the natural constraint set N_ε in at most one point and the value of $\mathcal{T}_\varepsilon$ along such a ray attains a strict maximum at this point

our motivation being the observation that the critical points of $\mathcal{T}_\varepsilon$ coincide with those of $\mathcal{T}_\varepsilon|_{N_\varepsilon}$. The natural constraint set has a geometrical interpretation (see Figure 4), namely that any ray in $B_M(0)$ intersects the natural constraint manifold N_ε in at most one point and the value of $\mathcal{T}_\varepsilon$ along such a ray attains a strict maximum at this point. This fact is readily established by a direct calculation for $\varepsilon = 0$ and deduced by a perturbation argument for $\varepsilon > 0$, and similar perturbative methods yield the existence of a a sequence $\{\zeta_n\} \subset B_{M-1}(0)$ with

$$\mathcal{T}_\varepsilon|_{N_\varepsilon} \to \inf \mathcal{T}_\varepsilon|_{N_\varepsilon} > 0, \qquad |\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n]|_{\tilde{Y}_\varepsilon \to \mathbb{R}} \to 0$$

as $n \to \infty$. The following theorem is established by applying weak continuity arguments to minimising sequences of the above kind.

Theorem 1.3 *Let $\{\zeta_n\} \subset B_{M-1}(0)$ be a minimising sequence for $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ with $|\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n]|_{\tilde{Y}_\varepsilon \to \mathbb{R}} \to 0$ as $n \to \infty$. There exists $\{w_n\} \subset \mathbb{Z}^2$ with the property that a subsequence of $\{\zeta_n(\cdot + w_n)\}$ converges weakly in $\tilde{Y}_\varepsilon$ to a nontrivial critical point ζ_∞ of $\mathcal{T}_\varepsilon$.*

The short proof of Theorem 1.3 does not show that the critical point ζ_∞ is a *ground state*, that is, a minimiser of $\mathcal{T}_\varepsilon$ over N_ε. This deficiency is removed in Section 5 with the help of an abstract version of the concentration-compactness principle due to Buffoni, Groves & Wahlén [3, Appendix A]. (That paper treats fully localised solitary waves in the Euler equations with weak surface tension using theory closely connected to ours.)

Theorem 1.4 *Let $\{\zeta_n\} \subset B_{M-1}(0)$ be a minimising sequence for $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ with $|\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n]|_{\tilde{Y}_\varepsilon \to \mathbb{R}} \to 0$ as $n \to \infty$. There exists $\{w_n\} \subset \mathbb{Z}^2$ with the property that a subsequence of $\{\zeta_n(\cdot + w_n)\}$ converges weakly, and strongly if $\varepsilon = 0$, in $\tilde{Y}_\varepsilon$ to a ground state ζ_∞.*

We prove Theorems 1.3 and 1.4 for $\varepsilon = 0$ and $\varepsilon > 0$ separately, in the latter case taking advantage of the relationship $\mathcal{I}_\varepsilon(u) = \varepsilon^3 \mathcal{T}_\varepsilon(\zeta)$, where $u = u_1(\zeta) + u_2(u_1(\zeta))$, and the fact that $\tilde{Y}_\varepsilon$ coincides with $H^s_\varepsilon(\mathbb{R}^2) := \chi_\varepsilon(\mathrm{D}) H^s(\mathbb{R}^2)$ for any $s > \frac{3}{2}$. The function $u_\infty = u_1(\zeta_\infty) + u_2(u_1(\zeta_\infty))$ given by these theorems is then a nontrivial critical point of $\mathcal{I}_\varepsilon$, which concludes the proof of Theorem 1.2. The discussion of the case $\varepsilon = 0$ does not contribute to this existence proof but shows that the KP ground states (that is, the ground states of $\mathcal{T}_0$) are characterised in the same way as the ground states of $\mathcal{T}_\varepsilon$ for $\varepsilon > 0$. Using this information, we show that the ground states of $\mathcal{T}_\varepsilon$ converge to those of $\mathcal{T}_0$ as $\varepsilon \to 0$ in the following sense.

Theorem 1.5 *Let* $c_\varepsilon = \inf_{N_\varepsilon} \mathcal{T}_\varepsilon$.

(i) One has that $\lim_{\varepsilon\to 0} c_\varepsilon = c_0$.

(ii) Let $\{\varepsilon_n\}$ *be a sequence with* $\lim_{n\to\infty} \varepsilon_n = 0$ *and* ζ^{ε_n} *be a ground state of* $\mathcal{T}_{\varepsilon_n}$. *There exists* $\{w_n\} \subset \mathbb{Z}^2$ *such that a subsequence of* $\{\zeta^{\varepsilon_n}(\cdot + w_n)\}$ *converges strongly in* $\tilde{Y}$ *to a ground state* $\zeta^\star$ *of* $\mathcal{T}_0$.

Our final result concerns convergence of FDKP-I solitary waves to KP-I solitary waves and is obtained as a corollary of Theorem 1.5.

Theorem 1.6 *Let* $\{\varepsilon_n\}$ *be a sequence with* $\lim_{n\to\infty} \varepsilon_n = 0$ *and* u^{ε_n} *be a critical point of* $\mathcal{I}_{\varepsilon_n}$ *with* $\mathcal{I}_{\varepsilon_n}(u^{\varepsilon_n}) = \varepsilon_n^3 c_{\varepsilon_n}$, *so that the formula* $u^{\varepsilon_n} = u_1(\zeta^{\varepsilon_n}) + u_2(u_1(\zeta^{\varepsilon_n}))$ *defines a ground state* ζ^{ε_n} *of* $\mathcal{T}_\varepsilon$. *There exists* $\{w_n\} \subset \mathbb{Z}^2$ *such that a subsequence of* $\{\zeta^{\varepsilon_n}(\cdot + w_n)\}$ *converges strongly in* $\tilde{Y}$ *to a ground state* $\zeta^\star$ *of* $\mathcal{T}_0$.

Defining $u_\varepsilon^\star(x,y) = \varepsilon^2\zeta^\star(\varepsilon x, \varepsilon^2 y)$, so that $u_\varepsilon^\star$ is a KP solitary wave with wave speed ε^2, we find that the difference $u^{\varepsilon_n} - u^\star_{\varepsilon_n}$ converges to zero in $(\tilde{Y}, |\cdot|_{\varepsilon_n})$ and in $H^{\frac{1}{2}}(\mathbb{R}^2)$ (see Remark 5.10). Although these functions are small, their difference converges to zero faster than the functions themselves), so that we also have convergence with respect to the original variables.

Remark 1.7 *The results presented in this paper apply with straightforward modifications to the generalised FDKP-I and KP-I equations obtained by replacing the nonlinear term* $(u^2)_x$ *by* $(u^p)_x$ *with* $2 \le p < 5$ *(see Proposition 2.2). The proof of the counterpart to Theorem 1.3 with* $\varepsilon = 0$ *also yields a concise variational existence theory for gKP-I solitary waves as an alternative to those already available in the literature (de Bouard & Saut [7], Pankov & Pflüger [19, 20], Willem [23, Ch 7], Wang, Ablowitz & Segur [21] and Liu & Wang [17]).*

2. Function spaces

In this section we introduce the function spaces (and basic properties thereof) which are used in the variational reduction and existence theory in Sections 3 and 4 below. For notational simplicity we generally omit the exact value of $\frac{1}{2}(\beta - \frac{1}{3})$ and treat it as being of unit size (without this simplification the term k_1^2 in the norm for $\tilde{Y}$ is multiplied by $\frac{1}{2}(\beta - \frac{1}{3})$, which does not affect the proof in any way.) Examining the quadratic parts of the variational functionals

$$\begin{aligned}\mathcal{I}(u) &= \mathcal{E}(u) - c\mathcal{M}(u)\\ &= \frac{1}{2}\int_{\mathbb{R}^2} \left((m(\mathrm{D})^{\frac{1}{2}}u)^2 - cu^2\right) \mathrm{d}x\,\mathrm{d}y + \frac{1}{3}\int_{\mathbb{R}^2} u^3 \,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

and

$$\mathcal{T}_0(\zeta) = \frac{1}{2}\int_{\mathbb{R}^2} |(\tilde{m}(\mathrm{D}))^{\frac{1}{2}}\zeta|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{1}{3}\int_{\mathbb{R}^2} \zeta^3 \,\mathrm{d}x\,\mathrm{d}y$$

for the steady FDKP-I and KP-I equations (3) and (11) shows that their natural energy spaces are the completions Y and $\tilde{Y}$ of

$$\partial_x\mathcal{S}(\mathbb{R}^2) = \{\partial_x f : f \in \mathcal{S}(\mathbb{R}^2)\},$$

where $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz space of rapidly decaying smooth functions, with respect to the norms

$$|u|_Y^2 = \int_{\mathbb{R}^2} \left(1 + \left|\frac{k_2}{k_1}\right| + \frac{|k|^{\frac{3}{2}}}{|k_1|}\right) |\hat{u}(k)|^2 \, \mathrm{d}k, \tag{12}$$

and

$$|u|_{\tilde{Y}}^2 = \int_{\mathbb{R}^2} \left(1 + \frac{k_2^2}{k_1^2} + k_1^2\right) |\hat{u}(k)|^2 \, \mathrm{d}k. \tag{13}$$

Here $\mathcal{F} : u \mapsto \hat{u}$ denotes the unitary Fourier transform on $\mathcal{S}(\mathbb{R}^2)$. (In defining $|\cdot|_Y$ we have used the fact that

$$m(k) \simeq 1 + \frac{|k_2|}{|k_1|}, \qquad |k| \leq \delta,$$

$$m(k) \simeq |k_1|^{\frac{1}{2}} + \frac{|k_2|^{\frac{3}{2}}}{|k_1|}, \qquad |k| \geq \delta,$$

for any $\delta > 0$.) We study $\mathcal{T}_0$ in the smaller space X defined as the completion of $\partial_x \mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$|u|_X^2 = \int_{\mathbb{R}^2} \left(1 + \frac{k_2^2}{k_1^2} + \frac{k_2^4}{k_1^2} + |k|^{2s}\right) |\hat{u}(k)|^2 \, \mathrm{d}k, \tag{14}$$

where the Sobolev index $s > \frac{3}{2}$ is fixed. Finally, we introduce the completion Z of $\partial_x \mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$|u|_Z^2 = \int_{\mathbb{R}^2} \left(1 + |k| + k_1^2 |k|^{2s-3}\right) |\hat{u}(k)|^2 \, \mathrm{d}k; \tag{15}$$

it follows from Lemma 2.1 and Remark 2.7 that $Z = m(D)X$.

Lemma 2.1

(i) One has the continuous embeddings

$$X \hookrightarrow \tilde{Y} \hookrightarrow Y \hookrightarrow L^2(\mathbb{R}^2), \quad H^{s-\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow Z \hookrightarrow L^2(\mathbb{R}^2), \quad X \hookrightarrow H^s(\mathbb{R}^2),$$

and in particular $X \hookrightarrow \mathrm{BC}(\mathbb{R}^2)$*, the space of bounded, continuous functions on* $\mathbb{R}^2$*.*

(ii) The Fourier multiplier $m(\mathrm{D})$ *maps* X *continuously into* Z*.*

Proof (i) The first and second chain of embeddings follow from the estimates

$$\begin{aligned} 1 \leq \; & 1 + \left|\frac{k_2}{k_1}\right| + \frac{|k_2|^{\frac{3}{2}}}{|k_1|} + |k_1|^{\frac{1}{2}} \\ \lesssim \; & 1 + \frac{k_2^2}{k_1^2} + k_1^2 + \frac{|k_2|^{\frac{3}{2}}}{|k_1|} \\ \lesssim \; & 1 + \frac{k_2^2}{k_1^2} + k_1^2 \\ \leq \; & 1 + \frac{k_2^2}{k_1^2} + \frac{k_2^4}{k_1^2} + |k|^{2s} \end{aligned}$$

(in the third step we multiply and divide the last term by $|k_1|^{\frac{1}{2}}$ and apply Young's inequality with $\frac{1}{4} + \frac{3}{4} = 1$), and

$$1 \leq 1 + |k| + k_1^2 |k|^{2s-3} \lesssim 1 + |k|^{2s-1},$$

while the third follows from the estimate

$$1+|k|^{2s} \le 1+\frac{k_2^2}{k_1^2}+\frac{k_2^4}{k_1^2}+|k|^{2s}.$$

The embedding of X into $\mathrm{BC}(\mathbb{R}^2)$ follows from $H^s(\mathbb{R}^2) \hookrightarrow \mathrm{BC}(\mathbb{R}^2)$ (because $s>\frac{3}{2}$).

(ii) Observe that

$$(1+|k|+k_1^2|k|^{2s-3})m(k)^2 \lesssim 1+\frac{k_2^2}{k_1^2}$$

for $|k| \le \delta$ and

$$\begin{aligned}(1+|k|+k_1^2|k|^{2s-3})m(k)^2 &\lesssim (1+|k|+k_1^2|k|^{2s-3})|k|\left(1+\frac{k_2^2}{k_1^2}\right)\\ &= \frac{|k|^4}{k_1^2}+|k|^{2s}\\ &\lesssim \frac{k_2^4}{k_1^2}+k_1^2+|k|^{2s}\\ &\lesssim \frac{k_2^4}{k_1^2}+|k|^{2s}\end{aligned}$$

for $|k| \ge \delta$ (because $(1+\beta|k|^2)|k|^{-1}\tanh|k| \gtrsim |k|$ for $|k| \ge \delta$), so that $|m(\mathrm{D})(\cdot)|_Z^2 \lesssim |\cdot|_X^2$. □

The space $\tilde{Y}$ admits a local representation: the map $w \mapsto u := w_x$ is an isometric isomorphism $A \to \tilde{Y}$, where A is the completion of $\partial_x \mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$|w|_A^2 = \int_{\mathbb{R}^2} \left(w_x^2+w_y^2+w_{xx}^2\right)\,\mathrm{d}x\,\mathrm{d}y.$$

In this spirit we can also define the localised space $A(Q_j)$, where

$$Q_j = \{(x,y) \in \mathbb{R}^2 : |x-j_1| < \tfrac{1}{2}, |y-j_2| < \tfrac{1}{2}\}$$

is the unit cube centered at the point $j=(j_1,j_2) \in \mathbb{Z}^2$, as the completion of $\partial_x C^\infty(\overline{Q}_j)$ with respect to the norm

$$|w|_{A(Q_j)}^2 = \int_{Q_j} (w_x^2+w_y^2+w_{xx}^2)\,\mathrm{d}x\,\mathrm{d}y,$$

and $\tilde{Y}(Q_j) = \partial_x A(Q_j)$ with $|u|_{\tilde{Y}(Q_j)} = |w|_{A(Q_j)}$. Note that $u|_{Q_j}$ belongs to $\tilde{Y}(Q_j)$ for each $u \in \tilde{Y}$ and

$$|u|_{\tilde{Y}}^2 = \sum_{j \in \mathbb{Z}^2} |u|_{\tilde{Y}(Q_j)}^2.$$

Proposition 2.2 *The space* $\tilde{Y}$ *is*

(i) continuously embedded in $L^p(\mathbb{R}^2)$ *for* $2 \le p \le 6$,

(ii) compactly embedded in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ *for* $2 \le p < 6$.

Furthermore, the space $\tilde{Y}(Q_j)$ *is continuously embedded in* $L^p(Q_j)$ *for* $2 \le p \le 6$.

Proof Part (i) and the assertion concerning $\tilde{Y}(Q_j)$ follow from Besov, Ilin & Nikolskii [1, Thm 15.7] (applied to the local representations). Part (ii) is an interpolation result between $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $L^6_{\mathrm{loc}}(\mathbb{R}^2)$; de Bouard & Saut [7, Lemma 3.3] show that the inclusion $\tilde{Y} \subset L^2_{\mathrm{loc}}(\mathbb{R}^2)$ is compact, and the inclusion $\tilde{Y} \subset L^6_{\mathrm{loc}}(\mathbb{R}^2)$ is continuous by (i). □

Our next results concern the functional $\mathcal{I}$ and its Euler–Lagrange equation.

Corollary 2.3

(i) *The formula* $u \mapsto -cu + m(\mathrm{D})u + u^2$ *maps* X *smoothly into* Z*.*

(ii) *The functional* $\mathcal{I}$ *maps* X *smoothly into* $\mathbb{R}$ *and its critical points are precisely the homoclinic solutions (in* X*) of equation (3).*

Proposition 2.4 *The formula* $u \mapsto -cu + m(\mathrm{D})u + u^2$ *defines a weakly continuous mapping* $X \to Z$*.*

Proof Suppose that $\{u_n\}$ converges weakly to u in X and hence weakly in $H^s(\mathbb{R}^2)$ and strongly in $L^4_{\mathrm{loc}}(\mathbb{R}^2)$. It follows that $\langle u_n^2, \phi\rangle_{L^2}$ converges to $\langle u^2, \phi\rangle_{L^2}$ for each $\phi \in C_0^\infty(\mathbb{R}^2)$, so that $\{u_n^2\}$ converges weakly to u^2 in $H^k(\mathbb{R}^2)$ for each integer $k \leq s$ and hence weakly in Z. □

We decompose $u \in L^2(\mathbb{R}^2)$ into the sum of functions u_1 and u_2 whose spectra are supported in the region

$$C = \left\{k \in \mathbb{R}^2 \colon |k| \leq \delta, \tfrac{|k_2|}{|k_1|} \leq \delta\right\} \tag{16}$$

and its complement (see Figure 3) by writing

$$u_1 = \chi(\mathrm{D})u, \qquad u_2 = (1 - \chi(\mathrm{D}))u,$$

where χ is the characteristic function of C. Since X is a subspace of $L^2(\mathbb{R}^2)$, the Fourier multiplier $\chi(\mathrm{D})$ induces an orthogonal decomposition

$$X = X_1 \oplus X_2,$$

where

$$X_1 = \chi(\mathrm{D})X, \qquad X_2 = (1 - \chi(\mathrm{D}))X,$$

with analogous decompositions for the spaces Y, $\tilde{Y}$ and Z; we henceforth use the subscripts $_1$ and $_2$ to denote the corresponding orthogonal projections.

Lemma 2.5 *The spaces* X_1, Y_1, $\tilde{Y}_1$ *and* Z_1 *all coincide with* $\chi(\mathrm{D})L^2(\mathbb{R}^2)$*, and the norms* $|\cdot|_{L^2}$, $|\cdot|_X$, $|\cdot|_Y$, $|\cdot|_{\tilde{Y}}$ *and* $|\cdot|_Z$ *are all equivalent norms for these spaces.*

Proof Observe that

$$\begin{aligned} \chi(\mathrm{D})L^2(\mathbb{R}^2) &= \{u \in L^2(\mathbb{R}^2) : \operatorname{supp} \hat{u} \subseteq C\}, \\ \chi(\mathrm{D})X &= \{u \in X : \operatorname{supp} \hat{u} \subseteq C\}, \end{aligned}$$

so that $X \subseteq L^2(\mathbb{R}^2)$ implies that $\chi(\mathrm{D})X \subseteq \chi(\mathrm{D})L^2(\mathbb{R}^2)$. Conversely, suppose that $u \in L^2(\mathbb{R}^2)$ with $\operatorname{supp} \hat{u} \subseteq C$, so that $|u|_X^2 \leq (1+2\delta^2)|u|_{L^2}^2$ and hence $u \in X$; it follows that $\chi(\mathrm{D})L^2(\mathbb{R}^2) \subseteq \chi(\mathrm{D})X$. The other equalities are established in the same way. □

Let us now consider the Fourier multipliers

$$n = m - 1, \qquad \tilde{n} = \tilde{m} - 1 \tag{17}$$

which arise in our study of solitary waves with near unit speed.

Lemma 2.6 *The mapping* $n(\mathrm{D})$ *is an isomorphism* $X_2 \to Z_2$*.*

Proof It follows from Lemma 2.1(ii) that $n(\mathrm{D}) = m(\mathrm{D}) - 1$ maps X continuously into Z and hence X_2 continuously into Z_2.

Writing

$$n(k) = \left((1+\beta|k|^2)^{\frac{1}{2}}\left(\frac{\tanh|k|}{|k|}\right)^{\frac{1}{2}}-1\right)\left(1+\frac{2k_2^2}{k_1^2}\right)^{\frac{1}{2}}+\left(1+\frac{2k_2^2}{k_1^2}\right)^{\frac{1}{2}}-1$$

and noting that $(1+\beta|k|^2)|k|^{-1}\tanh|k|-1 \gtrsim |k|$ for $|k|\geq\delta$, one finds that

$$n(k) \gtrsim \left((1+|k|)^{\frac{1}{2}}-1\right)\left(1+\frac{2k_2^2}{k_1^2}\right)^{\frac{1}{2}}+\left(1+\frac{2k_2^2}{k_1^2}\right)^{\frac{1}{2}}-1 \gtrsim |k|^{\frac{1}{2}}\left(1+\frac{k_2^2}{k_1^2}\right)^{\frac{1}{2}} = \frac{|k|^{\frac{3}{2}}}{|k_1|}$$

and therefore

$$\begin{aligned}\left(1+\frac{k_2^2}{k_1^2}+\frac{k_2^4}{k_1^2}+|k|^{2s}\right)n(k)^{-2} &\lesssim \frac{k_1^2}{|k|^3}+\frac{k_2^2}{|k|^3}+\frac{k_2^4}{|k|^3}+k_1^2|k|^{2s-3}\\ &\lesssim |k|+k_1^2|k|^{2s-3}\end{aligned}$$

for $|k|\geq\delta$. On the other hand, in the regime $|k|\leq\delta$, $\left|\frac{k_2}{k_1}\right|\geq\delta$ one has that

$$\left(1+\frac{k_2^2}{k_1^2}+\frac{k_2^4}{k_1^2}+|k|^{2s}\right)n(k)^{-2} \lesssim \left(1+\frac{k_2^2}{k_1^2}\right)\left(\left(1+\frac{k_2^2}{k_1^2}\right)^{\frac{1}{2}}-1\right)^{-2} \lesssim 1;$$

altogether we have established that $|n(\mathrm{D})^{-1}(\cdot)|_X^2 \lesssim |\cdot|_Z^2$. □

Remark 2.7 *A straightforward modification of the above proof shows that $m^{-1}(\mathrm{D})$ maps Z continuously into X, so that m is an isomorphism $X\to Z$. It is however rather the multiplier n that appears in our analysis.*

In view of the KP-scaling $(k_1,k_2)\mapsto(\varepsilon k_1,\varepsilon^2 k_2)$ it is convenient to work with the scaled norm

$$|u_1|_\varepsilon^2 = \int_{\mathbb{R}^2}\left(1+\varepsilon^{-2}\frac{k_2^2}{k_1^2}+\varepsilon^{-2}k_1^2\right)|\hat{u}_1(k)|^2\,\mathrm{d}k \tag{18}$$

for $\tilde{Y}_1$ (or, equivalently, for $\chi(\mathrm{D})L^2(\mathbb{R}^2)$, X_1, Y_1, Z_1).

Lemma 2.8 *The estimates*

$$|u_1|_{W^{m,\infty}(\mathbb{R}^2)} \lesssim \varepsilon|u_1|_\varepsilon, \qquad m\geq 0,$$

and

$$|u_1 v|_Z \lesssim \varepsilon|u_1|_\varepsilon|v|_X,$$

hold for all $u_1\in X_1$ and $v\in X$.

Proof Note that

$$|u_1|_{W^{m,\infty}(\mathbb{R}^2)} \lesssim |(1+|k|^m)\hat{u}_1|_{L^1} \lesssim |\hat{u}_1|_{L^1} \leq |u_1|_\varepsilon I^{\frac{1}{2}},$$

where

$$\begin{aligned} I &= \int_C \frac{1}{1+\varepsilon^{-2}\frac{k_2^2}{k_1^2}+\varepsilon^{-2}k_1^2}\,\mathrm{d}k\\ &\leq 4\int_0^\delta\int_0^\delta \frac{t_1}{1+\varepsilon^{-2}t_2^2+\varepsilon^{-2}t_1^2}\,\mathrm{d}t_2\,\mathrm{d}t_1\\ &= 4\varepsilon^3\int_0^{\delta/\varepsilon}\int_0^{\delta/\varepsilon}\frac{t_1}{1+t_2^2+t_1^2}\,\mathrm{d}t_2\,\mathrm{d}t_1\\ &\lesssim \varepsilon^2\end{aligned}$$

(because $C\subseteq\{(k_1,k_2): |k_1|, |\frac{k_2}{k_1}|\leq\delta\}$). Choosing $m>s$, one therefore finds that

$$|u_1 v|_Z \lesssim |u_1 v|_X \lesssim |u_1|_{W^{m,\infty}(\mathbb{R}^2)}|v|_X \lesssim \varepsilon|u_1|_\varepsilon|v|_X.$$

□

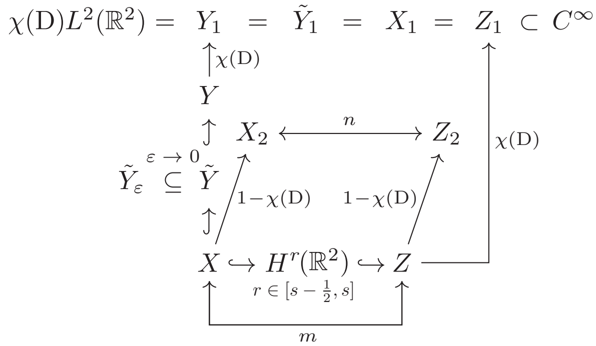

Figure 5. An overview of the spaces used in this paper. The spaces Y and $\tilde{Y}$ are the energy spaces for respectively the FDKP-I and KP-I equations. The operator $\chi(\mathrm{D})$ induces orthogonal decompositions $X = X_1 \oplus X_2$, $Z = Z_1 \oplus Z_2$, while $m(\mathrm{D})$, $n(\mathrm{D})$ define isomorphisms $X \to Z$ and $X_2 \to Z_2$. Finally, $\tilde{Y}_\varepsilon = \chi(\varepsilon\mathrm{D})\tilde{Y}$.

Finally, we introduce the space $\tilde{Y}_\varepsilon := \chi_\varepsilon(\mathrm{D})\tilde{Y}$, where $\chi_\varepsilon(k_1, k_2) = \chi(\varepsilon k_1, \varepsilon^2 k_2)$ (with norm $|\cdot|_{\tilde{Y}}$), noting the relationship

$$|u|_\varepsilon^2 = \varepsilon|\zeta|_{\tilde{Y}}^2, \qquad u(x,y) = \varepsilon^2\zeta(\varepsilon x, \varepsilon^2 y)$$

for $\zeta \in \tilde{Y}_\varepsilon$. Observe that $\tilde{Y}_\varepsilon$ coincides with $\chi_\varepsilon(\mathrm{D})X$, $\chi_\varepsilon(\mathrm{D})Y$, $\chi_\varepsilon(\mathrm{D})Z$ and $\chi_\varepsilon(\mathrm{D})L^2(\mathbb{R}^2)$ for $\varepsilon > 0$, while in the limit $\varepsilon \to 0$ we find that $\tilde{Y}_0 = \tilde{Y}$. We work in particular with the distinguished subsets $\{\zeta : |\zeta|_{\tilde{Y}} < M\}$ and $\{\zeta : |\zeta|_{\tilde{Y}} < M-1\}$ of $\tilde{Y}_\varepsilon$, denoting them by respectively $B_M(0)$ and $B_{M-1}(0)$.

We conclude this section with a result which is used in our analysis of the KP-I functional $\mathcal{T}_0$.

Corollary 2.9 *The functional $\mathcal{T}_0$ maps $\tilde{Y}$ smoothly into $\mathbb{R}$ and its critical points are precisely the homoclinic solutions of equation (11).*

Figure 5 summarises the various spaces and their relationships to each other.

3. Variational reduction

We proceed by making the Ansatz $c = 1 - \varepsilon^2$ and seeking critical points of the functional $\mathcal{I}_\varepsilon : X \to \mathbb{R}$ given by

$$\mathcal{I}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\varepsilon^2 u^2 + (n(\mathrm{D})^{\frac{1}{2}}u)^2\right) \mathrm{d}x\,\mathrm{d}y + \frac{1}{3}\int_{\mathbb{R}^2} u^3\,\mathrm{d}x\,\mathrm{d}y, \tag{19}$$

so that the critical points of $\mathcal{I}_\varepsilon$ are precisely the homoclinic solutions of (3) with $c = 1 - \varepsilon^2$.

Note that $u = u_1 + u_2 \in X_1 \oplus X_2$ is a critical point of $\mathcal{I}_\varepsilon$ if and only if

$$\mathrm{d}\mathcal{I}_\varepsilon[u_1 + u_2](w_1) = 0, \qquad \mathrm{d}\mathcal{I}_\varepsilon[u_1 + u_2](w_2) = 0$$

for all $w_1, w_2 \in X$, which equations are equivalent to the system

$$\begin{aligned} &\varepsilon^2 u_1 + n(\mathrm{D})u_1 + \chi(\mathrm{D})(u_1 + u_2)^2 = 0, && \text{in } Z_1, \\ &\varepsilon^2 u_2 + n(\mathrm{D})u_2 + (1 - \chi(\mathrm{D}))(u_1 + u_2)^2 = 0, && \text{in } Z_2. \end{aligned} \tag{20}$$

The next step is to solve (20) for u_2 as a function of u_1 using the following result, which is proved by a straightforward application of the contraction mapping principle.

Lemma 3.1 *Let W_1, W_2 be Banach spaces, $\overline{B}_1$ be a closed ball centred on the origin in W_1, r be a continuous function $\overline{B}_1 \to [0,\infty)$ and $F\colon \overline{B}_1 \times W_2 \to W_2$ be a smooth function satisfying*

$$|F(w_1,0)|_{W_2} \leq \tfrac{1}{2}r(w_1), \qquad |\mathrm{d}_2F[w_1,w_2]|_{W_2\to W_2} \leq \tfrac{1}{3}$$

for all $(w_1,w_2) \in \overline{B}_1 \times \overline{B}_{r(w_1)}(0)$. The fixed-point equation

$$w_2 = F(w_1,w_2)$$

has for each $w_1 \in \overline{B}_1$ a unique solution $w_2 = w_2(w_1) \in \overline{B}_{r(w_1)}(0)$. Moreover w_2 is a smooth function of w_1 and satisfies

$$|\mathrm{d}w_2[w_1]|_{W_1\to W_2} \lesssim |\mathrm{d}_1F[w_1,w_2]|_{W_1\to W_2},$$

and

$$\begin{aligned}|\mathrm{d}^2w_2[w_1]|_{W_1^2\to W_2} \lesssim{}& |\mathrm{d}_1^2F[w_1,w_2]|_{W_1^2\to W_2}\\ &+ |\mathrm{d}_2\mathrm{d}_2F[w_1,w_2]|_{W_1\times W_2\to W_2}\,|\mathrm{d}_1F[w_1,w_2]|_{W_1\to W_2}\\ &+ |\mathrm{d}_2^2F[w_1,w_2]|_{W_2^2\to W_2}\,|\mathrm{d}_1F[w_1,w_2]|_{W_1\to W_2}^2.\end{aligned}$$

Write (20) as

$$u_2 = G(u_1,u_2), \tag{21}$$

where

$$G(u_1,u_2) = -n(\mathrm{D})^{-1}(1-\chi(\mathrm{D}))\left(\varepsilon^2u_2 + (u_1+u_2)^2\right); \tag{22}$$

the following mapping property of G follows from Corollary 2.3 and Proposition 2.4.

Proposition 3.2 *Equation (22) defines a smooth and weakly continuous mapping $G: X_1 \times X_2 \to X_2$.*

Lemma 3.3 *Define $U = \{u_1 \in X_1 : |u_1|_\varepsilon \leq 1\}$. Equation (21) defines a map*

$$U \ni u_1 \mapsto u_2(u_1) \in X_2,$$

which satisfies

$$|\mathrm{d}^ku_2[u_1]|_{X_1^k\to X_2} \lesssim \varepsilon|u_1|_\varepsilon^{2-k}, \qquad k=0,1,2$$

(where by convention $|\mathrm{d}^ku_2[u_1]|_{X_1^k\to X_2}$ is interpreted as $|u_2(u_1)|_\varepsilon$ for $k=0$).

Proof We apply Lemma 3.1 to equation (21) with $W_1 = (X_1, |\cdot|_\varepsilon)$ $W_2 = (X_2, |\cdot|_X)$ and $F = G$. Note that

$$\begin{aligned}\mathrm{d}_1G[u_1,u_2](v_1) &= -n(\mathrm{D})^{-1}(1-\chi(\mathrm{D}))(2(u_1+u_2)v_1),\\ \mathrm{d}_2G[u_1,u_2](v_2) &= -n(\mathrm{D})^{-1}(1-\chi(\mathrm{D}))(\varepsilon^2v_2 + 2(u_1+u_2)v_2)\end{aligned}$$

and

$$|(n(\mathrm{D}))^{-1}(1-\chi(\mathrm{D}))z|_X \lesssim |z|_Z$$

(Lemma 2.6). Using Lemmata 2.1 and 2.8, we therefore find that

$$|G(u_1,0)|_X = |u_1^2|_Z \lesssim \varepsilon|u_1|_\varepsilon|u_1|_X \lesssim \varepsilon|u_1|_\varepsilon|u_1|_{L^2} \leq \varepsilon|u_1|_\varepsilon^2$$

and

$$\begin{aligned}|\mathrm{d}_2G[u_1,u_2](v_2)|_X &\lesssim \varepsilon^2|v_2|_Z + |u_1v_2|_Z + |u_2v_2|_Z\\ &\lesssim (\varepsilon^2 + \varepsilon|u_1|_\varepsilon + |u_2|_X)|v_2|_X.\end{aligned}$$

To satisfy the assumptions of Lemma 3.1, we choose $r(u_1) = \sigma\varepsilon|u_1|_\varepsilon^2$ for a sufficiently large value of $\sigma > 0$, so that

$$|u_2|_X \lesssim \tfrac{1}{2}r(u_1), \qquad |\mathrm{d}_2G[u_1,u_2]|_{X_2\to X_2} \lesssim \varepsilon$$

for $(u_1,u_2) \in U \times \overline{B}_{r(u_1)}(0)$. The lemma asserts the existence of a unique solution $u_2(u_1) \in \overline{B}_{r(u_1)}(0)$ of (21) for each $u_1 \in U$ which satisfies

$$|u_2(u_1)|_X \lesssim \varepsilon|u_1|_\varepsilon^2.$$

Observe that

$$\begin{aligned}|\mathrm{d}_1G[u_1,u_2](v_1)|_X &\lesssim |u_1v_1|_Z + |u_2v_1|_Z \\ &\lesssim \varepsilon(|u_1|_X + |u_2|_X)|v_1|_\varepsilon \\ &\lesssim \varepsilon(|u_1|_\varepsilon + \varepsilon|u_1|_\varepsilon^2)|v_1|_\varepsilon,\end{aligned}$$

and similarly

$$\begin{aligned}&|\mathrm{d}_1^2G[u_1,u_2](v_1,w_1)|_X \lesssim |v_1w_1|_Z \lesssim \varepsilon|v_1|_\varepsilon|w_1|_\varepsilon,\\ &|\mathrm{d}_2^2G[u_1,u_2](v_2,w_2)|_X \lesssim |v_2w_2|_Z \lesssim |v_2|_X|w_2|_X,\\ &|\mathrm{d}_1\mathrm{d}_2G[u_1,u_2](v_1,v_2)|_X \lesssim |v_1v_2|_Z \lesssim \varepsilon|v_1|_\varepsilon|v_2|_X.\end{aligned}$$

Combining these estimates in the fashion indicated in Lemma 3.1, one finds that

$$|u_1|_\varepsilon^{-2}|u_2(u_1)|_X + |u_1|_\varepsilon^{-1}|\mathrm{d}u_2[u_1]|_{X_1\to X_2} + |\mathrm{d}^2u_2[u_1]|_{X_1^2\to X_2} \lesssim \varepsilon. \qquad \square$$

Our next result shows in particular that $u = u_1 + u_2(u_1)$ belongs to $H^\infty(\mathbb{R}^2)$ for each $u_1 \in U_1$.

Proposition 3.4 *Any function $u = u_1 + u_2 \in X_1 \oplus X_2$ which satisfies (21) belongs to $H^\infty(\mathbb{R}^2)$.*

Proof Obviously $u_1 \in H^\infty(\mathbb{R}^2)$, and to show that u_2 is also smooth we indicate the regularity index s in the spaces X_2 and Z_2 explicitly. Since $H^s(\mathbb{R}^2)$ is an algebra for $s > \frac{3}{2}$ and $X_2^s \hookrightarrow (1-\chi(D))H^s(\mathbb{R}^2) \hookrightarrow Z_2^{s+\frac{1}{2}}$ (see Lemma 2.1(i)), the mapping

$$X_1 \oplus X_2^s \ni (u_1,u_2) \mapsto -(1-\chi(\mathrm{D}))\left(\varepsilon^2u_2 + (u_1+u_2)^2\right) \in Z_2^{s+\frac{1}{2}}$$

is continuous. It follows that $u_2 \in X_2^{s+\frac{1}{2}}$ because $n(\mathrm{D})$ is an isomorphism $X_2^{s+\frac{1}{2}} \to Z_2^{s+\frac{1}{2}}$ (see Lemma 2.6). Bootstrapping this argument yields $u_2 \in X_2^s \subset H^s(\mathbb{R}^2)$ for any $s \in \mathbb{R}$. $\square$

The (smooth) reduced variational functional $\mathcal{J}_\varepsilon : U \to \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{J}_\varepsilon(u_1) &:= \mathcal{I}_\varepsilon(u_1 + u_2(u_1)),\\ &= \frac{1}{2}\int_{\mathbb{R}^2}\left(\varepsilon^2u_1^2 + \varepsilon^2u_2(u_1)^2 + (n(\mathrm{D})^{\frac{1}{2}}u_1)^2 + (n(\mathrm{D})^{\frac{1}{2}}u_2(u_1))^2\right)\mathrm{d}x\,\mathrm{d}y\\ &\qquad + \frac{1}{3}\int_{\mathbb{R}^2}(u_1+u_2(u_1))^3\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

(recall that $\langle u_1, u_2(u_1)\rangle_{L^2} = 0)$), where $\mathrm{d}\mathcal{I}_\varepsilon[u_1+u_2(u_1)](v_2) = 0$ for all $v_2 \in X_2$ by construction. It follows that

$$\begin{aligned}\mathrm{d}\mathcal{J}_\varepsilon[u_1](v_1) &= \mathrm{d}\mathcal{I}_\varepsilon[u_1+u_2(u_1)](v_1) + \mathrm{d}\mathcal{I}_\varepsilon[u_1+u_2(u_1)](\mathrm{d}u_2[u_1](v_1))\\ &= \mathrm{d}\mathcal{I}_\varepsilon[u_1+u_2(u_1)](v_1)\end{aligned}$$

for all $v_1 \in X_1$, so that each critical point u_1 of $\mathcal{J}_\varepsilon$ defines a critical point $u_1 + u_2(u_1)$ of $\mathcal{I}_\varepsilon$. Conversely, each critical point $u = u_1 + u_2$ of $\mathcal{I}_\varepsilon$ with $u_1 \in U$ has the properties that $u_2 = u_2(u_1)$ and u_1 is a critical point of $\mathcal{J}_\varepsilon$.

Lemma 3.5 *The reduced functional $\mathcal{J}_\varepsilon : U \to \mathbb{R}$ satisfies*

$$\mathcal{J}_\varepsilon(u_1) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\varepsilon^2 u_1^2 + (n(\mathrm{D})^{\frac{1}{2}} u_1)^2\right) \mathrm{d}x\,\mathrm{d}y + \frac{1}{3}\int_{\mathbb{R}^2} u_1^3 \,\mathrm{d}x\,\mathrm{d}y + \mathcal{R}_\varepsilon(u_1),$$

where

$$|\mathrm{d}^k \mathcal{R}_\varepsilon(u_1)|_{X_1^k \to \mathbb{R}} \lesssim \varepsilon^2 |u_1|_\varepsilon^{4-k}, \qquad k = 0,1,2.$$

Proof Observe that

$$\mathcal{R}_\varepsilon(u_1) = \tfrac{1}{2}\varepsilon^2 K_1(u_1) + K_2(u_1) + K_3(u_1) + \tfrac{1}{3}K_4(u_1) + \tfrac{1}{2}K_5(u_1),$$

where

$$K_1(u_1) = |u_2(u_1)|_{L^2}^2, \quad K_2(u_1) = \langle u_1^2, u_2(u_1)\rangle_{L^2}, \quad K_3(u_1) = \langle u_1 u_2(u_1), u_2(u_1)\rangle_{L^2},$$

$$K_4(u_1) = \langle u_2(u_1)^2, u_2(u_1)\rangle_{L^2}, \quad K_5(u_1) = |(n(\mathrm{D}))^{\frac{1}{2}} u_2(u_1)|_{L^2}^2.$$

We investigate each of these quantities using the estimates

$$|\mathrm{d}^j u_2[u_1](\mathbf{v})|_{\mathrm{BC}\cap L^2} \lesssim |\mathrm{d}^j u_2[u_1](\mathbf{v})|_X \lesssim \varepsilon |u_1|_\varepsilon^{2-j} |v_1|_\varepsilon \cdots |v_j|_\varepsilon, \qquad j = 0,1,2,$$

$$|\mathrm{d}^j u_1[u_1](\mathbf{v})|_{\mathrm{BC}} \lesssim \varepsilon |u_1|_\varepsilon^{1-j} |v_1|_\varepsilon \cdots |v_j|_\varepsilon, \qquad j = 0,1,$$

$$|\mathrm{d}^j u_1[u_1](\mathbf{v})|_{L^2} \lesssim |u_1|_\varepsilon^{1-j} |v_1|_\varepsilon \cdots |v_j|_\varepsilon, \qquad j = 0,1,$$

where $\mathbf{v} = (v_1, \ldots, v_j)$ denotes a general element in X_1^j, and of course

$$\mathrm{d}^2 u_1[u_1] = 0.$$

Using Leibniz's rule, Hölder's inequality and the basic estimate $|\langle w\cdot, \cdot\rangle_{L^2}| \leq |w|_{\mathrm{BC}} |\langle \cdot, \cdot\rangle_{L^2}|$, one finds that

$$\begin{aligned}
\left|\mathrm{d}^k K_1[u_1](\mathbf{v})\right| &\lesssim \sum_{j=0}^{k} \left|\langle \mathrm{d}^j u_2[u_1](\mathbf{v}), \mathrm{d}^{k-j} u_2[u_1](\mathbf{v})\rangle_{L^2}\right| \\
&\lesssim \sum_{j=0}^{k} |\mathrm{d}^j u_2[u_1](\mathbf{v})|_{L^2} |\mathrm{d}^{k-j} u_2[u_1](\mathbf{v})|_{L^2} \\
&\lesssim \varepsilon^2 \sum_{j=0}^{k} |u_1|_\varepsilon^{2-j} |u_1|_\varepsilon^{2-(k-j)} |v_1|_\varepsilon \cdots |v_k|_\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\left|\mathrm{d}^k K_2[u_1](\mathbf{v})\right| &\lesssim \sum_{0\leq j+l\leq k} \left|\langle \mathrm{d}^j u_1[u_1](\mathbf{v}) \mathrm{d}^l u_1[u_1](\mathbf{v}), \mathrm{d}^{k-j-l} u_2[u_1](\mathbf{v})\rangle_{L^2}\right| \\
&\lesssim \varepsilon \sum_{0\leq j+l\leq k} |u_1|_\varepsilon^{1-j} |\mathrm{d}^l u_1[u_1](\mathbf{v})|_{L^2} |\mathrm{d}^{k-j-l} u_2[u_1](\mathbf{v})|_{L^2} |v_1|_\varepsilon \cdots |v_j|_\varepsilon \\
&\lesssim \varepsilon^2 |u_1|_\varepsilon^{4-k} |v_1|_\varepsilon \cdots |v_k|_\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\left|\mathrm{d}^k K_3[u_1](\mathbf{v})\right| &\lesssim \sum_{0\leq j+l\leq k} \left|\langle \mathrm{d}^j u_1[u_1](\mathbf{v}) \mathrm{d}^l u_2[u_1](\mathbf{v}), \mathrm{d}^{k-j-l} u_2[u_1](\mathbf{v})\rangle_{L^2}\right| \\
&\lesssim \varepsilon \sum_{0\leq j+l\leq k} |u_1|_\varepsilon^{1-j} |\mathrm{d}^l u_2[u_1](\mathbf{v})|_{L^2} |\mathrm{d}^{k-j-l} u_2[u_1](\mathbf{v})|_{L^2} |v_1|_\varepsilon \cdots |v_j|_\varepsilon \\
&\lesssim \varepsilon^3 |u_1|_\varepsilon^{5-k} |v_1|_\varepsilon \cdots |v_k|_\varepsilon
\end{aligned}$$

and

$$\begin{aligned}\left|\mathrm{d}^k K_4[u_1](\mathbf{v})\right| &\lesssim \sum_{0\leq j+l\leq k} \left|\langle \mathrm{d}^j u_2[u_1](\mathbf{v})\mathrm{d}^l u_2[u_1](\mathbf{v}), \mathrm{d}^{k-j-l}u_2[u_1](\mathbf{v})\rangle_{L^2}\right| \\ &\lesssim \varepsilon \sum_{0\leq j+l\leq k} |u_1|_\varepsilon^{2-j}|\mathrm{d}^l u_2[u_1](\mathbf{v})|_X|\mathrm{d}^{k-j-l}u_2[u_1](\mathbf{v})|_X|v_1|_\varepsilon\cdots|v_j|_\varepsilon \\ &\lesssim \varepsilon^3|u_1|_\varepsilon^{6-k}|v_1|_\varepsilon\cdots|v_k|_\varepsilon\end{aligned}$$

for $k=0,1,2$.

Finally, since u_2 solves (20), one obtains

$$\begin{aligned}K_5(u_1) &= |(n(\mathrm{D}))^{\frac{1}{2}}u_2(u_1)|_{L^2}^2 \\ &= \langle n(\mathrm{D})u_2(u_1), u_2(u_1)\rangle_{L^2} \\ &= -\varepsilon^2|u_2(u_1)|_{L^2}^2 - \langle (u_1+u_2(u_1))^2, u_2(u_1)\rangle_{L^2} \\ &= -\varepsilon^2 K_1(u_1) - K_2(u_1) - 2K_3(u_1) - K_4(u_1),\end{aligned}$$

all of which terms have been estimated. □

The next step is to convert $\mathcal{J}_\varepsilon$ into a perturbation of the KP-I functional, the main issue being the replacement of $n(k)$ by $\tilde{n}(k)$.

Proposition 3.6 *The Fourier multiplier* $(n/\tilde{n})^{\frac{1}{2}}$ *defines an isomorphism* $I_1 : \chi(\mathrm{D})L^2(\mathbb{R}^2) \to \chi(\mathrm{D})L^2(\mathbb{R}^2)$ *for sufficiently small values of* δ.

Proof Using the elementary estimates

$$n(k) = \tilde{n}(k) + \mathcal{O}(|(k_1, \tfrac{k_2}{k_1})|^4), \qquad \tilde{n}(k) \eqsim |(k_1, \tfrac{k_2}{k_1})|^2$$

as $(k_1, \frac{k_2}{k_1}) \to 0$, we find that

$$\left|\frac{n(k)}{\tilde{n}(k)} - 1\right| \lesssim \tilde{n}(k) \lesssim \delta^2,$$

and hence

$$\left(\frac{n(k)}{\tilde{n}(k)}\right)^{\frac{1}{2}} \eqsim 1$$

for $k \in C$, for sufficiently small values of δ. □

We now express the reduced functional in terms of $\tilde{u}_1 = (\frac{n}{\tilde{n}})^{\frac{1}{2}}u_1$; to this end define $\widetilde{\mathcal{J}}_\varepsilon(\tilde{u}_1) = \mathcal{J}_\varepsilon(u_1(\tilde{u}_1))$ and note that $\widetilde{\mathcal{J}}_\varepsilon$ is a smooth functional $\tilde{U} \to \mathbb{R}$, where $\tilde{U} = \{u_1 \in X_1 : |u_1|_\varepsilon \leq \tilde{\tau}\}$ and $\tilde{\tau} \in (0,1)$ is chosen so that $\tilde{U} \subseteq I_1[U]$.

Lemma 3.7 *The reduced functional* $\widetilde{\mathcal{J}}_\varepsilon : \tilde{U} \to \mathbb{R}$ *satisfies*

$$\tilde{\mathcal{J}}_\varepsilon(\tilde{u}_1) = \frac{1}{2}\int_{\mathbb{R}^2}\left(\varepsilon^2\tilde{u}_1^2 + (\tilde{n}(\mathrm{D})^{\frac{1}{2}}\tilde{u}_1)^2\right)\mathrm{d}x\,\mathrm{d}y + \frac{1}{3}\int_{\mathbb{R}^2}\tilde{u}_1^3\,\mathrm{d}x\,\mathrm{d}y + \tilde{\mathcal{R}}_\varepsilon(\tilde{u}_1),$$

where

$$|\mathrm{d}^k\tilde{\mathcal{R}}_\varepsilon[\tilde{u}_1]|_{X_1^k\to\mathbb{R}} \lesssim \varepsilon^2|u_1|_\varepsilon^{3-k} + \varepsilon^4|u_1|_\varepsilon^{2-k}, \qquad k=0,1,2.$$

Proof By construction

$$\int_{\mathbb{R}^2}(n(D)^{\frac{1}{2}}u_1)^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2}(\tilde{n}(D)^{\frac{1}{2}}\tilde{u}_1)^2\,\mathrm{d}x\,\mathrm{d}y,$$

and furthermore

$$|\mathrm{d}^k\mathcal{R}_\varepsilon[\tilde{u}_1]|_{X_1^k\to\mathbb{R}} \lesssim \varepsilon^2|\tilde{u}_1|_\varepsilon^{4-k}, \qquad k=0,1,2,$$

because $u_1 \mapsto \tilde{u}_1$ is an isomorphism $X_1 \to X_1$ (here we have abbreviated $\mathcal{R}_\varepsilon(u_1(\tilde{u}_1))$ to $\mathcal{R}_\varepsilon(\tilde{u}_1)$). It remains to estimate the differences

$$\int_{\mathbb{R}^2} u_1^2 \, \mathrm{d}x \, \mathrm{d}y - \int_{\mathbb{R}^2} \tilde{u}_1^2 \, \mathrm{d}x \, \mathrm{d}y, \qquad \int_{\mathbb{R}^2} u_1^3 \, \mathrm{d}x \, \mathrm{d}y - \int_{\mathbb{R}^2} \tilde{u}_1^3 \, \mathrm{d}x \, \mathrm{d}y$$

using the formulae

$$u_1(\tilde{u}_1) = \left(\frac{\tilde{n}(\mathrm{D})}{n(\mathrm{D})}\right)^{\frac{1}{2}} \tilde{u}_1, \qquad \mathrm{d}u_1[\tilde{u}_1](v_1) = \left(\frac{\tilde{n}(\mathrm{D})}{n(\mathrm{D})}\right)^{\frac{1}{2}} v_1, \qquad \mathrm{d}^2 u_1[\tilde{u}_1] = 0.$$

Observe that

$$\left| \left|1 - \frac{n(\mathrm{D})}{\tilde{n}(\mathrm{D})}\right|^{\frac{1}{2}} w_1 \right|_{L^2}^2 \lesssim |\tilde{n}(D)^{\frac{1}{2}} w_1|_{L^2}^2 \lesssim \varepsilon^2 |w_1|_\varepsilon^2,$$

for $w_1 \in \chi(\mathrm{D}) L^2(\mathbb{R}^2)$, so that

$$\left| \left|1 - \frac{n(\mathrm{D})}{\tilde{n}(\mathrm{D})}\right|^{\frac{1}{2}} \mathrm{d}^j u_1[\tilde{u}_1](\mathbf{v}) \right|_{L^2} \lesssim \varepsilon |\tilde{u}_1|_\varepsilon^{1-j} |v_1|_\varepsilon \cdots |v_j|_\varepsilon, \qquad j = 0, 1.$$

It follows that

$$K_6(\tilde{u}_1) := \int_{\mathbb{R}^2} \left(u_1^2 - \tilde{u}_1^2\right) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^2} \left(\left|1 - \frac{n(\mathrm{D})}{\tilde{n}(\mathrm{D})}\right|^{\frac{1}{2}} u_1 \right)^2 \mathrm{d}x \, \mathrm{d}y$$

satisfies

$$\begin{aligned} \left|\mathrm{d}^k K_6[\tilde{u}_1](\mathbf{v})\right| &\leq \sum_{j=0}^{k} \left| \left\langle \left|1 - \frac{n(D)}{\tilde{n}(D)}\right|^{\frac{1}{2}} \mathrm{d}^j u_1[\tilde{u}_1](\mathbf{v}), \left|1 - \frac{n(\mathrm{D})}{\tilde{n}(\mathrm{D})}\right|^{\frac{1}{2}} \mathrm{d}^{k-j} u_1[\tilde{u}_1](\mathbf{v}) \right\rangle_{L^2} \right| \\ &\lesssim \varepsilon^2 |\tilde{u}_1|_\varepsilon^{2-k} |v_1|_\varepsilon \cdots |v_k|_\varepsilon, \qquad k = 0, 1, 2. \end{aligned}$$

The term

$$K_7(\tilde{u}_1) := \int_{\mathbb{R}^2} \left(u_1^3 - \tilde{u}_1^3\right) \, \mathrm{d}x \, \mathrm{d}y = \sum_{m=0}^{2} \int_{\mathbb{R}^2} (u_1 - \tilde{u}_1) u_1^m \tilde{u}_1^{2-m} \, \mathrm{d}x \, \mathrm{d}y$$

is treated in a similar fashion. Using the estimate $|v_1|_{\mathrm{BC}} \lesssim \varepsilon |v_1|_\varepsilon$ (see Lemma 2.8), we find that

$$\begin{aligned} &\sum_{j+l=k} \left| \left\langle \left(1 - \left(\frac{n(\mathrm{D})}{\tilde{n}(\mathrm{D})}\right)^{\frac{1}{2}}\right) \mathrm{d}^j u_1[\tilde{u}_1](\mathbf{v}), \mathrm{d}^l \tilde{u}_1[\tilde{u}_1](\mathbf{v}) \, \mathrm{d}^{k-l-j} \tilde{u}_1[\tilde{u}_1](\mathbf{v}) \right\rangle_{L^2} \right| \\ &\quad \lesssim \sum_{j+l=k} |\mathrm{d}^l \tilde{u}_1[\tilde{u}_1](\mathbf{v})|_{\mathrm{BC}} \left| \left(1 - \left(\frac{n(\mathrm{D})}{\tilde{n}(\mathrm{D})}\right)^{\frac{1}{2}}\right) \mathrm{d}^j u_1[\tilde{u}_1](\mathbf{v}) \right|_{L^2} \left|\mathrm{d}^{k-j-l} \tilde{u}_1[\tilde{u}_1](\mathbf{v})\right|_{L^2} \\ &\quad \lesssim \sum_{j+l=k} \varepsilon |\tilde{u}_1|_\varepsilon^{1-l} \varepsilon |\tilde{u}_1|_\varepsilon^{1-j} |\tilde{u}_1|_\varepsilon^{1-(k-j-l)} |v_1|_\varepsilon \cdots |v_k|_\varepsilon, \qquad k = 0, 1, 2. \end{aligned}$$

with similar estimates for the summands with $m = 0$ and $m = 2$ in the formula for $K_7(\tilde{u})$. Altogether we find that

$$|\mathrm{d}^k K_7[\tilde{u}_1](\mathbf{v})| \lesssim \varepsilon^2 |\tilde{u}_1|_\varepsilon^{3-k} |v_1|_\varepsilon \cdots |v_k|_\varepsilon, \qquad k = 0, 1, 2. \qquad \square$$

Remark 3.8 *Using the simple expansion* $n(k) = \tilde{n}(k) + \mathcal{O}(|(k_1, \frac{k_2}{k_1})|^4)$ *for* $k \in C$ *leads to the insufficient estimate*

$$\int_{\mathbb{R}^2} \left((n(\mathrm{D}) - \tilde{n}(\mathrm{D}))^{\frac{1}{2}} u_1\right)^2 \mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} |n(k) - \tilde{n}(k)||\hat{u}_1|^2 \,\mathrm{d}k = \mathcal{O}(\varepsilon^2 |u_1|_\varepsilon^2)$$

(at the next step we use the KP scaling for u *and scale the functional by* ε^{-3}*).*

Finally, we use the KP-scaling

$$\tilde{u}_1(x, y) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 y).$$

The following proposition is immediate.

Proposition 3.9 *The mapping* $\tilde{u}_1 \mapsto \zeta$ *defines an isomorphism* $I_2 : \chi(D)L^2(\mathbb{R}) \to \chi_\varepsilon(\mathrm{D})L^2(\mathbb{R})$.

Note that $\tilde{u}_1 \in \chi(D)L^2(\mathbb{R}^2)$ has $\operatorname{supp} \hat{\tilde{u}}_1 \in C$, while $\chi_\varepsilon(\mathrm{D})L^2(\mathbb{R}^2)$ has $\operatorname{supp} \hat{\zeta} \in C_\varepsilon$, where

$$C_\varepsilon = \left\{(k_1, k_2) : |k| \le \tfrac{\delta}{\varepsilon}, \left|\tfrac{k_2}{k_1}\right| \le \tfrac{\delta}{\varepsilon}\right\}.$$

The formula $\mathcal{T}_\varepsilon(\zeta) := \varepsilon^{-3} \widetilde{\mathcal{J}}_\varepsilon(\tilde{u}_1(\zeta))$ therefore defines a smooth functional $B_M(0) \to \mathbb{R}$, where $B_M(0) = \{\zeta \in \tilde{Y}_\varepsilon : |\zeta| < M\}$ and $M > 1$ is chosen so that $B_M(0) \subseteq I_2[\tilde{U}]$, that is $M \lesssim \varepsilon^{-\frac{1}{2}} \tilde{\tau}$. (Recall that $\tilde{Y}_\varepsilon = \chi_\varepsilon(\mathrm{D})\tilde{Y}$ consists of those functions in $\tilde{Y}$ whose Fourier transforms are supported in C_ε; for $\varepsilon = 1$ it coincides with $\tilde{Y}_1$, and in the limit $\varepsilon \to 0$ it 'fills out' all of $\tilde{Y}$.)

Using Lemma 3.7 and the calculations

$$|\varepsilon \tilde{u}_1|_{L^2}^2 + |\tilde{n}(\mathrm{D})^{\frac{1}{2}} \tilde{u}_1|_{L^2}^2 = \varepsilon^3 |\zeta|_{\tilde{Y}}^2, \qquad |\tilde{u}_1|_\varepsilon = \varepsilon^{\frac{1}{2}} |\zeta|_{\tilde{Y}},$$

one finds that

$$\mathcal{T}_\varepsilon(\zeta) = \mathcal{Q}(\zeta) + \mathcal{S}(\zeta) + \varepsilon^{\frac{1}{2}} \mathcal{R}_\varepsilon(\zeta), \tag{23}$$

where

$$\mathcal{Q}(\zeta) = \frac{1}{2}|\zeta|_{\tilde{Y}}^2, \qquad \mathcal{S}(\zeta) = \frac{1}{3}\int_{\mathbb{R}^2} \zeta^3 \,\mathrm{d}x\,\mathrm{d}y,$$

and

$$|\mathrm{d}^k \mathcal{R}_\varepsilon[\zeta]|_{\tilde{Y}^k \to \mathbb{R}} = \varepsilon^{-\frac{7}{2} + \frac{k}{2}} |\mathrm{d}^k \tilde{\mathcal{R}}_\varepsilon[\tilde{u}_1]|_{X_1^k \to \mathbb{R}} \lesssim |\zeta|_{\tilde{Y}}^{2-k}, \qquad k = 0, 1, 2;$$

in particular, we note that

$$|\mathcal{R}_\varepsilon(\zeta)| + |\mathrm{d}\mathcal{R}_\varepsilon[\zeta](\zeta)| + |\mathrm{d}^2 \mathcal{R}_\varepsilon[\zeta](\zeta, \zeta)| \lesssim |\zeta|_{\tilde{Y}}^2.$$

Let $\zeta \in B_M(0)$ and define $u = u_1(\tilde{u}_1(\zeta)) + u_2(u_1(\tilde{u}_1(\zeta)))$. By construction one has that

$$\mathrm{d}\mathcal{I}_\varepsilon[u](w_1) = \varepsilon^3 \mathrm{d}\mathcal{T}_\varepsilon[\zeta](\varrho), \qquad \mathrm{d}\mathcal{I}_\varepsilon[u](w_2) = 0 \tag{24}$$

for each $w = w_1 + w_2 = X_1 \oplus X_2$, where $\rho = I_2(I_1(w_1))$ (see Propositions 3.6 and 3.9), so that in particular each critical point ζ_∞ of $\mathcal{T}_\varepsilon$ defines a critical point $u_\infty = u_1(\tilde{u}_1(\zeta_\infty)) + u_2(u_1(\tilde{u}_1(\zeta_\infty)))$ of $\mathcal{I}_\varepsilon$. Equation (24) also shows that each Palais–Smale sequence $\{\zeta_n\}$ for $\mathcal{T}_\varepsilon$ generates a Palais–Smale sequence $\{u_n\}$ with $u_n = u_1(\tilde{u}_1(\zeta_n)) + u_2(u_1(\tilde{u}_1(\zeta_n)))$ for $\mathcal{I}_\varepsilon$, and our next result confirms that weakly convergent sequences in $B_M(0) \subseteq \tilde{Y}_\varepsilon$ generate sequences which are weakly convergent in X.

Proposition 3.10 *Suppose that* $\{\zeta_n\} \subset B_M(0)$ *converges weakly in* $\tilde{Y}_\varepsilon$ *to* $\zeta_\infty \in B_M(0)$*. The corresponding sequence* $\{u_n\}$*, where*

$$u_n = u_1(\tilde{u}_1(\zeta_n)) + u_2(u_1(\tilde{u}_1(\zeta_n))),$$

converges weakly in X *to* $u_\infty = u_1(\tilde{u}_1(\zeta_\infty)) + u_2(u_1(\tilde{u}_1(\zeta_\infty)))$*.*

Proof Abbreviating $u_1(\tilde{u}_1(\zeta_n))$ to $u_{1,n}$, note that $\{u_{1,n}\} \subset U$ converges weakly in X_1 to $u_{1,\infty} = u_1(\tilde{u}_1(\zeta_\infty)) \in U$. Furthermore, $u_{2,n} = u_2(u_{1,n})$ is the unique solution in X_2 of equation (21) with $u_1 = u_{1,n}$, so that

$$u_{2,n} = G(u_{1,n}, u_{2,n}).$$

Observe that $\{u_{2,n}\}$ is bounded in X_2; the following argument shows that any weakly convergent subsequence of $\{u_{2,n}\}$ has weak limit $u_2(u_{1,\infty})$, so that $\{u_{2,n}\}$ itself converges weakly to $u_2(u_{1,\infty})$ in X_2. Suppose that (a subsequence of) $\{u_{2,n}\}$ converges weakly in X_2 to $u_{2,\infty}$. Because $G : X_1 \times X_2 \to X_2$ is weakly continuous (Proposition 3.2), we find that

$$u_{2,\infty} = G(u_{1,\infty}, u_{2,\infty}),$$

so that $u_{2,\infty} = u_2(u_{1,\infty})$ (the fixed-point equation $u_2 = G(u_{1,\infty}, u_2)$ has a unique solution in X_2).

Altogether we conclude that $\{u_{1,n} + u_{2,n}\}$ converges weakly in X to $u_\infty = u_{1,\infty} + u_{2,\infty}$. □

4. Existence theory

The functional $\mathcal{T}_\varepsilon : B_M(0) \to \mathbb{R}$ may be considered as a perturbation of the 'limiting' functional $\mathcal{T}_0 : \tilde{Y} \to \mathbb{R}$ with

$$\mathcal{T}_0(\zeta) = \mathcal{Q}(\zeta) + \mathcal{S}(\zeta).$$

More precisely $\varepsilon^{\frac{1}{2}}\mathcal{R}_\varepsilon \circ \chi_\varepsilon(\mathrm{D})$ (which coincides with $\varepsilon^{\frac{1}{2}}\mathcal{R}_\varepsilon$ on $B_M(0) \subset \tilde{Y}_\varepsilon$) converges uniformly to zero over $B_M(0) \subset \tilde{Y}$, and corresponding statements for its derivatives also hold. In this section we study $\mathcal{T}_\varepsilon$ by perturbative arguments in this spirit, choosing $M > 1$ sufficiently large that inequality (29) below holds for some $\zeta_0 \in \tilde{Y} \setminus \{0\}$.

We seek critical points of $\mathcal{T}_\varepsilon$ by considering its *natural constraint set*

$$N_\varepsilon = \{\zeta \in B_M(0)\colon \zeta \neq 0, \mathrm{d}\mathcal{T}_\varepsilon[\zeta](\zeta) = 0\},$$

noting the calculation

$$\mathrm{d}\mathcal{T}_\varepsilon[\zeta](\zeta) = 2\mathcal{Q}(\zeta) + 3\mathcal{S}(\zeta) + \varepsilon^{\frac{1}{2}}\mathrm{d}\mathcal{R}_\varepsilon[\zeta](\zeta), \tag{25}$$

which shows that

$$\begin{aligned} -\mathcal{S}(\zeta) &= \tfrac{2}{3}\mathcal{Q}(\zeta) + \tfrac{1}{3}\varepsilon^{\frac{1}{2}}\mathrm{d}\mathcal{R}_\varepsilon[\zeta](\zeta) \\ &= \tfrac{1}{3}|\zeta|_{\tilde{Y}}^2 + \mathcal{O}(\varepsilon^{\frac{1}{2}}|\zeta|_{\tilde{Y}}^2) \\ &\geq \tfrac{1}{6}|\zeta|_{\tilde{Y}}^2 \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}^2\mathcal{T}_\varepsilon[\zeta](\zeta,\zeta) &= 2\mathcal{Q}(\zeta) + 6\mathcal{S}(\zeta) + \varepsilon^{\frac{1}{2}}\mathrm{d}^2\mathcal{R}_\varepsilon[\zeta](\zeta,\zeta)^{\frac{1}{2}} \\ &= -2\mathcal{Q}(\zeta) - 2\varepsilon^{\frac{1}{2}}\mathrm{d}\mathcal{R}_\varepsilon[\zeta](\zeta) + \varepsilon^{\frac{1}{2}}\mathrm{d}^2\mathcal{R}_\varepsilon[\zeta](\zeta,\zeta) \\ &= -|\zeta|_{\tilde{Y}}^2 + \mathcal{O}(\varepsilon^{\frac{1}{2}}|\zeta|_{\tilde{Y}}^2) \\ &\leq -\tfrac{1}{2}|\zeta|_{\tilde{Y}}^2 \end{aligned}$$

(and in particular $\mathcal{S}(\zeta)<0$, $\mathrm{d}^2\mathcal{T}_\varepsilon[\zeta](\zeta,\zeta) < 0$) for points $\zeta \in N_\varepsilon$. Any nontrivial critical point of $\mathcal{T}_\varepsilon$ clearly lies on N_ε, and the following proposition shows that the converse is also true.

Proposition 4.1 *Any critical point of $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ is a (necessarily nontrivial) critical point of $\mathcal{T}_\varepsilon$.*

Proof Define $\mathcal{G}_\varepsilon : U_\varepsilon \setminus \{0\} \to \mathbb{R}$ by $\mathcal{G}_\varepsilon(\zeta) = \mathrm{d}\mathcal{T}_\varepsilon[\zeta](\zeta)$, so that $N_\varepsilon = \mathcal{G}_\varepsilon^{-1}(0)$ and $\mathrm{d}\mathcal{G}_\varepsilon[\zeta]$ does not vanish on N_ε (since $\mathrm{d}\mathcal{G}_\varepsilon[\zeta](\zeta) = \mathrm{d}^2\mathcal{T}_\varepsilon[\zeta](\zeta,\zeta) < 0$ for $\zeta \in N_\varepsilon$). There exists a Lagrange multiplier μ such that

$$\mathrm{d}\mathcal{T}_\varepsilon[\zeta^\star] - \mu \mathrm{d}\mathcal{G}_\varepsilon[\zeta^\star] = 0,$$

and applying this operator to $\zeta^\star$ we find that $\mu = 0$, whence $\mathrm{d}\mathcal{T}_\varepsilon[\zeta^\star] = 0$. □

There is a convenient geometrical interpretation of N_ε (see Figure 4).

Proposition 4.2 *Any ray in $(B_M(0) \setminus \{0\}) \cap \mathcal{S}^{-1}(-\infty, 0) \subset \tilde{Y}_\varepsilon$ intersects N_ε in at most one point and the value of $\mathcal{T}_\varepsilon$ along such a ray attains a strict maximum at this point.*

Proof Let $\zeta \in (B_M(0) \setminus \{0\}) \cap \mathcal{S}^{-1}(-\infty, 0) \subset \tilde{Y}_\varepsilon$ and consider the value of $\mathcal{T}_\varepsilon$ along the ray in $B_M(0) \setminus \{0\}$ through ζ, that is, the set $R_\zeta = \{\lambda\zeta : 0 < \lambda < M/|\zeta|_1\} \subset \tilde{Y}_\varepsilon$. The calculation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_\varepsilon(\lambda\zeta) = \mathrm{d}\mathcal{T}_\varepsilon[\lambda\zeta](\zeta) = \lambda^{-1}\mathrm{d}\mathcal{T}_\varepsilon[\lambda\zeta](\lambda\zeta)$$

shows that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_\varepsilon(\lambda\zeta) = 0$ if and only if $\lambda\zeta \in N_\varepsilon$; furthermore

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\mathcal{T}_\varepsilon(\lambda\zeta) &= 2\mathcal{Q}(\zeta) + 6\lambda^{-2}\mathcal{S}(\lambda\zeta) + \varepsilon^{\frac{1}{2}}\mathrm{d}^2\mathcal{R}_\varepsilon[\lambda\zeta](\zeta,\zeta) \\ &= -2\mathcal{Q}(\zeta) - 2\lambda^{-2}\varepsilon^{\frac{1}{2}}\mathrm{d}\mathcal{R}_\varepsilon[\lambda\zeta](\lambda\zeta) + \varepsilon^{\frac{1}{2}}\mathrm{d}^2\mathcal{R}_\varepsilon[\lambda\zeta](\zeta,\zeta) \\ &= -2\mathcal{Q}(\zeta) + \mathcal{O}(\varepsilon^{\frac{1}{2}}|\zeta|_{\tilde{Y}}^2) \\ &< 0\end{aligned}$$

for each ζ with $\lambda\zeta \in N_\varepsilon$. It follows that the critical points of $\mathcal{T}_\varepsilon|_{R_\zeta}$ are precisely the points at which R_ζ intersects N_ε and they are all strict local maxima; there is therefore at most one such point and it is a strict global maximum of $\mathcal{T}_\varepsilon|_{R_\zeta}$. □

Remark 4.3 *If $\varepsilon = 0$ we may take $M = \infty$, and in this case every ray in $\mathcal{S}^{-1}(-\infty, 0)$ intersects N_0 in precisely one point.*

In view of the above characterisation of nontrivial critical points of $\mathcal{T}_\varepsilon$ we proceed by seeking a 'ground state', that is, a minimiser $\zeta^\star$ of $\mathcal{T}_\varepsilon$ over N_ε. We make frequent use of the identities

$$\mathcal{T}_\varepsilon(\zeta) = \tfrac{1}{3}\mathcal{Q}(\zeta) + \tfrac{1}{3}\mathrm{d}\mathcal{T}_\varepsilon[\zeta](\zeta) + \varepsilon^{\frac{1}{2}}\left(\mathcal{R}_\varepsilon(\zeta) - \tfrac{1}{3}\mathrm{d}\mathcal{R}_\varepsilon[\zeta](\zeta)\right), \qquad (26)$$

$$\mathcal{T}_\varepsilon(\zeta) = -\tfrac{1}{2}\mathcal{S}(\zeta) + \tfrac{1}{2}\mathrm{d}\mathcal{T}_\varepsilon[\zeta](\zeta) + \varepsilon^{\frac{1}{2}}\left(\mathcal{R}_\varepsilon(\zeta) - \tfrac{1}{2}\mathrm{d}\mathcal{R}_\varepsilon[\zeta](\zeta)\right), \qquad (27)$$

which are obtained using (25) to eliminate respectively $\mathcal{S}(\zeta)$ and $\mathcal{Q}(\zeta)$ from (23), beginning with some *a priori* bounds for $\mathcal{T}_\varepsilon|_{N_\varepsilon}$.

Proposition 4.4 *Each $\zeta \in N_\varepsilon$ satisfies $\mathcal{T}_\varepsilon(\zeta) \geq \frac{1}{12}|\zeta|_{\tilde{Y}}^2$ and $|\zeta|_{\tilde{Y}} \gtrsim 1$. In particular, each $\zeta \in N_\varepsilon$ with $\mathcal{T}_\varepsilon(\zeta) < \frac{1}{12}(M-1)^2$ satisfies $|\zeta|_{\tilde{Y}} < M - 1$.*

Proof Let $\zeta \in N_\varepsilon$. Using (26), one finds that

$$\mathcal{T}_\varepsilon(\zeta) = \tfrac{1}{3}\mathcal{Q}(\zeta) + \mathcal{O}(\varepsilon^{\frac{1}{2}}|\zeta|_{\tilde{Y}}^2) = \tfrac{1}{6}|\zeta|_{\tilde{Y}}^2 + \mathcal{O}(\varepsilon^{\frac{1}{2}}|\zeta|_{\tilde{Y}}^2) \geq \tfrac{1}{12}|\zeta|_{\tilde{Y}}^2,$$

so that in particular $\mathcal{T}_\varepsilon(\zeta) < \frac{1}{12}(M-1)^2$ implies that $|\zeta|_{\tilde{Y}} < M-1$. Furthermore

$$|\zeta|_{\tilde{Y}}^2 = 2\mathcal{Q}(\zeta) = -3\mathcal{S}(\zeta) + \mathcal{O}(\varepsilon^{\frac{1}{2}}|\zeta|_{\tilde{Y}}^2) \lesssim |\zeta|_{\tilde{Y}}^3 + \varepsilon^{\frac{1}{2}}|\zeta|_{\tilde{Y}}^2,$$

where we have used (25) and the embedding $\tilde{Y} \hookrightarrow L^3(\mathbb{R}^2)$; it follows that $|\zeta|_{\tilde{Y}} \gtrsim 1$. □

Remark 4.5 *Let $c_\varepsilon := \inf_{N_\varepsilon} \mathcal{T}_\varepsilon$. It follows from Proposition 4.4 that $\liminf_{\varepsilon\to 0} c_\varepsilon \gtrsim 1$ and from equation (27) that $-\mathcal{S}(\zeta) \geq 2c_\varepsilon - \mathcal{O}(\varepsilon^{\frac{1}{2}}|\zeta|_{\tilde{Y}}^2)$ for all $\zeta \in N_\varepsilon$.*

The next result shows how points on N_0 may be approximated by points on N_ε.

Proposition 4.6 *Suppose that $\mathcal{S}(\zeta_0) < 0$ and $\lambda_0\zeta_0 \in B_{M-1}(0)$ is the unique point on the ray through $\zeta_0 \in \tilde{Y} \setminus \{0\}$ which lies on N_0. There exists $\xi_\varepsilon \in N_\varepsilon$ such that $\lim_{\varepsilon\to 0} |\xi_\varepsilon - \lambda_0\zeta_0|_{\tilde{Y}} = 0$.*

Proof Note that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_0(\lambda\zeta_0)\Big|_{\lambda=\lambda_0} = 0, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\mathcal{T}_0(\lambda\zeta_0)\Big|_{\lambda=\lambda_0} < 0. \tag{28}$$

Let $\zeta_\varepsilon = \chi_\varepsilon(\mathrm{D})\zeta_0$, so that $\zeta_\varepsilon \in \tilde{Y}_\varepsilon \subset \tilde{Y}$ with $\lim_{\varepsilon\to 0} |\zeta_\varepsilon - \zeta_0|_{\tilde{Y}} = 0$, and in particular

$$|\lambda_0\zeta_\varepsilon|_{\tilde{Y}} < M-1.$$

According to (28) we can find $\tilde{\gamma} > 1$ such that $\tilde{\gamma}|\lambda_0\zeta_\varepsilon|_{\tilde{Y}} < M$ (so that $\tilde{\gamma}\lambda_0\zeta_\varepsilon \in U_\varepsilon$) and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_0(\lambda\zeta_0)\Big|_{\lambda=\tilde{\gamma}^{-1}\lambda_0} > 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_0(\lambda\zeta_0)\Big|_{\lambda=\tilde{\gamma}\lambda_0} < 0,$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_\varepsilon(\lambda\zeta_\varepsilon)\Big|_{\lambda=\tilde{\gamma}^{-1}\lambda_0} > 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_\varepsilon(\lambda\zeta_\varepsilon)\Big|_{\lambda=\tilde{\gamma}\lambda_0} < 0$$

(the quantities on the left-hand sides of the inequalities on the second line converge to those on the first as $\varepsilon \to 0$). It follows that there exists $\lambda_\varepsilon \in (\widetilde{\gamma}^{-1}\lambda_0, \widetilde{\gamma}\lambda_0)$ with

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_\varepsilon(\lambda\zeta_\varepsilon)\Big|_{\lambda=\lambda_\varepsilon} = 0,$$

that is, $\xi_\varepsilon := \lambda_\varepsilon\zeta_\varepsilon \in N_\varepsilon$, and we conclude that this value of λ_ε is unique (see Proposition 4.2) and that $\lim_{\varepsilon\to 0} \lambda_\varepsilon = \lambda_0$. □

Corollary 4.7 *Any minimising sequence $\{\zeta_n\}$ of $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ satisfies*

$$\limsup_{n\to\infty} |\zeta_n|_{\tilde{Y}} < M-1.$$

Proof In view of Proposition 4.4 it sufficies to show that for each sufficiently large value of M (chosen independently of ε) there exists $\zeta^\star \in N_\varepsilon$ such that $\mathcal{T}_\varepsilon(\zeta^\star) < \frac{1}{12}(M-1)^2$. In fact, choose $\zeta_0 \in \tilde{Y} \setminus \{0\}$ and $M > 1$ such that

$$\mathcal{S}(\zeta_0) < 0, \qquad \frac{\mathcal{Q}(\zeta_0)^3}{\mathcal{S}(\zeta_0)^2} < \tfrac{27}{48}(M-1)^2. \tag{29}$$

The calculation

$$\mathrm{d}\mathcal{T}_0[\lambda_0\zeta_0](\lambda_0\zeta_0) = 2\lambda_0^2\mathcal{Q}(\zeta_0) + 3\lambda_0^3\mathcal{S}(\zeta_0)$$

then shows that $\lambda_0\zeta_0 \in N_0$, where

$$\lambda_0 = -\frac{2\mathcal{Q}(\zeta_0)}{3\mathcal{S}(\zeta_0)}.$$

It follows that $\lambda_0\zeta_0$ is the unique point on its ray which lies on N_0, and

$$\mathcal{T}_0(\lambda_0\zeta_0) = \tfrac{1}{3}\mathcal{Q}(\lambda_0\zeta_0) = \frac{4\mathcal{Q}(\zeta_0)^3}{27\mathcal{S}(\zeta_0)^2} < \tfrac{1}{12}(M-1)^2, \tag{30}$$

so that

$$|\lambda_0\zeta_0|_{\tilde{Y}} < M-1.$$

Proposition 4.6 asserts the existence of $\xi_\varepsilon \in N_\varepsilon$ with $\lim_{\varepsilon\to 0}|\xi_\varepsilon - \lambda_0\zeta_0|_{\tilde{Y}} = 0$. Using the limit

$$\lim_{\varepsilon\to 0}\mathcal{T}_\varepsilon(\xi_\varepsilon) = \mathcal{T}_0(\lambda_0\zeta_0)$$

and (30), we find that

$$\mathcal{T}_\varepsilon(\xi_\varepsilon) < \tfrac{1}{12}(M-1)^2.$$

□

The next step is to show that there is a minimising sequence for $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ which is also a Palais–Smale sequence.

Proposition 4.8 *There exists a minimising sequence $\{\zeta_n\} \subset B_{M-1}(0)$ of $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ such that*

$$\lim_{n\to\infty}|\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n]|_{\tilde{Y}_\varepsilon\to\mathbb{R}} = 0.$$

Proof Ekeland's variational principle for optimisation problems with regular constraints (Ekeland [12, Thm 3.1]) implies the existence of a minimising sequence $\{\zeta_n\}$ for $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ and a sequence $\{\mu_n\}$ of real numbers such that

$$\lim_{n\to\infty}|\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n] - \mu_n\,\mathrm{d}\mathcal{G}_\varepsilon[\zeta_n]|_{\tilde{Y}_\varepsilon\to\mathbb{R}} = 0.$$

Applying this sequence of operators to ζ_n, we find that $\mu_n \to 0$ as $n\to\infty$ (since $\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n](\zeta_n) = 0$ and $\mathrm{d}\mathcal{G}_\varepsilon[\zeta_n](\zeta_n) = \mathrm{d}^2\mathcal{T}_\varepsilon[\zeta_n](\zeta_n,\zeta_n) \lesssim -1$), whence $|\mathrm{d}\mathcal{T}|_\varepsilon[\zeta_n]_{\tilde{Y}_\varepsilon\to\mathbb{R}} \to 0$ as $n\to\infty$. □

The following lemma examines the convergence properties of more general Palais–Smale sequences.

Lemma 4.9

(i) Suppose that $\{\zeta_n\} \subset B_{M-1}(0)$ satisfies

$$\lim_{n\to\infty}\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n] = 0, \qquad \sup_{j\in\mathbb{Z}^2}|\zeta_n|_{L^2(Q_j)} \gtrsim 1.$$

There exists $\{w_n\} \subset \mathbb{Z}^2$ with the property that a subsequence of $\{\zeta_n(\cdot + w_n)\}$ converges weakly in $\tilde{Y}_\varepsilon$ to a nontrivial critical point ζ_∞ of $\mathcal{T}_\varepsilon$.

(ii) Suppose that $\varepsilon > 0$. The corresponding sequence of FDKP-solutions $\{u_n\}$, where

$$u_n = u_1(\tilde{u}_1(\zeta_n)) + u_2(u_1(\tilde{u}_1(\zeta_n)))$$

and we have abbreviated $\{\zeta_n(\cdot + w_n)\}$ to $\{\zeta_n\}$, converges weakly in X to $u_\infty = u_1(\tilde{u}_1(\zeta_\infty)) + u_2(u_1(\tilde{u}_1(\zeta_\infty)))$ (which is a nontrivial critical point of $\mathcal{I}_\varepsilon$).

Proof We can select $\{w_n\} \subset \mathbb{Z}^2$ so that

$$\liminf_{n\to\infty} |\zeta_n(\cdot + w_n)|_{L^2(Q_0)} \gtrsim 1.$$

The sequence $\{\zeta_n(\cdot + w_n)\} \subset B_{M-1}(0)$ admits a subsequence which converges weakly in $\tilde{Y}_\varepsilon$ and strongly in $L^2(Q_0)$ to $\zeta_\infty \in B_M(0)$; it follows that $|\zeta_\infty|_{L^2(Q_0)} > 0$ and therefore $\zeta_\infty \neq 0$. We henceforth abbreviate $\{\zeta_n(\cdot + w_n)\}$ to $\{\zeta_n\}$ and extract further subsequences as necessary.

We first treat the case $\varepsilon = 0$. For $w \in C_0^\infty(\mathbb{R}^2)$ we find that

$$\int_{\mathbb{R}^2} (\zeta_n^2 - \zeta_\infty^2) w \,\mathrm{d}x\,\mathrm{d}y \leq |\zeta_n - \zeta_\infty|_{L^3(|(x,y)|<R)} |\zeta_n + \zeta_\infty|_{L^3(|(x,y)|<R)} |w|_{L^3} \to 0$$

as $n \to \infty$, where R is chosen so that $\operatorname{supp} w \subset \{|(x,y)| < R\}$ ($\{\zeta_n\}$ converges strongly to ζ_∞ in $L^3(|(x,y)| < R)$. This result also holds for $w \in L^3(\mathbb{R}^2)$ (by density) and hence for all $w \in \tilde{Y}$ (because $\tilde{Y} \subset L^3(\mathbb{R}^2)$). Furthermore $\langle \zeta_n, w\rangle_{\tilde{Y}} \to \langle \zeta_\infty, w\rangle_{\tilde{Y}}$ as $n \to \infty$ for all $w \in \tilde{Y}$. By taking the limit $n \to \infty$ in the equation

$$\mathrm{d}\mathcal{T}_0[\zeta_n](w) = \langle \zeta_n, w\rangle_{\tilde{Y}} + \int_{\mathbb{R}^2} \zeta_n^2 w \,\mathrm{d}x\,\mathrm{d}y,$$

one therefore finds that

$$\langle \zeta_\infty, w\rangle_{\tilde{Y}} + \int_{|(x,y)|<R} \zeta_\infty^2 w \,\mathrm{d}x\,\mathrm{d}y = 0,$$

that is, $\mathrm{d}\mathcal{T}_0[\zeta_\infty](w) = 0$ for all $w \in \tilde{Y}$. It follows that $\mathrm{d}\mathcal{T}_0[\zeta_\infty] = 0$.

Now suppose that $\varepsilon > 0$. According to Proposition 3.10 the sequence $\{u_n\}$ converges weakly in X to u_∞, and the remarks below equation (24) show that

$$\lim_{n\to\infty} |\mathrm{d}\mathcal{I}_\varepsilon[u_n]|_{X\to\mathbb{R}} = 0.$$

Since $u \mapsto \varepsilon u + n(\mathrm{D})u + u^2$ is weakly continuous $X \mapsto L^2(\mathbb{R}^2)$ (see Proposition 2.4 and Lemma 2.1(i)), one finds that

$$\begin{aligned}
\mathrm{d}\mathcal{I}_\varepsilon[u_\infty](w) &= \int_{\mathbb{R}^2} \left(\varepsilon^2 u_\infty + n(\mathrm{D})u_\infty + u_\infty^2\right) w \,\mathrm{d}x\,\mathrm{d}y \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^2} \left(\varepsilon^2 u_n + n(\mathrm{D})u_n + u_n^2\right) w \,\mathrm{d}x\,\mathrm{d}y \\
&= \lim_{n\to\infty} \mathrm{d}\mathcal{I}_\varepsilon[u_n](w) \\
&= 0
\end{aligned}$$

for any $w \in \partial_x \mathcal{S}(\mathbb{R}^2)$, whence u_∞ is a critical point of $\mathcal{I}_\varepsilon$ (so that ζ_∞ is a critical point of $\mathcal{T}_\varepsilon$). □

It remains to show that the minimising sequence for $\mathcal{T}_\varepsilon$ over N_ε identified in Proposition 4.8 satisfies the 'nonvanishing' criterion in Lemma 4.9. This task is accomplished in Proposition 4.10 and Corollary 4.11 below.

Proposition 4.10 *The inequality*

$$\int_{\mathbb{R}^2} |\zeta||\xi|^2 \,\mathrm{d}x\,\mathrm{d}y \lesssim \sup_{j\in\mathbb{Z}^2} |\zeta|_{L^2(Q_j)}^{\frac{1}{6}} |\zeta|_{\tilde{Y}}^{\frac{5}{6}} |\xi|_{\tilde{Y}}^2$$

holds for all $\zeta, \xi \in \tilde{Y}$.

Proof This result follows from the calculation

$$\begin{aligned}\int_{\mathbb{R}^2} |\zeta||\xi|^2 \, \mathrm{d}x \, \mathrm{d}y &\lesssim |\zeta|_{L^3} |\xi|_{L^3}^2 \\ &\lesssim \left(\sum_{j\in\mathbb{Z}^2} |\zeta|_{L^3(Q_j)}^3 \right)^{\frac{1}{3}} |\xi|_{\tilde{Y}}^2 \\ &\lesssim \left(\sup_{j\in\mathbb{Z}^2} |\zeta|_{L^3(Q_j)} \sum_{j\in\mathbb{Z}^2} |\zeta|_{L^3(Q_j)}^2 \right)^{\frac{1}{3}} |\xi|_{\tilde{Y}}^2 \\ &\lesssim \left(\sup_{j\in\mathbb{Z}^2} |\zeta|_{L^2(Q_j)}^{\frac{1}{2}} |\zeta|_{\tilde{Y}(Q_j)}^{\frac{1}{2}} \sum_{j\in\mathbb{Z}^2} |\zeta|_{\tilde{Y}(Q_j)}^2 \right)^{\frac{1}{3}} |\xi|_{\tilde{Y}}^2 \\ &\lesssim \sup_{j\in\mathbb{Z}^2} |\zeta|_{L^2(Q_j)}^{\frac{1}{6}} |\zeta|_{\tilde{Y}}^{\frac{5}{6}} |\xi|_{\tilde{Y}}^2,\end{aligned}$$

where we have interpolated between $L^2(Q_j)$ and $L^6(Q_j)$ and used the embeddings $\tilde{Y} \hookrightarrow L^3(\mathbb{R}^2)$, $\tilde{Y}(Q_j) \hookrightarrow L^6(Q_j)$ and $l^\infty(\mathbb{Z}^2, \tilde{Y}(Q_j)) \hookrightarrow l^2(\mathbb{Z}^2, \tilde{Y}(Q_j)) = \tilde{Y}$. □

Corollary 4.11 *Any sequence* $\{\zeta_n\} \subset N_\varepsilon$ *satisfies*

$$\sup_{j\in\mathbb{Z}^2} |\zeta_n|_{L^2(Q_j)} \gtrsim 1.$$

Proof Using Proposition 4.10, one finds that

$$|\mathcal{S}(\zeta_n)| \le \int_{\mathbb{R}^2} |\zeta_n||\zeta_n|^2 \, \mathrm{d}x \, \mathrm{d}y \lesssim \sup_{j\in\mathbb{Z}^2} |\zeta|_{L^2(Q_j)}^{\frac{1}{6}} |\zeta|_{\tilde{Y}}^{\frac{17}{6}} \lesssim \sup_{j\in\mathbb{Z}^2} |\zeta|_{L^2(Q_j)}^{\frac{1}{6}}$$

(because $|\zeta_n|_{\tilde{Y}} < M$), and the result follows from this estimate and the fact that $-\mathcal{S}(\zeta_n) \ge 2c_\varepsilon - \mathcal{O}(\varepsilon^{\frac{1}{2}})$ with $\liminf_{\varepsilon\to 0} c_\varepsilon \gtrsim 1$ (see Remark 4.5). □

Theorem 4.12

(i) Let $\{\zeta_n\} \subset B_{M-1}(0)$ *be a minimising sequence for* $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ *with*

$$\lim_{n\to\infty} |\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n]|_{\tilde{Y}_\varepsilon\to\mathbb{R}} = 0.$$

There exists $\{w_n\} \subset \mathbb{Z}^2$ *such that a subsequence of* $\{\zeta_n(\cdot + w_n)\}$ *converges weakly in* $\tilde{Y}_\varepsilon$ *to a nontrivial critical point* ζ_∞ *of* $\mathcal{T}_\varepsilon$.

(ii) Suppose that $\varepsilon > 0$. *The corresponding sequence of FDKP-solutions* $\{u_n\}$, *where*

$$u_n = u_1(\tilde{u}_1(\zeta_n)) + u_2(u_1(\tilde{u}_1(\zeta_n)))$$

and we have abbreviated $\{\zeta_n(\cdot + w_n)\}$ *to* $\{\zeta_n\}$, *converges weakly in* X *to* $u_\infty = u_1(\tilde{u}_1(\zeta_\infty)) + u_2(u_1(\tilde{u}_1(\zeta_\infty)))$ *(which is a nontrivial critical point of* $\mathcal{I}_\varepsilon$*).*

5. Ground states

In this section we improve the result of Theorem 4.12 by showing that we can choose the sequence $\{w_n\}$ to ensure convergence to a ground state. For this purpose we use the following abstract concentration-compactness theorem, which is a straightforward modification of theory given by Buffoni, Groves & Wahlén [3, Appendix A].

Theorem 5.1 *Let H_0, H_1 be Hilbert spaces and H_1 be continuously embedded in H_0. Consider a sequence $\{x_n\}$ in $l^2(\mathbb{Z}^s, H_1)$, where $s \in \mathbb{N}$. Writing $x_n = (x_{n,j})_{j\in\mathbb{Z}^s}$, where $x_{n,j} \in H_1$, suppose that*

(i) $\{x_n\}$ *is bounded in* $l^2(\mathbb{Z}^s, H_1)$,

(ii) $S = \{x_{n,j} : n \in \mathbb{N}, j \in \mathbb{Z}^s\}$ *is relatively compact in* H_0,

(iii) $\limsup_{n\to\infty} |x_n|_{l^\infty(\mathbb{Z}^s,H_0)} \gtrsim 1$.

For each $\Delta > 0$ the sequence $\{x_n\}$ admits a subsequence with the following properties. There exist a finite number m of non-zero vectors $x^1, \ldots, x^m \in l^2(\mathbb{Z}^s, H_1)$ and sequences $\{w_n^1\}, \ldots, \{w_n^m\} \subset \mathbb{Z}^s$ satisfying

$$\lim_{n\to\infty} |w_n^{m''} - w_n^{m'}| \to \infty, \qquad 1 \le m'' < m' \le m$$

such that

$$T_{-w_n^{m'}} x_n \rightharpoonup x^{m'},$$

$$|x^{m'}|_{l^\infty(\mathbb{Z}^s,H_0)} = \lim_{n\to\infty} \left| x_n - \sum_{l=1}^{m'-1} T_{w_n^l} x^l \right|_{l^\infty(\mathbb{Z}^s,H_0)},$$

$$\lim_{n\to\infty} |x_n|^2_{l^2(\mathbb{Z}^s,H_1)} = \sum_{l=1}^{m'} |x^l|^2_{l^2(\mathbb{Z}^s,H_1)} + \lim_{n\to\infty} \left| x_n - \sum_{l=1}^{m'} T_{w_n^l} x^l \right|^2_{l^2(\mathbb{Z}^s,H_1)}$$

for $m' = 1, \ldots, m$,

$$\limsup_{n\to\infty} \left| x_n - \sum_{l=1}^{m} T_{w_n^l} x^l \right|_{l^\infty(\mathbb{Z}^s,H_0)} \le \Delta,$$

and

$$\lim_{n\to\infty} \left| x_n - T_{w_n^1} x^1 \right|_{l^\infty(\mathbb{Z}^s,H_0)} = 0$$

if $m = 1$. Here the weak convergence is understood in $l^2(\mathbb{Z}^s, H_1)$ and T_w denotes the translation operator $T_w(x_{n,j}) = (x_{n,j-w})$.

We proceed by using Theorem 5.1 to study Palais–Smale sequences for $\mathcal{T}_\varepsilon$, extracting subsequences where necessary for the validity of our arguments.

Lemma 5.2 *Suppose that $\{\zeta_n\} \subset B_{M-1}(0)$ satisfies*

$$\lim_{n\to\infty} \mathrm{d}\mathcal{T}_\varepsilon[\zeta_n] = 0, \qquad \sup_{j\in\mathbb{Z}^2} |\zeta_n|_{L^2(Q_j)} \gtrsim 1.$$

There exists $\{w_n\} \subset \mathbb{Z}^2$ and ζ_∞ such that $\zeta_n(\cdot + w_n) \rightharpoonup \zeta_\infty$ in $\tilde{Y}$, $\mathcal{S}(\zeta_n) \to \mathcal{S}(\zeta_\infty)$ as $n \to \infty$ and

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}^2} |\zeta_n(\cdot + w_n) - \zeta_\infty|_{L^2(Q_j)} = 0.$$

Proof Set $H_1 = \tilde{Y}(Q_0)$, $H_0 = L^2(Q_0)$, define $x_n \in l^2(\mathbb{Z}^2, H_1)$ for $n \in \mathbb{N}$ by

$$x_{n,j} = \zeta_n(\cdot + j)|_{Q_0} \in \tilde{Y}(Q_0), \qquad j \in \mathbb{Z}^2,$$

and apply Theorem 5.1 to the sequence $\{x_n\} \subset l^2(\mathbb{Z}^2, H_1)$, noting that

$$|x_n|_{l^2(\mathbb{Z}^2,H_1)} = |\zeta_n|_{\tilde{Y}}, \qquad |x_n|_{l^\infty(\mathbb{Z}^2,H_0)} = \sup_{j\in\mathbb{Z}^2} |\zeta_n|_{L^2(Q_j)}$$

for $n \in \mathbb{N}$. Assumption (ii) is satisfied because $\tilde{Y}$ is compactly embedded in $L^2(Q_0)$, while assumptions (i) and (iii) follow from the hypotheses in the lemma.

The theorem asserts the existence of a natural number m, sequences $\{w_n^1\}, \ldots, \{w_n^m\} \subset \mathbb{Z}^2$ with

$$\lim_{n\to\infty} |w_n^{m''} - w_n^{m'}| = \infty, \qquad 1 \le m'' < m' \le m,$$

and functions $\zeta^1, \ldots, \zeta^m \in B_M(0) \setminus \{0\}$ such that $\zeta_n(\cdot + w_n^{m'}) \rightharpoonup \zeta^{m'}$ in $\tilde{Y}$ as $n \to \infty$,

$$\limsup_{n\to\infty} \sup_{j\in\mathbb{Z}^2} \left| \zeta_n - \sum_{l=1}^m \zeta^l(\cdot - w_n^l) \right|_{L^2(Q_j)} \le \varepsilon^6,$$

$$\sum_{l=1}^m |\zeta^l|_{\tilde{Y}}^2 \le \limsup_{n\to\infty} |\zeta_n|_{\tilde{Y}}^2$$

and

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}^2} \left| \zeta_n - \zeta^1(\cdot - w_n^1) \right|_{L^2(Q_j)} = 0 \tag{31}$$

if $m = 1$. It follows from Lemma 4.9(i) that $\mathrm{d}\mathcal{T}_\varepsilon[\zeta^l] = 0$, so that $\zeta^l \in N_\varepsilon$ and $\mathcal{T}_\varepsilon(\zeta^l) \ge c_\varepsilon \gtrsim 1$.

Define

$$\tilde{\zeta}_n = \sum_{l=1}^m \zeta^l(\cdot - w_n^l), \qquad n \in \mathbb{N},$$

and note that

$$\mathcal{S}(\tilde{\zeta}_n) \to \sum_{l=1}^m \mathcal{S}(\zeta^l) \tag{32}$$

as $n \to \infty$ (approximate $\zeta^l \in L^3(\mathbb{R}^2)$ by a sequence of functions in $C_0^\infty(\mathbb{R}^2)$ and use the fact that $|w_n^{l_1} - w_n^{l_2}| \to \infty$). Furthermore, from Proposition 4.10, one finds that

$$\begin{aligned}
&\limsup_{n\to\infty} |\mathcal{S}(\zeta_n) - \mathcal{S}(\tilde{\zeta}_n)| \\
&\quad \lesssim \limsup_{n\to\infty} \int_{\mathbb{R}^2} |\zeta_n - \tilde{\zeta}_n| (|\zeta_n|^2 + |\tilde{\zeta}_n|^2) \, \mathrm{d}x \, \mathrm{d}y \\
&\quad \lesssim \limsup_{n\to\infty} \sup_{j\in\mathbb{Z}^2} |\zeta_n - \tilde{\zeta}_n|_{L^2(Q_j)}^{\frac{1}{6}} \limsup_{n\to\infty} |\zeta_n - \tilde{\zeta}_n|_{\tilde{Y}}^{\frac{5}{6}} (|\zeta_n|_{\tilde{Y}}^2 + |\tilde{\zeta}_n|_{\tilde{Y}}^2) && (33) \\
&\quad \le \varepsilon \limsup_{n\to\infty} (|\zeta_n|_{\tilde{Y}}^2 + |\tilde{\zeta}_n|_{\tilde{Y}}^2)^{\frac{17}{12}} \\
&\quad \lesssim \varepsilon && (34)
\end{aligned}$$

uniformly in m. Combining (32), (34) and

$$-\mathcal{S}(\zeta^l) \ge 2c_\varepsilon - \mathcal{O}(\varepsilon^{\frac{1}{2}} |\zeta^l|_{\tilde{Y}}^2), \qquad l = 1, \ldots, m,$$

(see Remark 4.5) yields

$$-\limsup_{n\to\infty} \mathcal{S}(\zeta_n) \ge 2mc_\varepsilon - \mathcal{O}(\varepsilon^{\frac{1}{2}})$$

and hence

$$2c_\varepsilon \ge 2mc_\varepsilon - \mathcal{O}(\varepsilon^{\frac{1}{2}})$$

uniformly in m (because of (27)). It follows that $m = 1$ (recall that $\liminf_{\varepsilon\to 0} c_\varepsilon \gtrsim 1$).

The advertised result now follows from (31) (with $\zeta_\infty = \zeta^1$ and $w_n = w_n^1$) and (33) (since $\mathcal{S}(\tilde{\zeta}_1) = \mathcal{S}(\zeta^1)$). $\square$

We can now strengthen Theorem 4.12, dealing with the cases $\varepsilon = 0$ and $\varepsilon > 0$ separately.

Lemma 5.3 *Suppose that* $\{\zeta_n\} \subset B_{M-1}(0)$ *satisfies*

$$\lim |\mathrm{d}\mathcal{T}_0[\zeta_n]|_{\tilde{Y}\to\mathbb{R}} = 0, \qquad \sup_{j\in\mathbb{Z}^2} |\zeta_n|_{L^2(Q_j)} \gtrsim 1.$$

There exists $\{w_n\} \subset \mathbb{Z}^2$ *such that* $\{\zeta_n(\cdot + w_n)\}$ *converges strongly in* $\tilde{Y}$ *to a nontrivial critical point of* $\mathcal{T}_0$.

Proof Lemma 5.2 asserts the existence of $\{w_n\} \subset \mathbb{Z}^2$ and $\zeta_\infty \neq 0$ such that $\zeta_n(\cdot + w_n) \rightharpoonup \zeta_\infty$ in $\tilde{Y}$ and $\mathcal{S}(\zeta_n) \to \mathcal{S}(\zeta_\infty)$ as $n \to \infty$. Abbreviating $\{\zeta_n(\cdot + w_n)\}$ to $\{\zeta_n\}$, we find from (25) that

$$\mathcal{Q}(\zeta_n) = \tfrac{1}{2}\mathrm{d}\mathcal{T}_0[\zeta_n](\zeta_n) - \tfrac{3}{2}\mathcal{S}(\zeta_n) \to -\tfrac{3}{2}\mathcal{S}(\zeta_\infty) = \mathcal{Q}(\zeta_\infty),$$

that is, $|\zeta_n|_{\tilde{Y}}^2 \to |\zeta_\infty|_{\tilde{Y}}^2$ as $n \to \infty$. It follows that $\zeta_n \to \zeta_\infty$ in $\tilde{Y}$ as $n \to \infty$ and in particular that $\mathrm{d}\mathcal{T}_0[\zeta_\infty] = 0$. $\square$

We obtain the following existence result in the case $\varepsilon = 0$ as a direct corollary of Lemma 5.3.

Theorem 5.4 *Let* $\{\zeta_n\} \subset B_{M-1}(0)$ *be a minimising sequence for* $\mathcal{T}_0|_{N_0}$ *with*

$$\lim |\mathrm{d}\mathcal{T}_0[\zeta_n]|_{\tilde{Y}\to\mathbb{R}} = 0.$$

There exists $\{w_n\} \subset \mathbb{Z}^2$ *such that* $\{\zeta_n(\cdot + w_n)\}$ *converges strongly in* $\tilde{Y}$ *to a ground state of* $\mathcal{T}_0$.

Let us now turn to the case $\varepsilon > 0$. We begin with the following observation.

Proposition 5.5 *Suppose that* $u_n \rightharpoonup u_\infty$ *in* $H^s(\mathbb{R}^2)$ *as* $n \to \infty$. *The limit*

$$\lim_{n\to\infty} |u_n - u_\infty|_\infty = 0$$

holds if and only if $u_n(\cdot - j_n) \rightharpoonup 0$ *in* $H^s(\mathbb{R}^2)$ *as* $n \to \infty$ *for all unbounded sequences* $\{j_n\} \subset \mathbb{Z}^2$.

Proof Suppose that $u_n(\cdot - j_n) \rightharpoonup 0$ in $H^s(\mathbb{R}^2)$ for all unbounded sequences $\{j_n\} \subset \mathbb{Z}^2$, so that

$$|u_n - u_\infty|_{L^\infty(Q_{j_n})} \leq |u_n(\cdot - j_n)|_{L^\infty(Q_0)} + |u_\infty(\cdot - j_n)|_{L^\infty(Q_0)} \to 0$$

for all unbounded sequences $\{j_n\}$ (because $H^s(\mathbb{R}^2)$ is compactly embedded in $L^\infty(Q_0)$), and this result is also true for bounded sequences (in that case $\{j_n : n \in \mathbb{N}\}$ is a finite set, so that $\{j_n\}$ has a constant subsequence $\{j^\star\}$; the sequence $\{u_n(\cdot - j^\star) - u_\infty(\cdot - j^\star)\}$ obviously converges weakly to zero, and hence the same is true of the weakly convergent sequence $\{u_n(\cdot - j_n) - u_\infty(\cdot - j_n)\}$). By choosing j_n such that

$$|u_n - u_\infty|_{L^\infty(Q_{j_n})} \geq \sup_{j\in\mathbb{Z}^2} |u_n - u_\infty|_{L^\infty(Q_j)} - \tfrac{1}{n}$$

and letting $n \to \infty$ in this inequality, we find that

$$\lim_{n\to\infty} |u_n - u_\infty|_\infty = \lim_{n\to\infty} \sup_{j\in\mathbb{Z}^2} |u_n - u_\infty|_{L^\infty(Q_j)} = 0. \tag{35}$$

Suppose on the other hand that (35) holds. The inequality

$$|u_n(\cdot - j_n)|_{L^\infty(Q_j)} \leq \sup_{j\in\mathbb{Z}^2} |u_n - u_\infty|_{L^\infty(Q_j)} + |u_\infty(\cdot - j_n)|_{L^\infty(Q_j)}$$

for each $j \in \mathbb{Z}^2$ shows that $u_n(\cdot - j_n) \rightharpoonup 0$ in $H^3(\mathbb{R}^2)$ for all unbounded sequences $\{j_n\} \subset \mathbb{Z}^2$. □

Theorem 5.6 *Let $\varepsilon > 0$ and $\{\zeta_n\} \subset B_{M-1}(0)$ be a minimising sequence for $\mathcal{T}_\varepsilon|_{N_\varepsilon}$ with*

$$\lim_{n\to\infty} |\mathrm{d}\mathcal{T}_\varepsilon[\zeta_n]|_{\tilde{Y}\to\mathbb{R}} = 0.$$

There exists $\{w_n\} \subset \mathbb{Z}^2$ such that $\{\zeta_n(\cdot + w_n)\}$ converges weakly in $\tilde{Y}_\varepsilon$ to a ground state ζ_∞ of $\mathcal{T}_\varepsilon$. The corresponding sequence of FDKP-solutions $\{u_n\}$, where

$$u_n = u_1(\tilde{u}_1(\zeta_n)) + u_2(u_1(\tilde{u}_1(\zeta_n)))$$

and we have abbreviated $\{\zeta_n(\cdot + w_n)\}$ to $\{\zeta_n\}$, converges weakly in X and strongly in $L^\infty(\mathbb{R}^2)$ to $u_\infty = u_1(\tilde{u}_1(\zeta_\infty)) + u_2(u_1(\tilde{u}_1(\zeta_\infty)))$ (which is a nontrivial critical point of $\mathcal{I}_\varepsilon$).

Proof Lemma 5.2 asserts the existence of $\{w_n\} \subset \mathbb{Z}^2$ and $\zeta_\infty \neq 0$ such that $\zeta_n(\cdot + w_n) \rightharpoonup \zeta_\infty$ in $\tilde{Y}$ as $n \to \infty$ and

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}^2} |\zeta_n(\cdot + w_n) - \zeta_\infty|_{H^s(Q_j)} = 0,$$

where we have estimated

$$\begin{aligned} |\zeta_n(\cdot + w_n) - \zeta_\infty|^2_{H^s(Q_j)} &\lesssim |\zeta_n(\cdot + w_n) - \zeta_\infty|_{L^2(Q_j)} |\zeta_n(\cdot + w_n) - \zeta_\infty|_{H^{2s}(Q_j)} \\ &\lesssim |\zeta_n(\cdot + w_n) - \zeta_\infty|_{L^2(Q_j)} |\zeta_n(\cdot + w_n) - \zeta_\infty|_{H^{2s}(\mathbb{R}^2)} \\ &\lesssim |\zeta_n(\cdot + w_n) - \zeta_\infty|_{L^2(Q_j)} \end{aligned}$$

because $\{\zeta_n(\cdot + w_n) - \zeta_\infty\}$ is bounded in $\tilde{Y}_\varepsilon$ (which coincides with $H^{2s}_\varepsilon(\mathbb{R}^2)$). It follows that

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}^2} |\zeta_n - \zeta_\infty|_{L^\infty(Q_j)} = \lim_{n\to\infty} |\zeta_n - \zeta_\infty|_\infty = 0,$$

where have again abbreviated $\{\zeta_n(\cdot + w_n)\}$ to $\{\zeta_n\}$, and Proposition 5.5 shows that $\zeta_n(\cdot - j_n) \rightharpoonup 0$ in $H^s_\varepsilon(\mathbb{R}^2)$ and hence in $\tilde{Y}_\varepsilon$ as $n \to \infty$ for all unbounded sequences $\{j_n\} \subset \mathbb{Z}^2$.

Using Proposition 3.10, one finds that $u_n(\cdot - j_n) \rightharpoonup 0$ in X and hence in $H^s(\mathbb{R}^2)$ for all unbounded sequences $\{j_n\} \subset \mathbb{Z}^2$, so that $u_n \to u_\infty$ in $L^\infty(\mathbb{R}^2)$ as $n \to \infty$ (Proposition 5.5). It follows that $u_n \to u_\infty$ in $L^3(\mathbb{R}^2)$ and in particular that $\mathcal{S}(u_n) \to \mathcal{S}(u_\infty)$ as $n \to \infty$. Since $\mathrm{d}\mathcal{I}_\varepsilon[u_\infty] = 0$ and $\mathrm{d}\mathcal{I}_\varepsilon[u_n](u_n) \to 0$ as $n \to \infty$ (see the remarks below equation (24)), one finds from the identity

$$\mathcal{I}_\varepsilon(u) = \tfrac{1}{2}\mathrm{d}\mathcal{I}_\varepsilon[u](u) - \tfrac{1}{2}\mathcal{S}(u)$$

that $\mathcal{T}_\varepsilon(\zeta_n) \to \mathcal{T}_\varepsilon(\zeta_\infty)$ as $n \to \infty$, so that $\mathcal{T}_\varepsilon(\zeta_\infty) = c_\varepsilon$. □

Finally, we show that critical points of $\mathcal{T}_\varepsilon$ converge to critical points of $\mathcal{T}_0$ as $\varepsilon \to 0$. The first step is to establish the corresponding convergence result for the infima of these functionals over their natural constraint sets.

Lemma 5.7 *One has that $\lim_{\varepsilon\to 0} c_\varepsilon = c_0$.*

Proof Let $\{\varepsilon_n\}$ be a sequence with $\lim_{n\to\infty}\varepsilon_n = 0$ and ζ^{ε_n}, ζ^0 be ground states of respectively $\mathcal{T}_{\varepsilon_n}$ and $\mathcal{T}_0$.

Because $\varepsilon^{\frac{1}{2}}\mathcal{R}_\varepsilon \circ \chi_\varepsilon(\mathrm{D})$ and $\varepsilon^{\frac{1}{2}}\mathrm{d}\mathcal{R}_\varepsilon \circ \chi_\varepsilon(\mathrm{D})$ converge uniformly to zero over $B_{M-1}(0) \subset \tilde{Y}$ as $\varepsilon \to 0$, we find that

$$\mathcal{T}_{\varepsilon_n}(\zeta^{\varepsilon_n}) - \mathcal{T}_0(\zeta^{\varepsilon_n}) = o(1), \qquad \mathrm{d}\mathcal{T}_{\varepsilon_n}[\zeta^{\varepsilon_n}] - \mathrm{d}\mathcal{T}_0[\zeta^{\varepsilon_n}] = o(1)$$

as $n \to \infty$ and hence that

$$\lim_{n\to\infty} |\mathrm{d}\mathcal{T}_0[\zeta^{\varepsilon_n}]|_{\tilde{Y}\to\mathbb{R}} = 0.$$

Proposition 4.10 implies that

$$\mathcal{S}(\zeta^{\varepsilon_n}) \leq \int_{\mathbb{R}^2} |\zeta^{\varepsilon_n}||\zeta^{\varepsilon_n}|^2 \,\mathrm{d}x\,\mathrm{d}y \lesssim \sup_{j\in\mathbb{Z}^2} |\zeta^{\varepsilon_n}|_{L^2(Q_j)}^{\frac{1}{6}} |\zeta^{\varepsilon_n}|_{\tilde{Y}}^{\frac{17}{6}} \lesssim \sup_{j\in\mathbb{Z}^2} |\zeta^{\varepsilon_n}|_{L^2(Q_j)}^{\frac{1}{6}}$$

(because $|\zeta^{\varepsilon_n}|_{\tilde{Y}} < M$), and combining this estimate with $S(\zeta^{\varepsilon_n}) \geq c_\varepsilon - \mathcal{O}(\varepsilon^{\frac{1}{2}})$ and $\liminf_{\varepsilon\to 0} c_\varepsilon \gtrsim 1$ yields

$$\sup_{j\in\mathbb{Z}^2} |\zeta^{\varepsilon_n}|_{L^2(Q_j)} \gtrsim 1.$$

According to Lemma 5.3 there exists $\{w_n\} \subset \mathbb{Z}^2$ and $\zeta^\star \in N_0$ such that $\mathrm{d}\mathcal{T}_0[\zeta^\star] = 0$ and $\zeta^{\varepsilon_n}(\cdot + w_n) \to \zeta^\star$ in $\tilde{Y}$ as $n \to \infty$. It follows that

$$\begin{aligned} c_0 &\leq \mathcal{T}_0(\zeta^\star) \\ &= \lim_{n\to\infty} \mathcal{T}_0(\zeta^{\varepsilon_n}) \\ &= \lim_{n\to\infty} \big(\mathcal{T}_0(\zeta^{\varepsilon_n}) - \mathcal{T}_{\varepsilon_n}(\zeta^{\varepsilon_n})\big) + \lim_{n\to\infty} \big(\mathcal{T}_{\varepsilon_n}(\zeta^{\varepsilon_n}) - c_{\varepsilon_n}\big) + \liminf_{n\to\infty} c_{\varepsilon_n} \\ &= \liminf_{n\to\infty} c_{\varepsilon_n}. \end{aligned} \tag{36}$$

Proposition 4.6 (with $\lambda_0 = 1$ and $\zeta_0 = \zeta^0$) asserts the existence of $\xi_n \in N_{\varepsilon_n}$ with $\xi_n \to \zeta^0$ in $\tilde{Y}$ and hence $\mathcal{T}_0(\xi_n) \to \mathcal{T}_0(\zeta^0) = c_0$ as $n \to \infty$. Because $\varepsilon^{\frac{1}{2}}\mathcal{R}_\varepsilon \circ \chi_\varepsilon(\mathrm{D})$ converges uniformly to zero over $B_{M-1}(0) \subseteq \tilde{Y}$ as $\varepsilon \to 0$, one finds that

$$\mathcal{T}_\varepsilon(\xi_n) - \mathcal{T}_0(\xi_n) = o(1)$$

as $n \to \infty$, whence

$$\begin{aligned} \limsup_{n\to\infty} c_{\varepsilon_n} &\leq \limsup_{n\to\infty} \mathcal{T}_{\varepsilon_n}(\xi_n) \\ &= \lim_{n\to\infty} \big(\mathcal{T}_{\varepsilon_n}(\xi_n) - \mathcal{T}_0(\xi_n)\big) + \lim_{n\to\infty} \big(\mathcal{T}_0(\xi_n) - c_0\big) + c_0 \\ &= c_0. \end{aligned} \tag{37}$$

The stated result follows from inequalities (36) and (37). □

Corollary 5.8 *Let $\{\varepsilon_n\}$ be a sequence with $\lim_{n\to\infty}\varepsilon_n = 0$ and ζ^{ε_n} be a ground state of $\mathcal{T}_{\varepsilon_n}$. There exists $\{w_n\} \subset \mathbb{Z}^2$ and a ground state $\zeta^\star$ of $\mathcal{T}_0$ such that a subsequence of $\{\zeta^{\varepsilon_n}(\cdot + w_n)\}_n$ converges to $\zeta^\star$ in $\tilde{Y}$ as $n \to \infty$.*

Proof Continuing the arguments in the proof of Lemma 5.7, we find that it remains only to show that $\mathcal{T}_0(\zeta^\star) = c_0$. This fact follows from the calculations

$$\mathcal{T}_{\varepsilon_n}(\zeta^{\varepsilon_n}) - \mathcal{T}_0(\zeta^{\varepsilon_n}) = o(1), \qquad \mathcal{T}_{\varepsilon_n}(\zeta^{\varepsilon_n}) = c_{\varepsilon_n} \to c_0$$

and $\zeta^{\varepsilon_n}(\cdot + w_n) \to \zeta^\star$ in $\tilde{Y}$ as $n \to \infty$. □

Finally, we record the corresponding result for FDKP solutions.

Theorem 5.9 *Let $\{\varepsilon_n\}$ be a sequence with $\lim_{n\to\infty}\varepsilon_n = 0$ and u^{ε_n} be a critical point of $\mathcal{I}_{\varepsilon_n}$ with $\mathcal{I}_{\varepsilon_n}(u^{\varepsilon_n}) = \varepsilon_n^3 c_{\varepsilon_n}$, so that the formula $u^{\varepsilon_n} = u_1(\zeta^{\varepsilon_n}) + u_2(u_1(\zeta^{\varepsilon_n}))$ defines a ground state ζ^{ε_n} of $\mathcal{T}_\varepsilon$. There exists $\{w_n\} \subset \mathbb{Z}^2$ and a ground state $\zeta^\star$ of $\mathcal{T}_0$ such that a subsequence of $\{\zeta^{\varepsilon_n}(\cdot + w_n)\}$ converges to $\zeta^\star$ in $\tilde{Y}$ as $n \to \infty$.*

Remark 5.10 *Define $u_\varepsilon^\star(x,y) = \varepsilon^2\zeta^\star(\varepsilon x, \varepsilon^2 y)$, so that $u_\varepsilon^\star$ is a KP solitary wave with wave speed ε^2 (see the comments above equation (5)). Abbreviating $u_1(\zeta^\varepsilon(\cdot + w_n))$, $u_2(u_1(\zeta^\varepsilon(\cdot + w_n)))$ to u_1^ε, u_2^ε, one finds that the convergence $|\zeta^\star - \zeta^{\varepsilon_n}(\cdot + w_{\varepsilon_n})|_{\tilde{Y}} = o(1)$ translates to $|u_{\varepsilon_n}^\star - u_1^{\varepsilon_n}|_{\varepsilon_n} = o(\varepsilon_n^{\frac{1}{2}})$, and by Lemma 3.3, $|u_2^{\varepsilon_n}|_{\varepsilon_n} \lesssim \varepsilon_n |u_1^{\varepsilon_n}|_{\varepsilon_n}^2 \lesssim \varepsilon_n^3$ is negligible in comparison. It follows that $|u_{\varepsilon_n}^\star - u^{\varepsilon_n}|_{\varepsilon_n} = o(\varepsilon_n^{\frac{1}{2}})$, while $|u_{\varepsilon_n}^\star|_{\varepsilon_n}$, $|u^{\varepsilon_n}|_{\varepsilon_n}$ are $O(\varepsilon_n^{\frac{1}{2}})$, so that the functions themselves are larger than their difference. Young's inequality also implies the convergence $|u^{\varepsilon_n} - u_{\varepsilon_n}^\star|_{H^{\frac{1}{2}}(\mathbb{R}^2)} = o(\varepsilon_n^{\frac{1}{2}})$.*

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