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# Optimal Investment Strategies under Decision-Dependent Stochastic Environments

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# Problem description

*How do decision-dependent changes to a firm's stochastic environment affect optimal investment strategies and the role of uncertainty?*

*Paper 1: What is the optimal entry strategy for a firm that can alter the growth rate of a potential project through investment, when the project is subject to a stochastic environment? How is the strategy affected if this revenue-enhancing investment is undertaken prior to committing to launching the project?*

*Paper 2: What is the optimal investment strategy of a firm in switching from an established product to bringing a new product on the market, if this switch changes both the growth and volatility of the firm's profits? What is the role of the pre- and post-investment uncertainty on the optimal strategy?*

# Preface

This thesis serves as the concluding work for a Master of Science degree at the Norwegian University of Science and Technology (NTNU). The degree is a specialization in Financial Engineering at the Department of Industrial Economics and Technology Management.

I would like to give my sincerest acknowledgements to my supervisor Associate Professor Verena Hagspiel for excellent guidance, invaluable comments, and interesting discussions. Further, I would like to thank Claudia Nunes and Dharma Kwon for good advice and references on the more technical parts of this thesis. I am also grateful for the cordial welcome of the research group at the department, and all the interesting discussions with everyone in the group.

Tord Olsen
Trondheim, June 2018

# Summary

This thesis investigates the optimal investment decisions of a firm, when the characteristics of the firm's stochastic environment are dependent on these decisions. We model situations where the market the firm operates in responds to the investment conducted by the firm, and therewith affects the characteristics of the firm's stochastic profit flow. The thesis is comprised of two papers, which focus on different specifications of such investment problems.

The first paper considers the case where a firm has the opportunity to enter a novel market. Prior to that it has the possibility to undertake a revenue-enhancing pre-investment. We find that there is an incentive for the firm to invest in two discrete steps, undertaking the revenue-enhancement before the investment to enter the market. We find that the relationship between uncertainty and investment timing can be ambiguous. Increased uncertainty can both delay or accelerate the investment decision for the revenue-enhancing activity. This is due to two counteracting effects of uncertainty. On the one hand, higher uncertainty increases the value of waiting to invest, which delays investment. On the other hand, it increases the value of the embedded option, which accelerates the investment. Which effect dominates is dependent on the cost parameters, the parameters for the economic environment, and the degree to which the firm can affect the stochastic environment through its revenue-enhancing activity.

The second paper considers a firm who is currently operating in a market with an established product. It has a one-time opportunity to introduce a novel product, and therewith switch to a new product market. The profits of the new product have a different growth rate and are exposed to a different volatility than that of the initial product. Thus, investing to bring the new product to market changes the characteristics of the firm's stochastic environment. We find that the existence of a finite optimal investment threshold is dependent on the interplay of the relative change in growth and volatility for the two products. Further, the effect of the uncertainty of the initial product on the optimal investment strategy is ambiguous, and depending on two contradicting effects. Increasing the pre-investment volatility increases the option value of waiting, which delays investment. However, this also increases the relative attractiveness of the second product compared to the first product, which therefore accelerates investment. Lastly, we show that the problem of a one-time investment opportunity with changing characteristics of the stochastic environment and constant profit function, is equivalent to a problem of changing profit function and constant characteristics of the environment.

# Sammendrag

I denne oppgaven studerer vi den optimale investeringsstrategien for et firma, når disse investeringene påvirker firmaets usikre omgivelser. Vi modellerer situasjoner hvor et firmas marked responderer på de investeringene firmaet gjennomfører, og at dette påvirker firmaets inntjening i markedet. Oppgaven består av to forskningsartikler, som fokuserer på ulike spesifikasjoner av et slikt investeringsproblem.

I den første artikkelen tar vi for oss et firma som har muligheten til å gå inn i et nytt marked. I forkant av dette har firmaet mulighet til å gjennomføre en inntektsøkende førinvestering. Vi ser at det er et insentiv for firmaet å investere i to steg, hvor det investerer i å påvirke de potensielle inntektene før det faktisk investerer for å gå inn i markedet. Vi finner at forholdet mellom usikkerhet og optimalt investeringstidspunkt kan være tvetydig. Økt usikkerhet kan både utsette eller fremskynde investeringsbeslutningen. Dette stammer fra to motstridende effekter av usikkerhet. Høyere usikkerhet øker verdien av å vente, som utsetter investering, mens det også øker verdien av den underliggende opsjonen til å investere i neste steg, som fremskynder investeringen. Hvilken av disse effektene som dominerer er avhengig av kostnadsstrukturen, parameterne for firmaets usikre økonomiske omgivelser, og i hvilken grad firmaet kan påvirke omgivelsene med sine inntektsøkende aktiviteter.

I den andre artikkelen studerer vi et firma som er aktivt i et marked med et etablert produkt. Firmaet har muligheten til å introdusere et nytt produkt, og dermed bytte sitt produkttilbud i markedet. Inntektene fra det nye produktet har annen forventet vekst og er eksponert for en annen volatilitet en inntektene fra det gamle produktet. Investering i å introdusere dette produktet vil derfor endre firmaets usikre omgivelser. Vi viser at eksistensen av en optimal investeringsterskel er avhengig av samspillet mellom endringene i forventet vekst og endringen i volatilitet for produktene. Videre er forholdet mellom usikkerhet og den optimale investeringsstrategien tvetydig. Økt volatilitet i det gamle markedet øker opsjonsverdien av å vente, som utsetter investering, men øker også den relative attraktiviteten til det nye markedet, som fremskynder investering. Vi viser at en investeringsmulighet hvor beslutningen påvirker firmaets omgivelser, men ikke inntektsstrukturen, er ekvivalent med muligheten til å påvirke inntektsstrukturen gjennom investering, men ikke påvirke omgivelsene.

# Contents

# 1 Introduction

Most models that consider investment under uncertainty make the assumption that the stochastic environment of the firm is exogenously given, and that the characteristics of the environment do not change (see Dixit and Pindyck (1994) for a survey of the field). These studies view the firm as a passive actor with respect to its stochastic environment, subject to some given process that drives its profits. However, firms are heterogeneous and have idiosyncratic capabilities and opportunities to actively affect their success in their own markets. Firms adapt their strategies and product portfolios in face of changing market conditions and possibilities. A firm can undertake strategic steps to boost its demand and growth by exploiting its idiosyncratic competitive advantages (McGrath et al., 2004). Hence, a firm's strategic decisions and the market within which it operates, may not be independent. We argue that investment models should take this into consideration.

In this thesis, we strive to account for endogenous effects of a firm's actions on its prospective revenues, through decision-dependent changes to the characteristics of its stochastic environment. The thesis presents two papers which deal with the overarching theme in different contexts. The investment opportunity and decision-logic considered are particular to each paper. Specific motivational examples are given in each of the two papers, in relation to the problem setup considered within. From a modelling point of view, including such endogenous effects on a firm's stochastic environment allows for studying a wider set of realistic problem settings, relative to the case where such dependencies are omitted.

The first paper studies the case of a firm with the possibility of entering a new market. Here we consider that the firm can affect the characteristics of its profits in the market through some revenue-enhancing pre-investment, before actually committing to enter the market. This investment allows the firm to boost the expected growth of the firm's profits in the market, but does not affect the volatility of the profits. Such an effect on the market can, e.g., be achieved through activities like marketing, lobbyism, joint venture investments, bundling-contracts, etc. The paper looks at two cases for the change in the market characteristics. One where the change in drift is fixed, and one where the firm can choose the intensity of the revenue-enhancing activity they undertake, in effect optimizing the change induced in its market. This paper builds on the work undertaken in *TIØ4550 Financial Engineering, Specialization Project* during Fall 2017. We extend this work with important results regarding the optimal strategies for the case of controlled intensity of revenue-enhancement. Furthermore, extensive robustness tests are performed, where we consider different specifications of how the revenue-enhancing activities of the firm affect the market characteristics, and different cost specifications for those activities.

The second paper studies the optimal investment strategy for a firm that is currently active in the market with an initial product, but has the opportunity to introduce a new product. We assume that this introduction will change both the growth and volatility of the firm's profits. This paper represents an early effort to include decision-dependent changes to both growth and volatility. This expands earlier work wherein only the volatility is affected (Alvarez and Stenbacka, 2003), or when only the drift is altered (Kwon, 2010; Hagspiel et al., 2016).

Both papers in this thesis consider endogenous changes to a firm's stochastic environment resulting from the firm's actions. This allows us to study cases where the impact of the firm on its own market is endogenous, rather than the characteristics of the profits being independent of the firm's actions. The results from including these decision-dependent changes to the characteristics of the stochastic environment suggest that the connection between uncertainty and investment timing is more complex than the investment logic from traditional modelling approaches indicates.

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# 2 Revenue-enhancing pre-investment under uncertainty

**Abstract:**
Models of investment under uncertainty mostly concern the firm's stochastic environment as exogenously given and subject to constant characteristics. We consider a firm that can sequentially invest to alter the growth rate of a project through a revenue-enhancing pre-investment activity prior to entering a new market, both when the change is fixed and when the magnitude of the change can be optimally chosen by the firm. We find that this incentivises the firm to invest early in revenue-enhancing activities, and then wait to invest to enter the market. There is both an option value of waiting that delays investment in revenue-enhancing activities, as well as an accelerating effect from the change in growth rate. The overall effect on the investment thresholds from increased uncertainty is ambiguous. Which effect dominates is dependent on both the cost parameters and the magnitude of the change in the rate of growth. When the firm can optimally choose the amount of the revenue-enhancing activity, we find that the firm invests more in these activities when uncertainty is higher, but the effect of uncertainty can still be ambiguous. When the marginal cost of the activity increases, the firm both delays the investment and undertakes less revenue-enhancement, but the overall amount spent increases. We conclude that increasing the drift through revenue-enhancing pre-investments is very attractive for the firm, and that this affects the firm's optimal investment strategy.

## 2.1 Introduction

In 2014, The Panasonic Corporation entered into a joint-venture with Tesla Motors Inc. on the *Tesla Gigafactory* project. The venture is a strategic alliance and R&D effort between the two firms in order to position Panasonic for higher long-term growth in a novel market. The president of Panasonic, K. Tsuga, has stated that they *"see the rechargeable battery business as the biggest growth driver. So we are aggressively making an upfront and strategic investment here"*[1]. Through the alliance with Tesla, Panasonic has invested to obtain a favorable position for capturing higher profits in the potential future of high-volume production of lithium-ion batteries for electric vehicles (EVs) and household electricity storage. Therewith, Panasonic has taken an upfront and proactive stake in the development of the EV-market, possibly obtaining a larger profit growth in the future than they would have by waiting passively for the market to develop and only supply EV-producers with battery-cells.

Strategic alliances, marketing campaigns, lobbyism, standard-captures, and other pre-launch activities may influence the growth potential of a project. Thus, a firm can effectively be proactive in its existing and potential markets, influencing the expected growth of new projects before they are installed. The idea that a firm can enhance the potential revenue of a project by undertaking some strategic pre-investment actions, models what McGrath (1997) refers to as *amplifying pre-investments.* Such actions could be aimed at affecting the revenue potential of the product, the adoption rate, or the likelihood of imitation or competing products taking shares of the market.

In this paper, we focus on actions that increase the revenue potential through active investments, changing the firm's future environment favourably. We study two different scenarios: one where the revenue-potential is subject to a fixed change after the firm undertakes a pre-amplifying investment, and one where the firm can choose the intensity of this investment optimally, effectively deciding to what degree it should boost the revenue-potential in the market.

This paper contributes to the strategy literature by formally modelling revenue-enhancing pre-investment opportunities, and investigating their effects. Furthermore, we add to the modelling literature by including such dependencies of the stochastic environment on the firm actions. Regarding the second strand of literature, investment problems under uncertainty are widely studied using the real options approach. However, most models consider the underlying process driving the uncertainty as exogenously given. Dixit and Pindyck (1994) present many of the early models, while Trigeorgis (1996) presents models for portfolios comprised of different options on the same real asset, i.e. embedded options. In the strategy literature, real options reasoning has been used in decision-heuristics. Examples are the score-based questionnaire in McGrath and MacMillan (2000) of mapping a project's possibilities and threats, or the mixed decision-tree analysis and scoring of MacMillan et al. (2006). The field of real options analysis was initiated by Myers (1977), who noted that the presence of uncertainty in cash-flows affects corporate expenditure decisions. A common assumption of the uncertainty in real option

[1] Teslarati.com, 11.01.2016. The CEO of Panasonic comments on updated forecasts for projects in an earnings briefing to shareholders. https://www.teslarati.com/panasonic-tesla-battery-gigafactory-investment-growth-driver/, accessed 01.06.2018

models is that the underlying price or demand is following an exogenous stochastic process, often a geometric Brownian motion. Thus, the resulting resolution of uncertainty is purely a function of time, and beyond the control of the firm. Work on endogenous uncertainty is limited to problems of learning-type investments. In this strand of work, Pindyck (1993) regards projects with cost uncertainty and time-to-build, where the *technical uncertainty* can only be resolved through actually undertaking the project. Such uncertainty relates to the physical difficulty of completing a project, affecting, for example, the final amount of an input factor. Hence, technical uncertainty represents endogenous resolution of the uncertainty, dependent of the firm's action, where the uncertainty is not only resolved through time, but also through investment. Another approach in the literature to model endogenous actions of the firm is to allow the exogenous stochastic process to be partly unobservable. The firm must then undertake costly learning activities to assess the true state of the market. Kwon and Lippman (2011), for example, consider a firm that undertakes a small-scale pilot project to infer the full project's profitability. The firm observes a noisy profit flow from the pilot and from this updates the belief of the market state in a Bayesian fashion. The firm must then consider the decision to expand the pilot project or exit. Thijssen et al. (2004) consider a similar situation, where the firm at random times receives imperfect signals from the market and uses these signals in updating its beliefs. The trade-off for the firm is therefore between waiting longer to reduce the uncertainty of the market state, and investing immediately to reap potential profits. This approach is further investigated in a model by Harrison and Sunar (2015), where the firm can adopt different learning modes that affect the quality of the obtained market signals, incurring cost at different rates, dependent on the choice of learning mode. The papers presented above represent examples of earlier models and extensions for including endogenous actions of the firm w.r.t. the stochastic environment.

The aforementioned models are all characterized by endogenous revelation of an exogenous uncertainty process. The firm takes an active role in learning about the uncertainty, but has no means of actually affecting its own environment. There is a clear gap in the literature of real options modelling in studying endogenous influence on the stochastic environment, which is already noted by Adner and Levinthal (2004). Adner and Levinthal (2004) argue that firms take steps to affect the attractiveness of possibilities, either by changing the technical agenda of the project or altering the target market. They further argue that the assumption of exogeneity can be seen as a "wait-and-see" approach to the investment problem. This critique is also valid for the endogenous uncertainty resolution approaches mentioned above, as the firm has no means to change the market state or adapt to it should the market belief turn out unfavourable. In a response to Adner and Levinthal (2004), McGrath et al. (2004) argue that the real options heuristics utilized in the strategy literature can give insights into how upside potential can be enhanced by strategic actions or redirecting projects. Nevertheless, the authors concur with Adner and Levinthal (2004) that further work on endogenous uncertainty resolution and influence is important for advancing the real options approach to investment analysis. We translate this to our modelling approach to allowing for the decision-maker to affect the stochastic process the firm is subject to, through undertaking some specified investments.

In this paper, we aim to address the aforementioned shortcomings of dealing with endogeneity in a real options approach. We analyze how the opportunity for a firm to undertake strategic pre-investment actions to alter a project's growth potential affects the investment behaviour and profits of the firm. This represents a shift from seeing the firm as a passive actor, subject to an exogenous market process, to allowing the firm to proactive and effectively shape its own growth potential through strategic investments. The work on real options subject to stochastic processes with changing parameters is generally very limited, and to the best of our knowledge restricted to one exogenously specified fixed change after an investment. Kwon (2010) studies a firm producing an aging product subject to a downward trending demand, with the possibility to innovate once. The uncertainty is modelled as an arithmetic Brownian motion, with a fixed change in drift if the firm innovates. It might be optimal for the firm to cease operations and exit, or to innovate once to boost the profits. The new product would obtain a higher, but still negative, drift rate if undertaken, thus making an eventual exit of the market inevitable. The effect of uncertainty on the optimal strategy of the firm is found to be non-monotonic, which contradicts the standard result of investment under uncertainty, that higher uncertainty delays investment. Matomäki (2013, Article 1) extends Kwon (2010) for more general stochastic processes, as well as including changes to the volatility of the process, and the results regarding the effect of uncertainty are in line with Kwon (2010). Further, Hagspiel et al. (2016b) expands the setting of Kwon (2010) to allow for capacity choice for the new product, while still holding the change in drift rate for the stochastic demand process as exogenously given. Including capacity choice yields a monotonic effect of uncertainty on the optimal investment timing. The firm invests in larger capacity when uncertainty increases, which then gives an incentive to invest later. Another approach taken in the literature is wherein the volatility of the stochastic process is changed after a certain investment. Herein, Alvarez and Stenbacka (2003) consider a setting where the investment changes the volatility of the firm's environment, while keeping the drift unchanged. They find a non-monotonic effect of uncertainty on the investment threshold. However, allowing for change in the volatility of the process necessitates the use of more advanced mathematical tools. This is left for future work for our problem, as we study a sequential investment problem, with embedded options. This is a complication relative to the single-investment case studied in Alvarez and Stenbacka (2003).

In this paper, the market is characterized by an uncertain price. The price follows a geometric Brownian motion, with a change of drift at the time the firm undertakes the revenue-enhancing investment. We introduce two models: (1) where the change in drift is fixed, meaning that the drift rate is boosted to a specified level when the firm invests in the revenue-enhancing activity. (2) where the change is dependent on the amount of the revenue-enhancing activity the firm undertakes, where the intensity of the activity determines the degree to which the drift is boosted. The first model presents an optimal stopping problem subject to a changing stochastic process, while the second model is a joint optimal stopping and impulse problem, where the change in drift is controlled by the firm (Vollert, 2012). The problem concerns the optimal investment strategy of sequential investment under uncertainty, as the firm can invest in the revenue-enhancing activity and in entering the market at two separated points in time. We find that a fixed change in drift incentivises the firm to invest sequentially, i.e. to invest in revenue-enhancing activities initially and then wait and hold the option to actually finish the project. This is in contrast to the similar two-stage sequential model in Dixit and Pindyck (1994) with constant drift, who find that the firm will never invest sequentially when there is no time-to-build. The incentive is increasing with the magnitude of the boost in drift. We show that the effect of uncertainty is not straightforward. Increasing uncertainty can both delay or accelerate the investment in revenue-enhancing activities. In the case where the firm can optimally choose the magnitude of the change in drift, we find that the firm invests more in revenue-enhancement when uncertainty increases. Further, when the marginal cost of this activity increases, the firm undertakes less revenue-enhancement, while the total amount spent on boosting the drift increases. We check the robustness of these results, considering a more general specification of the effect of the revenue-enhancing activity on the drift rate, and both concave and convex cost function for the activity. Extensive numerical analysis confirms the robustness of the results of uncertainty on investment.

The rest of the paper is organized as follows. An investment model with a fixed change in the drift of the price process is presented in Section 2.2. In Section 2.3, we extend this approach by letting the firm control the change of drift through the size of the investment. In Section 2.4 we perform a robustness analysis of the impulse function and the cost of the revenue-enhancing activity. Section 2.5 summarizes the results. Additional derivations are presented in Appendix 2.A, while Appendix 2.B presents proofs of all propositions and corollaries.

## 2.2 Investment under fixed change in drift

We consider the investment decision of a monopolist firm with an opportunity to enter a novel market. The market is characterized by a stochastic price process, with a fixed change in market growth triggered by the firm's investment in revenue-enhancing activities. The firm is currently not active, but has the option to irreversibly invest in order to enter the market. The uncertainty of the investment opportunity is characterized by a pair of price processes following geometric Brownian motion, as given by Eq. (2.1).

$$dP_1 = \alpha_1 P_1 dt + \sigma P_1 dz, \tag{2.1a}$$
$$dP_2 = \alpha_2 P_2 dt + \sigma P_2 dz. \tag{2.1b}$$

In Eq. (2.1), $dz$ denotes the increment of a standard Wiener process and $\sigma$ the volatility, equal for both geometric Brownian motions. $\alpha_1$ and $\alpha_2$ are the drift parameters for the first and second process, respectively. The second price process, $P_2(t)$, starts at the moment of a specified investment action of the firm at time $\tau$, with initial value equal to $P_1(\tau)$, i.e. $P_2(0) = P_1(\tau)$. A sample path of the price process is illustrated in Fig. 2.1. After the change in drift has occurred, $P_1(t)$ is irrelevant. Further, we assume an appropriate discount rate, $\rho$, for the project and assume that $\alpha_1 \leq \alpha_2 < \rho$. This assumption allows us to disregard the trivial situation where it would never be optimal for the firm to enter the market, as the expected growth is larger than the discount factor and therefore postponing the investment decision would always be optimal. Our model is similar to that presented by Kwon (2010) and Hagspiel et al. (2016b), where a producing firm is subject to a declining market, with the possibility to innovate once and boost the drift. However, in the mentioned works the boost in drift only postpones the inevitable exit of the market. Conversely, our problem is concerned with the decision of entry rather than exit, and not restricted to a declining market. Further, the two-stage sequential investment model of Dixit and Pindyck (1994, Chapter 10) is a special case of the problem studied here, with $\alpha_1 = \alpha_2$ in Eq. (2.1).

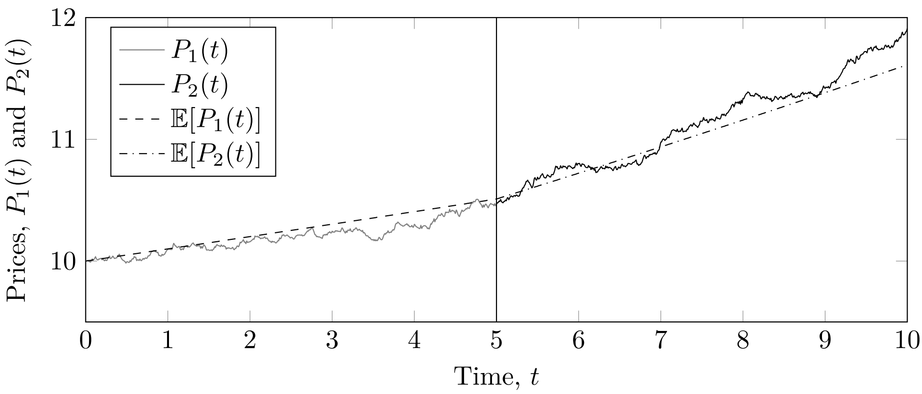



Figure 2.1: One sample path of $P_1(t)$ and $P_2(t)$, with $\alpha_1 = 0.01$, $\alpha_2 = 0.02$, $\sigma = 0.01$. Change of processes at $t = 5$.

In Section 2.2.1, we first outline a two-stage sequential investment problem, where the firm completes the project in two discrete steps. This describes a situation where the firm undertakes some initial investment, e.g. marketing or lobbyism, prior to actually completing the project. After that, the firm decides when to undergo the second investment to enter the market. Under certain assumptions, this model reduces to a single-investment problem, as outlined in Section 2.2.2, which represents a simplified case where all revenue-enhancing activities must be conducted at the same time as entering the market. We present this simplification as it helps to build intuition for the controlled single-stage investment problem presented in Section 2.3. In Section 2.2.3, results for the comparative statics analysis for both models are presented.

### 2.2.1 Two-stage investment with fixed change of drift

In a two-stage sequential investment model, the firm may first undertake an initial investment at a fixed cost $I_1$. The first-stage investment has the effect of increasing the drift from $\alpha_1$ to $\alpha_2$ by switching the price process in Eq. (2.1) from $P_1(t)$ to $P_2(t)$. The firm then obtains the option to invest in the second stage to complete the project at a fixed cost $I_2$. After the second investment is undertaken, the project is assumed to generate one unit of output per time period, at price $P_2(t)$, in perpetuity. Without loss of generality, we assume zero operating costs. Then the discounted value of a fixed operating cost can be incorporated into the investment cost, $I_2$. Thus, the per period profit is given by $\pi(P_2(t)) = P_2(t)$. The value of the firm can then be found as the solution to the following optimal stopping problem

$$\begin{aligned} F(P_1) = \sup_{\tau_1} \ \mathbb{E}^{P_1}\Bigg[ - I_1 e^{-\rho\tau_1} + \sup_{\tau_2 \mathbf{1}_{\{\tau_2>\tau_1\}}} \Bigg\{ \\ + e^{-\rho\tau_2} \times \mathbb{E}^{P_2}\left[\int_{\tau_2}^{\infty} e^{-\rho(t-\tau_2)}\pi(P_2(t-\tau_1))dt\right] - I_2 e^{-\rho\tau_2}\Bigg\}\Bigg]. \end{aligned} \tag{2.2}$$

In Eq. (2.2), $\tau_1$ denotes the optimal stopping time of undertaking the first-stage investment to improve the drift of the price process. $\tau_2$ denotes the optimal stopping time of the second-stage investment, at which the firm enters the market. Thenceforth, the firm earns the per period profit flow, $\pi(P_2(t))$. Further, the expectation operators denote that the expectations are conditional on the defined starting values, i.e. that $\mathbb{E}^{P_1} \equiv \mathbb{E}[\cdot|P_1(0) = P_1]$ and $\mathbb{E}^{P_2} \equiv \mathbb{E}[\cdot|P_2(0) = P_1(\tau_1)]$. The solution to the optimal stopping problem in Eq. (2.2) is characterized by the investment thresholds $P_1(\tau_1) = P_1^*$ and $P_2(\tau_2 - \tau_1) = P_2^*$ of the stochastic price process. The starting point for $P_2(t)$ is given by the value of the geometric Brownian motion $P_1(t)$ at the time of the first investment, $\tau_1$, i.e. $P_2(0) = P_1(\tau_1) =$

$P_1^*$. The three regions of the two-stage investment problem are illustrated in Fig. 2.2. The optimal stopping problem given by Eq. 2.2 can be split into three elements: the expected discounted value of the completed project at the time of completion, denoted by $V(P_2(t))$; the value of the option to undertake the second-stage investment, denoted by $F_2(P_2(t))$; and the value of the opportunity to invest in the project's first stage, denoted by $F_1(P_1(t))$.

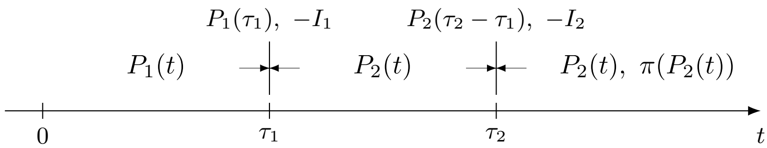



Figure 2.2: Regions for the two-stage sequential investment.

To find the value of the investment opportunity, $F_1(\cdot)$, we work backwards; first we derive the value of the completed project, $V(\cdot)$, and the option to invest in the second stage, $F_2(\cdot)$. Using conditions of continuity and smoothness of the value functions at their joint threshold, we obtain the second price threshold, $P_2(\tau_2 - \tau_1) = P_2^*$. Next, we find the value function of the first option, $F_1(P_1)$, and can therewith derive the optimal investment threshold of the first-stage investment, $P_1(\tau_1) = P_1^*$. The complete derivations of the value of the project are given in Appendix 2.A, while the option values, $F_1(\cdot)$ and $F_2(\cdot)$, are derived in Appendix 2.B.

The expected present value of the cash flows generated by the project, at the time of the second-stage investment, is given by

$$V(P_2) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} P_2(t) dt \mid P_2(0) = P_2\right] = \frac{P_2}{\rho - \alpha_2}, \tag{2.3}$$

where $P_2$ is the value of the price process at the time of investment. Thus, the completed project becomes more valuable if the drift-rate or the price at the time of investment increases. Proposition 2.2.1 presents the expression for the value of the option to invest in the second project stage.

**Proposition 2.2.1.** *The value of the option to undertake the second-stage investment, $F_2(P_2)$, is given by*

$$F_2(P_2) = \begin{cases} D_2 P_2^{\beta_{12}} & \text{if } P_2 < P_2^*, \\ V(P_2) - I_2 & \text{if } P_2 \geq P_2^*, \end{cases} \tag{2.4}$$

*where*

$$\beta_{12} = \frac{1}{2} - \frac{\alpha_2}{\sigma^2} + \sqrt{\left(\frac{\alpha_2}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}, \tag{2.5}$$

*and*

$$D_2 = \frac{1}{(\rho - \alpha_2)\beta_{12}} \left[\frac{\beta_{12}}{\beta_{12} - 1}(\rho - \alpha_2) I_2\right]^{1-\beta_{12}}. \tag{2.6}$$

*The optimal threshold for investing in the second project stage is given by*

$$P_2^* = \frac{\beta_{12}}{\beta_{12} - 1}(\rho - \alpha_2) I_2. \tag{2.7}$$

Proposition 2.2.1 shows that the value of the second-stage option is dependent on the price, $P_2$. If the price is lower than the threshold, the value stems from the option to invest in the second project stage at a later time, i.e. the firm is in the continuation region of the second option. If the price is higher than the threshold, the second project stage is undertaken, and the firm obtains the expected discounted value of the project, net investment cost. This represents the stopping region of the second-stage option.

Proposition 2.2.2 below presents the value of the option to invest in the first project stage. Here we have to distinguish between two cases. If the first investment threshold, $P_1^*$, is smaller than the second, $P_2^*$, the firm obtains the second-stage option to invest in the completed project if it exercises the first-stage option. However, if the first investment threshold is the largest, the firm would undertake both stages concurrently, and the firm receives the expected present value of the cash flows from exercising the first-stage option.

**Proposition 2.2.2.** *If the first investment threshold is lower than the second, i.e. $P_1^* < P_2^*$, where $P_2^*$ is given in Eq. (2.7), the value of the firm, $F_1(P_1)$, is equal to*

$$F_1(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^*, \\ D_2 P_1^{\beta_{12}} - I_1 & \text{if } P_1 \geq P_1^*, \end{cases} \tag{2.8}$$

*where $\beta_{12}$ and $D_2$ is given by Eq. (2.5) and Eq. (2.6), respectively, and*

$$\beta_{11} = \frac{1}{2} - \frac{\alpha_1}{\sigma^2} + \sqrt{\left(\frac{\alpha_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}, \tag{2.9}$$

$$D_1 = \frac{\beta_{12}}{\beta_{11}} D_2 \left[\frac{I_1}{(1 - \frac{\beta_{12}}{\beta_{11}})D_2}\right]^{1-\frac{\beta_{11}}{\beta_{12}}}. \tag{2.10}$$

*The first-stage investment threshold is then given by*

$$P_1^* = \left[\frac{I_1}{(1 - \frac{\beta_{12}}{\beta_{11}})D_2}\right]^{\frac{1}{\beta_{12}}}. \tag{2.11}$$

*If $P_1^* \geq P_2^*$, the value of the firm is given by*

$$F_1(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^*, \\ V(P_1) - I_1 - I_2 & \text{if } P_1 \geq P_1^*, \end{cases} \tag{2.12}$$

*where $\beta_{11}$ and $V(\cdot)$ is given by Eq. (2.9) and Eq. (2.3), respectively, and*

$$D_1 = \frac{1}{(\rho - \alpha_2)\beta_{11}} \left[\frac{\beta_{11}}{\beta_{11} - 1}(\rho - \alpha_2)(I_1 + I_2)\right]^{1-\beta_{11}}. \tag{2.13}$$

*Then, the first investment threshold is given by*

$$P_1^* = \frac{\beta_{11}}{\beta_{11} - 1}(\rho - \alpha_2)(I_1 + I_2). \tag{2.14}$$

The ordering of the investment threshold in Proposition 2.2.2 is dependent on all underlying parameters. The following corollary shows that the ordering of the thresholds is unique, which implies that only one of the cases $P_1^* < P_2^*$ and $P_1^* \geq P_2^*$ holds true for the stated expressions in Eq. (2.11) and Eq. (2.14), when compared to the expression of the second threshold in Eq. (2.7).

**Corollary 2.2.3.** *In the sequential investment problem in Proposition 2.3.3, only one of the cases for the ordering of the threshold give an admissible solution. Further, if it holds that*

$$I_1 < \frac{\beta_{12}}{\beta_{12} - 1} \times \left(\frac{1}{\beta_{12}} - \frac{1}{\beta_{11}}\right) \times I_2, \tag{2.15}$$

*then $P_1^* < P_2^*$.*

If the first threshold is the lowest, the firm will invest in the initial project stage as soon as the price process, $P_1(t)$, is larger than $P_1^*$, increasing the drift of the project. After this it is optimal to wait until the second price process, $P_2(t)$, reaches the second threshold, $P_2^*$, before investing in the last stage to complete the project. If the other case holds true, the firm will undertake both project stages at the same time as soon as the first price threshold is reached. Note that we assume no time-to-build, and that the investment is instantaneous. Corollary 2.2.4 show the dependence of the solution to the values of the drift parameters.

**Corollary 2.2.4.**

*i) If $\alpha_2 > \alpha_1$, then $\beta_{12} < \beta_{11}$, and Eq. (2.11) and Eq. (2.10) are well-behaved.*

*ii) If $\alpha_1 = \alpha_2$, then $\beta_{11} = \beta_{12}$, and $P_1^* > P_2^*$ always holds.*

Corollary 2.2.4 states that the expressions for the value function and the first threshold price are well-behaved for the cases considered in this paper, and take on positive and real values. Further, in the case of constant drift, the model reduces to the sequential investment problem presented by Dixit and Pindyck (1994, Ch. 10.1), albeit without suspension and operational costs. As Dixit and Pindyck (1994) find, under the assumption of no time-to-build, there is never an incentive to invest in two stages when the drift rate before and after revenue-enhancing investment does not change. We show that when the firm has the opportunity to boost the drift, there is an incentive to invest in two separate stages for certain parameter ranges.

## 2.2.2 Single-investment problem reduction

Note that if the second-stage investment cost is set to zero, the two-stage sequential investment problem reduces to that of a single-investment opportunity. We present this model simplifaction here as it will serve as a comparison for the single-investment problem with controlled increase in the drift, presented in Section 2.3.1. Let $I$ denoting the total cost of both revenue-enhancement and market entry. Then, the reduced model can be represented by the following optimal stopping problem

$$F(P_1) = \sup_{\tau} \mathbb{E}^{P_1}\left[-Ie^{\rho\tau} + e^{-\rho\tau} \times \mathbb{E}^{P_2}\left[\int_{\tau}^{\infty} e^{-\rho(t-\tau)}\pi(P_2(t-\tau))dt\right]\right]. \quad (2.16)$$

The solution to the reduced optimal stopping problem in Eq. (2.16) is also of a threshold-type, characterized by the investment threshold $P_1^*(\tau)$, where $\tau$ is the time of the investment. Now the increase in drift and the onset of the profit flow occur at the same time. The solution to this problem is given by Eq. (2.12)–(2.14) in Proposition 2.2.2, with $I_1 = I$ and $I_2 = 0$. The reduction of the problem is evident in Eq. (2.7) and Eq. (2.14): if $I_2 = 0$, then $P_2^* = 0$ and $P_1^*$ can never be smaller than $P_2^*$. This is intuitive. If the second-stage investment has zero cost, the project will be completed as soon as the threshold for the first-stage investment is reached.

### 2.2.3 Comparative statics results

In this section, we present the results of a comparative statics analysis for the optimal investment thresholds $P_1^*$ and $P_2^*$.

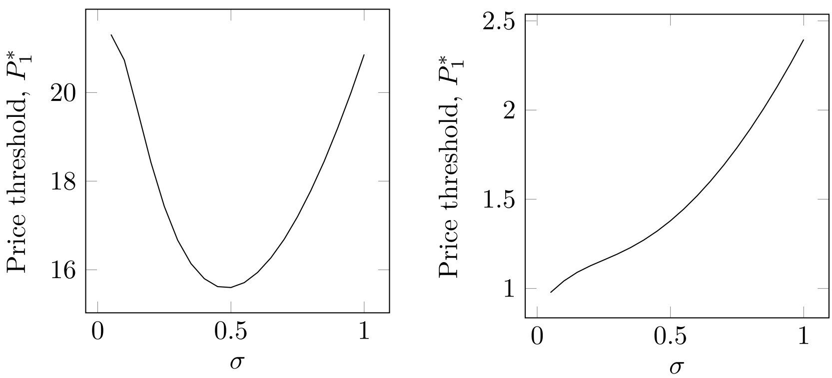
**Proposition 2.2.5.** *The optimal threshold of the option to invest in the second project stage, $P_2^*$, is increasing in $\sigma$. The first price threshold, $P_1^*$, is increasing in $\sigma$ if $P_1^* \geq P_2^*$. If $P_1^* < P_2^*$, then $P_1^*$ is increasing in $\sigma$ if the following condition holds,*

$$\frac{I_2}{I_1} < \frac{\beta_{11}(\beta_{12}-1)}{\beta_{11}-\beta_{12}} \exp\left[\frac{\beta_{11}}{\beta_{11}-\beta_{12}}\left(\frac{(\frac{1}{2}\sigma^2(2\beta_{12}-1)+\alpha_2)(\beta_{11}-1)}{(\frac{1}{2}\sigma^2(2\beta_{11}-1)+\alpha_1)(\beta_{12}-1)}-1\right)\right]. \tag{2.17}$$

*Otherwise, $P_1^*$ is decreasing in $\sigma$.*

Proposition 2.2.5 shows that the standard result in real options theory of the effect of increased uncertainty might not be true for the initial investment, if the condition in Eq. (2.17) does not hold. When it holds, the firm demands a higher price to invest in both stages when the uncertainty increases, which is the standard result of investment under uncertainty. We refer to this as the real options effect of uncertainty.

If the condition in Eq. (2.17) does not hold, a higher uncertainty accelerates the investment in the initial project stage. This is contradictory to the standard real options results (Dixit and Pindyck, 1994). Due to the complexity of the condition in Eq. (2.17), the impact of the different problem parameters cannot be determined easily. The condition is a function of the volatility, $\sigma$, both directly and via the dependence in $\beta_{11}$ and $\beta_{12}$. We see from numerical studies that when the first investment cost is very small compared to the cost of the second project stage, the effect of increased uncertainty on the first threshold is ambiguous. Further, this ambiguity seems only to be present when the difference in drift is smaller than some level. Fig. 2.3 presents two different cases: one for a relatively small change in drift, and one for a relatively large boost in drift. Hence, there are two opposing effects of uncertainty on the first threshold. The real options effect yields that higher uncertainty gives a higher value of waiting, and delays investment, while the change in drift incentivises the firm to invest the first project stage to boost the drift. The effect of uncertainty is dependent on which effect dominates.




(a) Small change in drift, $\alpha_2 = 0.04$ (b) Large change in drift, $\alpha_2 = 0.08$

Figure 2.3: Sensitivity of first price threshold, $P_1^*$, w.r.t. uncertainty, when the first investment cost is small compared to the second, for two levels of the boosted drift. (Parameters: $I_1 = 10$, $I_2 = 1000$, $\rho = 0.1$, $\alpha_1 = 0.01$.)

An explanation for this non-monotonic effect of uncertainty is that the option to invest in the second project stage becomes more valuable with increased uncertainty. This is a standard real options result. Thus, when the first investment is relatively inexpensive, this increased option value of the subsequent investment stage outweighs the increased value of the insurance embedded in the first-stage option. A higher uncertainty makes the insurance arising from the optionality of the first project stage more valuable, which the firm forfeits if it invests. The effects of changing the level of the initial drift on the optimal investment thresholds are presented in Proposition 2.2.6.

**Proposition 2.2.6.** *The optimal threshold to invest in the first project stage, $P_1^*$, is increasing in initial drift, $\alpha_1$, while the second threshold, $P_2^*$, is unaffected by $\alpha_1$.*

The threshold for investing in the second project stage is independent of the initial drift level. This is intuitive, as the first investment has already been undertaken, and therefore the first price process, $P_1(t)$, is irrelevant at the time of the decision to undertake the second investment. The drift has already been boosted, so the initial level is not relevant for the firm's decision. The first investment threshold is increasing in $\alpha_1$. If the level of the boosted drift is kept constant, an increase in the initial drift is effectively decreasing the magnitude of the change. Thus, the firm has less incentive to invest in the first project stage, and demands a higher price to undertake the revenue-enhancing activities. Proposition 2.2.7 presents the effect of changing the level of the boosted drift on the optimal investment thresholds.

**Proposition 2.2.7.** *The optimal threshold to invest in the second project stage, $P_2^*$, is decreasing in the boosted drift, $\alpha_2$, if the following condition holds:*

$$\sigma > \sqrt{\frac{\rho - \alpha_1\beta_{12}}{(\beta_{12}-1)(\beta_{12}-\frac{1}{2})}}. \tag{2.18}$$

*The first investment threshold, $P_1^*$, is decreasing in $\alpha_2$ when $P_1^* \geq P_2^*$.*

The threshold for investing to enter the market is decreasing in the boosted drift, $\alpha_2$. The firm invests in the market at the threshold $P_2^*$ if $P_1^* < P_2^*$, while investing at the threshold $P_1^*$ if $P_1^* \geq P_2^*$. Increasing the boosted drift thus incentivises the firm to enter the market earlier, as the expected discounted value of the project increases.

When the threshold for the first project stage is lower than the second, i.e. $P_1^* < P_2^*$, we refrain to numerical results in order to analyze the effect of increasing the boosted drift on $P_1^*$. Remark 2.2.8 presents the effect, while a numerical example is presented in Table 2.1.

**Remark 2.2.8.** *The optimal threshold to invest in the first project stage, $P_1^*$, is decreasing in the boosted drift, $\alpha_2$, when $P_1^* < P_2^*$.*

Our numerical analysis suggests that the effect of the boosted drift on the first-stage investment threshold when $P_1^* < P_2^*$ is equal to that presented in Proposition 2.2.7. This is as expected, as increasing the boosted drift increases the expected discounted value of the project, which increases the value of the option to invest in the final stage. When $P_1^* < P_2^*$, investing in the first project stage represents exchanging the option to invest in the initial stage with the option to invest in the second stage. Upon investing, the firm forgoes the option to invest in the first project stage at a later time, but receives option to invest in the last stage at a later time. Thus, when the value of the underlying project increases, the option to complete the project becomes more valuable, which motivates the firm to invest earlier to obtain this second-stage option.

| $\alpha_2$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 49.09 | 43.63 | 32.20 | 23.32 | 16.29 | 10.73 | 6.45 | 3.32 | 1.21 |
| $P_2^*$ | 45.00 | 43.42 | 42.00 | 40.75 | 39.65 | 38.70 | 37.87 | 37.16 | 36.54 |

Table 2.1: Effect of increasing the boosted drift on the optimal investment thresholds. (Parameter values: $\rho = 0.1$, $\alpha_1 = 0.0$, $I_1 = 50$, $I_2 = 300$.)

## 2.3 Investment under controlled change in drift

We now consider a situation where the firm can optimally choose the drift of the second price process by incurring some additional cost at the time of the first-stage investment. The problem constitutes a joint optimal stopping and optimal impulse problem (Vollert, 2012). In this case the firm has both discretion over the stopping times, as well as direct influence over the parameters of the stochastic diffusion process that characterizes the market environment. This represents an extension of the model presented in Section 2.2.1 to make the firm active in its own potential market, endogenously affecting the stochastic environment.

The drift term of the second price process is now modelled as dependent on the amount $K$, i.e. $\alpha_2 = \alpha_2(K)$ in Eq. (2.1). We denote $\alpha_2(K)$ as the *impulse function*, and $K$ the *control*. Note that the impulse function must be specified so that $\alpha_1 \leq \alpha_2 < \rho$ still hold. The control $K$ represents the amount invested by the firm in order to boost the growth of the project by e.g. marketing or lobbyism. We assume that the firm does not obtain any boost in the drift without investing in revenue-enhancing activities and incurring some extra cost, i.e. $\alpha_2(K = 0) = \alpha_1$. Further, we do not allow disinvestment in revenue-enhancement, where $K < 0$ and $\alpha_2 < \alpha_1$. We model the impulse function, $\alpha_2(K)$, by

$$\alpha_2(K) = \rho - \epsilon - \frac{\rho - \epsilon - \alpha_1}{1 + K}, \tag{2.19}$$

where $\epsilon > 0$ is an offset-value that ensures that the drift rate, $\alpha_2(K)$, can never approach the discount rate, $\rho$. This parameter is introduced in order to avoid that the expected discounted value of the cash flows from the project can approach infinity. Practically, this implies that there is a maximum obtainable value of $\alpha_2$ that the firm can approach, but never attain, by investing more in revenue-enhancing activities. An example plot of Eq. (2.19) is illustrated in Fig. 2.4. The choice of function for $\alpha_2$ is adopted to model diminishing marginal effects on the drift from increasing the amount of revenue-enhancing actions, in line with practical reasoning that increasing the intensity of an investment cannot increase the rate of profit indefinitely. In Section 2.4 we present an alternative impulse function, and perform a robustness check of our results w.r.t. this choice.

As the firm can now optimally decide the intensity of the revenue-enhancing activity, the cost of the first investment stage is given as a function of the control. We model the first investment stage as having both a fixed and variable part, denoted by $I_K(K) = I_1 + \xi K$. Here $I_1 > 0$ is a constant fixed cost, and $\xi$ the constant marginal cost of the activity. The fixed part represents, for example, the minimum required marketing that needs to be done prior to launching a product, for which the firm does not obtain any boost in the market. For the second project stage, similar to Section 2.2, there is a fixed investment cost $I_2 > 0$.

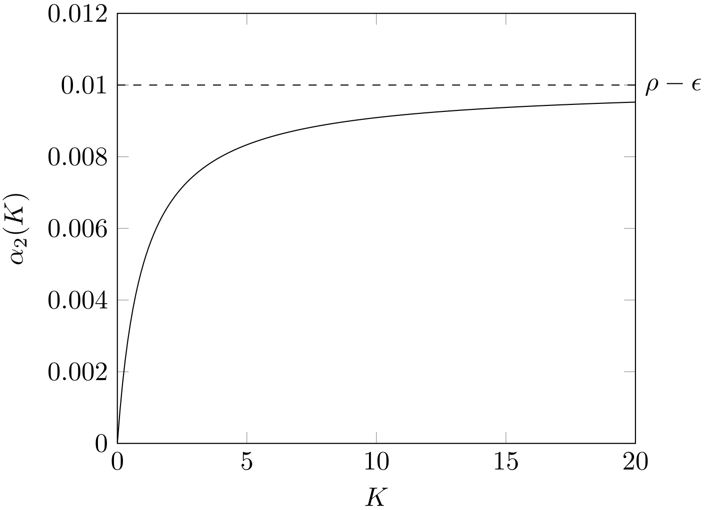



Figure 2.4: The assumed impulse function: the second drift-rate $\alpha_2$ plotted as a function of $K$ for parameters: $\rho = 0.04,\ \epsilon = 0.03,\ \alpha_1 = 0$.

In order to build intuition, we first consider a controlled single-investment problem in Section 2.3.1, where the firm undertakes the revenue-enhancing activities at the same time as committing to the project. This simplified problem will serve as a benchmark to understand how control over the change of drift affects threshold prices and values. Section 2.3.2 expands to the two-stage sequential investment with control over the drift at the time of the first project stage, i.e. the firm undertakes revenue-enhancing activities in the first project stage, but receives a flow of profits only after the second-stage investment. In Section 2.3.3, we present the comparative static analysis of the models.

### 2.3.1 Single-stage investment, controlled change of drift

We now consider a single-investment case, where the firm can choose the control $K$ optimally at the time of investment in order to boost the drift $\alpha_2$. The problem can be seen as a joint optimal stopping and control (impulse) problem, where the firm must decide when to invest and enter the market, and choose the amount of revenue-enhancing activities at the time of investment. The problem can then be

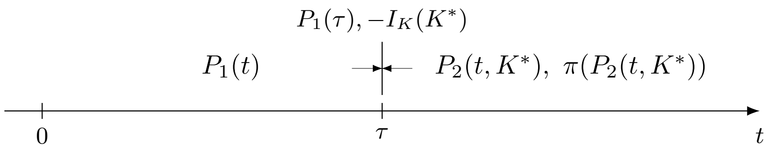



Figure 2.5: Regions for the single investment problem with controlled boost in drift.

written as

$$F(P_1) = \sup_{\tau} \mathbb{E}^{P_1}\Bigg[ - Ie^{-\rho\tau} + e^{\rho\tau} \times \max_{K\geq 0} \left\{ \mathbb{E}^{P_2}\left[ \int_{\tau}^{\infty} e^{-\rho(t-\tau)} \pi(P_2(t-\tau, K))dt - \xi K \right] \right\} \Bigg], \tag{2.20}$$

where the expectation operators are defined as $\mathbb{E}^{P_1} \equiv \mathbb{E}[\cdot|P_1(0) = P_1]$ and $\mathbb{E}^{P_2} \equiv \mathbb{E}[\cdot|P_2(0) = P_1(\tau)]$.

Fig. 2.5 illustrates the continuation and stopping regions of the investment problem. The problem is similar to the single investment problem presented in Section 2.2.2, with the difference that the firm can now decide on the optimal amount of the revenue-enhancing investment (the control variable), hereafter denoted by $K^*$. The solution procedure is as follows: we find the value function for a given $K$, then maximize the net project value with respect to $K$, for any given price $P_2$. Using the equation for the optimal value $K^*(P_2)$, we find the option value $F(P_1)$ and the threshold price $P_1^*$.

The expected value of the completed project, $V(P_2, K)$, is similar to before (see Eq. (2.3)), with the difference that the value function now becomes a function of $K$. Thus, the expected discounted value of the cash-flows generated by the project, at the time of investment, is given by

$$V(P_2, K) = \frac{1}{\rho - \alpha_2(K)} P_2. \tag{2.21}$$

At the time of the investment, the firm chooses the amount of the revenue-enhancing actions through the control $K$. Upon investment, the firm must pay the full investment cost of revenue-enhancement and market entry, i.e. $I_K(K) = I + \xi K$. Here $I = I_1 + I_2$ represent the fixed cost of both activities, and $\xi > 0$ the marginal cost of the revenue-enhancing activity. Thus, we want to find the value of $K$ that maximizes the value of the firm at the time of investment, given by

$$V(P_2, K) - I - \xi K = \frac{1}{\rho - \alpha_2(K)} P_2 - I - \xi K. \tag{2.22}$$

We find the optimal control $\hat{K}^*$ by the first-order optimality condition, and controlling that the second-order derivative is negative. Solving this maximization problem yields the value of the optimal revenue-enhancement and the resulting drift function, as functions of the price, $P_2$, presented in Proposition 2.3.1.

**Proposition 2.3.1.** *The optimal value of the control variable and the resulting optimal drift rate, as functions of the price $P_2$, are given by*

$$K^*(P_2) = \max\{0, \hat{K}^*\} = \max\left\{0, \frac{\xi(\alpha_1 - \rho) + \sqrt{P_2 \xi(\rho - \epsilon - \alpha_1)}}{\xi \epsilon}\right\}, \tag{2.23}$$

$$\alpha_2^*(P_2) = \alpha_2(K^*(P_2)) = \max\left\{0, \frac{(\epsilon - \rho)\sqrt{P_2 \xi(\rho - \epsilon - \alpha_1)} + \rho \xi(\rho - \epsilon - \alpha_1)}{\xi(\rho - \epsilon - \alpha_1) - \sqrt{P_2 \xi(\rho - \epsilon - \alpha_1)}}\right\}. \tag{2.24}$$

The optimal control is given as the maximum of zero and $\hat{K}^*$, as we assume that the control is bounded from below at zero. Thus, the resulting drift rate is the maximum of $\alpha_2^*(P_2)$ and $\alpha_1$, i.e. $\alpha_2(0) = \alpha_1$ in Eq. (2.24). Corollary 2.3.2 presents the condition for which the optimal control is positive.

**Corollary 2.3.2.** *The optimal control value $K^*$ is greater than zero if*

$$P_2 > \frac{\xi(\rho - \alpha_1)^2}{\rho - \epsilon - \alpha_1}. \tag{2.25}$$

Thus, the smaller the potential increase in the drift, the higher the price at the time of investment needs to be for the firm to undergo revenue-enhancing activities. We can now use the optimal control and drift as functions of the price, presented in Proposition 2.3.1, to find the value of the investment opportunity. The value of the firm is then presented in Proposition 2.3.3.

**Proposition 2.3.3.** *The optimal value of the firm, $F(P_1)$, is given by*

$$F(P_1) = \begin{cases} A P_1^{\beta_{11}} & \text{if } P_1 < P_1^*, \\ \frac{P_1}{\rho - \alpha_2^*(P_1)} - I - \xi K^*(P_1) & \text{if } P_1 \geq P_1^*, \end{cases} \tag{2.26}$$

*with $K^*(P_1)$ and $\alpha_2^*(P_1)$ as given in Proposition 2.3.1. If the condition in Corol-*

*lary 2.3.2 holds, so that $K^* > 0$, then*

$$\beta_{11} = \frac{1}{2} - \frac{\alpha_1}{\sigma^2} + \sqrt{\left(\frac{\alpha_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}, \tag{2.27}$$

$$A = \frac{\left((\epsilon + \alpha_1 - \rho)\xi + \sqrt{P_1^*\xi(\rho - \epsilon - \alpha_1)}\right) P_1^{*1-\beta_{11}}}{\epsilon\beta_{11}\sqrt{P_1^*\xi(\rho - \epsilon - \alpha_1)}}. \tag{2.28}$$

*Furthermore, the optimal investment threshold is given by*

$$\begin{aligned} P_1^* = &\frac{1}{2}\left(\frac{2\beta_{11} - 1}{\beta_{11} - 1}\right)^2 \xi(\rho - \epsilon - \alpha_1) + \frac{\beta_{11}}{\beta_{11} - 1}(I\epsilon - \xi(\rho - \alpha_1)) \\ &+ \frac{1}{2}\left(\frac{2\beta_{11} - 1}{\beta_{11} - 1}\right)\sqrt{\xi(\rho - \epsilon - \alpha_1)\left(\frac{\beta_{11}}{\beta_{11} - 1}(I\epsilon - \xi(\rho - \alpha_1) + \xi(\rho - \epsilon - \alpha_1)\right)}, \end{aligned} \tag{2.29}$$

*if the following condition holds,*

$$\frac{I}{\xi} \geq \frac{\rho - \alpha_1}{\epsilon}. \tag{2.30}$$

*If $K^* = 0$, then*

$$A = \frac{1}{\beta_{11}(\rho - \alpha_1)}(P_1^*)^{1-\beta_{11}}, \tag{2.31}$$

*and the optimal investment threshold is given by*

$$P_1^* = \frac{\beta_{11}}{\beta_{11} - 1}(\rho - \alpha_2)I. \tag{2.32}$$

Similar to the model in Section 2.2, the value of the option to invest is dependent on the current price level, $P_1$. If the current price is below the investment threshold, the firm holds the option to invest, with a value equal to the first case in Eq. (2.26). If the currently observed price is above the threshold, it is optimal to invest immediately, paying the investment cost $I(K^*) = I + \xi K^*(P_1)$, and therewith undertaking the amount of revenue-enhancing activities that maximizes the value of the project at the given price, $P_1$. The condition in Eq. (2.30) is necessary for the existence of a unique investment threshold[2]. We assume this

[2]Extensive numerical analysis shows that even if this condition does not hold, there is only one unique threshold $P_1^*$ that is admissible w.r.t. the assumption that $P_1^* \geq P_2^*$. We therefore disregard this for the rest of the section.

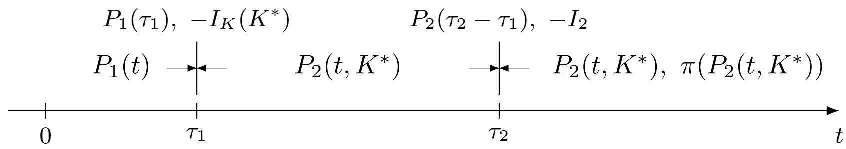



Figure 2.6: Regions for the sequential investment problem with controlled boost in drift.

condition to always hold, as the opposite case would imply an unrealistically high marginal cost of the activity compared to the fixed investment cost. The following corollary presents conditions for the existence of a real-valued threshold.

**Corollary 2.3.4.** *In Proposition 2.3.3, there exists a real-valued investment threshold, $P_1^*$, if the following condition holds*

$$I\epsilon \geq \xi(\rho-\alpha_1) - \xi(\rho-\epsilon-\alpha_1)\frac{\beta_{11}-1}{\beta_{11}}. \tag{2.33}$$

### 2.3.2 Two-stage investment, controlled change of drift

We now consider the situation where the firm can affect the expected growth of the price process before launching the project. This represents a two-stage sequential investment, where the increase in drift rate occurs after the initial investment. We assume that the firm incurs both a fixed and variable cost from the first-stage investment. Thus, we define the first-stage investment cost by $I_K(K) = I_1 + \xi K$. The joint optimal stopping and impulse control problem is then given by Eq. (2.34) with the corresponding regions illustrated in Fig. 2.6.

$$\begin{aligned} F(P_1) = \sup_{\tau_1} \mathbb{E}^{P_1}\Bigg[ & -I_1 e^{-\rho\tau_1} + \max_{K\geq 0} \mathbb{E}^{P_2}\Big\{ -\xi K e^{-\rho\tau_1} \\ & + \sup_{\tau_2 \mathbf{1}_{\{\tau_2>\tau_1\}}} \Big\{ e^{-\rho\tau_2}\Big[\int_{\tau_2}^{\infty} e^{-\rho(t-\tau_2)}\pi(P_2(t-\tau_2,K))dt\Big] - I_2 e^{-\rho\tau_2}\Big\}\Big\}\Bigg]. \end{aligned} \tag{2.34}$$

The solution approach is similar to that in Section 2.3.1, taking the control variable as given for any time $t > \tau_1$. Hence, the value function after investment is given by Eq. (2.21). The value of the option to invest in the second project stage is now a function of the control. The value function is given by the following proposition.

**Proposition 2.3.5.** *The value of the second-stage option, $F_2(P_2, K)$, is given by*

$$F_2(P_2, K) = \begin{cases} D_2(K) P_2^{\beta_{12}(K)} & \text{if } P_2 < P_2^*(K), \\ \frac{P_2}{\rho - \alpha_2(K)} - I_2 & \text{if } P_2 \geq P_2^*(K), \end{cases} \tag{2.35}$$

*where*

$$\beta_{12}(K) = \frac{1}{2} - \frac{\alpha_2(K)}{\sigma^2} + \sqrt{\left(\frac{\alpha_2(K)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}, \tag{2.36}$$

$$P_2^*(K) = \frac{\beta_{12}(K)}{\beta_{12}(K) - 1}(\rho - \alpha_2(K)) I_2, \tag{2.37}$$

$$D_2(K) = \frac{1}{(\rho - \alpha_2(K))\beta_{12}(K)} P_2^{*1-\beta_{12}(K)}. \tag{2.38}$$

The option value of the first project stage is, similar to the case presented in Section 2.2.1, dependent on the ordering of the first and second investment thresholds. The maximization with respect to $K$ is different for the two cases, dependent on whether exercise of the first-stage option represents entering the continuation or the stopping region of the second-stage option. Note that the boosted drift is a function of the investment $K$ (see Eq. (2.19)). Proposition 2.3.6 presents the value of the firm and the price thresholds for the two cases.

**Proposition 2.3.6.** *If $P_1^* < P_2^*$, the value of the firm, $F_1(P_1)$, is given by*

$$F_1(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^*(K^*) \\ D_2(K^*) P_1^{\beta_{12}(K^*)} - I_1 - \xi K^* & \text{if } P_1 \geq P_1^*(K^*) \end{cases} \tag{2.39}$$

*where*

$$\beta_{11} = \frac{1}{2} - \frac{\alpha_1}{\sigma^2} + \sqrt{\left(\frac{\alpha_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}, \tag{2.40}$$

$$K^* = K^*(P_1) = \underset{K \geq 0}{\arg\max} \left\{ D_2(K) P_1^{\beta_{12}(K)} - I_1 - \xi K \right\}, \tag{2.41}$$

$$D_1 = \frac{\beta_{12}(K^*)}{\beta_{11}} D_2(K^*) P_1^{*\beta_{12}(K^*) - \beta_{11}}, \tag{2.42}$$

*and $P_1^*$ is implicitly given as the solution to the following equation*

$$(\beta_{12}(K^*) - \beta_{11}) D_2(K^*) P_1^{*\beta_{12}(K^*)} + (I_1 + \xi K^*)\beta_{11} = 0. \tag{2.43}$$

*If $P_1^* \geq P_2^*$, the solution is the same as presented in Proposition 2.3.1 and*

| $\sigma$ | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 18.20 | 19.95 | 21.76 | 23.66 | 25.66 | 27.81 | 30.14 | 32.67 |
| $P_2^*$ | 30.77 | 32.88 | 35.99 | 39.88 | 44.46 | 49.69 | 55.54 | 62.03 |
| $K^*$ | 14.25 | 15.15 | 16.01 | 16.87 | 17.74 | 18.64 | 19.57 | 20.54 |
| $\alpha_2^*$ | 0.0474 | 0.0475 | 0.0476 | 0.0478 | 0.0479 | 0.0480 | 0.0481 | 0.0481 |

Table 2.2: Effect of increasing uncertainty on the optimal investment thresholds, the optimal amount of revenue-enhancing investment, and the resulting boosted drift. (Parameter values: $\rho = 0.1$, $\epsilon = 0.05$, $\alpha_1 = 0.01$, $\xi = 1$, $I_1 = 50$, $I_2 = 300$.)

*Proposition 2.3.3, with $I = I_1 + I_2$.*

In Proposition 2.3.6, the first threshold price must be found implicitly by solving Eq. (2.43) numerically. Unlike to the sequential investment problem with fixed change in drift, presented in Proposition 2.2.2, we cannot ex-ante determine the admissible threshold. Therefore, both cases of the threshold ordering must be considered, and the case that is admissible is adopted.

### 2.3.3 Comparative statics result

In this section, we present the comparative statics results for the optimal investment strategy when the firm has control over the change in drift. An extensive numerical analysis is conducted to examine how the investment thresholds and optimal drift rate change with the model parameters when analytic results are not obtainable. The effect of $\sigma$ on the initial investment threshold for a single-stage problem reduction is presented in Proposition 2.3.7.

**Proposition 2.3.7.** *The optimal threshold for the first stage investment, $P_1^*$, increases in $\sigma$, if $P_1^* \geq P_2^*$.*

From Proposition 2.3.7, we see that the result from the investment problem under fixed change in drift is still valid under controlled change. If the threshold for the initial investment is larger than the second threshold, the investment decision reduces to a single-stage investment, undertaken when the price becomes larger than $P_1^*$. Increased uncertainty would then delay the investment, consistent with the results from Section 2.2. The effects of $\sigma$ on the thresholds for the other cases are presented in Remark 2.3.8. Numerical results are given in Table 2.2 and Table 2.3.

**Remark 2.3.8.** *The optimal threshold for the second stage investment, $P_2^*$, increases in $\sigma$. The effect of $\sigma$ on the first-stage threshold, $P_1^*$, is ambiguous if $P_1^* < P_2^*$ and depends on the cost-parameters in relation to the potential increase in drift. The optimal control value, $K^*(P_1^*)$, increases in $\sigma$, yielding an increase in the optimal drift rate, $\alpha_2^*(P_1^*)$.*

| $\sigma$ | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 | 0.55 | 0.65 | 0.75 | 0.85 |
|---|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 21.40 | 19.70 | 17.60 | 16.30 | 15.80 | 15.90 | 16.40 | 17.40 | 18.70 |
| $P_2^*$ | 103.00 | 122.13 | 151.71 | 189.62 | 235.83 | 290.62 | 354.34 | 427.23 | 509.49 |
| $K^*$ | 784.01 | 837.28 | 876.32 | 916.26 | 961.11 | 1010.3 | 1061.3 | 1120.5 | 1182.5 |
| $\alpha_2^*$ | 0.0400 | 0.0400 | 0.0400 | 0.0400 | 0.0400 | 0.0400 | 0.0400 | 0.0400 | 0.0400 |

(a) Low variable cost, $\xi = 0.0001$

| $\sigma$ | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 | 0.55 | 0.65 | 0.75 | 0.85 |
|---|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 32.60 | 32.10 | 31.00 | 31.10 | 32.40 | 35.00 | 38.70 | 43.60 | 49.90 |
| $P_2^*$ | 103.20 | 122.93 | 152.84 | 190.91 | 237.19 | 292.01 | 355.70 | 428.55 | 510.76 |
| $K^*$ | 9.57 | 10.50 | 11.23 | 12.08 | 13.04 | 14.14 | 15.36 | 16.71 | 18.23 |
| $\alpha_2^*$ | 0.0372 | 0.0374 | 0.0375 | 0.0377 | 0.0379 | 0.0380 | 0.0382 | 0.0383 | 0.0384 |

(b) Medium variable cost, $\xi = 1$

| $\sigma$ | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 | 0.55 | 0.65 | 0.75 | 0.85 |
|---|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 53.20 | 58.50 | 64.20 | 72.50 | 84.20 | 100.10 | 121.20 | 148.60 | 183.50 |
| $P_2^*$ | 103.47 | 123.87 | 154.05 | 192.18 | 238.43 | 293.16 | 356.75 | 429.48 | 511.58 |
| $K^*$ | 3.87 | 4.46 | 5.03 | 5.68 | 6.43 | 7.31 | 8.31 | 9.45 | 10.73 |
| $\alpha_2^*$ | 0.0338 | 0.0345 | 0.0350 | 0.0355 | 0.0360 | 0.0364 | 0.0368 | 0.0371 | 0.0374 |

(c) High variable cost, $\xi = 10$

Table 2.3: Effect of increasing uncertainty on the optimal investment thresholds, the optimal amount of revenue-enhancing investment, and the resulting boosted drift, for three different levels of the variable cost parameter, $\xi$. (Parameter values: $\rho = 0.1$, $\epsilon = 0.06$, $\alpha_1 = 0.01$, $I_1 = 10$, $I_2 = 1000$.)

Table 2.2 shows that the firm invests at a higher threshold price for the second project stage when the uncertainty is higher. However, the effect of increasing uncertainty on $P_1^*$ when $P_1^* < P_2^*$ is ambiguous, as shown in Table 2.3. This is similar to the case of fixed change in drift, presented in Section 2.2.3, where uncertainty has an ambiguous effect on the initial investment threshold, dependent on the problem parameters. However, now there is an additional cost-parameter for the first-stage investment: the marginal cost of the control, $\xi$. From our numerical analysis, we see that in case where the fixed portion of the cost, $I_1$, is low, and the potential boost in drift is small, the first threshold can decrease in $\sigma$. This is however dependent on the value of $\xi$, as presented in Table 2.3. If the variable part of the cost is high, it mitigates the effect of the low fixed cost, such that the threshold is increasing in the uncertainty (see Table 2.3c). Since the variable cost is high, the first stage investment becomes more costly for the firm for a given level of the control. The firm chooses the optimal amount of revenue-enhancing activities, which results in the total cost of the first project stage to become large enough for the option value of waiting to dominate under increased uncertainty. However, the other way around is not observed, i.e. that very small values of $\xi$ reverse the effect of a high value of $I_1$. This can be seen as it is optimal for the firm to invest in revenue-enhancing activities, even when $\xi$ is large, to obtain a valuable boost in drift. Thus, the extra cost the firm incurs from boosting the drift, mitigates the low fixed cost $I_1$, and the value of waiting dominates the effect of uncertainty on the threshold. In the case where $\xi$ is small, the solution approaches the model with a fixed-change in drift in Section 2.2.1, as seen in Table 2.3a. For the cases when the first investment threshold decreases, the same reasoning as in Section 2.2.3 could hold. As the non-monotonic behaviour of the optimal strategy persists under controlled change in drift, more analysis of this ambiguous effect is warranted in future research.

Nonetheless, the amount of investment in revenue-enhancing activities always increases in $\sigma$ (for all values of the cost parameters), increasing the resulting boosted drift. We also find that the firm generally waits longer to invest in the market when the product market is more uncertain, i.e. $P_2^*$ increases in $\sigma$, but on the other hand conducts more revenue-enhancing activities to attain higher growth. Remark 2.3.9 presents how the cost of the control affect the optimal strategy for the firm.

**Remark 2.3.9.** *The investment threshold for the first project stage, $P_1^*$, increases in $\xi$. The second threshold, $P_2^*$, also increases, but at a lower rate than $P_1^*$. The optimal control $K^*$ and the resulting drift rate $\alpha_2^*$ decreases in $\xi$.*

Increasing the cost of the revenue-enhancing activities may lead to the threshold of the first project stage becoming larger than the second threshold, as seen in Table 2.4. The firm has a lower incentive to invest early in boosting the drift when such investments are more expensive, as the firm forgoes the opportunity to

| $\xi$ | 1 | 10 | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 23.70 | 35.70 | 41.54 | 44.06 | 46.06 | 47.72 | 49.12 | 50.32 |
| $P_2^*$ | 39.88 | 40.29 | 40.52 | 40.73 | 40.92 | 41.09 | 41.25 | 41.41 |
| $K^*$ | 16.89 | 5.70 | 3.96 | 3.05 | 2.49 | 2.11 | 1.82 | 1.59 |
| $\alpha_2^*$ | 0.0478 | 0.0440 | 0.0419 | 0.0401 | 0.0385 | 0.0371 | 0.0358 | 0.0346 |

Table 2.4: Effect of increased variable cost of revenue-enhancement on the optimal investment thresholds, the optimal amount of revenue-enhancing investment, and the resulting boosted drift. (Parameter values: $\rho = 0.1$, $\epsilon = 0.05$, $\alpha_1 = 0.01$, $\sigma = 0.2$, $I_1 = 50$, $I_2 = 300$.)

| $\alpha_1$ | 0.000 | 0.005 | 0.010 | 0.015 | 0.020 | 0.025 | 0.030 | 0.035 | 0.040 |
|---|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 21.65 | 22.54 | 23.66 | 25.07 | 26.91 | 29.41 | 32.95 | 38.42 | 41.58 |
| $P_2^*$ | 39.93 | 39.91 | 39.88 | 39.86 | 39.83 | 39.81 | 39.78 | 39.75 | 39.73 |
| $K^*$ | 17.75 | 17.32 | 16.87 | 16.40 | 15.89 | 15.32 | 14.67 | 13.86 | 11.70 |
| $\alpha_2^*$ | 0.0473 | 0.0475 | 0.0478 | 0.0480 | 0.0482 | 0.0485 | 0.0487 | 0.0490 | 0.0492 |

Table 2.5: Effect of increased initial drift rate on the optimal investment thresholds, the optimal amount of revenue-enhancing investment, and the resulting boosted drift. (Parameter values: $\rho = 0.1$, $\epsilon = 0.05$, $\sigma = 0.2$, $\xi = 1$, $I_1 = 50$, $I_2 = 300$.)

invest later. Also, the firm invests less in revenue-enhancing activities, obtaining a lower drift for the project. However, the overall amount paid in revenue-enhancing activities is greater. The firm is willing to pay more overall to obtain the boosted drift, even when it is more expensive per unit of change, as can be seen from multiplying $\xi$ and $K^*$ in Table 2.4. Since an increased marginal cost of the activity leads to a delay in investment, the firm also to pays more overall in revenue-enhancing activities at this time. Remark 2.3.10 presents the effect of changing the initial drift, $\alpha_1$, while numerical results are given in Table 2.5.

**Remark 2.3.10.** *The investment threshold for the first project stage, $P_1^*$, increases in $\alpha_1$, while the second threshold, $P_2^*$, decreases (albeit at a low rate). The optimal control, $K^*$, decreases in $\alpha_1$, while the resulting drift rate, $\alpha_2^*$, increases in $\alpha_1$.*

The level of the initial drift, $\alpha_1$, must be seen in relation to the maximum obtainable drift, given as $\rho - \epsilon$, which is assumed constant when changing $\alpha_1$. Increasing $\alpha_1$ thus makes the region of $\alpha_2(K)$ smaller, as $\alpha_2(0) = \alpha_1$ and $\alpha_2(\infty) = \rho - \epsilon$, and we assume $K \geq 0$. Remark 2.3.10 notes that the first threshold increases, while the second threshold decreases slightly. When the initial drift is greater, the firm demands a higher price before investing. When the difference in drift,

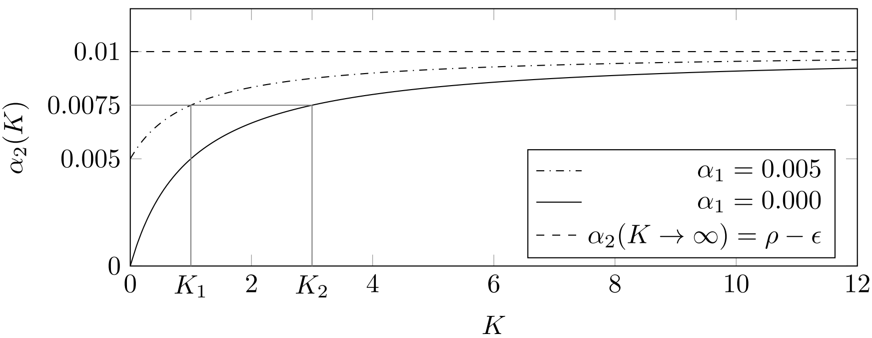



Figure 2.7: The effect of increasing $\alpha_1$ on $\alpha_2(K)$. (Parameters: $\rho = 0.04$, $\epsilon = 0.03$.)

$\rho - \epsilon - \alpha_1$, is small enough, we see that the ordering of the thresholds changes, and the incentive to invest in two stages diminishes. The firm undertakes less revenue-enhancing activities when the initial drift increases, but the resulting boosted drift rate, $\alpha_2(K)$, increases. However, the change in drift decreases when $\alpha_1$ increases, as seen in Table 2.5 by $\alpha_2^* - \alpha_1$.

Since only the lower bound of the range of $\alpha_2(K)$ increases with higher $\alpha_1$, the firm can obtain the same boosted drift with less effort. This observation can be explained by the example plot in Figure 2.7. As the initial drift, $\alpha_1$, is increased, the firm can obtain the same boosted drift, $\alpha_2(K)$, at a lower value of the control, $K$. This effect is evident from comparing $K_1$ and $K_2$ in Figure 2.7. Thus, the firm can obtain a higher boosted drift from an increase in the initial drift, even though the firm undertakes less revenue-enhancing activities. This effect arises since increasing $\alpha_1$ represents making the initial market conditions more favourable, while keeping the maximum obtainable drift for the firm constant. The effect of increasing $\epsilon$ on the optimal investment strategy of the firm is presented in Remark 2.3.11, with numerical results given in Table 2.6.

**Remark 2.3.11.** *The first-stage investment threshold $P_1^*$ increases in $\epsilon$. The second threshold $P_2^*$ also increases, but at a lower rate than $P_1^*$. The optimal control $K^*$ and the resulting drift rate $\alpha_2^*$ decrease in $\epsilon$.*

Increasing $\epsilon$ is effectively lowering the upper bound of $\alpha_2(K)$, i.e. the maximum obtainable drift rate, as the upper bound is defined as $\rho - \epsilon$. We see from Table 2.6 that the firm invests at a higher threshold price when the upper bound is lowered, and that the ordering of the thresholds depends on the range of obtainable drift, similar to increasing the initial drift, $\alpha_1$. The resulting boosted drift rate is now lowered. Thus, when the maximum obtainable drift is lowered, the firm invests later and undertakes less revenue-enhancing activities. Since an increase in $\epsilon$ makes the available magnitude of the boost lower, there is less incentive for the firm

| $\epsilon$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 2.80 | 6.16 | 10.66 | 16.42 | 23.66 | 32.82 | 43.16 | 48.37 |
| $P_2^*$ | 36.66 | 37.32 | 38.07 | 38.92 | 39.88 | 40.97 | 42.21 | 43.59 |
| $K^*$ | 37.44 | 27.13 | 22.41 | 19.30 | 16.87 | 14.68 | 11.99 | 7.57 |
| $\alpha_2^*$ | 0.0879 | 0.0775 | 0.0674 | 0.0575 | 0.0478 | 0.0381 | 0.0285 | 0.0188 |

Table 2.6: Effect of decreasing the maximum obtainable drift on the optimal investment thresholds, the optimal amount of revenue-enhancing investment, and the resulting boosted drift. (Parameter values: $\rho = 0.1$, $\alpha_1 = 0.01$, $\sigma = 0.2$, $\xi = 1$, $I_1 = 50$, $I_2 = 300$.)

to invest in the revenue-enhancing activity, which increases the first investment threshold. Further, the smaller obtainable boost makes the expected present value of the project's cash flow after revenue-enhancement lower, which decreases the attractiveness of the project, and increases the second investment threshold.

## 2.4 Robustness testing

In this section, we perform a robustness test of the results of the combined optimal stopping and impulse problem presented in Section 2.3.3. We derive the results assuming an alternate specification of the impulse function in Section 2.4.1, and introduce a non-linear cost function for the control in Section 2.4.2. We perform a comparative static analysis of uncertainty for both cases, and compare to the earlier findings presented in Section 2.3.3.

### 2.4.1 Impulse function

We now introduce a more flexible specification of the impulse function. Nonetheless, the basic assumptions made in Section 2.3 are upheld. I.e. we assume diminishing marginal returns of the value-enhancing activity and assuring that the resulting drift rate is lower than the discount rate. We utilize an inverse exponential function as given by Eq. (2.44), with example plots of the impulse function given in Fig. 2.8. Varying $\lambda$ changes the slope of the impulse, yielding a more refined specification of how the value-enhancing investment affects the drift rate of the stochastic process. However, the downside is that under this impulse function there exists no closed form solutions for the optimal control value and investment thresholds, in any of the cases. These values can easily be computed numerically.

$$\alpha_2(K) = \rho - \epsilon - (\rho - \epsilon - \alpha_1)e^{-\lambda K} \tag{2.44}$$

The optimal stopping problem is the same as presented in Eq. (2.34). The value

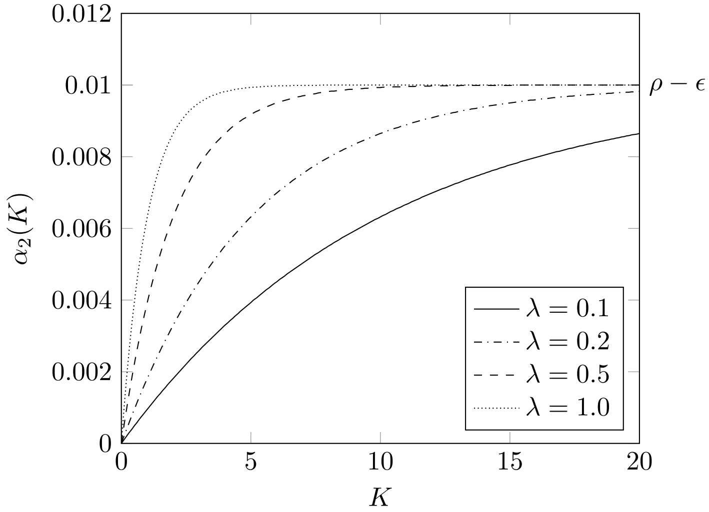



Figure 2.8: Inverse exponential impulse function as given by Eq. (2.44). The second drift-rate is plotted as a function of the control. (Parameters: $\rho = 0.04,\ \epsilon = 0.03,\ \alpha_1 = 0$).

of the options to invest in the first and second stage is equal to that given by Eq. (2.35)-(2.43). However, if $P_1^* \geq P_2^*$, there is no closed-form solution available, and the value of the option to invest is given by the following proposition.

**Proposition 2.4.1.** *If $P_1^* \geq P_2^*$, the value of the investment opportunity is given by*

$$F(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^*, \\ \frac{P_1}{\rho - \alpha_2(K^*)} - I_1 - I_2 - \xi K^*(P_1) & \text{if } P_1 \geq P_1^*, \end{cases} \tag{2.45}$$

*where*

$$\beta_{11} = \frac{1}{2} - \frac{\alpha_1}{\sigma^2} + \sqrt{\left(\frac{\alpha_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}, \tag{2.46}$$

$$K^* = K^*(P_1) = \arg\max_{K \geq 0} \left\{ \frac{P_1}{\rho - \alpha_2(K)} - I_1 - I_2 - \xi K^*(P_1) \right\}, \tag{2.47}$$

$$D_1 = \frac{(P_1^*)^{1-\beta_{11}}}{(\rho - \alpha_2(K^*))\beta_{11}}, \tag{2.48}$$

*and $P_1^*$ is implicitly given as the solution to the following equation*

$$(\beta_{11}-1)\frac{P_1^*}{\rho-\alpha_2(K^*(P_1^*))}-\beta_{11}(I_1+I_2+\xi K^*(P_1^*))=0. \tag{2.49}$$

Conducting a numerical comparative statics analysis of the investment thresholds, we compare the results to those obtained in Section 2.3.3. The effect of volatility on the optimal investment thresholds, optimal control value, and resulting drift rate, under the new impulse function are given in Table 2.7. We see in Table 2.7a and Table 2.7b that under a steep impulse function, which resembles the specification in Section 2.3.3, the initial investment threshold can be both increasing and decreasing in volatility. Similar to before, when the cost of the initial investment is significantly lower than that of the second investment, we observe a non-monotonic effect of uncertainty on the initial investment threshold. In Table 2.7c and Table 2.7d, the impulse function is less steep, with $\lambda = 0.1$. The effect of uncertainty can also be non-monotonic here, but this necessitates the marginal cost of the impulse, $\xi$, to be low. Thus, there is an interplay between the effect of the value-enhancing activity on the drift rate, and the marginal cost of the activity, in whether there is a non-monotonic effect of uncertainty. This is in line with Table 2.3 in Section 2.3.3, wherein a high marginal cost of the control makes the effect of uncertainty monotonic. Thus, we can confirm that our earlier results are robust to the specification of the impulse function, providing that the underlying assumptions of the firm's effect on the drift rate is upheld.

## 2.4.2 Non-linear cost of impulse

We now investigate the role of the specification of the cost for the value-enhancing activity. We introduce a more general, non-linear, investment cost function for the impulse, so that the cost structure can be either convex or concave, depending on parameter values. This is similar to the studies performed in capacity choice models under uncertainty, like Dangl (1999) and Hagspiel et al. (2016a). A concave cost function indicates a decreasing marginal cost of investment, in line with a situation of economies of scale for the investment, while a convex cost function represent diseconomies of scale. Here we assume that the impulse function is given as in Eq. (2.19), to investigate the case of changed investment cost compared to the base case controlled model with linear cost. The cost of the value-enhancing activity undertaken at the first investment stage is a function of the control $K$, and defined as

$$I_K(K)=I_1+\xi K^{\eta}, \tag{2.50}$$

| $\sigma$ | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 15.50 | 16.83 | 18.16 | 19.49 | 20.88 | 22.34 | 23.84 | 25.50 |
| $P_2^*$ | 30.74 | 32.77 | 35.82 | 39.67 | 44.21 | 49.41 | 55.25 | 61.73 |
| $K^*$ | 5.34 | 5.45 | 5.54 | 5.62 | 5.71 | 5.79 | 5.86 | 5.94 |
| $\alpha_2^*$ | 0.0498 | 0.0498 | 0.0498 | 0.0499 | 0.0499 | 0.0499 | 0.0499 | 0.0499 |

(a) $\xi = 1,\ I_1 = 50,\ I_2 = 300,\ \lambda = 1.$

| $\sigma$ | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 | 0.55 | 0.65 | 0.75 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 14.34 | 14.11 | 13.25 | 12.79 | 12.81 | 13.25 | 14.05 | 15.19 |
| $P_2^*$ | 102.46 | 119.45 | 147.44 | 184.24 | 229.62 | 283.78 | 346.99 | 419.47 |
| $K^*$ | 4.57 | 4.72 | 4.82 | 4.93 | 5.03 | 5.14 | 5.26 | 5.38 |
| $\alpha_2^*$ | 0.0496 | 0.0496 | 0.0497 | 0.0497 | 0.0497 | 0.0498 | 0.0498 | 0.0498 |

(b) $\xi = 1,\ I_1 = 10,\ I_2 = 1000,\ \lambda = 1.$

| $\sigma$ | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 19.88 | 21.93 | 24.15 | 26.50 | 28.98 | 31.62 | 34.49 | 37.56 |
| $P_2^*$ | 30.75 | 32.82 | 35.90 | 39.76 | 44.31 | 49.51 | 55.35 | 61.83 |
| $K^*$ | 33.20 | 34.37 | 35.47 | 36.51 | 37.51 | 38.47 | 39.42 | 40.36 |
| $\alpha_2^*$ | 0.0486 | 0.0487 | 0.0488 | 0.0490 | 0.0491 | 0.0491 | 0.0492 | 0.0493 |

(c) $\xi = 1,\ I_1 = 50,\ I_2 = 300,\ \lambda = 0.1.$

| $\sigma$ | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 | 0.55 | 0.65 | 0.75 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 11.96 | 11.51 | 10.52 | 9.90 | 9.70 | 9.86 | 10.30 | 10.98 |
| $P_2^*$ | 102.44 | 119.37 | 147.33 | 184.11 | 229.48 | 283.64 | 346.86 | 419.35 |
| $K^*$ | 66.03 | 67.48 | 68.41 | 69.34 | 70.30 | 71.32 | 72.38 | 73.46 |
| $\alpha_2^*$ | 0.0499 | 0.0500 | 0.0500 | 0.0500 | 0.0500 | 0.0500 | 0.0500 | 0.0500 |

(d) $\xi = 0.01,\ I_1 = 10,\ I_2 = 1000,\ \lambda = 0.1.$

Table 2.7: Effect of increasing uncertainty on the optimal investment thresholds, the optimal amount of revenue-enhancing investment, and the resulting boosted drift under an inverse exponential impulse function as given by Eq. (2.44). (General parameters: $\rho = 0.1$, $\epsilon = 0.05$, $\alpha_1 = 0.01$.)

where the constant $I_1$, $\xi$, and $\eta$ are larger than zero. $\eta = 1$ represents the case of linear cost studied in Section 2.3.1 and Section 2.3.2, while $\eta \in (0, 1)$ represents a concave and $\eta > 1$ a convex cost function. The value of the second-stage option is given by Eq. (2.35)-(2.38). Similar to before, there are two cases for the first-stage investment option which must be considered. The value of the first investment option is given in the following proposition.

**Proposition 2.4.2.** *If the first threshold is lower than the second, i.e.* $P_1^* < P_2^*$*, the value of the first investment option is given by*

$$F_1(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^*(K^*), \\ D_2(K^*) P_1^{\beta_{12}(K^*)} - I_1 - \xi (K^*)^\eta, & \text{if } P_1 \geq P_1^*(K^*), \end{cases} \tag{2.51}$$

*where* $\beta_{11}$ *is given by Eq. (2.9),* $D_1$ *by Eq. (2.42), and* $D_2(K)$ *by Eq. (2.38). The optimal control is given by the maximization*

$$K^* = K^*(P_1) = \underset{K \geq 0}{\arg\max} \left\{ D_2(K) P_1^{\beta_{12}(K)} - I_1 - \xi K^\eta \right\}, \tag{2.52}$$

*and* $P_1^*$ *is implicitly given as the solution to the following equation*

$$(\beta_{12}(K^*) - \beta_{11}) D_2(K^*) P_1^{*\beta_{12}(K^*)} + \beta_{11}(I_1 + \xi (K^*)^\eta) = 0. \tag{2.53}$$

*If* $P_1^* \geq P_2^*$*, the value of the first investment option is given by*

$$F(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^*, \\ \frac{P_1}{\rho - \alpha_2(K^*)} - I_1 - I_2 - \xi (K^*(P_1))^\eta & \text{if } P_1 \geq P_1^*, \end{cases} \tag{2.54}$$

*with* $D_1$ *given by Eq. (2.48) and the optimal control as the maximization*

$$K^* = K^*(P_1) = \underset{K \geq 0}{\arg\max} \left\{ \frac{P_1}{\rho - \alpha_2(K)} - I_1 - I_2 - \xi K^*(P_1) \right\}. \tag{2.55}$$

*The first investment threshold,* $P_1^*$*, is implicitly given as the solution to the equation given by*

$$(\beta_{11} - 1) \frac{P_1^*}{\rho - \alpha_2(K^*(P_1^*))} - \beta_{11}(I_1 + I_2 + \xi (K^*(P_1^*))^\eta) = 0. \tag{2.56}$$

Conducting a numerical comparative static analysis of the effect of uncertainty, the results under both concave and convex cost functions are given in Table 2.8. In Table 2.8a and Table 2.8b, we see that the earlier result of non-monotonicity w.r.t. uncertainty hold under a concave cost structure for the control. A relatively low initial investment cost may yield a non-monotonic effect of uncertainty on the investment cost. In Table 2.8c and Table 2.8d we see the same holds for a convex cost function. Thus, our previous results are robust under various specifications of the cost function. In a capacity choice model, Hagspiel et al. (2016a) investigate the case of both a concave and convex investment cost, and find that the firm invests significantly later under a convex than a concave cost function. Comparing the concave and convex cases in Table 2.8, we see that in our impulse problem, a convex investment cost also delays the initial investment decision relative to the concave case. The firm also invest in less value-enhancement when there are diseconomies of scale, as would be expected.

## 2.5 Conclusions

In this paper, we study the investment problem of a firm with an option to irreversibly invest to enter a novel market. The firm has the opportunity to undertake some amplifying pre-investments to boost the expected value of the profits from the project, e.g. through marketing or lobbyism. The stochastic market price is modelled as a geometric Brownian motion, subject to a change in drift following from the revenue-enhancing pre-investment. We consider a case with a fixed change in drift, and a situation where the magnitude of the change is influenced by the amount of revenue-enhancing activities undertaken by the firm. This makes the stochastic environment of the firm endogenous, as the firm can influence its potential profits through its actions.

We find that when a firm can change the drift rate of the cash flows from a project, it has an incentive to invest sequentially, and to boost the drift before committing to launching the project. This result is not dependent on including other complicating factors such as time-to-build, contrarily to what is shown by Dixit and Pindyck (1994), but is a pure effect of the change in the stochastic environment. The effect of uncertainty on the investment triggers is ambiguous. For the revenue-enhancing investment, increasing the uncertainty can both delay or accelerate the investment w.r.t. the threshold price, depending on the parameter values. The effect of the option value of waiting and the incentive to invest early to boost the drift are conflicting. Which effect dominates is dependent on the cost parameters, the magnitude of the change in drift, as well as the level of the uncertainty.

In the situation where the firm can optimally choose the magnitude of the boost in drift through the intensity of revenue-enhancing activities, we find that higher uncertainty leads to more investment. The firm invests more to boost the drift when the market is characterized by a higher volatility. Increasing the marginal cost of the revenue-enhancing activities decreases the intensity of the activity, but increases the total amount spent on boosting the drift. As a higher marginal cost

| $\sigma$ | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 14.96 | 16.18 | 17.31 | 18.43 | 19.58 | 20.78 | 22.11 | 23.53 |
| $P_2^*$ | 30.74 | 32.80 | 35.86 | 39.72 | 44.27 | 49.48 | 55.32 | 61.81 |
| $K^*$ | 60.18 | 64.52 | 68.36 | 72.13 | 76.00 | 79.98 | 84.22 | 88.62 |
| $\alpha_2^*$ | 0.0492 | 0.0492 | 0.0493 | 0.0493 | 0.0494 | 0.0494 | 0.0494 | 0.0494 |

(a) Concave cost function: $\xi = 1$, $I_1 = 50$, $I_2 = 300$, $\eta = 0.5$.

| $\sigma$ | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 | 0.55 | 0.65 | 0.75 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 14.70 | 14.26 | 13.22 | 12.62 | 12.56 | 12.97 | 13.77 | 14.91 |
| $P_2^*$ | 102.50 | 119.67 | 147.79 | 184.66 | 230.08 | 284.26 | 347.46 | 419.94 |
| $K^*$ | 36.25 | 40.20 | 42.93 | 45.94 | 49.42 | 53.46 | 58.00 | 63.10 |
| $\alpha_2^*$ | 0.0487 | 0.0488 | 0.0489 | 0.0489 | 0.0490 | 0.0491 | 0.0492 | 0.0492 |

(b) Concave cost function: $\xi = 1$, $I_1 = 10$, $I_2 = 1000$, $\eta = 0.5$.

| $\sigma$ | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 19.43 | 21.24 | 23.06 | 25.01 | 27.07 | 29.31 | 31.74 | 34.49 |
| $P_2^*$ | 30.83 | 33.04 | 36.25 | 40.22 | 44.86 | 50.12 | 56.01 | 62.52 |
| $K^*$ | 7.23 | 7.61 | 7.95 | 8.31 | 8.67 | 9.04 | 9.42 | 9.83 |
| $\alpha_2^*$ | 0.0439 | 0.0442 | 0.0444 | 0.0446 | 0.0448 | 0.0450 | 0.0452 | 0.0454 |

(c) Convex cost function: $\xi = 1$, $I_1 = 50$, $I_2 = 300$, $\eta = 1.5$.

| $\sigma$ | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 | 0.55 | 0.65 | 0.75 |
|---|---|---|---|---|---|---|---|---|
| $P_1^*$ | 24.71 | 24.51 | 23.72 | 23.86 | 25.20 | 27.65 | 31.41 | 36.62 |
| $P_2^*$ | 102.82 | 121.17 | 150.08 | 187.40 | 233.05 | 287.32 | 350.52 | 422.91 |
| $K^*$ | 5.87 | 6.38 | 6.77 | 7.23 | 7.77 | 8.39 | 9.12 | 9.95 |
| $\alpha_2^*$ | 0.0427 | 0.0432 | 0.0436 | 0.0439 | 0.0443 | 0.0447 | 0.0451 | 0.0454 |

(d) Convex cost function: $\xi = 1$, $I_1 = 10$, $I_2 = 1000$, $\eta = 1.5$.

Table 2.8: Effect of increasing uncertainty on the optimal investment thresholds, the optimal amount of revenue-enhancing investment, and the resulting boosted drift under a generalized investment cost function as given by Eq. (2.50). (General parameters: $\rho = 0.1$, $\epsilon = 0.05$, $\alpha_1 = 0.0$.)

delays the investment, the firm optimally incurs a larger total cost of this activity, even though the resulting effect on the drift is smaller.

This paper represents an early effort in including endogeneity in real options modelling, bridging the gap to the use of option reasoning in the decision-heuristic oriented strategy literature. The functional form of how the firm can affect the drift of the market, and the structure of the cost of this influence, is generally motivated, but not made to fit any specific practical actions. However, we see that the results are robust w.r.t. these specifications, suggesting that the results hold more generally for activities of this kind that a firm can undertake. Future research could investigate specific activities that a firm can undertake, basing the specification of the influence on literature on the type of activities considered, like marketing or standard-captures. This could also allow for empirical testing for the results. Further, considering changes in volatility could broaden the connection to practical investment problems, as the degree of risk-taking may be an important decision for a firm introducing a new product, like the case motivated in Alvarez and Stenbacka (2003). Lastly, future studies could compare the value added from the opportunity to boost the drift, to the case where the firm has no such affect on the market. This could give further insight into the nature of including such dependencies of the market characteristics on decisions of the firm.

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## 2.A Additional derivations

This section presents derivations of the value of the expected discounted cash flows of the project.

### 2.A.1 The value of the project

Assume that the profit flow per unit time period after the project is launched is given as $\pi(P_2, t) = P_2(t)$, where $P_2(t)$ follows Eq. (2.1b), and that the appropriate discount rate for the project is $\rho$. Then the value of the project $V(P_2)$ for a price $P_2$ at investment is found using a dynamic programming approach (Dixit and Pindyck, 1994). The per time period value of the installed project should be given by the profit flow, plus the change in project value (capital gains), so we have

$$\begin{aligned}\rho V(P_2)dt &= \pi(P_2)dt + \mathbb{E}[dV(P_2)] \\ &= \pi(P_2)dt + \left[\alpha_2 P_2 \frac{\partial V(P_2)}{\partial P_2} + \frac{1}{2}\sigma^2 P_2^2 \frac{\partial^2 V(P_2)}{\partial P_2^2}\right] dt\end{aligned} \tag{2.57}$$

Thus, the value of the completed project, $V(P_2)$, must satisfy the differential equation

$$\alpha_2 P_2 \frac{\partial V(P_2)}{\partial P_2} + \frac{1}{2}\sigma^2 P_2^2 \frac{\partial^2 V(P_2)}{\partial P_2^2} - \rho V(P_2) + \pi(P_2) = 0. \tag{2.58}$$

The last term leads to a particular solution, since $\pi(P_2) = P_2$. The value function $V(P_2)$ is given by the combination of a homogeneous and particular solution, and equal to

$$V(P_2) = B_1 P_2^{\beta_{12}} + B_2 P_2^{\beta_{22}} + \frac{P_2}{\rho - \alpha_2}. \tag{2.59}$$

In Eq. (2.59), $\beta_{12}$ and $\beta_{22}$ are the positive and negative solutions, respectively, of the fundamental quadratic

$$\mathcal{Q} \equiv \frac{1}{2}\sigma^2 \beta_2(\beta_2 - 1) + \alpha_2 \beta_2 - \rho = 0, \tag{2.60}$$

giving $\beta_{12} > 1$ and $\beta_{22} < 0$ (Dixit and Pindyck, 1994). The boundary condition $\lim_{P_2 \to 0} V(P_2) = 0$, gives that $B_2 = 0$ must hold, since $\beta_{22} < 0$. Further, assuming no speculative bubbles as in Dixit and Pindyck (1994), we have $B_1 = 0$ as well, and the project value is given its the fundamental value only:

$$V(P_2) = \frac{P_2}{\rho - \alpha_2}. \tag{2.61}$$

## 2.B Proofs

This section presents proofs of all propositions and corollaries.

### 2.B.1 Proof of Proposition 2.2.1

The discounted change in value of the option to undertake the second-stage investment, $F_2(P_2)$, for a given value $P_2$ of the price process is equal to the capital gains of the option, as there is no profit flow from holding the option. The value function must satisfy the Bellman equation

$$\rho F_2(P_2)dt = \mathbb{E}[dF_2(P_2)] = \left[\alpha_2 P_2 \frac{\partial F_2(P_2)}{\partial P_2} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 F_2(P_2)}{\partial P_2^2}\right] dt, \tag{2.62}$$

where the right-hand-side follows from Itô's Lemma. This gives the differential equation

$$\alpha_2 P_2 \frac{\partial F_2(P_2)}{\partial P_2} + \frac{1}{2}\sigma^2 P_2^2 \frac{\partial^2 F_2(P_2)}{\partial P_2^2} - \rho F_2(P_2) = 0. \tag{2.63}$$

From the boundary condition $F(0) = 0$, we obtain the solution form $F_2(P_2) = D_2 P_2^{\beta_{12}}$, where $\beta_{12}$ is the positive root of the characteristic quadratic equation

$$\frac{1}{2}\sigma^2 \beta_2(\beta_2 - 1) + \alpha_2 \beta_2 - \rho = 0. \tag{2.64}$$

The remaining boundary conditions for the value function are

$$\begin{aligned} F_2(P_2^*) &= V(P_2^*) - I_2, \\ F_2'(P_2^*) &= V'(P_2^*). \end{aligned} \tag{2.65}$$

Eq. (2.65) represent the value-matching and smooth-pasting conditions at the optimal threshold price for investing in the second project stage, with the value of the completed project $V(P_2)$ given by Eq. (2.61). At the investment threshold, the value of the option and the completed project must be continuous and smooth (Dixit and Pindyck, 1994). From these conditions the threshold price, $P_2^*$, and the constant, $D_2$, are given by

$$P_2^* = \frac{\beta_{12}}{\beta_{12}-1}(\rho-\alpha_2)I_2, \tag{2.66}$$

$$D_2 = \frac{1}{(\rho-\alpha_2)\beta_{12}}\left[\frac{\beta_{12}}{\beta_{12}-1}(\rho-\alpha_2)I_2\right]^{1-\beta_{12}}. \tag{2.67}$$

Hence, the value of the option to invest in the second-stage is given by

$$F_2(P_2) = \begin{cases} D_2 P_2^{\beta_{12}} & \text{if } P_2 < P_2^*, \\ V(P_2) - I_2 & \text{if } P_2 \geq P_2^*, \end{cases} \tag{2.68}$$

where $V(P_2)$ is given in Eq. (2.61). □

## 2.B.2 Proof of Proposition 2.2.2

The value of the option to invest in the first stage, $F_1(P_1)$, and thus the value of the firm, for a given price $P_1$, is given by the Bellman equation and Itô's Lemma as

$$\rho F_1(P_1)dt = \mathbb{E}[dF_1(P_1)] = \left[\alpha_1 P_1 \frac{\partial F_1(P_1)}{\partial P_1} + \frac{1}{2}\sigma^2 P_1^2 \frac{\partial^2 F_1(P_1)}{\partial P_1^2}\right] dt, \tag{2.69}$$

which gives the differential equation

$$\alpha_1 P_1 \frac{\partial F_1(P_1)}{\partial P_1} + \frac{1}{2}\sigma^2 P_1 \frac{\partial^2 F_1(P_1)}{\partial P_1^2} - \rho F_1(P_1) = 0. \tag{2.70}$$

With the boundary condition $F_1(0) = 0$, we obtain the value function $F_1(P_1) = D_1 P_1^{\beta_{11}}$ where $\beta_{11}$ is the positive solution to the characteristic equation

$$\frac{1}{2}\sigma^2 \beta_1(\beta_1 - 1) + \alpha_1\beta_1 - \rho = 0. \tag{2.71}$$

Similar to Section 2.B.1, the option value should satisfy the value-matching and smooth-pasting boundary conditions at the investment threshold $P_1^*$. However, we must check for both cases of $P_1^* < P_2^*$ and $P_1^* \geq P_2^*$, as the value of $P_1^*$ determines if we enter the continuation region or stopping region of $F_2(P)$ in Eq. (2.68).

**If $\mathbf{P_1^* < P_2^*}$** the boundary conditions become

$$\begin{aligned} D_1 P_1^{*\beta_{11}} &= D_2 P_1^{*\beta_{12}} - I_1, \\ \beta_{11} D_1 P_1^{*\beta_{11}-1} &= \beta_{12} D_2 P_1^{*\beta_{12}-1}, \end{aligned} \tag{2.72}$$

where $D_2$ is given by Eq. (2.67). This gives the solutions of $D_1$ and $P_1^*$,

$$D_1 = \frac{\beta_{12}}{\beta_{11}} D_2 \left[ \frac{I_1}{(1 - \frac{\beta_{12}}{\beta_{11}}) D_2} \right]^{1 - \frac{\beta_{11}}{\beta_{12}}}, \tag{2.73}$$

$$P_1^* = \left[ \frac{I_1}{(1 - \frac{\beta_{12}}{\beta_{11}}) D_2} \right]^{\frac{1}{\beta_{12}}}, \tag{2.74}$$

and we have the value of the firm given by

$$F_1(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^*, \\ D_2 P_1^{\beta_{12}} - I_1 & \text{if } P_1 \geq P_1^*. \end{cases} \tag{2.75}$$

**If $\mathbf{P_1^* \geq P_2^*}$** the boundary conditions become

$$\begin{aligned} D_1 P_1^{*\beta_{11}} &= \frac{P_1^*}{\rho - \alpha_2} - I_2 - I_1, \\ \beta_{11} D_1 P_1^{*\beta_{11} - 1} &= \frac{1}{\rho - \alpha_2}, \end{aligned} \tag{2.76}$$

which gives the expressions for $D_1$ and $P_1^*$

$$P_1^* = \frac{\beta_{11}}{\beta_{11} - 1} (\rho - \alpha_2) (I_1 + I_2), \tag{2.77}$$

$$D_1 = \frac{1}{(\rho - \alpha_2)\beta_{11}} \left[ \frac{\beta_{11}}{\beta_{11} - 1} (\rho - \alpha_2) (I_1 + I_2) \right]^{1 - \beta_{11}}, \tag{2.78}$$

and the value of the firm becomes

$$F_1(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^*, \\ V(P_1) - I_1 - I_2 & \text{if } P_1 \geq P_1^*, \end{cases} \tag{2.79}$$

where $V(P_1)$ is given in Eq. (2.61). □

### 2.B.3 Proof of Corollary 2.2.3

Taking the inequality $P_1^* \geq P_2^*$, with $P_1^*$ and $P_2^*$ being given by Eq. (2.14) and Eq. (2.7), respectively. Reordering the terms give that $P_1^* \geq P_2^*$ if the following inequality holds

$$I_1 \geq \frac{\beta_{12}}{\beta_{12}-1}\left(\frac{1}{\beta_{12}} - \frac{1}{\beta_{11}}\right) \times I_2, \tag{2.80}$$

where the right-hand side is positive, since $1 < \beta_{12} < \beta_{11}$ if $\alpha_2 > \alpha_1$ from Corollary 2.2.4. Now evaluating the inequality $P_1^* < P_2^*$, where $P_1^*$ is given by Eq. (2.14) and $P_2^*$ by Eq. (2.7). Then the inequality becomes

$$\left[\frac{I_1}{\left(1 - \frac{\beta_{12}}{\beta_{11}} D_2\right)}\right]^{\frac{1}{\beta_{12}}} < \frac{\beta_{12}}{\beta_{12}-1}(\rho - \alpha_2) I_2, \tag{2.81}$$

where $D_2$ is given in Proposition 2.2.1. Inserting the expression for $D_2$, we obtain by direct rearrangement that $P_1^* < P_2^*$ yields the inequality

$$I_1 < \frac{\beta_{12}}{\beta_{12}-1}\left(\frac{1}{\beta_{12}} - \frac{1}{\beta_{11}}\right) \times I_2. \tag{2.82}$$

Thus, the ordering of the thresholds is unique, and whether the inequality in Eq. (2.82) holds determines if $P_1^* < P_2^*$. □

## 2.B.4 Proof of Corollary 2.2.4

### 2.B.4.1 Part i)

Define the characteristic equation as $\mathbb{Q}(x, \rho, \sigma, \alpha) = \frac{1}{2}\sigma^2 x(x-1) + \alpha x - \rho = 0$, and let $\beta_{12}$ and $\beta_{11}$ represent the positive roots of the characteristic equations $\mathbb{Q}(\beta_2, \rho, \sigma, \alpha_2)$ and $\mathbb{Q}(\beta_1, \rho, \sigma, \alpha_1)$, respectively. Then $\beta_{12} < \beta_{11}$ gives

$$\frac{1}{2} - \frac{\alpha_2}{\sigma^2} + \sqrt{\left(\frac{\alpha_2}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} < \frac{1}{2} - \frac{\alpha_1}{\sigma^2} + \sqrt{\left(\frac{\alpha_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}$$
$$\Rightarrow \alpha_2 - \alpha_1 > \sigma^2 \left(\sqrt{\left(\frac{\alpha_2}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} - \sqrt{\left(\frac{\alpha_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}\right)$$

If we assume that $\alpha_2 - \alpha_1 > 0$, then we obtain that

$$\sqrt{\left(\frac{\alpha_2}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} - \sqrt{\left(\frac{\alpha_1}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} > 0,$$

and since this inequality holds, we know that $\alpha_2 > \alpha_2$ implies that $\beta_{12} < \beta_{11}$.

#### 2.B.4.2 Part ii)

If the drift rate is constant for the two price processes $P_1(t)$ and $P_2(t)$, i.e. $\alpha_1 = \alpha_2$, then $\beta_{12} = \beta_{11}$. From this, the value-matching and smooth-pasting boundary conditions when $P_1^* < P_2^*$, given by Eq. (2.72), are contradictory. Thus, $P_1^* \geq P_2^*$ always holds when $\alpha_1 = \alpha_2$. □

### 2.B.5 Proof of Proposition 2.2.5

To prove Proposition 2.2.5–2.2.7, we first present an auxiliary result outlining the sign of the partial derivatives of $\beta_{12}$ and $\beta_{11}$, where $\beta_{12}$ and $\beta_{11}$ denote the positive roots of the quadratic equations in Eq. (2.64) and Eq. (2.71), respectively. We know that $\beta_{12}, \beta_{11} > 1$ (Dixit and Pindyck, 1994).

**Lemma 2.B.1.** *The sign of the partial derivatives of $\beta_{12}$ are given as*

$$\frac{\partial \beta_{12}}{\partial \sigma} < 0, \qquad \frac{\partial \beta_{12}}{\partial \alpha_2} < 0. \tag{2.83}$$

*From this, the sign of the partial derivatives of the fraction $\frac{\beta_{12}}{\beta_{12}-1}$ are given as*

$$\frac{\partial}{\partial \sigma}\frac{\beta_{12}}{\beta_{12}-1} > 0, \qquad \frac{\partial}{\partial \alpha_2}\frac{\beta_{12}}{\beta_{12}-1} > 0. \tag{2.84}$$

*The same holds for the partial derivatives of $\beta_{11}$ and the fraction $\frac{\beta_{11}}{\beta_{11}-1}$ w.r.t. $\sigma$ and $\alpha_1$.*

*Proof of Lemma 2.B.1.* Evaluate the total derivative of the quadratic equation, $\mathbb{Q}$, in Eq. (2.64) w.r.t. $\sigma$ and $\alpha_2$ at the positive root $\beta_{12}$:

$$\frac{\partial \mathbb{Q}}{\partial \beta_{12}}\frac{\partial \beta_{12}}{\partial \sigma} + \frac{\partial \mathbb{Q}}{\partial \sigma} = 0, \qquad \frac{\partial \mathbb{Q}}{\partial \beta_{12}}\frac{\partial \beta_{12}}{\partial \alpha_2} + \frac{\partial \mathbb{Q}}{\partial \alpha_2} = 0. \tag{2.85}$$

Thus we have that the partial derivatives given by

$$\frac{\partial \beta_{12}}{\partial \sigma} = -\frac{\partial \mathbb{Q}/\partial \sigma}{\partial \mathbb{Q}/\partial \beta_{12}}, \qquad \frac{\partial \beta_{12}}{\partial \alpha_2} = -\frac{\partial \mathbb{Q}/\partial \alpha_2}{\partial \mathbb{Q}/\partial \beta_{12}}. \tag{2.86}$$

The partial derivative of the quadratic w.r.t. $\beta_{12}$ is given as $\partial \mathbb{Q}/\partial \beta_{12} = \frac{1}{2}\sigma^2(2\beta_{12} - 1) + \alpha_2$. Since we know $\beta_{12} > 1$, and $\sigma,\ \alpha_2 > 0$, we have $\partial \mathbb{Q}/\partial \beta_{12} > 0$. Further we have the partial derivatives

$$\frac{\partial \mathbb{Q}}{\partial \sigma} = \sigma \beta_{12}(\beta_{12} - 1) > 0, \qquad \frac{\partial \mathbb{Q}}{\partial \alpha_2} = \beta_{12} > 0. \tag{2.87}$$

From this we have that Eq. (2.85) holds. Further, have the derivative of the fraction given as

$$\frac{\partial}{\partial\beta_{12}}\frac{\beta_{12}}{\beta_{12}-1}=\frac{-1}{(\beta_{12}-1)^2}<0 \tag{2.88}$$

and the related derivatives of the fraction in Eq. (2.84) are true. The same can be shown for $\beta_{11}$. □

From Proposition 2.2.1 we have that the second investment threshold given by

$$P_2(\tau_2)=P_2^*=\frac{\beta_{12}}{\beta_{12}-1}(\rho-\alpha_2)I_2. \tag{2.89}$$

Trivially, we see that $P_2^*$ is unaffected by $\alpha_1$ and $I_1$, as well as increasing in $I_2$. Using Lemma 2.B.1 and noting that $\beta_{12}>1$, the partial derivative of $P_2^*$ w.r.t. $\sigma$ give

$$\frac{\partial}{\partial\sigma}P_2^*=(\rho-\alpha_2)I_2\frac{\partial}{\partial\sigma}\frac{\beta_{12}}{\beta_{12}-1}\quad>0 \tag{2.90}$$

with the assumption that $\rho>\alpha_2$. For the first investment threshold, it is given by Proposition 2.2.2 as

$$P_1^*=\begin{cases}\frac{\beta_{11}}{\beta_{11}-1}(\rho-\alpha_2)(I_1+I_2) & \text{if } P_1^*\geq P_2^*\\ \left[\frac{I_1}{\left(1-\frac{\beta_{12}}{\beta_{11}}\right)D_2}\right]^{\frac{1}{\beta_{12}}} & \text{if } P_1^*<P_2^*\end{cases} \tag{2.91}$$

where $D_2$ is given in Proposition 2.2.1. For the case when $P_1^*\geq P_2^*$, using Lemma 2.B.1, the sign of the partial derivative w.r.t. $\sigma$ is given by

$$\frac{\partial}{\partial\sigma}P_1^*=(\rho-\alpha_2)(I_1+I_2)\frac{\partial}{\partial\sigma}\frac{\beta_{11}}{\beta_{11}-1}>0 \tag{2.92}$$

For the case when $P_1^*<P_2^*$, the partial derivative w.r.t. $\sigma$ becomes

$$\begin{aligned}\frac{\partial}{\partial\sigma}P_1^*=&\underbrace{\left[\frac{I_1}{(1-\frac{\beta_{12}}{\beta_{11}})D_2}\right]^{\frac{1}{\beta_{12}}}}_{>0}\left(\frac{1}{\beta_{12}}\left(\frac{I_1}{(1-\frac{\beta_{12}}{\beta_{11}})D_2}\right)^{-1}\frac{\partial}{\partial\sigma}\frac{I_1}{(1-\frac{\beta_{12}}{\beta_{11}})D_2}\right.\\&\left.+\ln\left[\frac{I_1}{(1-\frac{\beta_{12}}{\beta_{11}})D_2}\right]\frac{\partial}{\partial\sigma}\frac{1}{\beta_{12}}\right).\end{aligned} \tag{2.93}$$

Using the partial derivatives of $D_2$, and $\frac{\beta_{12}}{\beta_{11}}$ as given by

$$\frac{\partial}{\partial\sigma}P_2^* = (\rho-\alpha_2)I_2\frac{\sigma\beta_{12}(\beta_{12}-1)}{0.5\sigma^2(2\beta_{12}-1)+\alpha_2}\frac{1}{(\beta_{12}-1)^2} \tag{2.94}$$

$$\begin{aligned}\frac{\partial}{\partial\sigma}D_2 &= \frac{(P_2^*)^{-\beta_{12}}}{(\rho-\alpha_2)\beta_{12}^2}\left[-P_2^*\left(\frac{\partial}{\partial\sigma}\beta_{12}\right)(\beta_{12}\ln(P_2^*)+1)-(\beta_{12}-1)\beta_{12}\left(\frac{\partial}{\partial\sigma}P_2^*\right)\right]\\ &= \frac{(P_2^*)^{-\beta_{12}}}{(\rho-\alpha_2)\beta_{12}^2}\times\frac{\sigma\beta_{12}(\beta_{12}-1)}{0.5\sigma^2(2\beta_{12}-1)+\alpha_2}\times P_2^*\times\beta_{12}\ln(P_2^*)\end{aligned} \tag{2.95}$$

$$\frac{\partial}{\partial\sigma}\frac{\beta_{12}}{\beta_{11}} = \frac{1}{\beta_{11}^2}\left[\beta_{12}\frac{\sigma\beta_{11}(\beta_{11}-1)}{0.5\sigma^2(2\beta_{11}-1)+\alpha_1}-\beta_{11}\frac{\sigma\beta_{12}(\beta_{12}-1)}{0.5\sigma^2(2\beta_{12}-1)+\alpha_2}\right] \tag{2.96}$$

we obtain that

$$\frac{\partial}{\partial\sigma}P_1^* = \frac{P_1^*}{\beta_1(\beta_{11}-\beta_{12})}\left[(1+(\beta_{11}-\beta_{12})\ln\left(\frac{P_2^*}{P_1^*}\right)\frac{d}{d\sigma}\beta_{12}-\frac{\beta_{12}}{\beta_{11}}\frac{d}{d\sigma}\beta_{11}\right]. \tag{2.97}$$

Thus, $\frac{\partial P_1^*}{\partial\sigma} > 0$ implies that the following inequality must hold:

$$(1+(\beta_{11}-\beta_{12})\ln(\frac{P_2^*}{P_1^*})\frac{d}{d\sigma}\beta_{12}-\frac{\beta_{12}}{\beta_{11}}\frac{d}{d\sigma}\beta_{11} > 0.$$

Rearranging, and using the fact that

$$\ln\left(\frac{P_2^*}{P_1^*}\right) = \ln\left(\frac{(\beta_{11}-\beta_{12})I_2}{\beta_{11}(\beta_{12}-1)I_1}\right)^{\frac{1}{\beta_{12}}},$$

we obtain that $\frac{d}{d\sigma}P_1^* > 0$ if the following condition holds

$$I_2 < \frac{\beta_{11}(\beta_{12}-1)}{\beta_{11}-\beta_{12}}\times\exp\left\{\frac{\beta_{12}}{\beta_{11}-\beta_{12}}\left(\frac{\beta_{12}}{\beta_{11}}\frac{d\beta_{11}/d\sigma}{d\beta_{12}/d\sigma}-1\right)\right\}\times I_1$$

Inserting the expressions for $\frac{d\beta_{11}/d\sigma}{d\beta_{12}/d\sigma}$ from Lemma 2.85, we obtain the stated condition. □

### 2.B.6 Proof of Proposition 2.2.6

The second investment threshold, $P_2^*$, is given in Proposition 2.2.1, while the first threshold is given by Proposition 2.2.2. For the sensitivity to the initial drift rate,

$\alpha_1$, the partial derivative of the second investment threshold is trivially zero, as all terms are independent of $\alpha_1$.
For the case when $P_1^* \geq P_2^*$, the sign of the partial derivative of $P_1^*$ w.r.t. $\alpha_1$ is given by

$$\frac{\partial}{\partial \alpha_1} P_1^* = (\rho - \alpha_2)(I_1 + I_2) \frac{\partial}{\partial \alpha_1} \frac{\beta_{11}}{\beta_{11} - 1} > 0, \tag{2.98}$$

where the sign is given by Lemma 2.B.1 and the assumptions that $\rho > \alpha_2$, $I_1, I_2 > 0$. For the case when $P_1^* < P_2^*$, we have that

$$\begin{aligned} \frac{\partial}{\partial \alpha_1} P_1^* &= \left[\frac{I_1}{D_2}\right]^{\frac{1}{\beta_{12}}} \frac{\partial}{\partial \alpha_1} \left[\frac{1}{1 - \frac{\beta_{12}}{\beta_{11}}}\right]^{\frac{1}{\beta_{12}}} \\ &= \left[\frac{I_1}{D_2}\right]^{\frac{1}{\beta_{12}}} \frac{1}{\beta_{12}} \left[\frac{1}{1 - \frac{\beta_{12}}{\beta_{11}}}\right]^{\frac{1}{\beta_{12}} - 1} \frac{\partial}{\partial \alpha_1} \frac{1}{1 - \frac{\beta_{12}}{\beta_{11}}}. \end{aligned} \tag{2.99}$$

Since we know that $\beta_{12} < \beta_{11}$ from Corollary 2.2.4, and that $D_2 > 0$, and assume that $I_1 > 0$, the sign of the derivative is dependent on the last term. Using the fact that

$$\frac{\partial}{\partial \alpha_1} \left(1 - \frac{\beta_{12}}{\beta_{11}}\right)^{-1} = \frac{-\beta_{12}}{\beta_{11}^2 \left(1 - \frac{\beta_{12}}{\beta_{11}}\right)^2} \frac{\partial}{\partial \alpha_1} \beta_{11}, \tag{2.100}$$

we know that $\frac{\partial P_1^*}{\partial \alpha_1} > 0$, since Lemma 2.B.1 states that $\frac{\partial \beta_{11}}{\partial \alpha_1} < 0$. □

## 2.B.7 Proof of Proposition 2.2.7

The expressions for the second investment threshold, $P_2^*$, and the first threshold, $P_1^*$, are given by Proposition 2.2.1 and Proposition 2.2.2, respectively. The sensitivity to the boosted drift rate, $\alpha_2$, is then given for $P_2^*$ as

$$\frac{\partial}{\partial \alpha_2} P_2^* = I_2 \frac{\partial}{\partial \alpha_2} (\rho - \alpha_2) \frac{\beta_{12}}{\beta_{12} - 1} = I_2 \left[\frac{-\beta_{12}}{\beta_{12} - 1} + (\rho - \alpha_2) \frac{\partial}{\partial \alpha_2} \frac{\beta_{12}}{\beta_{12} - 1}\right]. \tag{2.101}$$

From Lemma 2.B.1, we have that the last term in the bracket is positive, meaning that the overall effect is ambiguous. The last term is given as

$$\frac{\partial}{\partial \alpha_2} \frac{\beta_{12}}{\beta_{12} - 1} = \frac{-\partial \beta_{12} / \partial \alpha_2}{(\beta_{12} - 1)^2} = \frac{\beta_{12}}{\frac{1}{2}\sigma^2 (2\beta_{12} - 1) + \alpha_2} \frac{1}{(\beta_{12} - 1)^2} \tag{2.102}$$

Thus, the derivative of the threshold becomes

$$\frac{1}{I_2}\frac{\partial}{\partial \alpha_2}P_2^* = \left[-1+(\rho-\alpha_2)\frac{1}{\frac{1}{2}\sigma^2(2\beta_{12}-1)+\alpha_2}\frac{1}{\beta_{12}-1}\right]\frac{\beta_{12}}{\beta_{12}-1}. \quad (2.103)$$

Noting that the last term is always positive, the sign of the derivative is dependent on the term in the brackets. Finding the negative region as

$$-1+(\rho-\alpha_2)\frac{1}{\frac{1}{2}\sigma^2(2\beta_{12}-1)+\alpha_2}\frac{1}{\beta_{12}-1} < 0, \quad (2.104)$$

which implies

$$\sigma^2 > \frac{\rho-\alpha_1\beta_{12}}{(\beta_{12}-1)(\beta_{12}-\frac{1}{2})}. \quad (2.105)$$

Thus, $\partial P_2^*/\partial\alpha_2$ is negative as long as Eq (2.105) holds. □

## 2.B.8 Proof of Proposition 2.3.1

The firm maximizes the expected net present value function at the time of investment, given by

$$V(P_2,K)-I-\xi K = \frac{P_2}{\rho-\alpha_2(K)} - I - \xi K, \quad (2.106)$$

where $\alpha_2(K)$ is given by Eq. (2.19). The first- and second-order derivatives of this value function are given by

$$\frac{\partial}{\partial K}\left[V(P_2,K)-\xi K\right] = \frac{P_2\frac{\partial}{\partial K}\alpha_2(K)}{(\rho-\alpha_2(K))^2} - \xi, \quad (2.107)$$

and

$$\frac{\partial^2}{\partial K^2}\left[V(P_2,K)-\xi K\right] = \frac{P_2\left[(\rho-\alpha_2(K))\frac{\partial^2}{\partial K^2}\alpha_2(K)+2\left(\frac{\partial}{\partial K}\alpha_2(K)\right)^2\right]}{(\rho-\alpha_2(K))^3}. \quad (2.108)$$

Setting the first-order derivative to zero, and rearranging, give the positive solution

$$\hat{K}^*(P_2) = \frac{\xi(\alpha_1-\rho)+\sqrt{P_2\xi(\rho-\epsilon-\alpha_1)}}{\xi\epsilon}, \quad (2.109)$$

and $K^*(P_2^*) = \max\{0, \hat{K}^*(P_2)\}$. Further the second-order derivative is given by

$$\frac{\partial^2}{\partial K^2} V(P_2, K) - \xi K = \frac{-2\epsilon(\rho - \epsilon - \alpha_1)}{(K\epsilon + \rho - \alpha_1)^3} P_2. \tag{2.110}$$

This is always negative, as we assume that $\rho - \epsilon - \alpha_1 > 0$, and $K$ is bounded by below by zero. The price process $P_2(t)$ is following a geometric Brownian Motion, so it will never be negative. Hence, $K^*(P_2)$ is the global maximum. The resulting optimal drift rate $\alpha_2^*(P_2)$ is given by

$$\begin{aligned} \alpha_2^*(P_2) = \alpha_2^*(K^*(P_2)) &= \rho - \epsilon - \frac{\rho - \epsilon - \alpha_1}{1 + K^*(P_2)} \\ &= \rho - \epsilon - \frac{\xi\epsilon(\rho - \epsilon - \alpha_1)}{\xi(-\rho + \epsilon + \alpha_1) + \sqrt{P_2\xi(\rho - \epsilon - \alpha_1)}} \\ &= \frac{(\epsilon - \rho)\sqrt{P_2\xi(\rho - \epsilon - \alpha_1)} + \rho\xi(\rho - \epsilon - \alpha_1)}{\xi(\rho - \epsilon - \alpha_1) - \sqrt{P_2\xi(\rho - \epsilon - \alpha_1)}} \end{aligned} \tag{2.111}$$

□

## 2.B.9 Proof of Corollary 2.3.2

Using the equation for $\hat{K}^*(P_2)$ in Proposition 2.3.1, and finding the inequality $\hat{K}^*(P_2) > 0$, we obtain

$$\frac{\xi(\alpha_1 - \rho) + \sqrt{P_2\xi(\rho - \epsilon - \alpha_1)}}{\xi\epsilon} > 0, \tag{2.112}$$

which after rearranging yields the inequality

$$P_2 > \frac{\xi(\rho - \alpha_1)^2}{\rho - \epsilon - \alpha_1}, \tag{2.113}$$

where we have used the assumptions that $\rho - \epsilon - \alpha_1 > 0$, $\epsilon > 0$, and $\xi > 0$. □

## 2.B.10 Proof of Proposition 2.3.3

Before investment, the value of the opportunity to invest follows from a Bellman equation similar to Proposition 2.2.2. Solving the resulting ordinary differential equation, and letting $\beta_{11}$ be the positive solution to the quadratic equation $\mathbb{Q} = \frac{1}{2}\sigma^2\beta_1(\beta_1 - 1) + \alpha_1\beta_1 - \rho = 0$, the value of the firm is then given as in Eq. (2.26). The the value-matching and smooth-pasting boundary conditions at the investment threshold $P_1^*$ are given by,

$$\begin{aligned} A P_1^{*\beta_{11}} &= \frac{P_1^*}{\rho - \alpha_2^*(P_1^*)} - I - \xi K^*(P_1^*) \\ \beta_{11} A P_1^{*\beta_{11}-1} &= \frac{d}{dP_1}\left[\frac{P_1}{\rho - \alpha_2(K_1^*(P_1))} - I - \xi K^*(P_1)\right]_{P_1 = P_1^*}. \end{aligned} \tag{2.114}$$

Defining the term in brackets as $f(P_1, K(P_1))$, we obtain that the total derivative is given as

$$A P_1^{*\beta_{11}} \frac{\beta_{11}}{P_1^*} = \frac{\partial}{\partial P_1} f(P_1, K(P_1)) + \frac{\partial f(P_1, K(P_1))}{\partial K} \frac{\partial K(P_1)}{\partial P_1} \tag{2.115}$$

However, by the construction of the first-order maximization of $K$, we have that $\partial f(\cdot)/\partial K = 0$, and we obtain

$$A P_1^{*\beta_{11}} = \frac{P_1^*}{\beta_{11}} \frac{1}{\rho - \alpha_2(K_1^*(P_1^*))} \tag{2.116}$$

Subtracting the equations from the conditions of continuity and smoothness, we obtain

$$\frac{\beta_{11} - 1}{\beta_{11}} \frac{P_1^*}{\rho - \alpha_2(K^*(P_1^*))} - I - \xi K^*(P_1^*) = 0. \tag{2.117}$$

Using the equations for $K_1^*(P_1)$ and $\alpha_2^*(P_1)$, this becomes the second-order polynomial for $\sqrt{P_1^*}$ given by

$$P_1^* - \left(\frac{\beta_{11}}{\beta_{11} - 1} + 1\right) \sqrt{\xi(\rho - \epsilon - \alpha_1)} \sqrt{P_1^*} + \frac{\beta_{11}}{\beta_{11} - 1}(\xi(\rho - \alpha_1) - I\epsilon) = 0. \tag{2.118}$$

Solving for $\sqrt{P_1^*}$, we obtain

$$\begin{aligned} \sqrt{P_1^*} &= \frac{1}{2}\left(\frac{\beta_{11}}{\beta_{11} - 1} + 1\right) \sqrt{\xi(\rho - \epsilon - \alpha_1)} \\ &\pm \sqrt{\frac{1}{4}\left(\frac{\beta_{11}}{\beta_{11} - 1} + 1\right)^2 \xi(\rho - \epsilon - \alpha_1) + \frac{\beta_{11}}{\beta_{11} - 1}(I\epsilon - \xi(\rho - \alpha_1))}. \end{aligned} \tag{2.119}$$

Squaring this and rearranging, we obtain the solution for $P_1^*$, as given by

$$P_1^* = \frac{1}{2}\left(\frac{2\beta_{11}-1}{\beta_{11}-1}\right)^2 \xi(\rho-\epsilon-\alpha_1) + \frac{\beta_{11}}{\beta_{11}-1}(I\epsilon - \xi(\rho-\alpha_1))$$
$$\pm \frac{1}{2}\left(\frac{2\beta_{11}-1}{\beta_{11}-1}\right)\sqrt{\xi(\rho-\epsilon-\alpha_1)\left(\frac{\beta_{11}}{\beta_{11}-1}(I\epsilon-\xi(\rho-\alpha_1)+\xi(\rho-\epsilon-\alpha_1))\right)}. \tag{2.120}$$

Taking the minus part of Eq. (2.119), we derive when this is smaller than zero. This yields a non-admissible solution, as the square root of the threshold cannot be negative. Setting $\sqrt{P_1^*} < 0$, we obtain

$$\frac{\beta_{11}}{\beta_{11}-1}(I\epsilon - \xi(\rho-\alpha_1)) > 0. \tag{2.121}$$

We know the fraction is greater than zero, since $\beta_{11} > 1$. Thus, there is only one non-negative root for the investment threshold, i.e. $\sqrt{P_1^*} > 0$, if the following condition holds

$$I\epsilon > \xi(\rho-\alpha_1). \tag{2.122}$$

□

### 2.B.11 Proof of Corollary 2.3.4

For Eq. (2.120) to have real-valued solutions, the term in the square root must be greater than or equal to zero. When rearranging this term, we obtain the condition

$$I\epsilon \geq \xi(\rho-\alpha_1) - \xi(\rho-\epsilon-\alpha_1)\frac{\beta_{11}-1}{\beta_{11}}. \tag{2.123}$$

□

### 2.B.12 Proof of Proposition 2.3.5

As seen in Section 2.B.1, the value of the second-stage option is dependent on the value of the drift, and therefore becomes a function of the control $K$ in the controlled case. However, the choice of $K$ is undertaken before the second-stage option exists, and $K$ can therefore be considered a constant in this region. Thus, the proof follows analogously the proof in Section 2.B.1, with the terms being functions of $K$ through the dependence of $\alpha_2(K)$ on $K$. □

### 2.B.13 Proof of Proposition 2.3.6

For the case with controlled change in drift, the value of the option to invest in the first-stage is, similarly to the fixed case presented in Proposition 2.2.2, dependent on the ordering of the threshold. The value of the option to invest in the first stage, $F_1(P_1)$, for a given price $P_1$, is given by the Bellman equation in Appendix 2.B.2. The value function is therefore given as $F_1(P_1) = D_1 P_1^{\beta_{11}}$, where $D_1$ is a parameter to be decided and $\beta_{11}$ the positive solution to the fundamental equation in Eq. (2.71). At the investment threshold, the firm must chose optimal value of $K$, and pay the investment cost. The value-matching and boundary conditions is dependent on the ordering of the thresholds.

**If $\mathbf{P_1^* \geq P_2^*}$** the value of the option to invest is given by

$$F(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^* \\ \max_K \left\{ \frac{P_1}{\rho - \alpha_2(K)} - I_1 - I_2 - \xi K \right\} & \text{if } P_1 \geq P_1^* \end{cases} \tag{2.124}$$

This is the same situation as in Proposition 2.3.3, with $I = I_1 + I_2$, proven in Appendix 2.B.10

**If $\mathbf{P_1^* < P_2^*}$** the value of the option to invest is given by

$$F(P_1) = \begin{cases} D_1 P_1^{\beta_{11}} & \text{if } P_1 < P_1^* \\ \max_K \left\{ D_2(K) P_1^{\beta_{12}(K)} - I_1 - \xi K \right\} & \text{if } P_1 \geq P_1^* \end{cases} \tag{2.125}$$

where $D_2(K)$ is given by Eq. 2.6. Defining $K^*$ as the maximizing argument, the value-matching and smooth-pasting boundary conditions at the threshold becomes

$$\begin{aligned} D_1 P_1^{*\beta_{11}} &= D_2(K^*) P_1^{*\beta_{12}(K^*)} - I_1 - \xi K^*, \\ \beta_{11} D_1 P_1^{*\beta_{11}-1} &= \beta_{12}(K^*) D_2(K^*) P_1^{*\beta_{12}(K^*)-1}, \end{aligned} \tag{2.126}$$

Multiplying the second equation by $\frac{P_1^*}{\beta_{11}}$, and subtracting the left- and right-hand-sides of the equations yields that $P_1^*$ must satisfy the equation given by

$$(\beta_{12}(K^*) - \beta_{11}) D_2(K^*) P_1^{*\beta_{12}(K^*)} + (I_1 + \xi K^*)\beta_{11} = 0. \tag{2.127}$$

□

### 2.B.14 Proof of Proposition 2.3.7

Taking the value of the first investment threshold, $P_1^*$, given by Eq. (2.120), we compute the derivative $\frac{dP_1^*}{d\sigma}$. We obtain that

$$\frac{dP_1^*}{d\sigma} = \frac{1}{2}\xi(\rho-\epsilon-\alpha_1)\frac{d}{d\sigma}\left(\frac{2\beta_{11}-1}{\beta_{11}-1}\right)^2 + (I\epsilon-\xi(\rho-\alpha_1))\frac{d}{d\sigma}\frac{\beta_{11}}{\beta_{11}-1} + \frac{d}{d\sigma}\left[\frac{1}{2}\frac{2\beta_{11}-1}{\beta_{11}-1}\sqrt{\xi(\rho-\epsilon-\alpha_1)(\xi(\rho-\epsilon-\alpha_1)+\frac{\beta_{11}}{\beta_{11}-1}(I\epsilon-\xi(\rho-\alpha_1)))}\right] \tag{2.128}$$

The sign of the first term is given as

$$\frac{d}{d\sigma}\left(\frac{2\beta_{11}-1}{\beta_{11}-1}\right)^2 = \frac{4\beta_{11}-2}{\beta_{11}-1} \times \frac{-d\beta_{11}/d\sigma}{(\beta_{11}-1)^2} > 0. \tag{2.129}$$

This is larger than zero, since we know from Lemma 2.B.1 that $d\beta_{11}/d\sigma < 0$. The sign of the second term in Eq. (2.128) is given by $\frac{d}{d\sigma}\frac{\beta_{11}}{\beta_{11}-1}$, which we know from Lemma 2.B.1 is larger than zero. The sign of the last term in Eq. (2.128) is given by

$$\frac{d}{d\sigma}[\cdots] = \sqrt{\cdots} \times \frac{d}{d\sigma}\left[\frac{1}{2}\frac{2\beta_{11}-1}{\beta_{11}-1}\right] + \frac{1}{2}\frac{2\beta_{11}-1}{\beta_{11}-1} \times \frac{d}{d\sigma}\sqrt{\cdots}. \tag{2.130}$$

In this equation, we know from earlier that the first term is positive, as long as the square-root is well-defined. For the second term, the sign is determined by the derivative of the square-root, which is given by

$$\frac{d}{d\sigma}\sqrt{\cdots} = \frac{1}{2\sqrt{\cdots}} \times (I\epsilon-\xi(\rho-\alpha_1)) \times \frac{d}{d\sigma}\frac{\beta_{11}}{\beta_{11}-1}. \tag{2.131}$$

We know that $\frac{d}{d\sigma}\frac{\beta_{11}}{\beta_{11}-1} > 0$. Therefore, if $I\epsilon - \xi(\rho-\alpha_1) > 0$, the overall term is positive. This is exactly the necessary condition for a unique threshold, given in Section 2.B.10. Since all the evaluated derivatives are postive, we know that $dP_1^*/d\sigma > 0$. □

### 2.B.15 Proof of Proposition 2.4.1

This result is similar to the case proved in Section 2.B.10, with a different impulse function. However, now the maximization does not yield an analytical result for $K^*$. Thus, the value of the optimal control is given implicitly as the maximizing argument. Further, using conditions of continuity and smoothness of the value function at the investment threshold, similar to Section (2.B.2), it is straightforward to see that the investment threshold, $P_1^*$, must satisfy the expression given in Eq. (2.49). □

## 2.B.16 Proof of Proposition 2.4.2

This result follows the case proved in Section 2.B.10, with the investment cost for the first project stage given as $I_K(K) = I_1 + \xi K^{\eta}$. The value of the optimal control is given implicitly as the maximizing argument for each case of the ordering of the thresholds $P_1^*$ and $P_2^*$. Further, using conditions of continuity and smoothness of the value function at the investment threshold, similar to Section (2.B.2), $P_1^*$, must satisfy the expression given in Eq. (2.56). □

# 3 Optimal switching between projects with different profitability and uncertainty characteristics

**Abstract:**

We consider a firm that is currently producing an established product subject to a stochastic environment. The firm has the one-time opportunity to undertake an irreversible investment to switch to a new product, which changes the drift and volatility of the firm's underlying stochastic profit flow. We show that it is optimal to invest in the new product if an increase in the expected growth outweighs the increase in risk from switching. We find that the effect of uncertainty on the optimal investment strategy is not straightforward. The overall effect of uncertainty is determined by the interplay between the value of waiting and the effect of Jensen's inequality. Contrary to the standard real options result, that higher uncertainty delays investment, we show that an increase in volatility in the old market can both accelerate or delay the investment. We perform an analysis of the antecedents of this non-monotonic effect of uncertainty, and provide extensive economical reasoning for the results. Further, we show that an investment opportunity with changing characteristics of the stochastic environment and constant profit function, can be transformed to a case of changing profit function and constant parameters of the stochastic process.

## 3.1 Introduction

For a firm currently active in an established product market, switching to a new product is characterized by a change in the firm's stochastic environment. In deciding when to invest in the new product, the firm should take into account how the decision will affect the development of its expected profits, and the related uncertainty from changing the product market within which it operates. Highly innovative products might have the potential to become blockbusters, but also come with additional uncertainty stemming from risks related to how consumers respond to the new product. In the automotive industry, a pressing issue for traditional automobile manufacturers has been whether to enter the novel market for electric vehicles (EVs), and if so, when to enter. The industry incumbents have well-defined markets for their traditional products, but might be missing out on the growing profits from green transportation[1]. The EV market is growing faster than the market for traditional cars[2,3]. However, the younger EV market is exposed to higher uncertainty due to demanding technological challenges[4] and uncertain adoption rates of consumers[5]. Therefore, entering the EV market might change both the expected growth and uncertainty of the firm's profits.

This paper studies the investment problem of a monopolistic firm that has the one-time opportunity to switch from one stochastic profit flow to another. Such investment problems under uncertainty can be characterized as real options (Dixit and Pindyck, 1994; Trigeorgis, 1996), recognizing that the firm has discretion of the investment timing. However, a common assumption in modelling real options problems is that the uncertainty faced by the firm is both independent of the firm's actions and constant. In that, the fundamental uncertainty is modelled by an exogenous stochastic process, often a geometric Brownian motion with constant drift and volatility parameters. Acknowledging that the investment decisions of the firm can alter the characteristics of its uncertain profits, we need to account for the fact that the parameters of the stochastic process can be endogenously changed by the decision of the firm. We model the switch in products as a change from one stochastic regime to another. We model the underlying stochastic process as a geometric Brownian motion, but with different drift and volatility coefficients for each regime. Thus, the firm changes the characteristics of its stochastic environment from investing in the new product.

Most earlier work on investment decisions under uncertainty is limited to models where, upon investment, the functional form of the profit function change, but not the parameters of the underlying stochastic process (Dixit and Pindyck, 1994). One of the contributions here is Alvarez and Stenbacka (2001), who studies a

[1] The traditional German car manufacturers see a stagnation in their traditional markets, and set lofty goals for their EV portfolio. https://www.economist.com/news/business/21737534-coddled-successive-governments-industry-dogged-dieselgate-lagging-electric

[2] Boston Consulting Groups predictions on the EV market going forward: https://www.bloomberg.com/news/articles/2017-11-02/battery-powered-cars-to-be-half-of-global-auto-market-by-2030

[3] Statistics on the sales volumes of EVs: http://www.ev-volumes.com/

[4] How the technological uncertainty of the EV market affects the forecasts for the market: https://www.bcg.com/publications/2018/electric-car-tipping-point.aspx

[5] On the challenges in the EV market from consumer behaviour: https://www.bloomberg.com/news/features/2017-12-19/the-near-future-of-electric-cars-many-models-few-buyers

setting where the profit function can be improved by adopting a new technology at a later stage. Their model does however assume that the long-term growth and the volatility driving the profits remain constant, and find that increased uncertainty delays the decision to invest. In contrast, our paper allows the parameters of the stochastic process driving the profits to be changed after investment. Under this assumption, increased uncertainty might not delay investment, but accelerate it. Another strand of the literature allows the stochastic process that the firm faces to be subject to different regimes, and that the process alternates between these regimes over time. Bensoussan et al. (2012) introduce a model where the firm is exposed to exogenous and random switches of regimes, in which the regimes differ in the growth and volatility of the underlying stochastic process. Thus, the firm acknowledges that it might be facing different structures in its stochastic environment, but has no way of influencing the switching process itself. Like other studies on investment under uncertainty, Bensoussan et al. (2012) find that the optimal strategy of the firm is characterized by an investment threshold, where the firm invests as soon as it observes the exogenous process to reach the threshold. In a similar vein, Bollen (1999) studies the life-cycle problem of products, where the demand for a firm's output is randomly switched from an increasing to a decreasing regime. The author shows that the assumption of a constant regime for the stochastic demand can yield significant errors in the value of an investment opportunity. The aforementioned models allow the stochastic process to switch regimes, but do not allow the firm to affect these switches endogenously. Thus, they still assume that the firm is passive w.r.t. its stochastic environment. One attempt on making the underlying stochastic dependent on the firms actions is presented in Busch et al. (2013). They extend the setting of Bensoussan et al. (2012), by allowing the firm to alter the probability of a switch taking place through its investments. This represents an indirect influence of the firm on the stochastic environment, while in our paper we aim to model this explicitly.

For models considering a firm's possibility to directly influence the underlying stochastic process, the available literature is sparse. Kwon (2010) is one of the first to account for changing parameters of the underlying stochastic process in the investment decision. Kwon (2010) studies a firm currently producing an aging product, with the one-time opportunity to invest in a new product. He models the firm's initial profits by an arithmetic Brownian motion with negative drift. Investing in the new product increases the drift of the process. Assuming the resulting drift to still be negative, the paper also considers an exit option after investment, finding that increased uncertainty can both delay or speed up investment in the new product. Matomäki (2013, Article 1) generalizes the problem of Kwon (2010) by allowing for general specifications of the stochastic process, as well as including changes in volatility. He confirms the results of Kwon (2010), that increased uncertainty can both delay or expedite investment. However, both these papers include an exit option in addition to the investment option. Thus, the reason for the non-monotonic effect of uncertainty in these studies is that increasing the volatility increases the value of the exit option, which then makes it more attractive to invest sooner. Contrarily, the value of waiting from the option to invest in the new product increases in uncertainty, which therefore induces a delay in investment. Further, Hagspiel et al. (2016) extends Kwon (2010) by allowing for capacity discretion for the new product, while keeping the volatility constant and only switching the drift. This induces a monotonic effect of uncertainty, where increasing the volatility leads to an increase of the investment threshold. In the work of Kwon (2010), increased uncertainty makes it more likely that the exit option will be used, as the firm has a fixed capacity of production of the second product and cannot adjust to the increased uncertainty in the market. In Hagspiel et al. (2016), the possibility of optimizing the capacity restores the monotonic effect of uncertainty, which suggests that the firm can adjust the capacity to the change in volatility, and therefore making the increased value of the exit option smaller than the increased value of the option to invest in the second product. The aforementioned models with direct changes to the parameters of the stochastic process all include an exit option, which we do not consider. Our work is in line with Alvarez and Stenbacka (2003), as they look at a situation where the firm can invest in a new product, with changing parameters of the stochastic environment, and not including other embedded options. Upon investment, they assume that the volatility of the underlying stochastic process changes, while the drift is not affected. They find that uncertainty can indeed both delay and accelerate investment, under both concave and convex profit functions. This non-monotonic behaviour is a pure effect of the change in parameters of the underlying stochastic process, and does not arise from the presence of a compound option, like an option to exit.

In solving our problem, we utilize the theoretical framework of stochastic optimal control (Pham, 2009), relying on a viscosity solution approach to solving stochastic optimal control problems (Crandall et al., 1992). We apply the results of Ly Vath and Pham (2007) for a general switching problem, which permit multiple switches between various regimes. Our problem is a special case of the multidimensional problem studied therein. Thus, we employ well-defined mathematical results to investigate a novel investment problem, in addition analyzing how the optimal strategy depends on the underlying parameters. We show that it is optimal to invest in the new product if an increase in the expected growth from switching regimes outweighs the increase in risk. Our results confirm the findings of Alvarez and Stenbacka (2003) under cases where the drift is not changed from investment. Also allowing for a potential change in drift, we find that the effect of uncertainty on the optimal strategy is more complex and multifaceted than the results of Alvarez and Stenbacka (2003) imply. We show that increasing the uncertainty in the profits from the initial product can both delay or accelerate the investment. Further, we provide extensive economic reasoning for the determinants of this non-monotonic behaviour. Finally, we show that the posed problem can be transformed to a problem of constant parameters of the stochastic process, but with a change in the profit function after investment.

The rest of this paper is structured as follows. Section 2 presents the model, using the mathematical results of Ly Vath and Pham (2007) of optimal switching. Section 3 presents the comparative statics of the resulting investment threshold. Section 4 presents how a problem of changed profit function but constant parameters of the stochastic process can be transformed to a problem of constant profit function but changing process parameters. Section 5 provides some concluding remarks and possibilities for future work, while Appendix A presents proofs of all propositions.

## 3.2 Model

The firm currently operates in a market with an established product, denoted as regime $i = 1$. It holds a one time opportunity to invest in a new product, therewith switching to a regime $i = 2$. Switching incurs an irreversible investment cost. The profitability of the firm is uncertain, and assumed to follow a stochastic process given as a geometric Brownian motion with regime-dependent diffusion parameters. We define a stochastic process $\theta_i^{\vartheta}$ taking values in $\mathbb{R}_+^* = (0, \infty)$ which satisfies the stochastic differential equation

$$d\theta_i(t) = \alpha_i \theta_i(t) dt + \sigma_i \theta_i(t) dW_t, \qquad \theta_i(0) = \vartheta, \; i = 1, 2, \tag{3.1}$$

where $W$ is a Wiener process, and $\alpha_i$ and $\sigma_i$ are drift and volatility coefficients of the process $\theta$ in regime $i$ at time $t$, respectively. The drift parameter of each regime represents the general drift of the market of the given product, while the volatility parameter represents the uncertainty inherent in the given market. We assume that the driving Wiener process $W$ is the same in both regimes, and only the growth and diffusion parameters change. This accounts for a case where the firm changes its product offering when switching regime, but stays within the same general market structure or consumer base for its products. The profit flow per time period for the firm in regime $i$ is given by $\pi(\theta_i(t)) = \eta \theta_i^{\gamma}(t)$, where $\eta > 0$ and $\gamma \in (0, 1]$ are constants. We assume without loss of generality that there are no variable or fixed cost for producing the product. The profit flow is Hölder continuous and thus satisfy linear growth conditions with $\gamma \in (0, 1]$, where $\gamma = 1$ implies Lipschitz continuity[6]. This condition on the value function is necessary

[6]Hölder continuity implies that a function $f$ satisfies $\mid f(x) - f(y) \mid^{\alpha} \leq C \mid x - y \mid^{\alpha}$ for some finite constant $C$ and $\alpha \in (0, 1)$. Lipschitz continuity arises when $\alpha = 1$ in this expression. This also assures that the expectation of the form $\mathbb{E}\left[\int_t^T f(\theta(t), t) dt\right]$ is well defined (Pham, 2005), where $\theta(t)$ is a continuous-time stochastic process.

to utilize the approach of Ly Vath and Pham (2007), and ensures that the profit function is continuous and cannot explode in value.

The value function of the investment problem, given that the firm is currently in regime $i = 1, 2$, is denoted by $v_i$. These are the unique continuous viscosity solutions with linear growth conditions on $(0, \infty)$ and boundary conditions $v_i(0^+) = (-g_{ij})_+$, $i \neq j$ to the system of variational inequalities (Ly Vath and Pham, 2007, Thm. 3.4)

$$\begin{aligned} &\min\{rv_1 - \mathcal{L}_1 v_1 - \pi_1,\ v_1 - (v_2 - g_{12})\} = 0, \\ &\min\{rv_2 - \mathcal{L}_2 v_2 - \pi_2,\ v_2 - (v_1 - g_{21})\} = 0. \end{aligned} \tag{3.2}$$

Here $r$ is the discount rate, $g_{ij}$ is the cost of switching from regime $i$ to $j$, and $\mathcal{L}_i$ the second-order operator associated with the diffusion $\theta$ when the system is in regime $i$, which for an arbitrary $C^2$-function $\varphi$ on $(0, \infty)$ is given as $\mathcal{L}_i \varphi = \frac{1}{2}\sigma_i^2 \theta^2 \frac{\partial^2}{\partial \theta^2}\varphi + \alpha_i \theta \frac{\partial}{\partial \theta}\varphi$. We assume that $r > \alpha_i\ \forall i$, so that the drift in each region is not higher than the discount factor. This assures that investment might be optimal for the firm. Otherwise, the optimal strategy would always be to wait, as the profits would grow at a higher rate than the discount factor. The regions of $\theta$ where it is optimal for the firm to switch from regime $i$ to $j$ are denoted by $\mathcal{S}_i = \mathcal{S}_{ij}$ and defined as

$$\mathcal{S}_i = \{\vartheta > 0\colon v_i(\vartheta) = v_j(\vartheta) - g_{ij}\}, \quad i, j = 1, 2, \quad i \neq j. \tag{3.3}$$

The firm would switch from regime $i$ to $j$ for any realizations of the diffusion process for which the optimal value function in the regime equals the value of the other regime, minus the switching cost. This is refereed to as the *stopping regions* of the investment problem. We define an upper and lower threshold of $\theta$ where it is optimal to switch from regime $i$ as $\overline{\vartheta}_i^* = \sup \mathcal{S}_i \in [0, \infty]$ and $\underline{\vartheta}_i^* = \inf \mathcal{S}_i \in [0, \infty]$, where the convention $\inf \emptyset = \infty$ ensures that $\underline{\vartheta}_i^* = \infty$ if $\mathcal{S}_i = \emptyset$. I.e., if the stopping region is empty, it is never optimal to switch.

The first term of the maximization in the system of variational inequalities in Eq. (3.2) denotes the value function for $i = 1, 2$ when the firm would optimally stay in regime $i$ and not switch, i.e. the *continuation region*. This term yields a second-order ordinary differential equation for $i = 1, 2$ given by

$$rv_i - \mathcal{L}_i v_i - \pi_i = 0. \tag{3.4}$$

Omitting the particular term stemming from the running profit function $\pi_i$, this ODE has the homogeneous solution given by

$$v_i(\vartheta) = A\vartheta^{\beta_1^i} + B\vartheta^{\beta_2^i}, \tag{3.5}$$

for some constants $A$ and $B$, and

$$\beta_1^i = -\frac{\alpha_i}{\sigma_i^2} + \frac{1}{2} + \sqrt{\left(-\frac{\alpha_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}}, \tag{3.6}$$

$$\beta_2^i = -\frac{\alpha_i}{\sigma_i^2} + \frac{1}{2} - \sqrt{\left(-\frac{\alpha_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}}. \tag{3.7}$$

It can be shown that $\beta_1^i > 1$ and $\beta_2^i < 0$ (see e.g. Dixit and Pindyck (1994)). Further denote the particular solution to the ODE in Eq. (3.4) as $\hat{V}_i(\vartheta)$, subject to the boundary condition $\hat{V}_i(0^+) = \pi(0) = 0$. $\hat{V}_i$ is then defined as

$$\hat{V}_i(\vartheta) = \mathbb{E}\left[\int_0^\infty e^{-rt}\pi(\hat{\theta}_i^\vartheta(t))\right] = \frac{\eta\vartheta^\gamma}{r - \alpha_i\gamma + \frac{1}{2}\sigma_i^2\gamma(1-\gamma)} = K_i\eta\vartheta^\gamma, \tag{3.8}$$

where $\hat{\theta}_i^\vartheta$ is the solution to the SDE $d\hat{\theta}(t) = \alpha_i\hat{\theta}(t)dt + \sigma_i\hat{\theta}(t)dW_t$, with $\hat{\theta}(0) = \vartheta$, and we define the perpetual multiplier $K_i$ as

$$K_i = \frac{1}{r - \alpha_i\gamma + \frac{1}{2}\sigma_i^2\gamma(1-\gamma)} > 0. \tag{3.9}$$

The particular solution for each regime represents the value of never switching, i.e. the value of the firm given that it produces product $i = 1, 2$ forever.

The model outlined above follows the work of Ly Vath and Pham (2007) and would generally allow for multiple switches between the two regimes. However, we focus on a single switch case from an initial regime $i = 1$ to a regime $i = 2$. Switching is costly and irreversible, so that $g_{12} > 0$. Switching back is not possible in our case. Using Theorem 4.1 in Ly Vath and Pham (2007), the existence of a finite switching threshold $\underline{\vartheta}_1^*$ depends on the difference of the value of staying in the regimes in perpetuity, i.e. $\hat{V}_2 - \hat{V}_1$, given that there is no salvage value of switching back. Formally, the result says that for $i, j = 1, 2$, $i \neq j$, assuming $g_{ij} > 0$ and $g_{ji} \geq 0$, if $\hat{V}_i = \hat{V}_j$, then $\mathcal{S}_i = \emptyset$ and $\underline{\vartheta}_1^* = 1$. However, if $\hat{V}_j > \hat{V}_i$, then $\mathcal{S}_i = [\underline{\vartheta}_i^*, \infty)$ with $\underline{\vartheta}_i^* \in (0, \infty)$ and $\mathcal{S}_j = \emptyset$. This assures that after switching from the initial regime $i = 1$, switching back is not optimal, as long as there is a non-negative cost of switching back. The intuitive explanation of this is that if the value of producing the two products forever are equal, the firm has no incentive to pay a positive investment cost to switch products, and the switching threshold is infinite. Conversely, if one of the products has a higher value of producing in perpetuity, there is a finite threshold at which the firm should pay an investment cost to obtain the more profitable profit stream. Given no salvage value there is never an incentive to switch back to the less valuable regime. We can now investigate the optimality of switching, for different combinations of the diffusion parameters. The following proposition presents the conditions on the diffusion parameters for the existence of an optimal finite switching threshold.

**Proposition 3.2.1.**

(i) *If $\alpha_1 = \alpha_2$ and $\sigma_1 = \sigma_2$, then $K_1 = K_2$ and $\mathcal{S}_1 = \emptyset$. Therefore, it is never optimal to switch regime in this case.*

(ii) *If $\alpha_1 \neq \alpha_2$ and $\sigma_1 = \sigma_2$, then it is optimal to switch for some threshold $\underline{\vartheta}_i^* \in (0, \infty)$ if $\alpha_1 < \alpha_2$, but never optimal otherwise.*

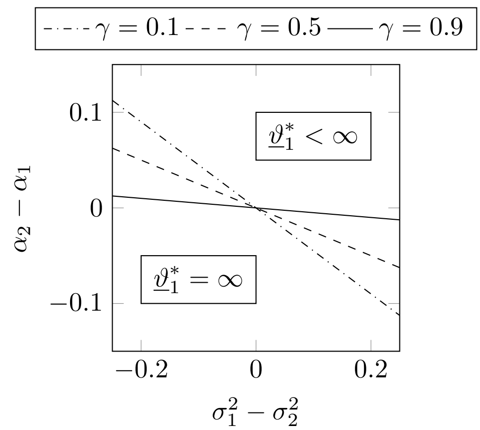
(iii) *If $\alpha_1 = \alpha_2$ and $\sigma_1 \neq \sigma_2$, then it is optimal to switch for some finite threshold if $\gamma \in (0, 1)$ and $\sigma_1 > \sigma_2$, but never optimal if $\gamma = 1$ or if $\sigma_1 < \sigma_2$.*

(iv) *If $\alpha_1 \neq \alpha_2$ and $\sigma_1 \neq \sigma_2$, the existence of an optimal switching threshold depends both on the difference in drift and volatility for the regimes. There exists a finite investment threshold $\underline{\vartheta}_1^*$ if $K_2 > K_1$, which is equivalent to the condition*

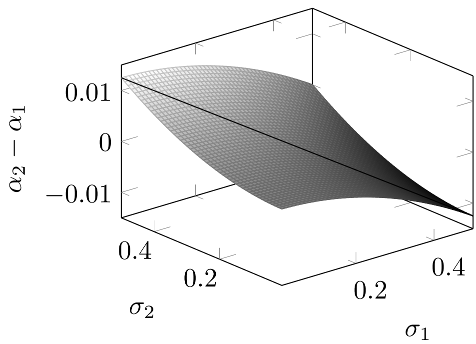
$$\alpha_2 - \alpha_1 + \frac{1}{2}(1 - \gamma)(\sigma_1^2 - \sigma_2^2) > 0. \tag{3.10}$$

We see from Proposition 3.2.1 (iii) that the effect of uncertainty is dependent on whether $\gamma = 1$ or $\gamma \in (0, 1)$. If $\gamma = 1$, the profit functions become linear in the diffusion process and the uncertainty of the regime does not affect the expected present value of the perpetuity. Thus, the existence of a finite switching threshold is solely dependent on the difference in drift, i.e. the sign of $\alpha_2 - \alpha_1$. However, if $\gamma \in (0, 1)$, the profit function becomes concave in the diffusion process, and a change in volatility alone can induce switching. The intuition for why a concave profit function leads to dependence on the volatility for the value of the perpetuity, is that a downward shock has a larger negative effect on the firms profit than the positive effect of an equal upward shock. Since the negative effect dominates, an increase in the uncertainty yields a lower value of the perpetuity since shocks (both negative and positive) are more likely with increased uncertainty of the diffusion process. In the case of $\gamma = 1$, the effects of downward and upward shocks of equal size on the profit are equal, so the net effect is zero, making the expected value independent of uncertainty.

If both the drift and the volatility in the regimes are different, Proposition 3.2.1 (iv) indicates that the optimality of switching depends on the interplay between the parameters of the two regimes. Even if the drift of the second regime is higher than for the first, it is still not optimal to switch if $\sigma_2$ is sufficiently high compared to $\sigma_1$. If $\alpha_1 > \alpha_2$, there is an upper bound on $\sigma_2$ for switching to be optimal,




(a) Minimum change in drift $\alpha_2 - \alpha_1$ necessary for the existence of a finite investment threshold $\underline{\vartheta}_1^*$, as a function of the change in squared volatility $\sigma_1^2 - \sigma_2^2$, for $\gamma \in \{0.1, 0.5, 0.9\}$.




(b) Minimum change in drift $\alpha_2 - \alpha_1$ necessary for the existence of a finite investment threshold $\underline{\vartheta}_1^*$, as a function of the volatilities $\sigma_1$ and $\sigma_2$, assuming $\gamma = 0.9$. The black line represents $\gamma = 0.9$ in Fig. 3.1a.

Figure 3.1: Minimum required change in drift $\alpha_2 - \alpha_1$ for the existence of a finite investment threshold $\underline{\vartheta}_1^*$.

given by

$$\sigma_2^2 < \sigma_1^2 - \frac{2(\alpha_1 - \alpha_2)}{1-\gamma}. \tag{3.11}$$

Indeed, if $\alpha_1 > \alpha_2$ and $0 < \sigma_1 < \sqrt{\frac{2(\alpha_1-\alpha_2)}{1-\gamma}}$, it can never be optimal for the firm to switch, for any positive volatility $\sigma_2$. Figure 3.1 shows the minimum required change in drift from investing necessary for the existence of a finite threshold $\underline{\vartheta}_1^*$, as a function of $\sigma_1$ and $\sigma_2$. We see that the required change in drift increases in $\sigma_2$ and decreases in $\sigma_1$, as a decrease in the change of volatility $\sigma_1 - \sigma_2$ from investing makes switching less attractive. In this, the firm requires a higher boost in drift from introducing the new product for switching to be optimal. Figure 3.1a shows how the steepness of the threshold for the required boost in drift increases when $\gamma$ decreases. This is due to the fact that a lower $\gamma$ yields a more concave profit function, so that the asymmetric effect of shocks discussed in the last paragraph is strengthened. Given that a finite optimal threshold exists, the following proposition presents the optimal value function of the firm and the investment threshold.

**Proposition 3.2.2.** *Assuming that the conditions in Proposition 3.2.1 for the existence of a finite switching threshold hold, i.e.* $K_2 > K_1$*, the value of the firm, given that it is currently in regime* $i = 1$*, is given by*

$$v(\vartheta) = v_1(\vartheta) = \begin{cases} A\vartheta^{\beta_1^1} + \hat{V}_1(\vartheta), & \vartheta < \underline{\vartheta}_1^*, \\ \hat{V}_2(\vartheta) - g_{12}, & \vartheta \geq \underline{\vartheta}_1^*, \end{cases} \tag{3.12}$$

*where* $\hat{V}_1(\vartheta)$ *and* $\hat{V}_2(\vartheta)$ *are given by Eq. (3.8). The constant* $A$ *and the threshold* $\underline{\vartheta}_1^*$ *are obtained from the value-matching and smooth-fit property of* $v_1(\underline{\vartheta}_1^*)$*. These boundary conditions give the explicit expressions*

$$\underline{\vartheta}_1^* = \left[\frac{\beta_1^1}{\beta_1^1 - \gamma}\frac{1}{K_2 - K_1}\frac{g_{12}}{\eta}\right]^{\frac{1}{\gamma}}, \tag{3.13}$$

$$A = \frac{\eta\gamma}{\beta_1^1}(K_2 - K_1)(\underline{\vartheta}_1^*)^{\gamma-\beta_1^1}. \tag{3.14}$$

Proposition 3.2.2 states that the optimal investment strategy of the firm is to invest in the new product if the value of the stochastic process $\theta_1(t)$ is higher than a certain threshold, i.e. $\vartheta \geq \underline{\vartheta}_1^*$. Otherwise, the firm continues to produce the initial product, and holds on to the the option to invest at some later time, which is done as soon as $\theta_1(t)$ reaches $\underline{\vartheta}_1^*$. For any current value $\vartheta$ of the stochastic process $\theta(t)$, the value of the firm is given by Eq. (3.12). If $\vartheta < \underline{\vartheta}_1^*$, the firm's value is given as the value of producing in regime $i = 1$ forever, plus an option value from the opportunity to switch. If $\vartheta \geq \underline{\vartheta}_1^*$, the firm switches to regime $i = 2$ immediately, and the value of the firm is given as the expected discounted cash flows from producing in regime $i = 2$, minus the investment cost.

## 3.3 Comparative statics

This section presents a comparative static analysis on the optimal investment threshold $\underline{\vartheta}_1^*$ with respect to the drift and volatility of each regime. Throughout this section we assume that the conditions in Proposition 3.2.1 for the existence of a finite switching threshold hold.

**Proposition 3.3.1.** *The optimal switching threshold $\underline{\vartheta}_1^*$ is decreasing in $\alpha_2$ and increasing in $\alpha_1$.*

Intuitively, if the positive change in drift from switching product is increasing (decreasing), the firm has more (less) incentive to switch, and thus the threshold where the firm would optimally pay the investment cost to obtain a more valuable profit flow decreases (increases). The following proposition presents the effect of uncertainty on the optimal investment threshold.

**Proposition 3.3.2.**

*(i) The optimal switching threshold $\underline{\vartheta}_1^*$ is strictly increasing in $\sigma_2$. It is increasing in $\sigma_1$ if the following condition holds,*

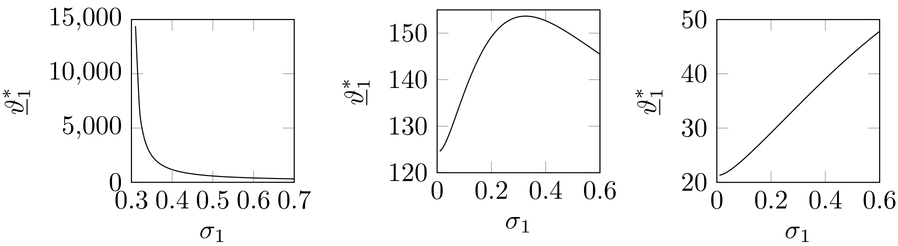
$$\frac{\beta_1^1 - 1}{\beta_1^1 - \gamma} \times \frac{1}{\frac{1}{2}\sigma_1^2(2\beta_1^1 - 1) + \alpha_1} - (1 - \gamma)\frac{K_1^2}{K_2 - K_1} > 0, \tag{3.15}$$

*and decreasing otherwise.*

*(ii) If $\sigma_2 = \sigma_1 = \sigma$, then the optimal switching threshold $\underline{\vartheta}_1^*$ is increasing in $\sigma$.*

We find that the investment threshold is always increasing in the post-investment volatility $\sigma_2$. If the volatility in the new market increases, the firm has less incentive to switch. Therefore, the firm demands a higher expected profit to switch to the new market and the threshold increases, which is a standard real options result. This result is similar to that of Alvarez and Stenbacka (2003) for a concave profit function. For the effect of uncertainty on investment timing, they point to the interplay of two opposing effects: (1) *the real options effect*, and (2) *the Jensen's inequality effect*[7]. Since increased post-investment uncertainty will decrease the expected cumulative profits after investment under a concave profit function (from Jensen's inequality), it will increase the value of waiting (the real options effect), and hence delay investment.

[7] From Jacod and Protter (2004): Jensen's inequality states that if $X$ is a random variable and $\phi$ is a convex (resp. concave) function, then $\phi(\mathbb{E}(X)) \leq$ (resp. $\geq$ ) $\mathbb{E}(\phi(X))$




(a) $\alpha_2 = \alpha_1 = 0.02$

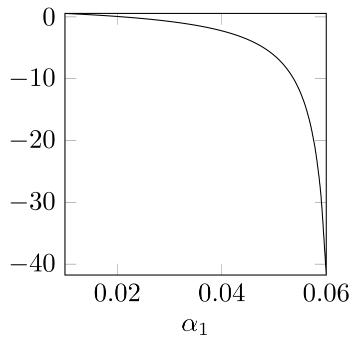
(b) $\alpha_2 = 0.035$

(c) $\alpha_2 = 0.06$

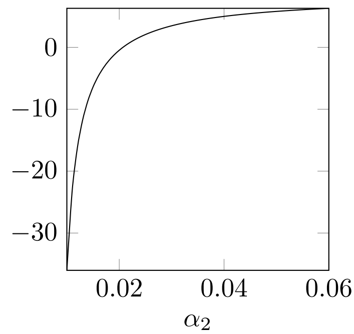
Figure 3.2: Effect of initial uncertainty $\sigma_1$ on investment threshold $\underline{\vartheta}_1^*$ for three different values of $\alpha_2$, capturing different levels of the change in drift from switching. General parameter values: $\alpha_1 = 0.02$, $r = 0.1$, $\sigma_2 = 0.3$, $g_{12} = 100$, $\gamma = 0.9$.

However, the effect of the initial volatility $\sigma_1$ is ambiguous, and a higher volatility can both accelerate or delay investment. The condition in Eq. (3.15) is rather involved, as it depends on most of the underlying parameters. Changing the initial volatility affects both the value of the option of switching and the value of producing in the initial regime in perpetuity, relative to the new regime. We refer to these two effects as the *option effect* and the *perpetuity effect*. An increase in the initial volatility would through the option effect increase the value of waiting, which increases the threshold. Reversely, it is also decreasing the attractiveness of the initial regime relative to the second regime through the perpetuity effect, which increases the incentive of switching and lowers the threshold. Figure 3.2 plots the threshold $\underline{\vartheta}_1^*$ as a function of pre-investment volatility $\sigma_1$ for three different levels of $\alpha_2$, showing that for different values of $\alpha_2$ the effect of initial volatility on the investment threshold can be various. In Figure 3.2c, we see that increasing $\sigma_1$ can delay investment if the change in drift is relatively large. Figure 3.2b shows a non-monotonic behaviour for small levels of $\alpha_2$ relative to $\alpha_1$, so that marginally increasing $\sigma_1$ can delay investment for relative low levels of uncertainty, and accelerate investment for higher levels. In the case where the drift of the two regimes is equal, Figure 3.2a shows that increased initial volatility decreases the investment threshold monotonically, suggesting that the adverse change in value of staying in the initial regime is larger than the increased option value of switching. However, in Figure 3.2a we see that the investment threshold is very large when $\sigma_1$ is close to $\sigma_2$. In this case, the benefit of switching to the new product is small. When the change in volatility from switching becomes larger, the threshold drastically decreases in value, as the firm would get a larger benefit from switching to the second regime with lower volatility. In Alvarez and Stenbacka (2003), where the drift is equal for the two regimes, they also find that the effect of the initial uncertainty is unclear, but state that their numerical results suggest that the threshold should decrease in the initial uncertainty. Our analysis confirms the results of Alvarez and Stenbacka (2003). We show that also accounting for a potential change in drift leads to a more complex picture of the effect of uncertainty when the drift are changed from investing as well.

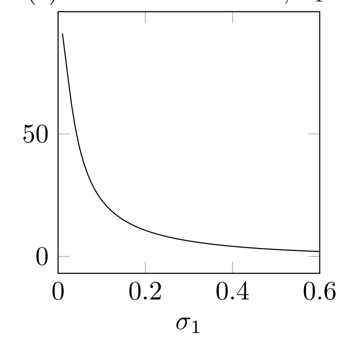
Investigating the antecedents of the two opposing effects, that is, the real options effect and the perpetuity effect, w.r.t. higher initial volatility, we look at how the left-hand side of Eq. (3.15) varies in the underlying parameters, *ceteris paribus*. In Figure 3.3, Eq. (3.15) is plotted as a function of the drift in each regime, the volatilities, the discount rate, and the level of concavity in the profit function. Note that we always assure that the conditions in Proposition 3.2.1 hold, so that a finite switching threshold exists. From Figure 3.3a and Figure 3.3b we find that the drift of the first and second regime have the opposite effect, as this represents changing the difference in drift obtained by switching regime, in each direction. An increase in the change of drift, i.e. $\alpha_2 - \alpha_1$, makes the value of staying in the first regime relatively lower than being in the second regime. From this, the perpetuity effect is relatively weaker and the option effect is more dominant. In Figure 3.3c we see that increased level of the initial uncertainty makes the marginal effect of a further increase in volatility smaller. Increasing the initial volatility strengthens both the option effect and the perpetuity effect. However, Figure 3.3c shows that increasing initial volatility would make the perpetuity effect more dominant. Economically, this suggests that for a higher level of uncertainty, a change in the initial uncertainty has a larger effect on the value of continuing to produce the first product, than on the value of waiting inherent in the option to invest in the new product. This explains the non-monotonic behaviour observed in Figure 3.2b. For the post-investment uncertainty, we see from Figure 3.3d that this also makes the perpetuity effect more dominant. Increasing the uncertainty in the second regime makes the attractiveness of the new product lower, and therefore weakens the option effect. The initial regime is not affected by the post-investment volatility, and therefore becomes relatively more attractive. Thus, increasing the initial uncertainty has more effect from the impact it has on the value of producing the initial product, the perpetuity effect, than on the value of waiting. In Figure 3.3e, we see that increasing the discount rate makes the perpetuity effect relatively more dominant compared to the options effect. A higher discount rate makes future profits less significant than profits today. Thus, the options effect from increased initial volatility is less dominant, as this represents the value of waiting from the option to invest in the new product in the future. For $\gamma$, which represents the degree of concavity of the profit function, Figure 3.3f shows that the perpetuity effect is less dominant when $\gamma$ increases. This is intuitive, as the initial uncertainty has less effect on the value of producing the first product for a lower degree of concavity, resulting from a higher $\gamma$, and therefore the option effect is more dominant.



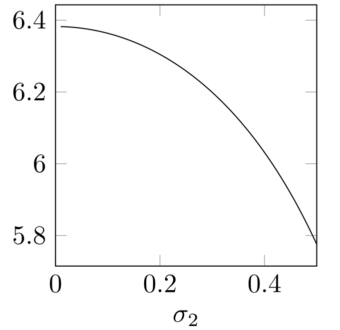

(a) Effect of initial drift, $\alpha_1$.



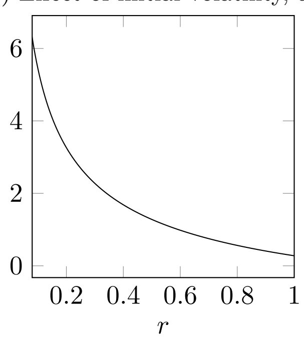

(b) Effect of second drift, $\alpha_2$.



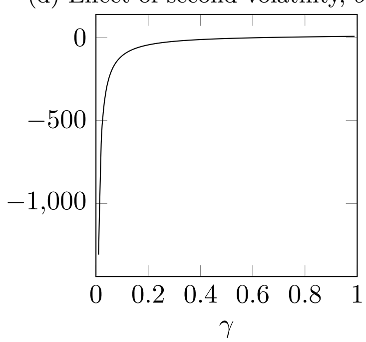

(c) Effect of initial volatility, $\sigma_1$.




(d) Effect of second volatility, $\sigma_2$.




(e) Effect of discount rate, $r$.




(f) Effect of concavity factor, $\gamma$.

Figure 3.3: Value of the left hand side of Eq. (3.15) as a function of the underlying parameters. General parameter values: $\alpha_1 = 0.01$, $\alpha_2 = 0.06$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $r = 0.08$, $\gamma = 0.9$. These plots show the dependence on the underlying parameters for the condition that increased initial uncertainty will increase the investment threshold. A value above (below) zero indicates that increased initial uncertainty increases (decreases) the investment threshold for the given parameters.

## 3.4 Improved profit flow and identical diffusion

Now assume that the initial profit flow is similar to before, but with different multiplicative constants $\eta_i$ so that $\pi_i(\theta(t)) = \eta_i \theta^\gamma(t)$ for the two regimes $i = 1, 2$. The profit flow in the second regime has a structural improvement, so that $\eta_2 > \eta_1$ in this regime. This can be seen as an upward boost in the profit flow from investing, as this yields that $\pi_1(\vartheta) = \eta_1 \vartheta^\gamma < \pi_2(\vartheta) = \eta_2 \vartheta^\gamma$. However, now we assume that the diffusion parameters are identical in the two regimes, i.e. $\alpha_1 = \alpha_2$ and $\sigma_1 = \sigma_2$. Proposition 3.4.1 presents the optimal switching strategy for a firm, given that the drift and volatility coefficients are constant and denoted by $\alpha_1$ and $\sigma_1$, respectively.

**Proposition 3.4.1.** *Given identical diffusions for the two regimes, but a structural multiplicative improvement of the profit flow in regime $i = 2$ over regime $i = 1$, the value of the firm is given as in Eq. (3.12) in Proposition 3.2.2, with constant $A$ and optimal switching threshold $\underline{\vartheta}_1^*$ given by*

$$\underline{\vartheta}_1^* = \left[ \frac{\beta_1^1}{\beta_1^1 - \gamma} \frac{g_{12}}{K_1(\eta_2 - \eta_1)} \right]^{\frac{1}{\gamma}}, \tag{3.16}$$

$$A = \frac{(\eta_2 - \eta_1)\gamma}{\beta_1^1} K_1 (\underline{\vartheta}_1^*)^{\gamma - \beta_1^1}. \tag{3.17}$$

From Proposition 3.4.1, we can see that an investment problem of investing to boost the profit function can be transformed to a problem of investing to change the coefficients of the stochastic process. We refer to the former problem as a *profit switching* problem, and to the latter as a *regime switching* problem. The following corollary denotes how one problem can be transformed into the other.

**Corollary 3.4.2.** *For a regime switching problem with $\pi_i(\theta(t)) = \eta_1 \theta_i^\gamma(t)$ and $\theta_i(t)$ defined as in Eq. (3.1) with $i = 1, 2$, the equivalent profit switching problem with $\pi_i(\theta(t)) = \eta_i \theta_1^\gamma(t)$, with $\eta_2 > \eta_1$ and $\theta_1(t)$ defined as in Eq. (3.1) with $i = 1$ is obtained by setting*

$$\frac{\eta_2}{\eta_1} = \frac{K_2}{K_1}, \tag{3.18}$$

*where $K_i$ is the perpetual multiplier in regime $i = 1, 2$ as defined by Eq. (3.9).*

Corollary 3.4.2 implies that a firm is indifferent between the opportunity of investing to change the coefficients of the stochastic regime and keeping the profit structure unchanged, and the opportunity to boost the profit structure without the ability to affect the stochastic process, as long as the problem parameters satisfy Eq. (3.18). This result shows that a problem of singular optimal switching with structurally identical profit flows and different diffusion parameters can be transformed to a problem with changing profit structure but identical coefficients of the stochastic process. Here structurally equal profit flows means that for any realized value of the stochastic process the profit flow is equal, i.e. $\pi_1(\vartheta) = \pi_2(\vartheta)$. This suggests that one could specify the effect of an investment as either changing the coefficients of the underlying stochastic process, or changing the form of the profit functions, and obtain the same results, as long as the earlier assumptions hold. These two different problem specifications have different reasoning. Therefore, the fact that these problem statements are connected could give important insight into similarities between different rationales of investing. More rigorous investigation should be performed on this connection in future work.

## 3.5 Conclusions

This paper studies the investment problem of a monopolistic firm subject to a concave profit function, for which the profits are dependent on a stochastic process, specified by a geometric Brownian motion. The firm is currently active in the market with an established product, with the opportunity to invest in a new product. The two products represents different regimes for the firm's stochastic environment, which we model by different coefficients for the drift and volatility of the geometric Brownian motion in each regime. We utilize the results of Ly Vath and Pham (2007) to solve the problem, using as a viscosity solution approach.

We find that increased uncertainty in the second market always delays investment, but that increasing the uncertainty of the initial market can both delay or speed up the investment, dependent on the change in the drift it obtains in the new market relative to the current one. Further, the effect of initial uncertainty is shown to be dependent on two effects: (1) the option effect and (2) the perpetuity effect, which arises from the concave profit function of the firm. We study how the drift and volatility in the two regimes, the discount rate, and the degree of concavity in the profit function influence these two effects, and give economic reasoning for which effect dominates in each case. We confirm the results of Alvarez and Stenbacka (2003), while also showing that the dependence of the optimal investment strategy on the initial uncertainty is more multifaceted when we account for a concurrent change in drift as well as in volatility. Alvarez and Stenbacka (2003) remark that the non-monotonic effect of uncertainty might explain why empirical studies that assume the drift and volatility to be constant have difficulties assessing empirically if the relationship between uncertainty and investment is positive or negative.

A possible extension of this work is to account for strategic interactions in the investment problem. For an exogenous switching problem, Bensoussan et al. (2017) study the case of a duopoly game of Stackelberg competition. In the endogenous switching option that we study in this paper, the decisions of the firms in the game to invest might also affect the regime for the other firms in the market.

This paper is an initial effort of extending the study of investment problems under endogenous effects on the stochastic process, similar to the problems studied in Kwon (2010), Hagspiel et al. (2016), and Matomäki (2013, Article 1). Including decision-dependent changes in the stochastic process yields novel results that partly contradict the well established results of earlier work on investment under uncertainty, and give exciting new possibilities for future research. Especially looking into the connection between changes in the stochastic process and changes in the profit functions is promising. Our results indicate that these two classes of problems for a singular investment decision can be transformed into one another. If this result holds in general, it might give further insight into the reasoning for investments under changing profit regimes.

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# 3.A Proofs

## 3.A.1 Proof of Proposition 3.2.1

Defining the perpetual discount factor $K_i$ as Eq. (3.9), Theorem 4.1 in Ly Vath and Pham (2007) states that the condition for the existence of a finite investment threshold in a two-regime case depends on the sign of $K_2 - K_1$, and the sign of the switching costs. Part (i) follows from Theorem 4.1 (1) in Ly Vath and Pham (2007) directly, while part (ii)–(iv) are found by solving for $K_2 > K_1$. □

## 3.A.2 Proof of Proposition 3.2.2

Noting that the profit functions in each regime are structurally equal, i.e. $\pi_1(\vartheta) = \pi_2(\vartheta) = \eta\vartheta^\gamma$, and satisfying Hölder continuity and linear growth ($\gamma \in (0, 1]$), the assumptions of Ly Vath and Pham (2007) are fulfilled. Thus, the value function are directly found by Theorem 4.1 in Ly Vath and Pham (2007), as $g_{12}, g_{21} \geq 0$ (i.e. there is no salvage value). The constant $A$ and threshold $\underline{\vartheta}_1^*$ is found by direct derivations of the continuity and smoothness of the value function at the threshold. □

## 3.A.3 Proof of Proposition 3.3.1

Given that the threshold $\underline{\vartheta}_1^*$ is given by Eq. (3.13), the partial derivatives w.r.t. $\alpha_2$ and $\alpha_1$ are given by

$$\frac{\partial \underline{\vartheta}_1^*}{\partial \alpha_2} = -\frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12}}{\eta} \times \left(\frac{K_1}{K_2 - K_1}\right)^2 \times \frac{\beta_1^1}{\beta_1^1 - \gamma} < 0, \tag{3.19}$$

$$\frac{\partial \underline{\vartheta}_1^*}{\partial \alpha_1} = \frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12} \beta_1^1}{(K_2 - K_1)\eta} \times \left[\frac{1}{\frac{1}{2}\sigma_1^2(2\beta_1^1 - 1) + \alpha_1} + \frac{1}{\beta_1^1 - \gamma}\frac{K_1^2}{K_2 - K_1}\right] > 0, \tag{3.20}$$

which always hold as long as the conditions for existence in Proposition 3.2.1 are fulfilled, i.e. $K_2 > K_1$ and $g_{12} > 0$. □

### 3.A.4 Proof of Proposition 3.3.2

(i) Given that the threshold $\underline{\vartheta}_1^*$ is given by Eq. (3.13), the partial derivatives w.r.t. $\sigma_2$ and $\sigma_1$ are given by

$$\frac{\partial \underline{\vartheta}_1^*}{\partial \sigma_2} = \frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12}(1-\gamma)\sigma_2}{\eta} \times \left(\frac{K_1}{K_2 - K_1}\right)^2 \times \frac{\beta_1^1}{\beta_1^1 - \gamma} > 0 \tag{3.21}$$

$$\frac{\partial \underline{\vartheta}_1^*}{\partial \sigma_1} = \frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12}\sigma_1\beta_1^1}{(K_2 - K_1)\eta(\beta_1^1 - \gamma)} \times \left[\frac{\beta_1^1 - 1}{\beta_1^1 - \gamma} \times \frac{1}{\frac{1}{2}\sigma_1^2(2\beta_1^1 - 1) + \alpha_1} - \frac{(1-\gamma)K_1^2}{K_2 - K_1}\right]. \tag{3.22}$$

Equation (3.15) arises from the bracketed term in Eq. (3.22), which determines the sign of the derivative.

(ii) Given that the threshold $\underline{\vartheta}_1^*$ is given by Eq. (3.13), and $\sigma_1 = \sigma_2 = \sigma$ the partial derivative w.r.t. $\sigma$ is given by

$$\begin{aligned} \frac{\partial \underline{\vartheta}_1^*}{\partial \sigma} = \frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12}\sigma\beta_1^1}{(K_2 - K_1)\eta(\beta_1^1 - \gamma)} \times \Bigg[&(1-\gamma)(K_2 + K_1) \\ &+ \frac{\beta_1^1 - 1}{\beta_1^1 - \gamma} \times \frac{1}{\frac{1}{2}\sigma^2(2\beta_1^1 - 1) + \alpha_1}\Bigg] > 0, \end{aligned} \tag{3.23}$$

which always hold as long as the conditions for existence in Proposition 3.2.1 are fulfilled.

□

### 3.A.5 Proof of Proposition 3.4.1

Define the profit functions for each regime to be $\pi_i(\theta(t)) = \eta_i\theta(t)$, $i = 1, 2$, with $\eta_2 > \eta_1$, and that the stochastic process following a geometric Brownian motion has constant parameters, $\alpha$ and $\sigma$. Then the expected discounted value of producing in perpetuity in a regime $i$, for any realized value $\vartheta$ of the process $\theta(t)$, is given by Eq. (3.8) and given as $\hat{V}_i(\vartheta) = K\eta_i\vartheta^\gamma$, with $K$ defined as in Eq. (3.9). In the stopping region, the value of the firm is given by $\hat{V}_2(\vartheta)$, while the value in the continuation region is given by the Bellman equation

$$rV(\vartheta)dt = \eta_1\vartheta^\gamma dt + \mathbb{E}[dV(\vartheta)]. \tag{3.24}$$

Expanding the expectation using Itô's Lemma, we obtain the differential equation

$$\frac{1}{2}\sigma^2\vartheta^2\frac{\partial^2 V(\vartheta)}{\partial\vartheta^2}+\alpha\vartheta\frac{\partial V(\vartheta)}{\partial\vartheta}-rV(\vartheta)+\eta_1\vartheta^\gamma=0. \tag{3.25}$$

The particular solution to this second-order differential equation is given by $V^P(\vartheta) = \hat{V}_1(\vartheta)$, while the homogeneous part (excluding the last term) is given by $V^H(\vartheta) = A_1\vartheta^{\beta_1} + A_2\vartheta^{\beta_2}$, where $\beta_1$ and $\beta_2$ are the positive and negative solution to the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha\beta-r=0$. The boundary condition that $V^H(\vartheta \to 0^+) = 0$ dictates that $A_2 = 0$. Given that the firm is currently in regime $i = 1$ and there is a switching cost $g_{12}$ to switch to regime $i = 2$, the value of the firm is given as in Eq. (3.12). Noting that the value function $v(\vartheta)$ should be continuous and smooth at the investment threshold $\vartheta = \underline{\vartheta}_1^*$ (Dixit and Pindyck, 1994; Alvarez and Stenbacka, 2003), we obtain the stated values of the constant $A_1$ and threshold $\underline{\vartheta}_1^*$ by direct derivation. □

### 3.A.6 Proof of Corollary 3.4.2

We compare the solution for the regime switching problem given in Proposition 3.2.2 and the solution for the profit switching problem given in Proposition 3.4.1. Equate the expressions for the investment threshold in each problem, given for the regime switching problem by Eq. (3.13), with $\eta = \eta_1$, and for the profit switching problem by Eq. (3.16). By straightforward rearrangement, we find that the stated condition in Eq. (3.18) makes the investment thresholds equal. Next, comparing the constant term for the solution of each problem, given by Eq. (3.14) and Eq. (3.17), we find directly by substitution that these are equal when Eq. (3.18) holds. □