Non-Existence of Classical Solutions with Finite Energy to the Cauchy Problem of the Compressible Navier-Stokes Equations

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Abstract

The well-posedness of classical solutions with finite energy to the compressible Navier-Stokes equations (CNS) subject to arbitrarily large and smooth initial data is a challenging problem. In the case when the fluid density is away from vacuum (strictly positive), this problem was first solved for the CNS in either one-dimension for general smooth initial data or multi-dimension for smooth initial data near some equilibrium state (i.e., small perturbation) [1,24,25,30–32]. In the case that the flow density may contain vacuum (the density can be zero at some space-time point), it seems to be a rather subtle problem to deal with the well-posedness problem for CNS. The local well-posedness of classical solutions containing vacuum was shown in homogeneous Sobolev space (without the information of velocity in L^2-norm) for general regular initial data with some compatibility conditions being satisfied initially [2, 4–6], and the global existence of classical solution in the same space is established under additional assumption of small total initial energy but possible large oscillations [19]. However, it was shown that any classical solutions to the compressible Navier-Stokes equations in finite energy (inhomogeneous Sobolev) space can not exist globally in time since it may blow up in finite time provided that the density was compactly supported [38]. In this paper, we investigate the well-posedess of classical solutions to the Cauchy problem of Navier-Stokes equations, and prove that the classical solution with finite energy does not exist in the inhomogeneous Sobolev space for any short time under some natural assumptions on initial data near the vacuum. This implies in particular that the homogeneous Sobolev space is crucial as studying the well-posedness for the Cauchy problem of compressible Navier-Stokes equations in the presence of vacuum at far fields even locally in time.

1 Introduction and Main Results

The motion of a n-dimensional compressible viscous, heat-conductive, Newtonian polytropic fluid is governed by the following full compressible Navier-Stokes system:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u, \\ \partial_t(\rho e) + \operatorname{div}(\rho e u) + p\operatorname{div} u = \dfrac{\mu}{2}|\nabla u + (\nabla u)^*|^2 + \lambda(\operatorname{div} u)^2 + \dfrac{\kappa(\gamma-1)}{R}\Delta e, \end{cases} \tag{1.1}$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, ρ, u, p and e denote the density, velocity, pressure and internal energy, respectively. μ and λ are the coefficient of viscosity and the second coefficient of viscosity respectively and κ denotes the coefficient of heat conduction, which satisfy

$$\mu > 0, \quad 2\mu + n\lambda \geq 0, \quad \kappa \geq 0.$$

The equation of state for polytropic gases satisfies

$$p = (\gamma-1)\rho e, \quad p = A\exp\big(\frac{(\gamma-1)S}{R}\big)\rho^\gamma, \tag{1.2}$$

where $A > 0$ and $R > 0$ are positive constants, $\gamma > 1$ is the specific heat ratio, S is the entropy, and we set $A = 1$ in this paper for simplicity. The initial data is given by

$$(\rho, u, e)(x,0) = (\rho_0, u_0, e_0)(x), \quad x \in \mathbb{R}^n \tag{1.3}$$

and is assumed to be continuous. In particular, the initial density is compactly supported on an open bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary, i.e.,

$$\operatorname{supp}_x \rho_0 = \bar{\Omega}, \quad \rho_0(x) > 0, \quad x \in \Omega \tag{1.4}$$

and the initial internal energy e_0 is assumed to be nonnegative but not identical to zero in Ω to avoid the trivial case.

When the heat conduction can be neglected and the compressible viscous fluids are isentropic, the compressible Navier-Stokes equations (1.1) can be reduced to the following system

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u, \end{cases} \tag{1.5}$$

for $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, where the equation of state satisfies

$$p = A\rho^\gamma \tag{1.6}$$

and the initial data are given by

$$(\rho, u)(x,0) = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^n \tag{1.7}$$

with the initial density being compactly supported, i.e., the assumption (1.4) holds.

It is an important issue to study the global existence (well-posedness) of classical/strong solution to CNS (1.1) and (1.5), and many significant progress have been made recently on this and related topics, such as the global existence and asymptotic behaviors of solutions to (1.1) and (1.5). For instance, in the case when the flow density is strictly away from the vacuum ($\inf_\Omega \rho > 0$), the short time existence of classical solution was shown for general regular initial data [23], the global existence of solutions problems were proved in spatial one-dimension by Kazhikhov et al. [1, 24, 25] for sufficiently smooth data and by Serre [35, 36] and Hoff [14] for discontinuous initial data. The key point here behind the strategies to establish the global existence of strong solutions lies in the fact that if the flow density is strictly positive at the initial time, so does for any later-on time. This is also proved to be true for weak solutions to the compressible Navier-Stokes equations (1.1) in one space dimension, namely, weak solution does not exhibit vacuum states in any finite time provided that no vacuum is present initially [17]. The corresponding multidimensional problems were also investigated as the flow density is away from the vacuum, for instance, the short time well-posedness of classical solution was shown by Nash and Serrin for general smooth initial data [33, 37], and the global existence of unique strong solution was first proved by Matsumura and Nishida [30–32] in the energy space (inhomogeneous Sobolev space)

$$\begin{cases} \rho - \bar\rho \in C\big(0,T; H^3(\mathbb{R}^3)\big) \bigcap C^1\big(0,T; H^2(\mathbb{R}^3)\big), \\ u,\ e - \bar e \in C\big(0,T; H^3(\mathbb{R}^3)\big) \bigcap C^1\big(0,T; H^1(\mathbb{R}^3)\big), \end{cases} \tag{1.8}$$

with $\bar\rho > 0$ and $\bar e > 0$ for any $T \in (0, \infty]$, where the additional assumption of small oscillation is required on the perturbation of initial data near the non-vacuum equilibrium state $(\bar\rho, 0, \bar e)$. The global existence of non-vacuum solution was also solved by Hoff for discontinuous initial data [15], and by Danchin [9] who set up the framework based on the Besov type space (a functional space invariant by the natural scaling of the associated equations) to obtain existence and uniqueness of global solutions, where the small oscillations on the perturbation of initial data near some non-vacuum equilibrium state is also required. It should be mentioned here that above smallness of the initial oscillation on the perturbation of initial data near the non-vacuum equilibrium state and the uniformly a-priori estimates established on the classical solutions to CNS (1.1) or (1.5) are sufficient to establish the strict positivity and uniform bounds of flow density, which is essential to prove the global existence of solutions with the flow density away from vacuum in the inhomogeneous Sobolev space (1.8) or other function spaces [9, 15]. However, recently, this assumption on the small oscillations on the initial perturbation of a non-vacuum state can be removed at least for the isentropic case by Huang-Li-Xin in [19] provided that the initial total mechanical energy is suitable small which is equivalent to that the mean square norm of the initial difference from the non-vacuum state is small so that the perturbation may contain large oscillations and vacuum state (see also [10]).

In the case when the flow density may contain vacuum (the flow density is non-negative), it is rather difficult and challenging to investigate the global existence (well-posedness) of classical/strong solutions to CNS (1.1) and CNS (1.5), corresponding to the well-posedness theory of classical solutions [30–32], and the possible appearance of vacuum in the flow density (i.e., the flow density is zero) is one of the essential difficulties in the analysis of the well-posedness and related problems [2, 4–6, 10, 14, 16, 17, 34, 35, 38–40]. Indeed, as it is well-known that (1.1) and (1.5) are strongly coupled systems of hyperbolic-parabolic type, the density $\rho(x,t)$ can be determined by its initial value $\rho_0(x_0)$ by Eq. $(1.5)_1$ along the particle path $x(t)$ satisfying $x = x(t)$ and $x(0) = x_0$ provided that the flow velocity $u(x,t)$ is a-priorily regular enough. Yet, the flow velocity can only be solved by Eq. $(1.5)_2$ which is uniformly parabolic so long as the density is a-priorily strictly positive and uniformly bounded function. However, the appearance of vacuum leads to the strong degeneracy of the hyperbolic-parabolic system and the behaviors of the solution may become singular, such as the ill-posedness and finite blow-up of classical solutions [3, 16, 35, 38, 39]. Recently, the global existence of weak solutions with finite energy to the isentropic system (1.5) subject to general initial data with finite initial energy (initial data may include vacuum states) by Lions [26–28], Jiang-Zhang [22] and Feireisl et al. [11], where the exponent γ may be required to be large and the flow density is allowed to vanish. Despite the important progress, the regularity, uniqueness and behavior of these weak solutions remain largely open. As emphasized before [3, 16, 35, 38, 39], the possible appearance of vacuum is one of the major difficulties when trying to prove global existence and strong regularity results. Indeed, Xin [38] first shows that it is impossible to obtain the global existence of finite energy classical solution to the Cauchy problem for (1.1) in the inhomogeneous Sobolev space (1.8) for any smooth initial data with initial flow density compactly supported and similar phenomena happens for the isentropic system (1.5) for a large class of smooth initial data with compactly supported density. To be more precise, if there exists any solution $(\rho, u, e) \in C^1(0, T; H^2(\mathbb{R}^3))$ for some time $T > 0$, then it must hold $T < +\infty$, which also implies the finite time blow-up of solution $(\rho, u, e) \in C^1(0, T; H^2(\mathbb{R}^3))$ if existing in the presence of the vacuum. Yet, Cho et al. [2, 4–6] proved the local well-posedness of classical solutions to the Cauchy problem for isentropic compressible Navier-Stokes equations (1.5) and full Navier-Stokes equations (1.1) with the initial density containing vacuum for some $T > 0$ in the homogeneous energy space

$$\begin{cases} \rho \in C\big(0, T; H^3(\mathbb{R}^3)\big) \bigcap C^1\big(0, T; H^2(\mathbb{R}^3)\big), \\ u,\, e \in C\big(0, T; D^3(\mathbb{R}^3)\big) \bigcap L^2\big(0, T; D^4(\mathbb{R}^3)\big), \end{cases} \tag{1.9}$$

where $D^k(\mathbb{R}^3) = \{ f \in L^1_{\rm loc}(\mathbb{R}^3) : \ \nabla f \in H^{k-1}(\mathbb{R}^3) \}$, under some additional compatibility conditions as (1.14) on u and similar compatibility condition on e. Moreover, under additional smallness assumption on initial energy, the global existence and uniqueness of classical solutions to the isentropic system (1.5) established by Huang-Li-Xin in homogeneous Sobolev space [19]. Interestingly, such a theory of global in time existence of classical solutions to the full CNS (1.1) fails to be true due to the blow-up results Xin-Yan [39] where they show that any classical solutions to (1.1) will blow-up in finite time as long as the initial density has an isolated mass group. Note that the blow-up results in [39] is independent of the spaces the solutions may be and whether they have small or large data. It should be noted that the main difference of the homogeneous Sobolev space (1.9) from the inhomogeneous Sobolev space (1.8) lies that there is no any estimates on the term $\|u\|_{L^2}$ for the velocity. Thus, it is natural and important to show whether or not the classical solution to the Cauchy problem for the CNS (1.1) and CNS (1.5) exits in the inhomogeneous Sobolev space (1.8) for some small time.

We study the well-posedess of classical solutions to the Cauchy problem for the full compressible Navier-Stokes equations (1.1) and the isentropic Navier-Stokes equations (1.5) in the inhomogeneous Sobolev space (1.8) in the present paper, and we prove that there does not exist any classical solution in the inhomogeneous Sobolev space (1.8) for any small time (refer to Theorems 1.1–1.3 for details). These imply that the homogeneous Sobolev spaces such as (1.8), are crucial in the study of the well-posedness theory of classical solutions to the Cauchy problem of compressible Navier-Stokes equations in the presence of vacuum at far fields.

The main results in this paper can be stated as follows:

Theorem 1.1 *The one-dimensional isentropic Navier-Stokes equations* (1.5)-(1.7) *with the initial density satisfying* (1.4) *with* $\Omega \triangleq I = (0, 1)$ *has no solution* (ρ, u) *in the* *inhomogeneous Sobolev space* $C^1\big([0,T];H^m(\mathbb{R})\big)$*,* $m>2$ *for any positive time* T*, if the initial data* (ρ_0,u_0) *satisfy one of the following two conditions in the interval* I*: there exist positive numbers* $\lambda_i,\ i=1,2,3,4$ *with* $0<\lambda_3,\lambda_4<1$ *such that*

$$\begin{cases} \dfrac{(\rho_0)_x}{\rho_0}\geq \lambda_1, & \text{in } (0,\lambda_3),\\ u_0(\lambda_3)<0,\ u_0\leq 0, & \text{in } (0,\lambda_3), \end{cases} \tag{1.10}$$

or

$$\begin{cases} \dfrac{(\rho_0)_x}{\rho_0}\leq -\lambda_2, & \text{in } (\lambda_4,1),\\ u_0(\lambda_4)>0,\ u_0\geq 0, & \text{in } (\lambda_4,1). \end{cases} \tag{1.11}$$

The following remark is helpful for understanding the conditions (1.10)-(1.11) and Theorem 1.1.

Remark 1.1 *The set of initial data* (ρ_0,u_0) *satisfying the condition* (1.10) *or* (1.11) *is non-empty. For example, for any given positive integers* k *and* l*. Set*

$$\rho_0(x)=\begin{cases} x^k(1-x)^k, & \text{for } x\in\ [0,1],\\ 0, & \text{for } x\in\mathbb{R}\setminus[0,1] \end{cases} \tag{1.12}$$

and

$$u_0(x)=\begin{cases} -\,x^l, & \text{for } x\in\ [0,\frac{1}{4}],\\ \text{smooth connection}, & \text{for } x\in\ (\frac{1}{4},\frac{3}{4}),\\ (1-x)^l, & \text{for } x\in\ [\frac{3}{4},1],\\ 0, & \text{for } x\in\mathbb{R}\setminus[0,1], \end{cases} \tag{1.13}$$

then (ρ_0,u_0) *satisfies both* (1.10) *and* (1.11)*.*

It is known that the system (1.5)-(1.7) *is well-posed in the homogeneous Sobolev space in classical sense if and only if* ρ_0 *and* u_0 *satisfy the following compatibility condition (see [5])*

$$\begin{cases} -\,\mu\Delta u_0-(\mu+\lambda)\nabla\mathrm{div}u_0+\nabla p_0=\rho_0 g,\\ g\in D^1,\quad \sqrt{\rho_0}g\in L^2. \end{cases} \tag{1.14}$$

In one-dimensional case, for (ρ_0, u_0) given by (1.12) and (1.13), we have

$$g = \begin{cases} \mathcal{O}(x^{l-k-2}) + \mathcal{O}(x^{l-k-1}) + \mathcal{O}(x^{k(\gamma-1)-1}), & \text{for } x \in \ [0, \frac{1}{4}], \\ \text{smooth connection}, & \text{for } x \in \ (\frac{1}{4}, \frac{3}{4}), \\ \mathcal{O}\big((1-x)^{l-k-2}\big) + \mathcal{O}\big((1-x)^{l-k-1}\big) & \\ \qquad\qquad + \mathcal{O}\big((1-x)^{k(\gamma-1)-1}\big), & \text{for } x \in \ [\frac{3}{4}, 1], \\ 0, & \text{for } x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

Direct calculations show (ρ_0, u_0) satisfy (1.14) if and only if

$$\begin{cases} k > \dfrac{3}{2(\gamma-1)}, \\ l > k + \dfrac{5}{2}. \end{cases} \tag{1.15}$$

For the initial data (ρ_0, u_0) given by (1.12) and (1.13) with (1.15), the system (1.5)-(1.7) is well-posed in homogeneous Sobolev space but has no solution in $C^1\big([0,T]; H^m(\mathbb{R})\big)$, $m > 2$ for any positive time T. Therefore, the solution constructed in [5] doesn't have finite energy in $C^1\big([0,T]; H^m(\mathbb{R})\big)$, $m > 2$ for any positive time T even if the initial data has finite energy in $H^m(\mathbb{R})$. Precisely, even if

$$\int_{\mathbb{R}} u_0^2(x)\,\mathrm{d}x < \infty,$$

but it holds that

$$\int_{\mathbb{R}} u^2(x,t)\,\mathrm{d}x = \infty, \quad \text{for any } t > 0.$$

Theorem 1.2 *The one-dimensional full Navier-Stokes equations (1.1)-(1.3) with zero heat conduction and the initial density satisfying (1.4) with $\Omega \triangleq I = (0,1)$ has no solution (ρ, u, e) in the inhomogeneous Sobolev space $C^1\big([0,T]; H^m(\mathbb{R})\big)$, $m > 2$ for any positive time T, if the initial data (ρ_0, u_0, e_0) satisfy one of the following two conditions in the interval I: there exist positive numbers λ_i, $i = 5, 6, 7, 8$ with $0 < \lambda_7, \lambda_8 < 1$ such that*

$$\begin{cases} \dfrac{(\rho_0)_x}{\rho_0} + \dfrac{(e_0)_x}{\rho_0} \geq \lambda_5, & \text{in } (0, \lambda_7), \\ u_0(\lambda_7) < 0, \ u_0 \leq 0, & \text{in } (0, \lambda_7), \end{cases} \tag{1.16}$$

or

$$\begin{cases} \dfrac{(\rho_0)_x}{\rho_0} + \dfrac{(e_0)_x}{\rho_0} \leq -\lambda_6, & \text{in } (\lambda_8, 1), \\ u_0(\lambda_8) > 0, \ u_0 \geq 0, & \text{in } (\lambda_8, 1). \end{cases} \tag{1.17}$$

Huang and Li [18] proved the well-posedness to the Cauchy problem of the n-dimensional full compressible Navier-Stokes equations (1.1)-(1.3) with positive heat conduction in Sobolev space, but the entropy function $S(t,x)$ is infinite in vacuum domain (see Remark 4.2 in [39]). If the entropy function $S(t,x)$ is required to be finite in vacuum domains, then we have the following non-existence result:

Theorem 1.3 *The n-dimensional full compressible Navier-Stokes equations* (1.1)-(1.3) *with positive heat conduction and the initial density satisfying* (1.4) *has no solution* (ρ, u, e) *in the inhomogeneous Sobolev space* $C^1\big([0,T]; H^m(\mathbb{R}^n)\big)$, $m > [\frac{n}{2}] + 2$ *with finite entropy* $S(t,x)$ *for any positive time* T.

To prove Theorem 1.1-Theorem 1.3, we will carry out the following steps. First we reduce the original Cauchy problem to an initial-boundary value problem, which then can be reduced further to an integro-differential system with degeneracy for t-derivative by the Lagrangian coordinates transformation, and one can then define a linear parabolic operator from the integro-differential system and establish the Hopf's lemma and a strong maximum principle for the resulting operator, and finally we prove that the resulting system is over-determined by contradiction. Because the linear parabolic operator here degenerates for t-derivative due to that the initial density vanishes on boundary, one needs careful analysis to deduce a localized version strong maximum principle on some rectangle away from boundaries.

We should stress that our method is based on maximum principle for parabolic operator, therefore we shall deal with one-dimensional isentropic case in Section 2, one-dimensional zero heat conduction case in Section 3 and n-dimensional positive heat conduction case in Section 4 separately, we define parabolic operators from momentum equation near the degenerate boundary in the Lagrangian coordinates by adding some conditions on initial data for the first two cases and the energy equation in the whole domain for the last case, respectively.

2 Proof of Theorem 1.1

2.1 Reformulation of Theorem 1.1

Suppose that $n = 1$. Let $(\rho, u) \in C^1\big([0,T]; H^m(\mathbb{R})\big)$, $m > 2$ be a solution to the system (1.5)-(1.7) with the initial density satisfying (1.4). Let $a(t)$ and $b(t)$ be the particle paths stating from 0 and 1, respectively. The following argument is due to Xin [38]. Following from the first equation of (1.5), we see $\operatorname{supp}_x \rho = [a(t), b(t)]$. It follows from the second equation of (1.5) that

$$u_{xx}(x,t) = 0, \quad \forall\, x \in \mathbb{R} \setminus [a(t), b(t)],$$

which gives

$$u(x,t) = \begin{cases} u(b(t),t) + (x - b(t))u_x(b(t),t), & \text{if } x > b(t), \\ u(a(t),t) + (x - a(t))u_x(a(t),t), & \text{if } x < a(t). \end{cases}$$

Since $u(\cdot,t) \in H^m(\mathbb{R}),\ m > 2$, then one has

$$u(x,t) = u_x(x,t) = 0, \quad \forall\, x \in \mathbb{R} \setminus [a(t), b(t)], \tag{2.1}$$

which implies $[a(t), b(t)] = [0,1]$, i.e., $\operatorname{supp}_x \rho(x,t) = [0,1]$. We should remark that the above argument doesn't apply to homogeneous Sobolev spaces since we have no control on L^2-norm of the velocity.

Therefore, by the above argument, to study the well-posedness of the system (1.5)-(1.7) with the initial density satisfying (1.4), we need only to study the well-posedness of the following initial-boundary value problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } I \times (0,T], \\ (\rho u)_t + (\rho u^2 + p)_x = \nu u_{xx}, & \text{in } I \times (0,T], \\ (\rho, u) = (\rho_0, u_0), & \text{on } I \times \{t = 0\}, \\ \rho = u = u_x = 0, & \text{on } \partial I \times (0,T], \end{cases} \tag{2.2}$$

where $\nu = 2\mu + \lambda$.

To prove the non-existence of Cauchy problem (1.5)-(1.7) in $C^1\big([0,T]; H^m(\mathbb{R})\big)$, $m > 2$, it suffices to show the non-existence of the initial-boundary value problem (2.2) in $C^{2,1}(\bar{I} \times [0,T])$, which denotes the set of functions that are C^2 in space and C^1 in time in the space-time domain $\bar{I} \times [0,T]$ hereafter. Thus, in order to prove Theorem 1.1, one needs only to show the following:

Theorem 2.1 *The initial-boundary value problem* (2.2) *has no solution* (ρ, u) *in* $C^{2,1}(\bar{I} \times [0,T])$ *for any positive time* T*, if the initial data* (ρ_0, u_0) *satisfy the condition* (1.10) *or* (1.11).

Let $\eta(x,t)$ denote the position of the gas particle starting from x at time $t=0$ satisfying

$$\begin{cases} \eta_t(x,t) = u(\eta(x,t),t), \\ \eta(x,0) = x. \end{cases} \tag{2.3}$$

ϱ and v are the Lagrangian density and velocity given by

$$\begin{cases} \varrho(x,t) = \rho(\eta(x,t),t), \\ v(x,t) = u(\eta(x,t),t). \end{cases}$$

Then the system (2.2) can be rewritten in the Lagrangian coordinates as

$$\begin{cases} \varrho_t + \dfrac{\varrho v_x}{\eta_x} = 0, & \text{in } I\times(0,T], \\ \eta_x \varrho v_t + (\varrho^\gamma)_x = \nu(\dfrac{v_x}{\eta_x})_x, & \text{in } I\times(0,T], \\ \eta_t(x,t) = v(x,t), & \\ (\varrho, v, \eta) = (\rho_0, u_0, x), & \text{on } I\times\{t=0\}, \\ \varrho = v = v_x = 0, & \text{on } \partial I\times(0,T]. \end{cases} \tag{2.4}$$

The first equation of (2.4) implies that

$$\varrho(x,t) = \frac{\rho_0(x)}{\eta_x(x,t)}.$$

Regarding ρ_0 as a parameter, then one can reduce the system (2.4) further to

$$\begin{cases} \rho_0 v_t + (\dfrac{\rho_0^\gamma}{\eta_x^\gamma})_x = \nu(\dfrac{v_x}{\eta_x})_x, & \text{in } I\times(0,T], \\ \eta_t(x,t) = v(x,t), & \\ (v,\eta) = (u_0, x), & \text{on } I\times\{t=0\}, \\ v = v_x = 0, & \text{on } \partial I\times(0,T]. \end{cases} \tag{2.5}$$

The condition (1.10) or (1.11) on the initial data (ρ_0, u_0) takes the following form in the Lagrangian coordinates

$$\begin{cases} \dfrac{(\rho_0)_x}{\rho_0} \ge \lambda_1, & \text{in } (0,\lambda_3), \\ v_0(\lambda_3) < 0,\ v_0 \le 0, & \text{in } (0,\lambda_3), \end{cases} \tag{2.6}$$

or

$$\begin{cases} \dfrac{(\rho_0)_x}{\rho_0} \le -\lambda_2, & \text{in } (\lambda_4,1), \\ v_0(\lambda_4) > 0,\ v_0 \ge 0, & \text{in } (\lambda_4,1). \end{cases} \tag{2.7}$$

The non-existence of the initial-boundary value problem (2.2) follows from the non-existence of the initial-boundary value problem (2.5) in $C^{2,1}(\bar{I} \times [0,T])$. Thus, Theorem 2.1 is a consequence of the following:

Theorem 2.2 *The problem (2.5) has no solution (v,η) in $C^{2,1}(\bar{I}\times[0,T])$ for any positive time T, if the initial data (ρ_0,u_0) satisfy the condition (2.6) or (2.7).*

2.2 Proof of Theorem 2.2

Given a sufficiently small positive time T^*, we let $(v,\eta) \in C^{2,1}(\bar{I}\times[0,T^*])$ be a solution of the system (2.5) with (2.6) or (2.7). Define the linear parabolic operator $\rho_0\partial_t + L$ by

$$\rho_0\partial_t + L := \rho_0\partial_t - \frac{\nu}{\eta_x}\partial_{xx} + \frac{\nu\eta_{xx}}{\eta_x^2}\partial_x,$$

where

$$\eta_x = 1 + \int_0^t v_x \,\mathrm{d}s \quad \text{and} \quad \eta_{xx} = \int_0^t v_{xx}\,\mathrm{d}s.$$

Then, it follows from the first equation of (2.5) that

$$\rho_0 v_t + Lv = -\big(\frac{\rho_0^\gamma}{\eta_x^\gamma}\big)_x. \tag{2.8}$$

Let M be a positive constant such that

$$\rho_0 + |v_0| + |(v_0)_x| + |(v_0)_{xx}| < M.$$

It follows from the continuity on time that for short time, it holds that

$$|v| + |v_x| + |v_{xx}| \leq M, \quad \text{in } I\times(0,T^*].$$

Taking a positive time $T < T^*$ sufficiently small such that $T \leq \frac{1}{2M}$, then one has

$$\left|\int_0^t v_x\,\mathrm{d}s\right| \leq MT \leq \frac{1}{2}, \quad \text{in } I\times(0,T].$$

This implies

$$\frac{1}{2} \leq \eta_x \leq \frac{3}{2} \quad \text{and} \quad , \quad \text{in } I\times(0,T]. \tag{2.9}$$

Thus, (2.5) is a well-defined integro-differential system with degeneracy for t-derivative due to that the initial density ρ_0 vanishes on the boundary ∂I.

Restrict T further such that $T \leq \frac{\lambda_1}{4M}$. Then, (2.9) together with (2.6) implies

$$-\Big(\frac{\rho_0^\gamma}{\eta_x^\gamma}\Big)_x = -\frac{\gamma\rho_0^\gamma}{\eta_x^\gamma}\Big[\frac{(\rho_0)_x}{\rho_0} - \frac{\eta_{xx}}{\eta_x}\Big] \leq -\frac{\gamma\rho_0^\gamma}{\eta_x^\gamma}\Big(\lambda_1 - \frac{\lambda_1}{2}\Big) < 0, \quad \text{in } (0,\lambda_3)\times(0,T]. \tag{2.10}$$

Thus, under the assumption (2.6), it follows from (2.8) and (2.10) that v satisfies the following differential inequality

$$\rho_0 v_t + Lv \leq 0, \quad \text{in } (0,\lambda_3)\times(0,T]. \tag{2.11}$$

Similarly, under the condition (2.7), v instead satisfies

$$\rho_0 v_t + Lv \geq 0, \quad \text{in } (\lambda_4,1)\times(0,T]. \tag{2.12}$$

In the rest of this section, we will establish the Hopf's lemma and strong maximum principle for a general function w satisfying the differential inequality (2.11) or (2.12). First recall the definition of the parabolic boundary (see [13]) of a bounded domain D of $\mathbb{R}^n \times \mathbb{R}^+$. The parabolic boundary $\partial_p D$ of D consists of points $(x_0,t_0) \in \partial D$ such that $B_r(x_0)\times(t_0 - r^2, t_0]$ contains points not in D, for any $r > 0$. Suppose that Q is a bounded domain of $\mathbb{R}^n$, we use the notation $Q_T := Q \times (0,T]$ to denote a cylinder in $(0,\lambda_3)\times(0,T]$. We first state the weak maximum principle in Q_T.

Lemma 2.1 *Suppose that $w \in C^{2,1}(Q_T)\bigcap C(\bar{Q}_T)$ satisfies (2.11) in Q_T. Then w attains its maximum on the parabolic boundary of Q_T.*

Proof. We first prove the statement under a stronger hypothesis instead of (2.11) that

$$\rho_0 w_t + Lw < 0, \quad \text{in } Q_T. \tag{2.13}$$

Assume w attains its maximum at an interior point (x_0,t_0) of the domain Q_T. Therefore

$$w_t(x_0,t_0) \geq 0, \quad w_x(x_0,t_0) = 0, \quad w_{xx}(x_0,t_0) \leq 0,$$

which implies $\rho_0 w_t + Lw \geq 0$, this contradicts (2.13). Next, define the auxiliary function

$$\varphi^\varepsilon = w - \varepsilon t,$$

for a positive number ε. Then

$$\rho_0\varphi_t^\varepsilon + L\varphi^\varepsilon = \rho_0 w_t + Lw - \varepsilon\rho_0 < 0, \quad \text{in } Q_T.$$

Thus φ^ε attains its maximum on the parabolic boundary of Q_T, which proves the assertion of Lemma 2.1 by letting ε go to zero. □

The result in Lemma 2.1 can be extended to a general domain D contained in $(0,\lambda_3)\times(0,T]$ (see [12]).

Lemma 2.2 *Suppose that $w \in C^{2,1}(D)\bigcap C(\bar{D})$ satisfies (2.11) in D. Then w attains its maximum on the parabolic boundary of D.*

We next present the Hopf's lemma that is crucial to prove Theorem 2.2.

Proposition 2.1 *Suppose that $w \in C^{2,1}\big((0,\lambda_3)\times(0,T]\big)\bigcap C([0,\lambda_3]\times[0,T])$ satisfies (2.11) and there exits a point $(0,t_0)\in\{0\}\times(0,T]$ such that $w(x,t)<w(0,t_0)$ for any point (x,t) in a neighborhood D of the point $(0,t_0)$, where*

$$D =: \big\{(x,t):\ (x-r)^2+(t_0-t)<r^2,\ 0<x<\tfrac{r}{2},\ 0<t\le t_0\big\},$$
$$0<r<\lambda_3,\quad t_0-\tfrac{3r^2}{4}>0.$$

Then it holds that

$$\frac{\partial w(0,t_0)}{\partial \vec{n}}>0,$$

where $\vec{n}:=(-1,0)$ is the outer unit normal vector at the point $(0,t_0)$.

Proof. For positive constants α and ε to be determined, set

$$q(\alpha,x,t)=e^{-\alpha[(x-r)^2+(t_0-t)]}-e^{-\alpha r^2}$$

and

$$\varphi(\varepsilon,\alpha,x,t)=w(x,t)-w(0,t_0)+\varepsilon q(\alpha,x,t).$$

First, we determine ε. The parabolic boundary $\partial_p D$ consists of two parts Σ_1 and Σ_2 given by

$$\Sigma_1=\big\{(x,t):\ (x-r)^2+(t_0-t)<r^2,\ x=\frac{r}{2},\ 0<t\le t_0\big\}$$

and

$$\Sigma_2=\big\{(x,t):\ (x-r)^2+(t_0-t)=r^2,\ 0\le x\le\frac{r}{2},\ 0<t\le t_0\big\}.$$

On $\bar{\Sigma}_1$, $w(x,t)-w(0,t_0)<0$, and hence $w(x,t)-w(0,t_0)<-\varepsilon_0$ for some $\varepsilon_0>0$. Note that $q\le 1$ on Σ_1. Then for such an ε_0, $\varphi(\varepsilon_0,\alpha,x,t)<0$ on Σ_1. For $(x,t)\in\Sigma_2$, $q=0$ and $w(x,t)\le w(0,t_0)$. Thus, $\varphi(\varepsilon_0,\alpha,x,t)\le 0$ for any $(x,t)\in\Sigma_2$ and $\varphi(\varepsilon_0,\alpha,0,t_0)=0$. One concludes that

$$\begin{cases}\varphi(\varepsilon_0,\alpha,x,t)\le 0,\ \text{on } \partial_p D,\\ \varphi(\varepsilon_0,\alpha,0,t_0)=0.\end{cases}\tag{2.14}$$

Next, we choose α. Since w satisfies (2.11), it follows that

$$\begin{aligned}
&\rho_0\varphi_t(\varepsilon_0,\alpha,x,t)+L\varphi(\varepsilon_0,\alpha,x,t)\\
&=\rho_0 w_t(x,t)+Lw(x,t)+\varepsilon_0\big[\rho_0 q_t(\alpha,x,t)+Lq(\alpha,x,t)\big]\\
&\le \varepsilon_0\big[\rho_0 q_t(\alpha,x,t)+Lq(\alpha,x,t)\big].
\end{aligned}\tag{2.15}$$

A direct calculation yields

$$\begin{aligned}
&e^{\alpha[(x-r)^2+(t_0-t)]}\big[\rho_0 q_t(\alpha,x,t)+Lq(\alpha,x,t)\big]\\
&=-\frac{4\nu(x-r)^2}{\eta_x}\alpha^2+\big[\rho_0+\frac{2\nu}{\eta_x}+\frac{2\nu\eta_{xx}(r-x)}{\eta_x^2}\big]\alpha\\
&\le -\frac{2\nu r^2}{3}\alpha^2+(M+4\nu+8\nu Mr)\alpha.
\end{aligned}$$

Therefore, there exists a positive number $\alpha_0=\alpha_0(\nu,r,M)$ such that

$$\rho_0 q_t(\alpha_0,x,t)+Lq(\alpha_0,x,t)\le 0,\quad \text{in } D. \tag{2.16}$$

Thus, it follows from (2.15) and (2.16) that

$$\rho_0\varphi_t(\varepsilon_0,\alpha_0,x,t)+L\varphi(\varepsilon_0,\alpha_0,x,t)\le 0,\quad \text{in } D. \tag{2.17}$$

In conclusion, in view of (2.14) and (2.17), one has

$$\begin{cases}
\rho_0\varphi_t(\varepsilon_0,\alpha_0,x,t)+L\varphi(\varepsilon_0,\alpha_0,x,t)\le 0, & \text{in } D,\\
\varphi(\varepsilon_0,\alpha_0,x,t)\le 0, & \text{on } \partial_p D,\\
\varphi(\varepsilon_0,\alpha_0,0,t_0)=0.
\end{cases}$$

This, together with Lemma 2.2 yields

$$\varphi(\varepsilon_0,\alpha_0,x,t)\le 0,\ \text{ in } D.$$

Therefore, the function $\varphi(\varepsilon_0,\alpha_0,\cdot,\cdot)$ attains its maximum at the point $(0,t_0)$ in D. In particular, it holds that

$$\varphi(\varepsilon_0,\alpha_0,x,t_0)\le\varphi(\varepsilon_0,\alpha_0,0,t_0),\quad \text{for all } x\in(0,\frac{r}{2}).$$

This implies

$$\frac{\partial\varphi(\varepsilon_0,\alpha_0,0,t_0)}{\partial\vec{n}}\ge 0.$$

Finally, we get

$$\frac{\partial w(0,t_0)}{\partial \vec{n}} \geq -\varepsilon_0 \frac{\partial q(\alpha_0,0,t_0)}{\partial \vec{n}} = 2\varepsilon_0\alpha_0 r e^{-\alpha_0 r^2} > 0.$$

□

In order to establish the strong maximum principle, we need to study the t-derivative of interior maximum point. The main ideas in the following lemmas come from [12].

Lemma 2.3 *Let* $w \in C^{2,1}\big((0,\lambda_3)\times(0,T]\big)\bigcap C([0,\lambda_3]\times[0,T])$ *satisfy* (2.11) *and have a maximum* M_0 *in the domain* $(0,\lambda_3)\times(0,T]$*. Suppose that* $(0,\lambda_3)\times(0,T]$ *contains a closed solid ellipsoid*

$$\Omega^\sigma := \big\{(x,t):\ (x-x_*)^2+\sigma(t-t_*)^2 \leq r^2\big\}, \quad \sigma > 0$$

and $w(x,t) < M_0$ *for any interior point* (x,t) *of* Ω^σ *and* $w(\bar{x},\bar{t}) = M_0$ *at some point* $(\bar{x},\bar{t})$ *on the boundary of* Ω^σ*. Then* $\bar{x} = x_*$*.*

Proof. It is easy to see that one may choose a smaller closed ellipsoid $\tilde{\Omega}^\delta$ with the center of the form $(x_*,\tilde{t}_*)$ such that it lies in the domain Ω^σ and has only two isolated boundary points in common. By the assumption of the Lemma 2.3, in $\tilde{\Omega}^\delta$, w attains the maximum M_0 at no more than two isolated boundary points on $\partial\tilde{\Omega}^\delta$. Therefore, without loss of generality, we may replace Ω^σ by $\tilde{\Omega}^\delta$, namely assuming that w attains the maximum M_0 in Ω^σ at no more than two isolated points $(\bar{x},\bar{t})$ and $(\tilde{x},\tilde{t})$ on $\partial\Omega^\sigma$. We prove the desired result by contradiction. Suppose that $\bar{x} \neq x_*$. Applying Lemma 2.2 on the domain $[0,\lambda_3]\times[0,T]$, one shows that $\bar{t} < T$. Choose a closed ball D with center $(\bar{x},\bar{t})$ and radius $\tilde{r} < \min\{|\bar{x}-x_*|, |\bar{x}-\tilde{x}|\}$ contained in $(0,\lambda_3)\times(0,T]$. Then $|x-x_*| \geq |\bar{x}-x_*| - \tilde{r} =: \hat{r}$ for any point $(x,t)\in D$. The parabolic boundary of D is composed of a part Σ_1 lying in Ω^σ and a part Σ_2 lying outside Ω^σ.

For positive constants α and ε to be determined, set

$$q(\alpha,x,t) = e^{-\alpha[(x-x_*)^2+\sigma(t-t_*)^2]} - e^{-\alpha r^2}$$

and

$$\varphi(\varepsilon,\alpha,x,t) = w(x,t) - M_0 + \varepsilon q(\alpha,x,t).$$

We first determine the value of ε. Note that $q(\alpha,x,t) > 0$ in the interior of Ω^σ, $q(\alpha,x,t) = 0$ on $\partial\Omega^\sigma$ and $q(\alpha,x,t) < 0$ outside Ω^σ. So, it holds that $\varphi(\varepsilon,\alpha,\bar{x},\bar{t}) = 0$. On Σ_1, $w(x,t) - M_0 < 0$, and hence $w(x,t) - M_0 < -\varepsilon_0$ for some $\varepsilon_0 > 0$. Note that $q(\alpha,x,t) \leq 1$

on Σ_1. Then for such an ε_0, $\varphi(\varepsilon_0,\alpha,x,t)<0$ on Σ_1. For $(x,t)\in\Sigma_2$, $q(\alpha,x,t)<0$ and $w(x,t)-M_0\le 0$. Thus, $\varphi(\varepsilon_0,\alpha,x,t)<0$ for any $(x,t)\in\Sigma_2$. One concludes that

$$\begin{cases}\varphi(\varepsilon_0,\alpha,x,t)<0, \ \text{on } \partial_p D,\\ \varphi(\varepsilon_0,\alpha,\bar{x},\bar{t})=0.\end{cases} \tag{2.18}$$

Next, we estimate $\rho_0 q_t(\alpha,x,t)+Lq(\alpha,x,t)$. One calculates that for the point $(x,t)\in D$,

$$\begin{aligned} &e^{\alpha[(x-x_*)^2+\sigma(t-t_*)^2]}\big[\rho_0 q_t(\alpha,x,t)+Lq(\alpha,x,t)\big]\\ &=-\frac{4\nu(x-x_*)^2}{\eta_x}\alpha^2+\Big[2\sigma\rho_0(t_*-t)+\frac{2\nu}{\eta_x}+\frac{2\nu\eta_{xx}(x_*-x)}{\eta_x^2}\Big]\alpha\\ &\le-\frac{8\nu\hat{r}^2}{3}\alpha^2+(2\sigma M+4\nu+8\nu M r)\alpha. \end{aligned}$$

Therefore, there exists a positive number $\alpha_0=\alpha_0(\nu,r,\hat{r},\sigma,M)$ such that

$$\rho_0 q_t(\alpha_0,x,t)+Lq(\alpha_0,x,t)\le 0, \quad \text{in } D. \tag{2.19}$$

Thus, it follows from (2.15) and (2.19) that

$$\rho_0\varphi_t(\varepsilon_0,\alpha_0,x,t)+L\varphi(\varepsilon_0,\alpha_0,x,t)\le 0, \quad \text{in } D. \tag{2.20}$$

In conclusion, it follows from (2.18) and (2.20) that

$$\begin{cases}\rho_0\varphi_t(\varepsilon_0,\alpha_0,x,t)+L\varphi(\varepsilon_0,\alpha_0,x,t)\le 0, & \text{in } D,\\ \varphi(\varepsilon_0,\alpha_0,x,t)<0, & \text{on } \partial_p D,\\ \varphi(\varepsilon_0,\alpha_0,\bar{x},\bar{t})=0.\end{cases}$$

However, Lemma 2.2 implies that

$$\varphi(\varepsilon_0,\alpha_0,x,t)<0, \quad \text{in } D,$$

which contradicts $\varphi(\varepsilon_0,\alpha_0,\bar{x},\bar{t})=0$ due to $(\bar{x},\bar{t})\in D$. □

Based on Lemma 2.3, it is standard to prove the following lemma. For details, please refer to Lemma 3 of Chapter 2 in [12].

Lemma 2.4 *Suppose that $w\in C^{2,1}\big((0,\lambda_3)\times(0,T]\big)\bigcap C([0,\lambda_3]\times[0,T])$ satisfies (2.11). If w has a maximum in an interior point $P_0=(x_0,t_0)$ of $(0,\lambda_3)\times(0,T]$, then $w(P)=w(P_0)$ for any point of the form $P=(x,t_0)$ in $(0,\lambda_3)\times(0,T]$.*

We first prove a localized version strong maximum principle in a rectangle $\mathcal{R}$ of the domain $(0,\lambda_3)\times(0,T]$.

Lemma 2.5 *Suppose that $w \in C^{2,1}\big((0,\lambda_3)\times(0,T]\big)\bigcap C([0,\lambda_3]\times[0,T])$ satisfies (2.11). If w has a maximum in the interior point $P_0=(x_0,t_0)$ of $(0,\lambda_3)\times(0,T]$, then there exists a rectangle*

$$\mathcal{R}(P_0) := \big\{(x,t):\ x_0-a_1\le x\le x_0+a_1,\ t_0-a_0\le t\le t_0\big\}$$

in $(0,\lambda_3)\times(0,T]$ such that $w(P)=w(P_0)$ for any point P of $\mathcal{R}(P_0)$.

Proof. We prove the desired result by contradiction. Suppose that there exists an interior point $P_1=(x_1,t_1)$ of $(0,\lambda_3)\times(0,T]$ with $t_1<t_0$ such that $w(P_1)<w(P_0)$. Connect P_1 to P_0 by a simple smooth curve γ. Then there exists a point $P_*=(x_*,t_*)$ on γ such that $w(P_*)=w(P_0)$ and $w(\bar{P})<w(P_*)$ for all any point $\bar{P}$ of γ between P_1 and P_*. We may assume that $P_*=P_0$ and P_1 is very near to P_0. There exist a rectangle $\mathcal{R}(P_0)$ in $(0,\lambda_3)\times(0,T]$ with small positive numbers a_0 and a_1 (will be determined) such that P_1 lies on $t=t_0-a_0$. Since $\mathcal{R}(P_0)\setminus\{t=t_0\}\bigcap\{t=\bar{t}\}$ contains some point $\bar{P}=(\bar{x},\bar{t})$ of γ and $w(\bar{P})<w(P_0)$, we deduce $w(P)<w(P_0)$ for each point P in $\mathcal{R}(P_0)\setminus\{t=t_0\}\bigcap\{t=\bar{t}\}$ due to Lemma 2.4. Therefore, $w(P)<w(P_0)$ for each point P in $\mathcal{R}(P_0)\setminus\{t=t_0\}$.

For positive constants α and ε to be determined, set

$$q(\alpha,x,t)=t_0-t-\alpha(x-x_0)^2$$

and

$$\varphi(\varepsilon,\alpha,x,t)=w(x,t)-w(P_0)+\varepsilon q(\alpha,x,t).$$

Assume further that $P=(x_0-a_1,t_0-a_0)$ is on the parabola $q(\alpha,x,t)=0$, then one has

$$\alpha=\frac{a_0}{a_1^2}. \tag{2.21}$$

To choose α, one calculates that

$$\begin{aligned}\rho_0 q_t(\alpha,x,t)+Lq(\alpha,x,t) &= -\rho_0+\big[\frac{2\nu}{\eta_x}-\frac{2\nu\eta_{xx}(x-x_0)}{\eta_x^2}\big]\alpha\\ &\le -\rho_0+(4\nu+8\nu M a_1)\alpha.\end{aligned} \tag{2.22}$$

Since ρ_0 has a positive lower bound depending on x_0-a_1 in $\mathcal{R}(P_0)$, one may choose α_0 such that

$$\alpha_0<\frac{\rho_0}{4\nu+8\nu M a_1}. \tag{2.23}$$

Thus (2.22) and (2.23) yield

$$\rho_0\varphi_t(\alpha_0,x,t)+L\varphi(\alpha_0,x,t)\le 0,\quad \text{in } \mathcal{R}(P_0). \tag{2.24}$$

One next fixes a_1 such that

$$a_1<\min\{x_0,\lambda_3-x_0\},$$

and then chooses a_0 by (2.21) and (2.23) as

$$a_0<\min\left\{t_0,\ \frac{a_1^2\rho_0}{2(4\nu+8\nu Ma_1)}\right\}.$$

Denote $\mathcal{S}=\{(x,t)\in\mathcal{R}(P_0):\ q(\alpha_0,x,t)\ge 0\}$. The parabolic boundary $\partial_p\mathcal{S}$ of $\mathcal{S}$ is composed of a part Σ_1 lying in $\mathcal{R}(P_0)$ and a part Σ_2 lying on $\mathcal{R}(P_0)\bigcap\{t=t_0-a_0\}$.

We now determine ε. Note that on Σ_2, $w(x,t)-M_0<0$, and $q(\alpha_0,x,t)$ is bounded, one can choose sufficiently small number ε_0 such that $\varphi(\varepsilon_0,\alpha_0,x,t)<0$ on Σ_2. On $\Sigma_1\setminus\{P_0\}$, $q(\alpha_0,x,t)=0$ and $w(x,t)-M_0<0$. Thus, $\varphi(\varepsilon_0,\alpha_0,x,t)<0$ on $\Sigma_1\setminus\{P_0\}$ and $\varphi(\varepsilon_0,\alpha_0,x_0,t_0)=0$. One concludes that

$$\begin{cases}\varphi(\varepsilon_0,\alpha_0,x,t)<0,\ \text{on } \partial_p\mathcal{S}\setminus\{P_0\},\\ \varphi(\varepsilon_0,\alpha_0,x_0,t_0)=0.\end{cases} \tag{2.25}$$

In conclusion, it follows from (2.24) and (2.25) that there exist ε_0, a_0 and a_1 such that

$$\begin{cases}\rho_0\varphi_t(\varepsilon_0,\alpha_0,x,t)+L\varphi(\varepsilon_0,\alpha_0,x,t)\le 0, & \text{in } \mathcal{S},\\ \varphi(\varepsilon_0,\alpha_0,x,t)<0, & \text{on } \partial_p\mathcal{S}\setminus\{P_0\},\\ \varphi(\varepsilon_0,\alpha_0,x_0,t_0)=0.\end{cases} \tag{2.26}$$

In view of Lemma 2.2 and (2.26), the function $\varphi(\varepsilon_0,\alpha_0,\cdot,\cdot)$ only attains its maximum at P_0 in $\bar{\mathcal{S}}$, thus

$$\frac{\partial\varphi(\varepsilon_0,\alpha_0,x_0,t_0)}{\partial t}\ge 0.$$

Note that q satisfies at P_0

$$\frac{\partial q(\alpha_0,x_0,t_0)}{\partial t}=-1.$$

Therefore

$$\frac{\partial w(x_0,t_0)}{\partial t}\ge\varepsilon_0. \tag{2.27}$$

But, by the assumption, w attains its maximum at P_0, it follows that

$$\rho_0 \frac{\partial w(x_0, t_0)}{\partial t} \leq -Lw(x_0, t_0) \leq 0,$$

which contradicts (2.27). $\square$

Now we can prove the following strong maximum principle.

Proposition 2.2 *Suppose that $w \in C^{2,1}\big((0, \lambda_3) \times (0, T]\big) \bigcap C([0, \lambda_3] \times [0, T])$ satisfies (2.11). If w attains its maximum at some interior point $P_0 = (x_0, t_0)$ of $(0, \lambda_3) \times (0, T]$, then $w(P) = w(P_0)$ for any point $P \in (0, \lambda_3) \times (0, t_0]$.*

Proof. We prove the desired result by contradiction. Suppose that $w \not\equiv w(P_0)$. Then there exists a point $P_1 = (x_1, t_1)$ of $(0, \lambda_3) \times (0, t_0]$ such that $w(P_1) < w(P_0)$. By Lemma 2.4, there must be $t_1 < t_0$.

Connect P_1 to P_0 by a straight line γ. There exists a point P_* on γ such that $w(P_*) = w(P_0)$ and $w(\bar{P}) < w(P_*)$ for any point $\bar{P}$ on γ lying between P_* and P_1. Denote by γ_0 the closed sub straight line of γ lying P_* and P_1. Construct a series of rectangles $\mathcal{R}_n, n = 1, 2, \cdots, N$ with small a_n and b_n such that $\gamma_0 \subset \bigcup_{n=1}^N \mathcal{R}_n$, $P_* \in \mathcal{R}_1$ and $P_1 \in \mathcal{R}_N$. Applying Lemma 2.5 on $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_N$ step by step it follows that $w = w(P_1)$ in $\bigcup_{n=1}^N \mathcal{R}_n$. Hence, one deduces $w(P_*) \equiv w(P_1)$ due to P_* lying on γ_0, which is a contradiction. $\square$

Let D be a domain contained in $(\lambda_4, 1) \times (0, T]$. Similar to Lemma 2.2, Proposition 2.1 and Proposition 2.2, we have the corresponding weak maximum principle, Hopf's lemma and strong minimum principle for w satisfying the differential inequality (2.12).

Lemma 2.6 *Suppose that $w \in C^{2,1}(D) \bigcap C(\bar{D})$ satisfies (2.12) in D. Then w attains its minimum on the parabolic boundary of D.*

Proposition 2.3 *Suppose that $w \in C^{2,1}\big((\lambda_4, 1) \times (0, T]\big) \bigcap C([\lambda_4, 1] \times [0, T])$ satisfies (2.12) and there exits a point $(1, t_0) \in \{1\} \times (0, T]$ such that $w(x, t) > w(1, t_0)$ for any point (x, t) in a neighborhood D of the point $(0, t_0)$, where*

$$D =: \left\{(x, y) : \ \big(x - (1 - r)\big)^2 + (t_0 - t) < r^2, \ 1 - \tfrac{r}{2} < x < 1, \ 0 < t \leq t_0\right\},$$
$$1 - r > \lambda_4, \quad t_0 - \tfrac{3r^2}{4} > 0.$$

Then it holds that

$$\frac{\partial w(1, t_0)}{\partial \vec{n}} < 0,$$

where $\vec{n} := (1, 0)$ is the outer unit normal vector at the point $(1, t_0)$.

Proposition 2.4 *Suppose that* $w \in C^{2,1}\big((\lambda_4,1)\times(0,T]\big)\bigcap C([\lambda_4,1]\times[0,T])$ *satisfies* (2.12)*. If* w *attains its minimum at some interior point* $P_0=(x_0,t_0)$ *of* $(\lambda_4,1)\times(0,T]$*, then* $w(P)=w(P_0)$ *for any point* P *of* $(\lambda_4,1)\times(0,t_0]$*.*

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. We first consider the case of the domain $(0,\lambda_3)\times(0,T]$. We establish the weak maximum principle, Hopf lemma and strong maximum principle for the general function w satisfying the differential inequality (2.11), which also apply to the solution v to (2.5) since v also enjoys (2.11). Since $v_0(\lambda_3)<0$, by continuity of v on time, then there exists a time $t_0>0$ such that $v(\lambda_3,\cdot)<0$ in $(0,t_0)$. By Lemma 2.1, v attains its maximum on the parabolic boundary $\{x=0\}\times(0,t_0]\bigcup\{x=\lambda_3\}\times(0,t_0]\bigcup[0,\lambda_3]\times\{t=0\}$. Since $v=0$ on the parabolic boundary $\{x=0\}\times(0,t_0]$ and $v_0\le 0$ in $[0,\lambda_3]$, by Proposition 2.2, v only attains its maximum on the set $\{x=0\}\times(0,t_0]\bigcup[0,\lambda_3]\times\{t=0\}$. Thus, $v(x,t)<v(0,t_0)(=0)$ for any point $(x,t)\in(0,\lambda_3)\times(0,t_0]$. Applying Proposition 2.1 shows that $\frac{\partial v(0,t_0)}{\partial\vec{n}}>0$, which contradicts $v_x(x,t)=0$ on $\partial I\times(0,T]$ of the system (2.5). The other case is similar. □

3 Proof of Theorem 1.2

3.1 Reformulation of Theorem 1.2

Suppose that $\kappa=0$ and $n=1$. Let $(\rho,u,e)\in C^1\big([0,T];H^m(\mathbb{R})\big)$, $m>2$ be a solution to the system (1.1)-(1.3) with the initial density satisfying (1.4). Let $a(t)$ and $b(t)$ be the particle paths stating from 0 and 1, respectively. Similar to (2.1), one can show that

$$\begin{cases}[a(t),b(t)]=[0,1],\\ u(x,t)=u_x(x,t)=0,\end{cases}$$

where $t\in(0,T^*)$ and $x\in[a(t),b(t)]^c$.

Therefore, to study the ill-posedness of the system (1.1)-(1.3) with the initial density satisfying (1.4), we need only to study that of the following initial-boundary value problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{in } I \times (0,T], \\ (\rho u)_t + (\rho u^2 + p)_x = \mu u_{xx}, & \text{in } I \times (0,T], \\ (\rho e)_t + (\rho e u)_x + p u_x = \mu u_x^2, & \text{in } I \times (0,T], \\ (\rho, u, e) = (\rho_0, u_0, e_0), & \text{on } I \times \{t=0\}, \\ \rho = u = u_x = 0, & \text{on } \partial I \times (0,T]. \end{cases} \tag{3.1}$$

To prove the non-existence of (1.1)-(1.3) in $C^1\big([0,T]; H^m(\mathbb{R})\big)$, $m > 2$, it suffices to show the non-existence of (3.1) in $C^{2,1}(\bar{I} \times [0,T])$. Thus, in order to prove Theorem 1.2, we need only to show the following:

Theorem 3.1 *The initial-boundary value problem* (3.1) *has no solution* (ρ, u, e) *in* $C^{2,1}(\bar{I}\times [0,T])$ *for any positive time* T, *if the initial data* (ρ_0, u_0, e_0) *satisfy the condition* (1.16) *or* (1.17).

Let $\eta(x,t)$ be the position of the gas particle starting from x at time $t = 0$ defined by (2.3). Let ϱ, v and $\mathfrak{e}$ be the Lagrangian density, velocity and internal energy respectively, which are defined by

$$\begin{cases} \varrho(x,t) = \rho(\eta(x,t),t), \\ v(x,t) = u(\eta(x,t),t), \\ \mathfrak{e}(x,t) = e(\eta(x,t),t). \end{cases} \tag{3.2}$$

Then the system (3.1) may be rewritten in the Lagrangian coordinates as

$$\begin{cases} \rho_0 v_t + (\frac{\rho_0 \mathfrak{e}}{\eta_x})_x = \mu(\frac{v_x}{\eta_x})_x, & \text{in } I \times (0,T], \\ \rho_0 \mathfrak{e}_t + (\gamma - 1)\frac{\rho_0 \mathfrak{e} v_x}{\eta_x} = \mu \frac{v_x^2}{\eta_x}, & \text{in } I \times (0,T], \\ \eta_t(x,t) = v(x,t), & \\ (v, \mathfrak{e}, \eta) = (u_0, e_0, x), & \text{on } I \times \{t=0\}, \\ v = v_x = 0, & \text{on } \partial I \times (0,T]. \end{cases} \tag{3.3}$$

In the Lagrangian coordinates, the condition (1.16) or (1.17) on the initial data $(\rho_0, u_0, \mathfrak{e}_0)$ becomes

$$\begin{cases} \frac{(\rho_0)_x}{\rho_0} + \frac{(\mathfrak{e}_0)_x}{\rho_0} \geq \lambda_5, & \text{in } (0, \lambda_7), \\ v_0(\lambda_7) < 0, v_0 \leq 0, & \text{in } (0, \lambda_7), \end{cases} \tag{3.4}$$

or

$$\begin{cases} \dfrac{(\rho_0)_x}{\rho_0} + \dfrac{(\mathfrak{e}_0)_x}{\rho_0} \le -\lambda_6, & \text{in } (\lambda_8, 1), \\ v_0(\lambda_8) > 0, v_0 \ge 0, & \text{in } (\lambda_8, 1), \end{cases} \tag{3.5}$$

respectively.

The non-existence of (3.3) in $C^{2,1}(\bar{I} \times [0,T])$ implies the non-existence of (3.1) in $C^{2,1}(\bar{I} \times [0,T])$. Thus, in order to prove Theorem 3.1, we need only to show the following:

Theorem 3.2 *The initial-boundary value problem* (3.3) *has no solution* $(v, \mathfrak{e}, \eta)$ *in* $C^{2,1}(\bar{I} \times [0,T])$ *for any positive time* T*, if the initial data* (ρ_0, u_0) *satisfy the condition* (3.4) *or* (3.5).

3.2 Proof of Theorem 3.2

Given sufficiently small positive time T^*. Let $(v, \mathfrak{e}, \eta) \in C^{2,1}(\bar{I} \times [0,T^*])$ be a solution of the system (3.3) with (3.4) or (3.5). Define the linear parabolic operator $\rho_0 \partial_t + L$ similar to Subsection 3.1 by

$$\rho_0 \partial_t + L := \rho_0 \partial_t - \frac{\mu}{\eta_x} \partial_{xx} + \frac{\mu \eta_{xx}}{\eta_x^2} \partial_x.$$

Then, it follows from the first equation of (3.3) that

$$\rho_0 v_t + Lv = -\Big(\frac{\rho_0 \mathfrak{e}}{\eta_x}\Big)_x. \tag{3.6}$$

Let M be a positive constant such that

$$\rho_0 + |v_0| + |(v_0)_x| + |(v_0)_{xx}| + |\mathfrak{e}_0| + |(\mathfrak{e}_0)_x| < M.$$

It follows from continuity on time that for suitably small T^* that

$$|v| + |v_x| + |v_{xx}| + |\mathfrak{e}| + |\mathfrak{e}_x| \le M, \quad \text{in } I \times (0, T^*]$$

and

$$\frac{(\rho_0)_x}{\rho_0} + \frac{\mathfrak{e}_x}{\rho_0} \ge \frac{\lambda_5}{2}, \quad \text{in } (0, \lambda_7) \times (0, T^*]. \tag{3.7}$$

Taking a positive time $T < T^*$ sufficiently small such that $T \le \frac{1}{2M}$, then one gets

$$\left| \int_0^t v_x \, \mathrm{d}s \right| \le MT \le \frac{1}{2}, \quad \text{in } I \times (0, T].$$

This implies

$$\frac{1}{2} \le \eta_x \le \frac{3}{2}, \quad \text{in } I \times (0,T]. \tag{3.8}$$

Thus, (3.3) is a well-defined integro-differential system with degeneracy for t-derivative due to that the initial density ρ_0 vanishes on the boundary ∂I.

Take T small further such that $T \le \frac{\lambda_5}{8M}$. Therefore, (3.4), (3.7) and (3.8) imply

$$\begin{aligned} &-\big(\frac{\rho_0 \mathfrak{e}}{\eta_x}\big)_x = -\frac{\rho_0 \mathfrak{e}}{\eta_x}\big[\frac{(\rho_0)_x}{\rho_0} + \frac{\mathfrak{e}_x}{\rho_0} - \frac{\eta_{xx}}{\eta_x}\big] \\ &\le -\frac{\rho_0 \mathfrak{e}}{\eta_x}\big(\frac{\lambda_5}{2} - \frac{\lambda_5}{4}\big) < 0, \quad \text{in } (0,\lambda_7) \times (0,T]. \end{aligned} \tag{3.9}$$

Thus, under the assumption (3.4), it follows from (3.6) and (3.9) that v satisfies the following differential inequality

$$\rho_0 v_t + Lv \le 0, \quad \text{in } (0,\lambda_7) \times (0,T].$$

Similarly, under the condition (3.5), v instead satisfies

$$\rho_0 v_t + Lv \ge 0, \quad \text{in } (\lambda_8, 1) \times (0,T].$$

The rest is the same as the proof of Theorem 2.2 in Subsection 2.2 and thus omitted.

4 Proof of Theorem 1.3

4.1 Reformulation of Theorem 1.3

Suppose that $\kappa > 0$. Let $(\rho, u, e) \in C^1\big([0,T]; H^m(\mathbb{R}^n)\big)$, $m > [\frac{n}{2}] + 2$ be a solution to the system (1.1)-(1.3) with the initial density satisfying (1.4). Denote by $X(x_0,t)$ the particle trajectory starting at x_0 when $t = 0$, that is,

$$\begin{cases} \partial_t X(x_0,t) = u(X(x_0,t),t), \\ X(x_0,0) = x_0. \end{cases}$$

Set

$$\Omega = \Omega(0) \quad \text{and} \quad \Omega(t) = \{x = X(x_0,t) :\ x_0 \in \Omega(0)\}.$$

It follows from the first equation of (1.1) that $\mathrm{supp}_x\,\rho = \Omega(t)$. Under the assumption that the entropy $S(t,x)$ is finite in the vacuum domain $\Omega(t)^c$, then one deduces from the equation of state (1.2) that

$$e(x,t) = 0, \qquad \text{for } x \in \Omega(t)^c.$$

Due to $e(\cdot,t) \in H^m(\mathbb{R}^n),\ m > [\frac{n}{2}] + 2$, one gets

$$e_{x_i}(x,t) = e_{x_ix_j}(x,t) = 0, \qquad \text{for } x \in \Omega(t)^c,\ i,j = 1,2,\cdots,n.$$

It follows from the third equation of (1.1) that

$$\frac{\mu}{2}|\nabla u + \nabla u^T|^2 + \lambda(\mathrm{div} u)^2 = 0, \qquad \text{for } x \in \Omega(t)^c. \tag{4.1}$$

Following the arguments in [38], one can calculate that

$$\frac{\mu}{2}|\nabla u + \nabla u^T|^2 + \lambda(\mathrm{div} u)^2 \geq \begin{cases} (2\mu + n\lambda)\displaystyle\sum_{i=1}^{n}(u_{x_i})^2 + \mu\sum_{i>j}^{n}(u_{x_i} + u_{x_j})^2, & \text{if } \lambda \leq 0, \\ 2\mu\displaystyle\sum_{i=1}^{n}(u_{x_i})^2 + \mu\sum_{i>j}^{n}(u_{x_i} + u_{x_j})^2, & \text{if } \lambda > 0, \end{cases} \tag{4.2}$$

this, together with (4.1) implies

$$\partial_i u_j + \partial_j u_i = 0, \qquad \text{for } x \in \Omega(t)^c,\ i,j = 1,2,\cdots,n.$$

Because of $u(\cdot,t) \in H^m(\mathbb{R}^n),\ m > [\frac{n}{2}] + 2$, it holds that

$$u(x,t) = u_{x_i}(x,t) = u_{x_ix_j}(x,t) = 0, \qquad \text{for } x \in \Omega(t)^c,\ i,j = 1,2,\cdots,n.$$

Furthermore, one has $\Omega(t) = \Omega(0)$.

One concludes that

$$\begin{cases} \Omega(t) = \Omega(0), \\ e(x,t) = e_{x_i}(x,t) = 0, \end{cases}$$

where $t \in (0,T^*)$ and $x \in \Omega(t)^c,\ i = 1,2,\cdots,n$.

Therefore, to study the ill-posedness of the system (1.1)-(1.3) with the initial density satisfying (1.4), one needs only to study the ill-posedness of the following initial-boundary value problem

$$
\begin{cases}
\partial_t \rho + \mathrm{div}(\rho u) = 0, & \text{in } \Omega \times (0,T], \\
\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mu \Delta u + (\mu+\lambda)\nabla \mathrm{div} u, & \text{in } \Omega \times (0,T], \\
\partial_t(\rho e) + \mathrm{div}(\rho e u) + p \mathrm{div} u = \dfrac{\mu}{2}|\nabla u + (\nabla u)^*|^2 & \\
\qquad\qquad + \lambda(\mathrm{div} u)^2 + \dfrac{\kappa(\gamma-1)}{R}\Delta e, & \text{in } \Omega \times (0,T], \\
(\rho, u, e) = (\rho_0, u_0, e_0), & \text{on } \Omega \times \{t=0\}, \\
e(x,t) = e_{x_i}(x,t) = 0, & \text{on } \partial\Omega \times (0,T].
\end{cases}
\tag{4.3}
$$

The non-existence of Cauchy problem (1.1)-(1.3) in $C^1\big([0,T]; H^m(\mathbb{R}^n)\big)$, $m > [\frac{n}{2}] + 2$ will follow from the non-existence of the initial-boundary value problem (4.3) in $C^{2,1}(\bar{\Omega} \times [0,T])$. Thus, in order to prove Theorem 1.3, we need only to show the following theorem:

Theorem 4.1 *The initial-boundary value problem (4.3) in the case of $\kappa > 0$ has no solution (ρ, u, e) in $C^{2,1}(\bar{\Omega} \times [0,T])$ for any positive time T.*

Let $\eta(x,t)$ denote the position of the gas particle starting from x at time $t = 0$ defined by (2.3). Let ϱ, v and $\mathfrak{e}$ be the Lagrangian density, velocity and internal energy, respectively, which are defined by (3.2). We will also use the following notations (see also [7, 8, 20, 21])

$$
\begin{cases}
J = \det D\eta, & \text{(Jacobian determinant)}, \\
B = [D\eta]^{-1}, & \text{(inverse of deformation tensor)}, \\
b = JB, & \text{(transpose of cofactor matrix)}.
\end{cases}
$$

We will always use the convention in this section that repeated Latin indices i, j, k, etc., are summed from 1 to n. Then the system (4.3) can be rewritten in the Lagrangian coordinates as

$$
\begin{cases}
\partial_t \varrho + \varrho B_i^j \partial_j v^i = 0, & \text{in } \Omega \times (0,T], \\
\varrho \partial_t v^i + (\gamma-1) B_i^j \partial_j(\varrho \mathfrak{e}) = \mu B_l^k \partial_k (B_l^j \partial_j v^i) & \\
\qquad\qquad + (\mu+\lambda) B_i^k \partial_k (B_l^j \partial_j v^l), & \text{in } \Omega \times (0,T], \\
\varrho \partial_t \mathfrak{e} + (\gamma-1)\varrho \mathfrak{e} B_i^j \partial_j v^i = \dfrac{\mu}{2}|B_l^j \partial_j v^i + (B_l^j \partial_j v^i)^*|^2 & \\
\qquad\qquad + \lambda (B_i^j \partial_j v^i)^2 + \dfrac{\kappa(\gamma-1)}{R} B_l^k \partial_k (B_l^j \partial_j \mathfrak{e}), & \text{in } \Omega \times (0,T], \\
\eta_t(x,t) = v(x,t), & \\
(\varrho, v, \mathfrak{e}, \eta) = (\rho_0, u_0, e_0, x), & \text{on } \Omega \times \{t=0\}, \\
\mathfrak{e}(x,t) = \mathfrak{e}_{x_i}(x,t) = 0, & \text{on } \partial\Omega \times (0,T].
\end{cases}
\tag{4.4}
$$

It follows from the first equation of (4.4) that

$$\varrho(x,t)=\frac{\rho_0(x)}{J(x,t)}.$$

Regarding the initial density ρ_0 as a parameter, one may rewrite (4.4) as

$$\begin{cases}\rho_0\partial_t v^i+(\gamma-1)b_i^j\partial_j(J^{-1}\rho_0\mathfrak{e})=\mu b_l^k\partial_k(J^{-1}b_l^j\partial_j v^i)\\ \qquad\qquad\qquad\qquad +(\mu+\lambda)b_i^k\partial_k(J^{-1}b_l^j\partial_j v^l), & \text{in } \Omega\times(0,T],\\ \rho_0\partial_t\mathfrak{e}+(\gamma-1)J^{-1}\rho_0\mathfrak{e}b_i^j\partial_j v^i=\frac{\mu}{2}J^{-1}|b_l^j\partial_j v^i+(b_l^j\partial_j v^i)^*|^2\\ \qquad\qquad +\lambda J^{-1}(b_i^j\partial_j v^i)^2+\frac{\kappa(\gamma-1)}{R}b_l^k\partial_k(J^{-1}b_l^j\partial_j\mathfrak{e}), & \text{in } \Omega\times(0,T],\\ \eta_t(x,t)=v(x,t),\\ (v,\mathfrak{e},\eta)=(u_0,e_0,x), & \text{on } \Omega\times\{t=0\},\\ \mathfrak{e}(x,t)=\mathfrak{e}_{x_i}(x,t)=0, & \text{on } \partial\Omega\times(0,T].\end{cases} \tag{4.5}$$

The non-existence of the initial-boundary value problem (4.3) will be a consequence of the non-existence of the initial-boundary value problem (4.5) in $C^{2,1}(\bar{\Omega}\times[0,T])$. Thus, in order to prove Theorem 4.1, we need only to show the following:

Theorem 4.2 *The problem (4.5) in the case of $\kappa>0$ has no solution $(v,\mathfrak{e},\eta)$ in $C^{2,1}(\bar{\Omega}\times[0,T])$ for any positive time T.*

4.2 Proof of Theorem 4.2

Let T^* be a given suitably small positive time. Let $(v,\mathfrak{e},\eta)\in C^{2,1}(\bar{\Omega}\times[0,T^*])$ be a solution of the system (4.5). Let M be a positive constant such that

$$\rho_0+\sum_{|\alpha|\le 2}|D^\alpha v_0|+\sum_{|\alpha|\le 2}|D^\alpha\mathfrak{e}_0|<M.$$

It follows from continuity on time that for short time T^*

$$\sum_{|\alpha|\le 2}|D^\alpha v|+\sum_{|\alpha|\le 2}|D^\alpha\mathfrak{e}|\le M,\quad \text{in } \Omega\times(0,T^*].$$

Due to (2.3), it holds that

$$\partial_j\eta^i(x,t)=\delta_j^i+\int_0^t\partial_j v^i(x,s)\,\mathrm{d}s.$$

Thus, $D\eta$ can be regarded as a small perturbation of the identity matrix, which implies both $D\eta$ and A are positive definite matrices. Thereby, there exist two positive numbers $\Lambda_1 \leq \Lambda_2$ such that

$$\Lambda_1|\xi|^2 \leq b_k^i b_k^j \xi_j \xi_i \leq \Lambda_2 |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } (x,t) \in \Omega \times (0, T^*]. \tag{4.6}$$

It follows from the definition of cofactor matrices that

$$|B_i^j| \leq (1 + MT)^{n-1}.$$

Note that (see [29])

$$J_t = J\mathrm{div}u.$$

The chain rule gives

$$J_t = JB_i^j \partial_j v^i = b_i^j \partial_j v^i.$$

Taking a positive time $T < T^*$ sufficiently small such that $T \leq \frac{1}{2^{n+1}M}$, then one has

$$\begin{aligned} |J(x,t) - 1| = \left| \int_0^t b_i^j(x,s) \partial_j v^i(x,s) \, \mathrm{d}s \right| &\leq JT|B_i^j||\partial_j v^i| \\ &\leq JMT(1+MT)^{n-1} \leq \frac{J}{4}, \quad \text{in } \Omega \times (0,T]. \end{aligned}$$

This implies

$$\frac{1}{2} < J(x,t) < \frac{3}{2}, \quad \text{in } \Omega \times (0,T]. \tag{4.7}$$

Direct calculations show (see also [7])

$$\partial_i J = b_k^j \partial_{ij} \eta^k$$

and

$$\partial_j b_i^k = J^{-1} \partial_{sj} \eta^r (b_r^s b_i^k - b_i^s b_r^k).$$

Therefore, one gets that

$$|\partial_i J| \leq \frac{3}{2}(1 + MT)^{n-1} MT \tag{4.8}$$

and

$$|\partial_j b_i^k| \leq 9(1 + MT)^{2n-2} MT. \tag{4.9}$$

Thus, (4.5) is a well-defined integro-differential system with a degeneracy for t-derivative since the initial density ρ_0 vanishes on the boundary $\partial\Omega$.

Define the linear parabolic operator $\rho_0\partial_t + L$ by

$$\rho_0\partial_t w + Lw := \rho_0\partial_t w - \frac{\kappa(\gamma-1)}{R}J^{-1}b_k^i b_k^j \partial_{ij} w \\ - \frac{\kappa(\gamma-1)}{R}b_k^i\partial_i(J^{-1}b_k^j)\partial_j w + (\gamma-1)J^{-1}\rho_0 b_i^j \partial_j v^i w.$$

Then, it follows from the second equation of (4.5) that

$$\rho_0\partial_t\mathfrak{e} + L\mathfrak{e} = \frac{\mu}{2}J^{-1}|b_l^j\partial_j v^i + (b_l^j\partial_j v^i)^*|^2 + \lambda J^{-1}(b_i^j\partial_j v^i)^2. \tag{4.10}$$

In the rest of this section, we will establish the Hopf's lemma and strong maximum principle for solutions of the following differential inequality

$$\rho_0\partial_t w + Lw \geq 0, \quad \text{in } \Omega\times(0,T]. \tag{4.11}$$

It follows from (4.10) and (4.2) that $\mathfrak{e}$ also satisfies (4.11).

We first give the weak maximum principle.

Lemma 4.1 *Suppose that $w \in C^{2,1}(Q_T)\bigcap C(\bar{Q}_T)$ satisfies (4.11). If $w \geq 0$ (> 0) on $\partial_p Q_T$, then $w \geq 0$ (> 0) in Q_T.*

Proof. Set

$$d = (\gamma-1)\max_{\bar{\Omega}\times[0,T]}\left|\frac{J_t}{J}\right|$$

and

$$\varphi = \exp(dt)w.$$

Define a new linear parabolic operator by

$$\rho_0\partial_t\varphi + \tilde{L}\varphi := \rho_0\partial_t\varphi + \tilde{L}\varphi - d\rho_0\varphi.$$

Direct calculation shows that

$$\rho_0\partial_t\varphi + \tilde{L}\varphi = \exp(dt)(\rho_0\partial_t w + Lw) \geq 0, \quad \text{in } Q_T.$$

We first prove the statement under a stronger hypothesis than (4.11) that

$$\rho_0\partial_t\varphi + \tilde{L}\varphi > 0, \quad \text{in } Q_T. \tag{4.12}$$

Assume that φ attains its non-negative minimum at an interior point (x_0, t_0) of the domain Q_T. Therefore

$$\partial_t \varphi(x_0, t_0) \leq 0, \quad \partial_j \varphi(x_0, t_0) = 0, \quad a_k^i a_k^j \partial_{ij} \varphi(x_0, t_0) \geq 0,$$

which implies $\rho_0 \partial_t \varphi + L\varphi \leq 0$, and this contradicts (4.12). Next, choose the auxiliary function

$$\psi^\varepsilon = \varphi + \varepsilon t,$$

for a positive number ε. One calculates

$$\rho_0 \partial_t \psi^\varepsilon + L\psi^\varepsilon = \rho_0 \varphi_t + L\varphi + \varepsilon \rho_0 > 0, \quad \text{in } Q_T.$$

Thus ψ^ε attains its non-negative minimum on $\partial_p Q_T$, which implies that φ also attains its non-negative minimum on $\partial_p Q_T$ by letting ε go to zero.

Since $w \geq 0$ (> 0) on $\partial_p Q_T$, so $\varphi \geq 0$ (> 0) on $\partial_p Q_T$ by the definition of φ, furthermore, $\varphi \geq 0$ (> 0) on Q_T. Therefore, $w \geq 0$ (> 0) on Q_T. $\square$

The result in Lemma 4.1 may be extended to a general domain D contained in $\Omega \times (0, T]$.

Lemma 4.2 *Suppose that $w \in C^{2,1}(D) \bigcap C(\bar{D})$ satisfies (4.11). If $w \geq 0$ (> 0) on $\partial_p D$, then $w \geq 0$ (> 0) in D.*

We next show the Hopf's lemma which is crucial to prove Theorem 4.2.

Proposition 4.1 *Suppose that $w \in C^{2,1}(\Omega \times (0, T]) \bigcap C(\bar{\Omega} \times [0, T])$ satisfies (4.11) and there exits a point $(x_0, t_0) \in \partial\Omega \times (0, T]$ such that $w(x, t) > w(x_0, t_0)$ for any point (x, t) in D, where*

$$D =: \left\{(x, t) : \ |x - \tilde{x}|^2 + (t_0 - t) < r^2, \ 0 < |x - x_0| < \tfrac{r}{2}, \ 0 < t \leq t_0\right\},$$
$$t_0 - \tfrac{3r^2}{4} > 0,$$

with $|x_0 - \tilde{x}| = r$ and $(x_0 - \tilde{x}) \perp \partial\Omega$ at x_0. Then it holds that

$$\frac{\partial w(x_0, t_0)}{\partial \vec{n}} < 0,$$

where $\vec{n} = \frac{x_0 - \tilde{x}}{|x_0 - \tilde{x}|}$.

Proof. For positive constants α and ε to be determined, set

$$q(\alpha, x, t) = -e^{-\alpha[|x-\tilde{x}|^2+(t_0-t)]} + e^{-\alpha r^2}$$

and

$$\varphi(\varepsilon, \alpha, x, t) = w(x,t) - w(x_0, t_0) + \varepsilon q(\alpha, x, t).$$

First, we determine ε. The parabolic boundary $\partial_p D$ consists of two parts Σ_1 and Σ_2 given by

$$\Sigma_1 = \big\{(x,t):\ |x-\tilde{x}|^2 + (t_0 - t) < r^2,\ |x - x_0| = \frac{r}{2},\ 0 < t \leq t_0\big\}$$

and

$$\Sigma_2 = \big\{(x,t):\ |x-\tilde{x}|^2 + (t_0 - t) = r^2,\ 0 \leq |x - x_0| \leq \frac{r}{2},\ 0 < t \leq t_0\big\}.$$

On $\bar{\Sigma}_1$, $w(x,t) - w(x_0,t_0) > 0$, and hence $w(x,t) - w(x_0,t_0) > \varepsilon_0$ for some $\varepsilon_0 > 0$. Note that $q \geq -1$ on Σ_1. Then for such an ε_0, $\varphi(\varepsilon_0, \alpha, x, t) > 0$ on Σ_1. For $(x,t) \in \Sigma_2$, $q = 0$ and $w(x,t) - w(x_0,t_0) \geq 0$. Thus, $\varphi(\varepsilon_0, \alpha, x, t) \geq 0$ for any $(x,t) \in \Sigma_2$ and $\varphi(\varepsilon_0, \alpha, x_0, t_0) = 0$. One concludes that

$$\begin{cases} \varphi(\varepsilon_0, \alpha, x, t) \geq 0, \text{ on } \partial_p D, \\ \varphi(\varepsilon_0, \alpha, x_0, t_0) = 0. \end{cases} \tag{4.13}$$

Next, we choose α. In view of (4.11), one has

$$\begin{aligned} &\rho_0 \partial_t \varphi(\varepsilon_0, \alpha, x, t) + L\varphi(\varepsilon_0, \alpha, x, t) \\ &= \rho_0 \partial_t w(x,t) + Lw(x,t) + \varepsilon_0 \big[\rho_0 \partial_t q(\alpha, x, t) + Lq(\alpha, x, t)\big] \\ &\geq \varepsilon_0 \big[\rho_0 \partial_t q(\alpha, x, t) + Lq(\alpha, x, t)\big]. \end{aligned} \tag{4.14}$$

A direct calculation yields

$$\begin{aligned} &e^{\alpha[|x-\tilde{x}|^2+(t_0-t)]}\big[\rho_0 \partial_t q(\alpha, x, t) + Lq(\alpha, x, t)\big] \\ &= \frac{4\kappa(\gamma-1)}{R} J^{-1} b_k^i b_k^j (x_i - \tilde{x}_i)(x_j - \tilde{x}_j)\alpha^2 - \big[\rho_0 + \frac{2\kappa(\gamma-1)}{R} J^{-1} b_k^i b_k^j \delta_{ij} \\ &\quad + \frac{2\kappa(\gamma-1)}{R} b_k^i \partial_i (J^{-1} b_k^j)(x_j - \tilde{x}_j)\big]\alpha - (\gamma-1) J^{-1} \rho_0 b_i^j \partial_j v^i \\ &\quad \times \big(1 - e^{\alpha[|x-\tilde{x}|^2+(t_0-t)-r^2]}\big). \end{aligned} \tag{4.15}$$

It follows from (4.6) and (4.7) that

$$\begin{aligned} &\frac{4\kappa(\gamma-1)}{R} J^{-1} b_k^i b_k^j (x_i - \tilde{x}_i)(x_j - \tilde{x}_j) \\ &\geq \frac{8\kappa(\gamma-1)\Lambda_1}{R}\big(|x_0 - \tilde{x}| - |x - x_0|\big)^2 \geq \frac{2\kappa(\gamma-1) r^2 \Lambda_1}{R}. \end{aligned}$$

The other terms on the right hand side of (4.15) may be estimated as

$$\left|\frac{2\kappa(\gamma-1)}{R}J^{-1}b_k^i b_k^j\delta_{ij}\right| \le \frac{4\kappa(\gamma-1)\Lambda_2}{R}, \tag{4.16}$$

$$\begin{aligned}\left|\frac{2\kappa(\gamma-1)}{R}b_k^i\partial_i(J^{-1}b_k^j)(x_j-\tilde{x}_j)\right| &\le \frac{81\kappa(\gamma-1)r}{R}(1+MT)^{3n-3}MT\\ &\le \frac{81\cdot 2^{2n-4}\kappa(\gamma-1)r}{R},\end{aligned} \tag{4.17}$$

$$\begin{aligned}\left|(\gamma-1)J^{-1}\rho_0 b_i^j\partial_j v^i\big(1-e^{\alpha[|x-\tilde{x}|^2+(t_0-t)-r^2]}\big)\right| &\le 3(\gamma-1)M^2(1+MT)^{n-1}\\ &\le 3\cdot 2^{n-1}(\gamma-1)M^2,\end{aligned} \tag{4.18}$$

where we have used (4.6)-(4.9). Finally, one obtains

$$\begin{aligned}&e^{\alpha[|x-\tilde{x}|^2+(t_0-t)]}\big[\rho_0\partial_t q(\alpha,x,t)+Lq(\alpha,x,t)\big]\\ &\ge \frac{2\kappa(\gamma-1)r^2\Lambda_1}{R}\alpha^2-\left(M+\frac{4\kappa(\gamma-1)\Lambda_2}{R}+\frac{81\cdot 2^{2n-4}\kappa(\gamma-1)r}{R}\right)\alpha\\ &\quad -3\cdot 2^{n-1}(\gamma-1)M^2.\end{aligned}$$

Thereby, there exists a positive number $\alpha_0=\alpha_0(\kappa,\gamma,r,R,M,\Lambda_1,\Lambda_2)$ such that

$$\rho_0\partial_t q(\alpha_0,x,t)+Lq(\alpha_0,x,t)\ge 0,\quad \text{in } D. \tag{4.19}$$

In conclusion, in view of (4.13), (4.14) and (4.19), one has

$$\begin{cases}\rho_0\partial_t\varphi(\varepsilon_0,\alpha_0,x,t)+L\varphi(\varepsilon_0,\alpha_0,x,t)\ge 0, & \text{in } D,\\ \varphi(\varepsilon_0,\alpha_0,x,t)\ge 0, & \text{on } \partial_p D,\\ \varphi(\varepsilon_0,\alpha_0,x_0,t_0)=0.\end{cases} \tag{4.20}$$

Lemma 4.2, together with (4.20), shows that

$$\varphi(\varepsilon_0,\alpha_0,x,t)\ge 0,\quad \text{in } D.$$

Therefore, the function $\varphi(\varepsilon_0,\alpha_0,\cdot,\cdot)$ attains its minimum at the point (x_0,t_0) in D. In particular, it holds that

$$\varphi(\varepsilon_0,\alpha_0,x,t_0)\ge\varphi(\varepsilon_0,\alpha_0,x_0,t_0),\qquad \text{for all } x\in\big\{x:|x-x_0|\le\frac{r}{2}\big\}.$$

This implies

$$\frac{\partial\varphi(\varepsilon_0,\alpha_0,x_0,t_0)}{\partial\vec{n}}\le 0.$$

Finally, one obtains

$$\frac{\partial w(x_0,t_0)}{\partial \vec{n}} \le -\varepsilon_0 \frac{\partial q(\alpha_0,x_0,t_0)}{\partial \vec{n}} = -2\varepsilon_0\alpha_0 r e^{-\alpha_0 r^2} < 0.$$

□

In order to establish the strong maximum principle, we first study the t-derivative at an interior minimum point.

Lemma 4.3 *Let $w \in C^{2,1}(\Omega\times(0,T])\bigcap C(\bar{\Omega}\times[0,T])$ satisfy (4.11) and have a minimum M_0 in the domain $\Omega\times(0,T]$. Suppose that $\Omega\times(0,T]$ contains a closed solid ellipsoid*

$$\Omega^\sigma := \big\{(x,t):\ |x-x_*|^2+\sigma(t-t_*)^2 \le r^2\big\}, \quad \sigma>0$$

and $w(x,t) > M_0$ for any interior point (x,t) of Ω^σ and $w(\bar{x},\bar{t}) = M_0$ at some point $(\bar{x},\bar{t})$ on the boundary of Ω^σ. Then $\bar{x} = x_*$.

Proof. It is easy to see that one may choose a smaller closed ellipsoid $\tilde{\Omega}^\delta$ with the center of the form $(x_*,\tilde{t}_*)$ such that it lies in the domain Ω^σ and has only two isolated boundary points in common. By the assumption of the Lemma 4.3, in $\tilde{\Omega}^\delta$, w attains the maximum M_0 at no more than two isolated boundary points on $\partial\tilde{\Omega}^\delta$. Therefore, without loss of generality, we may replace Ω^σ by $\tilde{\Omega}^\delta$, namely assuming that w attains the maximum M_0 in Ω^σ at no more than two isolated points $(\bar{x},\bar{t})$ and $(\tilde{x},\tilde{t})$ on $\partial\Omega^\sigma$. We prove the desired result by contradiction. Suppose that $\bar{x}\neq x_*$. Choose a closed ball D with center $(\bar{x},\bar{t})$ and radius $\tilde{r} < \min\{|\bar{x}-x_*|, |\bar{x}-\tilde{x}|\}$ contained in $\Omega\times(0,T]$. Then, one has

$$|x-x_*| \ge |\bar{x}-x_*| - \tilde{r} =: \hat{r}, \qquad \text{for } (x,t)\in D.$$

The parabolic boundary $\partial_p D = \partial D$ of D consists of a part Σ_1 lying in Ω^σ and a part Σ_2 lying outside Ω^σ.

For positive constants α and ε to be determined, set

$$q(\alpha,x,t) = -e^{-\alpha[|x-x_*|^2+\sigma(t-t_*)^2]} + e^{-\alpha r^2}$$

and

$$\varphi(\varepsilon,\alpha,x,t) = w(x,t) - M_0 + \varepsilon q(\alpha,x,t).$$

We first determine the value of ε. Note that $q(\alpha,x,t)<0$ in the interior of Ω^σ, $q(\alpha,x,t) = 0$ on $\partial\Omega^\sigma$ and $q(\alpha,x,t) > 0$ outside Ω^σ. So, it holds that $\varphi(\varepsilon,\alpha,\bar{x},\bar{t}) = 0$. On Σ_1, $w(x,t)-M_0>0$, and hence $w(x,t)-M_0>\varepsilon_0$ for some $\varepsilon_0>0$. Note that $q(\alpha,x,t)\ge -1$

on Σ_1. Then for such an ε_0, $\varphi(\varepsilon_0,\alpha,x,t)>0$ on Σ_1. For $(x,t)\in\Sigma_2$, we have $q(\alpha,x,t)>0$ and $w(x,t)-M_0\geq 0$. Thus, $\varphi(\varepsilon_0,\alpha,x,t)>0$ for any $(x,t)\in\Sigma_2$. One concludes that One concludes that

$$\begin{cases}\varphi(\varepsilon_0,\alpha,x,t)>0, \ \text{on } \partial_p D,\\ \varphi(\varepsilon_0,\alpha,\bar{x},\bar{t})=0.\end{cases} \tag{4.21}$$

Next, we choose α. We need only to estimate $\rho_0 q_t(\alpha,x,t)+Lq(\alpha,x,t)$ due to (4.14). One calculates that

$$\begin{aligned}&e^{\alpha[|x-x_*|^2+\sigma(t-t_*)^2]}\big[\rho_0\partial_t q(\alpha,x,t)+Lq(\alpha,x,t)\big]\\ &=\frac{4\kappa(\gamma-1)}{R}J^{-1}b_k^i b_k^j(x_i-(x_*)_i)(x_j-(x_*)_j)\alpha^2-\big[2\sigma\rho_0(t-t_*)\\ &\quad+\frac{2\kappa(\gamma-1)}{R}J^{-1}b_k^i b_k^j\delta_{ij}+\frac{2\kappa(\gamma-1)}{R}b_k^i\partial_i(J^{-1}b_k^j)(x_j-x_{*j})\big]\alpha\\ &\quad-(\gamma-1)J^{-1}\rho_0 b_i^j\partial_j v^i\big(1-e^{\alpha[|x-x_*|^2+\sigma(t-t_*)^2-r^2]}\big).\end{aligned}$$

Similar to (4.19), there exists $\alpha_0=\alpha_0(\kappa,\gamma,\sigma,r,\hat{r},R,M,\Lambda_1,\Lambda_2)>0$ such that

$$\rho_0\partial_t q(\alpha_0,x,t)+Lq(\alpha_0,x,t)\geq 0, \quad \text{in } D. \tag{4.22}$$

In conclusion, it follows from (4.14), (4.21) and (4.22) that

$$\begin{cases}\rho_0\partial_t\varphi(\varepsilon_0,\alpha_0,x,t)+L\varphi(\varepsilon_0,\alpha_0,x,t)\geq 0, & \text{in } D,\\ \varphi(\varepsilon_0,\alpha_0,x,t)>0, & \text{on } \partial_p D,\\ \varphi(\varepsilon_0,\alpha_0,\bar{x},\bar{t})=0.\end{cases} \tag{4.23}$$

Then Lemma 4.2 and (4.23) imply that

$$\varphi(\varepsilon_0,\alpha_0,x,t)>0, \quad \text{in } D.$$

which contradicts $\varphi(\varepsilon_0,\alpha_0,\bar{x},\bar{t})=0$ due to $(\bar{x},\bar{t})\in D$. □

Based on Lemma 4.3, it is standard to prove the following lemma. For details, one may refer to Lemma 3 of Chapter 2 in [12].

Lemma 4.4 *Suppose that* $w\in C^{2,1}(\Omega\times(0,T])\bigcap C(\bar{\Omega}\times[0,T])$ *satisfies* (4.11). *If* w *has a minimum in an interior point* $P_0=(x_0,t_0)$ *of* $\Omega\times(0,T]$, *then* $w(P)=w(P_0)$ *for any point of the form* $P=(x,t_0)$ *in* $\Omega\times(0,T]$.

Next, we prove a local strong minimum principle in a rectangle $\mathcal{R}$ of the domain $\Omega\times(0,T]$.

Lemma 4.5 *Suppose that* $w \in C^{2,1}(\Omega \times (0,T]) \bigcap C(\bar{\Omega} \times [0,T])$ *satisfies* (4.11). *If* w *has a minimum in the interior point* $P_0 = (x_0, t_0)$ *of* $\Omega \times (0,T]$, *then there exists a rectangle*

$$\mathcal{R}(P_0) := \big\{(x,t):\ (x_0)_i - c_i \le x_i \le (x_0)_i + c_i,\ t_0 - c_0 \le t \le t_0,\ 1 \le i \le n\big\}$$

in $\Omega \times (0,T]$ *such that* $w(P) = w(P_0)$ *for any point* P *of* $\mathcal{R}(P_0)$.

Proof. We prove the desired result by contradiction. Suppose that there exists an interior point $P_1 = (x_1, t_1)$ of $\Omega \times (0,T]$ with $t_1 < t_0$ such that $w(P_1) > w(P_0)$. Connect P_1 to P_0 by a simple smooth curve γ. Then there exists a point $P_* = (x_*, t_*)$ on γ such that $w(P_*) = w(P_0)$ and $w(\bar{P}) < w(P_*)$ for all any point $\bar{P}$ of γ between P_1 and P_*. We may assume that $P_* = P_0$ and P_1 is very near to P_0. There exists a rectangle $\mathcal{R}(P_0)$ in $\Omega \times (0,T]$ with small positive numbers c_0 and c_i (to be determined) such that P_1 lies on $t = t_0 - c_0$. Since $\mathcal{R}(P_0) \setminus \{t = t_0\} \bigcap \{t = \bar{t}\}$ contains some point $\bar{P} = (\bar{x}, \bar{t})$ of γ and $w(\bar{P}) > w(P_0)$, we deduce $w(P) > w(P_0)$ for each point P in $\mathcal{R}(P_0) \setminus \{t = t_0\} \bigcap \{t = \bar{t}\}$ due to Lemma 2.4. Therefore, $w(P) > w(P_0)$ for each point P in $\mathcal{R}(P_0) \setminus \{t = t_0\}$.

For positive constants α and ε to be determined, set

$$q(\alpha, x, t) = -t_0 + t + \alpha |x - x_0|^2$$

and

$$\varphi(\varepsilon, \alpha, x, t) = w(x,t) - w(P_0) + \varepsilon q(\alpha, x, t).$$

Assume further that $P = (x_0 - c, t_0 - c_0)$ is on the parabola $q(\alpha, x, t) = 0$, then one solves

$$\alpha = \frac{c_0}{|c|^2}, \tag{4.24}$$

where $|c| = (\sum_{i=1}^n |c_i|^2)^{\frac{1}{2}}$.

A direct calculation shows that

$$\begin{aligned} &\rho_0 \partial_t q(\alpha, x, t) + L q(\alpha, x, t) \\ &= -\alpha\Big[\frac{2\kappa(\gamma-1)}{R} J^{-1} b_k^i b_k^j \delta_{ij} + \frac{2\kappa(\gamma-1)}{R} b_k^i \partial_i\big(J^{-1} b_k^j\big)(x_j - (x_0)_j\big) \\ &\quad - (\gamma-1) J^{-1} \rho_0 b_i^j \partial_j v^i |x - x_0|^2\Big] + \rho_0\big[1 + (\gamma-1) J^{-1} b_i^j \partial_j v^i (-t_0 + t)\big]. \end{aligned} \tag{4.25}$$

The first three terms on the right hand side of (4.25) may be estimated in the same fashion as (4.16)-(4.18). For the last term, one has

$$|(\gamma-1) J^{-1} b_i^j \partial_j v^i (-t_0 + t)| \le 6(\gamma-1)(1 + MT)^{n-1} MT \le \frac{3}{2}(\gamma - 1).$$

Consequently, one gets

$$\begin{aligned}&\rho_0\partial_t q(\alpha,x,t)+Lq(\alpha,x,t)\\&\geq -\alpha\big[\frac{\kappa(\gamma-1)}{R}(4\Lambda_2+81\cdot 2^{2n-4}|c|)+3\cdot 2^{n-1}(\gamma-1)M^2|c|^2\big]\\&\quad+\frac{3\gamma-1}{2}\rho_0.\end{aligned}\tag{4.26}$$

Since ρ_0 has a positive lower bound depending on $x_0\pm c$ in $\mathcal{R}(P_0)$, one can choose α_0 such that

$$\alpha_0<\frac{(3\gamma-1)R\rho_0}{\kappa(\gamma-1)(8\Lambda_2+81\cdot 2^{2n-3}|c|)+3\cdot 2^n(\gamma-1)RM^2|c|^2},\tag{4.27}$$

then it follows from (4.25)-(4.27) that

$$\rho_0\partial_t\varphi(\alpha_0,x,t)+L\varphi(\alpha_0,x,t)\geq 0,\quad \text{in } \mathcal{R}(P_0).\tag{4.28}$$

Next, one first choses c such that

$$\big\{x\in\mathbb{R}^n:\ (x_0)_i-c_i\leq x_i\leq (x_0)_i+c_i,\ 1\leq i\leq n\big\}\subset\Omega,$$

and then further determines c_0 by (4.24) and (4.27) as

$$c_0<\min\Big\{t_0,\ \frac{(3\gamma-1)|c|^2R\rho_0}{\kappa(\gamma-1)(16\Lambda_2+81\cdot 2^{2n-2}|c|)+3\cdot 2^{n+1}(\gamma-1)RM^2|c|^2}\Big\}.$$

Denote $\mathcal{S}=\{(x,t)\in\mathcal{R}(P_0):\ q(x,t)\geq 0\}$. The parabolic boundary $\partial_p\mathcal{S}$ of $\mathcal{S}$ consists of a part Σ_1 lying in $\mathcal{R}(P_0)$ and a part Σ_2 lying on $\mathcal{R}(P_0)\bigcap\{t=t_0-c_0\}$.

Finally, one can choose ε. On Σ_2, $w(x,t)-M_0>0$. Note $q(\alpha,x,t)$ is bounded on Σ_2, one can choose ε_0 suitably small such that $\varphi(\varepsilon_0,\alpha_0,x,t)>0$ on Σ_2. On $\Sigma_1\setminus\{P_0\}$, $q(\alpha,x,t)=0$ and $w(x,t)-M_0>0$. Thus, $\varphi(\varepsilon_0,\alpha_0,x,t)>0$ on $\Sigma_1\setminus\{P_0\}$ and $\varphi(\varepsilon_0,\alpha_0,x_0,t_0)=0$. One concludes that

$$\begin{cases}\varphi(\varepsilon_0,\alpha_0,x,t)>0,\ \text{on } \partial_p\mathcal{S}\setminus\{P_0\},\\ \varphi(\varepsilon_0,\alpha_0,x_0,t_0)=0.\end{cases}\tag{4.29}$$

In conclusion, it follows from (4.28) and (4.29) that

$$\begin{cases}\rho_0\partial_t\varphi(\varepsilon_0,\alpha_0,x,t)+L\varphi(\varepsilon_0,\alpha_0,x,t)\geq 0, & \text{in } \mathcal{S},\\ \varphi(\varepsilon_0,\alpha_0,x,t)>0, & \text{on } \partial_p\mathcal{S}\setminus\{P_0\},\\ \varphi(\varepsilon_0,\alpha_0,x_0,t_0)=0.\end{cases}\tag{4.30}$$

In view of Lemma 4.2 and (4.30), the function $\varphi(\varepsilon_0, \alpha_0, \cdot, \cdot)$ attains its minimum at P_0 in $\bar{\mathcal{S}}$, thus

$$\frac{\partial \varphi(\varepsilon_0, \alpha_0, x_0, t_0)}{\partial t} \leq 0.$$

Note that q satisfies at P_0

$$\frac{\partial q(\alpha_0, x_0, t_0)}{\partial t} = 1.$$

Therefore

$$\frac{\partial w(x_0, t_0)}{\partial t} \leq -\varepsilon_0. \tag{4.31}$$

But, by the assumption, w attains its minimum at P_0, it follows that

$$\rho_0 \frac{\partial w(x_0, t_0)}{\partial t} \geq -Lw(x_0, t_0) \geq 0,$$

which contradicts (4.31). □

Now we come to the following global strong maximum principle which may be proved in a similar fashion as Proposition 2.2.

Proposition 4.2 *Suppose that* $w \in C^{2,1}(\Omega \times (0, T]) \bigcap C(\bar{\Omega} \times [0, T])$ *satisfies* (4.11)*. If* w *attains its minimum at some interior point* $P_0 = (x_0, t_0)$ *of* $\Omega \times (0, T]$*, then* $w(P) = w(P_0)$ *for any point* P *of* $\Omega \times (0, t_0]$*.*

We are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We establish the weak maximum principle, Hopf lemma and strong maximum principle for the general function w satisfying the differential inequality (4.11), which also apply to the solution $\mathfrak{e}$ to (4.5) since $\mathfrak{e}$ also enjoys (4.11). Since $\mathfrak{e}_0 \geq 0$ and $\mathfrak{e}_0 \not\equiv 0$ in Ω, and $\mathfrak{e} = 0$ on $\partial\Omega \times (0, t_0]$ due to (4.5), by Proposition 4.2, it holds that $\mathfrak{e} > 0$ in $\Omega \times (0, T]$. Taking any point (x_0, t_0) of $\partial\Omega \times (0, T]$, applying Proposition 4.1, we obtain $\frac{\partial \mathfrak{e}(x_0, t_0)}{\partial \vec{n}} < 0$, which contradicts $\mathfrak{e}_{x_i}(x_0, t_0) = 0$ on $\partial\Omega \times (0, T]$ due to (4.5). □

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